Project – Numerical methods for PDE in finance

Wealth distribution over the business cycle
A mean-field game approach

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Abstract

The standard economic framework with "heterogeneous agents" – the Aiyagari-Bewley model – has recently been reformulated as an example of Mean Field Game (MFG), by Achdou, Han, Lasry, Lions and Moll (2017). One of the key question in this type of model is to understand the transmission effects of aggregate shocks - on aggregate macro variables and on the shape of the wealth distribution. Such framework can thus be understood as a MFG with "common noise". We simulate this model using a discretization procedure for the common noise – with a finite number of shocks – and solving the MFG using specific finite-differences methods for the two PDEs. The main result shows that precautionary saving against aggregate fluctuation matters a lot quantitatively.

1 Introduction

One of the recent development of macroeconomics has been to integrate agent heterogeneity in credible fashion to study the income and wealth distributions of households. Since the contribution of Bewley (1986) and Aiyagari (1994), this "heterogenous agents" literature has brought progress for the study of numerous economic questions: the causes and dynamics of inequality, the implication of various market frictions to generate skewed wealth distributions or the distributional consequences of monetary or fiscal policies[1]. One aspect subject to a lack of understanding is the implication of aggregate shocks for income and wealth distribution. The main result of Krusell and Smith (1998) was to show that the standard framework with uninsurable idiosyncratic risk display what one call "approximate aggregation": the macroeconomic aggregates can be almost perfectly described using only the mean of the wealth distribution. Therefore, one is naturally inclined to wonder whether agent heterogeneity substantially changes the results at stake in "representative agent" models. This paper investigate this question.

In this project, I explore the implication of aggregate uncertainty in a standard Aiyagari-Bewley framework. However, unlike Krusell and Smith (1998) who approximate the wealth distribution using finite number of moments, or Reiter (2010) who use perturbation and projection methods – and thus linearization techniques – I use numerical methods developed in the Mean Field Game (MFG) literature by Y. Achdou.

The specificity and strength of most heterogenous agents models is that – when reformulated in continuous time – they can easily be seen as Mean-Field Games. A Mean-Field Game (MFG) is a game with a large number of "small" symmetric players: the interaction between them is only reflected by the interaction between each agent with the "distribution" (measure) of all the other agents. This (stochastic control problem) boils down to a system of two coupled partial differential equations:

- First a Hamilton-Jacobi-Bellman (HJB) equation – where the agents make their choice taking the distribution of agents as given
- Second a Fokker-Plank (or Kolmogorov Forward – KF) equation – where the distribution evolve, given optimal choices.

The standard Aiyagari-Bewley model has recently been reformulated as an example of Mean Field Game (MFG), by Achdou, Han, Lasry, Lions and Moll (2017). This model can be simulated by solving numerically this system of two equations, using a finite-difference scheme.

[1] One could mention for instance Gabaix, Lasry, Lions and Moll (2016), Kaplan and Violante (2014), Heathcote, Storesletten and Violante (2014), Benhabib, Bisin and Zhu (2011), Bhandari, Evans, Golosov and Sargent (2017)

To describe shortly the algorithm, the first step is to solve the HJB equation for a given time path of interest rate and the second is to solve the KF equation for the evolution of the joint distribution of income and wealth. Conveniently, after having solved the HJB equation, one obtains the time path of the distribution easily, because the KF equation is the "adjoint equation" of the HJB equation. The third step is to update the interest rate iterate and repeat the first two steps until an equilibrium fixed point for the time path of prices is found. This algorithm will be discussed in depth in section 3.

This project follows closely this approach[2], adding aggregate shocks, i.e. a "common noise", to the mean field game. Following the approach of Y. Achdou for crowd motion in movie theater and rationalized by Carmona, Delarue, Lacker et al. (2016), we approximate the common noise – that can be a two-state jump process or a Brownian motion – as a finite number of shocks. Between each shock, the MFG will be a standard system of two evolution PDEs, the HJB and the FP coupled by the interest rate. The main challenge will be to link the behavior of the MFG in function of the anticipation of future aggregate shocks and in function of the evolution of past state variables (i.e. saving). A way to include this is to change the terminal condition for the HJB – the final value function that will now be the expected value of future value function over the different path of the common noise – and the initial condition of the FP – the past wealth distribution before the realization of the common noise. With these two methods, one that compute the evolution of the MFG for different trajectories for the aggregate productivity – the noise in this setting.

However, one of the main challenge of this approach is when the discretization procedure is made finer to approximate the Brownian motion/Jump process of the common noise. The number of trajectories will grow exponentially and thus the computation time. A goal for further research will be to develop parallelization techniques to be able to reach a satisfying accuracy of the MFG with common noise.

I describe the theoretical model in the next section, and explain in depth the algorithm in the sections 3, 4, 5 and 6. In the section 7, I describe the strategy to handle common noise and show the main results.

[2] Note that part of the report is greatly inspired from the 'Numerical Appendix' written by B. Moll, describing the algorithm of the article Achdou, Han, Lasry, Lions and Moll (2017).

2 The economic model – a theoretical framework

2.1 Statement of stochastic control problem

The economy is composed of a continuum of households, who face idiosyncratic and uninsurable income shocks, and are subject to a credit (borrowing) constraint. They thus solve the following stochastic control problem:

$$
\begin{aligned}
&\max_{c_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \\
\text{subject to :} \quad & da_t = (z_t w_t + r_t a_t - c_t) dt && \text{(Budget constraint)} \\
\text{and} \quad & a_t \geq \underline{a} && \text{(Credit constraint)}
\end{aligned}
$$

where a is wealth – the state variable we'll focus on – c_t consumption, ρ rate of time preference, $u(\cdot)$ a utility function, supposed increasing and concave ($u' > 0, u'' < 0$). For the simulation, the utility will be chosen to adopt two specific functional form (i) Constant Relative Risk Aversion (CRRA) utility and (ii) Exponential utility:

$$
\text{(i)} \qquad u(c) = \frac{c^{1-\gamma}}{1-\gamma} \qquad\qquad \text{(ii)} \qquad u(c) = -\frac{1}{\theta} e^{-\theta c}
$$

The income is composed of a wage w_t and productivity factor z_t. The wage w_t, and interest rate r_t, will adjust in general equilibrium, considering the firm's side (cf. next subsection). The productivity z_t is subject to shocks that are (i) idiosyncratic (i.e. its law is drawn i.i.d.[3]) and (ii) uninsurable, (agents can not hedge/cover against this risk). z intuitively represents the state of the agent, for instance, employment $z_{high} = z_2$ and unemployment $z_{low} = z_1 = \frac{1}{2} z_2$. In this presentation, it is modeled simply as a jump processes with two states $\{z_1, z_2\}$ (with intensities λ_1, λ_2, the higher the intensity, the higher the proba to jump). However, one can generalize it any stochastic process: e.g. diffusion $dz_t = b(z_t)dt + \sigma(z_t)dB_t$, Poisson or Levy processes. The only numerical constraint is that it has a bounded domain[4].

One of the specificities of this problem is the credit constraint, which is a *state* constraints: $a_t \geq \underline{a}$. State constraint is a complicated problem for control theory. In our situation, intuitively, the optimal strategy of the agent might be (and will be) to move on the constraint ($\partial\Omega$) and stay there (poverty trap). Mathematically, it is not possible to find a PDE and a boundary condition on $\partial\Omega$ even in the sense of distribution.

[3]But the etymology implies that agents might react in their own way to a similar shock

[4]In case of diffusion, it will be reflected after a limit

According to Soner (1986), the mapping from the state to the set of admissible controls $a \mapsto \mathcal{A}_a$(and $c \in \mathcal{A}_a$) 'will have a complicated structure', and its regularity may not be insured in general. In our situation it will be the case, but it will implicitly impose a constraint on the derivative of the value function at the boundary.

This will result on both (i) a Dirac mass on the boundary and (ii) an explosion near the boundary. Economically, there is a lower bound on the value of this borrowing limit: $\underline{a} \geq -z_1 w_t / r_t$. The latter term represents the natural borrowing limit – the credit that an agent could repay if it would fall permanently in the low income state (and repaying its debt at rate r_t.

2.2 The firm – supply side

The supply side is driven by a representative firm, produce goods following a production function of aggregate productivity, capital and labor $F(A, K, L)$, supposed to be concave in the two last variables. Given δ the depreciation of capital and A productivity level, the firm is producing in a perfectly competitive environment, and thus maximize its profit:

$$\Pi(K_t, L_t) = \max_{K,L} \; F(A_t, K_t, L_t) - (r_t + \delta) K_t - w_t L_t$$

The price of demanded input factors – i.e. r_t interest rate, w_t wage – will be determined by the First-Order-Condition of this optimization problem:

$$r_t = \partial_K F(A_t, K_t, L_t) - \delta \qquad\qquad w_t = \partial_L F(A_t, K_t, L_t)$$

In this article, we suppose that (i) production function is simply given by the Cobb-Douglas production function: $F(A_t, K, L) = A_t \, K_t^{\alpha} \, L_t^{1-\alpha}$, and (ii) the effective labor supply is fixed and equal to the average productivity of households: $z_{av} = \frac{z_1 \lambda_2 + z_2 \lambda_1}{\lambda_1 + \lambda_2}$.

