Wealth distribution over the business cycle
A mean-field game approach

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Introduction

- Recent development in macroeconomics : the incorporation of agent heterogeneity in standard models :
  - The Aiyagari-Bewley-Huggett model,
    enriching the Brock-Mirman (1972) Stochastic Growth model
  - The HANK models
    enriching DSGE models, Woodford (2003), Gali (2008)
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  - Many others...
- Why is this important ?
  - Matching micro-data using macro models,
    e.g. the wealth and income distribution
  - Studying welfare implication of shocks and policies
  - Micro matters for macro :
    we reached the limits of representative agents model.

Introduction - toward mean-field games

- What are the limits of conventional HA theories ?
  - The shape of the distribution (wealth/income/consumption) is only a side effect of the models
  - No clear understanding of the evolution of the distribution

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  - Allow to study the dynamics of the distribution
- High entry cost (tools from functional analysis and stochastic calculus), but obtain new results easily.

This presentation : outline

Mean-Field Games – what is it ?

- A Nash equilibrium of a differential game, when the number of (symmetric and small) players become very large
  - Analogy with ”mean field” theory from particle physics
- It will consist of a system of two Partial Differential Equation (PDE) given by two building blocks :

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  - The evolution of the distribution of agents, when the agents control are given
    - One can characterize the agent distribution
    - Yield the second PDE : the Fokker-Planck (Kolmogorov-Forward) equation (FP)

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- I will do a brief lecture on these two points
  - Stochastic control and the HJB
  - Mean-Field theory and the FP.

The stochastic control problem – the HJB equation

- The aim of the agent is to maximize its objective function :

$$v(t_0, X_{t_0}) = \sup_{\{\alpha_t\}_{t_0}^T} \mathbb{E}_{t_0}\Big(\int_{t_0}^T L(t, X_t, \alpha_t)dt + g(X_T)\Big)$$

where v is the value function of the agent (at time t_0), L and G resp. the running gain and terminal gain.

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  where v is the value function of the agent (at time t_0), L and G resp. the running gain and terminal gain.
- α_t the (adapted) control variable and X_t is the state variable, (unique) solution of SDE :

$$\begin{cases} dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dB_t \\ X_{t_0} = x_0 \qquad\qquad (t_0, x_0) \in [0, T] \times \mathbb{R}^d \end{cases}$$

  where b is the drift, σ the variance and B_t a Brownian motion More on this.

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- Here, Bellman dynamic programming principle holds :

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- Use the Itô formula here to compute the value fct at time $t+h$:

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- This is the Hamilton Jacobi Bellman (HJB) PDE !

The stochastic control problem – the HJB equation

- The Hamilton-Jacobi-Bellman :

$$\partial_t v(t,x) + \sup_a \left\{ L(t,x,a) + \nabla_x v(t,x) \cdot b + \frac{1}{2} Tr\left(\sigma\sigma^T D^2_{xx} v(t,x)\right) \right\} = 0$$

- Sometimes, mathematicians write it with "Hamiltonian"

$$H(t,x,p,M) = \sup_a \left\{ L(t,x,a) + p \cdot b + \frac{1}{2} Tr\left(\sigma\sigma^T M\right) \right\} = 0$$

- and the HJB rewrites :

$$\partial_t v(t,x) + H(t,x,\nabla_x v, D^2_{xx} v) = 0$$

- The optimal control can be given in feedback form by the First-Order Conditions (FOC).

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- Plenty of different applications

The stochastic control problem – Applications

- Methods to find solutions :
  - Verification methods (guess and verify)
  - What if the fct v is not smooth ? (not $\mathcal{C}^{1,2}$)
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- Various applications in finance
  - One of the first applications : Merton portfolio selection problem
  - Optimal liquidation problems
  - Transaction costs and liquidity risk models
  - Applications in incomplete markets : super-replication of options (uncertain volatility models)

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- Many applications in economics !
  - Firm investment problems
  - Optimal investment/consumption strategies
  - Stochastic growth model and . . . RBC model !

Stochastic control – Applications – RBC model

$$v(t_0, k_0, z_0) = \sup_{\{c_t\}_{t \geq 0}} \mathbb{E}_{t_0}\Big(\int_{t_0}^{\infty} e^{-\rho t} u(c_t) dt \Big)$$

$$\begin{aligned} dk_t &= (z_t F(k_t) - \delta k_t - c_t) dt \\ dz_t &= \mu(z) dt + \sigma(z) dB_t \end{aligned}$$

- Applying the same methods, we can obtain the HJB :

$$\begin{aligned} \rho v(k, z) = \max_c u(c) &+ \partial_k v(k, z) \Big[zF(k) - \delta k - c \Big] \\ &+ \mu(z) \partial_z v(k, z) + \frac{\sigma(z)^2}{2} \partial^2_{zz} v(k, z) \end{aligned}$$

The evolution of the distribution – the Fokker-Planck equation

- The Fokker-Planck equation is known for a long time by physicists :
  - Used to compute the (probability) distribution of particles – e.g. fluid/gas – in a domain
  - Each particle is subject to shocks (e.g. diffusion).
  - In plasma physics, it corresponds to the Boltzmann equation.
- Knowing the initial distribution, one can compute the evolution of the distribution over time. It is forward in time.

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- After, I draw a direct link with stochastic calculus, through the use of the Feynman Kac formula
  - Feynman Kac is backward in time.
  - Also used a lot in option pricing, e.g. Black-Scholes model

The evolution of the distribution – the Fokker-Planck equation

- Suppose we consider N interacting particles X_t^i, $i = 1, \ldots, N$ subject to shocks (again a SDE) :

$$\begin{cases} dX_t^i = \frac{1}{N}\sum_{j=1}^N b(t, X_t^i, X_t^j)dt + \sigma dB_t^i \\ X_{t_0}^i = Y^i \end{cases}$$

with Y^i i.i.d. and B_t^i i.i.d. (independence is key !).

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with Y^i i.i.d. and B_t^i i.i.d. (independence is key !).
- What happen when $N \to \infty$?
  - 'Simply' use the law of large number !

$\frac{1}{N}\sum_{j=1}^N \varphi(Z^j) \to \int \varphi(z)m_Z(dz)$ where m_Z is the probability measure of the r.v. Z :

$$\begin{cases} dX_t^i = \int_{\mathbb{R}^d} b(t, X_t^i, y)m(dy)dt + \sigma dB_t^i \\ X_{t_0} = Y \end{cases}$$

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- From the evolution of these particles, one can obtain the Fokker-Planck :

$$\begin{cases} \partial_t m(t,x) - div\big(b\, m(t,x)\big) + \frac{\sigma^2}{2} D^2_{xx}\big(m(t,x)\big) = 0 \\ m(0,x) = m_0(x) \end{cases}$$

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  - $(b\nabla\cdot)^\star \equiv -div(b\cdot)$ and $(\sigma\sigma^T\Delta\cdot)^* \equiv D^2(\sigma\sigma^T\cdot)$

The evolution of the distribution – the Fokker-Planck equation – Link with Feynman-Kac

