

SOCIAL OPTIMA IN ECONOMIES WITH HETEROGENEOUS AGENTS
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Macro and beyond

MOTIVATION AND MATHEMATICAL FRAMEWORK

- *Continuous-time framework* to compute the *social optimum* in models with *heterogeneous* agents subject to *idiosyncratic shocks*.
- Reduce the problem to a *MFG*.
 - ▶ *HJB* coupled with a *KF*.
 - ▶ Market clearing conditions
 - ▶ Incomplete markets \Rightarrow restrictions on the control set.
- Mathematical ingredients:
 - ① *Hilbert space* (\approx scalar product) to compute infinite dimensional *Gateaux derivatives* (\approx directional derivative).
 - ② *Duality* approach to reduce the SP problem to a competitive problem through the use of a *Lagrange* multiplier (captures the externality)
- Use sparse matrices to compute the equilibrium (Achdou et al.(2017)).

ECONOMY: INDIVIDUAL PROBLEM

- Deterministic (no shocks!) aggregate variable $Z_t \in \mathbb{R}^p$
- Continuum of agent $i \in [0, 1]$ with constant death (and birth) rate η .
 - ▶ Controls $\mu(t, X_t^i) \in \mathbb{R}^m$
- Individual state $X_t^i \in \mathbb{R}^n$

$$dX_t^i = b\left(X_t^i, \mu\left(t, X_t^i\right), Z_t\right) dt + \sigma\left(X_t^i\right) dB_t^i \quad (1)$$

- Admissible set of control $\mu(t, x) \in \mathcal{M}(t, x)$ that solves uniquely (1)
 - ▶ Encompasses restriction on market completeness for example.

ECONOMY: MFG FORMULATION

- Value function $V(t, x)$ maximizes utility under (1):

$$V(t, x) = \max_{\mu(t, X_s) \in \mathcal{M}(t, X_s)} \mathbb{E}_t \left[\int_t^\infty e^{-(\rho+\eta)(s-t)} u(X_s, \mu) ds \middle| X_t = x \right]$$

- ▶ Transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} V(t, x) = 0$$

- Individual's problem summarized by the HJB:

$$\rho V(t, x) = \frac{\partial V(t, x)}{\partial t} + \max_{\mu(t, x) \in \mathcal{M}(t, x)} \{u(x, \mu(x, t)) + \mathcal{A}V\} \quad (2)$$

- ▶ *Infinitesimal generator* of the process (1):

$$\frac{\partial}{\partial t} + \mathcal{A} = \frac{\partial}{\partial t} + \sum_{i=1}^n b_i(x, \mu, Z) \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n [\sigma(x) \sigma(x)']_{i,k} \frac{\partial^2 V}{\partial x_i \partial x_k} - \eta V$$

ECONOMY: MFG FORMULATION

- Random initial state $x_0 \sim \psi(x)$
- Distribution $g(t, x)$ solves the KF equation:

$$\begin{cases} \frac{\partial g}{\partial t} &= \mathcal{A}^* g + \eta \psi \\ \int g(t, x) dx &= 1 \\ g(0, t) &= g_0(x) \end{cases} \quad (3)$$

- \mathcal{A}^* is the adjoint (\approx transpose through integration by part) of \mathcal{A} :

$$\mathcal{A}^* = \sum_{i=1}^n \frac{\partial}{\partial x_i} [b_i(x, \mu, Z) g(t, x)] + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n \frac{\partial^2}{\partial x_i \partial x_k} \left[[\sigma(x) \sigma(x)']_{i,k} g(t, x) \right] - \eta g$$

- Market clearing:

$$\forall k \in \{1, \dots, p\}, Z_k(t) = \int f_k(x, \mu) g(t, x) dx \quad (4)$$

COMPETITIVE EQUILIBRIUM

- A **competitive equilibrium** is a vector of *aggregate variables* $Z(t)$, a *value function* $V(t, x)$, a *control* $\mu(t, x)$ and a *density* $g(t, x)$ such that:
 - ① Given $Z(t)$, $V(t, x)$ is the solution of the HJB equation (2) and the optimal control is $\mu(t, x)$.
 - ② Given $\mu(t, x)$ and $Z(t)$, $g(t, x)$ is the solution to the KF system (3)
 - ③ Given $\mu(t, x)$ and $g(t, x)$, the aggregate variables $Z(t)$ satisfy the market clearing conditions (4)
- The recursive structure is helpful to solve numerically:

$$\{Z(t)\} \xRightarrow{(2)} \{Z(t), \mu(t, x)\} \xRightarrow{(3)} \{g(t, x), \mu(t, x)\} \xRightarrow{(4)} \{Z(t)\}$$

