Heterogeneous Agents models with Aggregate Shocks

Theory and Solution Methods

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Beyond Macro Reading Group

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Introduction – Motivation

- Many macro articles have to deal with agent heterogeneity and aggregate uncertainty
  - Incomplete Markets à la Bewley-Huggett-Aiyagari, Extension to HANKs
  - Pricing models à la Golosov-Lucas and Calvo+
  - Heterogeneous firms with lumpy invest^{nt} (Hopenhayn/Kahn-Thomas)
  - Intermediary asset pricing (He-Krishnamurty/Brunnermeier-Sannikov)
  - Search & Matching models (e.g OJS a la Robin, Shimer ...)
  - Network models with business cycles

  $\Rightarrow$ Any models where distribution of allocation matters for aggregates
- Most computational economics literature has tried to solve this insolvable problem for more than 20 years
  - Why unsolvable ?
  - Composition of aggregate and idiosyncratic uncertainty :

    $\Rightarrow$ need to keep track of all the histories of shocks
  - Infinite dimensional problem :

    $\Rightarrow$ need to keep track of the distribution of agents

  $\Rightarrow$ The literature has reduced the problem in different ways

Baseline model – Aiyagari without aggregate risk

- Let us recap the Aiyagari model
  - Will use it thoroughly as an example for the different algorithms
  - Continuous time version of the stationary case :
  - Household :
    - Two states : wealth a and labor prod. z ; control consumption : c
    - Idiosyncratic fluctuations in z (Pure jump/Jump-drift process)
    - State constraint (no borrowing) $a \geq \underline{a}$
    - Maximization :

$$\max_{c_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad da_t = (\underbrace{z_t w_t + r_t a_t - c_t}_{=s^\star(t,a,z)}) dt$$

  - Neoclassical firms : $Y_t = Z_t K_t^\alpha z_{av}^{1-\alpha}$
    - Interest rate : $r_t = \alpha Z_t K_t^{\alpha-1} z_{av}^{1-\alpha} - \delta$ & wage $w_t = (1-\alpha) Z_t K^\alpha z_{av}^{-\alpha}$
    - Capital demand $K_t(r) := \left(\frac{\alpha Z_t}{r_t + \delta}\right)^{\frac{1}{1-\alpha}} z_{av}$
  - Discrete time version here

Baseline model – Aiyagari without aggregate risk

- Equilibrium relations :
  - A Hamilton-Jacobi-Bellman : backward in time
    How the agent value/decisions change when distribution is given
  - A Kolmogorov-Forward (Fokker-Planck) : forward in time
    How the distribution changes, when agents control is given
  - These two relations are *coupled* :
    Through firm pricing (r_t & w_t) $\Rightarrow$ need to look for an eq. fixed point

$$-\partial_t v(t,a,z_j) + \rho v(t,a,z_j) = \max_c u(c) + \partial_a v(t,a,z_j) s(t,a,z_j) + \lambda_j \big(v(t,a,z_{-j}) - v(t,a,z_j)\big)$$

$$\partial_t g(t,a,z_j) = -\frac{d}{da}\big[s(t,a,z_j)\, g(t,a,z_j)\big] - \lambda_j g(t,a,z_j) + \lambda_{-j} g(t,a,z_{-j})$$

$$S_t(r) := \sum_{z_j} \int_a^\infty a\, g(t, da, z_j) = K_t(r)$$

Adding aggregate uncertainty

- What are the problems with aggregate risk ?
  - Aggregate shocks will affects the shape of the distribution
  - Agents needs to forecast its motion (of $g_t(\cdot)$) to make expectations about future prices ($r_t \ldots$) and value v_t
    - Only in case of strategic complementarity – coupling of HJB with KF.
  - The distribution $g(t, a, z_j)$, which is an infinite-dimensional object, becomes a state variable for each agent.
  - This changes for each path/history of aggregate shocks Z_t
- Examples :
  - AR(1)-change in agg. TFP Z_t : $dZ_t = \theta(\bar{Z} - Z_t)dt + \sigma dB_t$
  - Could also consider :
    - Shock to credit constraint $\underline{a}$ or to asset supply (gov^{nt} bond issuance)
    - Demand shocks/patience shock ρ
    - Change in idiosyncratic volatility $\sigma_z \equiv \mathbb{V}\mathrm{ar}(z)$ or transition probas λ

