

# The Inequality of Climate Change Heterogeneity, Optimal policy and Uncertainty

WORK IN PROGRESS

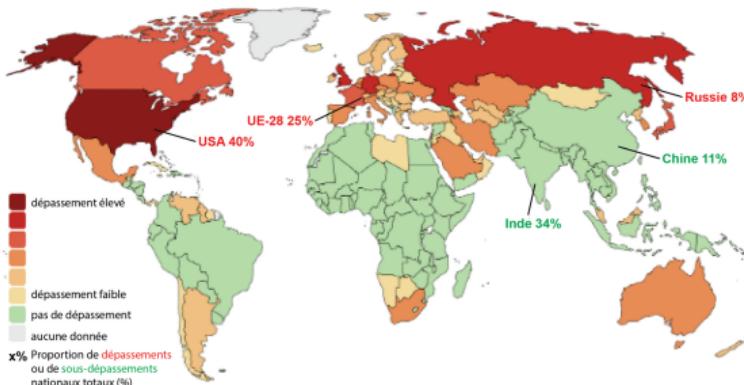
*Thomas Bourany*  
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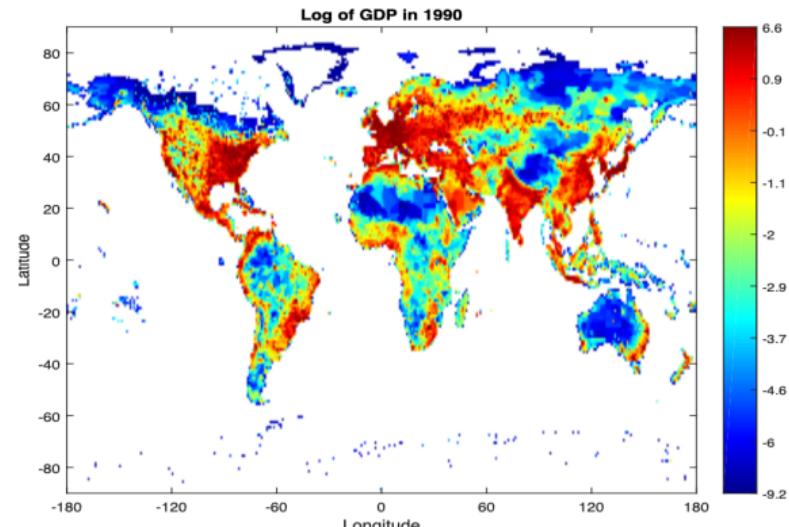
Feb 2023

## Introduction – Motivation

- ▶ Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
  - ***Unequal causes*** : Developed economies account for over 65% of cumulative GHG emissions (~ 25% each for the EU and the US)

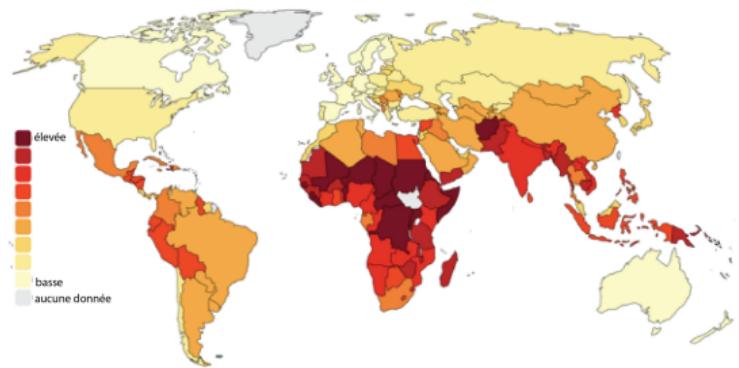


Source : Lancet planetary Health - Quantifying national responsibility for climate breakdown: an equality-based attribution approach for carbon dioxide emissions in excess of the planetary boundary - Jason Hickel 2020

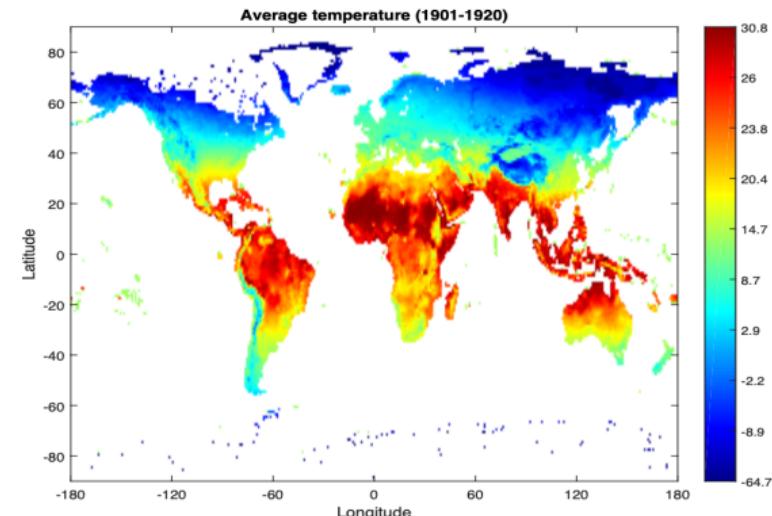


## Introduction – Motivation

- ▶ Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
  - ***Unequal consequences*** : Increase in temperatures will disproportionately affect developing countries where the climate is already warm



Source : Notre Dame Global Adaptation Initiative



## Introduction – this project

- ▶ Which countries will be affected the most by climate change ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - What is the optimal policy in presence of externalities *and* heterogeneities ?

