Nested CES Framework

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Overview

- Why nested CES
- Solution concepts
- Distortions
- Computing GE

Why

Advantages:

- Bassboosted industry equilibria
- Variable substitution elasticities
- Variable markups

Disadvantages:

- No closed-form solutions
- Strategic interaction at all levels

Applications

Nests can represent

- Industries (Atkeson & Burstein 2008, Edmond, Midrigan & Xu 2015)
- Geographical locations (Berger, Herkenhoff & Mongey 2019)
- Multiproduct firms (Midrigan 2011, Alvarez & Lippi 2014)
- Location-industry pairs
- Time periods

Solution Concepts

At two levels, need to choose:

- Cournot (treat others' prices as constant)
- Bertrand (treat others' output as constant)

Extensions:

- Stackelberg in any of those
- Multi-product firms within nests
- Multi-nest firms

Static Setup

Consumption aggregator:

$$C^{\frac{\eta-1}{\eta}} = \int q_j^{\frac{\eta-1}{\eta}} dj \text{ where } q_j^{\frac{\theta_j-1}{\theta_j}} = \sum_{i=1}^{n_j} q_{ij}^{\frac{\theta_j-1}{\theta_j}} \tag{1}$$

Elasticities $\theta_j > \eta$ within and across nests

Production technology $q_{ij} = z_{ij} l_{ij}$ with one input (later fixed):

$$\max_{p_{ij}, q_{ij}} p_{ij} q_{ij} - \frac{w}{z_{ij}} q_{ij} \text{s.t. industry conditions} \tag{2}$$

Static Setup

From consumer maximization:

$$q_{ij} = p_{ij}^{-\theta_j} p_j^{\theta_j - \eta} P^{\eta} C \tag{3}$$

$$p_{ij} = q_{ij}^{-\frac{1}{\theta_j}} q_j^{\frac{1}{\theta_j} - \frac{1}{\eta}} P C^{\frac{1}{\eta}} \tag{4}$$

Price indices are as usually

$$p_j^{1-\theta_j} = \sum_{i=1}^{n_j} p_{ij}^{1-\theta_j} \text{ and } P^{1-\eta} = \int p_j^{1-\eta} dj \tag{5}$$

Either

- treat $\{p_{kj}\}_{k \neq i}$ as constant and use (3)
- treat $\{q_{kj}\}_{k \neq i}$ as constant and use (4)

Shares

Revenue shares within industries:

$$s_{ij} \propto p_{ij}^{1-\theta_j} \propto q_{ij}^{\frac{\theta_j - 1}{\theta_j}} \tag{6}$$

Markups:

$$\mu_{ij} = \frac{\epsilon_{ij}}{\epsilon_{ij} - 1} \tag{7}$$

Elasticities:

$$\epsilon_{ij}^{C} = \left[s_{ij} \frac{1}{\eta} + (1 - s_{ij}) \frac{1}{\theta_j} \right]^{-1} \tag{8}$$

$$\epsilon_{ij}^{B} = s_{ij}\eta + (1 - s_{ij})\theta_j \tag{9}$$

Examples

perfect substitutes $\theta_j = \infty$

- zero markup in Bertrand
- variable markup $\mu_{ij} = \eta/(\eta - s_{ij})$ in Cournot

unit elasticity $\eta = 1$

- infinite markup in Bertrand
- variable markup $\mu_{ij} = \theta_j/[(\theta_j - 1)(1 - s_{ij})]$ in Cournot

same elasticity $\theta_j = \eta$

- constant markup $\mu_{ij} = \eta/(\eta - 1)$ in both

Solving for the Shares

Use two facts:

- $p_{ij} = \mu_{ij}(s_{ij})MC_{ij}$
- $s_{ij} \propto p_{ij}^{1-\theta_j}$

In equilibrium we must have

$$\frac{s_{ij}}{s_{kj}} = \left(\frac{\mu_{kj}(s_{kj})}{\mu_{ij}(s_{ij})}\right)^{\theta_j-1}\left(\frac{MC_{kj}}{MC_{ij}}\right)^{\theta_j-1} = \left(\frac{\mu_{kj}(s_{kj})}{\mu_{ij}(s_{ij})}\right)^{\theta_j-1}\left(\frac{z_{ij}}{z_{kj}}\right)^{\theta_j-1} \tag{10}$$

