The Inequality of Climate Change

Work in progress

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Capital Theory

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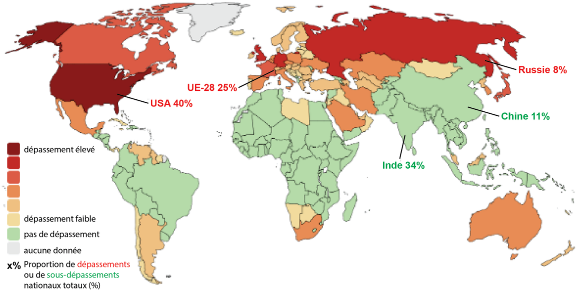
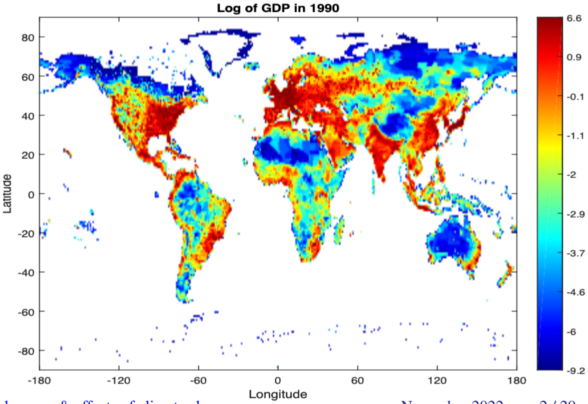
Introduction – Motivation

- Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
    - ***Unequal causes :*** Developed economies account for over 65% of cumulative GHG emissions ($\sim$ 25% each for the EU and the US)

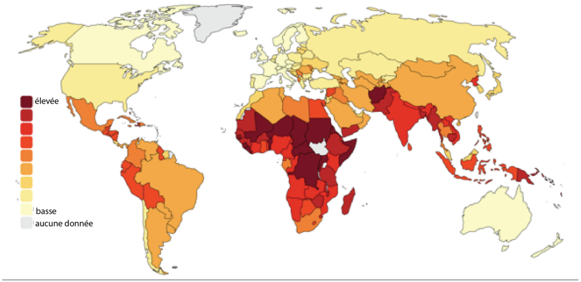


Source : Lancet planetary Health - Quantifying national responsibility for climate breakdown: an equality-based attribution approach for carbon dioxide emissions in excess of the planetary boundary - Jason Hickel 2020

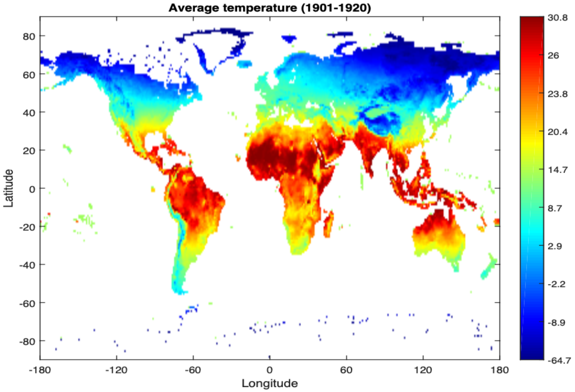


Introduction – Motivation

- Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
  - ***Unequal consequences :*** Increase in temperatures will disproportionately affect developing countries where the climate is already warm



Source : Notre Dame Global Adaptation Initiative



Introduction – this project

- Which countries will be affected the most by climate change ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - What is the optimal policy in presence of externalities *and* heterogeneities ?

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- Develop a simple and flexible model of climate economics
  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions

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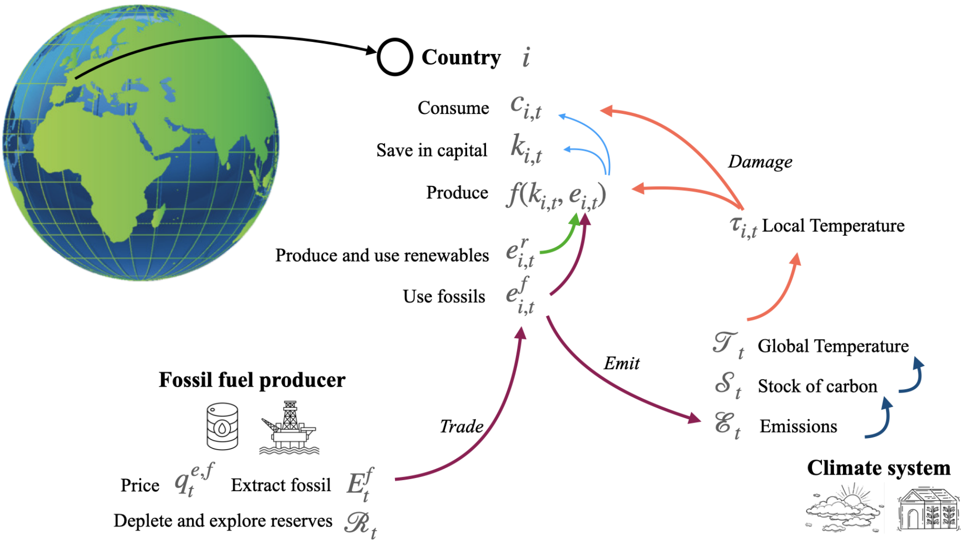
- Develop a simple and flexible model of climate economics
  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions
- Evaluate the heterogeneous welfare costs of global warming
- Provide analytical formulas and a numerical methodology to compute the cost of carbon
  - *CC* is linear in *GDP*/level of development and in temperature gaps
- Solve world optimal carbon policy with heterogeneous regions
  - Does the optimal carbon tax coincide with the social cost of carbon ?

Introduction – related literature

- Classic Integrated Assessment models (IAM) :
  - Nordhaus' Multi-regions DICE (2016), Golosov, Hassler, Krusell, Tsyvinski (2014)
- Models with analytical results :
  - Dietz, van der Ploeg, Rezai, Venmans (2021)
- Climate model with risk & uncertainty :
  - Cai, Lontzek, Judd (2019), Barnett, Brock and Hansen (2022)
- Macro (IAM) model with country heterogeneity :
  - Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021)
- Quantitative spatial models :
  - Cruz, Rossi-Hansberg (2021), Bilal, Rossi-Hansberg (2023), Rudik, Lyn, Tan, Ortiz-Bobea (2022)

Summary of the model



Model

- Neoclassical economy, in continuous time
- countries/regions $i \in \mathbb{I}$: ex-ante heterogeneous in dimensions $\underline{s}$
  - Here : $\underline{s}_i = \{p_i, z_i, \Delta_i\}$, relative heterogeneity doesn't change over time
  - Productivity grows at rate $\bar{g}$ and population grow at rate n
- regions heterogeneous ex-post $\bar{s}_i$
  - Here : capital and temperature $\bar{s}_i = \{k_i, \tau_i\}$
  - *Future :* could include z – endog. technical change – or p – migrations/demographics
  - *Renormalization :* all variables are values per unit of efficient labor $k_{i,t} = \frac{K_{i,t}}{p_{t_0}} e^{-(\bar{g}+n)t}$
- Aggregate variables : global $\mathcal{T}$emperature, carbon $\mathcal{S}$tocks in atmosphere, fossil energy $\mathcal{R}$eserves $S = \{\mathcal{T}, \mathcal{S}, \mathcal{R}\}$