Adding aggregate uncertainty : Ideas for solution

- Potential solutions :

1. Consider unexpected shocks ⇒ MIT shocks
2. Reduce the dimensionality of $g(\cdot)$ ⇒ Krusell-Smith
3. Simplify the problem (linearize it) ⇒ Perturbation à la Reiter
4. Combine 1 & 3 (linear combin. of MIT shocks) ⇒ BKM & ARS
5. Discretize the aggregate shocks ⇒ Achdou-Bourany
6. Keep the infinite-dimensionality ⇒ math-literature/Lions-Lasry

- Today (hopefully) : will cover 1, 2 and 3

MIT shocks : unexpected shocks

- MIT shocks are unexpected shocks : zero-probability events
  - Z_t is subject to a one-time shock on dB_t, i.e. normal $\mathcal{N}(0, \sigma)$
  - Then Z_t follows the OU-(AR(1)) drift process $dZ_t = \theta(\bar{Z} - Z_t)dt$
- *Main idea :*
  - Agents do no anticipate this and hence do not draw expectations
    - v_0 does not include the potentiality of such shocks
    - Once the shock is "revealed" there is no more uncertainty on the path of Z_t

$\Rightarrow$ Certainty equivalence **(CE)** :

- No influence of variance σ : only size of the shock matters
- CE typically holds in Linear-Quadratic model with (additive) shocks : quadratic utility/objective fct. and linear transition/policy functions
- (good approximation of more general models ?)

MIT shocks : unexpected shocks

- MIT shocks are unexpected shocks : zero-probability events
  - Z_t : One-time shock on dB_t then follows OU/AR(1) deterministically
- Solution method :
  - ▷ Almost no difference compared to deterministic case (cf recap here)
  1. Solve the HJB using backward induction : start from steady state v_T where T large (close to stationary)
  2. Solve the KF forward : start from the "before-shock" steady state g_0
  3. Find the equilibrium fixed-point, by iterating on the entire *path* of prices $\{r_t\}_{t\in[0,T]}$
- Method most commonly used as a starting point
  - Certainty equivalence and no anticipation
  - Often implies small GE effects (little price effects)

Krusell-Smith Algorithm

- Krusell & Smith (1998)
  - *Income & Wealth Heterogeneity in the Macroeconomy*, Journal of Pol. Econ.
  - over 2000 cites, a lot for a technical/computational econ paper !
- *Main idea :*
  - Reduce the dimensionality of the problem :
  - Dynamics of the infinite dimensional $g(t,a,z_j)$ – usually governed by the Kolmogorov Forward – will be simplified :
  - Agents perceive the law of motion to be log-linear in the aggregate variable
  - Only consider the first moment of g, i.e.

$$K_t \equiv S_t(r_t) = \sum_j \int_a a\, g(t,da,z_j)$$

Discrete time

Krusell-Smith Algorithm

- The agents take their decision (in HJB) by making expectation about the future path of interest rate $\{r_t\}_{t\in[0,T]}$, which depends on KF :

$$\partial_t g_{(t,a,z_j)} = H(g_t, Z_t, dZ_t) \qquad \forall (t, a, z_j)$$

- Krusell-Smith's method :
  - Bounded-rationality : agents do not anticipate the full complexity of this law of motion / KF
  - Replace $H(g_t, Z_t, dZ_t)$, function of g by $\widehat{H}$ a log linear function in a finite set of moment $m = (m_1, \dots, m_I)$
  - In practice, keep only the first moment $m_1 \equiv K \equiv S(r)$

$$d \log K_t = a_{(Z_t)}\, dt + b_{(Z_t)} \log K_t\, dt$$

  - Why ? for such model, the first moment is enough !

