The Inequality of Climate Change
Heterogeneity, Optimal policy and Uncertainty

WORK IN PROGRESS

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Macro Reading Group

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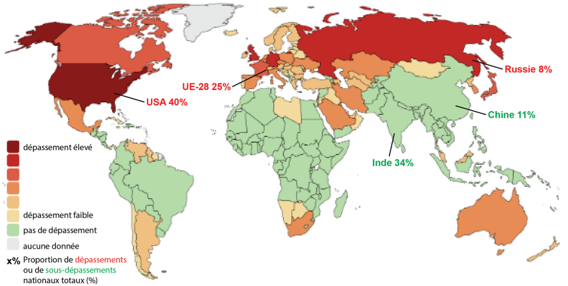
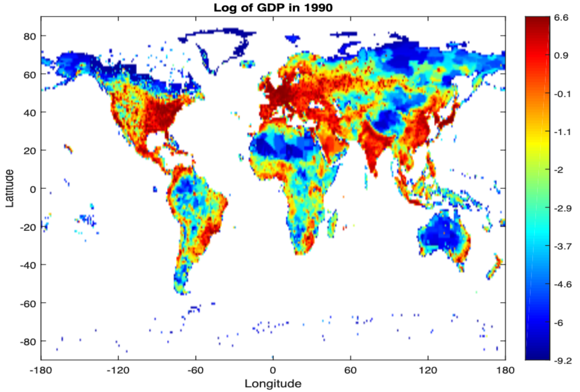
Introduction – Motivation

- Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
  - ***Unequal causes :*** Developed economies account for over 65% of cumulative GHG emissions ($\sim$ 25% each for the EU and the US)



Source : Lancet planetary Health - Quantifying national responsibility for climate breakdown: an equality-based attribution approach for carbon dioxide emissions in excess of the planetary boundary - Jason Hickel 2020



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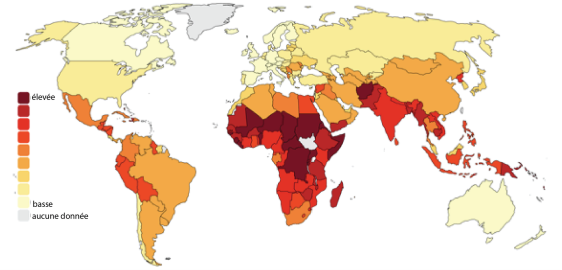
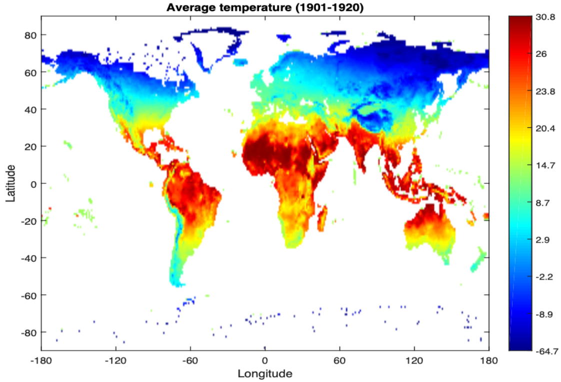
- Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
  - ***Unequal consequences :*** Increase in temperatures will disproportionately affect developing countries where the climate is already warm



Source : Notre Dame Global Adaptation Initiative



Introduction – this project

- Which countries will be affected the most by climate change ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - What is the optimal policy in presence of externalities *and* heterogeneities ?

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- Develop a simple and flexible model of climate economics
  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions

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- Develop a simple and flexible model of climate economics
  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions
- Evaluate the heterogeneous welfare costs of global warming
- Provide analytical formulas and a numerical methodology to compute the cost of carbon
  - *CC* is linear in *GDP*/level of development and in temperature gaps
- Solve world optimal carbon policy with heterogeneous regions
  - Does the optimal carbon tax coincide with the social cost of carbon ?

Introduction – related literature

- Classic Integrated Assessment models (IAM) :
  - Nordhaus' Multi-regions DICE (2016), Golosov, Hassler, Krusell, Tsyvinski (2014)
  - Dietz, van der Ploeg, Rezai, Venmans (2021), among others
- Macro (IAM) model with country heterogeneity :
  - Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021), among others
  - *This paper :* Studies the optimal policy with heterogeneity and externalities
- Climate model with risk & uncertainty :
  - Cai, Lontzek, Judd (2019), Barnett, Brock and Hansen (2022)
  - *This paper :* Includes heterogeneity and redistribution effects of climate & carbon taxation
- Quantitative spatial models :
  - Cruz, Rossi-Hansberg (2021), Bilal, Rossi-Hansberg (2023), Rudik et al (2022)
  - *This paper :* Considers forward-looking decision of agents & optimal policy
- Heterogeneous Agents models with optimal policy
  - Le Grand, Ragot (2018-), Davila, Schaab (2022), Bhandari Evans Golosov Sargent (2018-)
  - *This paper :* Studies climate externalities and Pigouvian taxation

Toy model – Energy market and Externality

- Consider two countries $i = N, S$
  - With different levels of capital, productivity, and population k_i, z_i, and p_i
  - Consuming and producing with capital and energy e
    - Energy producer with profit $\pi(E)$ owned by country i with share θ_i
  - Household problem :

$$\max_{c_i, e_i} U(c_i) p_i$$

$$c_i + q^e e_i = \mathcal{D}_i(\mathcal{S}) z_i F(k_i, e_i) + \theta_i \pi(E) \qquad [\lambda_i^k]$$

  - Subject to damage $\mathcal{D}_i(S)$ and climate externalities :

$$\mathcal{S} = \mathcal{S}_0 + \overbrace{\xi_i e_i + \xi_{-i} e_{-i}}^{=\text{GHG emissions}}$$

  - And consuming energy in a single energy market with price q^e

$$E = e_i + e_{-i} \qquad q^e = c'(E) \qquad \pi(E) = q^e E - c(E)$$

Toy model – Competitive equilibrium vs. Social Planner – Energy decision

- Competitive equilibrium Result :
  - Marginal Product of Energy = Energy Price

$$MPe_i = q^e \qquad with \qquad MPe_i := \mathcal{D}_i(\mathcal{S}) z_i F_e(k_i, e_i)$$

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- Comparison with Social planner :

$$\max_{\{c_i, e_i\}_i} \sum_{i=N,S} \omega_i U(c_i) p_i \qquad s.t \qquad c_i = \mathcal{D}_i(\mathcal{S}) z_i F(k_i, e_i) + \theta_i \bar{\kappa} L \qquad [\lambda_i^k] \qquad \forall\, i = N, S$$

$$E = e_i + e_{-i} \qquad [\lambda_i^E] \qquad \& \qquad \mathcal{S} = \mathcal{S}_0 + \xi_i e_i + \xi_{-i} e_{-i} \qquad E = f(L)$$

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- Social Planner : Results :
  - Marginal *Value* of Energy = Price + Social Cost of Carbon + Energy revenue Redistribution

$$\underbrace{\frac{\omega_i U'(c_i) p_i}{\frac{1}{2}\sum_j \omega_j U'(c_j) p_j}}_{=\lambda_i^k/\bar{\lambda}^k} MPe_i = \underbrace{q^e}_{=\lambda_i^E/\bar{\lambda}^k} + \xi_i\, SCC - \ Rent \qquad \text{with} \qquad SCC = -\sum_{j=N,S} \frac{\lambda_j^k}{\bar{\lambda}^k} \mathcal{D}_j'(\mathcal{S}) y_j$$

$$Rent = \sum_{j=N,S} \frac{\lambda_j^k}{\bar{\lambda}^k}\, \theta_j \pi'(E)$$

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  - "Average" Marg value of wealth $\bar{\lambda}^k = \frac{1}{2} \sum_{j=N,S} \omega_j U'(c_j) p_j$

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$$\text{low } z_i, k_i \quad \Rightarrow \quad \text{low } c_i, \text{ high } \lambda_i^k = \omega_i U'(c_i) \, p_i > \frac{1}{2}\sum_j \omega_j U'(c_j) p_j$$

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  - Energy rent redistribution :

$$Rent \propto \mathbb{C}\mathrm{ov}_j\Big(\omega_j U'(c_j) p_j, \theta_j \pi_j'(E)\Big) + \pi'(E)$$

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  - Climate externality :

$$SCC \propto -\mathbb{C}\text{ov}_j\Big(\omega_j U'(c_j) p_j, \mathcal{D}'_j(\mathcal{S}) y_j\Big) - \mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j]$$

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- Optimal policy : $\mathbf{t}_i$ redistributive tax/subsidy, $\mathbf{t}^R$: Energy rent subsidy and $\mathbf{t}^C$ Carbon tax

$$MPe_i = \frac{\frac{1}{2}\sum_j \omega_j U'(c_j) p_j}{\omega_i U'(c_i) p_i}\Big(q^e + \xi_i\, SCC - Rent\Big) = \mathbf{t}_i\, \left(q^e + \xi_i\, \mathbf{t}^C - \mathbf{t}^R\right)$$

Toy model – Effect of uncertainty

- Consider risks related to both
  (i) Economic growth $z_i(\epsilon_z)$
  (ii) Climate damage $\mathcal{D}_i(\mathcal{S}|\epsilon_d)$
  – Probability distribution $(\epsilon_z, \epsilon_d) =: \boldsymbol{\epsilon} \sim \phi(\boldsymbol{\epsilon})$

$$\max_{e_i} \int_{\mathcal{E}} \max_{c_i(\boldsymbol{\epsilon})} U(c_{i(\boldsymbol{\epsilon})}) d\phi_{(\boldsymbol{\epsilon})} \qquad \text{vs.} \qquad \max_{\{e_j\}_j} \int_{\mathcal{E}} \max_{\{c_j(\boldsymbol{\epsilon})\}_j} \sum_{j=N,S} \omega_j U(c_{j(\boldsymbol{\epsilon})}) d\phi_{(\boldsymbol{\epsilon})}$$