The optimality relation described above reduce to :

$$\begin{aligned} K(r) &:= \left(\frac{\alpha A}{r+\delta}\right)^{\frac{1}{1-\alpha}} z_{av} && \text{Capital demand} \\ w &= (1-\alpha)\, A \, K^{\alpha} z_{av}^{-\alpha} && \text{Wage} \\ r &= \alpha \, A \, K^{\alpha-1} z_{av}^{1-\alpha} - \delta && \text{Interest rate} \end{aligned}$$

Wage and interest rate are indeed constant when productivity A and all parameters are constant. Let us look at the Mean Field Game formulation[5] of the problem.

[5]Cf. appendix for a general setting

2.3 A mean field game formulation

The stochastic control problem boils down to a system of two partial differential equation: (i) a Hamilton-Jacobi-Bellman (HJB) equation and (ii) a Fokker-Plank (or Kolmogorov Forward – FP). When parameters are constant, the stationary equilibrium system is described by the equations:

$$\rho v_j(a) = \max_c u(c) + \partial_a v_j(a)(z_j w + ra - c) + \lambda_j(v_{-j}(a) - v_j(a)) \qquad \text{[HJB]}$$

$$0 = \frac{d}{da}[s_j(a)\, g_j(a)] + \lambda_j g_j(a) - \lambda_{-j} g_{-j}(a) \qquad \text{[FP]}$$

$$S(r) := \int_a^\infty a\, g_1(a)da + \int_a^\infty a\, g_2(a)da = K(r) \qquad \text{[Market clearing]}$$

where j indicate the z-state of the agent and $-j$ the opposite state.

However, when the system is subject to shocks – under either (i) perfect foresight/deterministic transitions, (ii) MIT shocks i.e. unanticipated (zero probability) shocks or (iii) anticipated common noises – the system dynamics is described by the two following PDEs:

$$\rho\, v_j(a,t) = \partial_t v_j(a,t) + \max_c u(c) + \partial_a v_j(a,t)\, s_j(a,t) + \lambda_j(v_{-j}(a,t) - v_j(a,t)) \qquad \text{[HJB]}$$

$$0 = \partial_t g_j(a,t) + \frac{d}{da}[s_j(a)\, g_j(a,t)] + \lambda_j g_j(a,t) - \lambda_{-j} g_{-j}(a,t) \qquad \text{[FP]}$$

$$S(r,t) := \int_{\underline{a}}^\infty a\, g_1(a,t)da + \int_{\underline{a}}^\infty a\, g_2(a,t)da = K(r,t) \qquad \text{[Market clearing]}$$

$$s_j(a,t) = z_j w_t + r_t\, a - c_j(a,t) \qquad c_j(a,t) = (u')^{-1}(\partial_a v_j(a,t)) \qquad \text{[FOC]}$$

$$v_j(\cdot,T) = v^\infty \qquad g_j(\cdot,0) = g^0 \qquad \partial_a v_j(\underline{a}) \geq u'(z^j w + r\underline{a}) \qquad \text{[Boundary conditions]}$$

In the two cases, the third equations describe the market clearing, i.e. the Walrasian adjustment of prices (i.e. interest rate) to equalize supply of saving $S(r)$ and demand for capital $K(r)$. The last equation represents the optimality condition of the control variable c. The HJB actually features an optimization problem, and, since the objective $u(c)-pc$ is concave in c, the optimum is reached for: $u'(c^\star) = p$ and thus $c^\star = (u')^{-1}(p)$.

Here the state-constraint does not show up in the HJB (as could be the case in discrete time) but it appears in the boundary condition. It comes from the optimality of the maximization in the HJB – holding everywhere, and in particular at $\underline{a}$ – and the FOC is given by $u'(c^j(\underline{a})) = \partial_a v^j(\underline{a})$, and the state-constraint affecting the control: $s^j(a) = z_t^j w + r\, a - c^j(a) \geq 0$. Since u is concave, its derivative is decreasing, yielding the boundary condition:
$u'(z^j w + r\underline{a})) \leq u'(c^j(\underline{a})) = \partial_a v^j(\underline{a})$.

3 Algorithm: an overview

The objective is to find equilibrium of the MFG i.e. the functions v^j and g^j $(j = 1, 2)$ solving the two PDEs and the interest rate r clearing the market. We describe a general structure to find the stationary equilibrium, iterating on ℓ:

1. Guess interest rate r^ℓ, compute capital demand $K(r^\ell)$ & wages $w(K)$
2. Solve the HJB using finite differences (semi-implicit method): obtain $s_j^\ell(a)$ and then v_j^ℓ, by a system of sort: $\rho\,\mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; r)\mathbf{v}$
   - In the stationary equilibrium, solve for v_∞. In the transition case, compute the path of v^n, starting from the *terminal* condition $v^N = v_\infty$ and iterating *backward.*
3. Using $\mathbf{A}^T$, solve the FP equation (finite diff. system: $\mathbf{A}(\mathbf{v}; r)^T\mathbf{g} = 0$), and obtain g_j
   - In the stationary equilibrium, solve for g_∞. In the transition case, compute the path of g^n, starting from the *initial* condition $g^0 = g_0$ and iterating *forward.*
4. Compute the capital supply $S(\mathbf{g}, r) = \int_a^\infty a\, g_1(a)da + \int_a^\infty a\, g_2(a)da$
5. If $S(r) > K(r)$, decrease $r^{\ell+1}$, updating (using bisection method, and conversely, and come back to step 2.
6. Stop if $S(r) \approx K(r)$

In this problem, the FP is the adjoint equation of the HJB equation, cf. appendix for an informal presentation. As a result, the operator matrix in the Fokker Plank equation is the transpose (A^T) of the HJB operator matrix A – *taking optimal consumption as given.* Therefore, the most important would be to solve the HJB equation numerically and then to solve FP by 'simply' transposing the matrix A.

4 Solving the HJB: a finite-difference scheme

We use a finite difference method, an in particular an "implicit upwind scheme". We here provide a description of the numerical algorithm. The functions v_1, v_2, g_1, g_2 are approximated at I discrete points in the space dimension, $a_i, i = 1, ..., I$. Grids are equispaced, and we denote by h the distance between grid points in space and τ distance between grid points in time. We use the short-hand notation $v^n_{i,j} \equiv v_j(a_i, t_n)$. The derivative ${v^n_{i,j}}' = \partial_a v_j(a_i, t_n)$ is approximated with either a forward or a backward difference approximation:

$$
\begin{aligned}
\partial_a v_j(a_i, t_n) &\approx \frac{v^n_{i+1,j} - v^n_{i,j}}{h} \equiv {v^n_{i,j,F}}' \\
\partial_a v_j(a_i, t_n) &\approx \frac{v^n_{i-1,j} - v^n_{i,j}}{h} \equiv {v^n_{i,j,B}}'
\end{aligned}
$$

The aim of the upwind scheme is to use the proper scheme to approximate this first order term in the HJB equation. The $\partial_a v_j(a_i)$ is approximated with a forward difference ('décentrage à gauche') whenever the drift of the state variable $s^n_j(a_i)$ is positive and with a backward difference approximation ('décentrage à droite') whenever it is negative.

The implicit scheme will be a way to (i) iterate over the value function in the stationary case (from $n = 1$ until v^n is close enough from v_∞, to and (ii) solving for the value v^n at time t_n given the knowledge of v^{n+1} at t_{n+1}, since the HJB is running backward. The two corresponding implicit scheme will be the following :

(i) Stationary case: (1)

$$
\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\tau} + \rho v^{n+1}_{i,j} = u(c^n_{i,j}) + \underline{{v^{n+1}_{i,j}}'}(z_j w + r a_i - c^n_{i,j}) + \lambda_j (v^{n+1}_{i,-j} - v^{n+1}_{i,j})
$$

(ii) Time-varying case (running backward): (2)

$$
\frac{v^n_{i,j} - v^{n+1}_{i,j}}{\tau} + \rho v^n_{i,j} = u(c^{n+1}_{i,j}) + \underline{{v^n_{i,j}}'}(z_j w^{n+1} + r^{n+1} a_i - c^{n+1}_{i,j}) + \lambda_j (v^n_{i,-j} - v^n_{i,j})
$$

Several important remarks

Direction of time Since the second one is exactly the same scheme as the first, only by interchanging the direction of time and the iteration on n ($\tilde{t} \leftarrow T - t$ <u>and</u> $\tilde{n} \leftarrow N - n$), we will only focus on the first scheme for the description. All the results presented further will hold for the second case. Furthermore, it is surely more convenient to think of the numerical scheme as a search for v^{n+1} knowing v^n.