$\Rightarrow$ Phenomenon called approximate aggregation

Krusell-Smith Algorithm – Approximate aggregation

- Phenomenon called approximate aggregation :
  - Keeping the first moment $m_1 \equiv K_t = \sum_j \int_a a\, g(t,da,z_j)$ is enough
  - Compute the value function $v(t,a,z_j,K)$
    - Value funct^o iteration on $v(a,K)$ & approx^{ion} outside grid (cubic spline)
    - Given "perceived" log-linear law of motion of $\widehat{K}_t$
    - Monte Carlo on the employment status (5,000 agents and 10,000 periods)
  - Accuracy measure ?
    - Compare the aggregate K given all the decision of agents $s(t,a,z_j,K)$
    - Regress future aggregate capital on its past values (using these 10,000 values)
    - The "reality" K_t respects the perceived Law of Motion $\widehat{K}_t$
    - $R^2 > 0.9999$ and $\mathbb{V}\text{ar}(\varepsilon) < 0.004\%$ with $\varepsilon = K_t - \widehat{K}_t$

Krusell-Smith Algorithm – Extensions and issues

- Simplifications in Krusell-Smith :
  1. Take only the first moment :
     - Can be checked in adding more moments $(m_2, \ldots, m_I)$ in $\widehat{H}$ and regressing K_t on $\widehat{K}_t$
     - Usually R^2 is still very high for most models.
  2. Take a (log-) linear law of motion for these moments
     - Can take non-linear dynamics/ flexible functional form for $\widehat{H}$
     - Fernández-Villaverde, Hurtado, Nuño (2019, WP) use a non-linear approximation for $\widehat{H}$:
     - Agents infer/"learn" a non-linear $\widehat{H}$ using machine learning techniques (neural network)
- One (main !) problem remains :
  - Can we hope that this algorithm does not create "self-fulfilling" expectations ?
  - The agents may act in a linear / approximate-aggregated way because they expected the others to do so ?

Perturbation methods :

- A second literature rely on linearization and perturbation methods
  - For HA models : Reiter (2009)
  - *Solving heterogeneous agents models by projection and perturbation*, JEDC
  - Follows a large anterior literature
    - DSGE lit. (RBC/medium-scale NK), Schmitt-Grohe Uribe (2004)
    - Used heavily for estimation (MCMC), because very fast
  - Large literature following this :
    - Reiter (2010), Den Haan (2010), Algan-Allais-Den Haan (2008)
    - Winberry (2018) Quantit. Econ., Mongey-Williams (2017) JMP
    - Ahn, Kaplan, Moll, Winberry and Wolf (2017) NBER Macro Annual
- *Main idea :*
  - Linearize the model in the aggregate shock Z_t
    - Linear perturbation in Z_t around the stationary equilibrium
    - but keep the non-linearity in idiosyncratic shocks
    - Large linear system : nb of states $\approx$ nb of gridpoints
  - Projection to simplify the large system and go faster

Reiter Algorithm

- Consider the equilibrium relations as the following system :
  - HJB, KF, Def of prices, Mkt clearing, Dynamics of agg. shocks
  - States : $\Theta_t = (v_t, g_t, p_t)$, agg. shocks Z_t
  - Could have a formulation with present/future state/control var. here

$$\mathbb{E}_t[d\Theta_t] = F(\Theta_t, dZ_t, Z_t)$$

- Steps :
  1. Solve the stationary system :

$$\mathbb{E}_t[d\overline{\Theta}] = F(\overline{\Theta}, 0, \overline{Z})$$

  2. Linearize the system around it, perturbing in the agg. shock :

$$\mathbb{E}_t[d\widehat{\Theta}_t] = \mathcal{L}F := \partial_\Theta F(\overline{\Theta}, 0, \overline{Z}) \cdot \widehat{\Theta}_t dt + \partial_Z F_Z(\overline{\Theta}, 0, \overline{Z}) \cdot dZ$$

  3. Reduce the state-space, with projection : basis x for Θ

$$\Theta_t \approx X = \sum_j \gamma_{jt} x_j \qquad \Rightarrow \qquad \mathcal{L}F(\overline{\Theta}, \overline{Z}) \cdot [\widehat{\Theta}_t dt, dZ] \approx \widehat{\mathcal{L}F}(X, \overline{Z}) \cdot [\widehat{X}_t dt, dZ]$$

Reiter Algorithm, Linearization and issues

- What is lost due to linearization, and what is preserved ?
  1. Certainty equivalence in aggregate uncertainty :
     - No influence of variance σ : only size of the shock Z_t matters
     - Agents do not "change" their decisions with aggregate uncertainty
     - Perturbation methods (at least in first order) not suited for asset pricing/portfolio choice models
     - However, agents still account for idiosyncratic variance : valid method to study uncertainty shocks (c.f. Bloom (2014))
     - Break certainty equivalence with higher order perturbation (2nd, 4th)
  2. State dependence, in particular of the aggregate IRF to the distribution g_0
  3. Path dependence, different histories of shocks $\{Z_t\}_{t\in[0,T]}$ won't have the same final effects on aggregate K_T or C_T
  4. No sign and size dependence : linearity of the system in Z_t make the response of a λZ_0 shocks λ time larger than a Z_0-sized shock.