Model – Household and firm

- Household problem in country i :

$$\max_{\{c_t, e_t^f, e_t^r\}} \int_{t_0}^{\infty} e^{-(\rho-n)t} \, u_i(c_t, \tau_t) dt$$

- Dynamics of capital in every country i :

$$\dot{k}_t = \mathcal{D}(\tau_t) f(k_t, e_t) + \theta_i^R \pi_t^R - (n + \bar{g} + \delta) k_t - q_t^e e_t - c_t + \mathbf{t}_{i,t}$$

- Choices :
  - c_t consumption, e_t energy, with production fct :

$$f(k,e) = z\left((1-\varepsilon)^{\frac{1}{\sigma}} k^{\alpha \frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} (z^e e)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

  - Damage function $\mathcal{D}_i(\tau_t)$ affect country's production
  - Directed technical change z_t^e & energy mix e_t with fossil e_t^f vs. renewable e_t^r

Model – Energy markets

- Two sources of energy : fossil e_t^f and renewable e_t^r for every i

$$e_{i,t} = \left(\omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e-1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r))^{\frac{\sigma^e-1}{\sigma^e}}\right)^{\frac{\sigma^e}{\sigma^e-1}}$$

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- Fossil fuels energy producer :
  - Extended-Hotelling problem (depleting reserves with stock effects and exploration)

$$\max_{\{E_t^f, \mathcal{I}_t\}_t} \pi_t(E_t^f, \mathcal{R}_t) = \max_{\{E_t^f, \mathcal{I}_t\}_t} q_t^{e,f} E_t^f - \mathcal{C}^e(E_t^f, \mathcal{R}_t) - \mathcal{C}^i(\mathcal{I}_t, \mathcal{R}_t)$$

$$s.t. \qquad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_{i,t} di \qquad \dot{\mathcal{R}}_t = -E_t^f + \delta_R \mathcal{I}_t$$

  - Optimal pricing with finite-resources rents More details

$$q_t^{e,f} = \bar{\nu} \left(\frac{E_t^{f\star}}{\mathcal{R}_t} \right)^{\nu} + \lambda_t^R \qquad \bar{\mu} \left(\frac{\mathcal{I}_t^\star}{\mathcal{R}_t} \right)^{\mu} = \delta_R \lambda_t^R$$

Model – Energy markets

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$$e_{i,t} = \left(\omega^{\frac{1}{\sigma^e}} \left(e^f_{i,t}\right)^{\frac{\sigma^e - 1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} \left(e^r_{i,t}\right)^{\frac{\sigma^e - 1}{\sigma^e}} \right)^{\frac{\sigma^e}{\sigma^e - 1}}$$

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$$q^{e,f}_t = \bar{\nu}\left(\frac{E^{f\star}_t}{\mathcal{R}_t}\right)^{\nu} + \lambda^R_t \qquad \bar{\mu}\left(\frac{\mathcal{I}^\star_t}{\mathcal{R}_t}\right)^{\mu} = \delta_R \lambda^R_t$$

- Renewable energy as a substitute technology

$$e^r_{i,t} = z^r_{i,t} {k^r_{i,t}}^{\alpha} \qquad q^{e,r}_{i,t} = r_{i,t} / (z^r_{i,t} \alpha {k^r_{i,t}}^{\alpha - 1})$$

Fossil energy and externality

- Fossil energy input e^f_t causes climate externality
  - Change the world climate – global temperature $\mathcal{T}_t$ and cumulative GHG in atmosphere $\mathcal{S}_t$:

$$\mathcal{E}_t = \int_{\mathbb{I}} \xi \, e^f_{i,t} \, p_i \, di$$

$$\dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s \mathcal{S}_t \qquad\qquad \dot{\mathcal{T}}_t = \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)$$

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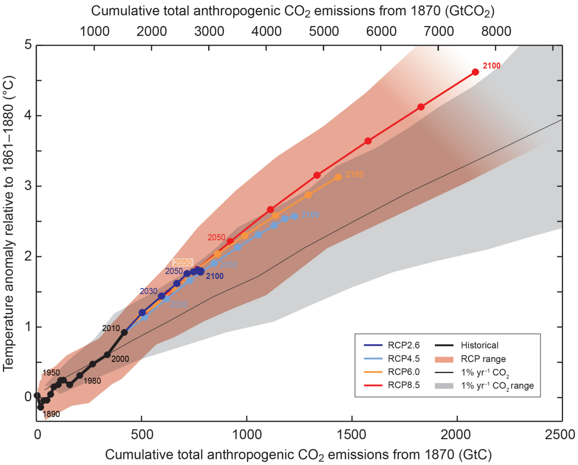
    – ζ is the inverse of persistence, so if $\zeta \to \infty$, we obtain a linear model :

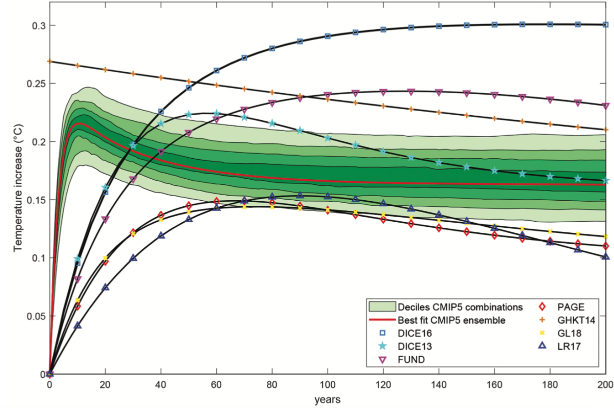
$$\mathcal{T}_t = \bar{\mathcal{T}} + \chi \mathcal{S}_t = \bar{\mathcal{T}} + \chi \int_{t_0}^{t} \int_{\mathbb{I}} \xi \, e_{i,s} \, di \, ds \Big|_{GtC}$$

  - The externality depends on policy $e^f_{i,t}$ as function of states $\{z, p, k, \tau\}$
    - Naturally, countries richer/more productive/with a larger population use more energy !
  - Impact of climate on country's local temperature :

$$\dot{\tau}_{i,t} = \Delta_i \dot{\mathcal{T}}_t$$

Temperature dynamics





Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

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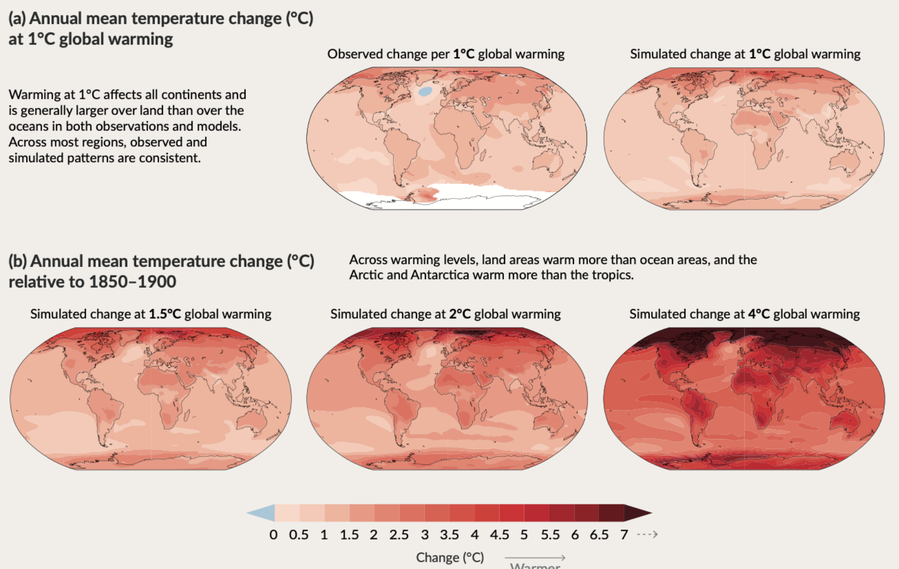