A semi-implicit scheme Concretely, this scheme is only *'semi implicit'* since the method do not approximate for the drift s^{n+1} but use the closed form solution for $s^n_{i,j} = z^j w^n + r^n a_i - c^n_{i,j}$ where $c^n_{i,j}$ is explicitely given by the first order condition:

$$c^n_{i,j} \approx c_j(a,t) = (u')^{-1}(\partial_a v_j(a,t)) \approx (u')^{-1}({v^n_{i,j}}') \tag{3}$$

The upwind scheme. It will aim at handling the first order underlined term above, to ensure the scheme is monotonous (cf. 'informal' theoretical results presented in the next two pages). This will imply to use a forward approximation ${v^n_{i,j,F}}'$ when the drift, i.e. the transport term, is positive $s^n_{i,j} > 0$, and conversely, use backward approximation ${v^n_{i,j,B}}'$ when the drift is negative $s^n_{i,j} < 0$. The drift will be mostly driven by consumption, and c.f. the previous remark, we will distinguish between two approximations for consumption:

$$\begin{aligned} c^n_{i,j,F} &= (u')^{-1}({v^n_{i,j,F}}') &\quad \Rightarrow \quad& s^n_{i,j,F} = z^j w^n + r^n a_i - c^n_{i,j,F} \\ c^n_{i,j,B} &= (u')^{-1}({v^n_{i,j,B}}') &\quad \Rightarrow \quad& s^n_{i,j,B} = z^j w^n + r^n a_i - c^n_{i,j,B} \end{aligned}$$

We therefore use the following approximation for the first order term:

$${v^n_{i,j}}'_{upwind} = {v^n_{i,j,B}}' \, \mathbb{1}_{\{s^n_{i,j,B}<0\}} + {v^n_{i,j,F}}' \, \mathbb{1}_{\{s^n_{i,j,F}>0\}} + {v^n_{i,j,0}}' \, \mathbb{1}_{\{s^n_{i,j,F} \leq 0 \leq s^n_{i,j,B}\}} \tag{4}$$

where some grids points fall in a set where the drift is approximately null, $s^n_{i,j,F} \leq 0 \leq s^n_{i,j,B}$. In this case, approximate the value function ${v^n_{i,j}}'$ by the utility when saving is null (i.e. consuming all income): ${v^n_{i,j,0}}' = u'(z_j w^n + r^n a_i)$.

Furthermore, the running gain $u(c^n_{i,j})$ is computed using the FOC eq. (3) and therefore $c^n_{i,j} = (u')^{-1}[{v^n_{i,j}}'_{upwind}]$ using the derivative of the value function obtained eq. (4)

The state constraint. Recall that theoretically – combining the FOC from HJB and the credit constraint – we obtained: $v^j(\underline{a}) \geq u'(z_j w + r\underline{a})$. Therefore, if the saving is negative at the boundary of the set, i.e. at a_1 in our discretized scheme, the upwind scheme imply to use a backward difference. A way to handle this state-contraint is to enforce it by setting:

$${v^n_{1,j,B}}' = u'(z_j w^n + r^n a_1)$$

This imply the state constraint is imposed whenever the backward difference approximation ${v^n_{1,j,B}}'$ would result in negative saving on the boundary. In the case the forward difference ${v^n_{1,j,F}}'$ is used – when $s_{1,j,F} > 0$ – the drift will move the agent away from the constraint and as a result the value function 'will never see the constraint'.

The initial guess. A simple initial guess for the value function is the value when savings is null and the agent never change state. The stationary HJB results in the guess:

$$v^0_{i,j} = \frac{u(z_j w + r a_i)}{\rho}$$

A matrix reformulation

With the notation $x^+ = max(x,0)$ and $x^- = min(x,0)$, the implicit scheme of the HJB eq. (1) rewrites:

$$\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\tau} + \rho\, v^{n+1}_{i,j} = u(c^n_{i,j}) \; + v^{n+1\,\prime}_{i,j,F}\, [z_j w + r a_i - c^n_{i,j,F}]^+ \; + \; v^{n+1\,\prime}_{i,j,B}\, [z_j w + r a_i - c^n_{i,j,B}]^- \; + \; \lambda_j (v^{n+1}_{i,-j} - v^{n+1}_{i,j})$$

$$\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\tau} \; + \; \rho\, v^{n+1}_{i,j} = u(c^n_{i,j}) \; + \; \frac{v^n_{i+1,j} - v^n_{i,j}}{h}\, [s^n_{i,j,F}]^+ \; + \; \frac{v^n_{i-1,j} - v^n_{i,j}}{h}\, [s^n_{i,j,B}]^- \; + \; \lambda_j (v^{n+1}_{i,-j} - v^{n+1}_{i,j})$$

where in the second line we only used the definition of $v^{n+1\,\prime}_{i,j,F}$ and $v^{n+1\,\prime}_{i,j,B}$. Collecting the terms with the same subscripts $(i-1, i, i+1)$, we can obtain:

$$\begin{aligned}
&\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\tau} \; + \; \rho v^{n+1}_{i,j} = u(c^n_{i,j}) \; + \; v^{n+1}_{i-1,j}\, x_{i,j} \; + \; v^{n+1}_{i,j}\, y_{i,j} \; + \; v^{n+1}_{i+1,j}\, z_{i,j} \; + \; v^{n+1}_{i,-j}\, \lambda_j \qquad (5)\\
&x_{i,j} = -\frac{(s^n_{i,j,B})^-}{h},\\
&y_{i,j} = -\frac{(s^n_{i,j,F})^+}{h} + \frac{(s^n_{i,j,B})^-}{h} - \lambda_j\\
&z_{i,j} = \frac{(s^n_{i,j,F})^+}{h}
\end{aligned}$$

Note that we obtained a system of $I \times 2$ equations described in eq. (5). An important point lies at the boundary, where $x_{1,j} = z_{I,j} = 0, (j = 1, 2)$. Thus $v^{n+1}_{0,j}$ and $v^{n+1}_{I+1,j}$ will never be used.

As a consequence of the various remarks, we can write the system with matrix notations:

$$\frac{v^{n+1} - v^n}{\tau} + \rho v^{n+1} = u + \mathbf{A}^n v^{n+1}$$

This system can in turn be written as

$$\mathbf{B}^n v^{n+1} = b^n, \qquad \text{with} \qquad \mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right) \mathbf{I} - \mathbf{A}^n, \quad b^n = u^n + \frac{1}{\Delta} v^n$$

This system of linear equations can be solved very efficiently using sparse matrix routines, since $\mathbf{A}^n$ and thus $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right) \mathbf{I} - \mathbf{A}^n$ are both tridiagonal by blocks.

In particular, we have the following form for matrix $\mathbf{A}$:

$$\mathbf{A}^n = \begin{bmatrix} y_{1,1} & z_{1,1} & 0 & \cdots & 0 & \lambda_1 & 0 & 0 & \cdots & 0 \\ x_{2,1} & y_{2,1} & z_{2,1} & 0 & \cdots & 0 & \lambda_1 & 0 & 0 & \cdots \\ 0 & x_{3,1} & y_{3,1} & z_{3,1} & 0 & \cdots & 0 & \lambda_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & x_{I,1} & y_{I,1} & 0 & 0 & 0 & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 & 0 & 0 & y_{1,2} & z_{1,2} & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & x_{2,2} & y_{2,2} & z_{2,2} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & x_{3,2} & y_{3,2} & z_{3,2} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & 0 & x_{I,2} & y_{I,2} \end{bmatrix}, \quad u^n = \begin{bmatrix} u(c^n_{1,1}) \\ \vdots \\ \vdots \\ u(c^n_{I,1}) \\ u(c^n_{1,2}) \\ \vdots \\ \vdots \\ u(c^n_{I,2}) \end{bmatrix}$$

Let us make few remarks on this matrix: First, one can notice it represents the infinitesimal generator of the stochastic process with drift s^n and jumping to the other state $-j$ with intensity λ_j. It is indeed a Markovian Jump process transition matrix on the discretized state space (a_i, z_j): all rows sum to zero and the diagonal terms are non-positive and off-diagonal terms are non-negative.

This intensity matrix will thus have nice properties for the numerical algorithm. We observe that if $\mathbf{A}^n$ has rows that sums to zeros, the matrix $-\mathbf{A}^n$ will be diagonally dominant (but not strict!)[6].

Consequently, provided that $\Delta < \infty, \rho > 0$ we obtain that $\mathbf{B}^n$ is a M-matrix – since $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n$. The scheme will thus will be monotonous.