Reiter Algorithm - Extensions

- Winberry (2018)
  - Use the technique developed in Algan-Allais-Den Haan (2008) to approximate the distrib. $g(a, z)$ with a parametric fct^{al} form :

$$\log g(a, z) \approx \sum_{k}^{n_g} \sum_{\ell}^{k} \gamma_k^{\ell} (z - m_1^z)^{k-\ell} (\log a - m_1^a)^{\ell}$$

  - Reduce the infinite dimensional object to a finite dim. one : n_g
  - Can compute the law of motion (replace the KF)
  - Use the same perturbation methods as in Reiter
  - Bayesian estimation of parameters
- Mongey and Williams (2017)
  - Use Reiter's algorithm and estimate it with aggregates time series and cross-sectional micro data :
  - Bayesian estimation and variance decomposition (4 different shocks)

Reiter Algorithm - Extensions

- Ahn, Kaplan, Moll, Winberry and Wolf (2018) combines :
  1. Continuous-time à la Achdou, Han, Lasry, Lions and Moll (2017)
     - Large speed gain for computing stationary equilibrium
  2. Algorithm à la Reiter (2009) for linearization and perturbation w.r.t. aggregate shocks
     - Automatic differentiation to linearize the system (more accurate than finite diff^o. / faster than symbolic diff^o)
  3. Clever dimensionality reduction (projection for g and v on a time invariant basis x)
     - More than tenfold speed for solving the linear system and IRFs
- Large literature using/developing these techniques for estimation...

Combining Linearization and MIT shocks : BKM

- Boppart, Krusell and Mitman (2018)
  - *Exploiting MIT shocks in heterogeneous-agent economies : the impulse response as a numerical derivative*, JEDC
  - Recent generalization by Auclert et al. (2019) and recent work by Kaplan-Moll-Violante
- *Main idea :*
  - Combining non-linearity of responses to MIT shocks
  - With linearity assumption to combine multiple shocks
  - IRF of an MIT shock is a derivative of the system :

    $\Rightarrow$ we ”just” need to “compute” it once !

Combining Linearization and MIT shocks : BKM

- More details on BKM
  - Sequential representation of heterogeneous agents models :
  - Express aggregate variables K_t (or C_t) as a fct of past shocks on Z_t
    - Sequence form :

$$dK_t = \mathcal{K}(\{dZ_s\}_{s\leq t}) \approx \mathcal{K}(dZ_t, dZ_{t-1}, \dots)$$

    - vs. Recursive form : $K_t = \widetilde{\mathcal{K}}(\Theta_t)$ with Θ_t states var. (v_t, g_t, p_t)
- Linearity assumption of the system :

$$\begin{aligned} dK_t &= \int_0^t \partial_{dZ_s}\mathcal{K}(0)dZ_s \\ &\approx \underbrace{\mathcal{K}(\varepsilon, 0, 0, \dots)}_{\substack{IRF \text{to a 1-time} \\ \varepsilon-\text{sized MIT shock} \\ \equiv \mathcal{K}_{dZ}(0)}} dZ_t + \mathcal{K}(0, \varepsilon, 0, \dots)dZ_{t-1} + \dots \end{aligned}$$

Combining Linearization and MIT shocks : BKM

- Solution method in practice :
  1. Simulate the IRF to a small (sized ε) MIT shocks :
     - Shock at date s gives IRF : $dK_t^s = \mathcal{K}(0, \dots, \varepsilon, 0, \dots)$
     - Such path represent the non-linear derivative $\partial_{dZ_s}\mathcal{K}(0)$ of the system to a shock
  2. Simulate a sequence of shocks $(\{dZ_s\}_{s\leq t}$
  3. Sum the IRF for different shock, rescaling by the size of the shock :

$$dK_t = \int_0^t \partial_{dZ_s}\mathcal{K}(0)dZ_s \approx \sum_s^t \tfrac{1}{\varepsilon} dK_t^s dZ_s$$

