The Inequality of Climate Change

Heterogeneity, Optimal policy and Uncertainty

WORK IN PROGRESS

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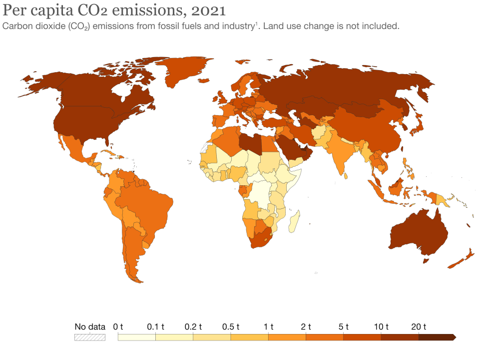
Capital Theory

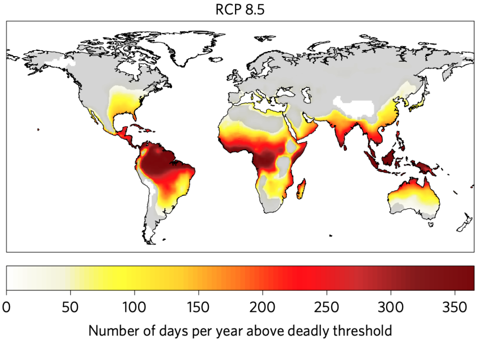
Mar 2023

Introduction – Motivation

Global warming is caused by greenhouse gas emissions (GHG) generated by human economic activity :

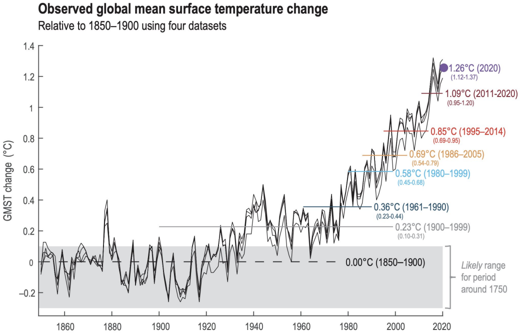
- ***Unequal causes :*** Developed economies account for over 65% of cumulative GHG emissions ($\sim$ 25% each for the EU and the US)
- ***Unequal consequences :*** Increase in temperatures disproportionately affects developing countries where the climate is already warm

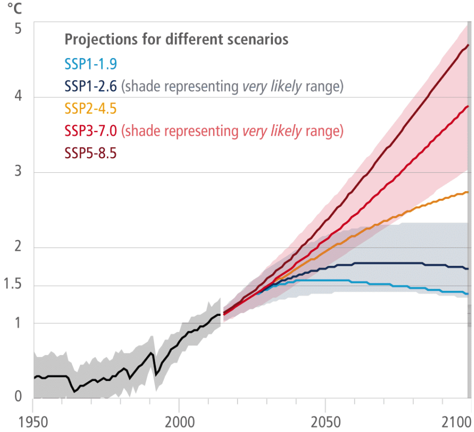




Introduction – Uncertainty

- However, the impact of climate change is uncertain for several reasons :
  - (i) Climate forecasts : temperature trajectories for a given path of emissions
  - (ii) Future growth : levels of future output for given damages
  - (iii) Path of emissions : Likelihood of pledges and mitigation policies to be actually implemented.





Introduction – this project

- Which countries will be affected the most by climate change and climate risk ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - Is the impact of climate risk quantitatively important ?

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  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions

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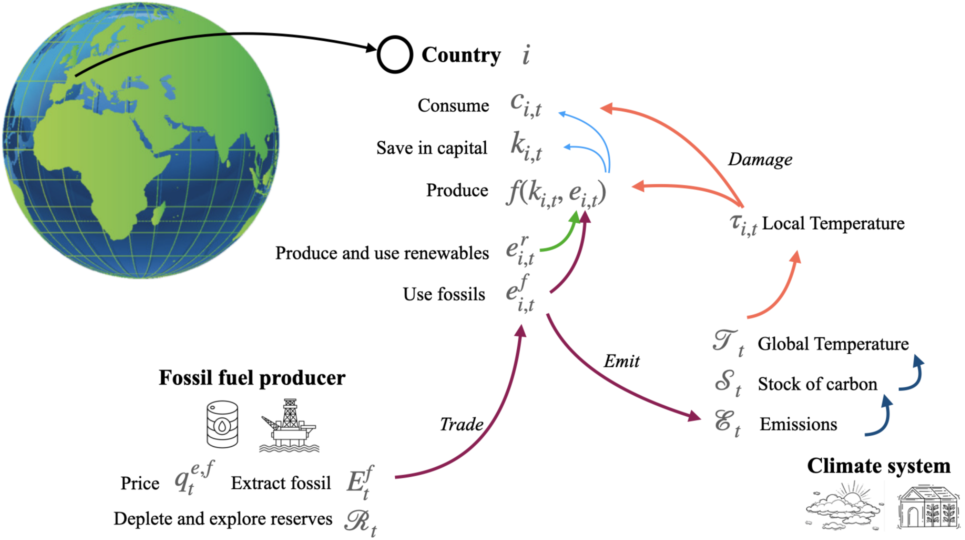
- Develop a simple and flexible model of climate economics
  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions
- Evaluate the heterogeneous welfare costs of global warming
  - Local Social Cost of Carbon can vary tenfold across countries
  - ... and $> 50\%$ across states of the world (within countries).
- Provide a new numerical methodology to :
  - Simulate globally – a sequentially – models with heterogeneous agents/countries
  - Handle aggregate shocks and different trajectories of temperatures
  - Potentially solve for optimal policy (Ramsey problem) in this environment

Introduction – related literature

- Classic Integrated Assessment models (IAM) :
  - Nordhaus' Multi-regions DICE (2016), Golosov Hassler Krusell Tsyvinski (2014)
  - Dietz van der Ploeg, Rezai, Venmans (2021), among others
- Macro (IAM) models with country heterogeneity :
  - Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021), among others
  - *This project :* Studies the uncertainty with heterogeneity and externalities
- Climate models with risk & uncertainty :
  - Cai, Lontzek, Judd (2019), Barnett, Brock and Hansen (2022)
  - *This project :* Includes heterogeneity and redistribution effects of climate & carbon taxation
- Quantitative spatial models :
  - Cruz, Rossi-Hansberg (2022), Bilal, Rossi-Hansberg (2023), Rudik et al (2022)
  - *This project :* Considers forward-looking decision of agents & optimal policy
- Heterogeneous Agents models with Aggregate Risk
  - Krusell-Smith (1998), Bhandari, Evans, Golosov, Evans (2018-), Proehl (2020), Schaab (2020), Fernandez-Villaverde et al (2022), Bilal (2022)
    Proba. approach to MFG : Carmona, Delarue (2018) and many more
  - *This project :* Studies climate externalities and Pigouvian taxation

Summary of the quantitative model



Model

- Neoclassical economy, in continuous time
- countries/regions $i \in \mathbb{I}$: ex-ante heterogeneous in dimensions $\underline{s}$
  - Here : $\underline{s}_i = \{p_i, z_i, \Delta_i\}$, relative heterogeneity doesn't change over time
  - Productivity grows at rate $\bar{g}$ and population grow at rate n
- regions heterogeneous ex-post $\bar{s}_i$
  - Here : capital and temperature $\bar{s}_i = \{k_i, \tau_i\}$
  - *Renormalization :* all variables are values per unit of efficient labor $k_{i,t} = \frac{K_{i,t}}{p_{t_0}} e^{-(\bar{g}+n)t}$
- Aggregate variables : global $\mathcal{T}$emperature, carbon $\mathcal{S}$tocks in atmosphere, fossil energy $\mathcal{R}$eserves $S = \{\mathcal{T}, \mathcal{S}, \mathcal{R}\}$

Model – Household and firm

- Household problem in country i :

$$\max_{\{c_{i,t},e^f_{i,t},e^r_{i,t}\}} \int_{t_0}^{\infty} e^{-(\rho-n)t}\, u(c_{i,t},\tau_{i,t})dt$$