Summary of Algorithm to solve the HJB equation.

1. Guess $v^0_{i,j}, i = 1, ..., I, j = 1, 2$ and for $n = 0, 1, 2, ...$ follow:
2. Compute $(v^n_{i,j})'$ using the current guess of the value function and the upwind scheme (forward difference when drift is positive, backward difference when drift is negative)
3. Compute c^n from $c^n_{i,j} = (u')^{-1}[(v^n_{i,j})']$
4. Find v^{n+1} by solving the system of linear equations involving the matrix $\mathbf{A}$ described above (implicit scheme)
5. If v^{n+1} is close enough to v^n: stop. Otherwise, go to step 2.

[6]A matrix B is diagonally dominant if $\forall i, |B_{ii}| \geq \sum_{j\neq i} |B_{ij}|$ (strictly dominant if the inequality is strict).
A matrix $\tilde{B}$ is a M-matrix if $\tilde{B_{ii}} > 0$ and $\tilde{B_{ij}} \leq 0, \forall j \neq i$ and $\forall i, B_{ii} > -\sum_{j\neq i} B_{ij}$
A matrix $\bar{B}$ is monotonous if it is invertible and its inverse is positive.

5 Solving the Kolmogorov Forward/ Fokker Planck equation

Recall the Fokker Planck in the evolution case:

$$0 = \partial_t g_j(a,t) + \frac{d}{da}[s_j(a)\, g_j(a,t)] + \lambda_j g_j(a,t) - \lambda_{-j} g_{-j}(a,t)$$

Subject to the 'constraint' $\int_{\underline{a}}^{\infty} g_1(a,t)da + \int_{\underline{a}}^{\infty} g_2(a,t)da = K(r,t)$ The discretization scheme – using the same grid as before – will aim at solving the equation:

$$0 = \frac{g_{i,j}^{n+1} - g_{i,j}^{n}}{\tau} + [s_{i,j}^n\ g_{i,j}^n]' + \lambda_j g_{i,j} - \lambda_{-j} g_{i,-j}$$

Conversely to the HJB, which is fully non-linear, the Fokker Planck equation is linear, and so will be the finite-difference scheme. The main advantage is thus that the solution g can be obtained in one step.

Upwind scheme. Similarly as in the previous case, we use a upwind scheme. However, since the direction of time if *forward*, the scheme is reversed: When drift is positive, the *backward* approximation is then used, and conversely, when the drift is negative one need to use the forward approximation:

$$0 = \frac{g_{i,j}^{n+1} - g_{i,j}^{n}}{\tau} + \frac{[s_{i,j,F}^n]^+\ g_{i,j}^n - [s_{i-1,j,F}^n]^+\ g_{i-1,j}^n}{h} + \frac{[s_{i+1,j,B}^n]^+\ g_{i+1,j}^n - [s_{i,j,B}^n]^+\ g_{i,j}^n}{h} + \lambda_j g_{i,j} - \lambda_{-j} g_{i,-j}$$

Note that because $g_{0,j}$ and $g_{I+1,j}$ are outside the state-space, the measure g will be null on these points, and thus $[s_{0,j}^n]^+$ and $[s_{I+1,j}^n]^-$ will never be used.

Matrix reformulation Rewriting the scheme, collecting the terms with the same subscripts $(i-1, i, i+1)$, we can obtain:

$$\begin{aligned}
&\frac{g_{i,j}^{n+1} - g_{i,j}^{n}}{\tau} \ +\ g_{i+1,j}^{n+1}\, x_{i+1,j} \ +\ g_{i,j}^{n+1}\, y_{i,j} \ +\ g_{i-1,j}^{n+1}\, z_{i-1,j} \ +\ g_{i,-j}^{n+1}\, \lambda_{-j} \\
&x_{i+1,j} = -\frac{(s_{i+1,j,B}^n)^-}{h}, \\
&y_{i,j} = -\frac{(s_{i,j,F}^n)^+}{h} + \frac{(s_{i,j,B}^n)^-}{h} - \lambda_j \\
&z_{i-1,j} = \frac{(s_{i-1,j,F}^n)^+}{h}
\end{aligned} \tag{6}$$

This discretization scheme yields the following system :

$$0 = \frac{g^{n+1} - g^n}{\tau} \ +\ (\mathbf{A}^n)^T g^{n+1}$$

Again, it can be solved implicitly:

$$\mathbf{C}^n g^{n+1} = g^n \qquad \text{with} \qquad \mathbf{C}^n = (\mathbf{I} - \tau(\mathbf{A}^n)^T) \tag{7}$$

Stationary distribution. In the stationary case, it amounts at solving the system:

$$0 = (\mathbf{A}^n)^T g^n$$

where the matrix $\mathbf{A}^T$ is the transposed of the intensity matrix $\mathbf{A}$ found in the HJB approximation. In such situation the distribution g^n can be found in one step.

However, one issue lies in the fact that the matrix $\mathbf{A}^T$ is singular and cannot be inverted. One need to 'fix' this by applying any value to a given (i, j), say $g_{i,j} = 0.1$. Since the stationary distribution of any Markovian process is unique modulo a multiplicative constant, we only need to renormalize to obtain: $\int_{\underline{a}}^{\infty} g_1(a,t)da + \int_{\underline{a}}^{\infty} g_2(a,t)da = 1$.
In practice, one solve this issue by setting the value of the diagonal of the matrix $(A^T)_{kk} = 1$ and $b_k = 0.1$ for any k, then solve for the value $\tilde{g}^n$ and the renormalization is $g^n_{i,j} = \widetilde{g_{i,j}}^n / (\sum_{i=1}^{I} \widetilde{g_{i,1}} h + \widetilde{g_{i,2}} h)$.

Moreover, this will allow to solve for the initial distribution when this one is unknown. More specifically, this will be the case in the application in the next section.

Transition case. In the case where shocks occur and economic fundamentals change, one can solve implicitly the system as in eq. (7), with τ the time step. Given the initial distribution g^0 computed above – as in the stationary case – computing the evolution will simply implies:

$$g^{n+1} = (\mathbf{C}^n)^{-1} g^n$$

FP as the adjoint of HJB The matrix $\mathbf{A}^T$ is the transposed of the intensity matrix found in the HJB. That is, once the HJB equation is solved, we basically get the Kolmogorov Forward equation "for free" ("two birds one stone", cf. B. Moll). These methods take advantage of the property of the Fokker-Planck being the adjoint of the HJB – when optimal control is reached. Since the matrix $\mathbf{A}^n$ is the discretized version of the infinitesimal generator $\mathcal{A}$ of the jump process, the matrix $\mathbf{A}^T$ will naturally be the discretized version of the adjoint operator $\mathcal{A}^*$ associated to the Fokker Planck equation (c.f. the appendix for details).

6 Equilibrium on capital market

The last step of the algorithm is to update the price and quantities for the market to clear. The market clearing conditions imposes :

$$\int_{\underline{a}}^{\infty} a\, g_1(a,t)da + \int_{\underline{a}}^{\infty} a\, g_2(a,t)da =: S(r,t) = K(r,t) := \Big(\frac{\alpha A}{r+\delta}\Big)^{\frac{1}{1-\alpha}} z_{av}$$

First, one of the theoretical result of Achdou, Han, Lasry, Lions and Moll (2017) provides a proof of unicity:
Under the assumption that IES $:= -\frac{u'(c)}{u''(c)c} \geq 1$ (i.e. $\frac{1}{RA} \leq 1$) and that the household is credit constrained $a \geq 0$, then we have:
(i) the optimal consumption $c_j(a,r) := (u')^{-1}(\partial_a v_j)$ is decreasing in interest rate r
(ii) the optimal saving $s(a,r) := s_j(a,t) = z_j w_t + r_t\, a - c_j(a,r)$ is increasing in r,
(iii) the stationary distribution shift rightward in r, i.e. the cdf $G_j(a;r)$ is strictly decreasing in r for all a in its support
(iv) $S(r)$ is strictly increasing and, since K(r) is clearly decreasing, the model has a unique stationary equilibrium.

In order to find iteratively this unique equilibrium, there exits several solutions:

One can iterate on quantities (B. Moll's codes). Given the supply of capital from household saving update the capital demanded by firms: $K^{new}(r,t) := \theta K^{old}(r,t) + (1-\theta) S(r,t)$ and then compute the interest rate as the marginal return of capital stock. The second option is to update the prices, i.e. interest rates, to change household saving behavior. Given the interest rate formula: $r(K) = \alpha\, A\, K^{\alpha-1} z_{av}^{1-\alpha} - \delta$, update the interest rate following the scheme:

$$r^{new} = \theta\; r(K(r^{old})) + (1-\theta)\; r(S(r^{old}))$$

The main advantage of this method is to be able to update in one step the *path* r_t for $t \in [0,T]$ given the *paths* K_t and S_t. When dealing with the stationary equilibrium however, another simple technique is the bisection method.