     - Possibility of testing the linearity assumption by changing the size/sign of ε

Linearization & MIT shocks – Extensions : SHADE

- Auclert, Bardóczy, Rognlie and Straub (2019)'s SHADE :
  - Equilibrium relations as the system :

$$H(K_t, Z_t) = 0$$

  - Linearizing :

$$H_K(\overline{K}, \overline{Z})dK_t + H_Z(\overline{K}, \overline{Z})dZ_t = 0$$

  - Path of capital as function of past shocks :

$$dK_t = \underbrace{-[\overline{H}_K]^{-1}\overline{H}_Z}_{\equiv \mathcal{K}_{dZ}(0)} dZ_t$$

- $\overline{H}_K$ and $\overline{H}_Z$ called "sequence space Jacobians"
  - Need to be computed once
  - Sufficient statistics : all we need, to know the agg. system response
  - Fast : used in estimation (of shock process dZ_s)

Linearization & MIT shocks – Extensions : SHADE

- These "sequence space Jacobians" :
  - Are the sufficient statistics :
    - $\overline{H}_K, \overline{H}_Z$ and $\mathcal{K}_{dZ} \equiv -[\overline{H}_K]^{-1}\overline{H}_Z$ as a $T \times T$ matrix
    - IRF for a path $\{dZ_t\}_t$: $\approx$ derivative of system in response to shocks
    - "News" of different horizons s shocks : s-th columns of $\mathcal{K}_{dZ}$
    - Include "under the hood" the underlying heterogeneity
  - Methods to compute it :
    - Direct methods (finite difference)
    - *Fake news* algorithm : linearize the underlying heterogeneous agents model and avoid recomputing several of the matrices
- Substantial speed gains :
  - Linearization and no need to recompute the Jacobian
  - Lots of clever methods :
    - Directed acyclic graph to exploit the sparsity of system : dimension reduction by composition of Jacobians along the blocks of this DAG
    - Likelihood-based estimation : feasible now for even large models

Other solution methods and optimal policies

- Linearization techniques to handle optimal policies/Ramsey plans
  - Bhandari, Evans, Golosov and Sargent (2018)
    - Linearization w.r.t all the variables/distribution (Fréchet derivative)
  - Comp. eq. vs. Constrained Efficiency vs. Pareto optimal ?
    Nuño (2017) and Nuño-Moll (2017)
  - "Major & minor agents" : Nuño and Thomas (2016)

  ⇒ Léo's presentation next week !

- Other methods involving "reduced heterogeneity" :
  - Ways to "summarize" heterogeneity : Ragot (2018)
  - History Representation of HA models : summarize the different paths of idiosyncratic shocks with "representative histories"
  - Possible to determine optimal fiscal-monetary policy : Le Grand, Ragot et al. (2017)

Tree structure for aggregate shocks : Achdou-Bourany

- Achdou-Bourany (2018)
  - Master thesis under supervision of Y. Achdou
- Main idea : approximate the process for the Z_t by a finite number of "simple" shocks :
  - Every ΔT (deterministic times), Z_t jumps stochastically to one of K outcomes
  - Repeat this : a finite M number of "wave" of uncertainty
  - This way, you can build a tree of K^M paths of Z_t with deterministic branches separated by stochastic shocks
  - Taking $\Delta T \to 0$, you can approximate any process (e.g. Donsker's theorem for Brownian motion)
  - Need to link the branches together in an appropriate way

Tree structure for aggregate shocks : Achdou-Bourany

- Grafting branches :
  - On each branch (between each shock), compute the evolution of the system : HJB and KF : $v(a,z_j,\tilde{Z})$ and $g(a,z_j,\tilde{Z})$
- To account for future and past shocks ?

  $\Rightarrow$ use boundary conditions of the PDEs !
  - t_m^- time before revelation of the shock ($Z_{t_m^-} = Z_m$)
  - t_m^+ : time when shocks hits ($Z_{t_m^+} = Z_{m+1}$ take K values)

$$v(a,z_j,Z_m) = \sum_{k|Z_{m+1}=Z_k} \mathbb{P}(Z_{m+1}|Z_m)\, v(a,z_j,Z_{m+1})$$

$$g(a,z_j,Z_m) = g(a,z_j,Z_{m+1})$$

  - Agents are forward looking, form expectations over the different future branches (paths of Z_t)
  - Continuity of $g(\cdot)$ in time t
- Loop to find eq. fixed point on the entire tree (all branches !)
  - Problem : computationally heavy/slow !