Hence,

$$s_{kj} = \left(\frac{z_{kj}}{\mu_{kj}(s_{kj})}\right)^{\theta_j-1}\left[\sum_{i=1}^{n_j}\left(\frac{z_{ij}}{\mu_{ij}(s_{ij})}\right)^{\theta_j-1}\right]^{-1} \tag{11}$$

Recursive Algorithm

Set up the recursion:

$$\hat{s}_{kj}^{(n+1)} = \left(\frac{z_{kj}}{\mu_{kj}\left(s_{kj}^{(n)}\right)} \right)^{\theta_j - 1} \left[\sum_{i=1}^{n_j} \left(\frac{z_{ij}}{\mu_{ij}\left(s_{ij}^{(n)}\right)} \right)^{\theta_j - 1} \right]^{-1} \tag{12}$$

$$s_{kj}^{(n+1)} = x s_{kj}^{(n)} + (1-x)\hat{s}_{kj}^{(n+1)} \tag{13}$$

Use $x \in (0, 1)$ to dampen fluctuations

Good in practice (no idea why)

$$x = 1 - 2^{-\theta_j} \tag{14}$$

Price Leadership

Extend model to allow Stackelberg competition

Firm i leader, internalizes effect on other prices

Optimal markup has same form, leader's elasticity now

$$\tilde{\epsilon}_{ij} = \tilde{s}_{ij}\eta + (1 - \tilde{s}_{ij})\theta_j \tag{15}$$

$$\tilde{s}_{ij} = s_{ij}\left[1 + \sum_{k \neq i}^{n_j} \frac{(\theta_j - \eta)(\theta_j - 1)s_{kj}^2}{\epsilon_{kj}(\epsilon_{kj} - 1) + (\theta_j - \eta)(\theta_j - 1)s_{kj}(1 - s_{kj})}\right] \tag{16}$$

As though her share was higher, so markup increased

Optimal Concentration

Social planner allocates quantities subject to labor constraint

Relative output levels

$$\frac{q_{ij}}{q_{kj}} = \left(\frac{z_{ij}}{z_{kj}}\right)^{\frac{1}{\theta_j}} \tag{17}$$

For shares this implies

$$\frac{s^o_{ij}}{s^o_{kj}} = \left(\frac{z_{ij}}{z_{kj}}\right)^{\theta_j - 1} \tag{18}$$