- Dynamics of capital in every country i :

$$dk_{i,t} = \Big(\mathcal{D}^y(\tau_{i,t})z_{i,t}f(k_{i,t},e_{i,t}) - (n+\bar{g}+\delta)k_{i,t} - q^e_t e_{i,t} - c_{i,t}\Big)dt$$

- Choices :
  - c_t consumption, e_t energy, with production fct :

$$f(k,e) = \Big(\big(1-\varepsilon\big)^{\frac{1}{\sigma}}k^{\alpha\frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}}\big(z^e e\big)^{\frac{\sigma-1}{\sigma}}\Big)^{\frac{\sigma}{\sigma-1}}$$

  - Damage function $\mathcal{D}^y(\tau_t)$ affect country's production
  - Directed technical change z^e_t & energy mix e_t with fossil e^f_t vs. renewable e^r_t

Model – Energy markets

- Two sources of energy : fossil e_t^f and renewable e_t^r for every i

$$e_{i,t} = \left(\omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e - 1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r)^{\frac{\sigma^e - 1}{\sigma^e}} \right)^{\frac{\sigma^e}{\sigma^e - 1}}$$

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- Energy producers :
  - *For now :* Exogenous prices of fossils and non-carbon energy
    - Price of fossil : $q^{e,f}_t$ growing the rate g^f
    - Price of renewable $q^{e,r}_t$ growing the rate g^r
    - Exogenous Directed technical change
    - Energy efficiency z^e_t growing at the rate g^e.
  - *Future :* Static problems (Not Hotelling)
    - Production/extraction E^f_t vs. reserves $\mathcal{R}_t$ problem for fossil producers with $q^{e,f}_t = c'(E^f_t/\mathcal{R}_t)$
    - Production/generation e^r_t vs. capacity $\mathcal{C}_{i,t}$ problem for non-carbon producers $q^{e,r}_{i,t} = c'(e^r_t/\mathcal{C}_{i,t})$

Climate model :

- Fossil energy input e_t^f causes climate externality
  - World climate – global temperature $\mathcal{T}_t$ and cumulative GHG in atmosphere $\mathcal{S}_t$:

$$\mathcal{E}_t = \int_{\mathbb{I}} \xi \, e_{i,t}^f \, p_i \, di$$
$$d\mathcal{S}_t = \big(\mathcal{E}_t - \delta_s \mathcal{S}_t\big) dt$$
$$d\mathcal{T}_t = \zeta(\chi \mathcal{S}_t - \mathcal{T}_t) dt$$

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  - Impact of climate on country's local temperature :

$$\dot{\tau}_{i,t} = \Delta_i \dot{\mathcal{T}}_t$$

Climate model : Extension

- Future : more sophisticated climate block – Nordhaus (2016), Cai, Lontzek, Judd (2018)
  - Emissions come from Land and Fossil

$$\mathcal{E}_t = \mathcal{E}_{\ell,t}(\mathcal{T}) + \mathcal{E}_{f,t}$$

  - World divided in "boxes" : AT : atmosphere, UO Upper Ocean+Biosphere, LO Lower Ocean

$$\mathcal{M}_t = (M_{AT,t}, M_{UO,t}, M_{LO,t}) \qquad \mathcal{T}_t = (T_{AT,t}, T_{LO,t})$$

  - Carbon Cycle, Radiative forcing and Temperature dynamics

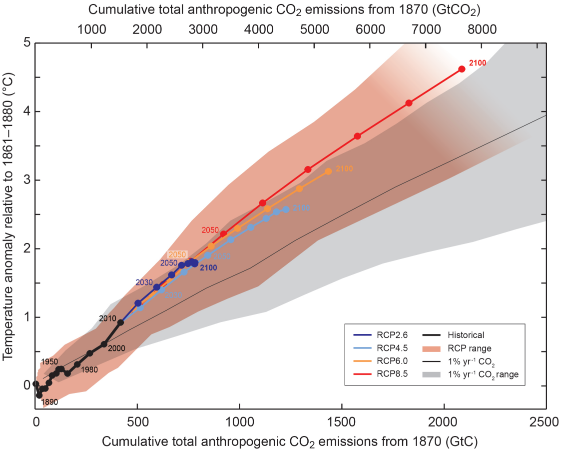
$$d\mathcal{M}_t = \left(\Phi_M \mathcal{M}_t + (\mathcal{E}_t, 0, 0)^T\right) dt$$

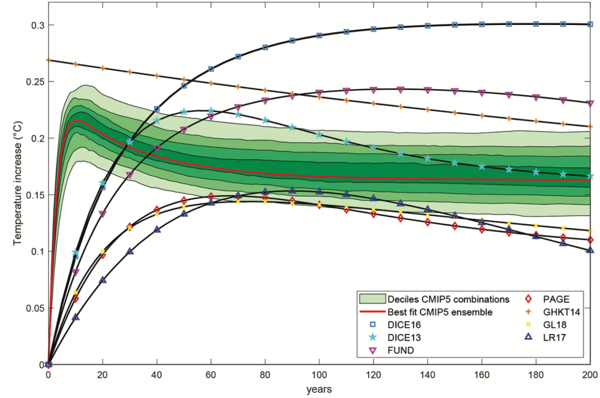
$$\mathcal{F}_t = \eta \log\left\{\frac{M_{AT,t}}{\bar{M}_{AT}}\right\} + \mathcal{F}_{ex,t}$$

$$d\mathcal{T}_t = \left(\Phi_T \mathcal{T}_t + (\zeta \mathcal{F}_t, 0)^T\right) dt$$

  with Φ_M and Φ_T Markovian matrices
- Adding 5-6 states variables : No challenge for the sequential method at hand !

Temperature dynamics





Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

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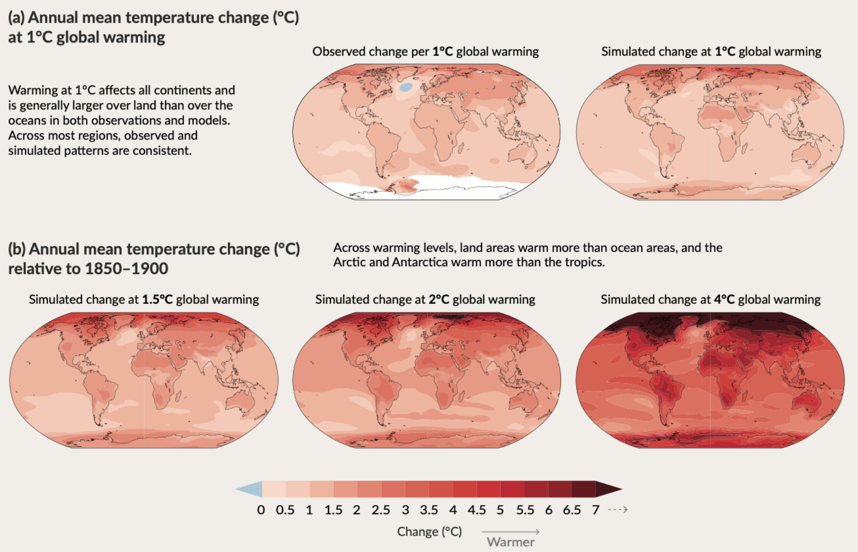

Model Solution :

- Mathematical theory : Draw from the sequential approach :
  - Pontryagin Maximum Principle (PMP)
  - Case with risk : Stochastic PMP, c.f. Yong, Zhou (1999)
  - Case with heterogeneous agents / idiosyncratic risk :
    - Probabilistic Approach to Mean Field Games : FBSDE
    - Social Planner $\Leftrightarrow$ Mean Field Control / McKean Vlasov FBSDE
    - Carmona, Delarue (2018, Vol 1) / Pham et al (2018-)
  - Case with heterogeneous agents / aggregate risk :
    c.f. Carmona, Delarue (2018, Vol 2)

- Method : Shooting algorithm
  - Extended to Heterogeneity and Aggregate Risk
  - Application to Economics
- Optimal Policy and Ramsey Problem :
  - McKean Vlasov Control problem : Social Planner