Existence & Uniqueness – Mathematical literature on MFG

⇒ Heterogeneous agents ≡ Mean Field Games (MFG)

- Cardaliaguet, Delarue, Lasry and Lions (2019)
  - Master equation in infinite-dimension :
    - Value $U_{(t,a,z_j,Z,g)} = v_{(t,a,z_j,Z)}$ definite along the characteristics of the system (v, g) for the dynamics of Z_t.
    - Equation (& U and $D_m U$) in Wasserstein space $g \in \mathcal{P}_{([0,T]\times[\underline{a},\infty],[\underline{Z},\overline{Z}])}$
- Carmona, Delarue and Lacker (2016)
  - Stochastic Partial Diff. equations (SPDE) :
    - Both HJB & KF equations become stochastic with aggreg. shocks Z_t
- Carmona and Delarue (2018)
  - Forward-Backward Stochastic Diff. equations (FBSDE) :
    - Stochastic Pontryagin Maximum Principle (Hamiltonian !)
    - Forward states variables K_t, g_t and Backward costates $\approx v_t$

⇒ Different approaches summarized in sect^o 3 of my master thesis here :

MFG literature exploding in the recent years !

Conclusion

- Challenging problem and many different methods
- No perfect solution – (un)fortunately ?
  - Every algorithm with its own way of bypassing difficulties
  - e.g. trade-off : Linearity/simplification for "speed"
    vs. Role for uncertainty/shape of distribution for "accuracy"
- Still lacks theoretical results on the strength of various methods
  - Global methods vs. Local (higher order) perturbation
  - Could compare them for various (closed-form) models
- Still large gains despite the fixed cost of entering in this literature
- THANK YOU FOR YOUR ATTENTION

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Aiyagari model without aggregate risk – discrete time

- Household :
  - Two states : wealth a and labor prod. z ; control consumption : c
  - Idiosyncratic fluctuation in z (Markov chain/AR(1) process)
  - State constraint (no borrowing) $a_t \geq \underline{a}$
  - Maximization :

$$\max_{c_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad\qquad c_t + a_{t+1} = z_t w_t + r_t\,(1 + a_t)$$

- Neoclassical firms : $Y_t = Z_t K_t^{\alpha} z_{av}^{1-\alpha}$
  - Interest rate : $r_t = \alpha\, Z_t K_t^{\alpha-1} z_{av}^{1-\alpha} - \delta$ & wage $w_t = (1-\alpha) Z_t K^{\alpha} z_{av}^{-\alpha}$
  - Capital demand $K_t(r) := \left(\frac{\alpha Z_t}{r_t+\delta}\right)^{\frac{1}{1-\alpha}} z_{av}$

Aiyagari model without aggregate risk – discrete time

- Equilibrium (recursive) relations :
  - A Bellman equation : backward in time
    How the agent value/decisions change when distribution is given
  - A Law of Motion of the distribution : forward in time
    How the distribution changes, when agents control is given
  - These two relations are *coupled* :
    Through firm pricing (r_t & w_t) $\Rightarrow$ need to look for an eq. fixed point

$$v_t(a,z) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[v_{t+1}(a',z')\middle|\sigma(z)\right]$$

$$s.t. \qquad c+a'=zw_t+r_t\,(1+a) \quad a' \geq \underline{a} \quad \Rightarrow \quad a'^{\star} = \mathscr{A}(a,z)$$

$$\forall\ \widetilde{A} \subset [\underline{a}, \infty) \qquad g_{t+1}(\widetilde{A}, z') = \sum_z \pi_{z'|z} \int \mathbb{1}\{\mathscr{A}(a,z)\in\widetilde{A}\} g_t(da,z)$$

$$S_t(r) := \sum_z \int_{\underline{a}}^{\infty} a\, g_t(da,z_j) = K_t(r)$$

Back

The algorithm : an overview

- Aim : find the stationary equilibria : i.e. the functions $v(a,z_j)$ and $g(a,z_j)$ and the interest rate r.
- General structure :
  1. Guess interest rate r^ℓ, compute capital demand $K(r^\ell)$ & wages $w(K)$
  2. Solve the HJB using finite differences (semi-implicit method) : obtain $s^\ell(a,z_j)$ and then $v^\ell(a,z_j)$, by a system of sort : $\rho\,\mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v};r)\mathbf{v}$
  3. Using $\mathbf{A}^T$, solve the FP equation (finite diff. system : $\mathbf{A}(\mathbf{v};r)^T\mathbf{g} = 0$), and obtain $g(a,z_j)$
  4. Compute the capital supply $S(\mathbf{g},r) = \sum_j \int_{\underline{a}}^{\infty} a\, g(a,z_j)da$
  5. If $S(r) > K(r)$, decrease $r^{\ell+1}$ (update using bisection method), and conversely, and come back to step 2.
  6. Stop if $S(r) \approx K(r)$

