

The Inequality of Climate Change

Heterogeneity, optimal energy policy and uncertainty

WORK IN PROGRESS

Thomas Bourany

THE UNIVERSITY OF CHICAGO

Econ Dynamics

May 2023

Introduction – this project

- ▶ Marginal damages of climate & temperature varies across countries
 - Vary with the damage function : non-linearity matters a lot !
- ▶ What is the optimal taxation of energy in the presence of climate externality *and* heterogeneities ?
 - In context where fossil fuels taxation and climate policy redistributes across countries

Introduction – this project

- ▶ Marginal damages of climate & temperature varies across countries
 - Vary with the damage function : non-linearity matters a lot !
- ▶ What is the optimal taxation of energy in the presence of climate externality *and* heterogeneities ?
 - In context where fossil fuels taxation and climate policy redistributes across countries
- ▶ Develop a simple and flexible model of climate economics
 - Standard NCG – IAM model with heterogeneous regions
 - Normative implications : Ramsey policy + possibility to study uncertainty

Introduction – this project

- ▶ Marginal damages of climate & temperature varies across countries
 - Vary with the damage function : non-linearity matters a lot !
- ▶ What is the optimal taxation of energy in the presence of climate externality *and* heterogeneities ?
 - In context where fossil fuels taxation and climate policy redistributes across countries
- ▶ Develop a simple and flexible model of climate economics
 - Standard NCG – IAM model with heterogeneous regions
 - Normative implications : Ramsey policy + possibility to study uncertainty
- Evaluate the heterogeneous welfare costs of global warming
- Provide analytical formulas and a numerical methodology to compute the cost of carbon
 - Heterogeneity increases the welfare cost of carbon
- Solve world optimal carbon policy with heterogeneous regions
 - Does the optimal carbon tax coincide with the social cost of carbon ?
⇒ Maybe not, depend on transfer policy : need to adjust for inequality level
 - What are the welfare gains of suboptimal policies ?

Toy model

- ▶ Consider two countries $i = N, S$, (North/South)
 - HH consuming good c_i and producing with energy e_i and productivity z_i
 - Energy producer with profit $\pi(E)$ owned by country i with share θ_i
- ▶ Household problem :

$$\mathcal{V}_i = \max_{c_i, e_i} U(c_i)$$

$$c_i + q^e e_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) \quad [\lambda_i^k]$$

- ▶ Subject to damage $\mathcal{D}_i(\mathcal{S})$ and climate externalities :

$$\mathcal{S} = \mathcal{S}_0 + \overbrace{\xi_N e_N + \xi_S e_S}^{\text{=GHG emissions}}$$

- ▶ And consuming energy in a single energy market with price q^e

$$E = e_N + e_S \qquad q^e = c'(E) \qquad \pi(E) = q^e E - c(E)$$

Toy model – Competitive equilibrium

► Three dimensions of heterogeneity :

1. Different levels of productivity $z_i : z_N > z_S$
 2. Different climate damage $\gamma_i = -\frac{\mathcal{D}'_i(S)}{S\mathcal{D}_i(S)}$, $\gamma_S > \gamma_N$
 3. Different energy rent $\theta_i : \theta_N > \theta_S$
- ⇒ Yields heterogeneity in consumption $c_N > c_S$

► Competitive equilibrium Result :

- Marginal Product of Energy = Energy Cost

$$MPe_i = q^e = c'(E) \quad \text{with} \quad MPe_i := \mathcal{D}_i(S)z_iF'(e_i)$$

- Inequality

$$\lambda_i^k = U'(c_i) \quad c_i = \mathcal{D}_i(S)z_iF(e_i) + \theta_i\pi(E) - q^e e_i$$

Toy model – First Best and Decentralization

- Comparison with Social planner with full transfers (First Best)

$$\begin{aligned}\mathbb{W} &= \max_{\{c_i, e_i\}_i} \sum_{i=N,S} \omega_i U(c_i) \\ \sum_{i=N,S} c_i + c(E) &= \sum_{i=N,S} \mathcal{D}_i(\mathcal{S}) z_i F(e_i) \quad [\lambda] \\ \mathcal{S} &:= \mathcal{S}_0 + \xi_N e_N + \xi_S e_S \quad E := e_N + e_S\end{aligned}$$

- Marginal Product of Energy = Energy Cost + Social Cost of Carbon

$$MPe_i = c'(E) + \xi_i \underbrace{\overline{SCC}}_{=\mathbf{t}^e} \quad \text{with} \quad \overline{SCC} := \sum_{i=N,S} \mathcal{D}'_i(\mathcal{S}) z_i F(e_i)$$

- Redistribution

$$\omega_S U(c_S) = \omega_N U(c_N) = \lambda$$

- Decentralization, needs to redistribute with *lump-sum transfers* $T_S = -T_N$

$$\Rightarrow c_i + (q^e + \mathbf{t}^e) e_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) + T_i$$

- Are lump-sum transfers feasible?

Toy model – Second Best - Ramsey Problem

- Assume now that *lump-sum transfers across countries* are prohibited

- Allow for carbon tax \mathbf{t}_i^e and lump-sum rebate $T_i = \mathbf{t}_i^e e_i$

$$\begin{aligned} \mathcal{W} &= \max_{\{c_i, e_i\}_i} \sum_{i=N,S} \omega_i U(c_i) \\ s.t \quad c_i + (q^e + \mathbf{t}_i^e) e_i &= \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) + T^i \quad [\phi_i] \\ \mathcal{S} &:= \mathcal{S}_0 + \xi_N e_N + \xi_S e_S \quad E := e_N + e_S \quad q^e = c'(E) \end{aligned}$$

- Ramsey policy result :
 - Planner's marginal value of wealth

$$\phi_i = \omega_i U'(c_i)$$

- Energy decision :

$$\phi_i [M P e_i - c'(E)] + \underbrace{\xi_i \sum_j \phi_j \mathcal{D}'_j(\mathcal{S}) z_j F(e_j)}_{\propto -\text{SCC}} + \underbrace{\pi'(E) \sum_j \phi_j \theta_j}_{= \text{rent redistribution}} - \underbrace{c''(E) \sum_j \phi_j^k e_j}_{= \text{cost redistribution}} = 0$$

Social Cost of Carbon with inequality

- Measure of inequality

$$\hat{\phi}_i = \frac{\phi_i}{\bar{\phi}} = \frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)} \leq 1 \qquad \bar{\phi} = \frac{1}{2} (\omega_N U'(c_N) + \omega_S U'(c_S))$$

- The SCC is exacerbated by heterogeneity

$$\begin{aligned} SCC &= - \sum_j \hat{\phi}_j \mathcal{D}'_j(\mathcal{S}) y_j \\ &= -\text{Cov}_j \left(\frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}, \mathcal{D}'_j(\mathcal{S}) y_j \right) - \mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] > -\mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] = \overline{SCC} \end{aligned}$$

- Why? Low-income countries tend to be warmer/more vulnerable to climate change

Social Cost of Carbon with inequality

- Measure of inequality

$$\hat{\phi}_i = \frac{\phi_i}{\bar{\phi}} = \frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)} \leq 1 \qquad \bar{\phi} = \frac{1}{2} (\omega_N U'(c_N) + \omega_S U'(c_S))$$