Model Solution with Heterogeneity & Aggregate Risk

- States variables :
  - Individual states : $x_{i,t} \in \underline{\mathbb{X}} \subset \mathbb{R}^d$ (possibly with state-constraints)
  - Aggregate states : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$
  - Distribution of individual states $P_{x,t}$
  - $x_{i,t}$ and X_t diffusion process with control $c^\star(x, \mathcal{X}, y, z) \in \mathbb{C}$

$$dx_{i,t} = b(x_{i,t}, \mathcal{X}_t, c^\star_{i,t})dt + \sigma(x_{i,t}, \mathcal{X}_t)dB^0_t$$

$$d\mathcal{X}_t = \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma}(\mathcal{X}_t, P_{x,t})dB^0_t$$

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- Hamiltonian :

$$\mathcal{H}(x, y, z, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \max_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y + \sigma(x, \mathcal{X}) * z \big) + \bar{b}(\mathcal{X}_t, P_{x,t}) \cdot \mathcal{Y} + \bar{\sigma}(\mathcal{X}_t, P_{x,t}) * \mathcal{Z}$$

- Optimal control $c^\star \in \mathrm{argmax}_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y \big)$

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- Optimal control $c^\star \in \text{argmax}_{c \in \mathbb{C}} \Big(u(x, c) + b(x, \mathcal{X}, c) \cdot y \Big)$
- Using the Stochastic Pontryagin maximum principle :

$$dy_{i,t} = -D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t})dt + z_{i,t} dB^0_t$$
$$d\mathcal{Y}_{i,t} = -D_X \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t})dt + \mathcal{Z}_{i,t} dB^0_t$$

Model solution : FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} & = D_y \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t}) dt + \sigma(x_{i,t}, \mathcal{X}_t) dB_t^0 \\ dy_{i,t} & = -D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t}) dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?

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- Question : What else do we need ?
  - The individual risk loading in the costate $z_{i,t}$:
    - Expectation error in the law of motion of $y_{i,t}$

$$z_{i,t}(x, \mathcal{X}, y) = \mathbb{E}^{\epsilon}\left[\frac{y_{i,t+dt}(\epsilon) - y_{i,t} + D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t) dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t^0, despite running backward.
      $\Rightarrow$ ***Intuition :*** even if agents are forward-looking, they can't know the future.
    - Advantage : Numerically Feasible via Monte Carlo or Tree Methods

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  - The initial condition y_0 as a function of y_0
  - A boundary condition of y_T or transversality $\lim_{t\to\infty} e^{-\rho t} x_t y_t = 0$

Discretizing the aggregate shocks

- Objective : Simulate the equilibrium $(\{x, y, z\}_{i,t}, \mathcal{X}_t)$ with *large* aggregate shocks
- Global method :
  - Aggregate risk may push the equilibrium far away from the steady state
  - Impossibility to use first/second order Taylor (local) approximation
- Complexity :
  - The number of trajectories for states/costates $(\{x_{i,t}, y_{i,t}\}, \mathcal{X}_t)$ grows with the number of trajectories, i.e. states of the world dB_t^0
  - Idea : Approximate/discretize the risk dB_t^0 to be able to follow the model *on each trajectory*

Method – Shooting algorithm

- Deterministic case / Representative agent :
  1. Start from initial condition X_{t_0} and the guess Y_{t_0}
  2. Simulate the sequence (X_t, Y_t) for $t \in [t_0, T]$ using the forward ODE system + finite diff.
  3. Update the guess Y_{t_0} to match the terminal condition Y_T
     – In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_0, T]$ and minimize $\int_{t_0}^{T} (\tilde{Y}_t - Y_t)^2 dt$

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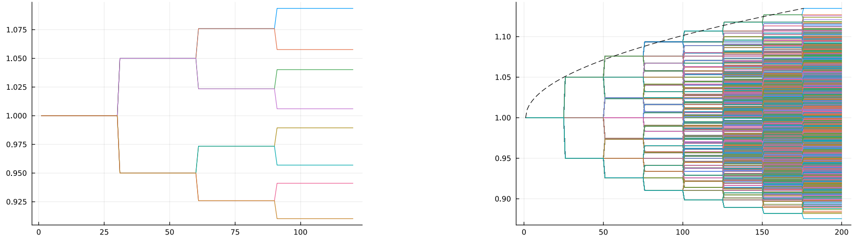
- Same method works with heterogeneity !
  - Difficulty : guess y_{i,t_0} for the (large ?) set of agents, $\forall i \in \mathbb{I}$

  $\Rightarrow$ Leverage optimization routines (and automatic differentiation)

  $\Rightarrow$ Can couple that with Monte Carlo and Deep Learning
  (but not stable at all in the neoclassical model)

Method – Shooting on a tree – 1

- Stochastic case :
- Build a *Tree* to approximate the Exogenous process : dB_t^0
  - Consist of a finite set of M “waves”, at dates $t_1, t_2, \dots, t_M$
  - Each wave consist of K “states of the world” ϵ for dB_t^0
  - Complexity $(B_t^0)_t$ is approximated with sequences of K^M values



Brownian motion approximated with a tree

$\Rightarrow$ Functional quantization methods : Can approximate any stochastic process with a tree !

Method – Shooting on a tree – 2

- Stochastic case : for a "wave" $k = 1$ to M
  1. Simulate the sequence (X_t, Y_t) for $t \in [t_{k-1}, t_k]$ using the forward ODE
  2. Simulate the sequence of costate $\tilde{Y}_{t(\epsilon)}$ for $t \in [t_k, T]$ after the realization of the shock – and for future waves – using the backward ODE.
  3. Update the initial condition $Y_{t_{k-1}}$ to match the terminal condition :

$$\overline{Y}_{t_k} = \mathbb{E}\left(\tilde{Y}_{t_k(\epsilon)} \mid \mathcal{F}_{t_{k-1}}\right)$$

     – In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_{k-1}, t_k]$ starting from $\overline{Y}_{t_k}$ and minimize

$$\min_{Y_{t_k}} \int_{t_{k-1}}^{t_k} (\tilde{Y}_t - Y_t)^2 dt$$

     – The expectation error is expressed : $Z_{t(\epsilon)} = \mathbb{E}_\epsilon \left[\frac{\tilde{Y}_{t_k(\epsilon)} - \overline{Y}_{t_k}}{\epsilon}\right]$
  4. Redo the Forward-Backward steps 1-3 for all the waves until convergence.

The *Business as Usual* is the standard neoclassical economy

- Using Pontryagin Max. Principle :
  - We obtain a system of coupled SDEs More details
- Back to the standard IAM - DICE Model from Nordhaus

$$\begin{cases} dc_{i,t} &= \frac{1}{\eta} c_{i,t}(r_{i,t} - \rho)dt - c_{i,t}\mathbf{z}_{i,t} dB_t^0 + \frac{1}{2} c_{it}(1+\eta)\mathbf{z_{i,t}}_t^2 dt \\ dk_{i,t} &= \mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e^f_{i,t}, e^r_{i,t}) - (n + \bar{g} + \delta)k_{i,t} - q^f_t e^f_{i,t} - q^r_{i,t} e^r_{i,t} - c_{i,t} \\ q^f_t &= MPe^f_{it} \qquad\qquad q^r_{it} = MPe^r_{it} \end{cases}$$

\+ Climate block for Carbon $\mathcal{S}_t$ and temperature $\tau_{i,t}$

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  - We obtain a system of coupled SDEs More details
- Back to the standard IAM - DICE Model from Nordhaus

$$\begin{cases} dc_{i,t} &= \frac{1}{\eta} c_{i,t}(r_{i,t} - \rho)dt - c_{i,t}\mathbf{z}_{i,t} dB_t^0 + \frac{1}{2} c_{it}(1+\eta)\mathbf{z_{i,t}}_t^2 dt \\ dk_{i,t} &= \mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e^f_{i,t}, e^r_{i,t}) - (n + \bar{g} + \delta)k_{i,t} - q^f_t e^f_{i,t} - q^r_{i,t} e^r_{i,t} - c_{i,t} \\ q^f_t &= MPe^f_{it} \qquad\qquad q^r_{it} = MPe^r_{it} \end{cases}$$