- Competitive equilibrium :
  - Almost no change in behavior : Expected Marginal Product of Energy = Energy Price

$$\int_{\mathcal{E}} MPe_{i(\boldsymbol{\epsilon})} \, d\phi_{(\boldsymbol{\epsilon})} = q^e$$

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$$\int_{\mathcal{E}} \frac{\omega_i U'(c_{i(\boldsymbol{\epsilon})}) p_i}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k) p_k]} MPe_{i(\boldsymbol{\epsilon})} d\phi_{(\boldsymbol{\epsilon})} = q^e + \int_{\mathcal{E}} SCC_{(\boldsymbol{\epsilon})}\, d\phi_{(\boldsymbol{\epsilon})} - \int_{\mathcal{E}} Rent_{(\boldsymbol{\epsilon})}\, d\phi(\boldsymbol{\epsilon})$$

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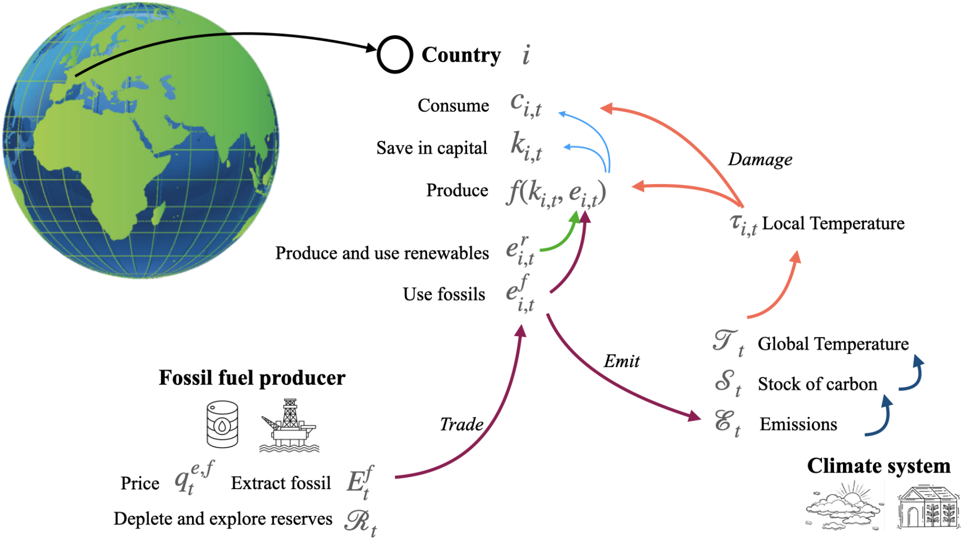
  - Optimal policy :

$$\mathbb{E}[MPe_{i(\epsilon)}] = \underbrace{\frac{\mathbb{E}_{\epsilon}[\omega_i U'(c_{i(\epsilon)})p_i]}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)p_k]}}_{=\mathbf{t}_i} \underbrace{\Big[1 + \mathbb{C}\text{orr}_{\epsilon}\big(\omega_i U'(c_{i(\epsilon)})p_i, MPe_{i(\epsilon)}\big)\Big]}_{=\widetilde{\mathbf{t}}_i} \Big(q^e + \underbrace{\mathbb{E}_{\epsilon}[SCC]}_{=\widetilde{\mathbf{t}}^C} - \underbrace{\mathbb{E}_{\epsilon}[Rent]}_{=\widetilde{\mathbf{t}}^R}\Big)$$

Summary of the quantitative model



Model

- Neoclassical economy, in continuous time
- countries/regions $i \in \mathbb{I}$: ex-ante heterogeneous in dimensions $\underline{s}$
  - Here : $\underline{s}_i = \{p_i, z_i, \Delta_i\}$, relative heterogeneity doesn't change over time
  - Productivity grows at rate $\bar{g}$ and population grow at rate n
- regions heterogeneous ex-post $\bar{s}_i$
  - Here : capital and temperature $\bar{s}_i = \{k_i, \tau_i\}$
  - *Future :* could include z – endog. technical change – or p – migrations/demographics
  - *Renormalization :* all variables are values per unit of efficient labor $k_{i,t} = \frac{K_{i,t}}{p_{t_0}} e^{-(\bar{g}+n)t}$
- Aggregate variables : global $\mathcal{T}$emperature, carbon $\mathcal{S}$tocks in atmosphere, fossil energy $\mathcal{R}$eserves $S = \{\mathcal{T}, \mathcal{S}, \mathcal{R}\}$

Model – Household and firm

- Household problem in country i :

$$\max_{\{c_{i,t},e^f_{i,t},e^r_{i,t}\}} \int_{t_0}^{\infty} e^{-(\rho-n)t}\, u(c_{i,t},\tau_{i,t})dt$$

- Dynamics of capital in every country i :

$$\dot{k}_{i,t} = \mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e_{i,t}) + \theta^R_i \pi^R_t - (n + \bar{g} + \delta)k_{i,t} - q^e_t e_{i,t} - c_{i,t}$$

- Choices :
  - c_t consumption, e_t energy, with production fct :

$$f(k,e) = \left((1-\varepsilon)^{\frac{1}{\sigma}} k^{\alpha\frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} (z^e e)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

  - Damage function $\mathcal{D}^y(\tau_t)$ affect country's production
  - Directed technical change z^e_t & energy mix e_t with fossil e^f_t vs. renewable e^r_t

Model – Energy markets

- Two sources of energy : fossil e_t^f and renewable e_t^r for every i

$$e_{i,t} = \left(\omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e - 1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r))^{\frac{\sigma^e - 1}{\sigma^e}}\right)^{\frac{\sigma^e}{\sigma^e - 1}}$$

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- Fossil fuels energy producer :
  - Extended-Hotelling problem (depleting reserves with stock effects and exploration)

$$\max_{\{E_t^f,\mathcal{I}_t\}_t} \int_0^\infty e^{-\rho t}\pi_t(E_t^f,\mathcal{R}_t)dt \qquad \text{with } \pi_t(E_t^f,\mathcal{R}_t) = q_t^{e,f}E_t^f - \mathcal{C}^e(E_t^f,\mathcal{R}_t) - \mathcal{C}^i(\mathcal{I}_t,\mathcal{R}_t)$$

$$s.t. \qquad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_{i,t} di \qquad \dot{\mathcal{R}}_t = -E_t^f + \delta_R \mathcal{I}_t$$

  - Optimal pricing with finite-resources rents More details

$$q_t^{e,f} = \bar{\nu}\left(\frac{E_t^{f\star}}{\mathcal{R}_t}\right)^{\nu} + \lambda_t^R \qquad \bar{\mu}\left(\frac{\mathcal{I}_t^\star}{\mathcal{R}_t}\right)^{\mu} = \delta_R \lambda_t^R$$

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- Renewable energy as a substitute technology

$$e_{i,t}^r = z_{i,t}^r k_{i,t}^r \qquad q_{i,t}^{e,r} = r_{i,t}/z_{i,t}^r$$

Fossil energy and externality

- Fossil energy input e^f_t causes climate externality
  - Change the world climate – global temperature $\mathcal{T}_t$ and cumulative GHG in atmosphere $\mathcal{S}_t$:

$$\mathcal{E}_t = \int_{\mathbb{I}} \xi \; e^f_{i,t} \, p_i \, di$$

$$\dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s \mathcal{S}_t \qquad\qquad \dot{\mathcal{T}}_t = \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)$$

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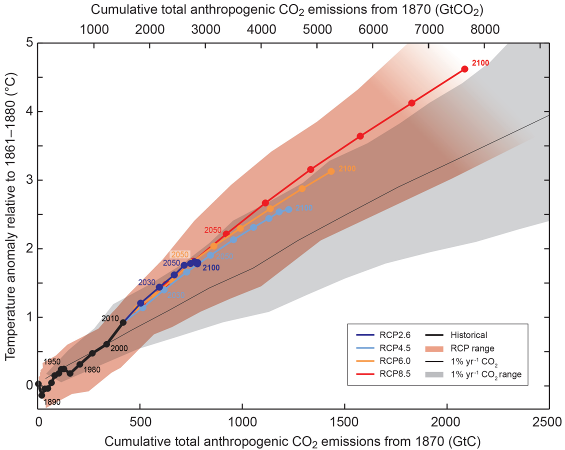
    – ζ is the inverse of persistence, so if $\zeta \to \infty$, we obtain a linear model :

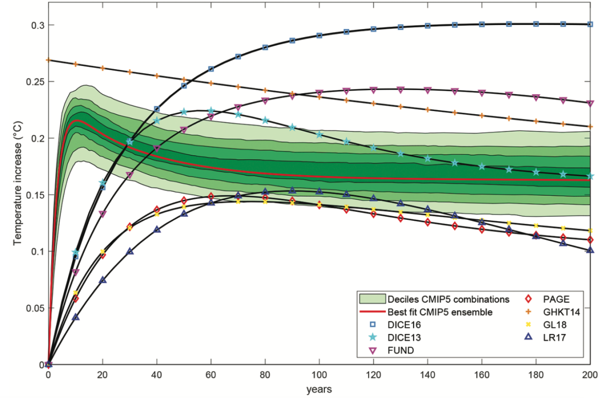
$$\mathcal{T}_t = \bar{\mathcal{T}} + \chi \mathcal{S}_t = \bar{\mathcal{T}} + \chi \int_{t_0}^{t} \int_{\mathbb{I}} \xi \, e_{i,s} \, di \, ds \Big|_{GtC}$$