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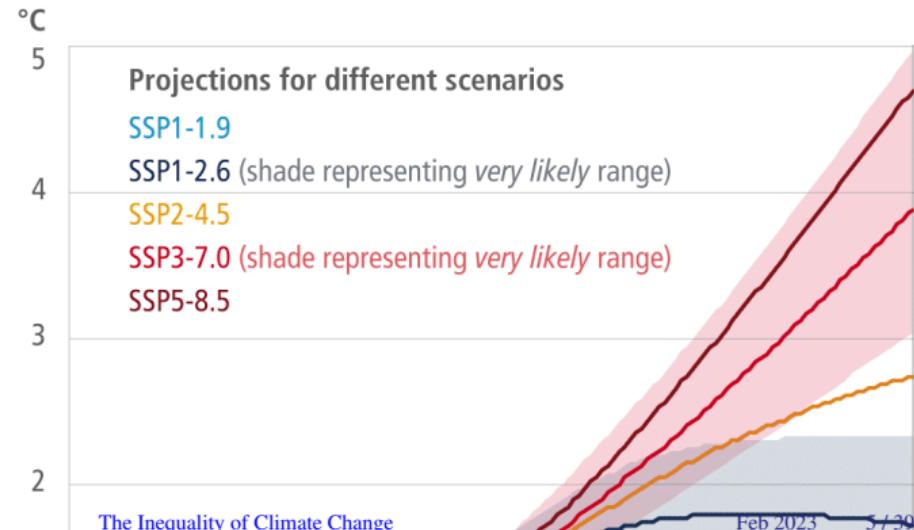
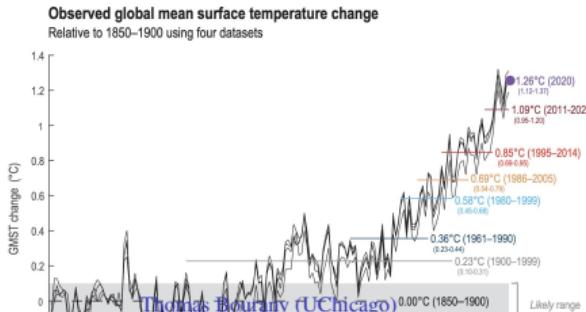
- ▶ Which countries will be affected the most by climate change ?
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- ▶ Develop a simple and flexible model of climate economics
  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions

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  - Standard NCG – IAM model with heterogeneous regions
  - Every country is small relative to global GHG – no incentives to curb emissions
- Evaluate the heterogeneous welfare costs of global warming
- Provide analytical formulas and a numerical methodology to compute the cost of carbon
  - $CC$  is linear in  $GDP$ /level of development and in temperature gaps
- Solve world optimal carbon policy with heterogeneous regions
  - Does the optimal carbon tax coincide with the social cost of carbon ?

## Introduction – Uncertainty

- However, the impact of climate change is uncertain for several reasons :
  - (i) Climate forecasts : temperature trajectories for a given path of emissions
  - (ii) Future growth : levels of future output for given damages
  - (iii) Path of emissions : Likelihood of pledges and mitigation policies to be actually implemented.



## Introduction – related literature

- ▶ Classic Integrated Assessment models (IAM) :
  - Nordhaus' Multi-regions DICE (2016), Golosov, Hassler, Krusell, Tsvyanski (2014)
  - Dietz, van der Ploeg, Rezai, Venmans (2021), among others
- ▶ Macro (IAM) model with country heterogeneity :
  - Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021), among others
    - *This paper* : Studies the optimal policy with heterogeneity and externalities
- ▶ Climate model with risk & uncertainty :
  - Cai, Lontzek, Judd (2019), Barnett, Brock and Hansen (2022)
    - *This paper* : Includes heterogeneity and redistribution effects of climate & carbon taxation
- ▶ Quantitative spatial models :
  - Cruz, Rossi-Hansberg (2021), Bilal, Rossi-Hansberg (2023), Rudik et al (2022)
    - *This paper* : Considers forward-looking decision of agents & optimal policy
- ▶ Heterogeneous Agents models with optimal policy
  - Le Grand, Ragot (2018-), Davila, Schaab (2022), Bhandari Evans Golosov Sargent (2018-)
    - *This paper* : Studies climate externalities and Pigouvian taxation

## Toy model – Energy market and Externality

- ▶ Consider two countries  $i = N, S$ 
  - With different levels of productivity  $z_i$
  - Consuming and producing with energy  $e$ 
    - Energy producer with profit  $\pi(E)$  owned by country  $i$  with share  $\theta_i$
  - Household problem :

$$\max_{c_i, e_i} U(c_i)$$

$$c_i + q^e e_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) \quad [\lambda_i^k]$$

- Subject to damage  $\mathcal{D}_i(\mathcal{S})$  and climate externalities :

$$\mathcal{S} = \mathcal{S}_0 + \overbrace{\xi_N e_N + \xi_S e_S}^{=\text{GHG emissions}}$$

- And consuming energy in a single energy market with price  $q^e$

$$E = e_N + e_S \quad q^e = c'(E) \quad \pi(E) = q^e E - c(E)$$

## Toy model – Competitive equilibrium vs. Social Planner – Energy decision

► Competitive equilibrium Result :