\+ Climate block for Carbon $\mathcal{S}_t$ and temperature $\tau_{i,t}$

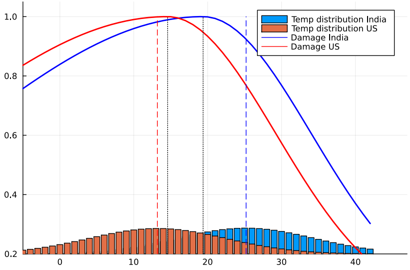
- What is the impact of aggregate risk ?
  1. Direct effect : Saving/consumption on impact $\mathbf{z_{i,t}}$
  2. Indirect effect : Precautionary saving motive : $\mathbf{z_{i,t}}_t^2$ and prudence $1+\eta$

Damage functions

- Climate change has two effects :
  - Affect firm productivity $\mathcal{D}(\tau_t)z$ as a shifter – as in Nordhaus DICE-2016

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma^{\oplus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma^{\ominus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

  - Deviation from ”ideal” temperature $\tau_i^\star = (1-\alpha)\tau^\star + \alpha\tau_{i,t_0}$, with $\tau^\star = 15.5°C$
  - Damage sensitivities γ are asymmetrical and can also be heterogeneous and uncertain



Marginal values of temperature

- Marginal values of the climate variables : $\lambda^S_{i,t}$ and $\lambda^\tau_{i,t}$

$$d\lambda^\tau_{i,t} = \Big[\lambda^\tau_{i,t}(\widetilde{\rho} + \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau_i^\star)\mathcal{D}^y(\tau_{i,t})}^{-\partial_\tau \mathcal{D}^y} f(k_{i,t}, e_{i,t})\lambda^k_{i,t}\Big] dt + z^\tau_{i,t} dB^0_t$$

$$d\lambda^S_{i,t} = \Big[\,\lambda^S_{i,t}(\widetilde{\rho} + \delta^s) - \zeta\,\chi\,\Delta_i\,\lambda^\tau_{i,t}\,\Big] dt + z^S_{i,t} dB^0_t$$

- Costate $\lambda^S_{i,t}$: marg. cost for country i of 1 extra *Mt* carbon in atmosphere. Increases with :
  - Temperature gaps $\tau_{i,t} - \tau_i^\star$ & damage sensitivity of TFP γ_i
  - Development level $f(k_{i,t}, e_{i,t})$
  - Climate params : χ climate sensitivity, Δ_i "catching up" of τ_i and ζ reaction speed
  - Aggregate risk $z^\tau_{i,t}$

Local Social cost of carbon

- The marginal "externality damage" or "local social cost of carbon" (SCC) for region i :

$$LSCC_{i,t} := -\frac{\partial \mathcal{W}_{i,t}/\partial \mathcal{S}_t}{\partial \mathcal{W}_{i,t}/\partial c_{i,t}} = -\frac{\lambda^S_{i,t}}{\lambda^k_{i,t}}$$

  - Ratio of marg. cost of carbon vs. the marg. value of consumption/capital
  - Thm : ***Stationary LSCC*** : $t \to \infty$, Balance growth path with $\mathcal{E}_t = \delta_s \mathcal{S}_t$ and $\tau_t \to \tau_\infty$

$$LSCC_{i,t} \equiv \frac{\chi}{\widetilde{\rho} + \delta^s} \Delta_i(\tau_{i,\infty} - \tau_i^\star)\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty}$$

  - What determine temperatures ? Details Temperature

Global Social cost of carbon

- The social planner considers a *"Global cost of carbon"* as marg. damage for ***all*** regions :

$$SCC_t = -\frac{\lambda_t^S}{\bar{\lambda}_t^k} = -\int_{i\in\mathbb{I}} \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} LSCC_{i,t} di$$

  - Important question : which util' unit $\bar{\lambda}_t^k$ to compute the SCC ? Average marginal utils ?

$$\bar{\lambda}_t^k = \int_{\mathbb{I}} \lambda_{j,t}^k dj = \int_{\mathbb{I}} \omega_j u'(c_{j,t}) p_j dj$$

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$$\bar{\lambda}_t^k = \int_{\mathbb{I}} \lambda_{j,t}^k dj = \int_{\mathbb{I}} \omega_j u'(c_{j,t}) p_j dj$$

  - Inequality measure :

$$\widehat{\lambda}_{i,t}^k := \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} = \frac{\omega_i u'(c_{i,t}) p_i}{\int_{\mathbb{I}} \omega_j u'(c_{j,t}) p_j dj} \lesseqgtr 1$$

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- In this case, the Global SCC is expressed as :

$$SCC_t \equiv \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^k \, LSCC_{i,t} di = \mathbb{E}^{\mathbb{I}}\left[LSCC_{i,t}\right] + \mathbb{C}\text{ov}^{\mathbb{I}}\left(\widehat{\lambda}_{i,t}^k, LSCC_{i,t}\right) > \mathbb{E}^{\mathbb{I}}\left[LSCC_{i,t}\right] =: \overline{SCC}_t$$

$\Rightarrow$ If damages are concentrated in high-$\widehat{\lambda}_{i,t}^k$ / poorer countries, it exacerbates the global SCC !
i.e. higher than the representative agent $SCC_t > \mathbb{E}_j[LSCC_{it}]$

Climate uncertainty and the Cost of Carbon :

- Stochastic case, for any shock ϵ with distribution $\epsilon \sim \varphi(\epsilon)$
- New – stochastic – measure for inequalities :

$$\widehat{\widehat{\lambda}}^k_{it}(\epsilon) = \frac{\lambda^k_{it}(\epsilon)}{\mathbb{E}_{k,\epsilon}[\lambda^k_{i,t}(\epsilon)]} = \frac{\omega_i u'(c_{i,t}(\epsilon))p_i}{\int_\epsilon \int_j \omega_j u'(c_{j,t}(\epsilon))p_j \, dj \, d\varphi(\epsilon)}$$

- Uncertainty-adjusted SCC writes :

$$\mathbb{E}_\epsilon[SCC] = \int_{\mathcal{E}} \int_{\mathbb{I}} \widehat{\widehat{\lambda}}^k_{it}(\epsilon) LSCC_{it}(\epsilon) d\varphi(\epsilon)$$

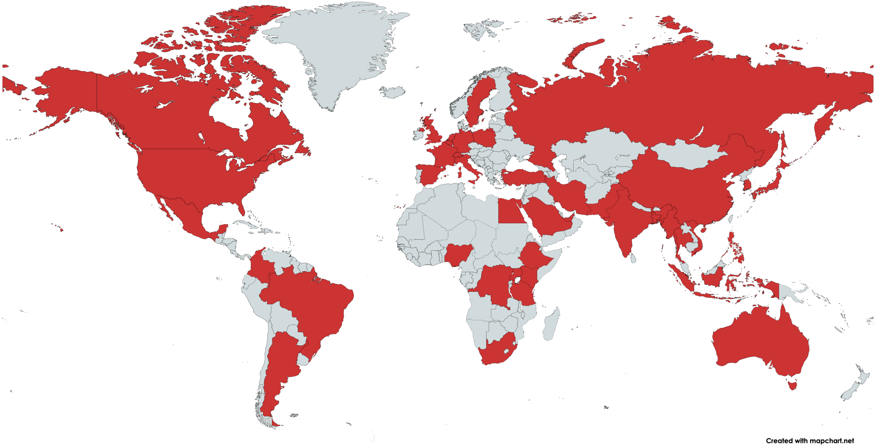
$$= \mathbb{E}_j\Big[\underbrace{\mathbb{C}\text{ov}_\epsilon\Big(\widehat{\widehat{\lambda}}^k_{it}(\epsilon), LSCC_{jt}(\epsilon)\Big)}_{=\text{effect of aggregate risk } \epsilon}\Big] + \underbrace{\mathbb{C}\text{ov}_j\Big[\mathbb{E}_\epsilon\Big(\widehat{\widehat{\lambda}}^k_{it}(\epsilon)\Big), \mathbb{E}_\epsilon\Big(LSCC_{jt}(\epsilon)\Big)\Big]}_{=\text{effect of heterogeneity across } j} + \underbrace{\mathbb{E}_{j,\epsilon}[LSCC_{jt}(\epsilon)]}_{=\text{average exp. damage}}$$

$$> \mathbb{E}_\epsilon[\overline{SCC}(\epsilon)] \qquad \& \qquad > SCC_t$$

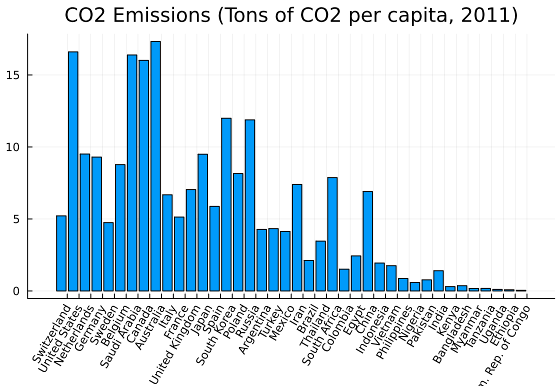
$\Rightarrow$ Climate uncertainty reinforces the unequal costs of climate change !