  - The externality depends on policy $e^f_{i,t}$ as function of states $\{z, p, k, \tau\}$
    - Naturally, countries richer/more productive/with a larger population use more energy !
  - Impact of climate on country's local temperature :

$$\dot{\tau}_{i,t} = \Delta_i \dot{\mathcal{T}}_t$$

Temperature dynamics





Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

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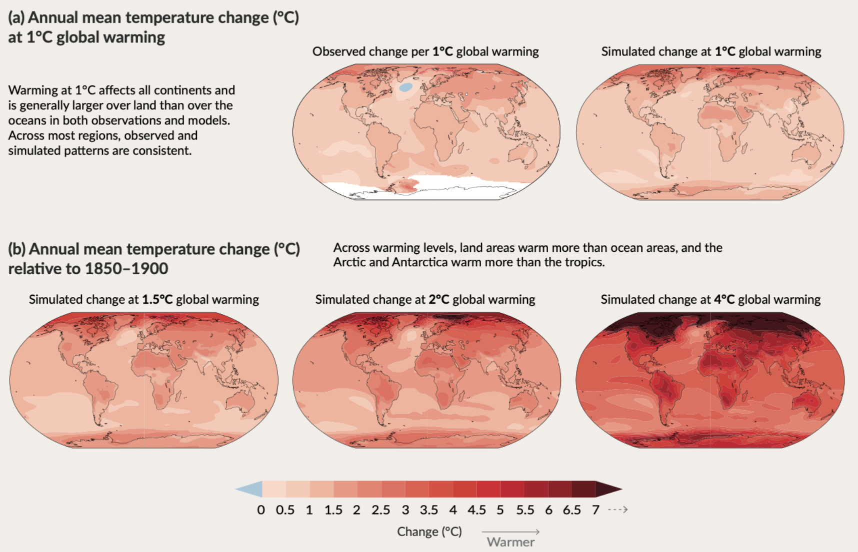


The Business as Usual equilibrium is the standard neoclassical economy

- Using Pontryagin Max. Principle :
  - We obtain a system of coupled ODEs More details
- Back to the standard IAM - DICE Model from Nordhaus

$$\begin{cases} \dot{c}_{i,t} &= c_{it}\frac{1}{\eta}(r_{it}-\rho) \\ \dot{k}_{i,t} &= \mathcal{D}^y(\tau_{i,t})z_{i,t}f(k_{i,t},e^f_{i,t},e^r_{i,t}) + \theta_i^R\pi_t^R - (n+\bar{g}+\delta)k_{i,t} - q_t^f e^f_{i,t} - q^r_{i,t}e^r_{i,t} - c_{i,t} \\ q_t^f &= MPe^f_{it} \qquad q^r_{it} = MPe^r_{it} \end{cases}$$

+ Climate blocks for Carbon $\mathcal{S}_t$ and temperature $\tau_{i,t}$
+ Dynamics of Hotelling rents λ_t^R for fossil price q_t^f.

The Business as Usual equilibrium is the standard neoclassical economy

- Using Pontryagin Max. Principle :
  - We obtain a system of coupled ODEs More details
- Back to the standard IAM - DICE Model from Nordhaus

$$\begin{cases} \dot{c}_{i,t} &= c_{it}\frac{1}{\eta}(r_{it}-\rho) \\ \dot{k}_{i,t} &= \mathcal{D}^y(\tau_{i,t})z_{i,t}f(k_{i,t},e^f_{i,t},e^r_{i,t}) + \theta^R_i\pi^R_t - (n+\bar{g}+\delta)k_{i,t} - q^f_t e^f_{i,t} - q^r_{i,t}e^r_{i,t} - c_{i,t} \\ q^f_t &= MPe^f_{it} \qquad q^r_{it} = MPe^r_{it} \end{cases}$$

+ Climate blocks for Carbon $\mathcal{S}_t$ and temperature $\tau_{i,t}$
+ Dynamics of Hotelling rents λ^R_t for fossil price q^f_t.

- Are we doomed to collapse ?
- What is the valuation of the (heterogeneous) economic damages of climate change
- What about policy ?

The Ramsey Problem

- Consider a Social Planner that care about aggregate welfare :

$$\max_{\{c,e^f,e^r,k,\tau,\mathcal{S},\mathcal{R},\mathcal{I}\}} \int_0^\infty \int_{\mathbb{I}} e^{-\bar{\rho}t}\, \omega_i\, \mathcal{D}(\tau_{i,t}) u(c_{i,t}) p_i \, di \, dt$$

  subject to

  - Optimality conditions of households, for c_i, e_i^f, e_i^r and k_i
  - Optimality conditions of the Fossil, for E^f, $\mathcal{I}$ and $\mathcal{R}$
  - Climate and temperature dynamics τ_i and $\mathcal{S}$

- Methodological contribution $\Rightarrow$ adapt the Pontryagin Maximum Principle for :
  - A continuum of heterogeneous agents : Mean Field Control / McKean Vlasov FBSDE – Carmona, Delarue (2018-) / Pham et al (2018-)
  - Case with Aggregate uncertainty (future plan) : Stochastic PMP, c.f. Yong, Zhou (1999)

$\Rightarrow$ Large scale system of mean-field ODE More details - Hamiltonian

Damage functions

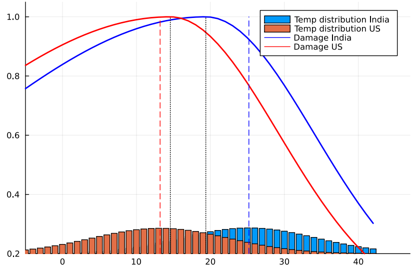
- Climate change has two effects :
  - Affect household utility function $u(c_t, \tau_t) = \mathcal{D}^u(\tau_t)\frac{c_t^{1-\eta}}{1-\eta}$

$$\mathcal{D}_u(\tau) = \begin{cases} e^{-\phi^{\oplus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\phi^{\ominus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

  - Affect firm productivity $\mathcal{D}(\tau_t)z$ as a shifter – as in Nordhaus DICE-2016

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma^{\oplus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma^{\ominus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

    - Deviation from ”ideal” temperature $\tau_i^\star$
    - Damage sensitivities γ and ϕ are asymmetrical and can also be heterogeneous and uncertain



Marginal values of temperature

- Marginal values of the climate variables : ψ^S_t and $\psi^\tau_{i,t}$

$$\dot{\psi}^\tau_{i,t} = \psi^\tau_{i,t}(\widetilde{\rho} + \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau^\star_i)\mathcal{D}^y(\tau_{i,t})}^{-\partial_\tau \mathcal{D}^y} f(k_{i,t}, e_{i,t})\psi^k_{i,t} + \overbrace{\phi_i(\tau_{i,t} - \tau^\star_i)\mathcal{D}^u(\tau_{i,t})}^{\partial_\tau \mathcal{D}^u} u(c_{i,t})$$

$$\dot{\psi}^S_t = \psi^S_t(\widetilde{\rho} + \delta^s) - \zeta\,\chi \int_{\mathbb{I}} \Delta_i\, \psi^\tau_{i,t}\, di$$

- Marg. cost for i of releasing carbon in atmosphere ψ^S_t increases with :
  - Temperature gap $\tau_{i,t} - \tau^\star_i$
  - Damage sensitivity to temperature for TFP γ_i and utility ϕ
  - The development level $f(k_{i,t}, e_{i,t})$ and $u(c_{i,t})$
  - The ”catching up” effect of temperature from cold location Δ_i

Social cost of carbon

- The marginal "externality damage" or "social cost of carbon" (SCC) can be expressed naturally :

$$SCC_t := -\frac{\partial W_t/\partial \mathcal{S}_t}{\partial W_t/\partial c_t} = -\frac{\psi_t^S}{\bar{\psi}_t^k}$$

  - Ratio of marg. cost of carbon vs. the marg. value of consumption/capital
  - Important question : which util' unit $\bar{\psi}_t^k$ to compute the SCC ? Average marginal utils ?

$$\bar{\psi}_t^k = \int_{\mathbb{I}} \psi_{j,t}^k dj \approx \int_{\mathbb{I}} \omega_i u'(c_{j,t}) p_j dj$$

- As a result :
  - Stationary value : $t \to \infty$, with $\mathcal{E}_t = \delta_s \mathcal{S}_t$ and $\tau_t \to \tau_\infty$

$$SCC_t \equiv \frac{1}{\bar{\psi}_t^k} \frac{\chi}{\widetilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_i(\tau_{i,\infty} - \tau_i^\star)\Big(\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} \psi_{i,\infty}^k + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \omega_i u(c_{i,\infty}) p_i\Big) di$$

Social cost of carbon

- One could also consider the *"Local cost of carbon"* as the marginal damage of the region :

$$LCC_{i,t} = \frac{\chi}{\widetilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_i(\tau_{i,\infty} - \tau_i^\star)\left(\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \frac{u(c_{i,\infty})}{u'(c_{i,\infty})}\right)$$