- The SCC is exacerbated by heterogeneity

$$\begin{aligned} SCC &= - \sum_j \hat{\phi}_j \mathcal{D}'_j(\mathcal{S}) y_j \\ &= - \text{Cov}_j \left(\frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}, \mathcal{D}'_j(\mathcal{S}) y_j \right) - \mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] > - \mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] = \overline{SCC} \end{aligned}$$

- Why? Low-income countries tend to be warmer/more vulnerable to climate change

- For the social value of rent (exporters) and energy cost (importer), it's the contrary !

$$SVR = \text{Cov}_j \left(\frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}, \theta_j \pi'_j(E) \right) + \pi'(E) < \pi'(E)$$

$$SCE = \text{Cov}_j \left(\frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}, e_j c''(E) \right) + c''(E) < c''(E) = \pi'(E)$$

Optimal energy policy

► Energy taxation :

$$MPe_i = c'(E) + \xi_i \mathbf{t}_i^e$$

$$\mathbf{t}_i^e = \frac{\frac{1}{2} \sum_j \omega_j U'(c_j)}{\omega_i U'(c_i)} [\textcolor{green}{SCC} - \textcolor{brown}{SVR} + \textcolor{red}{SCE}]$$

► Four motives with a single tax and lump-sum rebate

- Distribution : Tax is higher for poorer countries $\omega_S U'(c_S) > \omega_N U'(c_N) \Rightarrow \mathbf{t}_S^e > \mathbf{t}_N^e$
- Optimal tax level : Depends on
 - Distribution of climate damage in $\textcolor{green}{SCC}$
 - Distribution of energy rent in $\textcolor{brown}{SVR}$
 - Distribution of energy spending in $\textcolor{red}{SCE}$

Toy model – Effect of uncertainty

► Consider risks related to both

- (i) Climate damage $\mathcal{D}_i(\mathcal{S}|\epsilon_d)$
- (ii) Could also consider economic growth $z_i(\epsilon_z)$
 - Probability distribution $(\epsilon_z, \epsilon_d) =: \epsilon \sim \varphi(\epsilon)$

$$\max_{e_i} \int_{\mathcal{E}} \max_{c_i(\epsilon)} U(c_i(\epsilon)) d\varphi(\epsilon) \quad \text{vs.} \quad \max_{\{e_j\}_j} \int_{\mathcal{E}} \max_{\{c_j(\epsilon)\}_j} \sum_{j=N,S} \omega_j U(c_j(\epsilon)) d\varphi(\epsilon)$$

Toy model – Effect of uncertainty

► Consider risks related to both

- (i) Climate damage $\mathcal{D}_i(\mathcal{S}|\epsilon_d)$
- (ii) Could also consider economic growth $z_i(\epsilon_z)$
 - Probability distribution $(\epsilon_z, \epsilon_d) =: \epsilon \sim \varphi(\epsilon)$

$$\max_{e_i} \int_{\mathcal{E}} \max_{c_i(\epsilon)} U(c_i(\epsilon)) d\varphi(\epsilon) \quad \text{vs.} \quad \max_{\{e_j\}_j} \int_{\mathcal{E}} \max_{\{c_j(\epsilon)\}_j} \sum_{j=N,S} \omega_j U(c_j(\epsilon)) d\varphi(\epsilon)$$

► Competitive equilibrium :

- Almost no change in behavior : Expected Marginal Product of Energy = Energy Price

$$\int_{\mathcal{E}} MPe_i(\epsilon) d\varphi(\epsilon) = q^e$$

Toy model – Effect of uncertainty

► Consider risks related to both

- (i) Climate damage $\mathcal{D}_i(\mathcal{S}|\epsilon_d)$
- (ii) Could also consider economic growth $z_i(\epsilon_z)$
 - Probability distribution $(\epsilon_z, \epsilon_d) =: \epsilon \sim \varphi(\epsilon)$

$$\max_{e_i} \int_{\mathcal{E}} \max_{c_i(\epsilon)} U(c_i(\epsilon)) d\varphi(\epsilon) \quad \text{vs.} \quad \max_{\{e_j\}_j} \int_{\mathcal{E}} \max_{\{c_j(\epsilon)\}_j} \sum_{j=N,S} \omega_j U(c_j(\epsilon)) d\varphi(\epsilon)$$

► Competitive equilibrium :

- Almost no change in behavior : Expected Marginal Product of Energy = Energy Price

$$\int_{\mathcal{E}} MP_{e_i}(\epsilon) d\varphi(\epsilon) = q^e$$

► Ramsey planner :

- Taxes take uncertainty into account :

$$\mathbb{E}_{\epsilon}(MP_{e_i}(\epsilon)) = q^e + \underbrace{\frac{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}{\mathbb{E}_{\epsilon}(\omega_i U'(c_j(\epsilon)))}}_{=\text{redistributive effect w/ risk}} \left[\underbrace{-\text{Cov}_{\epsilon}\left(\frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, MP_{e_i}(\epsilon)\right)}_{=\text{effect of agg. risk } \epsilon \text{ on energy choice}} + \mathbb{E}_{\epsilon}[SCC(\epsilon) - SVR(\epsilon) + SCE(\epsilon)] \right]$$

Toy model – Social cost of carbon and Uncertainty

► Social cost of carbon

$$\begin{aligned}
 \mathbb{E}_{\epsilon}[SCC] &= \int_{\mathcal{E}} \sum_{j=N,S} \frac{\omega_j U'(c_j(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]} \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) d\varphi(\epsilon) \\
 &= -\text{Cov}_{j,\epsilon} \left(\frac{\omega_j U'(c_j(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) \right) - \mathbb{E}_{j,\epsilon}[\mathcal{D}'_j(\mathcal{S}) y_k] \\
 &= -\mathbb{E}_j \left[\underbrace{\text{Cov}_{\epsilon} \left(\frac{\omega_j U'(c_j(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) \right)}_{=\text{effect of agg. risk } \epsilon} \right] \\
 &\quad - \underbrace{\text{Cov}_j \left[\frac{\mathbb{E}_{\epsilon}(\omega_j U'(c_j(\epsilon)))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, \mathbb{E}_{\epsilon}(\mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)) \right]}_{=\text{effect of heterogeneity across } j} - \underbrace{\mathbb{E}_{j,\epsilon}[\mathcal{D}'_j(\mathcal{S}) y_k]}_{=\text{average exp. damage}} \\
 &> \mathbb{E}_{\epsilon}[\overline{SCC}(\epsilon)]
 \end{aligned}$$

Model

- ▶ Neoclassical economy, in continuous time
 - countries/regions $i \in \mathbb{I}$: ex-ante heterogeneous : productivity z_i and more
 - ex-post heterogeneity in capital and temperature $\{k_i, \tau_i\}$
- ▶ Representative household problem in each country i :

$$\mathcal{V}_{i,t_0} = \max_{\{c_{it}, e_{it}^f, e_{it}^r\}} \int_{t_0}^{\infty} e^{-\rho t} u(c_{it}, \tau_{it}) dt$$

- ▶ Dynamics of wealth of country i , [More details](#) with wealth $w_{it} = b_{it} + k_{it}$:

$$\dot{w}_{it} = r^* w_{it} + \mathcal{D}^y(\tau_{it}) z_{it} f(k_{it}, e_{it}) - (\bar{\delta} + r_t^*) k_{it} + \theta_i \pi_t^f + \pi_{it}^r - (q_t^f + \mathbf{t}_{it}^f) e_{it}^f - (q_t^r + \mathbf{t}_{it}^r) e_{it}^r - c_{it} + \mathbf{t}_{it}^{ls}$$