Concentration

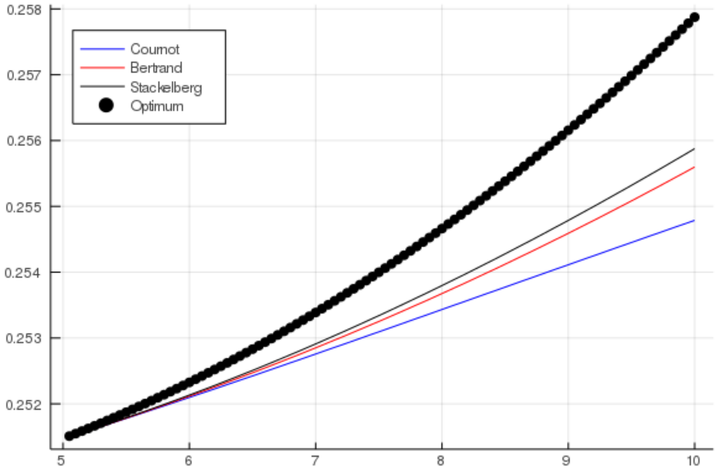


Figure: $\mathbf{z} = (1, 1.01, 1.02, 1.05)$

Concentration

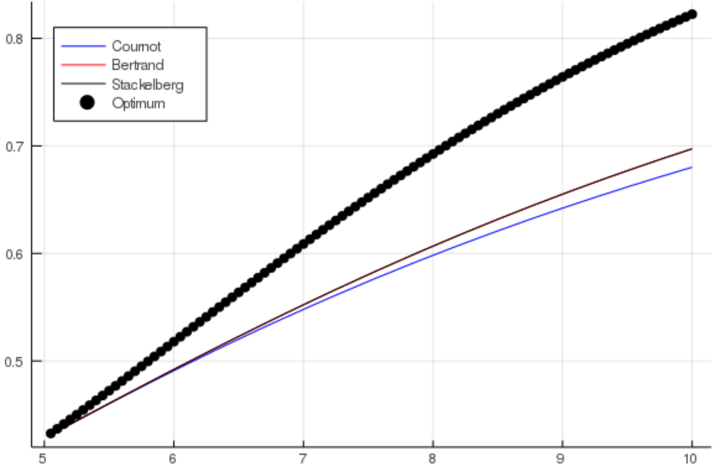


Figure: $\mathbf{z} = (1, 1.1, 1.2, 1.5)$

Concentration

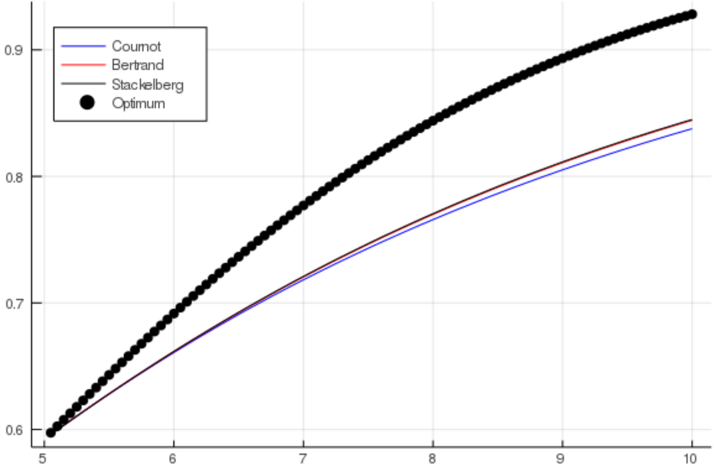


Figure: $\mathbf{z} = (1, 1.1, 1.5)$

Concentration

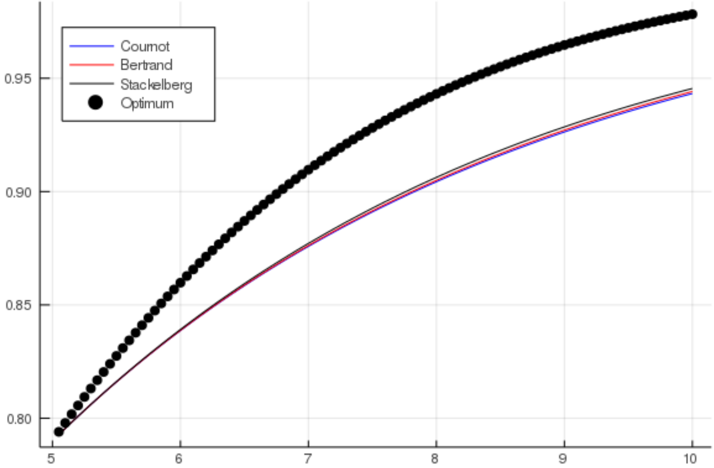


Figure: $\mathbf{z} = (1, 1.5)$

Distortions

Following Edmond, Midrigan & Xu (2015), define aggregate markup:

$$WL = \mathcal{M}^{-1} PC \tag{19}$$

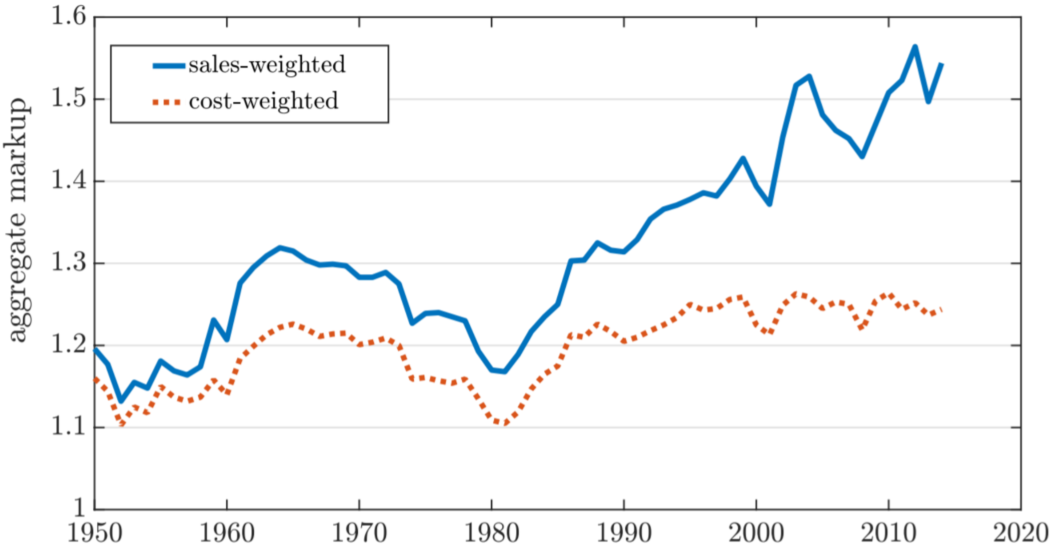
How to aggregate:

- **harmonic** average weighted with sales
- arithmetic average weighted with **costs**

In another paper they report $\mathcal{M} \sim \underline{1.25}$

Compare to sales-weighted arithmetic average of $\underline{1.6}$ in De Loecker & Eeckhout (2020)

Cost-weighted Markups



Distrortions

With CRS, industries under-concentrated:

- Highest markup firms are most productive
- Also face lowest (and decreasing elasticities)

Gains from correcting misallocation smaller than with CES:

- In CES high markup points to large misallocation
- In NCES planner runs into same diminishing returns

Intuition:

$$q_j = \left(\underbrace{q_{j-i}^{\frac{\theta_j - 1}{\theta_j}}}_{\text{intercept}} + q_{ij}^{\frac{\theta_j - 1}{\theta_j}} \right)^{\frac{\theta_j}{\theta_j - 1}} \tag{20}$$

Aggregate Markup

Why harmonic average

$$l_{ij} = \frac{1}{z_{ij}} p_{ij}^{-\theta_j} p_j^{\theta_j - \eta} P^{\eta} C = \frac{1}{z_{ij}} p_{ij}^{-\theta_j} p_j^{\theta_j - 1} \left(\frac{p_j}{P}\right)^{1-\eta} PC = \frac{s_{ij} s_j}{z_{ij} p_{ij}} PC = \frac{s_{ij} s_j}{\mu_{ij}} \frac{PC}{W} \tag{21}$$

$$L = \int \sum_{i=1}^{n_j} l_{ij} dj = \frac{PC}{W} \int \left(\sum_{i=1}^{n_j} s_{ij} \mu_{ij}^{-1} \right) s_j dj \tag{22}$$

Hence,

$$\mathcal{M} \cdot WL = PC \text{ where } \mathcal{M} = \left(\int \left(\sum_{i=1}^{n_j} s_{ij} \mu_{ij}^{-1} \right) s_j dj \right)^{-1} \tag{23}$$

With decreasing returns literally same up to multiplier

Aggregate Markup

Why weight with costs

$$\frac{\mu_{ij} \cdot Wl_{ij}}{\mathcal{M} \cdot WL} = \frac{p_{ij} q_{ij}}{PC} \tag{24}$$

Rearrange:

$$\mu_{ij} \frac{Wl_{ij}}{Wl_j} \frac{Wl_j}{WL} = \mathcal{M} \frac{p_{ij} q_{ij}}{p_j q_j} \frac{p_j q_j}{PC} \tag{25}$$

Integrate:

$$\int \left(\sum_{i=1}^{n_j} \mu_{ij} \bar{s}_{ij} \right) \bar{s}_j dj = \mathcal{M} \tag{26}$$

Here $\bar{s}_{ij}$ and $\bar{s}_j$ cost shares

Solving for GE

Consumption:

$$U(C, L) = \ln(C) - \frac{\chi L^{1+\varphi}}{1+\varphi} \tag{27}$$

Budget:

$$\int \left(\sum_{i=1}^{n_j} p_{ij} q_{ij} \right) dj = \int \left(\sum_{i=1}^{n_j} W l_{ij} \right) dj + \Pi \tag{28}$$

Production:

$$q_{ij} = z_{ij} l_{ij}^{\alpha} \tag{29}$$

Within Industries

The condition for shares:

$$\frac{s_{ij}}{s_{kj}} = \left(\frac{\mu_{kj}(s_{kj})}{\mu_{ij}(s_{ij})}\right)^{\frac{(\theta_j - 1)\alpha}{\alpha + (1-\alpha)\theta_j}} \left(\frac{z_{ij}}{z_{kj}}\right)^{\frac{\theta_j - 1}{\alpha + (1-\alpha)\theta_j}} \tag{30}$$