- Marginal Product of Energy = Energy Price

$$MPe_i = q^e \quad \text{with} \quad MPe_i := \mathcal{D}_i(\mathcal{S})z_i F_e(e_i)$$

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- ▶ Comparison with Social planner :

$$\begin{aligned} & \max_{\{c_i, e_i, q^e\}_i} \sum_{i=N,S} \omega_i U(c_i) \\ \text{s.t. } & c_i = \mathcal{D}_i(\mathcal{S})z_i F(e_i) + \theta_i \pi(E) \quad [\phi_i^k] \quad \lambda_i^k = U'(c_i) \quad [\phi_i^c] \\ & MPe_i = q^e \quad [\phi_i^e] \quad q^e = c'(E) \quad [\phi^E] \\ & \mathcal{S} := \mathcal{S}_0 + \xi_N e_N + \xi_{ses} \quad E := e_N + es \end{aligned}$$

## Ramsey planner optimality

- ▶ Planner's marginal value of wealth

$$\phi_i^k = \omega_i U'(c_i)$$

- Measure of inequality  $\phi_i^k / \bar{\phi}^k$  with  $\bar{\phi}^k = \frac{1}{2} \sum_j \omega_j U'(c_j)$  and  $\frac{\phi_i^k}{\bar{\phi}^k} = \frac{\omega_i U'(c_i)}{\frac{1}{2} \sum_j \omega_j U'(c_j)} \leq 1$

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- ▶ Energy decision :

$$\underbrace{\frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}}_{\text{redistribution term}} \left( \partial_e \tilde{F}(\mathcal{S}, e_i) - q^e \right) + \xi_i \underbrace{\sum_j \frac{\phi_j^k}{\bar{\phi}^k} \mathcal{D}'_j(\mathcal{S}) y_j}_{\text{α-social cost of carbon}} + \underbrace{\partial_E \pi(q^e, E) \sum_j \theta_j \frac{\phi_j^k}{\bar{\phi}^k}}_{\text{energy rent redistribution}} \\
 - \underbrace{\tilde{\phi}_i^E c''(E)}_{\text{effect on energy supply}} + \underbrace{\tilde{\phi}_i^e \partial_{ee}^2 \tilde{F}(\mathcal{S}, e_i)}_{\text{effect on energy choice}} [e_i]$$

## Social Cost of Carbon with inequality

- ▶ The SCC is exacerbated with heterogeneity

$$\begin{aligned} SCC &= - \sum_j \frac{\phi_j^k}{\phi} \mathcal{D}'_j(\mathcal{S}) y_j \\ &= -\text{Cov}_j \left( \frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}, \mathcal{D}'_j(\mathcal{S}) y_j \right) - \mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] \\ &> -\mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] \end{aligned}$$

- Why? Because low-income countries tend to be warmer/more vulnerable to climate change

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 &> -\mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j]
 \end{aligned}$$

- Why ? Because low-income countries tend to be warmer/more vulnerable to climate change
- ▶ For the social value of energy rent, it's the contrary !

$$Rent = \text{Cov}_j \left( \frac{\omega_j U'(c_j)}{\frac{1}{2} \sum_j \omega_j U'(c_j)}, \theta_j \pi'_j(E) \right) + \pi'(E) < \pi'(E)$$

## Toy model – Effect of uncertainty

- ▶ Consider risks related to both

- (i) Economic growth  $z_i(\epsilon_z)$
- (ii) Climate damage  $\mathcal{D}_i(\mathcal{S}|\epsilon_d)$ 
  - Probability distribution  $(\epsilon_z, \epsilon_d) =: \epsilon \sim \varphi(\epsilon)$

$$\max_{e_i} \int_{\mathcal{E}} \max_{c_i(\epsilon)} U(c_i(\epsilon)) d\varphi(\epsilon) \quad \text{vs.} \quad \max_{\{e_j\}_j} \int_{\mathcal{E}} \max_{\{c_j(\epsilon)\}_j} \sum_{j=N,S} \omega_j U(c_j(\epsilon)) d\varphi(\epsilon)$$

- ▶ Competitive equilibrium :

- Almost no change in behavior : Expected Marginal Product of Energy = Energy Price

$$\int_{\mathcal{E}} MPe_i(\epsilon) d\varphi(\epsilon) = q^e$$

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## Toy model – Social Planner and Uncertainty

- Energy decision for the planner :

$$\int_{\mathcal{E}} \frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]} (MPe_i(\epsilon) - q^e) d\varphi(\epsilon) = \int_{\mathcal{E}} SCC(\epsilon) d\varphi(\epsilon) - \int_{\mathcal{E}} SVR(\epsilon) d\varphi(\epsilon) \\ - \tilde{\phi}^E c''(E) + \int_{\mathcal{E}} \tilde{\phi}_i^e(\epsilon) \partial_{ee}^2 \tilde{F}(\mathcal{S}, e_i | \epsilon) d\varphi(\epsilon)$$

- Rewriting and using expectations formulas, we obtain :