- Damage $\mathcal{D}^y(\tau_{it})$ affect country's production and consumption $u(\cdot, \tau_{it})$
- Energy mix : $e_{it} = \mathcal{E}(e_{it}^f, e_{it}^r | \sigma_e)$ with fossil e_{it}^f – emitting carbon – vs. renewable e_{it}^r
- Energy **rents** redistributed : share θ_i for fossils / fully for local renew. firm.
- Prices, fossil q_t^f and non-carbon q_t^r (c.f. next slides)

Model – Energy markets

► Fossil fuels energy producer :

- Extended-Hotelling problem (depleting reserves with stock effects and exploration)

$$\max_{\{E_t^f, \mathcal{I}_t\}_t} \int_0^\infty e^{-\rho t} \pi_t^f(q_t^f, E_t^f, \mathcal{R}_t) dt \quad \text{with } \pi_t(E_t^f, \mathcal{R}_t) = q_t^{ef} E_t^f - \mathcal{C}^f(E_t^f, \mathcal{R}_t) - \mathcal{C}^i(\mathcal{I}_t, \mathcal{R}_t)$$

$$s.t. \quad E_t^f = \int_{\mathbb{I}} e_{it}^f di \quad \dot{\mathcal{R}}_t = -E_t^f + \delta_R \mathcal{I}_t$$

- Optimal pricing with finite-resources rents [More details](#)

$$q_t^{ef} = \mathcal{C}_E^f(E_t^f, \mathcal{R}_t) + \lambda_t^R \quad \mathcal{C}_{\mathcal{I}}^i(\mathcal{I}_t, \mathcal{R}_t) = \delta_R \lambda_t^R$$

Model – Energy markets

► Fossil fuels energy producer :

- Extended-Hotelling problem (depleting reserves with stock effects and exploration)

$$\max_{\{E_t^f, \mathcal{I}_t\}_t} \int_0^\infty e^{-\rho t} \pi_t^f(q_t^f, E_t^f, \mathcal{R}_t) dt \quad \text{with } \pi_t(E_t^f, \mathcal{R}_t) = q_t^{ef} E_t^f - \mathcal{C}^f(E_t^f, \mathcal{R}_t) - \mathcal{C}^i(\mathcal{I}_t, \mathcal{R}_t)$$

$$s.t. \quad E_t^f = \int_{\mathbb{I}} e_{it}^f di \quad \dot{\mathcal{R}}_t = -E_t^f + \delta_R \mathcal{I}_t$$

- Optimal pricing with finite-resources rents [More details](#)

$$q_t^{ef} = \mathcal{C}_E^f(E_t^f, \mathcal{R}_t) + \lambda_t^R \quad \mathcal{C}_{\mathcal{I}}^i(\mathcal{I}_t, \mathcal{R}_t) = \delta_R \lambda_t^R$$

► Renewable energy as a substitute technology *for each country i* (Static problem)

$$\pi_{it}^r = \max_{\{e_{it}^r\}} q_{it}^r e_{it}^r - \mathcal{C}^r(e_{it}^r) \quad \Rightarrow \quad q_{it}^r = \mathcal{C}_E^r(e_{it}^r)$$

- Problem with prod. fct instead of marg. costs

Climate model :

- ▶ Fossil energy input e_t^f causes climate externality

$$\mathcal{E}_t = \int_{\mathbb{I}} \xi_i e_{it}^f di$$

- ▶ World climate – cumulative GHG in atmosphere \mathcal{S}_t leads to increase in temperature

$$\dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s \mathcal{S}_t$$

- ▶ Impact of climate on country's local temperature :

$$\dot{\tau}_{i,t} = \zeta (\Delta_i \chi \mathcal{S}_t - (\tau_{i,t} - \bar{\tau}_{i,t_0}))$$

- Simple model : Climate sensitivity to carbon χ , Climate reaction/inertia ζ , Carbon content of fossils ξ , Country scaling factor Δ_i , Carbon exit for atmosphere δ_s

Model – Household Solution

- ▶ Household solves a consumption/saving/energy decision, as in the NCG
 - Using Pontryagin (PMP), we obtain a system of coupled ODEs [More details](#)

Model – Household Solution

► Household solves a consumption/saving/energy decision, as in the NCG

- Using Pontryagin (PMP), we obtain a system of coupled ODEs [More details](#)
- Consumption/Saving Euler equation (financial integration) :

$$\dot{c}_{it} = c_{it} \frac{1}{\eta} (r_t^* - \rho) \qquad MPk_{it} - \bar{\delta} = r_t^*$$

- Energy decisions :
Static demand for the two sources of energy : fossil e_{it}^f and renewable $e_{i,t}^r$ for every i , taking prices $\{q^f, q^r\}$ as given

$$\begin{aligned} MPe_{it} &= \mathcal{Q}(q_{i,t}^f + \mathbf{t}_{it}^f, q_{i,t}^r + \mathbf{t}_{it}^r) \\ \Rightarrow \quad e_{i,t}^f &= \mathcal{Q}_{q^f}(q_{i,t}^f + \mathbf{t}_{it}^f, q_{i,t}^r + \mathbf{t}_{it}^r) e_{i,t} \qquad e_{i,t}^r = \mathcal{Q}_{q^r}(q_{i,t}^f + \mathbf{t}_{it}^f, q_{i,t}^r + \mathbf{t}_{it}^r) e_{i,t} \end{aligned}$$

- with $MPk_{it} = \mathcal{D}(\tau_{it}) z_{i,tfk,it}$ and $MPe_{it} = \mathcal{D}^y(\tau_{i,t}) z_{i,tfe}(k_{i,t}, e_{i,t})$, and $\mathcal{Q}(\cdot)$ are aggregators functions (e.g. CES) and $\mathcal{Q}_{q^f}(\cdot)$ demand for fossil.

Impact of increase in temperature

- ▶ Using Damage fct $\mathcal{D}^y(\tau_{i,t}) = e^{-\frac{1}{2}\gamma_i(\tau_{i,t}-\tau_i^*)^2}$ and $u(c, \tau) = u(\mathcal{D}^u(\tau_{i,t})c)$, w/ $u(\hat{c}) = \frac{c^{1-\eta}}{1-\eta}$
- ▶ Marginal values of the climate variables : $\lambda_{i,t}^S$ and $\lambda_{i,t}^\tau$

$$\dot{\lambda}_{i,t}^\tau = \lambda_{i,t}^\tau(\rho + \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau_i^*)\mathcal{D}^y(\tau_{i,t})f(k_{i,t}, e_{i,t})\lambda_{i,t}^k}^{-\partial_\tau \mathcal{D}^y(\tau_{i,t})} + \overbrace{\phi_i(\tau_{i,t} - \tau_i^*)\mathcal{D}^u(\tau_{i,t})^{1-\eta}c_{i,t}^{1-\eta}}^{\partial_\tau u(c, \tau)}$$

$$\dot{\lambda}_{i,t}^S = \lambda_{i,t}^S(\rho + \delta^S) - \zeta \chi \Delta_i \lambda_{i,t}^\tau$$