- If $w(t,x)$ is a $\mathcal{C}^{1,2}$ function and has bounded derivative, $\nabla_x v \in L^\infty$, and is solution of :

$$\begin{cases} \partial_t w(t,x) + b \cdot \nabla_x w(t,x) + \frac{1}{2} Tr\left(\sigma\sigma^T D^2_{xx} w(t,x)\right) = 0 \\ w(T,x) = g(x) \end{cases}$$

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- Then, the Feynman Kac formula gives us the form of the solution :

$$w(t,x) = \mathbb{E}_{t_0}\Big(g(X_T^{t_0,x})\Big)$$

where X_T is the solution of the SDE :

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- The above PDE is called Feynman-Kac equation or "Kolmogorov Backward equation" A general Feynman Kac thm

The evolution of the distribution – the Fokker-Planck equation – Link with Feynman-Kac

- The Feynman-Kac/Kolmogorov Backward equation is

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- When one return the time, one may find the following "Kolmogorov Forward equation"

$$\begin{cases} -\partial_t p(t,x) - div\big(b\, p(t,x)\big) + \frac{1}{2} D^2_{xx}\big(\sigma\sigma^T p(t,x)\big) = 0 \\ p(0,x) = p_0(x) \end{cases}$$

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  - More formally, this equation is the ”adjoint” equation of the KBE
    More on adjoint operators
  - $(b\nabla\cdot)^\star \equiv -div(b\cdot)$ and $(\sigma\sigma^T\Delta\cdot)^* \equiv D^2(\sigma\sigma^T\cdot)$

MFG – a general formulation

- Mean field games take advantage of these two PDEs, it is a mixture of various elements :
  - Game theory : Nash equilibria when the number of players $N \to \infty$
  - Stochastic control : the HJB equation
  - Mean field theory : the FP equation

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  - Stochastic control : the HJB equation
  - Mean field theory : the FP equation
- Usual assumptions :
  - The agent control the drift of the diffusion, but not the variance
  - The agents are small enough, so that we do not consider inter-individual interactions
    - Without this, no Fokker-Planck equation !

MFG – a general formulation

- The optimal control problem :

$$\sup_{\{\alpha_t\}_t^T} \mathbb{E}_t\Big(\int_t^T L(X_s, m_s, \alpha_s) ds + g(X_T, m_T) \Big)$$

Controlling the SDE : $dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$

  - Writing the Hamiltonian :
    $H(x, m, \nabla v) = \sup_a \big(L(x, m, a) + a \cdot \nabla_x v(t, x) \big)$
  - If v is regular, the control is given by the feedback :

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$$\alpha^\star = -D_p H(t, x, \nabla_x v)$$

- This yields the system of PDEs

$$\begin{aligned}
&\text{(i)} && -\partial_t v - \nu \Delta v + H(t, x, \nabla_x v) = 0 && \quad in\ \mathbb{R}^d \times [0, T] \\
&\text{(ii)} && \partial_t m - \nu \Delta m - div\big(D_p H(t, x, \nabla_x v)\, m \big) = 0 && \quad in\ \mathbb{R}^d \times [0, T] \\
&\text{(iii)} && m(0, \cdot) = m_0(\cdot) \qquad\qquad v(x, T) = G(x, m_T) &&
\end{aligned}$$

MFG – a general formulation

- The "economists-friendly" formulation would be :

$$
\begin{aligned}
&\text{(i)} \quad -\partial_t v(t,x) - \nu Tr\big(D^2_{xx} v(t,x)\big) + \big(L(t,x,a^\star) + a^\star \cdot \nabla_x v(t,x)\big) = 0 \\
&\text{(ii)} \quad \partial_t m(t,x) - \nu Tr\big(D^2_{xx} m(x,t)\big) - \sum_i \partial_{x_i}\big(a^* \, m\big) = 0 \\
&\text{(iii)} \quad m(0,\cdot) = m_0(\cdot) \qquad\qquad v(x,T) = G(x,m_T)
\end{aligned}
$$

where $a^\star$ is the optimal control for problem at (t,x)

  - Remember : $div(f(x)) = \sum_i^d \partial_{x_i} f(x)$ and $\Delta f(x) = Tr\big(D^2_{xx} f(x)\big) = \sum_i^d \partial^2_{x_i x_i} f(x)$

Wrapping-up

- ”Solving het. agents models = Solving PDEs” (cf. B. Moll)
  - A Hamilton-Jacobi-Bellman : backward in time
    How the agent value/decisions change when distribution is given
  - A Fokker-Planck (Kolmogorov-Forward) : forward in time
    How the distribution changes, when agents control is given

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    How the agent value/decisions change when distribution is given
  - A Fokker-Planck (Kolmogorov-Forward) : forward in time
    How the distribution changes, when agents control is given
- Let's see a concrete example : the Aiyagari-Bewley model :
  - Reference : Achdou, Han, Lasry, Lions and Moll (2017)

MFG – the Aiyagari-Bewley model

- This model has become the workhorse model to study income and wealth distribution in Macroeconomics
- Households are heterogeneous (ex-post) in their wealth a and income y, and solve an analogous stochastic control problem.

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- This model has become the workhorse model to study income and wealth distribution in Macroeconomics
- Households are heterogeneous (ex-post) in their wealth a and income y, and solve an analogous stochastic control problem.
- Income y_t is the only stochastic process (for now !) : Poisson processes with two states $z_t \in \{z_1, z_2\}$ with intensities λ_1, λ_2 (the higher the intensity, the higher the proba to jump).
- Can be generalized to any process (diffusion, Poisson, Levy)
- We can analyze both the stationary case and the transition case (evolving in time).

MFG – the Aiyagari-Bewley model

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$$a_t \geq \underline{a}$$

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  - Mathematically, it is not possible to find a PDE and a boundary condition on $\partial\Omega$ even in the sense of distribution.

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  - Mathematically, it is not possible to find a PDE and a boundary condition on $\partial\Omega$ even in the sense of distribution.
- This will result on both (i) a Dirac mass on the boundary and (ii) an explosion near the boundary.

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- The stochastic control problem is the following :

$$\max_{c_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\begin{aligned} \text{subject to :} \quad & da_t = (z_t w + r\, a_t - c_t) dt & \text{(Budget constraint)} \\ \text{and} \quad & a_t \geq \underline{a} & \text{(Credit constraint)} \end{aligned}$$

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subject to : $da_t = (z_t w + r\, a_t - c_t)dt$ (Budget constraint)

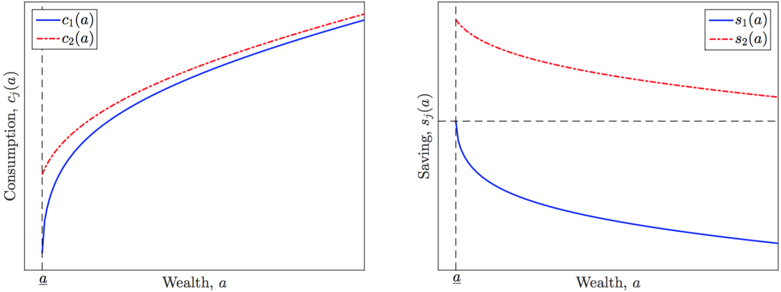
and $a_t \geq \underline{a}$ (Credit constraint)

- z_t Jump processes with two states $\{z_1, z_2\}$ (intensities λ_1, λ_2)
  - c consumption, ρ time pref., $u(\cdot)$ utility ($u' > 0, u'' < 0$).
  - $\underline{a} \geq -y_1/r_t$ natural borrowing limit.
  - r_t interest rate, w_t wage : adjust in general equilibrium.
- z_t idiosync. productivity can be generalized to diffusions : $dz_t = b(z_t)dt + \sigma(z_t)dB_t$.