7 Adding common noise (aggregate shocks)

One of the main interest for economists is to understand how the economy will react to a exogenous shock (true for any model, with representative or heterogenous agents, in partial or general equilibrium, in discrete or continuous time). Most of the time the most conventional exogenous shocks is a shock on the aggregate level of productivity (TFP: total factor productivity), affecting firms in their production process. In the MFG literature, such aggregate shocks is referred as a 'common noise'.

Question: What stochastic process will follow the aggregate productivity? The form of the process will matter a lot for the anticipations of agents. The simplest way would be to consider a simple Brownian motion dB_t starting at a initial level $\bar{A}_0$ and stopped at a deterministic stopping time T (assuming to be the stationary equilibrium). This will be the situation considered in the following.

However, for exogenous shocks, economists often consider stationary processes as $AR(1)$ processes ($X_{t+1} = \mu + \varphi X_t + \sigma\varepsilon_t$). The closest process in continuous time would be the Ornstein-Uhlenbeck, that mean-reverts to the value μ: $dX_t = \theta(\mu - X_t)dt + \ \sigma dB_t$. To avoid the productivity level to be negative (which could be the case for Brownian Motion), one could also consider Geometric Brownian motion: $dX_t = X_t\mu dt + \ X_t\sigma dB_t$.

In the original article by Krusell and Smith, the aggregate state is defined as a Markov Chain, which could be defined by a Poisson /Jump process $dX_t = dN_t$ in continuous time.

According to me, growth theory have made substantial progress in endogenizing the productivity level, and there is a great need to understand how this might affect the distribution of agents. This was the object of several articles in O. Guéant's PhD thesis. One of the key question in heterogeneous agent models will be to understand quantitatively the rise in inequalities: is it due to an increase in the reward of the entrepreneur or is it due to a endogenous process of increase in the skill-premium?

However, in our case, we only consider an exogenous change in the TFP level. How will affect households? Recall that in general equilibrium it will first impact the marginal returns of factors and thus prices:

$$\begin{aligned} w_t &= (1-\alpha)\, A_t\, K^{\alpha} z_{av}^{-\alpha} \\ r_t &= \alpha\, A_t\, K^{\alpha-1} z_{av}^{1-\alpha} - \delta \end{aligned}$$

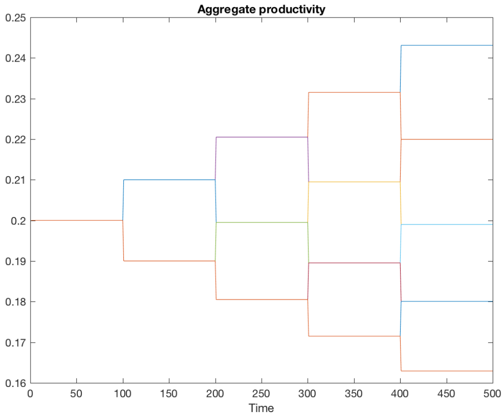
The consequence of this price change will change of labor (w_t) and capital (r_t) income of households. They will anticipate (through their value function v) the rise or fall of wages and interest, and change their saving behavior accordingly (s and thus g).

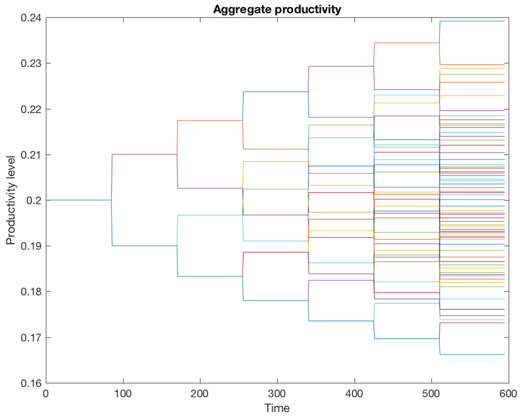
7.1 Numerical solutions – building and grafting trees

The main idea will be to approximate the process for the aggregate shock/common noise by a finite number M of 'simple' shocks. Since we consider a simple Brownian Motion, an obvious approximation would be to consider a process that rise or drop with probability 1/2. A justification lies in the Donsker theorem, that shows that a "normal"-random-walk converges in law toward a BM, when time increment goes to zero.

Building trees In our case, the process for productivity A_t, will start at $\bar{A}_0$ and every ΔT, it will switch between two values (or K values). This way, we will 'build' a *tree.* of different path of productivity, and from each node growing different branches. When taking $\Delta T \to 0$, you can approximate any process.

Two examples of trees, with $M = 4$ (qualitative example) and $M = 6$ (quantitative[7]). Left: 16 different paths, Right: 64 paths.





A way to 'solve' the MFG with common noise will be to compute the evolution of the MFG system (HJB and FP) on *each branch* (between each shock) for a given value $\tilde{A}$. We will obtain the equilibrium given by $v(a, j, t, \tilde{A})$, $g(a, j, t, \tilde{A})$ and $r_t(\tilde{A})$.

Grafting branches To integrate the evolution of the MFG during future and past shocks, we need to link the different branches together the appropriate way. To this purpose, we will use the boundary conditions of the system of PDEs. Recall the boundary conditions in the MFG:

$$v_j(\cdot, T) = v^\infty \qquad\qquad g_j(\cdot, 0) = g^0$$

The HJB runs backward and thus requires a terminal condition (the stationary value functions in our case), and the FP runs forward taking into account the initial distribution.

Suppose we have a finite numbers M of 'waves', for $m = 1, \dots, M$, t_m^- will be the time before revelation of the shock ($A_{t_m^-} = A_m$) and t_m^+ will be the time when the shock hits ($A_{t_m^+} = A_{m+1}$ take 2 (or K) values).

$$\begin{aligned} v(a, j, t_m^-, A_m) &= \sum_{k | A_{m+1} = A_k} \mathbb{P}(A_{m+1} | A_m)\, v(a, j, t_m^+, A_{m+1}) \\ g(a, j, t_m^-, A_m) &= g(a, j, t_m^+, A_{m+1}) \end{aligned}$$

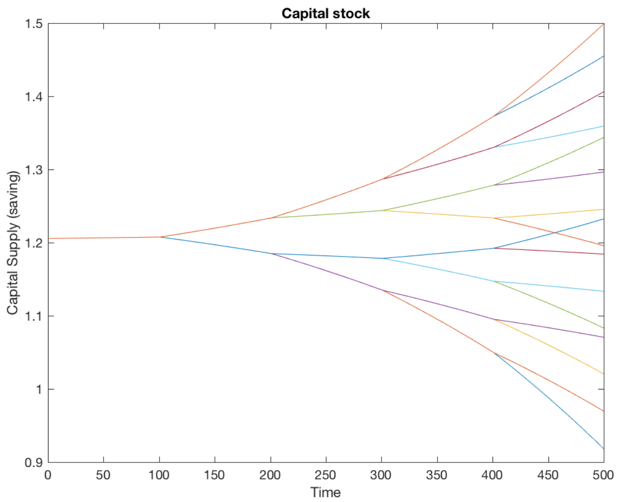
Since agents are forward looking (and rational!), they form expectations over the different set of future branches (accounted in the value function). The value function will 'jump' between t_m^- and t_m^+ to correct the (past) expectations. The distribution however will be backward-looking and will naturally account for the past drift values. It will be continuous in time t.

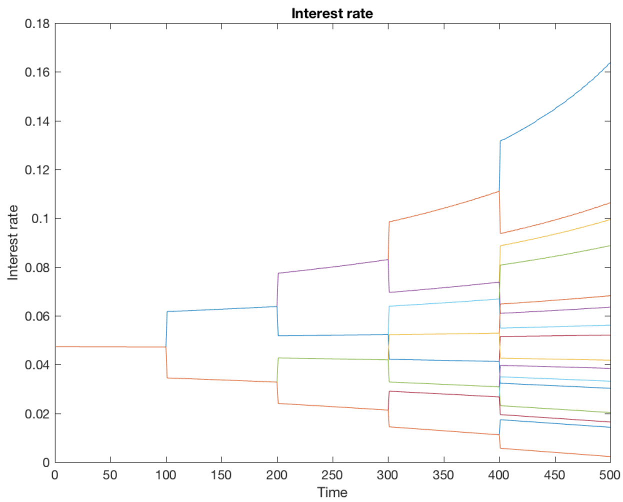
[7] my computer switched off when I tried 7 :'(

7.2 Results

The evolution of aggregate variables

The main interest of macroeconomists is to know whether microeconomic heterogeneity can have a impact on aggregate variables. In the following graph we plot the evolution of the main variables – the capital supply and interest rate – along each branch of the tree of productivity. As a result of the continuity of the wealth distribution g in time, the capital stock adjust continuously, unlike the interest rate that features jumps at each nodes of the tree. In particular, asset prices will adjust instantaneously at each node of the tree, and this for two reasons: (i) since the productivity rises, the capital will be more productive and the demand for capital from firms will bounce as well, rising the asset prices and decreasing the interest rate (ii) at the revelation of the common noise, there will be a shift in expectations of the households, and during the boom, they will reduce their precautionary saving, decreasing the demand for asset/capital and raising the interests (but relatively less than the previous motive).