- ▶ Costate $\lambda_{i,t}^S$: marg. cost of 1Mt carbon in atmosphere, for country i . Increases with :
 - Temperature gaps $\tau_{i,t} - \tau_i^*$ & damage sensitivity of TFP γ_i and utility ϕ_i
 - Development level $f(k_{i,t}, e_{i,t})$ and $c_{i,t}$
 - Climate params : χ climate sensitivity, Δ_i “catching up” of τ_i and ζ reaction speed

Local Social cost of carbon

- The marginal “externality damage” or “local social cost of carbon” (SCC) for region i :

$$LSCC_{i,t} := -\frac{\partial \mathcal{V}_{i,t} / \partial \mathcal{S}_t}{\partial \mathcal{V}_{i,t} / \partial c_{i,t}} = -\frac{\lambda_{i,t}^S}{\lambda_{i,t}^k}$$

- Ratio of marg. cost of carbon vs. the marg. value of consumption/capital

- Theorem : **Stationary LSCC** :

When $t \rightarrow \infty$ and for a BGP with $\mathcal{E}_t = \delta_s \mathcal{S}_t$ and $\tau_t \rightarrow \tau_\infty$, the LSCC is *proportional* to climate sensitivity χ , **marg. damage γ , ϕ , temperature**, and **output, consumption**.

$$LSCC_{i,t} \equiv \frac{\chi \Delta_i}{\tilde{\rho} + \delta^s} (\tau_{i,\infty} - \tau_i^*) [\gamma_i y_{i,\infty} + \phi_i c_{i,\infty}]$$

- More general formula : [Here](#), Proof : [Here](#) + What determine temperatures ? [Details Temperature](#)

Model – Equilibrium

- ▶ Two types of interactions
 - On climate (externality) + heterogeneous effects of temperatures
 - On energy market + redistribution effects of energy rent
 - No bilateral flows (eq. doesn't exist with continuum and trade or migration)

Model – Equilibrium

► Two types of interactions

- On climate (externality) + heterogeneous effects of temperatures
- On energy market + redistribution effects of energy rent
- No bilateral flows (eq. doesn't exist with continuum and trade or migration)

► Equilibrium

- Given, initial conditions $\{k_0, \tau_0\}$ and country-specific policies $\{\mathbf{t}_{it}^f, \mathbf{t}_{it}^r, \mathbf{t}_{it}^{ls}\}$, a competitive equilibrium is a continuum of sequences of states $\{k_{it}, \tau_{it}\}_{i,t}$ and $\{\mathcal{S}_t, \mathcal{T}_t, \mathcal{R}_t\}_t$ and policies $\{c_{i,t}, e_{i,t}^f, e_{i,t}^r\}_{i,t}$ and $\{E_t^f, \mathcal{E}_t, \mathcal{I}_t\}_t$, and price sequences $\{q_t^f, q_t^r\}$ such that :
 - Households choose policies $\{c_{i,t}, e_{i,t}^f, e_{i,t}^r\}_{i,t}$ to max utility s.t. budget constraint, giving \dot{k}_{it}
 - Renewable energy firm produce $\{e_{it}^r\}$ to max static profit
 - Fossil fuel firm extract and explore $\{E_t^f, \mathcal{I}_t\}$ to max profit, yielding $\dot{\mathcal{R}}_t$
 - Emissions \mathcal{E}_t affects climate $\{\mathcal{S}_t, \mathcal{T}_t\}_t$, & $\{\tau_{i,t}\}_{i,t}$.
 - Prices $\{q_t^f, q_t^r\}$ adjust to clear the markets : $E_t^f = \int_{\mathbb{I}} e_{it}^f di$ and $e_{it}^r = e_{it}^r$

Optimal policy

- ▶ Social planner, First best with a full set of instruments :
 - Solves world's inequality, using lump-sum transfers such that $\lambda_t = u'(c_{it}) = u'(c_{jt}), \forall i, j \in \mathbb{I}$
 - Pigouvian tax in RA economy with $\mathbf{t}_t^f = -\frac{\lambda_t^S}{\lambda_t^k} =: SCC_t$, c.f. GHKT (2014)
 - Imply cross-countries lump-sum transfers

Optimal policy

- ▶ Social planner, First best with a full set of instruments :
 - Solves world's inequality, using lump-sum transfers such that $\lambda_t = u'(c_{it}) = u'(c_{jt}), \forall i, j \in \mathbb{I}$
 - Pigouvian tax in RA economy with $\mathbf{t}_t^f = -\frac{\lambda_t^S}{\lambda_t^k} =: SCC_t$, c.f. GHKT (2014)
 - Imply cross-countries lump-sum transfers

- ▶ Second best / Ramsey planner :
 - Doesn't have access to redistribution / lump-sum transfers
 - Can only use region- i -specific distortive energy taxes : $\{\mathbf{t}_{it}^f, \mathbf{t}_{it}^r\}$
 - Redistribute lump sum the tax revenues : $\mathbf{t}_{it}^{ls} = \mathbf{t}_{it}^f e_{it}^f + \mathbf{t}_{it}^r e_{it}^r$

Optimal policy

- ▶ Social planner, First best with a full set of instruments :
 - Solves world's inequality, using lump-sum transfers such that $\lambda_t = u'(c_{it}) = u'(c_{jt}), \forall i, j \in \mathbb{I}$
 - Pigouvian tax in RA economy with $\mathbf{t}_t^f = -\frac{\lambda_t^S}{\lambda_t^k} =: SCC_t$, c.f. GHKT (2014)
 - Imply cross-countries lump-sum transfers
- ▶ Second best / Ramsey planner :
 - Doesn't have access to redistribution / lump-sum transfers
 - Can only use region- i -specific distortive energy taxes : $\{\mathbf{t}_{it}^f, \mathbf{t}_{it}^r\}$
 - Redistribute lump sum the tax revenues : $\mathbf{t}_{it}^{ls} = \mathbf{t}_{it}^f e_{it}^f + \mathbf{t}_{it}^r e_{it}^r$
- ▶ Questions :
 - Is the level of energy tax regions specific ?
 - ⇒ Yes ! depends on the distribution of wealth/consumption
 - What is the level of the Pigouvian tax ?
 - ⇒ \propto Welfare cost/climate damage : “social costate” for carbon S , i.e. ψ^S
 - ⇒ Inequality/Heterogeneity in damage change the *level* of this tax

The Ramsey Problem

- Consider a Social Planner that care about aggregate welfare :

$$\mathcal{W}_{t_0} = \max_{\{\mathbf{t}_{it}^f, \mathbf{t}_{it}^r, c_{it}, e_{it}^f, e_{it}^r, k_{it}, \lambda_{it}^k, \tau_{it}, S_t, \mathcal{R}_t, \mathcal{I}_t, \lambda_t^{\mathcal{R}}\}_{i,t}} \int_{t_0}^{\infty} \int_{\mathbb{I}} e^{-\bar{\rho}t} \omega_i u(c_{i,t}, \tau_{i,t}) di dt$$

subject to

- Optimality conditions of households, for c_i , e_i^f , e_i^r and k_i
- Optimality conditions of the Fossil firm, for E^f , \mathcal{I} and \mathcal{R}
- Optimality condition of the renewable firm, for e_i^r
- Climate and temperature dynamics τ_i and S
- Given Pareto weights ω_i

⇒ Large scale system of ODE More details - Hamiltonian

- A Ramsey plan is a set $\{\mathbf{t}_{it}^f, \mathbf{t}_{it}^r, \mathbf{t}_{it}^{ls}\}_{it}$ s.t. the competitive equilibrium is maximizing welfare