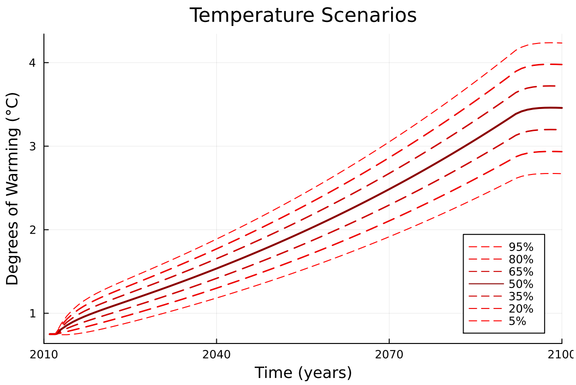
Numerical Application

- Data : 40 countries
- Temperature (of the *largest city*), GDP, energy, population
- Calibrate z to match the distribution of output per capita at steady state

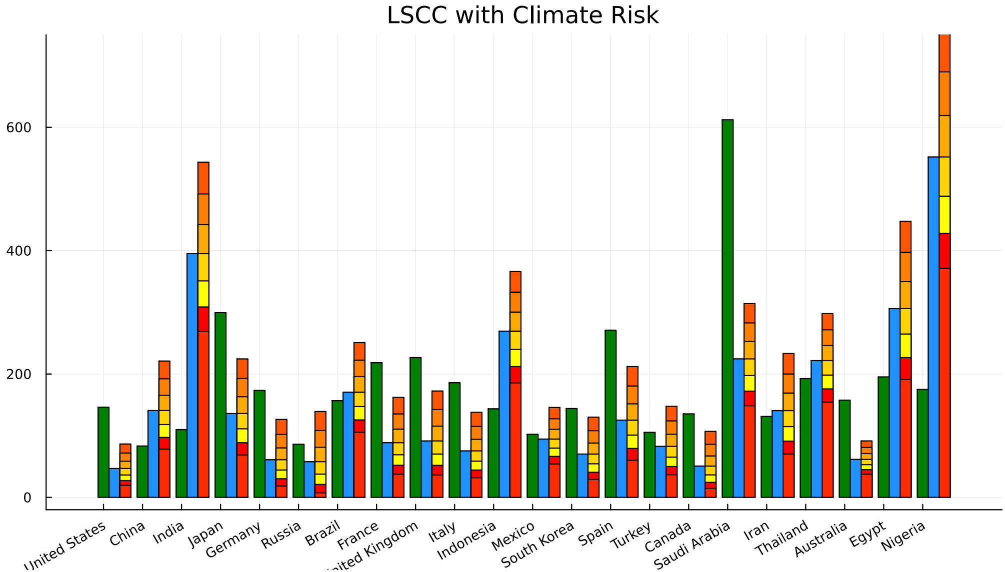


CO2 Emissions per capita and Climate risk





Distribution of carbon prices without and with uncertainty



Uncertainty and climate risk

Graph showing the effects of uncertainty on the SCC

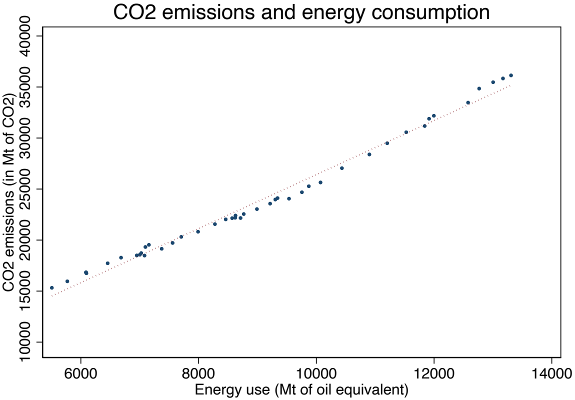
Graph showing the $z_{i,t}$ along the tree, to show the impact of the revelation of information

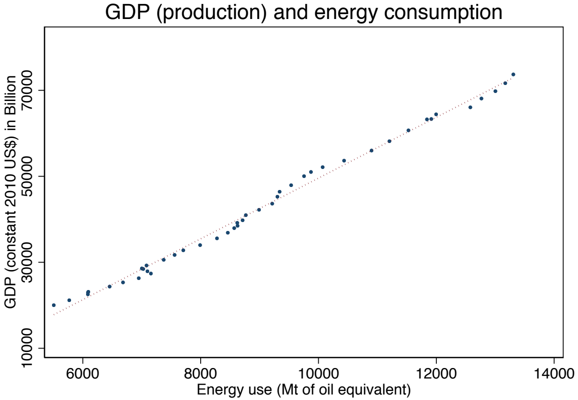
Conclusion

- Climate change have redistributive effects
  - Cost of carbon very heterogeneous across countries
  - Climate risk amplifies the inequality impact
- New methodology to simulate aggregate risk globally
  - Rely on the Sequential method and shooting algorithm
  - Adapt it to aggregate risk with discretization with a tree
- Future plans :
  - More developed climate model
  - Different sources of uncertainty,
  - growth in TFP z
  - fossil/renewable price difference g^f vs g^r.

Motivation

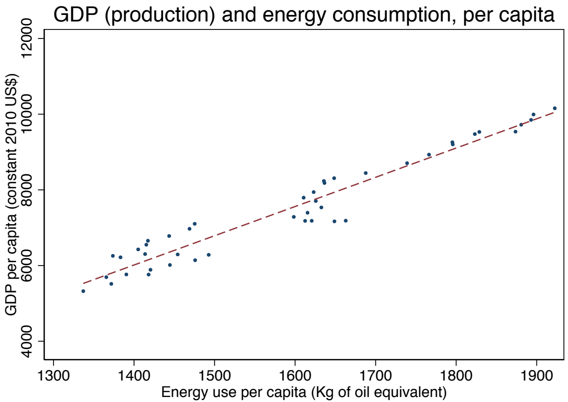
- CO_2 emissions correlate linearly with energy use
- Energy use (85% from fossils) correlates with output/growth

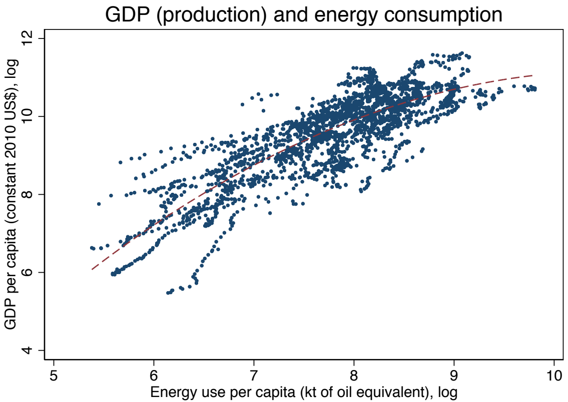




Introduction – Motivation

- Also true per capita and for the trajectory of individual countries





More details – PMP – Competitive equilibrium

- Household problem : State variables $s_{i,t} = (k_i, \tau_i, z_i, p_i, \Delta_i)$ Back
- Pontryagin Maximum Principle

$$\mathcal{H}(s, \{c, e^f, e^r\}, \{\lambda^k, \lambda^\tau, \lambda^s\}) = u(c_i, \tau_i) + \lambda^k_{i,t}\Big(\mathcal{D}(\tau_{it})f(k_t, e_t) - (n + \bar{g} + \delta)k_t - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big)$$
$$+ \lambda^\tau_{i,t}\zeta\Big(\Delta_i\, \chi\, \mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \lambda^S_{i,t}\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big)$$

$$[c_t] \qquad u'(c_{it}) = \lambda^k_{i,t}$$

$$[e^f_t] \qquad MPe^f_{it} = \mathcal{D}(\tau_{i,t})z\,\partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}} = q^f_t$$

$$[e^r_t] \qquad MPe^r_{it} = \mathcal{D}(\tau_{i,t})z\,\partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^r_{i,t}}{(1-\omega)e_{i,t}}\right)^{-\frac{1}{\sigma_e}} = q^r_{it}$$

$$[k_t] \qquad \dot{\lambda}^k_t = \lambda^k_t\big(\rho - \partial_k f(k_{i,t}, e_{i,t})\big)$$