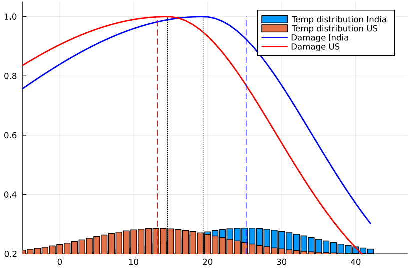
Damage functions

- Climate change has two effects :
  - Affect household utility function $u(c_t, \tau_t) = \mathcal{D}^u(\tau_t)\frac{c_t^{1-\eta}}{1-\eta}$

$$\mathcal{D}_u(\tau) = \begin{cases} e^{-\phi^{\oplus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\phi^{\ominus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

  - Affect firm productivity $\mathcal{D}(\tau_t)z$ as a shifter – as in Nordhaus DICE-2016

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma^{\oplus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma^{\ominus}\frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$



    – Deviation from ”ideal” temperature $\tau_i^\star = (1-\alpha)\tau^\star + \alpha\tau_{i,t_0}$, with $\tau^\star = 15.5°C$
    – Damage sensitivity γ and ϕ is asymmetrical and can also be heterogeneous and uncertain

Marginal values of temperature

- Using Pontryagin Max. Principle :
  - We obtain a system of coupled ODEs More details
- Marginal values of the climate variables : λ^S and λ^T

$$\dot{\lambda}^{\mathcal{T}}_{i,t} = \lambda^{\mathcal{T}}_{i,t}(\tilde{\rho} + \Delta_i \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau_i^\star)\mathcal{D}^y(\tau_{i,t})}^{-\partial_T \mathcal{D}^y} f(k_{i,t}, e_{i,t})\lambda^k_{i,t} + \overbrace{\phi(\tau_{i,t} - \tau_i^\star)\mathcal{D}^u(\tau_{i,t})}^{\partial_T \mathcal{D}^u} u(c_{i,t})$$

$$\dot{\lambda}^S_{i,t} = \lambda^S_{i,t}(\tilde{\rho} - \delta^s) - \Delta_i \, \zeta \, \chi \, \lambda^{\mathcal{T}}_{i,t}$$

- Marg. cost for i of releasing carbon in atmosphere $\lambda^S_{i,t}$ increases with :
  - Temperature gap $\tau_{i,t} - \tau_i^\star$
  - Damage sensitivity to temperature for TFP γ_i and utility ϕ
  - The development level $f(k_{i,t}, e_{i,t})$ and $u(c_{i,t})$

Social cost of carbon

- The marginal "externality damage" or "cost of carbon" can be expressed naturally :

$$CC_{i,t} := -\frac{\partial U_{i,t}/\partial \mathcal{S}_{i,t}}{\partial U_{i,t}/\partial c_{i,t}} = -\frac{\lambda^S_{i,t}}{\lambda^k_{i,t}}$$

Cost of country i of emitting one additional ton of CO_2 is

- Ratio of marg. cost of carbon vs. the marg. value of consumption/capital
- Stationary value : $t \to \infty$, with $\mathcal{E}_t = \delta_s \mathcal{S}_t$ and $\tau_t \to \tau_\infty$

$$CC_{i,t} \equiv \frac{\Delta \chi}{\widetilde{\rho} - \delta^s}(\tau_{i,\infty} - \tau_i^\star)\Big(\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} + \phi_i \mathcal{D}^u(\tau_{i,\infty})\frac{c_{i,\infty}}{1-\eta}\Big)$$

- Can integrate over time t & state-space i : $SCC_t = -\int_{\mathbb{S}} \frac{\lambda^S_{i,t}}{\lambda^k_{i,t}} di$
- Solution of the adjoint equation : Proof
- Uncertainty SCC with uncertainty

Social cost of carbon & temperature

- Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta\,\chi\,\xi\,\omega \int_{t_0}^{T} e^{(n+\bar{g})t-\delta_s(T-t)} q_t^{f\;-\sigma_e} \int_{j\in\mathbb{I}} \left(z_j z_{j,t}^e \mathcal{D}(\tau_{j,t})\right)^{\sigma-1} y_{j,t}\, q_{j,t}^{\sigma_e-\sigma}\, dj\, dt$$

  - Geographical factors determining warming Δ_i
  - Climate sensitivity χ & carbon exit from atmosphere δ_s
  - Growth of population n, aggregate productivity $\bar{g}$
  - Deviation of output from trend y_i & relative TFP z_j
  - Directed technical change z_t^e, elasticity of energy in output σ
  - Fossil energy price $q^{e,f}$ and Hotelling rent $g^{q^f} \approx \dot{\lambda}_t^R/\lambda_t^R = \rho$
  - Change in energy mix, renewable share ω, price q_t^r & elasticity of source σ_e

- Approximations at $T \equiv$ Generalized Kaya (or $I = PAT$) identity More details

$$\frac{\dot{\tau}_T}{\tau_T} \propto n + \bar{g}^y - (1-\sigma)\left(g^{z^e} - \widetilde{\gamma}\right) + (\sigma_e - \sigma)(1-\omega)g^{q^r} - (\sigma_e(1-\omega) + \sigma\omega)g^{q^f}$$

Optimal energy and emissions decisions

- To determine the temperature paths, we need to know the decisions of energy use :
  - Different choices of emissions depending on the level of externality !

1. ***Business as usual*** : each country optimizes without internalization :

$$\max_{\{c_{i,t}, e^f_{i,t}, e^r_{i,t}\}_{t>0}} U_{i,t_0} \qquad \forall i \in \mathbb{I}$$

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2. ***Social planner*** : a planner chooses the decisions of *all* countries $\imath \in \mathbb{I}$:

$$\max_{\left\{c_{i,t}, e^f_{i,t}, e^r_{i,t}\right\}_{i\in\mathbb{I}, t>0}} \int_{i\in\mathbb{I}} \omega_i \, U_{i,t_0} p_i di$$

- Details Details Social Planner

Optimal energy and emissions decisions

1. ***Business as usual*** :
   - *Fossil energy :* only private tradeoff : marg. product of energy = marg cost + Hotelling rent

$$[e^f_{i,t}] \qquad \underbrace{\mathcal{D}(\tau_{i,t})z\,\partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}}}_{MPe^f_{i,t}} = q^{e,f}_t = \bar{\nu}\left(\frac{E^f_t}{\mathcal{R}_t}\right)^{\nu} + \lambda^R_t$$