The Ramsey Problem – Measure of inequality

► Shadow value of capital/wealth

$$[c_{it}] \quad \psi_{it}^w = \underbrace{\omega_i u_c(c_i, \tau_{it})}_{=\text{direct effect}} + \underbrace{\phi_{it}^c u_{cc}(c_i, \tau_{it})}_{=\text{effect on savings}}$$

- Financial integration $\phi_{it}^c = 0$. With credit constraint, redistributive effects of interest rate/saving
- Depends on Pareto weights ω_i

► Measure of inequality and redistribution motive :

$$\widehat{\psi}_{it}^w = \frac{\psi_{it}^w}{\overline{\psi}_t^w} \leq 1$$

$$\text{low } z_i, k_i \quad \Rightarrow \quad \text{low } c_i, \text{ high } \psi_{i,t}^w \approx \omega_i u'(c_i) p_i > \overline{\psi}_t^w = \int_{\mathbb{I}} \psi_{i,t}^w di$$

The Ramsey Problem – Measure for Social Cost of Carbon

- Shadow value of carbon ψ_t^S and of temperature ψ_{jt}^τ

$$\dot{\psi}_t^S = (\tilde{\rho} + \delta^s) \psi_t^S - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi_{jt}^\tau dj$$

- Social planner summarizes climate damages in one variable $\psi_t^S = \frac{\partial \mathcal{W}_t}{\partial S}$
- Integrate all welfare costs of temperatures ψ_{jt}^τ , i.e. local costs of carbon/climate damages :

$$LSCC_{it}^{sp} = \frac{\psi_{it}^S}{\psi_{it}^k}$$

- Can reexpress the welfare cost of carbon WCC_t

$$WCC_t = -\frac{\dot{\psi}_t^S}{\frac{\psi_{it}^k}{\psi_{it}^S}} = \text{Cov}_j \left(\widehat{\psi}_{it}^k, LSCC_{j,t}^{sp} \right) + \mathbb{E}_j[LSCC_{j,t}^{sp}] > \mathbb{E}_j[LSCC_{j,t}^{sp}] = SCC_t$$

The Ramsey Problem – Measure for Fossil price redistribution

- Shadow value of fossil price ϕ_t^{Ef}

$$\begin{aligned} SVF_t &= \frac{\phi_t^{Ef}}{\psi_{it}^k} = \int_{\mathbb{I}} \hat{\psi}_{jt}^k e_{jt}^f dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \hat{\psi}_{jt}^k \theta_j dj \\ &= \mathbb{Cov}_j(\hat{\psi}_{jt}^k, e_{jt}^f) - E_t^f \mathbb{Cov}_j(\hat{\psi}_{jt}^k, \theta_{jt}^f) \end{aligned}$$

- SVF is the shadow value of changing (endogenously !) the fossil price q_t^f
- Low price q_t^f benefit fossil consumers, especially if they're poor / high $\hat{\psi}_{it}^k$
 - Low price q_t^f hurts the fossil firm owners θ_{jt}^f especially if they have high $\hat{\psi}_{it}^k$
 - In practice, $SVF_t > 0$

Optimal Fossil taxation

- Optimal policy for fossil energy, FOC of Ramsey planner :

$$\left(\frac{\mathcal{Q}^2}{f_{ee,it}} + \mathcal{Q}_{q^f q^f} \right) \left[-\xi_i \frac{\psi_t^S}{\psi_t^k} + \frac{\phi_t^{Ef}}{\psi_t^k} \mathcal{C}_{EE}^f(\cdot) - \psi_{it}^k \mathbf{t}_{it}^f \right] + \dots = 0$$

$$\Rightarrow \quad \widehat{\psi}_{it}^k \mathbf{t}_{it}^f = \xi_i WCC_t + SVF_t \mathcal{C}_{EE}^f \quad \& \quad \mathbf{t}_{it}^r = 0$$

- Pigouvian tax :

- Integrate several redistribution motives :
 - Changes in carbon/climate impacts in WCC_t
 - Changes in fossil fuel rent redistribution SVF along supply curve \mathcal{C}_{EE}^f of fossil firm
 - Depends on the country's income level $\widehat{\psi}_{it}^k$: set a lower tax on poorer, high $\widehat{\psi}_{it}^k$ countries

Optimal Fossil taxation

- Optimal policy for fossil energy, FOC of Ramsey planner :

$$\left(\frac{\mathcal{Q}_{f_{ee,it}}^2}{f_{ee,it}} + \mathcal{Q}_{q^f q^f}\right) \left[-\xi_i \frac{\psi_t^S}{\psi_t^k} + \frac{\phi_t^{Ef}}{\psi_t^k} \mathcal{C}_{EE}^f(\cdot) - \psi_{it}^k \mathbf{t}_{it}^f \right] + \dots = 0$$

$$\Rightarrow \quad \widehat{\psi}_{it}^k \mathbf{t}_{it}^f = \xi_i WCC_t + SVF_t \mathcal{C}_{EE}^f \quad \& \quad \mathbf{t}_{it}^r = 0$$

- Pigouvian tax :

- Integrate several redistribution motives :
 - Changes in carbon/climate impacts in WCC_t
 - Changes in fossil fuel rent redistribution SVF along supply curve \mathcal{C}_{EE}^f of fossil firm
 - Depends on the country's income level $\widehat{\psi}_{it}^k$: set a lower tax on poorer, high $\widehat{\psi}_{it}^k$ countries

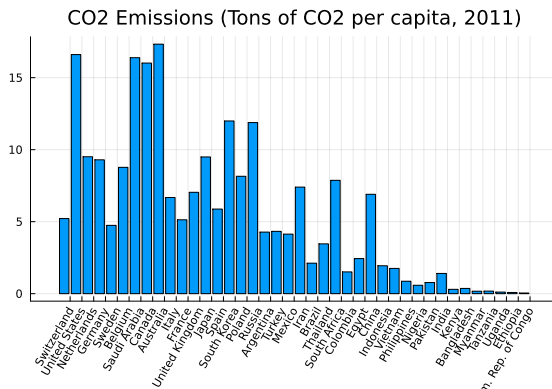
- Welfare costs of suboptimal taxes :

- Proportional to $\left(\frac{\mathcal{Q}_{f_{ee,it}}^2}{f_{ee,it}} + \mathcal{Q}_{q^f q^f}\right)$, i.e. depends on the curvature of production function f_{ee} , elasticity \mathcal{Q}_{q^f} and curvature $\mathcal{Q}_{q^f q^f}$ of fossil demand

Numerical Applications

Numerical Application

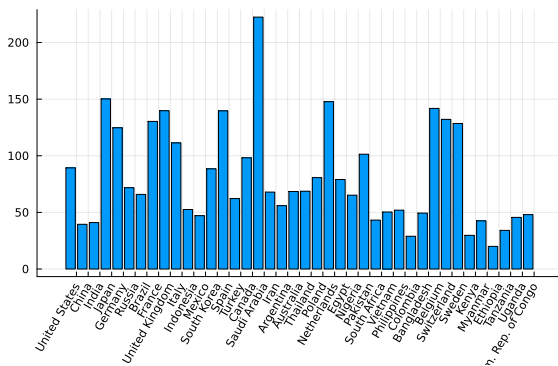
- Data : 40 countries
- Temperature (of the *largest city*), GDP, energy, population
- Calibrate z to match the distribution of output per capita at steady state