Aiyagari-Bewley model : the Household

$$\max_{c_t} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt \qquad\qquad da_t = (z_t w + r\, a_t - c_t) dt$$

- The agent controls the drift : here $s_j(a) = z_j w + r\, a - c_j(a)$
  Optimal saving policy function, given by the FOC in the HJB :
  $c_j(a) = (u')^{-1}(\partial_a v_j(a))$



Aiyagari-Bewley model : the Firm

- Beside the household, capital is used by a representative firm :
  - Use capital K to produce $F(K, L) = A\,K^{\alpha}\,z_{av}^{1-\alpha}$
  - Rent it at the interest r,
  - Hire households and pay the wage w.
- Capital demand is thus :

$$K(r) := \left(\frac{\alpha A}{r+\delta}\right)^{\frac{1}{1-\alpha}} z_{av}$$

  - δ depreciation of capital and A productivity level
  - z_{av} is the average productivity of households : $z_{av} = \frac{z_1\lambda_2 + z_2\lambda_1}{\lambda_1+\lambda_2}$

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  - δ depreciation of capital and A productivity level
  - z_{av} is the average productivity of households : $z_{av} = \frac{z_1\lambda_2 + z_2\lambda_1}{\lambda_1+\lambda_2}$
- If one have the capital stock K, you could easily compute the interest rate and the wage paid by the firm :

$$w = (1-\alpha)\,A\,K^{\alpha} z_{av}^{-\alpha}$$

$$r = \alpha\,A\,K^{\alpha-1} z_{av}^{1-\alpha} - \delta$$

Aiyagari-Bewley model : a MFG formulation

- Doing the same computation as above, one obtain the system of PDEs
- The stationary case :

$$\rho v_j(a) = \max_c u(c) + \partial_a v_j(a)(z_j w + ra - c) + \lambda_j(v_{-j}(a) - v_j(a)) \quad \text{[HJB]}$$

$$0 = \frac{d}{da}[s_j(a)\, g_j(a)] + \lambda_j g_j(a) - \lambda_{-j} g_{-j}(a) \quad \text{[FP]}$$

$$S(r) := \int_a^\infty a\, g_1(a) da + \int_a^\infty a\, g_2(a) da = K(r) \quad \text{[Market clearing]}$$

- For the stationary case, these equations are simply ODE...
- When one add transition dynamics, we obtain PDEs

Aiyagari-Bewley model : a MFG formulation

- When studying the dynamics of the system, we obtain :

$$\rho\, v_j(a,t) = \partial_t v_j(a,t) + \max_c u(c) + \partial_a v_j(a,t)\, s_j(a) + \lambda_j(v_{-j}(a,t) - v_j(a,t)) \quad \text{[HJB]}$$

$$0 = \partial_t g^j(a,t) + \frac{d}{da}[s_j(a)\, g_j(a)] + \lambda_j g_j(a) - \lambda_{-j} g_{-j}(a,t) \quad \text{[FP]}$$

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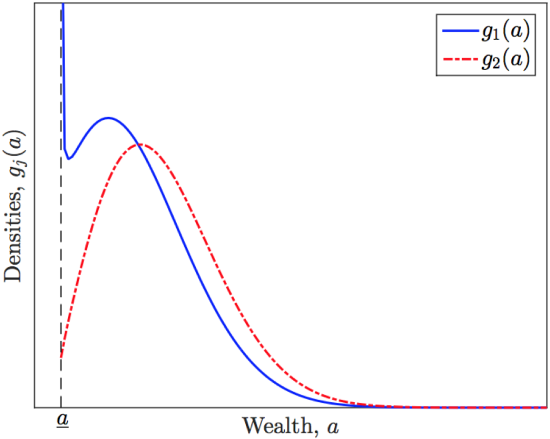
- Note :
  - We obtained as many PDEs as income states (z_1, z_2)
  - Idiosyncratic state j is a variable of value function. We could have written : $v(a,j,t)$
  - Will matter if z is a diffusion : adding a dimension is not free ...

Aiyagari-Bewley model : Stationary wealth distributions

- Two income states : Blue, poor agent, Red, rich agent
  - Dirac point mass at the borrowing constraint !



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  - Here it is a 'state' constraint, so it does show up in the HJB !
  - It would if it was a constraint on the control !

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- Why ?
  - FOC yields $u'(c^j(\underline{a})) = \partial_a v^j(\underline{a})$
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- v^j is determined by both (i) the HJB in the interior and (ii) the boundary condition $(*)$.
  - The junction may not be $\mathcal{C}^2$, $\rightarrow$ viscosity solutions.
  - Cf. any book on stochastic control or B. Moll slides on the topic

Aiyagari-Bewley model : theoretical results

- Achdou, Han, Lasry, Lions and Moll (2017) provide plenty of different theoretical results :
  1. Analysis of household decisions :
     - Full characterization of consumption and saving behavior :
     - Decision of the Poor, the Rich, close or far from the constraint
     - Time needed to hit the credit constraint
     - MPC and MPS (given by the Feynman-Kac formula !)

- A *general computational algorithm.*

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  4. Extension to 'soft'-borrowing constraint
     - Interest rate r higher is $a < 0$.
     - Theoretical characterization
     - Match empirical evidence : spike around zero net worth.
- A *general computational algorithm.*

Aiyagari-Bewley model : theoretical results

- An Euler equation :

$$(\rho - r)u'(c_j(a)) = u''(c_j(a))c_j'(a)s_j + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a)))$$

- Assumption 1 : Absolution risk aversion $R(c) = -u''(c)/u'(c)$ is finite when wealth a approaches the constraint $\underline{a}$.
- (Behavior of the poor) if $r < \rho$ and assumption 1 holds, then :
  - (Prop 1) $s_1(\underline{a}) = 0$ and $s_1(a) < 0$, they all decumulate assets except constrained individuals, who consume everything (poverty trap !).
  - (Cor. 1) Poor individuals hit the borrowing constraint in finite time, at a speed proportional to $\nu = (\rho - r)IES(c_1)c_1 + \lambda_1(c_2 - c_1)$

The algorithm : an overview

- Aim : find the equilibria : i.e. the functions v^j and g^j ($j = 1, 2$) and the interest rate r.
- General structure :
  1. Guess interest rate r^ℓ, compute capital demand $K(r^\ell)$ & wages $w(K)$
  2. Solve the HJB using finite differences (semi-implicit method) : obtain $s_j^\ell(a)$ and then v_j^ℓ, by a system of sort : $\rho\,\mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; r)\mathbf{v}$
  3. Using $\mathbf{A}^T$, solve the FP equation (finite diff. system : $\mathbf{A}(\mathbf{v}; r)^T\mathbf{g} = 0$), and obtain g_j
  4. Compute the capital supply $S(\mathbf{g}, r) = \int_{\underline{a}}^{\infty} a\, g_1(a)da + \int_{\underline{a}}^{\infty} a\, g_2(a)da$
  5. If $S(r) > K(r)$, decrease $r^{\ell+1}$ (update using bisection method), and conversely, and come back to step 2.
  6. Stop if $S(r) \approx K(r)$

