Social optima in economies with heterogeneous agents
Nuño & Moll
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Leo Aparisi de Lannoy

Beyond macro

Motivation and mathematical framework

- *Continuous-time framework* to compute the *social optimum* in models with *heterogeneous* agents subject to *idiosyncratic shocks*.
- Reduce the problem to a *MFG*.
  - *HJB* coupled with a *KF*.
  - Market clearing conditions
  - Incomplete markets $\Rightarrow$ restrictions on the control set.
- Mathematical ingredients:
  1. *Hilbert space* ($\approx$ scalar product) to compute infinite dimensional *Gateaux derivatives* ($\approx$ directional derivative).
  2. *Duality* approach to reduce the SP problem to a competitive problem through the use of a *Lagrange* multiplier (captures the externality)
- Use sparse matrices to compute the equilibrium (Achdou et al.(2017)).

▸ Math

Economy: Individual problem

- Deterministic (no shocks!) aggregate variable $Z_t \in \mathbb{R}^p$
- Continuum of agent $i \in [0, 1]$ with constant death (and birth) rate η.
  - Controls $\mu\left(t, X_t^i\right) \in \mathbb{R}^m$
- Individual state $X_t^i \in \mathbb{R}^n$

$$dX_t^i = b\left(X_t^i, \mu\left(t, X_t^i\right), Z_t\right) dt + \sigma\left(X_t^i\right) dB_t^i \tag{1}$$

- Admissible set of control $\mu(t, x) \in \mathcal{M}(t, x)$ that solves uniquely (1)
  - Encompasses restriction on market completeness for example.

Economy: MFG formulation

- Value function $V(t,x)$ maximizes utility under (1):

$$V(t,x) = \max_{\mu(t,X_s)\in\mathcal{M}(t,X_s)} \mathbb{E}_t\left[\int_t^{\infty} e^{-(\rho+\eta)(s-t)} u(X_s,\mu)\,ds \middle| X_t = x\right]$$

  - Transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} V(t,x) = 0$$

- Individual's problem summarized by the HJB:

$$\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \max_{\mu(t,x)\in\mathcal{M}(t,x)} \left\{u(x,\mu(x,t)) + \mathcal{A}V\right\} \tag{2}$$

  - *Infinitesimal generator* of the process (1):

$$\frac{\partial}{\partial t} + \mathcal{A}V = \frac{\partial}{\partial t} + \sum_{i=1}^{n} b_i(x,\mu,Z)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\sigma(x)\sigma(x)'\right]_{i,k}\frac{\partial^2 V}{\partial x_i \partial x_k} - \eta V$$

Economy: MFG formulation

- Random initial state $x_0 \sim \psi(x)$
- Distribution $g(t,x)$ solves the KF equation:

$$\begin{cases} \frac{\partial g}{\partial t} & = \mathcal{A}^* g + \eta\psi \\ \int g(t,x)\,dx & = 1 \\ g(0,t) & = g_0(x) \end{cases} \tag{3}$$

- $\mathcal{A}^*$is the adjoint ($\approx$ transpose through integration by part) of $\mathcal{A}$:

$$\mathcal{A}^* g = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[b_i(x,\mu,Z)\,g(t,x)\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k}\left[\left[\sigma(x)\,\sigma(x)'\right]_{i,k} g(t,x)\right] - \eta g$$

- Market clearing:

$$\forall k \in \{1,\ldots,p\},\, Z_k(t) = \int f_k(x,\mu)\,g(t,x)\,dx \tag{4}$$

Competitive equilibrium

- A **competitive equilibrium** is a vector of *aggregate variables* $Z(t)$, a *value function* $V(t,x)$, a *control* $\mu(t,x)$ and a *density* $g(t,x)$ such that:
  1. Given $Z(t)$, $V(t,x)$ is the solution of the HJB equation (2) and the optimal control is $\mu(t,x)$.
  2. Given $\mu(t,x)$ and $Z(t)$, $g(t,x)$ is the solution to the KF system (3)
  3. Given $\mu(t,x)$ and $g(t,x)$, the aggregate variables $Z(t)$ satisfy the market clearing conditions (4)
- The recursive structure is helpful to solve numerically:

$$\{Z(t)\} \overset{(2)}{\Rightarrow} \{Z(t), \mu(t,x)\} \overset{(3)}{\Rightarrow} \{g(t,x), \mu(t,x)\} \overset{(4)}{\Rightarrow} \{Z(t)\}$$