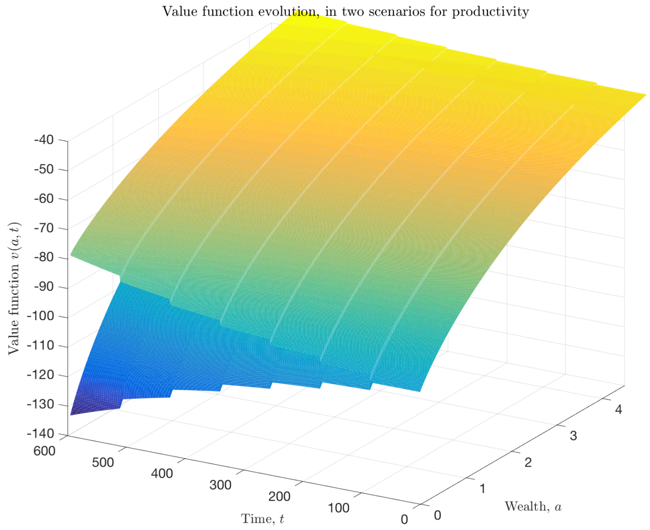


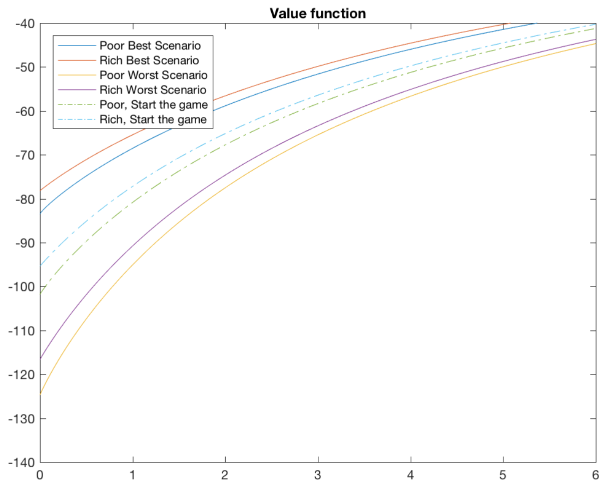
The value function and wealth distribution as a function of time

As described before, for each 'wave' of shock m, there will be a change of information about the productivity A_t between t_m^- and t_m^+. Therefore, the value function will be discontinuous at this point t_m to account for the revelation of the shock: this jump in the value function is link to the jump in asset price (interest rate) since the agents will evaluate differently the need of saving (and working in the case of endogenous labor). For two scenarios of productivity – one where the productivity only rises (the best case scenario) and one where it only falls (the worst case) – the next two graphs display the evolution of the value functions:

left: value function of the poor agents over time (case with $M = 6$ shocks),

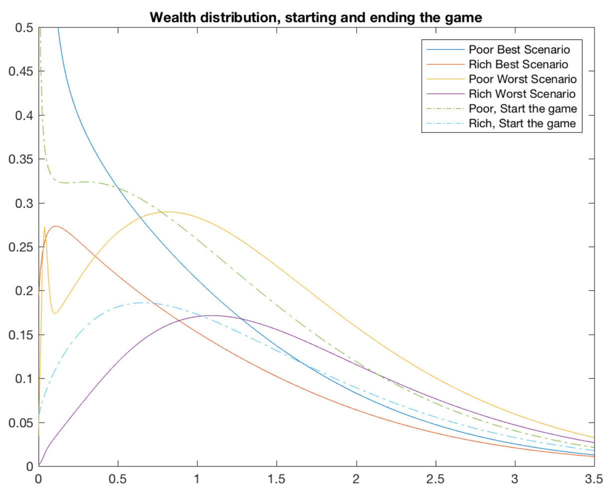
right: value function of both agents at the beginning and end of the game (case with $M = 4$),



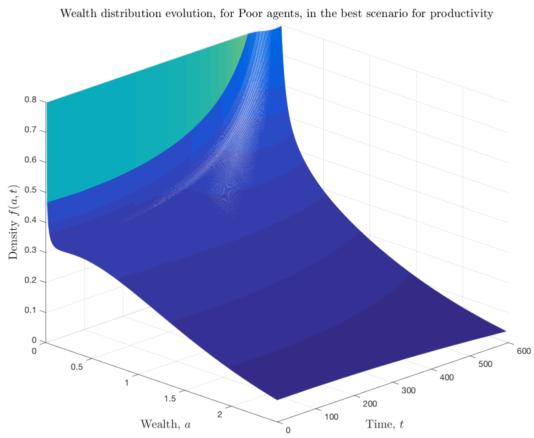


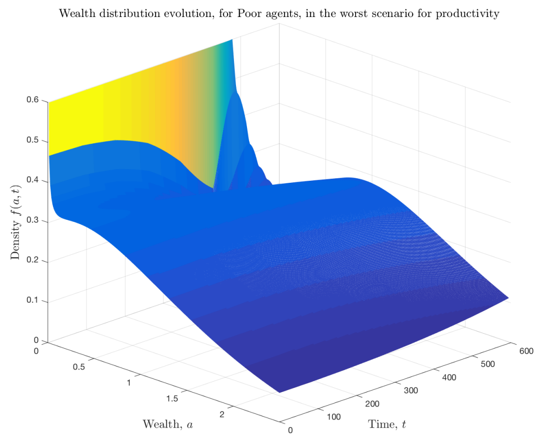
On the contrary, the wealth distribution is continuous over time. The main result of this experiment is the counter-cyclical nature of the wealth distribution, due to precautionary saving: when the productivity rises (in boom periods) agents decrease their wealth and the distribution is closer to the borrowing constraint. The reason for this is twofold: In period of economic growth, the wages rise, following the productivity. The income being higher that dampen the need to self-insure against future shocks. Moreover, the interest rate – as marginal return to capital – being lower, the agents will not find it interesting to hold asset as a source of income. These two motives discourage saving and both rich and poor agents will be concentrated in region with low wealth and wealth distribution will explode close to the constraint for the poor agents.

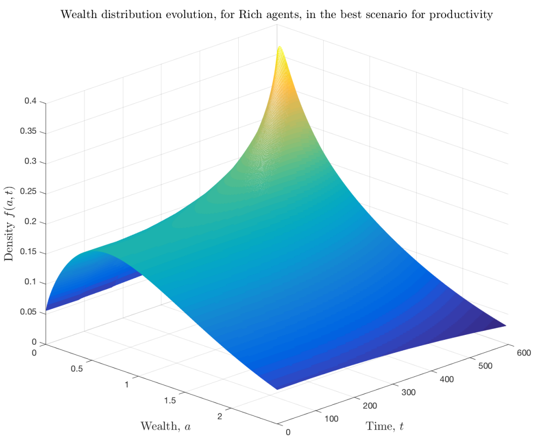
Wealth distribution at the beginning and at the end of the game (case with $M = 4$)

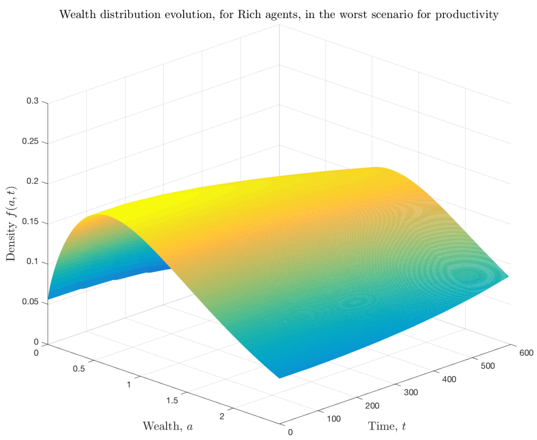


In particular, the next four graphs, the wealth distribution is represented as a function of time for the two scenario: best case (left) and worst-case scenario (right), for the poor agent (first line), or the rich (second line). Note hat the wealth distribution shift leftward in situation of recession (decrease in productivity), since the agents want to accumulate assets in order to self-insure against potential risk of further recessions or risk of falling into the low income state. In particular, for the poor agents, the value function is so low close to the constraint that the agents are incentivized to save a lot out of their income when they can, in order to avoid this 'poverty trap'. Therefore, the poor agents will drift more slowly and will try to stay away from the borrowing constraint: as a result, there will be no Dirac mass point at the state-constraint.