Stationary MFG equations

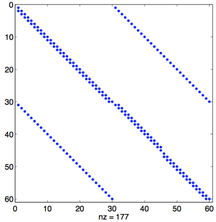
The algorithm : advantages relative to discrete time :

1. Borrowing constraint only appears in the boundary conditions
   - FOCs $u'(c_{(a,z_j)}) = \partial_a v_{(a,z_j)}$ and HJB eq. always holds with equality
   - No need to split the Bellman equation (constrained vs. unconstrained agents)
2. In continuous time there is no future (i.e. $t+1$) only present t !
   - Only involve contemporaneous variables (FOC are 'static')
   - No need to use costly root-finding to obtain optimal $c_{(a,z_j)}$.
3. The discretized system is easy to solve :
   - 'Simply' a matrix inversion
     (Finite differences : taught in 1st year in any engineering school).
   - Matrix is sparse (tridiagonal)
   - Continuous space : one step left or one step right
4. HJB and FP are coupled
   - The matrix to solve FP is the transpose of the one of HJB.
   - Why ? Operator in FP is simply the 'adjoint' of the operator in HJB : 'Two birds one stone'
   - Specificity of MFG !

The algorithm : Finite difference scheme

- Finite difference scheme : discretize the state-space a_i for $i = 1, \dots I$.

$$\partial_a v(a_i, z_j) \approx \frac{v_{i+1,j} - v_{i,j}}{\Delta a} \equiv v'_{i,j,F} \qquad \partial_a v(a_i, z_j) \approx \frac{v_{i-1,j} - v_{i,j}}{\Delta a} \equiv v'_{i,j,B}$$



- Vector form :
- Linear system to solve $\mathbf{A}$ is sparse.

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; r)\mathbf{v}$$
$$0 = \mathbf{A}(\mathbf{v}; r)^T \mathbf{g}$$
$$S(\mathbf{g}, r) = K(r)$$

The algorithm : theoretical results

- This numerical solution converges to the unique (viscosity) solution of the HJB, under some conditions :
  1. Monotonicity (invertible and inverse positive)
  2. Consistent (approx error is majored by powers of step sizes)
  3. Stability (iteration in k is bounded)
- Is the matrix monotonous ?
  - In the scheme for solving the HJB, one can distinguish if the drift is positive or negative :
  - that is the upwind scheme
  - When $s(a) > 0$ use $v'_{i,j,F}$, and $s(a) < 0$, use $v'_{i,j,B}$
  - This insures the convergence of the algorithm

The algorithm : transition dynamics

- The algo for transitions is a generalization :
  - Discretization : $v_{i,j}^n$ and $g_{i,j}^n$ stacked into v^n and g^n
  - Somehow, it is more specific to Mean Field Games :
- Take advantage of the backward-forward structure of the MFG
  - Make a guess r_t^ℓ $(t = 1, \dots, N)$ on the *path* interest rates.
  - Solve the HJB (implicit scheme), given terminal condition ;

$$\rho v^{n+1} = u^n + \mathbf{A}\left(v^{n+1}; r^n\right) v^{n+1} + \frac{v^{n+1} - v^n}{\Delta t}$$

$$v^N = v_\infty \qquad \text{(terminal condition = steady state)}$$

  - Solve the FP forward, given the initial condition

$$\frac{g^{n+1} - g^n}{\Delta t} = \mathbf{A}(v^n; r^n)^T g^{n+1}$$

$$g^1 = g_0 \qquad \text{(initial condition)}$$

  - Update the interest rates path

The algorithm : wrapping up

- This algorithm to compute the dynamics of the system will be used a lot when adding aggregate shocks.
  - HJB start from the end (what agent anticipate) and runs backward until the computation of the initial value function
  - FP start from the beginning (what wealth agents hold) and runs forward to compute the evolution of distributions.
  - If there are discrepancies between capital demand and capital supply, loop to correct the path of interest rate.
- Performance of the algorithm :
  - $\approx$ 1000 grid points in space, 400 in time :
  - Stationary equilibrium : 0.25-0.4 sec
  - Transition dynamics : around 30-50 secs
    - Perfect foresight or MIT shocks.
    - 10^{-6} error on the path of interest rate.
  - What about anticipated aggregate shocks ?