- As a result, the Social Cost of Carbon can be reexpressed as :

$$\begin{aligned} SCC_t \equiv \int_{\mathbb{I}} \frac{\psi_{i,t}^k}{\bar{\psi}_t^k} LCC_{i,t} di = \mathbb{E}^{\mathbb{I}}\left[LCC_{i,t}\right] + \mathbb{C}\mathrm{ov}^{\mathbb{I}}\left(\frac{\psi_{i,t}^k}{\bar{\psi}_t^k}, LCC_{i,t}\right) \\ > \mathbb{E}^{\mathbb{I}}\left[LCC_{i,t}\right] = \overline{SCC}_t \end{aligned}$$

- Solution of the adjoint equation : Proof
- Uncertainty SCC with uncertainty

Social cost of carbon & temperature

- Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta\,\chi\,\xi\,\omega \int_{t_0}^{T} e^{(n+\bar{g})t-\delta_s(T-t)} q_t^{f\ -\sigma_e} \int_{j\in\mathbb{I}} \left(z_j z_{j,t}^{e} \mathcal{D}(\tau_{j,t})\right)^{\sigma-1} y_{j,t}\, q_{j,t}^{\sigma_e-\sigma}\, dj\, dt$$

  - Geographical factors determining warming Δ_i
  - Climate sensitivity χ & carbon exit from atmosphere δ_s
  - Growth of population n, aggregate productivity $\bar{g}$
  - Deviation of output from trend y_i & relative TFP z_j
  - Directed technical change z_t^e, elasticity of energy in output σ
  - Fossil energy price $q^{e,f}$ and Hotelling rent $g^{q^f} \approx \dot{\lambda}_t^R / \lambda_t^R = \rho$
  - Change in energy mix, renewable share ω, price q_t^r & elasticity of source σ_e

- Approximations at $T \equiv$ Generalized Kaya (or $I = PAT$) identity More details

$$\frac{\dot{\tau}_T}{\tau_T} \propto\ n + \bar{g}^{y} - (1-\sigma)\left(g^{z^e} - \widetilde{\gamma}\right) + (\sigma_e - \sigma)(1-\omega)g^{q^r} - (\sigma_e(1-\omega) + \sigma\omega)g^{q^f}$$

Optimal energy and emissions decisions

1. ***Business as usual / Competitive equilibrium*** :

   - *Fossil energy :* only private tradeoff : marg. product of energy = marg cost + Hotelling rent

$$[e^f_{i,t}] \qquad \underbrace{\mathcal{D}(\tau_{i,t}) z\, \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}}}_{MPe^f_{i,t}} = q^f_t = \bar{\nu}\left(\frac{E^f_t}{\mathcal{R}_t}\right)^{\nu} + \lambda^R_t$$

Optimal energy and emissions decisions

1. ***Business as usual / Competitive equilibrium*** :
   - *Fossil energy :* only private tradeoff : marg. product of energy = marg cost + Hotelling rent

$$[e^f_{i,t}] \qquad \underbrace{\mathcal{D}(\tau_{i,t})z\,\partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}}}_{MPe^f_{i,t}} = q^f_t = \bar{\nu}\left(\frac{E^f_t}{\mathcal{R}_t}\right)^{\nu} + \lambda^R_t$$

2. ***Ramsey planner :***
   - *Fossil energy :*

$$[e^f_{i,t}] \qquad \underbrace{\psi^k_{i,t}}_{MV^k_{i,t}}\underbrace{\mathcal{D}(\tau_{i,t})z_{i,t}\,\partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}}}_{MPe^f_{i,t}} = \psi^k_{i,t}\, q^f_t + \xi\, SCC_t + \pi'(E^f_t)SVR_t$$

$$- \partial_{e^f}\widehat{\phi}^e_{it} + p_i\,\phi^{Ef}_t\,\partial_{EE}\mathcal{C}(E^f_t)$$

Optimal energy and emissions decisions

- Planner's optimal choice for fossil energy consumption :

$$\psi^k_{i,t} MPe^f_{i,t} = \psi^k_{i,t}\, q^f_t + \xi\, SCC_t + \pi'(E^f_t) SVR_t - \partial_{e^f} \widehat{\phi}^e_{it} + p_i\, \phi^{Ef}_t\, \partial_{EE} \mathcal{C}(E^f_t)$$

Optimal energy and emissions decisions

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– Redistribution motive : energy is more valuable in poorer countries !

$$\text{low } z_i, k_i \qquad \Rightarrow \quad \text{low } \; c_i, \; \text{high } \psi^k_{i,t} \approx \omega_i U'(c_i)\, p_i \; > \overline{\psi}^k_t = \int_{\mathbb{I}} \psi^k_{i,t} di$$

Optimal energy and emissions decisions

- Planner's optimal choice for fossil energy consumption :

$$\psi^k_{i,t} MPe^f_{i,t} = \psi^k_{i,t}\, q^f_t + \xi\, SCC_t + \pi'(E^f_t) SVR_t - \partial_{ef}\, \widehat{\phi}^e_{it} + p_i\, \phi^{Ef}_t\, \partial_{EE}\mathcal{C}(E^f_t)$$

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  – Climate externality :

$$SCC_t = \mathbb{C}\mathrm{ov}_j\left(\frac{\psi^k_{j,t}}{\overline{\psi}^k_t}, LCC_{j,t}\right) + \mathbb{E}_j[LCC_{j,t}] > \mathbb{E}_j[LCC_{j,t}]$$

Optimal energy and emissions decisions

- Planner's optimal choice for fossil energy consumption :

$$\psi^k_{i,t} MPe^f_{i,t} = \psi^k_{i,t}\, q^f_t + \xi\, SCC_t + \pi'(E^f_t) SVR_t - \partial_{e^f} \widehat{\phi}^e_{it} + p_i\, \phi^{Ef}_t\, \partial_{EE}\mathcal{C}(E^f_t)$$

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– Energy rent redistribution :

$$\pi'(E^f_t) SVR_t = \mathbb{C}\mathrm{ov}_j\left(\frac{\psi^k_{i,t}}{\overline{\psi}^k_t}, \theta_j \pi'_j(E)\right) + \pi'(E) < \pi'(E)$$

– Curvature of the output production function :

$$\partial_{e^f} \widehat{\phi}^e_{it} = \partial_{e^f}\left[MPe^f_t \phi^f_{j,t} + MPe^r_t \phi^r_{j,t}\right]$$

Optimal energy and emissions decisions

- Planner's optimal choice for fossil energy consumption :

$$\psi^k_{i,t} MPe^f_{i,t} = \psi^k_{i,t}\, q^f_t + \xi\, SCC_t + \pi'(E^f_t) SVR_t - \partial_{e^f} \widehat{\phi}^e_{it} + p_i\, \phi^{Ef}_t\, \partial_{EE}\mathcal{C}(E^f_t)$$

  – Redistribution motive : energy is more valuable in poorer countries !

$$\text{low } z_i, k_i \qquad \Rightarrow \quad \text{low } \; c_i, \; \text{high } \psi^k_{i,t} \approx \omega_i U'(c_i)\, p_i \; > \overline{\psi}^k_t = \int_{\mathbb{I}} \psi^k_{i,t} di$$

  – Climate externality :

$$SCC_t = \mathbb{C}\mathrm{ov}_j\left(\frac{\psi^k_{j,t}}{\overline{\psi}^k_t}, LCC_{j,t}\right) + \mathbb{E}_j[LCC_{j,t}] > \mathbb{E}_j[LCC_{j,t}]$$

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$$\partial_{e^f} \widehat{\phi}^e_{it} = \partial_{e^f}\left[MPe^f_t \phi^f_{j,t} + MPe^r_t \phi^r_{j,t}\right]$$

  – Curvature of the energy supply curve :

$$\phi^{Ef}_t\, \partial_{EE}\mathcal{C}(E^f_t, \mathcal{R}_t) = \phi^{Ef}_t\, \nu\bar{\nu}\frac{E^{\nu-1}}{\mathcal{R}^\nu_t}$$

Decentralization - 1

- With inequality $\psi^k_{i,t} \neq \psi^k_{j,t}$, it is unclear how to decentralize
- Allowing lump sum transfer across countries solves world inequality

$$\psi^k_{i,t} = \psi^k_{j,t} = \bar{\psi}^k_{i,t}$$

- Pigouvian taxation as in the Representative agent world = $\mathbf{t}^f_t = SCC_t = \mathbb{E}^{\mathbb{I}}\left(LCC_{j,t}\right)$
- Restricting to intra-country redistribution ?
  - Allow distortive taxes and lump-sum rebate in i only

  $\Rightarrow$ Planner cares about redistribution : $\lambda^k_{i,t}$ vs. $\bar{\lambda}^k_t = \int_j \lambda^k_{j,t} p_j dj$
  - Combination of distortive taxes and lump-sum rebate

Decentralization - 2

- Energy taxes :

$$MPe_{i,t}\frac{\partial e_{i,t}}{\partial e^f_{i,t}} = q^{e,f}_t + \underbrace{\frac{\bar{\psi}^k_t}{\psi^k_{i,t}}}_{=\text{redistribution}}\left(\mathbf{t}^S_t - \mathbf{t}^R_t - \mathbf{t}^e_{i,t} + \mathbf{t}^C_t\right)$$

- Combination :
  - Redistributive taxes/subsidy :

$$\mathbf{t}_{i,t} = \frac{\bar{\psi}^k_t}{\lambda^k_{i,t}} = \frac{\int_{\mathbb{I}} \omega_j u'(c_j)p_j dj}{\omega_i u'(c_i)p_i} \lessgtr 1$$