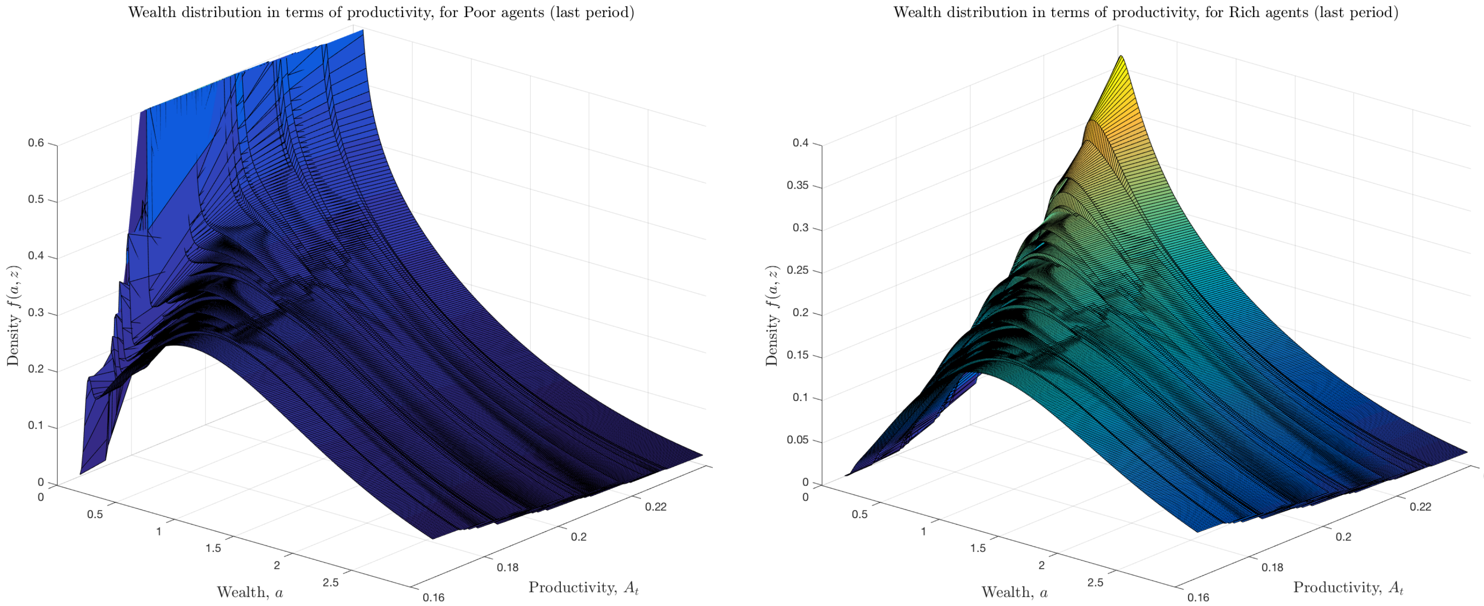


The wealth distribution as a function of productivity

With our approximation of the Brownian motion A, the main mathematical objective would be to compute $v(a, j, t, A)$ and $g(a, j, t, A)$ for <u>any</u> value of A. In our simulation we obtained the simulation of the MFG for a finite number of values of A. Still, we can draw the graph, at terminal time, of the wealth distribution. Note that for some values of A, there exists several values for the wealth distribution, corresponding to the different paths $(A_t)_t$ leading to the same value of A_T.

Typically, for high values, the wealth distribution will be higher when the shock is negative and then medium (the agent have saved to self insure), while when the TFP decreases after a boom, the agents will have consumed more. (think about 'la cigale et la fourmi').

Results – $g(a, 1, t, A)$ ***(left) and*** $g(a, 2, t, A)$ ***(right)***



Once again, the wealth distribution is concentrated closer to the credit constraint during booms in productivity, since the wages and thus income and consumption will be higher. Moreover, the capital stock being higher during booms, the interest rate – as marginal return to capital – will be lower. The agents – with both high and low income – will have little incentive to increase their precautionary saving since – even in the bad 'state of the world', their wage and interest will not encourage their saving. The logic is reversed in the case of recessions, where agents save more, and the wealth distribution is spread toward higher values.

Moreover, another interesting point lies in the fact that these recession episodes, while exacerbating the precautionary saving behavior, will 'remove' the Dirac mass point at the credit constraint. In a prolonged fear of unemployment during recessions – the wage being lower but interest higher – the household will save so much that none of them will be 'trapped' in the situation of null saving at the credit constraint.

Link with the master equation

This method is interesting to approximate the behavior of the MFG in presence of common noise. In particular, we *'discretized* the common noise in finitely many points. This approach is rationalized by the theoretical results of Carmona, Delarue, Lacker et al. (2016), where the authors also rely on an 'discretization' procedure in order to provide an existence proof of the MFG. In particular, when finding the measure of agents in the control problem, conditional on the common noise, the conditioning part seems to be problematic in the use of fixed-point arguments. Discretization seems to be an alternative approach and conditioning in a finite space can recover continuity that is necessary in the use of fixed-point theorem. Refining the discretization, the existence is proved at the limit. The proof however reads only as a 'weak equilibrium', due to several questions of adaptability of the solution to the filtration of the common noise.

Another approach would be to rely on infinite-dimensional equation, the master equation, as described in Cardaliaguet et al. (2015). This equation will be infinite-dimensional since the value function will encompass both the state variable and the measure of the agents over this state-space. This framework seems to be more flexible to include common-noise and associated numerical methods should be developed to avoid linearization methods as in Ahn, Kaplan, Moll, Winberry and Wolf (2017).

7.3 Quantitative comparison with representative agent model

The Aiyagari model is an extension with uninsurable shocks and credit frictions of the standard stochastic neoclassical growth model – introduced by Brock and Mirman (1972). In this section, I compare the two models with aggregate shocks, with 6 waves of change in productivity.

The supply-side (firm's side) is similar. The input are priced competitively and the demand for capital is expressed:

$$\begin{aligned}
K(r) &:= \Big(\frac{\alpha A}{r+\delta}\Big)^{\frac{1}{1-\alpha}} z_{av} && \text{Capital demand} \\
w &= (1-\alpha)\, A\, K^{\alpha} z_{av}^{-\alpha} && \text{Wage} \\
r &= \alpha\, A\, K^{\alpha-1} z_{av}^{1-\alpha} - \delta && \text{Interest rate}
\end{aligned}$$

In comparison, the demand (household) side is different: there is a single representative agent, optimizing its utility:

$$v(t_0, k_0, z_0) = \sup_{\{c_t\}_{t\ge 0}} \mathbb{E}_{t_0}\Big(\int_{t_0}^{\infty} e^{-\rho t} u(c_t) dt \Big)$$

$$dk_t = (z_t F(k_t) - \delta k_t - c_t) dt$$

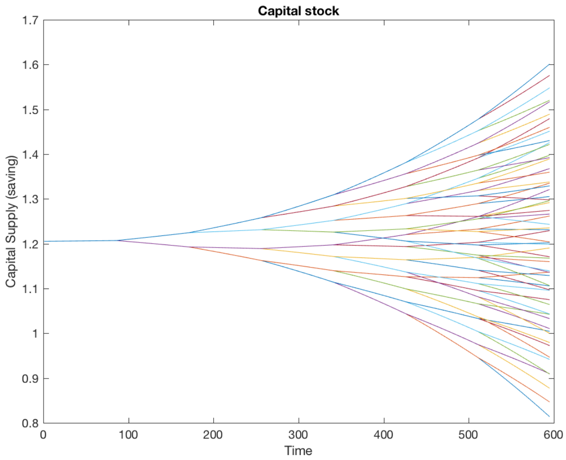
$$dz_t = \mu(z) dt + \sigma(z) dB_t$$

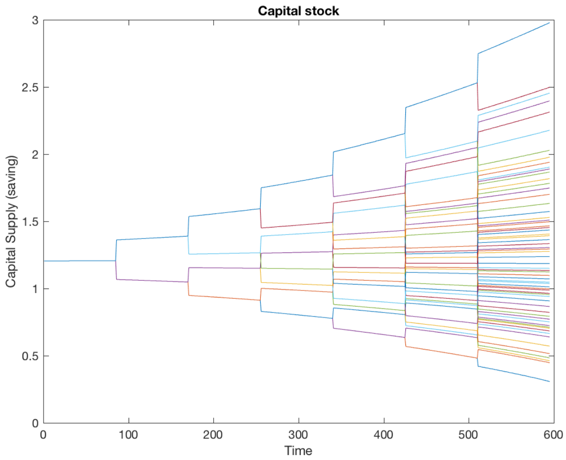
The Brownian motion is 'approximated' using the same discretization procedure as above, for the two models to be comparable. The control problem, using Dynamic Programming, yields the following HJB equation:

$$\rho v(k,z) = \max_c u(c) + \partial_k v(k,z)\big[zF(k) - \delta k - c\big] + \mu(z)\partial_z v(k,z) + \frac{\sigma(z)^2}{2}\partial^2_{zz} v(k,z)$$