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2. ***Social planner*** :
   - *Fossil energy :*

$$[e^f_{i,t}] \qquad \underbrace{\hat{\lambda}^k_{i,t}}_{MV^k_{i,t}} \underbrace{\mathcal{D}(\tau_{i,t}) z_{i,t}\, \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}}}_{MPe^f_{i,t}} = \underbrace{\hat{\lambda}^{e,f}_t + \hat{\lambda}^R_t}_{=\text{price of fossil energy}} - \underbrace{\int_{\mathbb{I}} \xi \omega_j \hat{\lambda}^S_{j,t} p_j dj}_{\hat{\lambda}^S_t = \text{externality cost}}$$

   - *Guess for carbon taxation ?* $\mathbf{t}^S_t = -\frac{\lambda^S_t}{\lambda^k_{i,t}}$

Dynamic distortion : Competitive equilibrium vs. Social planner

- Competitive equilibrium, state : capital k and costate λ^k

$$\frac{\dot{c}_{i,t}}{c_{i,t}} = \frac{1}{\eta}\big(r_{i,t} - \rho\big)$$

- Social planner allocation :
  - Impact of $\mathcal{E}_t = \int_{\mathbb{I}} e^f_{i,t} p_{i,t} di$

$$\frac{\dot{c}_{i,t}}{c_{i,t}} = \frac{1}{\eta}\big(r_{i,t} - \rho\big) - \frac{de^f_{i,t}}{dk_{i,t}} \left(\frac{\widehat{\lambda}^{e,f}_t + \widehat{\lambda}^R_t - \widehat{\lambda}^S_t}{\widehat{\lambda}^k_{i,t}} \right)$$

    - Marginal value of state λ_t decreases with the externality of $\bar{E}$
    - Consume more today because more capital in the future affects the choice of energy $e(s)$ through $\partial_s e(\tilde{s})$

$$\frac{de^f_{i,t}}{dk_{i,t}} = \underbrace{\frac{\partial e^f_{i,t}}{\partial e_{i,t}} \frac{de_{i,t}}{dk_{i,t}}}_{\text{energy demand effect}} + \underbrace{\frac{\partial e^f_{i,t}}{\partial q^r_{i,t}} \frac{dq^r_{i,t}}{dr_{i,t}} \frac{dr_{i,t}}{dk_{i,t}}}_{\text{substitution effect}}$$

  - Dynamic distortion : PMP for HA models

Decentralization - 1

- With inequality $\widehat{\lambda}^k_{i,t} \neq \widehat{\lambda}^k_{j,t}$, it is unclear how to decentralize
- Allowing lump sum transfer across countries solves world inequality

$$\lambda^k_{i,t} = \lambda^k_{j,t} = \bar{\lambda}^k_t$$

- Restricting to intra-country redistribution :
  - Allow distortive taxes and lump-sum rebate in i only

  $\Rightarrow$ Planner cares about redistribution : $\lambda^k_{i,t}$ vs. $\bar{\lambda}^k_t = \int_j \lambda^k_{j,t} p_j dj$
  - Combination of distortive taxes and lump-sum rebate

Decentralization - 2

- Energy taxes :

$$\widehat{\lambda}^k_{i,t}\, MPe_{i,t}\,\frac{\partial e_{i,t}}{\partial e^f_{i,t}} = \widehat{\lambda}^{ef}_t + \widehat{\lambda}^R_t + \widehat{\lambda}^S_t \qquad \Leftrightarrow \qquad MPe_{i,t}\,\frac{\partial e_{i,t}}{\partial e^f_{i,t}} = \underbrace{\frac{\bar{\lambda}^k_t}{\lambda^k_{i,t}}}_{=\text{redistribution}} \Big(\underbrace{q^{e,f}_t}_{=\frac{\widehat{\lambda}^{ef}_t + \widehat{\lambda}^R_t}{\bar{\lambda}^k_t}} + \underbrace{\mathbf{t}^S_t}_{=-\frac{\widehat{\lambda}^S_t}{\bar{\lambda}^k_t}} \Big)$$

- Combination :
  – Redistributive taxes/subsidy : $\mathbf{t}_{i,t} = \frac{\bar{\lambda}^k_t}{\lambda^k_{i,t}}$
  – The Pigouvian tax :$\mathbf{t}^f_S$ is flat rate accounting for the climate externality

$$\mathbf{t}^S_t = -\frac{\widehat{\lambda}^S_t}{\bar{\lambda}^k_t} = -\int_{\mathbb{I}} \frac{\lambda^S_{i,t}}{\bar{\lambda}^k_t} p_i di \neq CC_{i,t}$$

  – Everything is rebated lump sum to the household in each country i

Carbon taxation

- We saw two notions to prices and tax carbon emissions :

$$CC_{i,t} = -\frac{\lambda^S_{i,t}}{\lambda^k_{i,t}}$$

- The "social cost of carbon" is often expressed as :

$$SCC_t = \int_{\mathbb{I}} CC_{j,t} dj = -\int_{\mathbb{I}} \frac{\lambda^S_{j,t}}{\lambda^k_{j,t}} p_{j,t} di$$

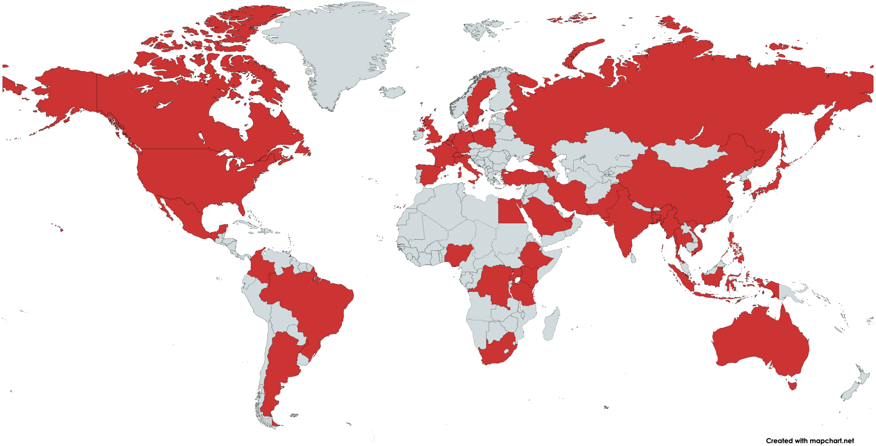
- We noticed that Pigouvian carbon taxes on fossils $e^f_{i,t}$ for country i are :

$$\mathbf{t}^S_{i,t} = -\frac{1}{\bar{\lambda}^k_{i,t}} \int_{\mathbb{I}} \omega_j \lambda^S_{j,t} p_j dj \neq -\int_{\mathbb{S}} \frac{\lambda^S_{j,t}}{\lambda^k_{j,t}} p_{j,t} di$$