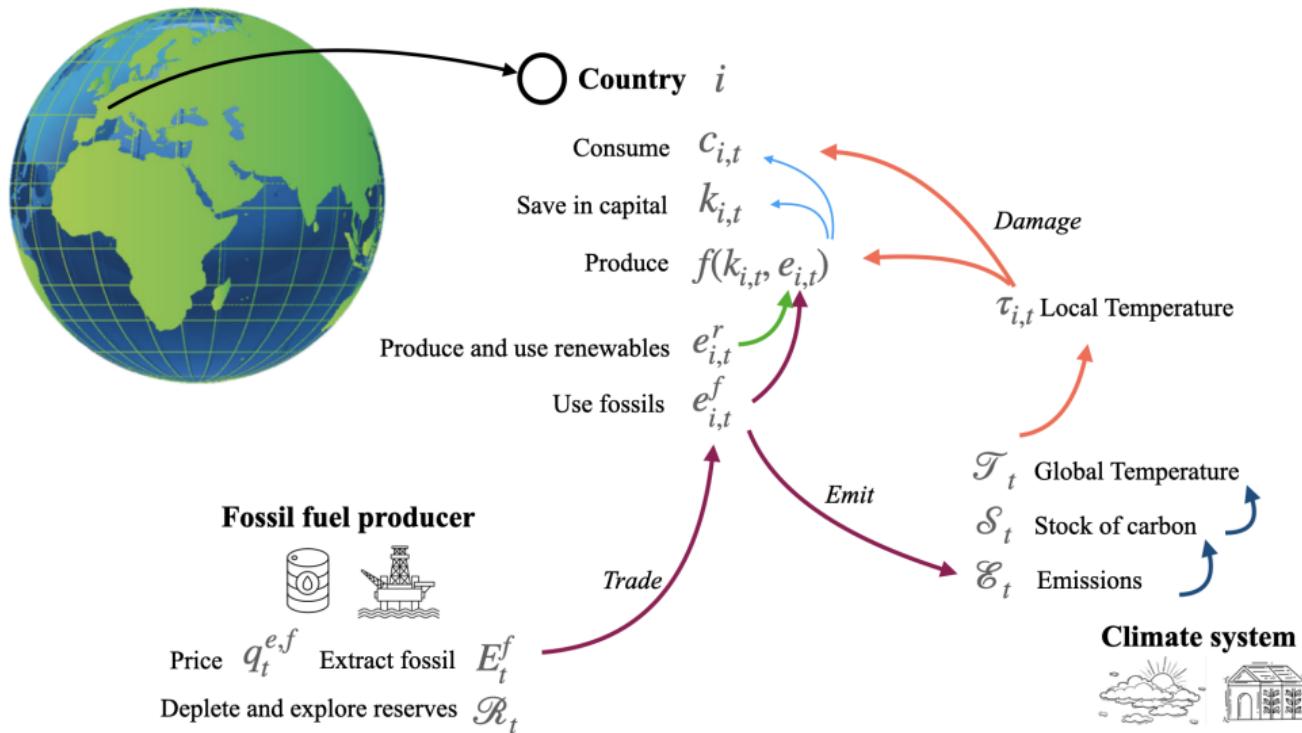
$$\mathbb{E}_\epsilon(MPe_i(\epsilon)) = q^e + \underbrace{\frac{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}{\mathbb{E}_\epsilon(\omega_i U'(c_j(\epsilon)))}}_{\text{redistributive effect}} \left[ \underbrace{-\text{Cov}_\epsilon\left(\frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, MPe_i(\epsilon)\right)}_{\text{effect of agg. risk } \epsilon \text{ on energy choice}} \right. \\ \left. + \mathbb{E}_\epsilon[SCC(\epsilon)] - \mathbb{E}_\epsilon[SVR(\epsilon)] - \tilde{\phi}^E c''(E) + \mathbb{E}_\epsilon[\tilde{\phi}_i^e(\epsilon) \partial_{ee}^2 \tilde{F}(\mathcal{S}, e_i | \epsilon)] \right]$$

## Toy model – Social cost of carbon and Uncertainty

- ▶ Social cost of carbon

$$\begin{aligned}
 \mathbb{E}_\epsilon[SCC] &= \int_{\mathcal{E}} \sum_{j=N,S} \frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]} \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) d\varphi(\epsilon) \\
 &= -\text{Cov}_{j,\epsilon} \left( \frac{\omega_j U'(c_j(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) \right) - \mathbb{E}_{j,\epsilon} [\mathcal{D}'_j(\mathcal{S}) y_k] \\
 &= -\mathbb{E}_j \left[ \underbrace{\text{Cov}_\epsilon \left( \frac{\omega_j U'(c_j(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) \right)}_{=\text{effect of agg. risk } \epsilon} \right] \\
 &\quad - \underbrace{\text{Cov}_j \left[ \frac{\mathbb{E}_\epsilon(\omega_j U'(c_j(\epsilon)))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, \mathbb{E}_\epsilon(\mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)) \right]}_{=\text{effect of heterogeneity across } j} - \underbrace{\mathbb{E}_{j,\epsilon} [\mathcal{D}'_j(\mathcal{S}) y_k]}_{=\text{average exp. damage}} \\
 &> \mathbb{E}_\epsilon[\overline{SCC}(\epsilon)]
 \end{aligned}$$