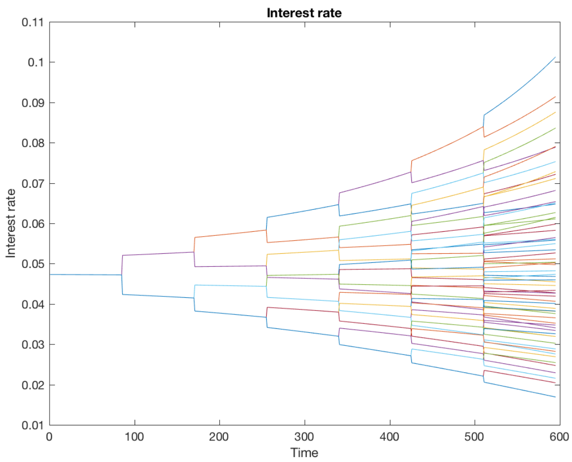
Since the supply side is similar, the dynamics are qualitatively similar, but the main interest is the quantitative difference. With the implication of the precautionary saving behavior of household in the first model – that do not exist in the second Brock Mirman model – the model will be *counter-cyclical.* It will *smooth* the aggregate fluctuations implied by the change in productivity, compared to the standard Neoclassical model.

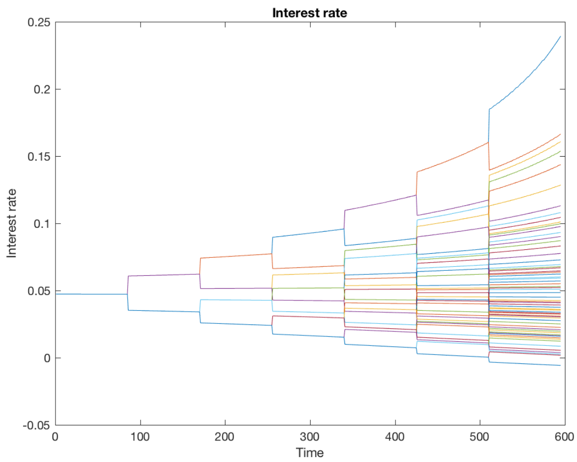
Capital supply On the left the Krusell/Smith model, on the right the Brock/Mirman





Interest rate On the left the Krusell/Smith model, on the right the Brock/Mirman





Precautionary savings reduces the fluctuation caused by productivity shocks. Quantitatively, it matters a lot: in the best-case scenario (where productivity only increases), the capital stock decreases from 3 to 1.6 in Krusell-Smith model. For interest rate the result is symmetrical. In the worst case scenario, where demand for capital is lower and interests are higher, the interest rate is smoothed in Krusell-Smith decreases to 10 % (compared with 22% in neoclassical model).

A final conclusion is the standard result is such type of model: precautionary saving with aggregate shocks is important quantitatively, and it matters for policy making, i.e. for monetary policy – to manage the path of interest rate – and fiscal policy – to act to redistribution.

7.4 Computational challenge

The main drawback of this method is to its computational complexity when the number of shocks increases. Approximating the Brownian motion with a discretization by a M-shocks random walk will raise the complexity by 2^n each time M increases by n. Therefore, I could not increase the number of shocks above $M > 7$. The following table displays the computational 'cost', in terms of number of computation of the H.J.B. and F.P. (the two steps where matrix inversion is needed).

Table 1: Summary – computational cost

Number of shocks ('waves')	Number of branches	Computing time (sec.)	(min.)	Number of computations H.J.B.	F.P.
2	4	70	1.6	285	210
3	8	240	4	578	403
4	16	510	8.5	1158	783
5	32	1196	19	2320	1545
6	64	2252	37	3071	2063

Note that the computing time increases exponentially (except between $M = 5$ and $M = 6$ where I found a way to shorten the loop on interest rate, shortening by 20% the computing time).

$$r^{new} = r\big(S(r^{old})\big) - \theta\Big[r\big(K(r^{old})\big) - r\big(S(r^{old})\big)\Big]$$

A easy step would be to try to accelerate this loop, by changing the step θ_k over the iterations.

In this table, note that I did not count the cost of storage of large $4-D$ arrays (may reach several Gb when $M \geq 7$) involved in the model : $v(t, a, j, A)$ and $g(t, a, j, A)$.

8 Conclusion

In this project, I described the numerical algorithm used in Achdou, Han, Lasry, Lions and Moll (2017), to simulate the Mean Field Game, composed of two PDEs: a Hamilton-Jacobi-Bellman describing the evolution of the value function and a Fokker-Planck describing the evolution of the distribution. These two PDEs were solved using a Finite-Difference Scheme. Given this basis, we added aggregate productivity shocks, i.e. common noise, in this MFG system. The main idea is to discretize the common noise and use the terminal condition for the HJB and the initial condition for the FP to link the different trajectories of the MFG system with common noise. The main result is to show that precautionary saving behavior of household against aggregate fluctuations is important quantitatively compared to the representative agent model.

Appendices

.1 Mean field games – a general framework

As described above, Mean field games is the equilibrium a system of two PDEs: a HJB and a FP equation. It is a mixture of various elements:

- Game theory: Nash equilibria when the number of players $N \to \infty$
- Stochastic control: the HJB equation
- Mean field theory: the FP equation

The usual assumptions of the (economic) framework allow to reduce the equilibrium to a system of PDEs. The agent control the drift of the diffusion, but not the variance. This allows the HJB to feature only first order terms in the Hamiltonian (the Laplacian goes out of H). Moreover, the agents should be small enough, so that one consider only inter-individual interactions through the interaction between an agent and the distribution. This will allow to derive the Fokker-Planck equation.

To obtain a general formulation of the typical MFG, one start from the stochastic control problem:

$$\sup_{\{\alpha_t\}_t^T} \mathbb{E}_t \Big(\int_t^T L(X_s, m_s, \alpha_s) ds + g(X_T, m_T) \Big)$$

Controlling the SDE: $\quad dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$

- Writing the Hamiltonian: $H(x, m, \nabla v) = \sup_a \big(L(x, m, a) + a \cdot \nabla_x v(t, x) \big)$
- If v is regular, the control is given by the feedback:

$$\alpha^\star = -D_p H(t, x, \nabla_x v)$$

This yields the system of PDEs

$$\begin{aligned}
&\text{(i)} \quad -\partial_t v - \nu \Delta v + H(t, x, \nabla_x v) = 0 \qquad in\ \mathbb{R}^d \times [0, T] \\
&\text{(ii)} \quad \partial_t m - \nu \Delta m - div\big(D_p H(t, x, \nabla_x v)\, m\big) = 0 \qquad in\ \mathbb{R}^d \times [0, T] \\
&\text{(iii)} \quad m(0, \cdot) = m_0(\cdot) \qquad\qquad u(x, T) = G(x, m_T)
\end{aligned}$$

.2 Operator theory – note on adjoints

If one recall the Feynman-Kac (or Kolmogorov Backward) equation (indirectly given by the Itô's formula):

$$\begin{cases} \partial_t w(t, x) + b \cdot \nabla_x w(t, x) + \frac{1}{2} Tr\big(\sigma\sigma^T D^2_{xx} w(t, x)\big) = 0 \\ v(T, x) = g(x) \end{cases}$$

When take the expectation (involving the measure m), one may find the following Fokker Planck (or Kolmogorov Forward) equation, which is 'simply' the adjoint equation of the one above:

$$\begin{cases} -\partial_t m(t, x) - \text{div}\big(b\, m(t, x)\big) + \frac{1}{2} D^2_{xx}\big(\sigma\sigma^T m(t, x)\big) = 0 \\ m(0, x) = m_0(x) \end{cases}$$

In particular, (and in rather heuristic way) we can say that the HJB given by the optimal control will "behave like" a Kolmogorov Backward equation. The two equations will therefore be coupled.

Recall some notes on adjoint operators, with $\varphi \in \mathcal{C}_c^\infty$ and $m \in \mathcal{D}'$:

- The gradient: $\langle \nabla\varphi, m\rangle = -\langle f, \mathrm{div}(m)\rangle$
- "Scaled" gradient: $\langle b\nabla\varphi, m\rangle = -\langle \varphi, \mathrm{div}(b\, m)\rangle$
- The Laplacian Δ is self-adjoint ($\approx$ symmetrical): $\langle \Delta\varphi, m\rangle = \langle \varphi, \Delta m\rangle$
- "Scaled Laplacian: $\langle \sigma\sigma^T\Delta\varphi, m\rangle = \langle \varphi, D^2_{xx}(\sigma\sigma^T m)\rangle$

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