Stationary MFG equations

The algorithm : advantages relative to discrete time :

1. Borrowing constraint only appears in the boundary conditions
   - FOCs $u'(c(a)) = \partial_a v^j(a)$ and HJB eq. always holds with equality
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3. The discretized system is easy to solve :
   - 'Simply' a matrix inversion
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   - Matrix is sparse (tridiagonal)
   - Continuous space : one step left or one step right

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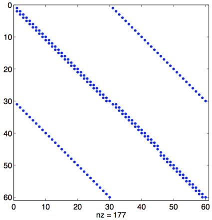
   - Matrix is sparse (tridiagonal)
   - Continuous space : one step left or one step right
4. HJB and FP are coupled
   - The matrix to solve FP is the transpose of the one of HJB.
   - Why ? Operator in FP is simply the 'adjoint' of the operator in HJB : 'Two birds one stone'
   - Specificity of MFG !

The algorithm : Finite difference scheme

- Finite difference scheme : discretize the state-space a_i for $i = 1, \dots I$.

$$\partial_a v_j(a_i) \approx \frac{v_{i+1,j} - v_{i,j}}{\Delta a} \equiv v'_{i,j,F} \qquad \partial_a v_j(a_i) \approx \frac{v_{i-1,j} - v_{i,j}}{\Delta a} \equiv v'_{i,j,B}$$



- Vector form :
- Linear system to solve $\mathbf{A}$ is sparse.

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; r)\mathbf{v}$$

$$0 = \mathbf{A}(\mathbf{v}; r)^T \mathbf{g}$$

$$S(\mathbf{g}, r) = K(r)$$

The algorithm : theoretical results

- This numerical solution converges to the unique (viscosity) solution of the HJB, under some conditions :
  1. Monotonicity (invertible and inverse positive)
  2. Consistent (approx error is majored by powers of step sizes)
  3. Stability (iteration in k is bounded)
- Is the matrix monotonous ?
  - In the scheme for solving the HJB, one can distinguish if the drift is positive or negative :
  - that is the upwind scheme
  - When $s(a) > 0$ use $v'_{i,j,F}$, and $s(a) < 0$, use $v'_{i,j,B}$
  - This insures the convergence of the algorithm

The algorithm : transition dynamics

- The algo for transitions is a generalization :
  - Discretization : $v_{i,j}^n$ and $g_{i,j}^n$ stacked into v^n and g^n
  - Somehow, it is more specific to Mean Field Games :

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- Take advantage of the backward-forward structure of the MFG
  - Make a guess r^ℓ_t $(t = 1, \dots, N)$ on the *path* interest rates.
  - Solve the HJB (implicit scheme), given terminal condition ;

$$\rho v^{n+1} = u^n + \mathbf{A}(v^{n+1}; r^n)\, v^{n+1} + \frac{v^{n+1} - v^n}{\Delta t}$$

$$v^N = v_\infty \qquad \text{(terminal condition = steady state)}$$

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  - Solve the FP forward, given the initial condition

$$\frac{g^{n+1} - g^n}{\Delta t} = \mathbf{A}(v^n; r^n)^T g^{n+1}$$

$$g^1 = g_0 \qquad \text{(initial condition)}$$

  - Update the interest rates path

The algorithm : wrapping up

- This algorithm to compute the dynamics of the system will be used a lot when adding aggregate shocks.
  - HJB start from the end (what agent anticipate) and runs backward until the computation of the initial value function
  - FP start from the beginning (what wealth agents hold) and runs forward to compute the evolution of distributions.
  - If there are discrepancies between capital demand and capital supply, loop to correct the path of interest rate.

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  - If there are discrepancies between capital demand and capital supply, loop to correct the path of interest rate.
- Performance of the algorithm :
  ($\approx$ 1000 grid points in space, 400 in time) :
  - Stationary equilibrium : 0.25-0.4 sec
  - Transition dynamics : around 50 secs
    - MIT shocks or perfect foresight.
    - 10^{-6} error on the path of interest rate.
  - What about anticipated shocks ?

Adding aggregate shocks

- That is where things start to complicate !
  - MFG literature, aggregate shocks referred as ’common noise’
- We still have the same household problem Don’t remember ?
- We suppose that the path of productivity A_t follow a stochastic process :

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- We still have the same household problem Don't remember ?
- We suppose that the path of productivity A_t follow a stochastic process :
  - Brownian motion dB_t
  - Ornstein-Uhlenbeck (the closest to AR(1) : mean revert) $dX_t = \theta(\mu - X_t)dt + \ \sigma dB_t$
  - Geometric Brownian motion (stay > 0) : $dX_t = X_t \mu dt + \ X_t \sigma dB_t$
  - Poisson /Jump process $dX_t = dN_t$
  - Will matter a lot !
  - Need your opinion on this !

Adding aggregate shocks

- That is where things start to complicate !
  - MFG literature, aggregate shocks referred as 'common noise'
- We still have the same household problem Don't remember ?
- We suppose that the path of productivity A_t follow a stochastic process :
  - Brownian motion dB_t
  - Ornstein-Uhlenbeck (the closest to AR(1) : mean revert) $dX_t = \theta(\mu - X_t)dt + \sigma dB_t$
  - Geometric Brownian motion (stay > 0) : $dX_t = X_t \mu dt + X_t \sigma dB_t$
  - Poisson /Jump process $dX_t = dN_t$
  - Will matter a lot !
  - Need your opinion on this !
    - According to me : need to endogenize it ;)

Adding aggregate shocks

- Why will it matter ? (for household ?)
- Affect firm's production, capital demand and ... interest rate and wages !

$$w_t = (1-\alpha)\,A_t\,K^{\alpha} z_{av}^{-\alpha}$$
$$r_t = \alpha\,A_t\,K^{\alpha-1} z_{av}^{1-\alpha} - \delta$$

- Household will anticipate (through v) the rise or fall of wages, and change their saving behavior accordingly (s and thus g)

Aggregate shocks – building trees

- Idea : approximate the process for the shock/common noise by a finite number M of 'simple' shocks :
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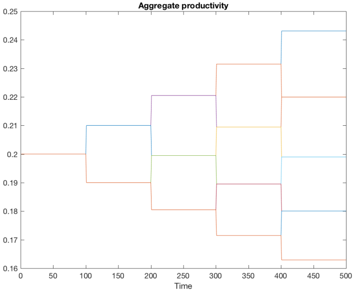
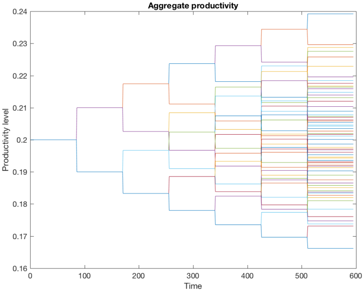
- Need to link the different branches together in the appropriate way

Aggregate shocks – building trees

- Two examples of trees, with $M = 4$ (qualitative example) and $M = 6$ (quantitative [1]).
- left : 16 different paths, right : 64 paths.