FBSDE for McKean Vlasov systems – general formulation

- Let us consider the Social Planner :

$$\mathcal{W}_{t_0} = \max_{\{c_i\}_i} \int_{t_0}^{\infty} e^{-\rho t} \int_{i \in \mathbb{I}} \omega_i \, u_i(x_{i,t}, c_{i,t}) di \, dt$$

s.t. individual *and* aggregate dynamics, and controlling $c_i, \forall i \in \mathbb{I}$.

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s.t. individual *and* aggregate dynamics, and controlling $c_i, \forall i \in \mathbb{I}$.

- Set up the Social Planner Hamiltonian :

$$\bar{\mathcal{H}}^{SP}(\{x,y,z\}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_x) = \max_{c\in\mathbb{C}} \int_{x\in\mathbb{X}} \Big[\omega u(x,c) + b(x,\mathcal{X},c)\cdot y + \sigma(x,\mathcal{X}) * z\Big] P_x(dx) + \bar{b}(\mathcal{X}, P_x)\cdot\mathcal{Y} + \bar{\sigma}(\mathcal{X}, P_x) * \mathcal{Z}$$

- Optimal control $c^\star \in \operatorname{argmax}_{x\in\mathbb{C}} \bar{\mathcal{H}}(\cdot)$

FBSDE for McKean Vlasov systems – general formulation

- Using the Stochastic Pontryagin maximum principle :

$$dy_{i,t} = -D_x\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \tilde{z}_{i,t}dB_t^0 - \widetilde{\mathbb{E}}\left[D_\mu\bar{\mathcal{H}}^{SP}(\{\tilde{x},\tilde{y},\tilde{z}\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)(x_{i,t})\right]dt$$
$$d\mathcal{Y}_t = -D_X\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \mathcal{Z}_t dB_t^0$$

Two effects internalized by the social planner :

1. Effect on Aggregate variables $\mathcal{X}_t$
2. Effect on the distribution P_x :

FBSDE for McKean Vlasov systems – general formulation

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$$d\mathcal{Y}_t = -D_X\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \mathcal{Z}_t dB_t^0$$

  Two effects internalized by the social planner :

1. Effect on Aggregate variables $\mathcal{X}_t$
2. Effect on the distribution P_x :
   - Intuition : shifting the distribution of states x for all other agents $\tilde{x}$
   - $D_\mu H$ is the L-derivative w.r.t the measure $\mu \equiv P_{x,t}$
   - Idea : lifting of the function $H(x,\mu) = \widehat{H}(x,\widehat{X})$ where $\widehat{X} \sim \mu$ and hence $D_\mu H(x,\mu)(\widehat{X}) = D_{\widehat{x}}\widetilde{H}(x,\widehat{X})$
   - Probabilistic approach : easy to compute $\widetilde{\mathbb{E}}[D_\mu H(\tilde{x}_t,\mu)] = \widetilde{\mathbb{E}}[D_{\widehat{x}}H(\tilde{x}_t,\widehat{X})]$
   - Here : effects are homogeneous for all agents : interaction with measure P_x is non-local !
   - Back

More details – PMP – Ramsey Optimal Allocation

- Hamiltonian :

$$\begin{aligned}
\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\},\{\psi\}) = \int_{\mathbb{I}} \omega_i \mathcal{D}^u(\tau_{it}) u(c_i) p_i di + \\
+ \psi^k_{i,t}\Big(\mathcal{D}(\tau_{it}) f(k_t,e_t) - (n+\bar{g}+\delta)k_t + \theta_i \pi(E^f_t,\mathcal{I}_t,\mathcal{R}_t) - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big) \\
+ \psi^S_t\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big) + \psi^\tau_{it}\zeta\Big(\Delta_i\,\chi\,\mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \psi^{\mathcal{R}}_{it}\Big(-E^f_t + \delta^R \mathcal{I}_t\Big) \\
+ \psi^{\lambda k}_{i,t}\Big(\lambda^k_t(\rho - r_t)\Big) + \psi^{\lambda R}_t\Big(\rho\lambda^R_t + \mathcal{C}^f_{\mathcal{R}}(E^f_t,\mathcal{I}_t,\mathcal{R}_t)\Big) \\
+ \phi^c_{it}\big(\mathcal{D}^u(\tau_{it})u'(c_i) - \lambda^k_{it}\big) + \phi^{ef}_{it}\Big(MPe^f_{it} - q^f_t\Big) + \phi^r_{it}\Big(MPe^r_{it} - q^r_{it}\Big) \\
+ \phi^{Ef}_t\Big(q^f_t - \mathcal{C}^f_E(\cdot) - \lambda^{\mathcal{R}}_t\Big) + \phi^{\mathcal{I}f}_t\big(\delta\lambda^{\mathcal{R}}_t - \mathcal{C}^f_{\mathcal{I}}(\cdot)\big)
\end{aligned}$$

- Back

Ramsey Optimal Allocation - FOCs

- FOCs

$$[c_{it}] \qquad \psi_{it}^{k} = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{=\text{direct effect}} + \underbrace{\phi_{it}^{c} \mathcal{D}^u(\tau_{it}) u''(c_i)}_{=\text{effect on savings}}$$

$$\text{Define :} \qquad \widehat{\phi}_{it}^{e} = \phi_{it}^{f} MPe_t^f + \phi_{it}^{r} MPe_t^r$$

$$[e_{it}^f] \qquad \psi_{i,t}^{k}\left(MPe_{it}^f - q_t^f\right) + \xi_{it} p_i \psi_t^S + p_i \partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi_{jt}^k dj + \partial_{e^f} \widehat{\phi}_{it}^{e} - p_i \phi_t^{Ef} \partial_{EE} \mathcal{C}(\cdot) = 0$$

$$[e_{it}^r] \qquad \psi_{i,t}^{k}\left(MPe_{it}^r - q_{it}^r\right) + \partial_{e^r} \widehat{\phi}_{it}^{e} = 0$$

$$[\mathcal{I}_t] \qquad \delta\, \psi_t^{\mathcal{R}} + \partial^2_{\mathcal{R}\mathcal{I}}\, \mathcal{C}(\cdot)\, \psi_t^{\lambda,\mathcal{R}} - \phi_t^{\mathcal{I}} \partial^2_{\mathcal{I}\mathcal{I}}\, \mathcal{C}(\cdot) = 0$$

$$[q_t^f] \qquad \phi_t^{Ef} = \int_{\mathbb{I}} e_{it}^f \psi_{jt}^k dj + \int_{\mathbb{I}} \phi_{jt}^f dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi_{jt}^k dj$$