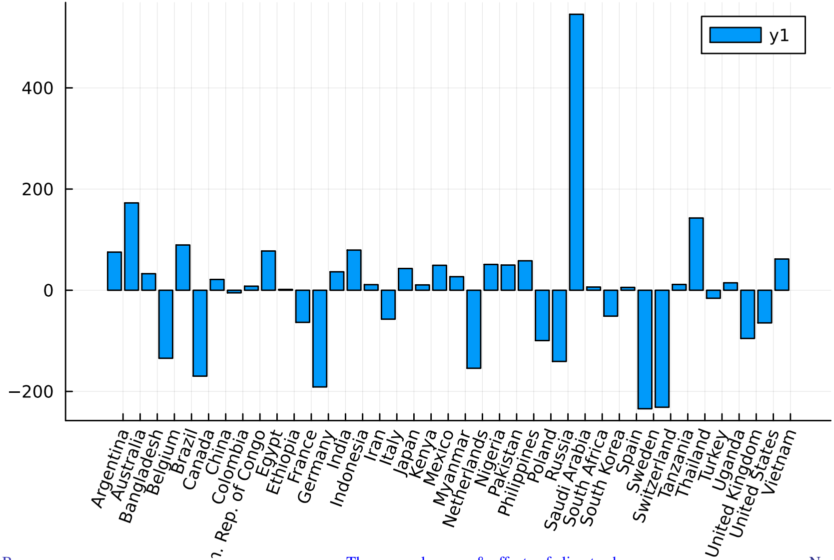
  - In particular, rich/developed countries would pay larger taxes (lower $\lambda^{k,i}_t$)
  - Redistribution motive to correct from unequal marginal costs of emitting

Application

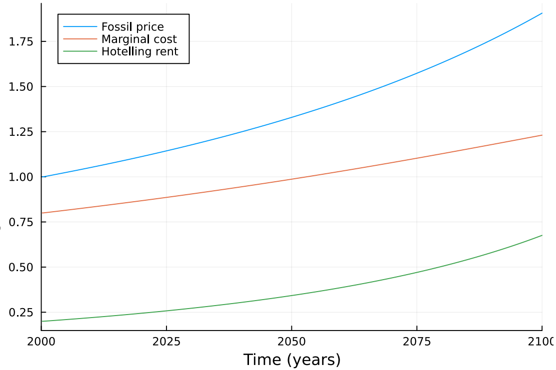
- Data : 40 countries
- Temperature (of the largest city), GDP, energy, population
- Calibrate z to match the distribution of output per capita at steady state

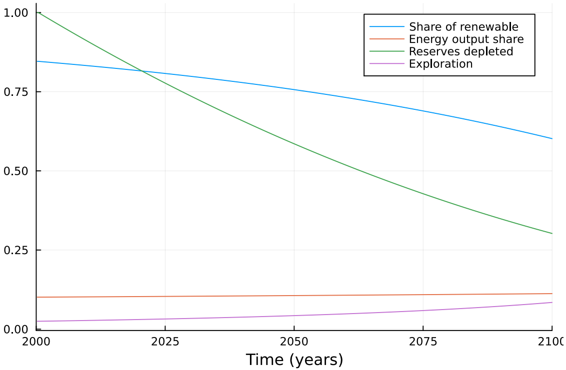


Distribution of carbon prices :

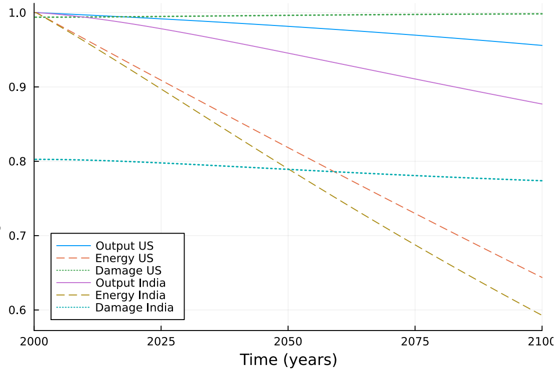


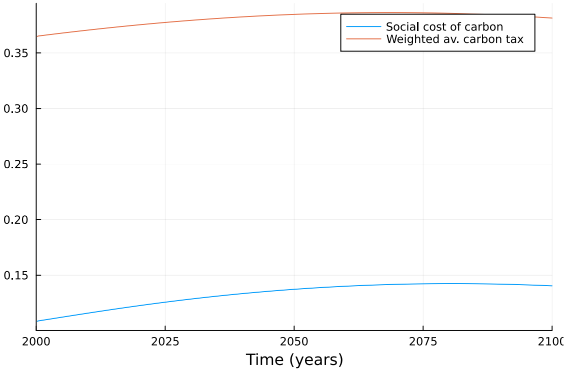
Energy


Fossil price
Marginal cost
Hotelling rent
1.75
1.50
1.25
1.00
0.75
0.50
0.25
2000
2025
2050
2075
2100
Time (years)


Share of renewable
Energy output share
Reserves depleted
Exploration
1.00
0.75
0.50
0.25
0.00
2000
2025
2050
2075
2100
Time (years)

Output and carbon price





Extensions - 1 - Endogeneous growth

- As of now, TFP z_t and directed technical change z_t^e are exogenous – growing at $\bar{g}^y$ and g^{z^e}
- Could easily nest an endogenous growth block in this model
  - Household / firm in country i chooses an amount $x_{i,t}$ of R&D to be allocated to increase TFP at rate ω_t^z or energy technology (efficiency)
  - Cost $c(x_{i,t})$

$$\dot{z}_t = h^y(\omega^z x_t) \qquad \dot{z}_t^e = h^e\big((1-\omega^z)x_t\big)$$

  - As a result, the marginal value of an investment in R&D is ”priced” on the costates :

$$-\dot{\lambda}_t^z + \rho\lambda_t^z = \lambda_t^k \mathcal{D}(\tau_t) f(k_t, e_t) \qquad \text{Recall : } y_t = z_t \mathcal{D}(\tau_t) f(k_t, e_t)$$

$$-\dot{\lambda}_t^{z^e} + \rho\lambda_t^{z^e} = \lambda_t^k z_t \mathcal{D}(\tau_t) \partial_{z^e} f(k_t, e_t)$$

  - And optimal decisions depend on this shadow value ;

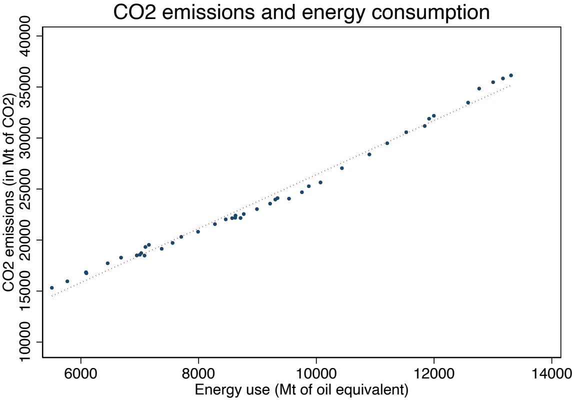
$$\omega^z h'^y(\omega^z x_t)\,\frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t}) \qquad (1-\omega^z) h'^e\big((1-\omega^z)x_t\big)\,\frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t})$$