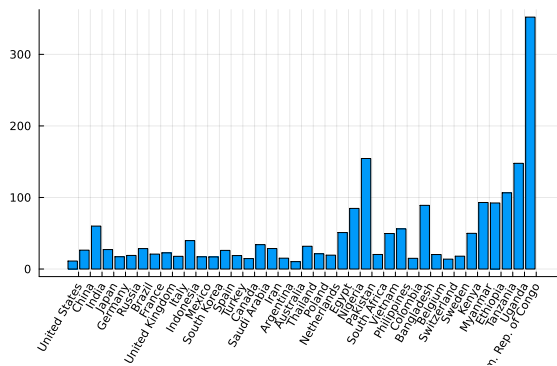
Local Cost of Carbon

► Difference $LSCC_i = \psi_{it}^S / \psi_{it}^k$ and $LWCC_{it} = \hat{\psi}_{it}^k LSCC_{it} = \psi_{it}^S / \bar{\psi}_{it}^k$

LSCC (USD PPP 2011)



LWCC (USD PPP 2011)



Conclusion

- ▶ Climate change has redistributive effects & heterogeneous impacts
- ▶ Redistributive effects of policy
 - Pigouvian tax that covers aggregate marginal damages
 - Can account for inequality both for heterogeneous welfare costs of climate and redistributive effects of energy price, for importers and exporters
- ▶ Study suboptimal policies
 - If carbon taxes are unfeasible : renewable subsidy ?
- ▶ Future plans
 - Dynamics on the capacity of renewable ?
 - Endogenous growth in TFP/energy saving technology
Learning-by-doing : positive externality ?
 - Uncertainty (simple tree ?)

Appendices

More details – Capital market

- In each country, the agent can save in two assets, capital k_{it} and bonds b_{it} :

$$\begin{cases} \dot{k}_{it} &= \mathcal{D}_i^y(\tau_{it})z_{if}(k_{it}, e_{it}) - (\delta + n + \bar{g})k_{it} + \iota_{it} \\ \dot{b}_{it} &= r^*b_{it} + \theta_i\pi_t^f + \pi_{it}^r - (q_t^f + \mathbf{t}_{it}^f)e_{it}^f - (q_t^r + \mathbf{t}_{it}^r)e_{it}^r - \iota_{it} - c_{it} + \mathbf{t}_{it}^{ls} \\ b_{it} &\geq -\vartheta k_{it} \end{cases}$$

- Combining, substituting ι_{it} and defining wealth $w_{it} = k_{it} + b_{it}$, we obtain the main equation

$$\dot{w}_{it} = r^*w_{it} + \mathcal{D}_i^y(\tau_{it})z_{if}(k_{it}, e_{it}) - (\bar{\delta} + r_t^*)k_{it} + \theta_i\pi_t^f + \pi_{it}^r - (q_t^f + \mathbf{t}_{it}^f)e_{it}^f - (q_t^r + \mathbf{t}_{it}^r)e_{it}^r - c_{it} + \mathbf{t}_{it}^{ls}$$

$$k_{it} \leq \frac{1}{1 - \vartheta} w_{it}$$

- Two polar cases :

- $\vartheta \rightarrow 0$, full autarky (no trade), and $w_{it} = k_{it}$
- $\vartheta \rightarrow 1$, full financial integration :

$$k_{it} \quad s.t. \quad MPk_{it} - \bar{\delta} = \mathcal{D}_i^y(\tau_{it})z_i\partial_k f(k_{it}, e_{it}) - (\delta + n + \bar{g}) = r_t^*$$

More details – Energy market

- Fossil fuel producer : price the Hotelling rent with the maximum principle :

$$\mathcal{H}^m(\mathcal{R}_t, \lambda_t^R, E_t, \mathcal{I}_t^e) = \pi_t(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) + \lambda_t^R(\delta^R \mathcal{I}_t^e - E_t)$$

- Rent λ_t^R grows with interest ρ and with the marginal gain of increasing reserves

$$\begin{aligned}\dot{\lambda}_t^R &= \rho \lambda_t^R - \partial_R \mathcal{C}(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) \\ &= \rho \lambda_t^R + \frac{\bar{\nu} \nu}{1 + \nu} \left(\frac{E_t^*}{\mathcal{R}_t} \right)^{1+\nu} + \frac{\bar{\mu} \mu}{1 + \mu} \left(\frac{I_t^*}{\mathcal{R}_t} \right)^{1+\mu} \\ \dot{\lambda}_t^R &= \rho \lambda_t^R + \frac{\bar{\nu}^{-1/\nu} \nu}{1 + \nu} (q^{ef} - \lambda_t^R)^{1+1/\nu} + \frac{\bar{\mu}^{-1/\mu} \mu}{1 + \mu} (\delta^R \lambda_t^R)^{1+1/\mu}\end{aligned}$$

- Because of decreasing return to scale and Hotelling rents : profits are > 0

$$\pi_t(E_t^f, \mathcal{R}_t, \lambda_t^R) = \frac{1 + \nu - \bar{\nu}}{1 + \nu} \left(\frac{E_t^f}{\mathcal{R}_t} \right)^{1+\nu} \mathcal{R}_t + \lambda_t^R E_t^f - \frac{\bar{\mu}^{-1/\mu}}{1 + \mu} (\delta^R \lambda_t^R)^{1+1/\mu}$$

More details – PMP – Competitive equilibrium

- ▶ Household problem : State variables $s_{i,t} = (k_i, \tau_i, z_i, p_i, \Delta_i)$
- ▶ Pontryagin Maximum Principle

[Back](#)

$$\mathcal{H}^{hh}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}) = u(c_i, \tau_i) + \lambda_{i,t}^k \left(\mathcal{D}(\tau_{it}) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - q_t^f e_{it}^f - q_{it}^r e_{it}^r - c_t \right) + \lambda_{i,t}^\tau \zeta \left(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{i0}) \right) + \lambda_{i,t}^S \left(\mathcal{E}_t - \delta^S \mathcal{S}_t \right)$$

$$[c_t] \quad u'(c_{it}) = \lambda_{i,t}^k$$

$$[e_t^f] \quad MPE_{it}^f = \mathcal{D}(\tau_{i,t}) z \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e_{i,t}^f}{\omega e_{i,t}} \right)^{-\frac{1}{\sigma_e}} = q_t^f$$

$$[e_t^r] \quad MPE_{it}^r = \mathcal{D}(\tau_{i,t}) z \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e_{i,t}^r}{(1 - \omega) e_{i,t}} \right)^{-\frac{1}{\sigma_e}} = q_t^r$$

$$[k_t] \quad \dot{\lambda}_t^k = -\lambda_t^k \left(\mathcal{D}(\tau_{i,t}) \partial_k f(k_{i,t}, e_{i,t}) - \delta - \bar{g} - n - \rho \right)$$