Aggregate shocks – grafting branches

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Aggregate shocks – grafting branches

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    t_m^+ : time when shocks hits ($A_{t_m^+} = A_{m+1}$ take 2 (or K) values)

$$v(a, j, t_m^-, A_m) = \sum_{k | A_{m+1} = A_k} \mathbb{P}(A_{m+1} | A_m) \, v(a, j, t_m^+, A_{m+1})$$

$$g(a, j, t_m^-, A_m) = g(a, j, t_m^+, A_{m+1})$$

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- Continuity of m in time t

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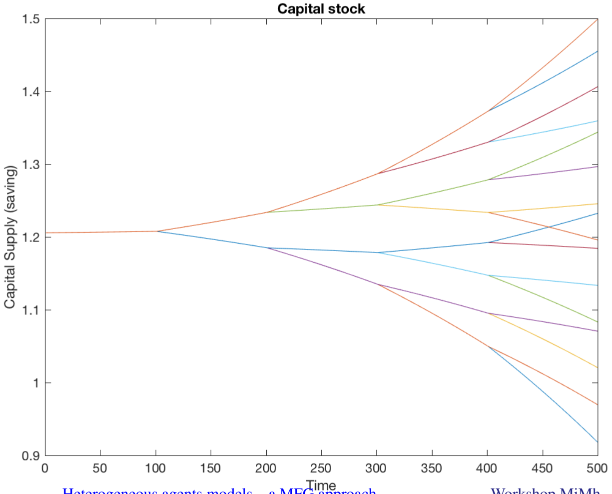
    - Nice, since the FP runs forward, no ?
- In practice, this loop on prices may take a long time.

Results – aggregate variables

- For each branch, one can compute capital stock and interest rates



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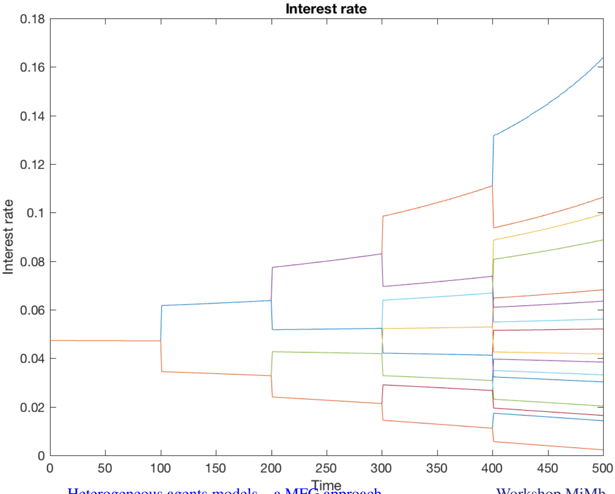

Results – v solution to HJB

- The value function evolves across time, with productivity
- Movie ?

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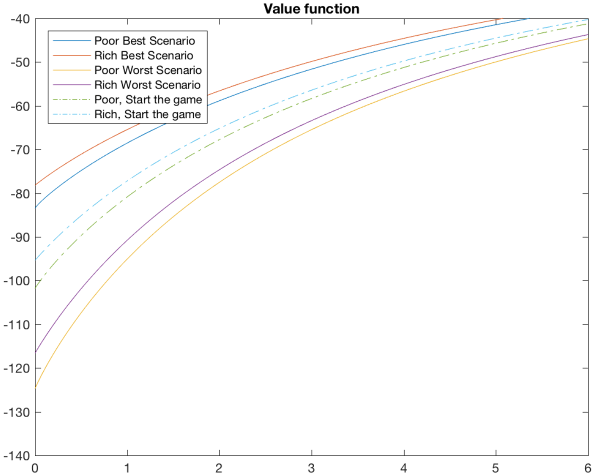
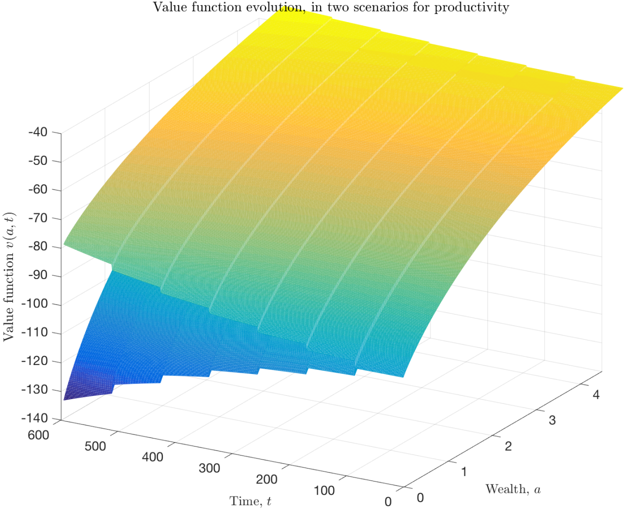


Results – jump in v

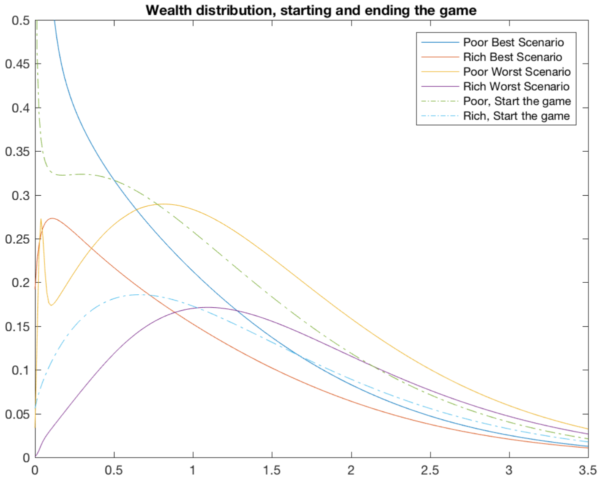


Results – g solution to FP

- The wealth distribution evolves across time, with productivity
- Movie ?

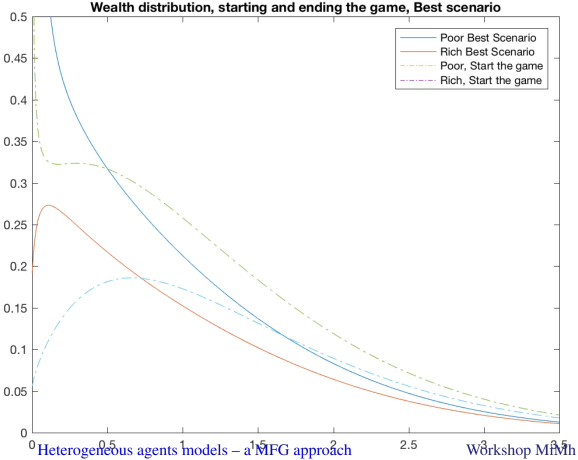
Results – g solution to FP

- The wealth distribution evolves across time, with productivity



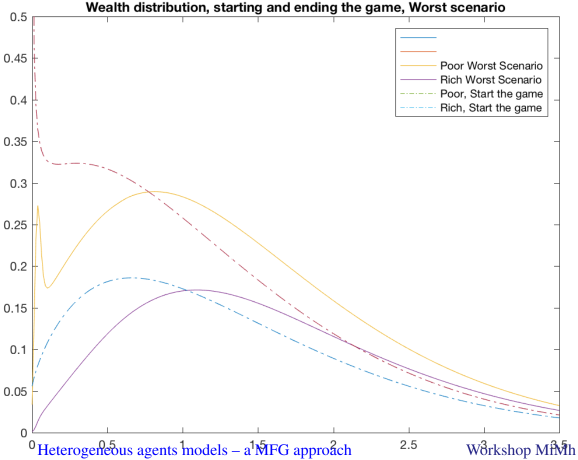
Results – g solution to FP

- The wealth distribution evolves across time, in the best case scenario (i.e. productivity increases !)