  - Pigouvian tax :$\mathbf{t}^S$ is flat rate accounting for the climate externality

$$\mathbf{t}^S_t = -\int_{\mathbb{I}} \frac{\lambda^k_{j,t}}{\bar{\lambda}^k_t}\frac{\lambda^S_{j,t}}{\lambda^k_{j,t}} p_j dj = \mathbb{C}\text{ov}_j\left(\frac{\lambda^k_{j,t}}{\bar{\lambda}^k_t}, LCC_{j,t}\right) + \overline{SCC}_t > \overline{SCC}_t$$

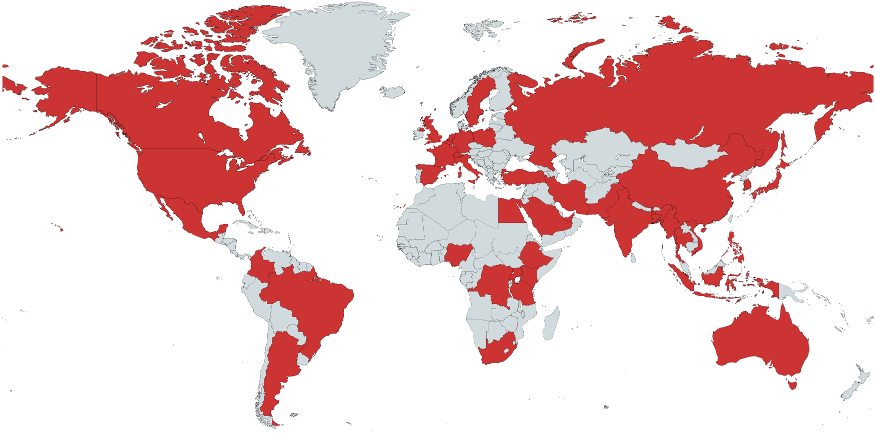
  - Fossil rent : $\mathbf{t}^R$ is flat rate accounting for the rent redistribution

$$\mathbf{t}^R_t = -\int_{\mathbb{I}} \frac{\lambda^k_{j,t}}{\bar{\lambda}^k_t}\theta_j\, \pi_e(E^f_t, \mathcal{R}_t)dj = \left(1 + \mathbb{C}\text{ov}_j\left(\frac{\lambda^k_{j,t}}{\bar{\lambda}^k_t}, \theta_j\right)\right)\pi_e(E^f_t, \mathcal{R}_t) < \pi_e(E^f_t, \mathcal{R}_t)$$

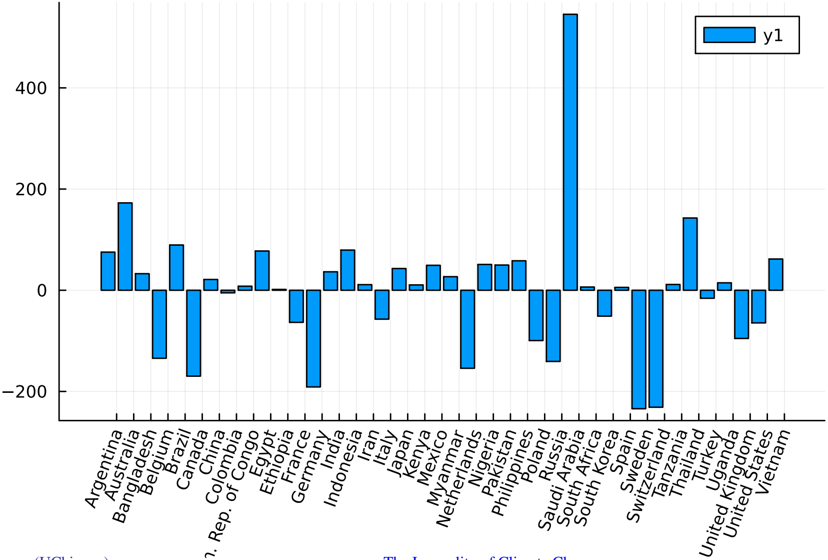
  - Everything is rebated lump sum to the household in each country i

Numerical Application

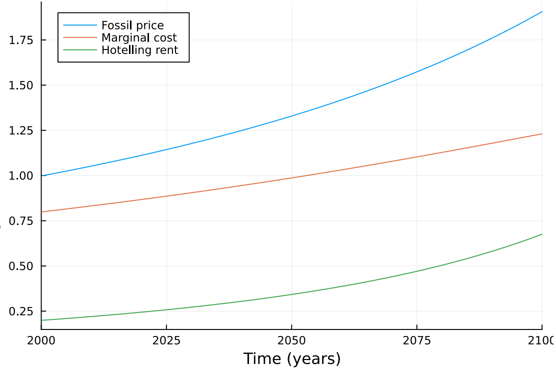
- Data : 40 countries
- Temperature (of the largest city), GDP, energy, population
- Calibrate z to match the distribution of output per capita at steady state

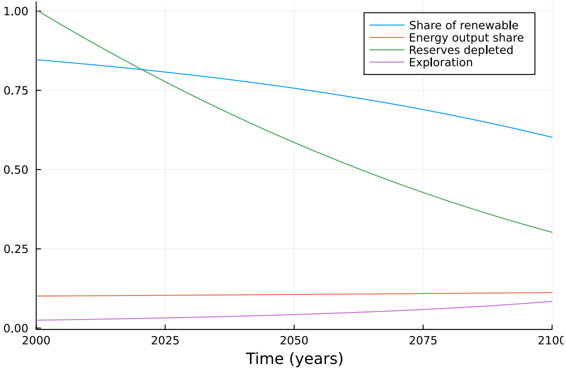


Distribution of carbon prices :

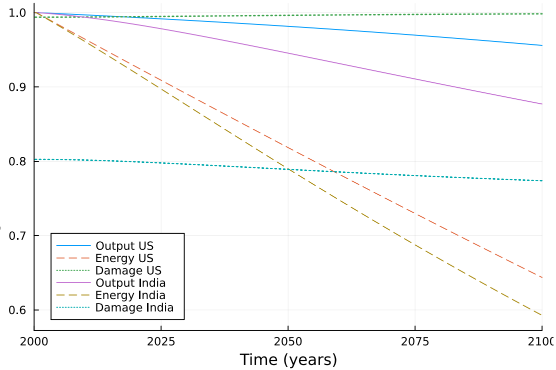


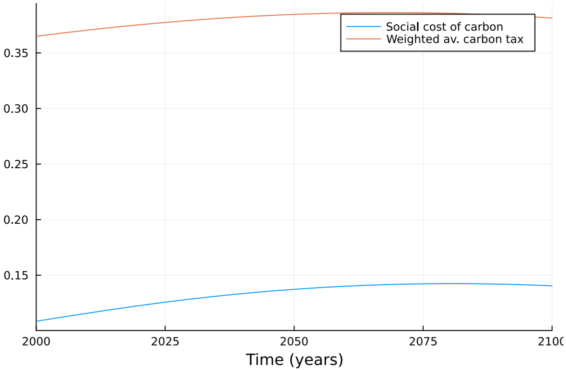
Energy


Fossil price
Marginal cost
Hotelling rent
1.75
1.50
1.25
1.00
0.75
0.50
0.25
2000
2025
2050
2075
2100
Time (years)


Share of renewable
Energy output share
Reserves depleted
Exploration
1.00
0.75
0.50
0.25
0.00
2000
2025
2050
2075
2100
Time (years)

Output and carbon price





Extensions - 1 - Endogeneous growth

- As of now, TFP z_t and directed technical change z_t^e are exogenous – growing at $\bar{g}^y$ and g^{z^e}
- Could easily nest an endogenous growth block in this model
  - Household / firm in country i chooses an amount $x_{i,t}$ of R&D to be allocated to increase TFP at rate ω_t^z or energy technology (efficiency)
  - Cost $c(x_{i,t})$

$$\dot{z}_t = h^y(\omega^z x_t) \qquad \dot{z}_t^e = h^e\big((1-\omega^z)x_t\big)$$

  - As a result, the marginal value of an investment in R&D is "priced" on the costates :

$$-\dot{\lambda}_t^z + \rho\lambda_t^z = \lambda_t^k \mathcal{D}(\tau_t) f(k_t, e_t) \qquad \text{Recall : } y_t = z_t \mathcal{D}(\tau_t) f(k_t, e_t)$$

$$-\dot{\lambda}_t^{z^e} + \rho\lambda_t^{z^e} = \lambda_t^k z_t \mathcal{D}(\tau_t) \partial_{z^e} f(k_t, e_t)$$

  - And optimal decisions depend on this shadow value ;

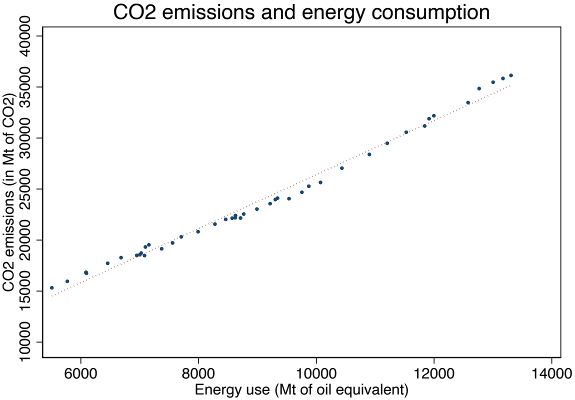
$$\omega^z h'^y\big(\omega^z x_t\big)\frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t}) \qquad (1-\omega^z)h'^e\big((1-\omega^z)x_t\big)\frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t})$$