## Summary of the quantitative model



# Model

- ▶ Neoclassical economy, in continuous time
  - countries/regions  $i \in \mathbb{I}$  : ex-ante heterogeneous in dimensions  $\underline{s}$ 
    - Here :  $\underline{s}_i = \{p_i, z_i, \Delta_i\}$ , relative heterogeneity doesn't change over time
    - Productivity grows at rate  $\bar{g}$  and population grow at rate  $n$
  - regions heterogeneous ex-post  $\bar{s}_i$ 
    - Here : capital and temperature  $\bar{s}_i = \{k_i, \tau_i\}$
    - Future : could include  $z$  – endog. technical change – or  $p$  – migrations/demographics
    - Renormalization : all variables are values per unit of efficient labor  $k_{i,t} = \frac{K_{i,t}}{p_{t_0}} e^{-(\bar{g}+n)t}$
  - Aggregate variables : global Temperature, carbon Stocks in atmosphere, fossil energy Reserves  $S = \{\mathcal{T}, \mathcal{S}, \mathcal{R}\}$

## Model – Household and firm

- ▶ Household problem in country  $i$  :

$$\max_{\{c_{i,t}, e_{i,t}^f, e_{i,t}^r\}} \int_{t_0}^{\infty} e^{-(\rho-n)t} u(c_{i,t}, \tau_{i,t}) dt$$

- ▶ Dynamics of capital in every country  $i$  :

$$\dot{k}_{i,t} = \mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e_{i,t}) + \theta_i^R \pi_t^R - (n + \bar{g} + \delta) k_{i,t} - q_t^e e_{i,t} - c_{i,t}$$

- ▶ Choices :

- $c_t$  consumption,  $e_t$  energy, with production fct :

$$f(k, e) = \left( (1 - \varepsilon)^{\frac{1}{\sigma}} k^{\alpha \frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} (z^e e)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- Damage function  $\mathcal{D}^y(\tau_t)$  affect country's production
- Directed technical change  $z_t^e$  & energy mix  $e_t$  with fossil  $e_t^f$  vs. renewable  $e_t^r$

## Model – Energy markets

- ▶ Two sources of energy : fossil  $e_t^f$  and renewable  $e_t^r$  for every  $i$

$$e_{i,t} = \left( \omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e - 1}{\sigma^e}} + (1 - \omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r)^{\frac{\sigma^e - 1}{\sigma^e}} \right)^{\frac{\sigma^e}{\sigma^e - 1}}$$

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- ▶ Fossil fuels energy producer :
  - Extended-Hotelling problem (depleting reserves with stock effects and exploration)

$$\max_{\{E_t^f, \mathcal{I}_t\}_t} \int_0^\infty e^{-\rho t} \pi_t(E_t^f, \mathcal{R}_t) dt \quad \text{with } \pi_t(E_t^f, \mathcal{R}_t) = q_t^{e,f} E_t^f - \mathcal{C}^e(E_t^f, \mathcal{R}_t) - \mathcal{C}^i(\mathcal{I}_t, \mathcal{R}_t)$$

$$\text{s.t.} \quad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_{i,t} di \quad \dot{\mathcal{R}}_t = -E_t^f + \delta_R \mathcal{I}_t$$

- Optimal pricing with finite-resources rents More details

$$q_t^{e,f} = \bar{\nu} \left( \frac{E_t^f}{\mathcal{R}_t} \right)^\nu + \lambda_t^R \quad \bar{\mu} \left( \frac{\mathcal{I}_t^*}{\mathcal{R}_t} \right)^\mu = \delta_R \lambda_t^R$$

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- ▶ Renewable energy as a substitute technology

$$e_{i,t}^r = z_{i,t}^r k_{i,t}^r \quad q_{i,t}^{e,r} = r_{i,t} / z_{i,t}^r$$

## Fossil energy and externality

- ▶ Fossil energy input  $e_t^f$  causes climate externality
  - Change the world climate – global temperature  $\mathcal{T}_t$  and cumulative GHG in atmosphere  $\mathcal{S}_t$  :

$$\begin{aligned}\mathcal{E}_t &= \int_{\mathbb{I}} \xi e_{i,t}^f p_i di \\ \dot{\mathcal{S}}_t &= \mathcal{E}_t - \delta_s \mathcal{S}_t \quad \quad \quad \dot{\mathcal{T}}_t = \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)\end{aligned}$$