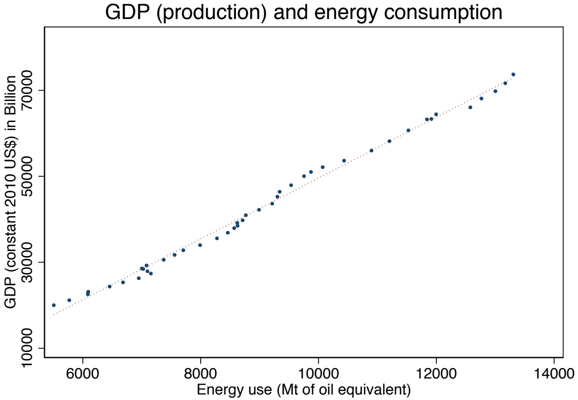
Conclusion

- Climate change is induced by externality
  - Energy/Emission choice doesn't include the impact on other countries
  - Cause strengthened by heterogeneity in wealth (capital/productivity)
  - Effect strengthened by heterogeneity in impact (temperature/damage)
- Social planner allocation correct for these different dimensions
  - Both Static correction $\equiv$ modified Pigouvian carbon taxation
  - And dynamic : through the marginal value of states
- Future plans :
  - Simulation of the three equilibria $CE/tax/SP$
  - Match the distribution of k using dynamics over 1960-2020
  - Social cost of carbon including heterogeneity and model uncertainty

Motivation

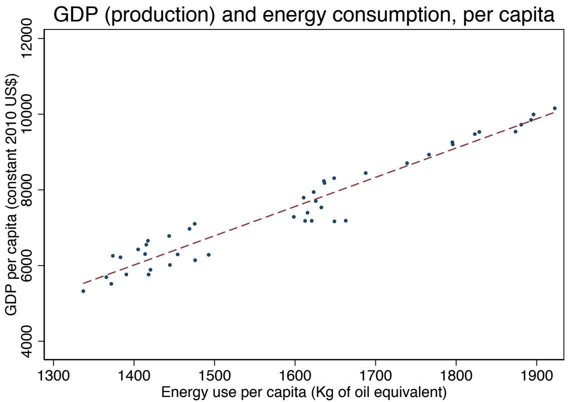
- CO_2 emissions correlate linearly with energy use
- Energy use (85% from fossils) correlates with output/growth

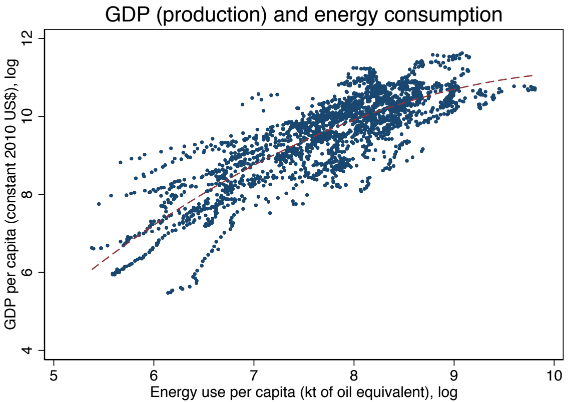




Introduction – Motivation

- Also true per capita and for the trajectory of individual countries





More details – Energy market

- Fossil fuel producer : price the Hotelling rent with the maximum principle :
- Rent λ_t^R grows with interest ρ and with the marginal gain of increasing reserves

$$\dot{\lambda}_t^R = \rho\lambda_t^R + \frac{\bar{\nu}\nu}{1+\nu}\left(\frac{E_t^\star}{R_t}\right)^{1+\nu} + \frac{\bar{\mu}\mu}{1+\mu}\left(\frac{I_t^\star}{R_t}\right)^{1+\mu}$$

$$\dot{\lambda}_t^R = \rho\lambda_t^R + \frac{\bar{\nu}^{-1/\nu}\nu}{1+\nu}\left(q^{e,f} - \lambda_t^R\right)^{1+1/\nu} + \frac{\bar{\mu}^{-1/\mu}\mu}{1+\mu}\left(\delta^R\lambda_t^R\right)^{1+1/\mu}$$

- Alternative energy price q_t^C

$$\begin{cases} \vartheta_t = 0 & \textit{if} \quad q_t^C < q_t^e \\ \vartheta_t = 1 & \textit{if} \quad q_t^C > q_t^e \end{cases}$$

More details – PMP

- State variables $s = (p, z, \delta, k, \tau, \mathcal{T}, \mathcal{S}, \mathcal{R})$ and three controls (c, e, ϑ)

$$\dot{k}_t = \mathcal{D}(\tau_t) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - c_t - q_t^e e_t - \Lambda_t(\vartheta_t) e_t$$

$$\mathcal{E}_t = e^{(n+\bar{g})t} \int_{\mathbb{S}} \xi (1 - \vartheta_t(s)) e_t^f(s) p_t(s) ds$$

$$\dot{\tau}_t = \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t) \qquad \dot{\mathcal{S}}_t = \mathcal{E}_t - \delta^s \mathcal{S}_t$$

$$\dot{R}_t = -E_t + \delta_R \mathcal{I}_t$$

$$s_0 = (z_{0,i}, k_{0,i}, R_0, T_{0,i})$$

- Pontryagin Maximum Principle

$$\mathcal{H}(s, c, e, \varphi, \{\lambda\}) = u(c, \tau) + \lambda^k \dot{k} + \lambda^S \dot{S} + \lambda^T \dot{T}$$

$$\partial_c \mathcal{H}(\cdot) = 0 \qquad \partial_e \mathcal{H}(\cdot) = 0 \qquad - \dot{\lambda}_t^x + \widetilde{\rho} \lambda_t^x = \partial_x \mathcal{H}(\cdot)$$

  - Hamiltonian :

$$\mathcal{H}(s, c, \lambda) = u(c, \tau) + \lambda^k \Big(\mathcal{D}(\tau_t) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - c_t - q_t^e e_t \Big) + \lambda_t^S \Big(\int_{\mathbb{S}} \xi e_t^f p_t ds - \delta^s \mathcal{S}_t \Big) + \lambda_t^\tau \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t)$$

Cost of carbon / Marginal value of temperature

- Solving for the cost of carbon and temperature $\Leftrightarrow$ solving ODE

$$\dot{\lambda}_t^T = \lambda_t^T(\widetilde{\rho} + \Delta\zeta) + \gamma(T - T^\star)\mathcal{D}^y(T)f(k,e)\lambda_t^k + \phi(T - T^\star)\mathcal{D}^u(T)u(c)$$
$$\dot{\lambda}_t^S = {\lambda^S}_t(\widetilde{\rho} - \delta^s) - \Delta\zeta\chi\lambda_t^T$$

- Solving for λ_t^T and λ_t^S, in stationary equilibrium $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$

$$\begin{aligned}
{\lambda^S}_t &= -\int_t^\infty e^{-(\widetilde{\rho}-\delta^s)\tau}\Delta\zeta\chi\lambda_\tau^T d\tau \\
\lambda_t^T &= -\int_t^\infty e^{-(\widetilde{\rho}+\Delta\zeta)\tau}(T_\tau - T^\star)\Big(\gamma\mathcal{D}^y(T_\tau)y_\tau\lambda_\tau^k + \phi\mathcal{D}^u(T_\tau)u(c_\tau)\Big)d\tau \\
\lambda_t^T &= -\frac{1}{\widetilde{\rho} + \Delta\zeta}(T - T^\star)\Big(\gamma\mathcal{D}^y(T_\infty)y_\infty\lambda_\infty^k + \phi\mathcal{D}^u(T_\infty)u(c_\infty)\Big) \\
\lambda_t^S &= \frac{1}{\widetilde{\rho} - \delta^s}\Delta\zeta\chi\lambda_\infty^T \\
&= -\frac{\Delta\chi}{\widetilde{\rho} - \delta^s}\frac{\zeta}{\widetilde{\rho} + \Delta\zeta}(T - T^\star)\Big(\gamma\mathcal{D}^y(T_\infty)y_\infty\lambda_\infty^k + \phi\mathcal{D}^u(T_\infty)u(c_\infty)\Big) \\
\lambda_t^S &\xrightarrow[\zeta\to\infty]{} -\frac{\Delta\chi}{\widetilde{\rho} - \delta^s}(T - T^\star)\Big(\gamma\mathcal{D}^y(T_\infty)y_\infty\lambda_\infty^k + \mathcal{D}^u(T_\infty)u(c_\infty)\Big)
\end{aligned}$$