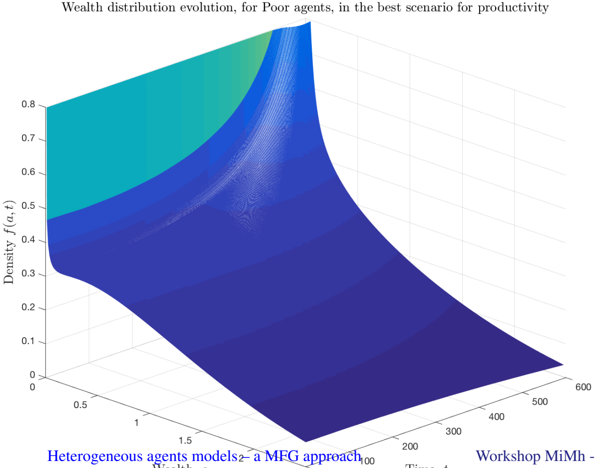
Results – g solution to FP

- The wealth distribution evolve across time, in the *worst* case scenario (i.e. productivity decreases !)



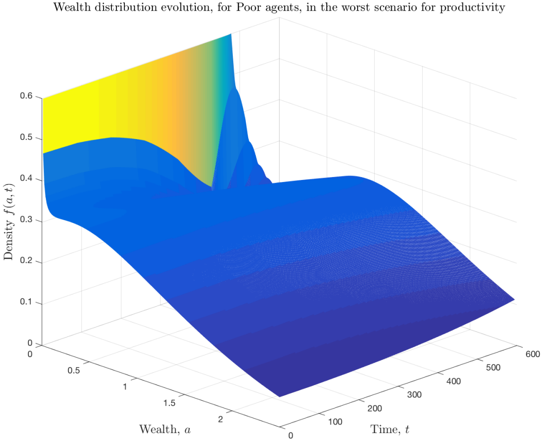
Results – g solution to FP

- The wealth distribution evolve across time, in the *best* case scenario (i.e. productivity increases !) [poor]



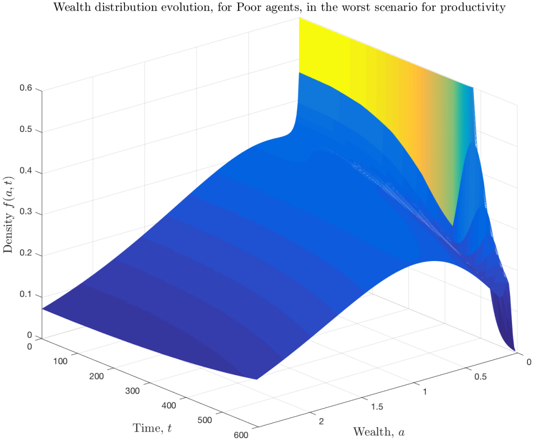
Results – g solution to FP

- The wealth distribution evolve across time, in the *worst* case scenario (i.e. productivity decreases !) [poor]



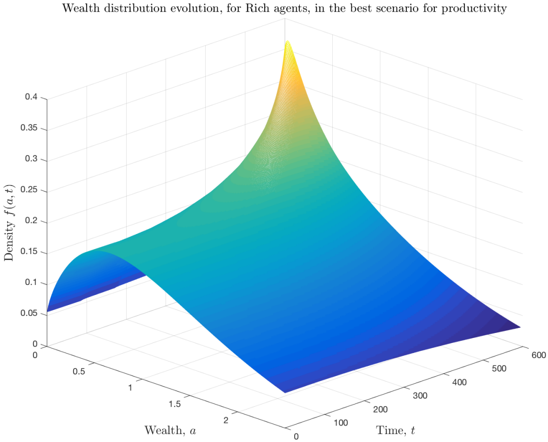
Results – g solution to FP

- The wealth distribution evolve across time, in the *worst* case scenario (i.e. productivity decreases !) [from behind]



Results – g solution to FP

- The wealth distribution evolve across time, in the *best* case scenario (i.e. productivity increases !) [rich]



Results – g solution to FP

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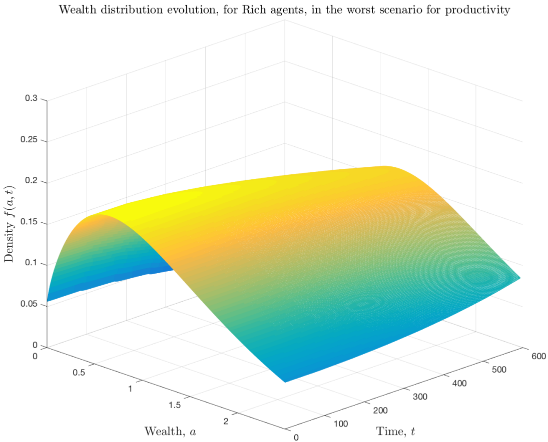
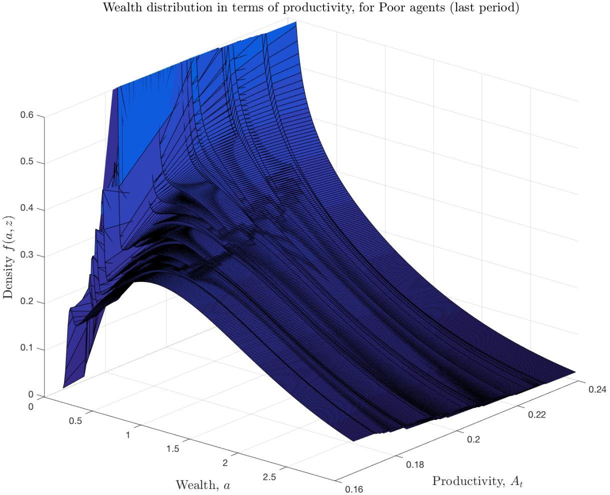