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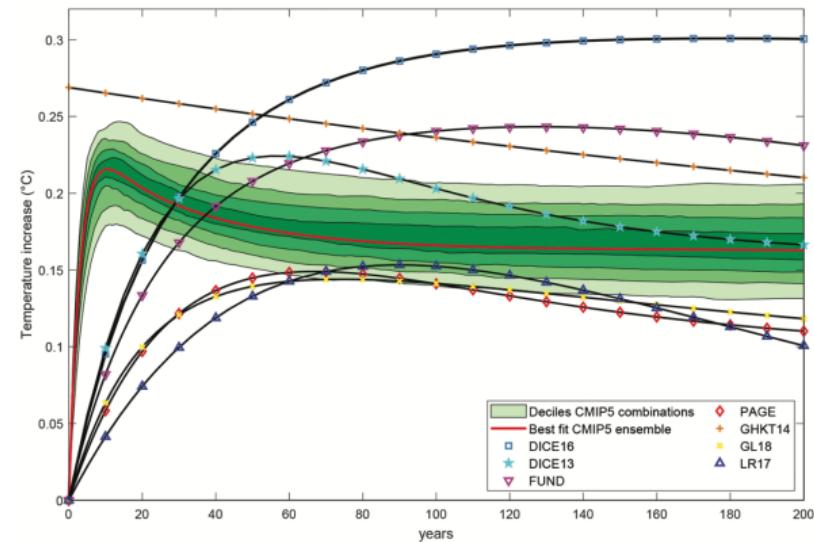
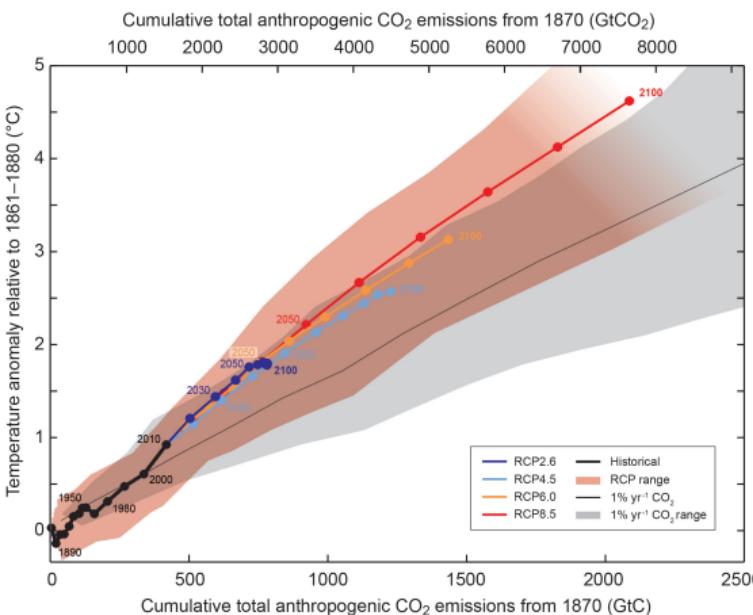
- $\zeta$  is the inverse of persistence, so if  $\zeta \rightarrow \infty$ , we obtain a linear model :

$$\mathcal{T}_t = \bar{\mathcal{T}} + \chi \mathcal{S}_t = \bar{\mathcal{T}} + \chi \int_{t_0}^t \int_{\mathbb{I}} \xi e_{i,s} di ds \Big|_{GtC}$$

- The externality depends on policy  $e_{i,t}^f$  as function of states  $\{z, p, k, \tau\}$ 
  - Naturally, countries richer/more productive/with a larger population use more energy !
- Impact of climate on country's local temperature :

$$\dot{\tau}_{i,t} = \Delta_i \dot{\mathcal{T}}_t$$

# Temperature dynamics



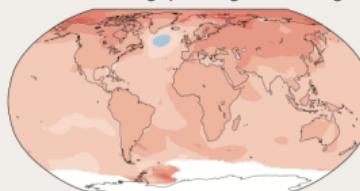
Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

# Temperature dynamics

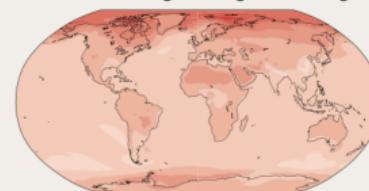
(a) Annual mean temperature change ( $^{\circ}\text{C}$ )  
at  $1^{\circ}\text{C}$  global warming

Warming at  $1^{\circ}\text{C}$  affects all continents and is generally larger over land than over the oceans in both observations and models. Across most regions, observed and simulated patterns are consistent.

Observed change per  $1^{\circ}\text{C}$  global warming



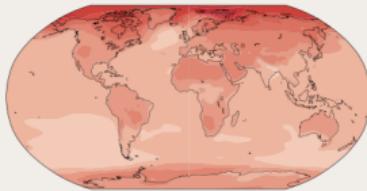
Simulated change at  $1^{\circ}\text{C}$  global warming



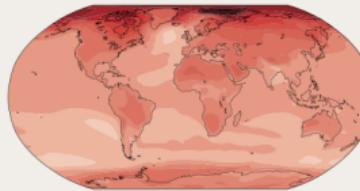
(b) Annual mean temperature change ( $^{\circ}\text{C}$ )  
relative to 1850–1900

Across warming levels, land areas warm more than ocean areas, and the Arctic and Antarctica warm more than the tropics.

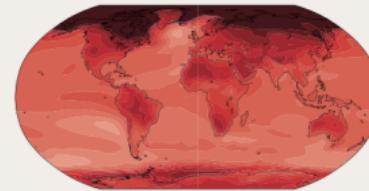
Simulated change at  $1.5^{\circ}\text{C}$  global warming



Simulated change at  $2^{\circ}\text{C}$  global warming



Simulated change at  $4^{\circ}\text{C}$  global warming



## Climate Models - 1

- ▶ Integrated Assessment Models can be more detailed :
  - Complete Picture of the Atmosphere-Ocean and Temperatures relationships
- 1. Carbon System :
  - World is divided into “boxes” :  $AT$  : atmosphere,  $UO$  Upper Ocean+Biosphere, and  $LO$  Lower Ocean
  - Emissions come from Land and Industrie  $\mathcal{E}_t = \mathcal{E}_{land,t} + \mathcal{E}_{ind,t}$

$$\mathcal{M}_t = (M_{AT,t}, M_{UO,t}, M_{LO,t})$$

$$\dot{\mathcal{M}}_t = \Phi_M \mathcal{M}_t + (\mathcal{E}_t, 0, 0)^T$$

with  $\Phi_M$  Markovian matrix

In particular, carbon doesn't flow to lower atmosphere instantaneously

- 2. Radiative Forcing :

$$\mathcal{F}_t = \eta \log \left\{ \frac{M_{AT,t}}{\bar{M}_{AT}} \right\} + \mathcal{F}_{ex,t}$$