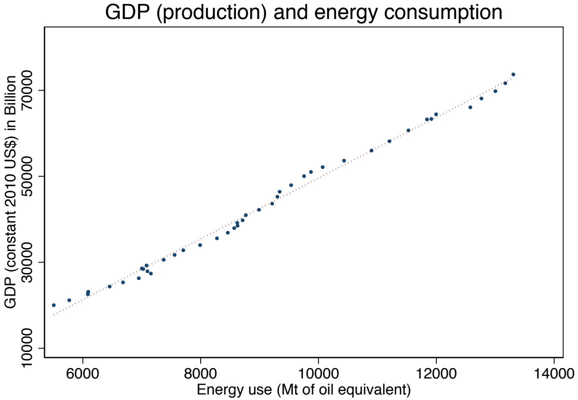
Conclusion

- Climate change is induced by externality
  - Energy/Emission choice doesn't include the impact on other countries
  - Cause strengthened by heterogeneity in wealth (capital/productivity)
  - Effect strengthened by heterogeneity in impact (temperature/damage)
- Social planner allocation correct for these different dimensions
  - Both Static correction $\equiv$ modified Pigouvian carbon taxation
  - And dynamic : through the marginal value of states
- Future plans :
  - Simulation of the three equilibria $CE/tax/SP$
  - Match the distribution of k using dynamics over 1960-2020
  - Social cost of carbon including heterogeneity and model uncertainty

Motivation

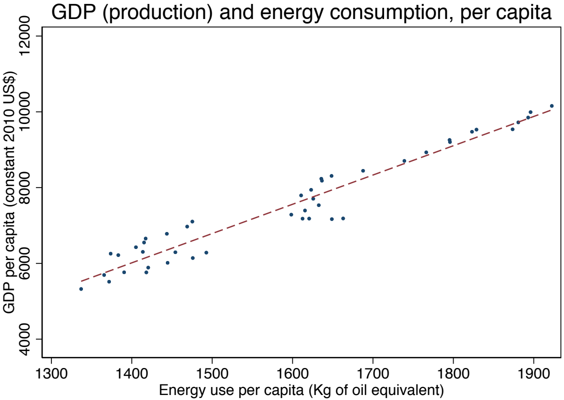
- CO_2 emissions correlate linearly with energy use
- Energy use (85% from fossils) correlates with output/growth

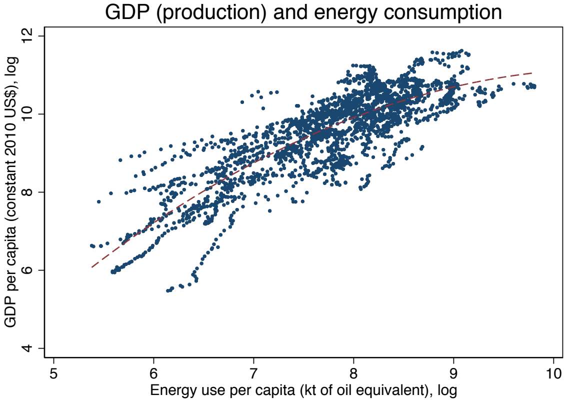




Introduction – Motivation

- Also true per capita and for the trajectory of individual countries





More details – Energy market

- Fossil fuel producer : price the Hotelling rent with the maximum principle :

$$\mathcal{H}^m(\mathcal{R}_t, \lambda_t^R, E_t, \mathcal{I}_t^e) = \pi_t(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) + \lambda_t^R(\delta^R \mathcal{I}_t^e - E_t)$$

- Rent λ_t^R grows with interest ρ and with the marginal gain of increasing reserves

$$\begin{aligned}\dot{\lambda}_t^R &= \rho\lambda_t^R - \partial_R \mathcal{C}(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) \\ &= \rho\lambda_t^R + \frac{\bar{\nu}\nu}{1+\nu}\left(\frac{E_t^\star}{R_t}\right)^{1+\nu} + \frac{\bar{\mu}\mu}{1+\mu}\left(\frac{I_t^\star}{R_t}\right)^{1+\mu} \\ \dot{\lambda}_t^R &= \rho\lambda_t^R + \frac{\bar{\nu}^{-1/\nu}\nu}{1+\nu}\left(q^{e,f} - \lambda_t^R\right)^{1+1/\nu} + \frac{\bar{\mu}^{-1/\mu}\mu}{1+\mu}\left(\delta^R\lambda_t^R\right)^{1+1/\mu}\end{aligned}$$

- Because of decreasing return to scale and Hotelling rents : profits are > 0

$$\pi_t(E_t^f, \mathcal{R}_t, \lambda_t^R) = \frac{1+\nu-\bar{\nu}}{1+\nu}\left(\frac{E_t^f}{\mathcal{R}_t}\right)^{1+\nu}\mathcal{R}_t + \lambda_t^R E_t^f - \frac{\bar{\mu}^{-1/\mu}}{1+\mu}\left(\delta^r\lambda_t^R\right)^{1+\frac{1}{\mu}}$$

back

More details – PMP – Competitive equilibrium

- Household problem : State variables $s_{i,t} = (k_i, \tau_i, z_i, p_i, \Delta_i)$ Back
- Pontryagin Maximum Principle

$$\mathcal{H}^{hh}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}) = u(c_i, \tau_i) + \lambda^k_{i,t}\Big(\mathcal{D}(\tau_{it})f(k_t, e_t) - (n + \bar{g} + \delta)k_t - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big)$$

$$[c_t] \qquad u'(c_{it}) = \lambda^k_{i,t}$$

$$[e^f_t] \qquad MPe^f_{it} = \mathcal{D}(\tau_{i,t}) z\, \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}} = q^f_t$$

$$[e^r_t] \qquad MPe^r_{it} = \mathcal{D}(\tau_{i,t}) z\, \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e^r_{i,t}}{(1-\omega) e_{i,t}}\right)^{-\frac{1}{\sigma_e}} = q^r_{it}$$

$$[k_t] \qquad \dot{\lambda}^k_t = \lambda^k_t\big(\rho - \partial_k f(k_{i,t}, e_{i,t})\big)$$

- Fossil Energy Monopoly :

$$\mathcal{H}^m(\mathcal{R}_t, \lambda^R_t, E_t, \mathcal{I}^e_t) = \pi_t(E^f_t, \mathcal{I}^f_t, \mathcal{R}_t) + \lambda^R_t(\delta^R \mathcal{I}^e_t - E_t)$$

$$[\mathcal{R}_t] \qquad \dot{\lambda}^R_t = \rho\lambda^R_t + \frac{\bar{\nu}\nu}{1+\nu}\left(\frac{E^\star_t}{R_t}\right)^{1+\nu} + \frac{\bar{\mu}\mu}{1+\mu}\left(\frac{I^\star_t}{R_t}\right)^{1+\mu}$$

$$[E^f_t] \qquad q^{e,f}_t = \nu_E(E, \mathcal{R}) + \lambda^R_t = \bar{\nu}\left(\frac{E_t}{\mathcal{R}_t}\right)^{\nu} + \lambda^R_t$$

$$[\mathcal{I}_t] \qquad \lambda^R_t \delta^R = \mu_I(I_t, R_t) = \bar{\mu}\left(\frac{I_t}{\mathcal{R}_t}\right)^{\mu} \qquad I_t = R_t\Big(\frac{\lambda^R_t \delta}{\bar{\mu}}\Big)^{1/\mu}$$

More details – PMP – Ramsey Optimal Allocation

- Hamiltonian :

$$
\begin{aligned}
\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\},\{\psi\}) = \int_{\mathbb{I}} \omega_i \mathcal{D}^u(\tau_{it}) u(c_i) p_i di + \\
+ \psi^k_{i,t}\Big(\mathcal{D}(\tau_{it}) f(k_t, e_t) - (n+\bar{g}+\delta)k_t + \theta_i \pi(E^f_t, \mathcal{I}_t, \mathcal{R}_t) - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big) \\
+ \psi^S_t\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big) + \psi^{\mathcal{T}}_{it}\zeta\Big(\Delta_i\, \chi\, \mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \psi^{\mathcal{R}}_{it}\Big(- E^f_t + \delta^R \mathcal{I}_t\Big) \\
+ \psi^{\lambda k}_{i,t}\Big(\lambda^k_t(\rho - r_t)\Big) + \psi^{\lambda R}_t\Big(\rho\lambda^R_t + \mathcal{C}^f_{\mathcal{R}}(E^f_t, \mathcal{I}_t, \mathcal{R}_t)\Big) \\
+ \phi^c_{it}\big(\mathcal{D}^u(\tau_{it}) u'(c_i) - \lambda^k_{it}\big) + \phi^{ef}_{it}\Big(MPe^f_{it} - q^f_t\Big) + \phi^r_{it}\Big(MPe^r_{it} - q^r_{it}\Big) \\
+ \phi^{Ef}_t\Big(q^f_t - \mathcal{C}^f_E(\cdot) - \lambda^{\mathcal{R}}_t\Big) + \phi^{\mathcal{I}f}_t\big(\delta\lambda^{\mathcal{R}}_t - \mathcal{C}^f_{\mathcal{I}}(\cdot)\big)
\end{aligned}
$$