Cost of carbon / Marginal value of temperature

- Closed form solution for CC :
  - In stationary equilibrium : $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
  - Fast temperature adjustment $\zeta \to \infty$
  - no internalization of externality (business as usual)

$$CC_t(s) \equiv \frac{\Delta\chi}{\widetilde{\rho} - \delta^s}(T_\infty - T^\star)\left(\gamma\mathcal{D}^y(T_\infty)y_\infty + \phi\mathcal{D}^u(T_\infty)\frac{c_\infty}{1-\eta}\right)$$

- Heterogeneity + uncertainty about models

Uncertainty about models :

- In our model, we rely strongly on model specification :
  - Parameters θ of models :
    - Climate system and damages : $(\xi, \chi, \zeta, \delta^s, \gamma, \phi)$
    - Economic model : $\varepsilon, \nu, \bar{g}, n$ or extended : $\omega, \sigma, \sigma^e, \nu, \mu$
    - Models with probability weight $\pi(\theta)$
  - Social cost of carbon, weighted for model uncertainty :

$$SCC_t(\theta) = -\int_{\mathbb{S}} \frac{\lambda_t^S(s,\theta)}{\lambda_t^k(s,\theta)} g(s) ds$$

$$S\bar{C}C_t = \int_{\Theta} SCC_t(\theta)\pi(\theta)d\theta = \int_{\Theta}\int_{\mathbb{S}} \frac{\lambda_t^S(s,\theta)}{\lambda_t^k(s,\theta)} g(s) ds \pi(\theta) d\theta$$

  - Counterfactual computation of SCC
    - Representative country / no uncertainty $\frac{\lambda_t^S}{\lambda_t^k}$
    - With heterogeneous regions / no uncertainty $SCC_t(\bar{\theta})$
    - No heterogeneity / model uncertainty $\int_{\Theta} \frac{\lambda_t^S(\bar{s},\theta)}{\lambda_t^k(\bar{s},\theta)} \pi(\theta) d\theta$
    - With heterogeneous regions / with model uncertainty $S\bar{C}C_t$
  - back

Long term temperature

- Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i(\mathcal{T}_T - \mathcal{T}_{t_0}) = \Delta_i \chi \int_{t_0}^{T} e^{-\delta_s(T-t)} \mathcal{E}_t dt$$
$$= \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e^f_{j,t}(1 - \vartheta_{j,t}) dj \, dt$$

  - Use CES demands : $e^f_{j,t} = \omega e_{j,t} {q^f_t}^{-\sigma_e} q_t^{\sigma_e}$ for energy and $e_t = (z z^e_t \mathcal{D}(\tau_{j,t}))^{\sigma-1} q_t^{-\sigma}$
  - Moreover, CES price index $q_t = (\omega {q^f_t}^{1-\sigma_e} + (1-\omega) {q^r_t}^{1-\sigma_e})^{1/(1-\sigma_e)}$, so first order approximation : $g^q = \omega g^{q^f} + (1-\omega) g^{q^r}$ with growth for q^f and q^r as well as $z^e_t = e^{g^e t}$
  - Gives :

$$e^f_{j,t} = \omega {q^f_t}^{-\sigma_e} q_{j,t}^{\sigma_e} (z z^e_t \mathcal{D}(\tau_{j,t}))^{\sigma-1} q_{j,t}^{-\sigma}$$

Temperature dynamics

- Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j\in\mathbb{I}} e^f_{j,t}(1-\vartheta_{j,t}) dj\, dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} q_t^{f\ -\sigma_e} \int_{j\in\mathbb{I}} \omega(zz^e\mathcal{D}(\tau_{j,t}))^{\sigma-1} e^{g^e t} q_{j,t}^{\sigma_e-\sigma}(1-\vartheta_{j,t}) dj\, dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \omega \int_{t_0}^{T} e^{(n+\bar{g})t - \delta_s(T-t)} e^{(-\sigma_e + (\sigma_e - \sigma)\omega) g^{q^f} t}\, e^{(\sigma_e-\sigma)(1-\omega) g^{q^r} t}$$
$$\times \int_{j\in\mathbb{I}} \left(zz_t^e \mathcal{D}(\tau_{j,t})\right)^{\sigma-1} y_{j,t}(1-\vartheta_{j,t}) dj\, dt$$

- back

Social Planner allocation

- Solving the social planner allocation : Hamiltonian

$$\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\}) = \int_{\mathbb{I}} \omega_i\, u(c_i,\tau_i)p_i di - wL_t^f + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^k \Big(\mathcal{D}(\tau_t)f(k_t,e_t) - (n+\bar{g}+\delta)k_t - c_t\Big)p_i di$$
$$+ \widehat{\lambda}_t^S\Big(\int_{\mathbb{I}} \xi^f e_t^f p_i di - \delta^s \mathcal{S}_t\Big) + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^{\mathcal{T}} \Delta_i \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)p_i di$$
$$+ \widehat{\lambda}_t^R\Big(-E_t^f + \delta^R \mathcal{I}_t\Big) + \widehat{\lambda}_t^{e^f}\Big(\widetilde{\mathcal{F}}(L_t^f,\mathcal{R}_t) - E_t^f\Big) + \int_{\mathbb{I}} \widehat{\lambda}_t^{e^r}\Big(z_{i,t}^r k_{i,t}^{r\ \alpha} - e_t^r\Big)p_i di$$

with $\quad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_i di \quad$ and $\quad e_t = \left(\omega^{\frac{1}{\sigma^e}}(e_{i,t}^f)^{\frac{\sigma^e-1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}}(e_{i,t}^r))^{\frac{\sigma^e-1}{\sigma^e}}\right)^{\frac{\sigma^e}{\sigma^e-1}}$

- Results :

$$\omega_i\, u_c(c_i,\tau_i) = \widehat{\lambda}_{i,t}^k$$
$$\widehat{\lambda}_{i,t}^k\, MPe_{i,t}\frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S$$
$$\widehat{\lambda}_{i,t}^k\, MPe_{i,t}\frac{\partial e_{i,t}}{\partial e_{i,t}^r} = \widehat{\lambda}_t^{e^r}$$

- Back

Social Planner allocation

- Moreover, using Pontryagin principle, accounting for the distribution, we should have an adjustment for the state dynamics :
  - Valuation of the state changes when the planner knows if after the externality
- Back