    $\Rightarrow$ Very different speeds for different algos ! Back

Krusell-Smith Algorithm in Discrete time

- Model in discrete time :
  - Using the discrete time Aiyagari model
  - Add a jump/AR(1) process for aggregate productivity Z_t

$$v_t(a,z;g,Z) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[v_{t+1}(a',z';g',Z'))\middle|\sigma(z,Z)\right]$$
$$s.t. \qquad c + a' = z w_t(K,Z) + r_t(K,Z)\,(1+a) \quad a' \geq \underline{a}$$

$$g' = H(g,Z) = \Pi_{(g,v,K,Z)} \cdot g$$
$$S(r) := \textstyle\sum_j \int_a^\infty a\, g(da, z_j) = K(r)$$

  - The agents take their decision (in Bellman eq.) by making expectation about the future path of prices $\{r_t, w_t\}_{t\in[0,T]}$, which depends on the Law of Motion of the distribution
    - Law of Motion $H(\cdot)$ is "perceived" to be log linear in the first aggregate moment K

Krusell-Smith Algorithm in Discrete time

- Krusell-Smith's method : change the "perceived" law of motion :
  - Bounded-rationality : agents do not anticipate the full complexity of this law of motion / KF
  - Replace $H(g, Z)$, function of g...

$$g' = H(g, Z) = \Pi_{(g,v,K,Z)} \cdot g \qquad \Rightarrow \qquad K' = f(K; g, v, Z)$$

... by $\widehat{H}$ a log linear function in a finite set of moment $m = (m_1 \dots m_I)$

  - In practice, keep only the first moment $m_1 \equiv K \equiv S(r)$

$$m = \widehat{H}(m, Z) \qquad \Rightarrow \qquad \log K' = a(z) + b(z) \log K$$

  - Why ? for such model, the first moment is enough !

$\Rightarrow$ Phenomenon called approximate aggregation

Back

Krusell-Smith Algorithm

- Krusell-Smith results on approximate aggregation

$$\log \bar{k}' = 0.095 + 0.962 \log \bar{k}; \quad R^2 = .999998,\ \hat{\sigma} = 0.0028\%,$$

in good times and

$$\log \bar{k}' = 0.085 + 0.965 \log \bar{k}; \quad R^2 = .999998,\ \hat{\sigma} = 0.0036\%$$

in bad times.[10]

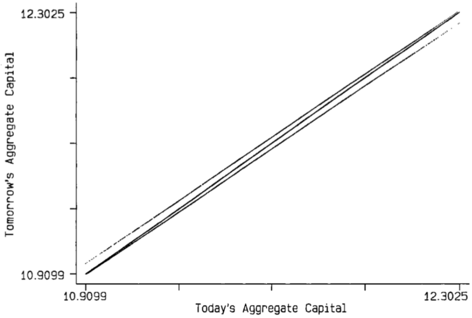


FIG. 1.—Tomorrow's vs. today's aggregate capital (benchmark model)

Back

Perturbation methods in discrete time : Reiter

- Equilibrium relations of Krusell-Smith model in discrete time :
  - Euler equation, Law of motion of distribution (discretized as an histogram), Price/TFP dynamics
    - ε_t Exog. shocks on Z_t and η_t expectation error.

$$H(\Theta_{t+1}, \Theta_t, \eta_{t+1}, \varepsilon_{t+1}) = 0$$

  - Stationary equilibrium :

$$H(\overline{\Theta}, \overline{\Theta}, 0, 0) = 0$$

  - Linearization (finite diff^o) :

$$H_1(\overline{\Theta}, \overline{\Theta}, 0, 0)\widehat{\Theta}_{t+1} + H_2(\overline{\Theta}, \overline{\Theta}, 0, 0)\widehat{\Theta}_t + H_3\eta_{t+1} + H_4\varepsilon_{t+1} = 0$$

Back