- Back

Ramsey Optimal Allocation - FOCs

- FOCs

$$[c_{it}] \qquad \psi^k_{it} = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{=\text{direct effect}} + \underbrace{\phi^c_{it} \mathcal{D}^u(\tau_{it}) u''(c_i)}_{=\text{effect on savings}}$$

$$\text{Define}: \qquad \widehat{\phi}^e_{it} = \phi^f_{it} MPe^f_t + \phi^r_{it} MPe^r_t$$

$$[e^f_{it}] \qquad \psi^k_{i,t}\left(MPe^f_{it} - q^f_t\right) + \xi_{it} p_i \psi^S_t + p_i \partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj + \partial_{e^f} \widehat{\phi}^e_{it} - p_i \phi^{Ef}_t \partial_{EE} \mathcal{C}(\cdot) = 0$$

$$[e^r_{it}] \qquad \psi^k_{i,t}\left(MPe^r_{it} - q^r_{it}\right) + \partial_{e^r} \widehat{\phi}^e_{it} = 0$$

$$[\mathcal{I}_t] \qquad \delta\, \psi^{\mathcal{R}}_t + \partial^2_{\mathcal{R}\mathcal{I}} \mathcal{C}(\cdot)\, \psi^{\lambda,\mathcal{R}}_t - \phi^{\mathcal{I}}_t \partial^2_{\mathcal{I}\mathcal{I}} \mathcal{C}(\cdot) = 0$$

$$[q^f_t] \qquad \phi^{Ef}_t = \int_{\mathbb{I}} e^f_{it} \psi^k_{jt} dj + \int_{\mathbb{I}} \phi^f_{jt} dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj$$

- Back

Ramsey Optimal Allocation - FOCs

- Backward equations for planner's costates

$$[k_i] \qquad \dot{\psi}^k_{it} = \psi^k_{it}(\tilde{\rho} - r_{it} + \partial_k MPk_i)\psi^k_{it} - \partial_k \widehat{\phi}^e_{it}$$

$$[\mathcal{S}_i] \qquad \dot{\psi}^{\mathcal{S}}_t = \left(\tilde{\rho} + \delta^s\right)\psi^{\mathcal{S}}_t - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi^{\tau}_{jt} dj$$

$$[\tau_i] \qquad \dot{\psi}^{\tau}_t = \left(\tilde{\rho} + \zeta\right)\psi^{\tau}_t - \left(\omega_i \mathcal{D}'(\tau_{it})u(c_{it}) + \psi^k_{it}\mathcal{D}'(\tau_{it})f(k_{it}, e_{it}) + \phi^c_{it}\mathcal{D}'(\tau_{it})u'(c_i) + \partial_\tau \widehat{\phi}^e_{it}\right)$$

$$[\mathcal{R}] \qquad \dot{\psi}^{\mathcal{R}}_t = \psi^{\mathcal{R}}_t\left(\tilde{\rho} - \partial^2_{\mathcal{R}\mathcal{R}}\mathcal{C}(\cdot)\right) - \phi^{Ef}_t \partial^2_{\mathcal{R}E}\mathcal{C}(\cdot)$$

$$[\lambda^k_i] \qquad \dot{\psi}^{\lambda,k}_t = \tilde{\rho}\psi^{\lambda,k}_t - (\rho - r_{i,t})\psi^k_t + \phi^c_{i,t}$$

$$[\lambda^{\mathcal{R}}_i] \qquad \dot{\psi}^{\lambda,\mathcal{R}}_t = \psi^{\lambda,\mathcal{R}}_t(\tilde{\rho} - \rho) + \phi^{Ef}_t - \delta\phi^{\mathcal{I}f}_t$$

- Back

Cost of carbon / Marginal value of temperature

- Solving for the cost of carbon and temperature $\Leftrightarrow$ solving ODE

$$\dot{\lambda}^{\mathcal{T}}_{i,t} = \lambda^{\mathcal{T}}_t(\widetilde{\rho} + \Delta\zeta) + \gamma(\tau - \tau^\star)\mathcal{D}^y(\tau)f(k,e)\lambda^k_t + \phi(\tau - \tau^\star)\mathcal{D}^u(\tau)u(c)$$

$$\dot{\lambda}^S_t = \lambda^S{}_t(\widetilde{\rho} + \delta^s) - \int_{\mathbb{I}} \Delta_i \zeta \chi \lambda^{\mathcal{T}}_{i,t}$$

- Solving for $\lambda^{\mathcal{T}}_t$ and $\lambda^{\mathcal{S}}_t$, in stationary equilibrium $\dot{\lambda}^S_t = \dot{\lambda}^{\mathcal{T}}_t = 0$

$$\lambda^{\mathcal{T}}_{i,t} = -\int_t^\infty e^{-(\widetilde{\rho}+\zeta)u}(\tau_u - \tau^\star)\Big(\gamma\mathcal{D}^y(\tau_u)y_{\mathcal{T}}\lambda^k_u + \phi\mathcal{D}^u(\tau_u)u(c_u)\Big)du$$

$$\lambda^{\mathcal{T}}_{i,t} = -\frac{1}{\widetilde{\rho} + \Delta\zeta}(\tau_\infty - \tau^\star)\Big(\gamma\mathcal{D}^y(\tau_\infty)y_\infty\lambda^k_\infty + \phi\mathcal{D}^u(\tau_\infty)u(c_\infty)\Big)$$

$$\begin{aligned}
\lambda^S_t &= -\int_t^\infty e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\int_{\mathbb{I}}\Delta_j\lambda^{\mathcal{T}}_{j,u}dj\,du \\
&= \frac{1}{\widetilde{\rho}+\delta^s}\zeta\chi\int_{\mathbb{I}}\Delta_j\lambda^{\mathcal{T}}_{j,\infty} \\
&= -\frac{\chi}{\widetilde{\rho}+\delta^s}\frac{\zeta}{\widetilde{\rho}+\zeta}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^\star)\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_\infty\lambda^k_{j,\infty} + \phi\mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)dj \\
\lambda^S_t &\xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho}+\delta^s}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^\star)\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_{j,\infty}\lambda^k_{j,\infty} + \mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)dj
\end{aligned}$$

Cost of carbon / Marginal value of temperature

- Closed form solution for CC :
  - In stationary equilibrium : $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
  - Fast temperature adjustment $\zeta \to \infty$
  - no internalization of externality (business as usual)

$$LCC_{i,t} \equiv \frac{\Delta_i \chi}{\widetilde{\rho} + \delta^s}(\tau_\infty - \tau^\star)\left(\gamma \mathcal{D}^y(\tau_\infty) y_\infty + \phi \mathcal{D}^u(\tau_\infty)\frac{c_\infty}{1-\eta}\right)$$

- Heterogeneity + uncertainty about models

Uncertainty about models :

- In our model, we rely strongly on model specification :
  - Parameters θ of models :
    - Climate system and damages : $(\xi, \chi, \zeta, \delta^s, \gamma, \phi)$
    - Economic model : $\varepsilon, \nu, \bar{g}, n$ or extended : $\omega, \sigma, \sigma^e, \nu, \mu$
    - Models with probability weight $\pi(\theta)$
  - Social cost of carbon, weighted for model uncertainty :

$$SCC_t(\theta) = -\int_{\mathbb{S}} \frac{\lambda_t^S(s,\theta)}{\lambda_t^k(s,\theta)} g(s) ds$$

$$S\bar{C}C_t = \int_{\Theta} SCC_t(\theta)\pi(\theta)d\theta = \int_{\Theta}\int_{\mathbb{S}} \frac{\lambda_t^S(s,\theta)}{\lambda_t^k(s,\theta)} g(s) ds \pi(\theta) d\theta$$

  - Counterfactual computation of SCC
    - Representative country / no uncertainty $\frac{\lambda_t^S}{\lambda_t^k}$
    - With heterogeneous regions / no uncertainty $SCC_t(\bar{\theta})$
    - No heterogeneity / model uncertainty $\int_{\Theta} \frac{\lambda_t^S(\bar{s},\theta)}{\lambda_t^k(\bar{s},\theta)} \pi(\theta) d\theta$
    - With heterogeneous regions / with model uncertainty $S\bar{C}C_t$
  - back

Long term temperature

- Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i(\mathcal{T}_T - \mathcal{T}_{t_0}) = \Delta_i \chi \int_{t_0}^{T} e^{-\delta_s(T-t)} \mathcal{E}_t dt$$
$$= \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e^f_{j,t}(1 - \vartheta_{j,t}) dj\, dt$$

  - Use CES demands : $e^f_{j,t} = \omega e_{j,t} q^{f\ -\sigma_e}_t q_t^{\sigma_e}$ for energy and $e_t = (z z^e_t \mathcal{D}(\tau_{j,t}))^{\sigma-1} q_t^{-\sigma}$
  - Moreover, CES price index $q_t = (\omega q^{f\ 1-\sigma_e}_t + (1-\omega) q^{r\ 1-\sigma_e}_t)^{1/(1-\sigma_e)}$, so first order approximation : $g^q = \omega g^{q^f} + (1-\omega) g^{q^r}$ with growth for q^f and q^r as well as $z^e_t = e^{g^e t}$
  - Gives :

$$e^f_{j,t} = \omega q^{f\ -\sigma_e}_t q_{j,t}^{\sigma_e} (z z^e_t \mathcal{D}(\tau_{j,t}))^{\sigma-1} q_{j,t}^{-\sigma}$$

Temperature dynamics

- Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e^f_{j,t}(1 - \vartheta_{j,t}) dj\, dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} q_t^{f\;-\sigma_e} \int_{j \in \mathbb{I}} \omega (z z^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} e^{g^e t} q_{j,t}^{\sigma_e - \sigma} (1 - \vartheta_{j,t}) dj\, dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \omega \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} e^{(-\sigma_e + (\sigma_e - \sigma)\omega) g^{q^f} t}\; e^{(\sigma_e - \sigma)(1-\omega) g^{q^r} t}$$
$$\times \int_{j \in \mathbb{I}} \left(z z_t^e \mathcal{D}(\tau_{j,t}) \right)^{\sigma - 1} y_{j,t} (1 - \vartheta_{j,t}) dj\, dt$$