- Back

Ramsey Optimal Allocation - FOCs

- Backward equations for planner's costates

$$[k_i] \qquad \dot{\psi}^k_{it} = \psi^k_{it}(\tilde{\rho} - r_{it} + \partial_k MPk_i)\psi^k_{it} - \partial_k \widehat{\phi}^e_{it}$$

$$[\mathcal{S}_i] \qquad \dot{\psi}^{\mathcal{S}}_t = \left(\tilde{\rho} + \delta^s\right)\psi^{\mathcal{S}}_t - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi^{\tau}_{jt} dj$$

$$[\tau_i] \qquad \dot{\psi}^{\tau}_t = \left(\tilde{\rho} + \zeta\right)\psi^{\tau}_t - \Big(\omega_i \mathcal{D}'(\tau_{it}) u(c_{it}) + \psi^k_{it} \mathcal{D}'(\tau_{it}) f(k_{it}, e_{it}) + \phi^c_{it} \mathcal{D}'(\tau_{it}) u'(c_i) + \partial_\tau \widehat{\phi}^e_{it}\Big)$$

$$[\mathcal{R}] \qquad \dot{\psi}^{\mathcal{R}}_t = \psi^{\mathcal{R}}_t \Big(\tilde{\rho} - \partial^2_{\mathcal{R}\mathcal{R}} \mathcal{C}(\cdot)\Big) - \phi^{Ef}_t \partial^2_{\mathcal{R}E} \mathcal{C}(\cdot)$$

$$[\lambda^k_i] \qquad \dot{\psi}^{\lambda,k}_t = \tilde{\rho}\psi^{\lambda,k}_t - (\rho - r_{i,t})\psi^k_t + \phi^c_{i,t}$$

$$[\lambda^{\mathcal{R}}_i] \qquad \dot{\psi}^{\lambda,\mathcal{R}}_t = \psi^{\lambda,\mathcal{R}}_t(\tilde{\rho} - \rho) + \phi^{Ef}_t - \delta \phi^{\mathcal{I}f}_t$$

- Back

Cost of carbon / Marginal value of temperature

- Solving for the cost of carbon and temperature $\Leftrightarrow$ solving ODE

$$\dot{\lambda}^{\mathcal{T}}_{i,t} = \lambda^{\mathcal{T}}_t(\widetilde{\rho} + \Delta\zeta) + \gamma(\tau - \tau^{\star})\mathcal{D}^y(\tau)f(k,e)\lambda^k_t + \phi(\tau - \tau^{\star})\mathcal{D}^u(\tau)u(c)$$

$$\dot{\lambda}^S_t = \lambda^S{}_t(\widetilde{\rho} + \delta^s) - \int_{\mathbb{I}} \Delta_i \zeta \chi \lambda^{\mathcal{T}}_{i,t}$$

- Solving for $\lambda^{\mathcal{T}}_t$ and $\lambda^{\mathcal{S}}_t$, in stationary equilibrium $\dot{\lambda}^S_t = \dot{\lambda}^{\mathcal{T}}_t = 0$

$$\lambda^{\mathcal{T}}_{i,t} = -\int_t^{\infty} e^{-(\widetilde{\rho}+\zeta)u}(\tau_u - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_u)y_{\mathcal{T}}\lambda^k_u + \phi\mathcal{D}^u(\tau_u)u(c_u)\Big)du$$

$$\lambda^{\mathcal{T}}_{i,t} = -\frac{1}{\widetilde{\rho} + \Delta\zeta}(\tau_{\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{\infty})y_{\infty}\lambda^k_{\infty} + \phi\mathcal{D}^u(\tau_{\infty})u(c_{\infty})\Big)$$

$$\begin{aligned}
\lambda^S_t &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi \int_{\mathbb{I}} \Delta_j \lambda^{\mathcal{T}}_{j,u} dj\, du \\
&= \frac{1}{\widetilde{\rho} + \delta^s}\zeta\chi \int_{\mathbb{I}} \Delta_j \lambda^{\mathcal{T}}_{j,\infty} \\
&= -\frac{\chi}{\widetilde{\rho} + \delta^s}\frac{\zeta}{\widetilde{\rho} + \zeta}\int_{\mathbb{I}} \Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_{\infty}\lambda^k_{j,\infty} + \phi\mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)dj
\end{aligned}$$

$$\lambda^S_t \xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho} + \delta^s}\int_{\mathbb{I}} \Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_{j,\infty}\lambda^k_{j,\infty} + \mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)dj$$

Cost of carbon / Marginal value of temperature

- Closed form solution for CC :
  - In stationary equilibrium : $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
  - Fast temperature adjustment $\zeta \to \infty$
  - no internalization of externality (business as usual)

$$LCC_{i,t} \equiv \frac{\Delta_i \chi}{\widetilde{\rho} + \delta^s}(\tau_\infty - \tau^\star)\left(\gamma \mathcal{D}^y(\tau_\infty) y_\infty + \phi \mathcal{D}^u(\tau_\infty)\frac{c_\infty}{1-\eta}\right)$$

- Heterogeneity + uncertainty about models

Uncertainty about models :

- In our model, we rely strongly on model specification :
  - Parameters θ of models :
    - Climate system and damages : $(\xi, \chi, \zeta, \delta^s, \gamma, \phi)$
    - Economic model : $\varepsilon, \nu, \bar{g}, n$ or extended : $\omega, \sigma, \sigma^e, \nu, \mu$
    - Models with probability weight $\pi(\theta)$
  - Social cost of carbon, weighted for model uncertainty :

$$SCC_t(\theta) = -\int_{\mathbb{S}} \frac{\lambda_t^S(s,\theta)}{\lambda_t^k(s,\theta)} g(s) ds$$

$$S\bar{C}C_t = \int_{\Theta} SCC_t(\theta)\pi(\theta)d\theta = \int_{\Theta}\int_{\mathbb{S}} \frac{\lambda_t^S(s,\theta)}{\lambda_t^k(s,\theta)} g(s) ds \pi(\theta) d\theta$$

  - Counterfactual computation of SCC
    - Representative country / no uncertainty $\frac{\lambda_t^S}{\lambda_t^k}$
    - With heterogeneous regions / no uncertainty $SCC_t(\bar{\theta})$
    - No heterogeneity / model uncertainty $\int_{\Theta} \frac{\lambda_t^S(\bar{s},\theta)}{\lambda_t^k(\bar{s},\theta)} \pi(\theta) d\theta$
    - With heterogeneous regions / with model uncertainty $S\bar{C}C_t$
  - back

Social cost of carbon & temperature

- Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta\, \chi\, \xi\, \omega \int_{t_0}^{T} e^{(n+\bar{g})t-\delta_s(T-t)} q_t^{f\;-\sigma_e} \int_{j\in\mathbb{I}} \left(z_j z_{j,t}^e \mathcal{D}(\tau_{j,t})\right)^{\sigma-1} y_{j,t}\, q_{j,t}^{\sigma_e-\sigma} dj\, dt$$

  - Geographical factors determining warming Δ_i
  - Climate sensitivity χ & carbon exit from atmosphere δ_s
  - Growth of population n, aggregate productivity $\bar{g}$
  - Deviation of output from trend y_i & relative TFP z_j
  - Directed technical change z_t^e, elasticity of energy in output σ
  - Fossil energy price $q^{e,f}$ and Hotelling rent $g^{q^f} \approx \dot{\lambda}_t^R / \lambda_t^R = \rho$
  - Change in energy mix, renewable share ω, price q_t^r & elasticity of source σ_e

- Approximations at $T \equiv$ Generalized Kaya (or $I = PAT$) identity More details

$$\frac{\dot{\tau}_T}{\tau_T} \propto\; n + \bar{g}^y - (1-\sigma)\left(g^{z^e} - \widetilde{\gamma}\right) + (\sigma_e - \sigma)(1-\omega)g^{q^r} - (\sigma_e(1-\omega) + \sigma\omega)g^{q^f}$$

Back

Long term temperature

- Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i(\mathcal{T}_T - \mathcal{T}_{t_0}) = \Delta_i \chi \int_{t_0}^{T} e^{-\delta_s(T-t)} \mathcal{E}_t dt$$
$$= \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e^f_{j,t}(1 - \vartheta_{j,t}) dj\, dt$$

  - Use CES demands : $e^f_{j,t} = \omega e_{j,t} q^{f}_{t}{}^{-\sigma_e} q_t^{\sigma_e}$ for energy and $e_t = (z z^e_t \mathcal{D}(\tau_{j,t}))^{\sigma-1} q_t^{-\sigma}$
  - Moreover, CES price index $q_t = (\omega q^{f}_{t}{}^{1-\sigma_e} + (1-\omega) q^{r}_{t}{}^{1-\sigma_e})^{1/(1-\sigma_e)}$, so first order approximation : $g^q = \omega g^{q^f} + (1-\omega) g^{q^r}$ with growth for q^f and q^r as well as $z^e_t = e^{g^e t}$
  - Gives :

$$e^f_{j,t} = \omega q^{f}_{t}{}^{-\sigma_e} q_{j,t}^{\sigma_e} (z z^e_t \mathcal{D}(\tau_{j,t}))^{\sigma-1} q_{j,t}^{-\sigma}$$