Social planner (SP) problem

- Infinitely lived SP maximizes a Social Welfare Function (SWF) with Pareto weights $\omega(t,x)$ to get a value function $J(g_0(x))$:

$$\max_{g(t,x),Z(t),\mu(t,x)\in\mathcal{M}(t,x)} \int_{t=0}^{\infty} e^{-\rho t}\left[\int_{x\in\mathcal{X}} \omega(t,x)\,u(x,\mu)\,g(t,x)\,dx\right] dt$$

$$\text{s.t}\begin{cases}\frac{\partial g}{\partial t} & = \mathcal{A}^{*}g + \eta\psi\ [\boldsymbol{j}\,(\boldsymbol{t},\boldsymbol{x})] \\ \int g(t,x)\,dx & = 1 \\ g(0,t) & = g_0(x)\end{cases} \quad (3)$$

$$\forall k\in\{1,\dots,p\},\ Z_k(t) = \int f_k(x,\mu)\,g(t,x)\,dx\ (4)\ [\boldsymbol{\lambda_k}\,(\boldsymbol{t})]$$

Necessary conditions: from SP to Competitive

- After a lot of Gateaux derivatives... The main result!

Proposition 1: Necessary conditions

- If a solution to the SP problem exists with $(e^{-\rho t}g, e^{-\rho t}\mu, e^{-\rho t}j) \in L^2 \Rightarrow \exists$ Competitive equilibrium (HJB, complementary slackness, SP objective from HJB):

$$\rho j\left(t,x\right) = \max_{\mu(t,x)\in\mathcal{M}(t,x)} \omega\left(t,x\right) u\left(x,\mu\right) + \sum_{k=1}^{p} \lambda_k\left(t\right)\left[f_k\left(x,\mu\right) - Z_k\right] + \mathcal{A}j\left(t,x\right) + \frac{\partial j\left(t,x\right)}{\partial t}$$

$$\forall k \in \{1,\ldots,p\}\, \lambda_k\left(t\right) = \sum_{i=1}^{n} \int \frac{\partial j}{\partial x_i}\frac{\partial b_i}{\partial Z_k} g\left(t,x\right) dx$$

$$J\left(g\left(0,x\right)\right) = \int j\left(0,x\right) g\left(0,x\right) dx + \eta \int_{t=0}^{\infty} \left[\int e^{-\rho t} j\left(t,x\right) \psi\left(x\right) dx\right] dt$$

- Discrete grids of asset a (x before) and idiosyncratic productivity z (B before):

$$(a, z) \in \{a_1, a_2 + \Delta a, \ldots, a_I\} \times \{z_1, z_2 + \Delta z, \ldots, z_J\}$$

- Planner value function (j before) is $V_{ij} \equiv V(a_i, z_j)$
- Saving function s_{ij}

$$s_{i,j} = wz_j + (r + \eta) a_i - c_{ij}$$

- Define *forward* and *backward* derivatives in asset a:

$$\partial_{a,F} V_{ij} = \frac{V_{i+1,j} - V_{i,j}}{\Delta a}$$

$$\partial_{a,B} V_{ij} = \frac{V_{i,j} - V_{i-1,j}}{\Delta a}$$

- Forward derivative for idiosyncratic productivity z:

$$\partial_z V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{\Delta z}$$

$$\partial_{zz} V_{i,j} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta z)^2}$$

Upwind scheme

- The utility, including the multiplier is $u\left(c\right)=\frac{c^{1-\gamma}}{1-\gamma}+\lambda\left(a-K\right)$
- The discretized HJB is:

$$\begin{aligned}\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\rho V_{i,j}^{n+1}=&u\left(c_{i,j}^{n}\right)+\partial_{a,F}V_{i,j}^{n+1}s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_{a,B}V_{i,j}^{n+1}s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}\\&+\theta\left(\hat{z}-z_j\right)\partial_z V_{i,j}^{n+1}+\frac{\sigma_z^2 z_j}{2}\partial_{zz}V_{i,j}^{n+1}-\eta V_{i,j}^{n+1}\end{aligned}$$

- With savings given by:

$$\begin{aligned}s_{ij,F}^{n}&=wz_j+(r+\eta)a_i-\left(u'\right)^{-1}\left(\partial_{a,F}V_{i,j}^{n}\right)\\s_{ij,B}^{n}&=wz_j+(r+\eta)a_i-\left(u'\right)^{-1}\left(\partial_{a,B}V_{i,j}^{n}\right)\end{aligned}$$

Solving the HJB: Sparse matrices

- Can rewrite the previous HJB as:

$$\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n}}{\Delta}+\rho\mathbf{V}^{n+1}=\mathbf{u}^{n}+\mathbf{A}^{n}\mathbf{V}^{n+1}$$