- back

Social Planner allocation

- Solving the social planner allocation : Hamiltonian

$$\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\}) = \int_{\mathbb{I}} \omega_i\, u(c_i,\tau_i)p_i di - wL_t^f + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^k \Big(\mathcal{D}(\tau_t)f(k_t,e_t) - (n+\bar{g}+\delta)k_t - c_t\Big)p_i di$$
$$+ \widehat{\lambda}_t^S\Big(\int_{\mathbb{I}} \xi^f e_t^f p_i di - \delta^s \mathcal{S}_t\Big) + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^{\mathcal{T}} \Delta_i \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)p_i di$$
$$+ \widehat{\lambda}_t^R\Big(-E_t^f + \delta^R \mathcal{I}_t\Big) + \widehat{\lambda}_t^{ef}\Big(\widetilde{\mathcal{F}}(L_t^f,\mathcal{R}_t) - E_t^f\Big) + \int_{\mathbb{I}} \widehat{\lambda}_t^{er}\Big(z_{i,t}^r {k_{i,t}^r}^{\alpha} - e_t^r\Big)p_i di$$

with $\quad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_i di \quad$ and $\quad e_t = \left(\omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e-1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r))^{\frac{\sigma^e-1}{\sigma^e}}\right)^{\frac{\sigma^e}{\sigma^e-1}}$

- Results :

$$\omega_i\, u_c(c_i,\tau_i) = \widehat{\lambda}_{i,t}^k$$
$$\widehat{\lambda}_{i,t}^k\, MPe_{i,t}\, \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{ef} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S$$
$$\widehat{\lambda}_{i,t}^k\, MPe_{i,t}\, \frac{\partial e_{i,t}}{\partial e_{i,t}^r} = \widehat{\lambda}_t^{er}$$

- Back

Social Planner allocation

- Moreover, using Pontryagin principle, accounting for the distribution, we should have an adjustment for the state dynamics :
  - Valuation of the state changes when the planner knows if after the externality
- Back

Decentralization

- With inequality $\widehat{\lambda}^k_{i,t} \neq \widehat{\lambda}^k_{j,t}$, it's unclear how to decentralize
- Allowing lump sum transfer across countries solves world inequality $\lambda^k_{i,t} = \lambda^k_{j,t} = \bar{\lambda}^k_t$, as a result :

$$\widehat{\lambda}^k_{i,t}\, MPe_{i,t} \frac{\partial e_{i,t}}{\partial e^f_{i,t}} = \widehat{\lambda}^{ef}_t + \widehat{\lambda}^R_t + \widehat{\lambda}^S_t \qquad \Leftrightarrow \qquad MPe_{i,t} \frac{\partial e_{i,t}}{\partial e^f_{i,t}} = \Big(\underbrace{\mathcal{C}'(E^f_t) + \lambda^R_t}_{=q^{e,f}_t = \frac{\widehat{\lambda}^{ef}_t + \widehat{\lambda}^R_t}{\bar{\lambda}^k_t}} + \underbrace{T^f_t}_{=\frac{\widehat{\lambda}^S_t}{\bar{\lambda}^k_t}} \Big)$$

- The climate $\mathbf{t}^f_t$ is flat rate accounting for the climate externality

$$\mathbf{t}^f_t = \frac{\widehat{\lambda}^S_t}{\bar{\lambda}^k_t} = \int_{\mathbb{I}} \frac{\lambda^S_{i,t}}{\bar{\lambda}^k_t} p_i di \neq CC_{i,t}$$

Optimal abatement of emissions decisions

1. ***Business as usual*** :
   - *Abatement :*

$$[\vartheta_t] \qquad \partial_\vartheta \Lambda(\vartheta_{i,t}) e^f_{i,t} = 0 \qquad \Rightarrow \vartheta_{i,t} = 0$$

Optimal abatement of emissions decisions

1. ***Business as usual*** :
   - *Abatement :*

$$[\vartheta_t] \qquad \partial_\vartheta \Lambda(\vartheta_{i,t}) e^f_{i,t} = 0 \qquad \Rightarrow \vartheta_{i,t} = 0$$

2. ***Social planner*** :
   - *Abatement :*

$$[\vartheta_t] \qquad \partial_\vartheta \Lambda(\vartheta^i_t) = \bar{\theta}(\vartheta_{i,t})^\theta = \underbrace{-\frac{1}{\lambda^k_{i,t}} \int_{\mathbb{I}} \omega_j \lambda^S_{j,t} p_j dj}_{=\text{carbon tax for } i}$$

FBSDE for MFG systems – general formulation

- State $X_t \equiv (a_t, z_t) \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), and X diffusion process with control $\alpha^\star(t, X, P_X, Y) \equiv c_t^\star$

$$dX_t = b(X_t, P_{X_t}, \alpha_t^\star)dt + \sigma dB_t$$

- Set up the Hamiltonian :

$$\mathcal{H}(t, x, P_X, y) = \max_{\alpha \in \mathbb{A}} \left(u(\alpha) + b(x, p_X, \alpha) \cdot y \right)$$

- Optimal control $c^\star \in \text{argmax}_{\alpha \in \mathbb{A}} \left(u(\alpha) + b(x, p_X, \alpha) \cdot y \right)$
- Using the Pontryagin maximum principle :

$$dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t$$

back

FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t \end{cases}$$

- Question : What else do we need ?

FBSDE system for MFG

- Coupled FBSDE system for each agent

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- Question : What else do we need ?
  - The risk loading in the costate $\widetilde{Z}_t$:
    - Intuitions : expectation error in the law of motion of Y_t

$$\widetilde{Z}_t(x) = \mathbb{E}\left[\frac{dY_t + D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo

FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t \end{cases}$$

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$$\widetilde{Z}_t(x) = \mathbb{E}\left[\frac{dY_t + D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo
  - The initial condition Y_0 as a function of X_0

FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t \end{cases}$$

- Question : What else do we need ?
  - The risk loading in the costate $\widetilde{Z}_t$:
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$$\widetilde{Z}_t(x) = \mathbb{E}\Big[\frac{dY_t + D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt}{dB_t^0}\Big]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo
  - The initial condition Y_0 as a function of X_0
  - A boundary condition of Y_T or transversality $\lim_{t\to\infty} e^{-\rho t}X_tY_t = 0$

back

Social planner : Mean Field Control/McKean Vlasov

- Suppose now that the social planner controls the dynamic of each agents ... *accounting for its effect on the distribution*
- Get an additional term :

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t - \widetilde{\mathbb{E}}\left[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)\right] \end{cases}$$

- Effect on the distribution :

Social planner : Mean Field Control/McKean Vlasov

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- Get an additional term :

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t - \widetilde{\mathbb{E}}\left[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)\right] \end{cases}$$

- Effect on the distribution :
  - Social planner internalize the externality coming from heterogeneity
    - $D_\mu H$ is the L-derivative w.r.t the measure $\mu \equiv \mathbb{P}_{X_t}$
    - Idea : lifting of the function $H(x, \mu) = \widehat{H}(x, \widehat{X})$ where $\widehat{X} \sim \mu$ and hence $D_\mu H(x, \mu)(\widehat{X}) = D_{\widehat{x}}\widetilde{H}(x, \widehat{X})$
    - Intuition : shift the distribution of states $\widehat{X}$ for all agents
    - Probabilistic approach : easy to compute $\widetilde{\mathbb{E}}[D_\mu H(\tilde{X}_t, \mu)] = \widetilde{\mathbb{E}}[D_{\widehat{x}} H(\tilde{X}_t, \widehat{X})]$
  - Here : the effect is homogeneous for all agents : the interaction with the measure is non-local !

back

Sequential approach FBSDE w/ Agg. shocks

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t + \widetilde{\sigma} dB_t^0 \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t + \widetilde{Z}_t^0 dB_t^0 \end{cases}$$

- Question : What else do we need ?

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  - The risk loading on idiosyncratic shocks $\widetilde{Z}_t$:
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- Question : What else do we need ?
  - The risk loading on idiosyncratic shocks $\widetilde{Z}_t$:
  - The risk loading on aggregate shocks $\widetilde{Z}_t^0$:
  - The initial conditions $Y_0(X_0)$ and boundary condition on Y_T or transversality $\lim_{t\to\infty} e^{-\rho t} X_t Y_t = 0$

MFG system : Recursive approach w/ Agg. shocks

- Here : recursive w.r.t. idiosyncratic shocks, but sequential w.r.t. aggregate shocks.
- System for v and g :

$$-\partial_t v + \rho v = \max_{\alpha} u(\alpha) + \mathcal{A}(v)v + Z_t^0 dB_t^0$$
$$\partial_t g = \mathcal{A}^*(v)g + \partial_x[\sigma g]dB_t^0$$

- Solve the PDE system :
  - Finite difference, upwinding scheme
  - View that as a non-linear system : use Quasi Newton methods
  - New part : forcing terms $\partial_x[\sigma g]dB_t^0$ and $Z_t^0 dB_t^0$
  - Initial and terminal conditions

$$v_T = v^\infty \qquad\qquad g_0 = g^\infty$$

    - Direct effect of uncertainty on measure
    - Indirect effect through agent expectations : shadow price of aggregate risk Z_t^0