Temperature dynamics

- Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s (T-t)} \int_{j \in \mathbb{I}} e^{f}_{j,t} (1 - \vartheta_{j,t}) dj \, dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s (T-t)} q_t^{f \; -\sigma_e} \int_{j \in \mathbb{I}} \omega (z z^e \mathcal{D}(\tau_{j,t}))^{\sigma - 1} e^{g^e t} q_{j,t}^{\sigma_e - \sigma} (1 - \vartheta_{j,t}) dj \, dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \omega \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s (T-t)} e^{(-\sigma_e + (\sigma_e - \sigma)\omega) g^{q^f} t} e^{(\sigma_e - \sigma)(1-\omega) g^{q^r} t}$$
$$\times \int_{j \in \mathbb{I}} \left(z z_t^e \mathcal{D}(\tau_{j,t}) \right)^{\sigma - 1} y_{j,t} (1 - \vartheta_{j,t}) dj \, dt$$

- back

Social Planner allocation

- Solving the social planner allocation : Hamiltonian

$$\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\}) = \int_{\mathbb{I}} \omega_i\, u(c_i,\tau_i)p_i di - wL_t^f + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^k \Big(\mathcal{D}(\tau_t)f(k_t,e_t) - (n+\bar{g}+\delta)k_t - c_t\Big)p_i di$$
$$+ \widehat{\lambda}_t^S\Big(\int_{\mathbb{I}} \xi^f e_t^f p_i di - \delta^s \mathcal{S}_t\Big) + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^{\mathcal{T}} \Delta_i \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)p_i di$$
$$+ \widehat{\lambda}_t^R\Big(-E_t^f + \delta^R \mathcal{I}_t\Big) + \widehat{\lambda}_t^{e^f}\Big(\widetilde{\mathcal{F}}(L_t^f,\mathcal{R}_t) - E_t^f\Big) + \int_{\mathbb{I}} \widehat{\lambda}_t^{e^r}\Big(z_{i,t}^r k_{i,t}^{r\ \alpha} - e_t^r\Big)p_i di$$

$$\text{with} \qquad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_i di \qquad \text{and} \qquad e_t = \left(\omega^{\frac{1}{\sigma^e}}\left(e_{i,t}^f\right)^{\frac{\sigma^e-1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}}\left(e_{i,t}^r\right)\right)^{\frac{\sigma^e-1}{\sigma^e}}\Bigg)^{\frac{\sigma^e}{\sigma^e-1}}$$

- Results :

$$\omega_i\, u_c(c_i,\tau_i) = \widehat{\lambda}_{i,t}^k$$
$$\widehat{\lambda}_{i,t}^k\, MPe_{i,t}\,\frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S$$
$$\widehat{\lambda}_{i,t}^k\, MPe_{i,t}\,\frac{\partial e_{i,t}}{\partial e_{i,t}^r} = \widehat{\lambda}_t^{e^r}$$

- Back

FBSDE for MFG systems – general formulation

- State $X_t \equiv (a_t, z_t) \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), and X diffusion process with control $\alpha^\star(t, X, P_X, Y) \equiv c_t^\star$

$$dX_t = b(X_t, P_{X_t}, \alpha_t^\star)dt + \sigma dB_t$$

- Set up the Hamiltonian :

$$\mathcal{H}(t, x, P_X, y) = \max_{\alpha \in \mathbb{A}} \left(u(\alpha) + b(x, p_X, \alpha) \cdot y \right)$$

- Optimal control $c^\star \in \operatorname{argmax}_{\alpha \in \mathbb{A}} \left(u(\alpha) + b(x, p_X, \alpha) \cdot y \right)$
- Using the Pontryagin maximum principle :

$$dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t$$

back

FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t \end{cases}$$

- Question : What else do we need ?

FBSDE system for MFG

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- Question : What else do we need ?
  - The risk loading in the costate $\widetilde{Z}_t$:
    - Intuitions : expectation error in the law of motion of Y_t

$$\widetilde{Z}_t(x) = \mathbb{E}\Big[\frac{dY_t + D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt}{dB_t^0}\Big]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo

FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t \end{cases}$$

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    - Can compute that by Monte Carlo
  - The initial condition Y_0 as a function of X_0

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    - Can compute that by Monte Carlo
  - The initial condition Y_0 as a function of X_0
  - A boundary condition of Y_T or transversality $\lim_{t\to\infty} e^{-\rho t} X_t Y_t = 0$

back

Social planner : Mean Field Control/McKean Vlasov

- Suppose now that the social planner controls the dynamic of each agents ... *accounting for its effect on the distribution*
- Get an additional term :

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t - \widetilde{\mathbb{E}}\left[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)\right] \end{cases}$$

- Effect on the distribution :

Social planner : Mean Field Control/McKean Vlasov

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$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t - \widetilde{\mathbb{E}}\left[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)\right] \end{cases}$$

- Effect on the distribution :
  - Social planner internalize the externality coming from heterogeneity
    - $D_\mu H$ is the L-derivative w.r.t the measure $\mu \equiv \mathbb{P}_{X_t}$
    - Idea : lifting of the function $H(x, \mu) = \widehat{H}(x, \widehat{X})$ where $\widehat{X} \sim \mu$ and hence $D_\mu H(x, \mu)(\widehat{X}) = D_{\widehat{x}}\widetilde{H}(x, \widehat{X})$
    - Intuition : shift the distribution of states $\widehat{X}$ for all agents
    - Probabilistic approach : easy to compute $\widetilde{\mathbb{E}}[D_\mu H(\tilde{X}_t, \mu)] = \widetilde{\mathbb{E}}[D_{\widehat{x}}H(\tilde{X}_t, \widehat{X})]$
  - Here : the effect is homogeneous for all agents : the interaction with the measure is non-local !

back

Sequential approach FBSDE w/ Agg. shocks

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t + \widetilde{\sigma} dB_t^0 \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t + \widetilde{Z}_t^0 dB_t^0 \end{cases}$$

- Question : What else do we need ?

Sequential approach FBSDE w/ Agg. shocks

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t + \widetilde{\sigma} dB_t^0 \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t + \widetilde{Z}_t^0 dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The risk loading on idiosyncratic shocks $\widetilde{Z}_t$:
  - The risk loading on aggregate shocks $\widetilde{Z}_t^0$:

Sequential approach FBSDE w/ Agg. shocks

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t + \widetilde{\sigma} dB_t^0 \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t + \widetilde{Z}_t^0 dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The risk loading on idiosyncratic shocks $\widetilde{Z}_t$:
  - The risk loading on aggregate shocks $\widetilde{Z}_t^0$:
  - The initial conditions $Y_0(X_0)$ and boundary condition on Y_T or transversality $\lim_{t\to\infty} e^{-\rho t} X_t Y_t = 0$

MFG system : Recursive approach w/ Agg. shocks

- Here : recursive w.r.t. idiosyncratic shocks, but sequential w.r.t. aggregate shocks.
- System for v and g :

$$-\partial_t v + \rho v = \max_{\alpha} u(\alpha) + \mathcal{A}(v)v + Z_t^0 dB_t^0$$
$$\partial_t g = \mathcal{A}^*(v)g + \partial_x[\sigma g]dB_t^0$$

- Solve the PDE system :
  - Finite difference, upwinding scheme
  - View that as a non-linear system : use Quasi Newton methods
  - New part : forcing terms $\partial_x[\sigma g]dB_t^0$ and $Z_t^0 dB_t^0$
  - Initial and terminal conditions

$$v_T = v^\infty \qquad\qquad g_0 = g^\infty$$

    - Direct effect of uncertainty on measure
    - Indirect effect through agent expectations : shadow price of aggregate risk Z_t^0