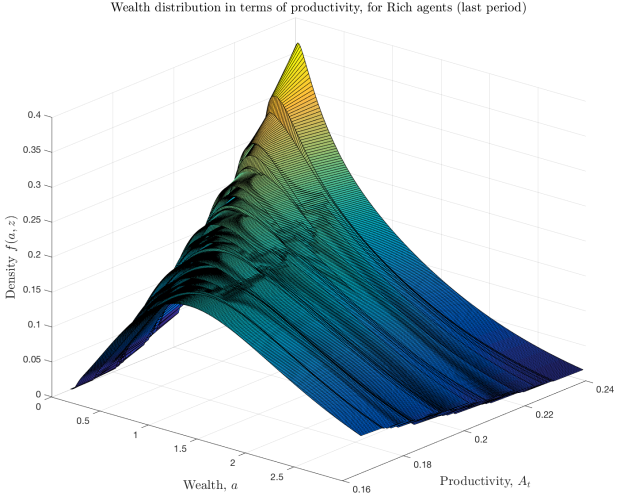
Results – mathematical objective

- The main idea, mathematically, is to be able to compute $v(a, j, t, A)$ and $g(a, j, t, A)$ for <u>any</u> value of A.
- Solving infinite-dimensional equation, i.e. the master-equation.
- Here, discretization procedure inspired by Carmona, Delarue and Lacker
- (btw : only 'weak equilibrium', question of adaptability of the solution)
- However, can still have a good approximation :

Results – objective – $g(a, 1, t, A)$



Results – objective – $g(a, 2, t, A)$

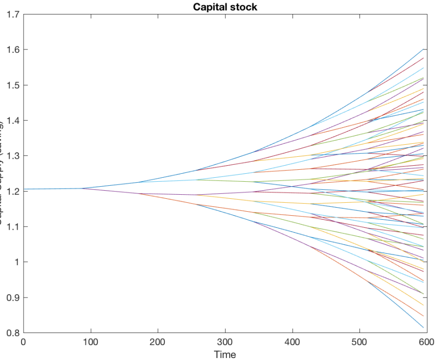


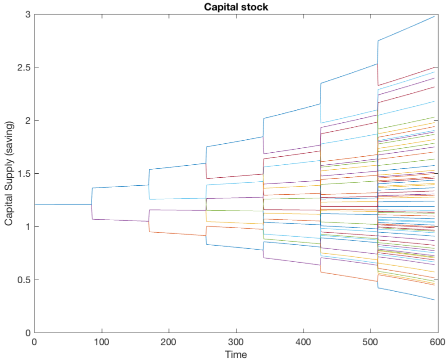
Results – comparison – RBC

- Need to compare the heterogeneous agent model with the Brock-Mirman (72) model
  - i.e. stochastic growth model, or RBC when adding endogenous labor supply
- I made use of the deterministic neoclassical growth model
- I build an approximation scheme for the Brownian motion, as before
- Solve the RBC (B/M) model on each branch of the tree
- Compare the graph quantitatively (with 6 shocks)

Results – comparison – RBC

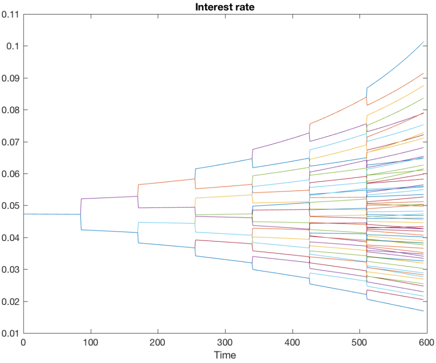
- On the left the Krusell/Smith model, on the right the Brock/Mirman

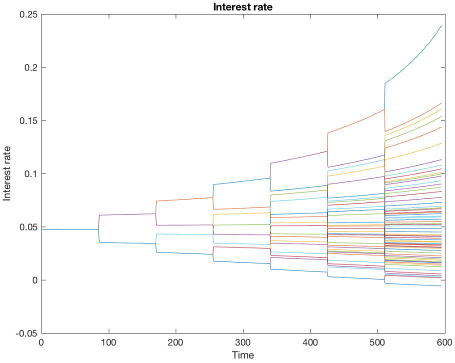




Results – comparison – RBC

- What about interest rate ? (left K/S, right B/M)





Results – comparison – RBC

- Precautionary savings reduce the fluctuation caused by productivity shocks
  - Capital : Best case scenario : decrease from 3 to 1.6 the aggregate level of capital
  - Interest rate : Worst case scenario : decrease from 22% to 10 % the interest rate.
- Smooth the business cycle !
- Well-known fact in such type of models !
- Precautionary saving with aggregate shocks : important quantitatively

Computational challenge

TABLE – Summary – computational cost

Number of shock ('waves')	Number of branches	Computing time (sec.)	(min.)	Number of computations H.J.B.	F.P.
2	4	70	1.6	285	210
3	8	240	4	578	403
4	16	510	8.5	1158	783
5	32	1196	19	2320	1545
6	64	2252	37	3071	2063

This without counting the cost of storage of large $4 - D$ arrays (may reach several Gb when $M \geq 7$)

Several limitation and future research

1. Computing time may be quite long, for $M \geq 7$
    - Solution : parallelize the algo, code it in C++ (internship : task 1)
    - Code it in Julia/Fortran (faster ?), use cloud computing (planned)
2. What about endogenous labor supply ?
    - With controls on c, s and ℓ : more heterogeneity
    - Solution : loop over wages to clear labor markets (algo ready, internship : task 2)
3. Idiosyncratic shocks follow 2 states process (boring ?) What about income as diffusion ?
    - Solution : Done in stationary equilibrium, need to study common noise (internship : task 3)
4. What if idiosyncratic shock is correlated to aggregate state
    - Solution : λ_j (or b/σ if diffusion) change with A_t (internship : task 4)

Several limitation and future research

5. Is it better than Reiter and Winberry's algorithm to study aggregate uncertainty ?
   - Avoid linearization, can use large shocks
   - Comparison with discrete time methods (planned)
6. And Krusell/Smith ? Does it feature approximate aggregation ?
   - Comparison with their algo (planned)
7. Extension to fat-tailed wealth distribution :
   - Wealth in illiquid asset/wealth hand-to-mouth behavior (Kaplan/Violante)
   - Need to add one dimension (internship : task 5 maybe)
8. What about the data ? Does it fit the business cycle time series ? the micro data ?
9. Extension with a demand side ? HANK ?
10. Fiscal/Monetary policy ?
11. Any suggestion ?

Conclusion

- MFG : high entry cost (need to study PDEs) but numerical algorithm more or less straightforward.
- Relevant framework to study evolution of wealth distributions along aggregate fluctuations
- Powerful tool with great adaptability / generalization of other models

- *Thank you for your attention !*

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Brownian motion

- This is the *"continuous-time"* stochastic process which is the closest to a random-walk.
- We define as a ***Brownian motion*** the continuous process W_t valued in $\mathbb{R}$ such that :
  1. The function $t \mapsto W_t(\omega)$ is continuous on $\mathbb{R}_+$
  2. For all $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\sigma(W_u, u \leq s)$
  3. For all $t \geq s \geq 0$, $W_t - W_s$ follows the normal distribution $\mathcal{N}(0, \sigma^2)$

Go back

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  - The brownian motion is "standard" if $W_0 = 0$ and $\sigma = 1$.
  - Here, the Brownian motion is a martingale
  - It is used to model any "small" shock in a continuous-time finance/macro models.
- *By Donsker theorem, one can show that a "normal"-random-walk converges in law toward a BM, when time increment goes to zero.*