- Rearranging:

$$\left[\left(\frac{1}{\Delta}+\rho\right)\mathbf{I}^{n}-\mathbf{A}^{n}\right]\mathbf{V}^{n+1}=\mathbf{u}^{n}+\frac{\mathbf{V}^{n}}{\Delta}$$

- Algorithm:
  - Starts with $V^0_{i,j}=\frac{u(ra_i+wz_j)}{\rho}$
  1. Compute $\partial_{a,F}V_{ij}\quad\partial_{a,B}V_{ij}\quad\partial_z V_{i,j}\quad\partial_{zz}V_{i,j}$ using finite diff.
  2. Compute $c^n_{i,j}$ using:

$$c^n_{i,j}=(u')^{-1}\left(\partial_{a,F}V^n_{i,j}\mathbf{1}_{s^n_{i,j,F}>0}+\partial_{a,B}V^n_{i,j}\mathbf{1}_{s^n_{i,j,B}}<0+u'\left(wz_j+ra_i\right)\mathbf{1}_{s^n_{i,j,F}<0,s^n_{i,j,B}>0}\right)$$

  3. Solve the previous system to find $\mathbf{V}^{n+1}$
  4. Iterates until $\left|\mathbf{V}^{n+1}-\mathbf{V}^{n}\right|<\epsilon$

Solving the KF and finding the aggregate capital

- Discretized KF can be written taking advantage of the self adjoint property as:

$$\mathbf{A}^{'}\mathbf{g} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\lim_{n\to\infty} \mathbf{A}^{n} = \mathbf{A}$$

  - We solve the system and renormalize:

$$g_{ij} = \frac{g_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J} g_{ij}\Delta a \Delta z}$$

- Finding the equilibrium aggregate capital is the standard Aiyagari algorithm.

Aiyagari

Finding the Lagrange multiplier

- Choose a constant $\vartheta \in (0, 1)$, begin with an initial guess of the Lagrange multiplier $\lambda^0 = 0$, set $m = 0$:

1. Compute the value function V^m, consumption C^m, density g^m and aggregate capital K^m given λ^m.
2. Update $\tilde{\lambda}^{m+1}$:

$$\tilde{\lambda}^{m+1} = \frac{\alpha(1-\alpha)}{(K^m)^{2-\alpha}} \sum_{i=1}^{I} \sum_{j=1}^{J} \left[g_{i,j}^m + a_i \frac{g_{i+1,j}^m - g_{i,j}^m}{\Delta a} - K^m z_j \frac{g_{i+1,j}^m - g_{i,j}^m}{\Delta a} \right] V_{i,j}^m \Delta a \Delta z$$

3. If $\tilde{\lambda}^{m+1} \neq \lambda^m$, set $\lambda^{m+1} = \vartheta \tilde{\lambda}^{m+1} + (1-\vartheta)\lambda^m$
4. If $\left|\lambda^{m+1} - \lambda^m\right| > \epsilon$, keep iterating

Aiyagari: Finding aggregate capital

- Choose $\nu \in (0,1)$, and K^0

1. Compute r^n and w^n as a function of K^n
2. Given r^n and w^n solve the HJB numerically to get V^n and c^n
3. Given c^n solves the KF to get g^n
4. Compute aggregate capital $S^n = \sum_{(i,j)} a_i g_{ij} \Delta a \Delta z$
5. Update $K^{n+1} = \nu S^n + (1-\nu) K^n$
6. f $\left|K^{n+1} - K^n\right| > \epsilon$, keep iterating

back

- Choose as Hilbert *space* the space of Lebesgue-integrable functions L^2 with the inner product:

$$\langle f, g\rangle = \int f(x)\, g(x)\, dx$$

- Let's T an operator, its *adjoint* T^* is defined by:

$$\langle Tf, g\rangle = \langle f, T^* g\rangle$$

  - An operator is *self-adjoint* if $T = T^*$

- Let $J(g)$ be a functional and let h be arbitrary in L^2. The *Gateaux derivative* of J in the direction h is defined as:

$$\delta J(g; h) = \lim_{\alpha \to 0} \frac{J(g + \alpha h) - J(g)}{\alpha}$$

back

Math

- Let $H : L^2 \mapsto \mathbb{R}^n$.
- If J has a continuous Fréchet differential, a necessary condition for J to have an extremum at g under the constraint $H(g) = 0$ is:

$$\exists \lambda \in L^2 \Rightarrow \begin{cases} \delta \mathcal{L}(g; h) & = 0 \\ \mathcal{L}(g) & = J(g) + \langle \lambda, H(g) \rangle \end{cases}$$

back