- ▶ Fossil Energy Monopoly :

$$\mathcal{H}^m(\mathcal{R}_t, \lambda_t^R, E_t^f, \mathcal{I}_t) = \pi_t(E_t^f, \mathcal{I}_t, \mathcal{R}_t) + \lambda_t^R (\delta^R \mathcal{I}_t - E_t^f)$$

$$[\mathcal{R}_t] \quad \dot{\lambda}_t^R = \rho \lambda_t^R + \frac{\bar{\nu} \nu}{1 + \nu} \left(\frac{E_t^*}{R_t} \right)^{1+\nu} + \frac{\bar{\mu} \mu}{1 + \mu} \left(\frac{I_t^*}{R_t} \right)^{1+\mu}$$

$$[E_t^f] \quad q_t^{ef} = \nu_E(E, \mathcal{R}) + \lambda_t^R = \bar{\nu} \left(\frac{E_t}{\mathcal{R}_t} \right)^\nu + \lambda_t^R$$

$$[\mathcal{I}_t] \quad \lambda_t^R \delta^R = \mu_I(I_t, R_t) = \bar{\mu} \left(\frac{\mathcal{I}_t}{\mathcal{R}_t} \right)^\mu \quad \mathcal{I}_t = R_t \left(\frac{\lambda_t^R \delta}{\bar{\mu}} \right)^{1/\mu}$$

Cost of carbon / Marginal value of temperature

- Solving for the cost of carbon and temperature \Leftrightarrow solving ODE

$$\dot{\lambda}_{i,t}^{\tau} = \lambda_t^{\tau}(\tilde{\rho} + \Delta\zeta) + \gamma(\tau - \tau^*)\mathcal{D}^y(\tau)f(k, e)\lambda_t^k + \phi(\tau - \tau^*)\mathcal{D}^u(\tau)u(c)$$

$$\dot{\lambda}_t^S = \lambda_t^S(\tilde{\rho} + \delta^s) - \int_{\mathbb{I}} \Delta_i \zeta \chi \lambda_{i,t}^{\tau}$$

- Solving for λ_t^{τ} and λ_t^S , in stationary equilibrium $\dot{\lambda}_t^S = \dot{\lambda}_t^{\tau} = 0$

$$\lambda_{i,t}^{\tau} = - \int_t^{\infty} e^{-(\tilde{\rho} + \zeta)u} (\tau_u - \tau^*) \left(\gamma \mathcal{D}^y(\tau_u) y_{\tau} \lambda_u^k + \phi \mathcal{D}^u(\tau_u) u(c_u) \right) du$$

$$\lambda_{i,t}^{\tau} = - \frac{1}{\tilde{\rho} + \Delta\zeta} (\tau_{\infty} - \tau^*) \left(\gamma \mathcal{D}^y(\tau_{\infty}) y_{\infty} \lambda_{\infty}^k + \phi \mathcal{D}^u(\tau_{\infty}) u(c_{\infty}) \right)$$

$$\lambda_t^S = - \int_t^{\infty} e^{-(\tilde{\rho} + \delta^s)u} \zeta \chi \int_{\mathbb{I}} \Delta_j \lambda_{j,u}^{\tau} dj du$$

$$= \frac{1}{\tilde{\rho} + \delta^s} \zeta \chi \int_{\mathbb{I}} \Delta_j \lambda_{j,\infty}^{\tau}$$

$$= - \frac{\chi}{\tilde{\rho} + \delta^s} \frac{\zeta}{\tilde{\rho} + \zeta} \int_{\mathbb{I}} \Delta_j (\tau_{j,\infty} - \tau^*) \left(\gamma \mathcal{D}^y(\tau_{j,\infty}) y_{j,\infty} \lambda_{j,\infty}^k + \phi \mathcal{D}^u(\tau_{j,\infty}) u(c_{j,\infty}) \right) dj$$

$$\lambda_t^S \xrightarrow{\zeta \rightarrow \infty} - \frac{\chi}{\tilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_j (\tau_{j,\infty} - \tau^*) \left(\gamma \mathcal{D}^y(\tau_{j,\infty}) y_{j,\infty} \lambda_{j,\infty}^k + \mathcal{D}^u(\tau_{j,\infty}) u(c_{j,\infty}) \right) dj$$

Cost of carbon / Marginal value of temperature

► Closed form solution for CC :

- In stationary equilibrium : $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
- Fast temperature adjustment $\zeta \rightarrow \infty$
- no internalization of externality (business as usual)

$$LSCC_{i,t} \equiv \frac{\Delta_i \chi}{\rho - n + \bar{g}(\eta - 1) + \delta^s} (\tau_\infty - \tau^*) \left(\gamma \mathcal{D}^y(\tau_\infty) y_\infty + \phi \mathcal{D}^u(\tau_\infty) c_\infty \right)$$

► Heterogeneity + uncertainty about models [Back](#)

Social cost of carbon & temperature

- Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta \chi \xi \omega \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} q_t^f - \sigma_e \int_{j \in \mathbb{I}} (z_j z_{j,t}^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} y_{j,t} q_{j,t}^{\sigma_e - \sigma} dj dt$$

- Geographical factors determining warming Δ_i
- Climate sensitivity χ & carbon exit from atmosphere δ_s
- Growth of population n , aggregate productivity \bar{g}
- Deviation of output from trend y_i & relative TFP z_j
- Directed technical change z_t^e , elasticity of energy in output σ
- Fossil energy price q^{ef} and Hotelling rent $g^{qf} \approx \dot{\lambda}_t^R / \lambda_t^R = \rho$
- Change in energy mix, renewable share ω , price q_t^r & elasticity of source σ_e

- Approximations at $T \equiv$ Generalized Kaya (or $I = PAT$) identity [More details](#)

$$\frac{\dot{\tau}_T}{\tau_T} \propto n + \bar{g}^y - (1 - \sigma)(g^{z^e} - \tilde{\gamma}) + (\sigma_e - \sigma)(1 - \omega)g^{q^r} - (\sigma_e(1 - \omega) + \sigma\omega)g^{q^f}$$

More details – PMP – Ramsey Optimal Allocation

► Hamiltonian :

$$\begin{aligned}
 \mathcal{H}^{sp}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}, \{\psi\}) = & \int_{\mathbb{I}} \omega_i u(c_i, \tau_i) p_i di \\
 & + \psi_{i,t}^k \left(\mathcal{D}(\tau_{it}) f(k_{it}, e_{it}) - (n + \bar{g} + \delta) k_t + \theta_i \pi(E_t^f, \mathcal{I}_t, \mathcal{R}_t) + \pi_{it}^r(e_{it}^r) - (q_t^f + \mathbf{t}_{it}^f) e_{it}^f - (q_{it}^r + \mathbf{t}_{it}^r) e_{it}^r - c_t + \mathbf{t}_t^{ls} \right) \\
 & + \psi_t^S \left(\mathcal{E}_t - \delta^S \mathcal{S}_t \right) + \psi_{it}^\tau \zeta \left(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{i0}) \right) + \psi_{it}^{\mathcal{R}} \left(-E_t^f + \delta^R \mathcal{I}_t \right) \\
 & + \psi_{i,t}^{\lambda k} \left(\lambda_t^k (\rho - r_t) \right) + \psi_t^{\lambda R} \left(\rho \lambda_t^R + \mathcal{C}_{\mathcal{R}}^f(E_t^f, \mathcal{I}_t, \mathcal{R}_t) \right) + \phi_{it}^c (u_c(c_i, \tau_{it}) - \lambda_{it}^k) \\
 & + \phi_{it}^{ef} \left(e_{it}^f - \mathcal{Q}_{q^f}(q_t^f + \mathbf{t}_{it}^f, q_t^r + \mathbf{t}_{it}^r) e_{it} \right) + \phi_{it}^{er} \left(e_{it}^r - \mathcal{Q}_{q^r}(q_t^f + \mathbf{t}_{it}^f, q_t^r + \mathbf{t}_{it}^r) e_{it} \right) \\
 & + \phi_{it}^e \left(f_e(k_{it}, e_{it}) - \mathcal{Q}(q_t^f + \mathbf{t}_{it}^f, q_t^r + \mathbf{t}_{it}^r) \right) + \phi_t^{Ef} (q_t^f - \mathcal{C}_E^f(\cdot) - \lambda_t^{\mathcal{R}}) + \phi_{it}^{Er} (q_{it}^r - \mathcal{C}_e^r(\cdot)) + \phi_t^{\mathcal{I}f} (\delta \lambda_t^{\mathcal{R}} - \mathcal{C}_{\mathcal{I}}^f(\cdot))
 \end{aligned}$$