## Climate Models - 2

### 3. Temperature System :

- Also divided into “boxes” :  $AT$  : atmosphere,  $LO$  Lower Ocean

$$\mathcal{T}_t = (T_{AT,t}, T_{LO,t})$$

$$\dot{\mathcal{T}}_t = \Phi_T \mathcal{T}_t + (\zeta \mathcal{F}_t, 0)^T$$

with  $\Phi_T$  Markovian matrix

### 4. Tipping Elements

- Markov Process that has much long persistence : e.g. ice cap

$$\dot{\mathcal{J}}_t = \Phi_J(\mathcal{T}_t, \mathcal{M}_t, \mathcal{J}_t, \epsilon_t)$$

- Example : if the ice cap melts, it will take hundreds of years to reappear.
- Direct impact on the damage function :

$$\mathcal{D}(\tau, \mathcal{J}) = \mathcal{D}_T(\tau) \mathcal{D}_J(\mathcal{J})$$

## Damage functions

- ▶ Climate change has two effects :

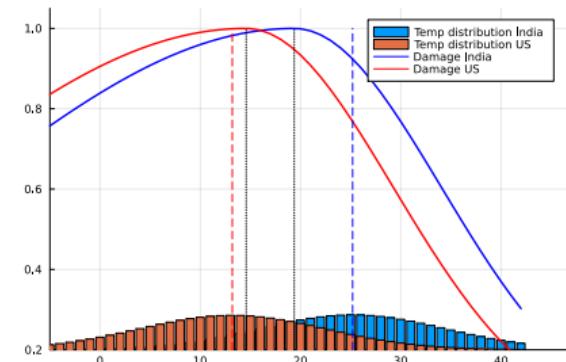
- Affect household utility function  $u(c_t, \tau_t) = \mathcal{D}^u(\tau_t) \frac{c_t^{1-\eta}}{1-\eta}$

$$\mathcal{D}_u(\tau) = \begin{cases} e^{-\phi^{\oplus} \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau > \tau_i^* \\ e^{-\phi^{\ominus} \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau < \tau_i^* \end{cases}$$

- Affect firm productivity  $\mathcal{D}(\tau_t)z$  as a shifter – as in Nordhaus DICE-2016

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma^{\oplus} \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau > \tau_i^* \\ e^{-\gamma^{\ominus} \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau < \tau_i^* \end{cases}$$

- Deviation from "ideal" temperature  $\tau_i^*$
- Damage sensitivities  $\gamma$  and  $\phi$  are asymmetrical and can also be heterogeneous and uncertain



# The Business as Usual equilibrium is the standard neoclassical economy

- ▶ Using Pontryagin Max. Principle :
  - We obtain a system of coupled ODEs [More details](#)
- ▶ Back to the standard IAM - DICE Model from Nordhaus

$$\begin{cases} \dot{c}_{i,t} &= c_{it} \frac{1}{\eta} (r_{it} - \rho) \\ \dot{k}_{i,t} &= \mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e_{i,t}^f, e_{i,t}^r) + \theta_i^R \pi_t^R - (n + \bar{g} + \delta) k_{i,t} - q_t^f e_{i,t}^f - q_{i,t}^r e_{i,t}^r - c_{i,t} \\ \dot{q}_t^f &= MPe_{it}^f \quad q_{it}^r = MPe_{it}^r \end{cases}$$

- + Climate blocks for Carbon  $\mathcal{S}_t$  and temperature  $\tau_{i,t}$
- + Dynamics of Hotelling rents  $\lambda_t^R$  for fossil price  $q_t^f$ .

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- + Climate blocks for Carbon  $\mathcal{S}_t$  and temperature  $\tau_{i,t}$
- + Dynamics of Hotelling rents  $\lambda_t^R$  for fossil price  $q_t^f$ .
- ▶ Are we doomed to collapse ?
- ▶ What is the valuation of the (heterogeneous) economic damages of climate change
- ▶ What about policy ?

## The Ramsey Problem

- ▶ Consider a Social Planner that care about aggregate welfare :

$$\max_{\{c, e^f, e^r, k, \tau, \mathcal{S}, \mathcal{R}, \mathcal{I}\}} \int_0^\infty \int_{\mathbb{I}} e^{-\bar{\rho}t} \omega_i \mathcal{D}(\tau_{i,t}) u(c_{i,t}) p_i \, di \, dt$$

subject to

- Optimality conditions of households, for  $c_i$ ,  $e_i^f$ ,  $e_i^r$  and  $k_i$
- Optimality conditions of the Fossil, for  $E^f$ ,  $\mathcal{I}$  and  $\mathcal{R}$
- Climate and temperature dynamics  $\tau_i$  and  $\mathcal{S}$

- ▶ Methodological contribution ⇒ adapt the Pontryagin Maximum Principle for :
  - A continuum of heterogeneous agents : Mean Field Control / McKean Vlasov FBSDE – Carmona, Delarue (2018-) / Pham et al (2018-)
  - Case with Aggregate uncertainty (future plan) : Stochastic PMP, c.f. Yong, Zhou (1999)