Go back

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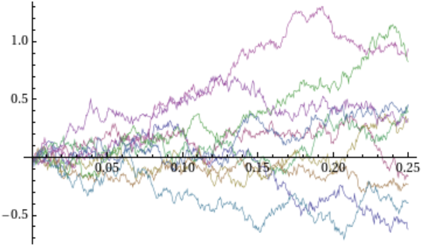

Itô's formula

- For any X_t Itô process :
$$dX_t = b_t\,dt + \sigma_t\,dB_t$$
and any $\mathcal{C}^{1,2}$ scalar function $f(t,x)$ of two real variables t and x, one has :
$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + b_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}\,dB_t$$
- For vector-valued processes $\mathbf{X}_t = (X_t^1, X_t^2, \ldots, X_t^n)$
$$d\mathbf{X}_t = \boldsymbol{b}_t\,dt + \sigma_t\,d\mathbf{B}_t$$
- The Itô formula rewrites :
$$\begin{aligned} df(t,\mathbf{X}_t) &= \frac{\partial f}{\partial t}(t,X_t)\,dt + \sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(t,X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 f}{\partial x_i\partial x_j}(t,X_t)d < X^i, X^j >_t \\ &= \partial_t f\,dt + \nabla_x f\cdot d\mathbf{X}_t + \frac{1}{2}Tr\left(\sigma_t\sigma_t^T D_{xx}^2 f\right)\,dt, \\ &= \left\{\partial_t f + \nabla_x f\cdot \boldsymbol{b}_t + \frac{1}{2}Tr\left(\sigma_t\sigma_t^T D_{xx}^2 f\right)\right\}dt + \nabla_x f\cdot\sigma_t\,d\mathbf{B}_t \end{aligned}$$

Go back

Feyman Kac - a general formula

- Consider the function

$$v(t_0,x_0) = \mathbb{E}_{t_0}\Big[\int_{t_0}^{T} e^{-\int_{t_0}^{s} r(u,X_u)du} f(s,X_s)ds + e^{-\int_{t_0}^{T} r(u,X_u)du} g(X_T)\Big] \qquad \forall (t,x) \in [0,T] \times$$

Supposing that X follows the SDE :

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_{t_0} = x_0 \qquad\qquad (t_0,x_0) \in [0,T] \times \mathbb{R}^d \end{cases}$$

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- The Feynman-Kac formula tells us that v is solution to the PDE :

$$\begin{cases} r(t,x)\, v(t,x) - \partial_t v(t,x) - \nabla_x v(t,x) \cdot b - \frac{1}{2} Tr\big(\sigma\sigma^T D^2_{xx} v(t,x)\big) = f(t,x) \\ v(T,.) = g \end{cases}$$

Feyman Kac - a general formula

- Consider the function

$$v(t_0,x_0) = \mathbb{E}_{t_0}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{s} r(u,X_u)du} f(s,X_s)ds + e^{-\int_{t_0}^{T} r(u,X_u)du} g(X_T)\right] \qquad \forall (t,x) \in [0,T] \times$$

Supposing that X follows the SDE :

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_{t_0} = x_0 \qquad\qquad (t_0,x_0) \in [0,T] \times \mathbb{R}^d \end{cases}$$

- The Feynman-Kac formula tells us that v is solution to the PDE :

$$\begin{cases} r(t,x)\,v(t,x) - \partial_t v(t,x) - \nabla_x v(t,x)\cdot b - \frac{1}{2}Tr\left(\sigma\sigma^T D^2_{xx} v(t,x)\right) = f(t,x) \\ v(T,.) = g \end{cases}$$

- Moreover, if $w(t,x)$ is $\mathcal{C}^{1,2}$ and has bounded derivative, then $w(t,x) = v(t,x)$, i.e. admits the representation above.
  - Intuitions : a function v of X subject to a diffusion can be represented by the expected future value g, adding running gain f and discounting r
- Used a lot in finance to compute option prices (Black-Scholes)
  - One can compute w using Monte-Carlo methods for instance

Operators - a primer

- If Operators are the infinite-dimensional version of matrices, Adjoint operator are the "equivalent" of transpose matrices.
- Most of the time an operator is a function applied on function :
  - Example : $\nabla : \mathcal{C}^1 \to \mathcal{C}^0, f \mapsto \nabla f$
- The basic idea of linear algebra extend to functional spaces :

$$\langle Mv_1, v_2 \rangle = \langle v_1, M^T v_2 \rangle$$

- The only difference is that inner product is "replaced" by duality brackets. For conventional functional spaces, it is "defined" as follow :

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\, g(x) dx$$

- The nice thing is that you get more flexibility : f or g can be much less regular : it can be probability measure $\mathcal{P}(\mathbb{R}^d)$ or "distributions" $\mathcal{D}'(\mathbb{R}^d)$ for instance.

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Operators - a primer

- This flexibility has a cost : one of the two functions should be regular enough to compensate for the irregularity of the other.
  - For instance, $f = \varphi \in \mathcal{C}_c$ and $g = m \in \mathcal{P}$:

$$\langle L\varphi, m\rangle = \int_{\mathbb{R}^d} L[\varphi](x)\, m(dx)$$

- Let's transpose an operator ! For our first example, $f \in \mathcal{C}^1$ and $g \in \mathcal{C}_c$ (compact support). Then, we already knew the result, actually (by integration by part) :

$$\begin{aligned}
\langle \nabla f, g\rangle &= \int_{\mathbb{R}^d} \nabla f(x) g(x) dx = \sum_i^d \int_{\mathbb{R}} \partial_{x^i} f(x^i) g(x^i) dx^i \\
&= \sum_i \left[f\, g\right]_{-\infty}^{\infty} - \sum_i^d \int_{\mathbb{R}} f(x^i) \partial_{x^i} g(x^i) dx^i \\
&= -\int_{\mathbb{R}^d} f(x) \nabla g(x) dx \\
&= -\langle f, \nabla g\rangle
\end{aligned}$$

- This can be generalized, even if $f \notin \mathcal{C}^1$. (Important, e.g. if the measure/ distribution of agents has (Dirac) mass points (at the credit constraint in our case).

Operators - a primer

- Following this technique, one can find the adjoints of common operators.
- Here are a few of them, with $\varphi \in \mathcal{C}_c^\infty$ and $m \in \mathcal{D}'$:
  - The gradient is given above : $\langle \nabla \varphi, m \rangle = -\langle f, \nabla g \rangle$
  - "Scaled" gradient : $\langle b \nabla \varphi, m \rangle = -\langle \varphi, div(b\, m) \rangle$
  - The Laplacian Δ is self-adjoint ($\approx$ symmetrical) : $\langle \Delta \varphi, m \rangle = \langle \varphi, \Delta m \rangle$
  - "Scaled Laplacian : $\langle \sigma \sigma^T \Delta \varphi, m \rangle = \langle \varphi, D^2_{xx}(\sigma \sigma^T m) \rangle$
- Remember : $div[f] = \sum_i^d \partial_{x_i} f$ and $\Delta f = Tr\left(D^2_{xx} f\right)$
- These formulas should be useful for most cases found in economics.
  - Still not convinced ?
- When we discretize the operators numerically (in the finite difference scheme), this will yield matrices that we can transpose without problem... Go back