Decentralization

- With inequality $\widehat{\lambda}^k_{i,t} \neq \widehat{\lambda}^k_{j,t}$, it's unclear how to decentralize
- Allowing lump sum transfer across countries solves world inequality $\lambda^k_{i,t} = \lambda^k_{j,t} = \bar{\lambda}^k_t$, as a result :

$$\widehat{\lambda}^k_{i,t}\, MPe_{i,t} \frac{\partial e_{i,t}}{\partial e^f_{i,t}} = \widehat{\lambda}^{ef}_t + \widehat{\lambda}^R_t + \widehat{\lambda}^S_t \qquad \Leftrightarrow \qquad MPe_{i,t} \frac{\partial e_{i,t}}{\partial e^f_{i,t}} = \Big(\underbrace{\mathcal{C}'(E^f_t) + \lambda^R_t}_{=q^{e,f}_t = \frac{\widehat{\lambda}^{ef}_t + \widehat{\lambda}^R_t}{\bar{\lambda}^k_t}} + \underbrace{T^f_t}_{=\frac{\widehat{\lambda}^S_t}{\bar{\lambda}^k_t}} \Big)$$

- The climate $\mathbf{t}^f_t$ is flat rate accounting for the climate externality

$$\mathbf{t}^f_t = \frac{\widehat{\lambda}^S_t}{\bar{\lambda}^k_t} = \int_{\mathbb{I}} \frac{\lambda^S_{i,t}}{\bar{\lambda}^k_t} p_i di \neq CC_{i,t}$$

Optimal abatement of emissions decisions

1. ***Business as usual*** :
   - *Abatement :*

$$[\vartheta_t] \qquad \partial_\vartheta \Lambda(\vartheta_{i,t}) e^f_{i,t} = 0 \qquad\qquad \Rightarrow \vartheta_{i,t} = 0$$

Optimal abatement of emissions decisions

1. ***Business as usual*** :
   - *Abatement :*

$$[\vartheta_t] \qquad \partial_\vartheta \Lambda(\vartheta_{i,t}) e^f_{i,t} = 0 \qquad\qquad \Rightarrow \vartheta_{i,t} = 0$$

2. ***Social planner*** :
   - *Abatement :*

$$[\vartheta_t] \qquad \partial_\vartheta \Lambda(\vartheta^i_t) = \bar{\theta}(\vartheta_{i,t})^\theta = \underbrace{-\frac{1}{\lambda^k_{i,t}} \int_{\mathbb{I}} \omega_j \lambda^S_{j,t} p_j dj}_{=\text{carbon tax for } i}$$

FBSDE for MFG systems – general formulation

- State $X_t \equiv (a_t, z_t) \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), and X diffusion process with control $\alpha^\star(t, X, P_X, Y) \equiv c_t^\star$

$$dX_t = b(X_t, P_{X_t}, \alpha_t^\star)dt + \sigma dB_t$$

- Set up the Hamiltonian :

$$\mathcal{H}(t, x, P_X, y) = \max_{\alpha \in \mathbb{A}} \big(u(\alpha) + b(x, P_X, \alpha) \cdot y\big)$$

- Optimal control $c^\star \in \operatorname{argmax}_{\alpha \in \mathbb{A}} \big(u(\alpha) + b(x, P_X, \alpha) \cdot y\big)$
- Using the Pontryagin maximum principle :

$$dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t$$

back

FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \widetilde{Z}_t dB_t \end{cases}$$

- Question : What else do we need ?

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- Question : What else do we need ?
  - The risk loading in the costate $\widetilde{Z}_t$:
    - Intuitions : expectation error in the law of motion of Y_t

$$\widetilde{Z}_t(x) = \mathbb{E}\Big[\frac{dY_t + D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt}{dB_t^0}\Big]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo

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    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo
  - The initial condition Y_0 as a function of X_0

FBSDE system for MFG

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$$\widetilde{Z}_t(x) = \mathbb{E}\Big[\frac{dY_t + D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt}{dB_t^0}\Big]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t, despite running backward)
    - Can compute that by Monte Carlo
  - The initial condition Y_0 as a function of X_0
  - A boundary condition of Y_T or transversality $\lim_{t\to\infty} e^{-\rho t}X_tY_t = 0$

back

Social planner : Mean Field Control/McKean Vlasov

- Suppose now that the social planner controls the dynamic of each agents ... *accounting for its effect on the distribution*
- Get an additional term :

$$\begin{cases} dX_t &= D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t \\ dY_t &= -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t - \widetilde{\mathbb{E}}\left[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)\right] \end{cases}$$

- Effect on the distribution :

Social planner : Mean Field Control/McKean Vlasov

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- Effect on the distribution :
  - Social planner internalize the externality coming from heterogeneity
    - $D_\mu H$ is the L-derivative w.r.t the measure $\mu \equiv \mathbb{P}_{X_t}$
    - Idea : lifting of the function $H(x, \mu) = \widehat{H}(x, \widehat{X})$ where $\widehat{X} \sim \mu$ and hence $D_\mu H(x, \mu)(\widehat{X}) = D_{\widehat{x}}\widetilde{H}(x, \widehat{X})$
    - Intuition : shift the distribution of states $\widehat{X}$ for all agents
    - Probabilistic approach : easy to compute $\widetilde{\mathbb{E}}[D_\mu H(\tilde{X}_t, \mu)] = \widetilde{\mathbb{E}}[D_{\widehat{x}} H(\tilde{X}_t, \widehat{X})]$
  - Here : the effect is homogeneous for all agents : the interaction with the measure is non-local !

back

Sequential approach FBSDE w/ Agg. shocks

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t + \widetilde{\sigma} dB_t^0 \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t + \widetilde{Z}_t^0 dB_t^0 \end{cases}$$

- Question : What else do we need ?

Sequential approach FBSDE w/ Agg. shocks

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- Question : What else do we need ?
  - The risk loading on idiosyncratic shocks $\widetilde{Z}_t$:
  - The risk loading on aggregate shocks $\widetilde{Z}_t^0$:

Sequential approach FBSDE w/ Agg. shocks

- Coupled FBSDE system for each agent

$$\begin{cases} dX_t & = D_y\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + \sigma dB_t + \widetilde{\sigma} dB_t^0 \\ dY_t & = -D_x\mathcal{H}(t, X_t, P_{X_t}, Y_t)dt + Z_t dB_t + \widetilde{Z}_t^0 dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The risk loading on idiosyncratic shocks $\widetilde{Z}_t$:
  - The risk loading on aggregate shocks $\widetilde{Z}_t^0$:
  - The initial conditions $Y_0(X_0)$ and boundary condition on Y_T or transversality $\lim_{t\to\infty} e^{-\rho t} X_t Y_t = 0$

MFG system : Recursive approach w/ Agg. shocks

- Here : recursive w.r.t. idiosyncratic shocks, but sequential w.r.t. aggregate shocks.
- System for v and g :

$$-\partial_t v + \rho v = \max_{\alpha} u(\alpha) + \mathcal{A}(v)v + Z_t^0 dB_t^0$$
$$\partial_t g = \mathcal{A}^*(v)g + \partial_x[\sigma g]dB_t^0$$

- Solve the PDE system :
  - Finite difference, upwinding scheme
  - View that as a non-linear system : use Quasi Newton methods
  - New part : forcing terms $\partial_x[\sigma g]dB_t^0$ and $Z_t^0 dB_t^0$
  - Initial and terminal conditions

$$v_T = v^\infty \qquad\qquad g_0 = g^\infty$$

    - Direct effect of uncertainty on measure
    - Indirect effect through agent expectations : shadow price of aggregate risk Z_t^0