Can be solver for without aggregate quantities

Set up the recursion and iterate

Within Industries

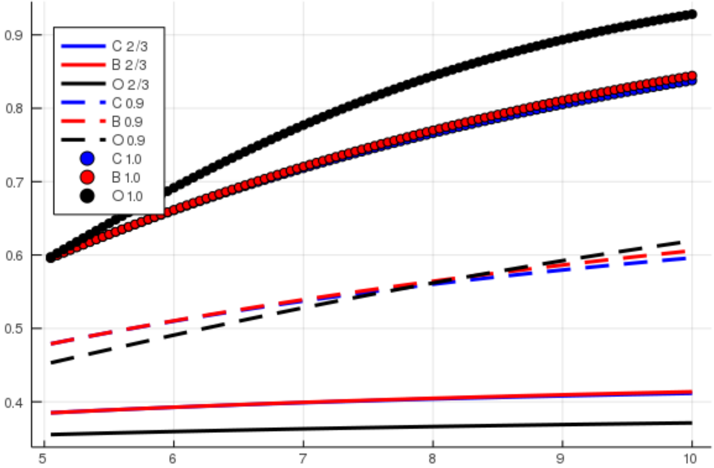


Figure: $\mathbf{z} = (1, 1.1, 1.5)$

Across Industries

Make use of the consumption-leisure tradeoff:

$$W = \chi L^{\varphi} P C \tag{31}$$

Start from q_{ij}:

$$q_{ij} = p_{ij}^{-\theta_j} p_j^{\theta_j - \eta} P^{\eta} C = p_{ij}^{-\eta} s_{ij}^{\frac{\theta_j - \eta}{\theta_j - 1}} P^{\eta} C = z_{ij}^{\frac{\eta}{\alpha}} \mu_{ij}^{-\eta} s_{ij}^{\frac{\theta_j - \eta}{\theta_j - 1}} \left(\frac{\alpha P}{W} \right)^{\eta} q_{ij}^{-\frac{(1-\alpha)\eta}{\alpha}} C \tag{32}$$

$$q_{ij} = \left(z_{ij}^{\frac{\eta}{\alpha}} \alpha^{\eta} \mu_{ij}^{-\eta} s_{ij}^{\frac{\theta_j - \eta}{\theta_j - 1}} \right)^{\frac{\alpha}{\alpha + (1-\alpha)\eta}} \left(\frac{P}{W} \right)^{\frac{\alpha\eta}{\alpha + (1-\alpha)\eta}} C^{\frac{\alpha}{\alpha + (1-\alpha)\eta}} = \xi_{ij} \left(\frac{P C^{\frac{1}{\eta}}}{W} \right)^{\frac{\alpha\eta}{\alpha + (1-\alpha)\eta}} \tag{33}$$

Integrating,

$$C = \xi \left(\frac{P}{W} \right)^{\frac{\alpha}{1-\alpha}} \tag{34}$$

Across Industries

Do the same for l_{ij} to obtain L:

$$l_{ij} = \left(z_{ij}^{\eta-1} \left(\frac{\alpha}{\mu_{ij}} \right)^{\eta} s_{ij}^{\frac{\theta_j-\eta}{\theta_j-1}} \right)^{\frac{1}{\alpha+(1-\alpha)\eta}} \left(\frac{PC^{\frac{1}{\eta}}}{W} \right)^{\frac{\eta}{\alpha+(1-\alpha)\eta}} = \zeta_{ij} \left(\frac{PC^{\frac{1}{\eta}}}{W} \right)^{\frac{\eta}{\alpha+(1-\alpha)\eta}} \tag{35}$$

Integrating,

$$L = \zeta \left(\frac{PC^{\frac{1}{\eta}}}{W} \right)^{\frac{\eta}{\alpha+(1-\alpha)\eta}} \tag{36}$$

Equilibrium

Now we have the system:

$$\omega = \chi L^{\varphi} C \tag{37}$$

$$C = \xi \omega^{\frac{\alpha}{\alpha-1}} \tag{38}$$

$$L = \zeta \left(C \omega^{-\eta} \right)^{\frac{1}{\alpha + (1-\alpha)\eta}} \tag{39}$$

Up to finitely many mistakes in arithmetics,

$$L = \left(\frac{\zeta}{\chi \xi} \right)^{\frac{1}{\varphi+1}} \text{ with } C = \xi^{\frac{1+\varphi-\alpha}{1+\varphi}} \chi^{\frac{\alpha}{1+\varphi}} \zeta^{-\frac{\varphi\alpha}{1+\varphi}} \text{ and } \omega = \zeta^{\frac{\varphi(1-\alpha)}{1+\varphi}} (\chi \xi)^{\frac{1-\alpha}{1+\varphi}} \tag{40}$$