SOCIAL PLANNER (SP) PROBLEM

- Infinitely lived SP maximizes a Social Welfare Function (SWF) with Pareto weights $\omega(t, x)$ to get a value function $J(g_0(x))$:

$$\begin{aligned} \max_{g(t,x), Z(t), \mu(t,x) \in \mathcal{M}(t,x)} & \int_{t=0}^{\infty} e^{-\rho t} \left[\int_{x \in \mathcal{X}} \omega(t, x) u(x, \mu) g(t, x) dx \right] dt \\ \text{s.t } \rho V(t, x) &= \frac{\partial V(t, x)}{\partial t} + \max_{\mu(t,x) \in \mathcal{M}(t,x)} \{u(x, \mu(x, t)) + \mathcal{A}V\} \quad (2) \end{aligned}$$

$$\begin{cases} \frac{\partial g}{\partial t} &= \mathcal{A}^* g + \eta \psi[j(t, x)] \\ \int g(t, x) dx &= 1 \\ g(0, t) &= g_0(x) \end{cases} \quad (3)$$

$$\forall k \in \{1, \dots, p\}, Z_k(t) = \int f_k(x, \mu) g(t, x) dx \quad (4) [\lambda_k(t)]$$

NECESSARY CONDITIONS: FROM SP TO COMPETITIVE

- After a lot of Gateaux derivatives... The main result!

PROPOSITION 1: NECESSARY CONDITIONS

- If a solution to the SP problem exists with $(e^{-\rho t}g, e^{-\rho t}\mu, e^{-\rho t}j) \in L^2 \Rightarrow \exists$ Competitive equilibrium (HJB, complementary slackness, SP objective from HJB):

$$\rho j(t, x) = \max_{\mu(t, x) \in \mathcal{M}(t, x)} \omega(t, x) u(x, \mu) + \sum_{k=1}^p \lambda_k(t) [f_k(x, \mu) - Z_k] + \mathcal{A}j(t, x) + \frac{\partial j(t, x)}{\partial t}$$

$$\forall k \in \{1, \dots, p\} \lambda_k(t) = \sum_{i=1}^n \int \frac{\partial j}{\partial x_i} \frac{\partial b_i}{\partial Z_k} g(t, x) dx$$

$$J(g(0, x)) = \int j(0, x) g(0, x) dx + \eta \int_{t=0}^{\infty} \left[\int e^{-\rho t} j(t, x) \psi(x) dx \right] dt$$

FINITE DIFFERENCE SCHEME: AIYAGARI EXAMPLE

- Discrete grids of asset a (x before) and idiosyncratic productivity z (B before):

$$(a, z) \in \{a_1, a_2 + \Delta a, \dots, a_I\} \times \{z_1, z_2 + \Delta z, \dots, z_J\}$$

- Planner value function (j before) is $V_{ij} \equiv V(a_i, z_j)$
- Saving function s_{ij}

$$s_{i,j} = wz_j + (r + \eta) a_i - c_{ij}$$

- Define *forward* and *backward* derivatives in asset a :

$$\partial_{a,F} V_{ij} = \frac{V_{i+1,j} - V_{i,j}}{\Delta a}$$

$$\partial_{a,B} V_{ij} = \frac{V_{i,j} - V_{i-1,j}}{\Delta a}$$

- Forward derivative for idiosyncratic productivity z :

$$\partial_z V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{\Delta z}$$

$$\partial_{zz} V_{i,j} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta z)^2}$$

UPWIND SCHEME

- The utility, including the multiplier is $u(c) = \frac{c^{1-\gamma}}{1-\gamma} + \lambda(a - K)$
- The discretized HJB is:

$$\begin{aligned} \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = & u(c_{i,j}^n) + \partial_{a,F} V_{i,j}^{n+1} s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n > 0} + \partial_{a,B} V_{i,j}^{n+1} s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n < 0} \\ & + \theta(\hat{z} - z_j) \partial_z V_{i,j}^{n+1} + \frac{\sigma_z^2 z_j}{2} \partial_{zz} V_{i,j}^{n+1} - \eta V_{i,j}^{n+1} \end{aligned}$$

- With savings given by:

$$\begin{aligned} s_{ij,F}^n &= wz_j + (r + \eta)a_i - (u')^{-1} \left(\partial_{a,F} V_{i,j}^n \right) \\ s_{ij,B}^n &= wz_j + (r + \eta)a_i - (u')^{-1} \left(\partial_{a,B} V_{i,j}^n \right) \end{aligned}$$

SOLVING THE HJB: SPARSE MATRICES

- Can rewrite the previous HJB as:

$$\frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta} + \rho \mathbf{V}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{V}^{n+1}$$

- Rearranging:

$$\left[\left(\frac{1}{\Delta} + \rho \right) \mathbf{I} - \mathbf{A}^n \right] \mathbf{V}^{n+1} = \mathbf{u}^n + \frac{\mathbf{V}^n}{\Delta}$$

- Algorithm:

▶ Starts with $V_{i,j}^0 = \frac{u(r a_i + w z_j)}{\rho}$

① Compute $\partial_{a,F} V_{ij}$ $\partial_{a,B} V_{ij}$ $\partial_z V_{i,j}$ $\partial_{zz} V_{i,j}$ using finite diff.