► [Back](#)

Ramsey Optimal Allocation - FOCs

► FOCs w.r.t. $\{c_{it}, e_{it}, e_{it}^f, e_{it}^r, \mathcal{I}_t\}$, prices $\{q_t^f, q_t^r\}$ and taxes, denoting $\tilde{q}_{it} = q_t + \mathbf{t}_{it}$

$$[c_{it}] \quad \psi_{it}^k = \underbrace{\omega_i u_c(c_i, \tau_{it}) p_i}_{=\text{direct effect}} + \underbrace{\phi_{it}^c u_{cc}(c_i, \tau_{it})}_{=\text{effect on savings}}$$

$$[e_{it}] \quad \psi_{it}^k f_{e,it} + \phi_{it}^e f_{ee,it} - \phi_{it}^{ef} \mathcal{Q}_{q^f} - \phi_{it}^{er} \mathcal{Q}_{q^r} = 0 \quad \Rightarrow \quad \phi_{it}^e = \frac{1}{f_{ee,it}} (\phi_{it}^{ef} \mathcal{Q}_{q^f} + \phi_{it}^{er} \mathcal{Q}_{q^r} - \psi_{it}^k f_{e,it})$$

$$[e_{it}^f] \quad \phi_{it}^{ef} = \psi_{it}^k \tilde{q}_t^f - \psi_{it}^k \mathbf{t}_{it}^f - \xi \psi_t^S p_i + \phi_t^{Ef} \mathcal{C}_{EE}^f(\cdot) \quad [e_{it}^r] \quad \phi_{it}^{er} = \psi_{it}^k \tilde{q}_t^r - \psi_{it}^k \mathbf{t}_{it}^r + \phi_{it}^{Er} \mathcal{C}_{er}^r(\cdot)$$

$$[\tilde{q}_{it}^f] \quad \phi_{it}^e \mathcal{Q}_{q^f} + \phi_{it}^{ef} \mathcal{Q}_{q^f q^f} + \phi_{it}^{er} \mathcal{Q}_{q^r q^f} = 0$$

$$\Rightarrow \quad \left(\frac{\mathcal{Q}_{q^f}^2}{f_{ee,it}} + \mathcal{Q}_{q^f q^f} \right) [-\xi \psi_t^S p_i + \phi_t^{Ef} \mathcal{C}_{EE}^f(\cdot) - \psi_{it}^k \mathbf{t}_{it}^f] + \left(\frac{\mathcal{Q}_{q^f} \mathcal{Q}_{q^r}}{f_{ee,it}} + \mathcal{Q}_{q^r q^f} \right) [\phi_{it}^{Er} \mathcal{C}_{er}^r(\cdot) - \psi_{it}^k \mathbf{t}_{it}^r]$$

$$[\tilde{q}_{it}^r] \quad \phi_{it}^e \mathcal{Q}_{q^r} + \phi_{it}^{ef} \mathcal{Q}_{q^f q^r} + \phi_{it}^{er} \mathcal{Q}_{q^r q^r} = 0$$

$$\Rightarrow \quad \left(\frac{\mathcal{Q}_{q^f} \mathcal{Q}_{q^r}}{f_{ee,it}} + \mathcal{Q}_{q^f q^r} \right) [-\xi \psi_t^S p_i + \phi_t^{Ef} \mathcal{C}_{EE}^f(\cdot) - \psi_{it}^k \mathbf{t}_{it}^f] + \left(\frac{\mathcal{Q}_{q^r}^2}{f_{ee,it}} + \mathcal{Q}_{q^r q^r} \right) [\phi_{it}^{Er} \mathcal{C}_{er}^r(\cdot) - \psi_{it}^k \mathbf{t}_{it}^r] = 0$$

$$[q_{it}^f] \quad \phi_{it}^{Ef} = \int_{\mathbb{I}} \psi_{jt}^k e_{jt}^f dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi_{jt}^k dj \quad [q_{it}^r] \quad \phi_{it}^{Er} = \psi_{it}^k e_{it}^r - \psi_{it}^k \partial_{q^r} \pi_{it}^r = 0$$

$$[\mathcal{I}_t] \quad \delta \psi_t^{\mathcal{R}} + \partial_{\mathcal{R}\mathcal{I}}^2 \mathcal{C}(\cdot) \psi_t^{\lambda, \mathcal{R}} - \phi_t^{\mathcal{I}} \partial_{\mathcal{I}\mathcal{I}}^2 \mathcal{C}(\cdot) = 0$$



Ramsey Optimal Allocation - FOCs

► Backward equations for planner's costates

$$[k_i] \quad \dot{\psi}_{it}^k = \psi_{it}^k (\tilde{\rho} - r_{it}) + \psi_{it}^{\lambda k} \lambda_{it}^k \partial_k MPk_i + \frac{f_{ek,it}}{f_{ee,it}} \left[-\xi \psi_t^S p_i + \phi_t^{Ef} C_{EE}^f(\cdot) - \psi_{it}^k \mathbf{t}_{it}^f \right]$$

$$[S_i] \quad \dot{\psi}_t^S = (\tilde{\rho} + \delta^s) \psi_t^S - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi_{jt}^\tau dj$$

$$[\tau_i] \quad \dot{\psi}_t^\tau = (\tilde{\rho} + \zeta) \psi_t^\tau - \left(u_i \tau_t(c_{it}, \tau_{it}) + \psi_{it}^k \mathcal{D}'(\tau_{it}) f(k_{it}, e_{it}) + \phi_{it}^c u_{c,\tau}(c_{it}, \tau_{it}) + \mathcal{D}'(\tau_{it}) f_e \phi_{it}^e \right)$$

$$[\mathcal{R}] \quad \dot{\psi}_t^{\mathcal{R}} = \psi_t^{\mathcal{R}} \left(\tilde{\rho} - \partial_{\mathcal{R}}^2 \mathcal{C}(\cdot) \right) - \phi_t^{Ef} \partial_{\mathcal{R}E}^2 \mathcal{C}(\cdot)$$

$$[\lambda_i^k] \quad \dot{\psi}_t^{\lambda,k} = \psi_t^{\lambda,k} [\tilde{\rho} - (\rho - r_{i,t})] + \phi_{i,t}^c$$

$$[\lambda_i^{\mathcal{R}}] \quad \dot{\psi}_t^{\lambda,\mathcal{R}} = \psi_t^{\lambda,\mathcal{R}} (\tilde{\rho} - \rho) + \phi_t^{Ef} - \delta \phi_t^{\mathcal{I}f}$$

► [Back](#)