② Compute $c_{i,j}^n$ using:

$$c_{i,j}^n = (u')^{-1} \left(\partial_{a,F} V_{i,j}^n \mathbf{1}_{s_{i,j,F}^n > 0} + \partial_{a,B} V_{i,j}^n \mathbf{1}_{s_{i,j,B}^n < 0} + u' (w z_j + r a_i) \mathbf{1}_{s_{i,j,F}^n < 0, s_{i,j,B}^n > 0} \right)$$

③ Solve the previous system to find \mathbf{V}^{n+1}

④ Iterates until $|\mathbf{V}^{n+1} - \mathbf{V}^n| < \epsilon$

SOLVING THE KF AND FINDING THE AGGREGATE CAPITAL

- Discretized KF can be written taking advantage of the self adjoint property as:

$$\mathbf{A}' \mathbf{g} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\lim_{n \rightarrow \infty} \mathbf{A}^n = \mathbf{A}$$

- ▶ We solve the system and renormalize:

$$g_{ij} = \frac{g_{ij}}{\sum_{i=1}^I \sum_{j=1}^J g_{ij} \Delta a \Delta z}$$

- Finding the equilibrium aggregate capital is the standard Aiyagari algorithm.

FINDING THE LAGRANGE MULTIPLIER

- Choose a constant $\vartheta \in (0, 1)$, begin with an initial guess of the Lagrange multiplier $\lambda^0 = 0$, set $m = 0$:
- ① Compute the value function V^m , consumption C^m , density g^m and aggregate capital K^m given λ^m .
- ② Update $\tilde{\lambda}^{m+1}$:

$$\tilde{\lambda}^{m+1} = \frac{\alpha(1-\alpha)}{(K^m)^{2-\alpha}} \sum_{i=1}^I \sum_{j=1}^J \left[g_{i,j}^m + a_i \frac{g_{i+1,j}^m - g_{i,j}^m}{\Delta a} - K^m z_j \frac{g_{i+1,j}^m - g_{i,j}^m}{\Delta a} \right] V_{i,j}^m \Delta a \Delta z$$

- ③ If $\tilde{\lambda}^{m+1} \neq \lambda^m$, set $\lambda^{m+1} = \vartheta \tilde{\lambda}^{m+1} + (1 - \vartheta) \lambda^m$
- ④ If $|\lambda^{m+1} - \lambda^m| > \epsilon$, keep iterating

AIYAGARI: FINDING AGGREGATE CAPITAL

- Choose $\nu \in (0, 1)$, and K^0
- ① Compute r^n and w^n as a function of K^n
- ② Given r^n and w^n solve the HJB numerically to get V^n and c^n
- ③ Given c^n solves the KF to get g^n
- ④ Compute aggregate capital $S^n = \sum_{(i,j)} a_i g_{ij} \Delta a \Delta z$
- ⑤ Update $K^{n+1} = \nu S^n + (1 - \nu) K^n$
- ⑥ f $|K^{n+1} - K^n| > \epsilon$, keep iterating

- Choose as Hilbert *space* the space of Lebesgue-integrable functions L^2 with the inner product:

$$\langle f, g \rangle = \int f(x) g(x) dx$$

- Let's T an operator, its *adjoint* T^* is defined by:

$$\langle Tf, g \rangle = \langle f, T^*g \rangle$$

► An operator is *self-adjoint* if $T = T^*$

- Let $J(g)$ be a functional and let h be arbitrary in L^2 . The *Gateaux derivative* of J in the direction h is defined as:

$$\delta J(g; h) = \lim_{\alpha \rightarrow 0} \frac{J(g + \alpha h) - J(g)}{\alpha}$$

- Let $H : L^2 \mapsto \mathbb{R}^n$.
- If J has a continuous Fréchet differential, a necessary condition for J to have an extremum at g under the constraint $H(g) = 0$ is:

$$\exists \lambda \in L^2 \Rightarrow \begin{cases} \delta \mathcal{L}(g; h) &= 0 \\ \mathcal{L}(g) &= J(g) + \langle \lambda, H(g) \rangle \end{cases}$$