⇒ Large scale system of mean-field ODE More details - Hamiltonian

## Marginal values of temperature

- Marginal values of the climate variables :  $\psi_t^S$  and  $\psi_{i,t}^\tau$

$$\dot{\psi}_{i,t}^\tau = \psi_{i,t}^\tau(\tilde{\rho} + \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau_i^*) \mathcal{D}^y(\tau_{i,t}) f(k_{i,t}, e_{i,t}) \psi_{i,t}^k}^{-\partial_\tau \mathcal{D}^y} + \overbrace{\phi_i(\tau_{i,t} - \tau_i^*) \mathcal{D}^u(\tau_{i,t}) u(c_{i,t})}^{\partial_\tau \mathcal{D}^u}$$

$$\dot{\psi}_t^S = \psi_t^S(\tilde{\rho} + \delta^s) - \zeta \chi \int_{\mathbb{I}} \Delta_i \psi_{i,t}^\tau di$$

- Marg. cost for  $i$  of releasing carbon in atmosphere  $\psi_t^S$  increases with :

- Temperature gap  $\tau_{i,t} - \tau_i^*$
- Damage sensitivity to temperature for TFP  $\gamma_i$  and utility  $\phi$
- The development level  $f(k_{i,t}, e_{i,t})$  and  $u(c_{i,t})$
- The "catching up" effect of temperature from cold location  $\Delta_i$

## Social cost of carbon

- ▶ The marginal “externality damage” or “social cost of carbon” (SCC) can be expressed naturally :

$$SCC_t := -\frac{\partial W_t / \partial S_t}{\partial W_t / \partial c_t} = -\frac{\psi_t^S}{\bar{\psi}_t^k}$$

- Ratio of marg. cost of carbon vs. the marg. value of consumption/capital
- Important question : which util' unit  $\bar{\psi}_t^k$  to compute the SCC ? Average marginal utils ?

$$\bar{\psi}_t^k = \int_{\mathbb{I}} \psi_{j,t}^k dj \approx \int_{\mathbb{I}} \omega_i u'(c_{j,t}) p_j dj$$

- ▶ As a result :

- Stationary value :  $t \rightarrow \infty$ , with  $\mathcal{E}_t = \delta_s S_t$  and  $\tau_t \rightarrow \tau_\infty$

$$SCC_t \equiv \frac{1}{\bar{\psi}_t^k} \frac{\chi}{\tilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_i (\tau_{i,\infty} - \tau_i^*) \left( \gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} \psi_{i,\infty}^k + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \omega_i u(c_{i,\infty}) p_i \right) di$$

## Social cost of carbon

- One could also consider the “*Local cost of carbon*” as the marginal damage of the region :

$$LCC_{i,t} = \frac{\chi}{\tilde{\rho} + \delta^s} \Delta_i(\tau_{i,\infty} - \tau_i^*) \left( \gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \frac{u(c_{i,\infty})}{u'(c_{i,\infty})} \right)$$

- As a result, the Social Cost of Carbon can be reexpressed as :

$$\begin{aligned} SCC_t &\equiv \int_{\mathbb{I}} \frac{\psi_{i,t}^k}{\bar{\psi}_t^k} LCC_{i,t} di = \mathbb{E}^{\mathbb{I}}[LCC_{i,t}] + \text{Cov}^{\mathbb{I}}\left(\frac{\psi_{i,t}^k}{\bar{\psi}_t^k}, LCC_{i,t}\right) \\ &> \mathbb{E}^{\mathbb{I}}[LCC_{i,t}] = \overline{SCC}_t \end{aligned}$$

⇒ Correlation between local damage and poverty increases the Social Cost of Carbon !

- Solution of the adjoint equation : [Proof](#)
- Uncertainty [SCC with uncertainty](#)

## Social cost of carbon & temperature

- ▶ Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta \chi \xi \omega \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} q_t^{f-\sigma_e} \int_{j \in \mathbb{I}} (z_j z_{j,t}^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} y_{j,t} q_{j,t}^{\sigma_e - \sigma} dj dt$$

- Geographical factors determining warming  $\Delta_i$
- Climate sensitivity  $\chi$  & carbon exit from atmosphere  $\delta_s$
- Growth of population  $n$ , aggregate productivity  $\bar{g}$
- Deviation of output from trend  $y_i$  & relative TFP  $z_j$
- Directed technical change  $z_t^e$ , elasticity of energy in output  $\sigma$
- Fossil energy price  $q_t^{e,f}$  and Hotelling rent  $g^{q^f} \approx \lambda_t^R / \lambda_t^F = \rho$
- Change in energy mix, renewable share  $\omega$ , price  $q_t^r$  & elasticity of source  $\sigma_e$

- ▶ Approximations at  $T \equiv$  Generalized Kaya (or  $I = PAT$ ) identity More details

$$\frac{\dot{\tau}_T}{\tau_T} \propto n + \bar{g}^y - (1 - \sigma)(g^{z^e} - \tilde{\gamma}) + (\sigma_e - \sigma)(1 - \omega)g^{q^r} - (\sigma_e(1 - \omega) + \sigma\omega)g^{q^f}$$

# Optimal energy and emissions decisions

## 1. *Business as usual / Competitive equilibrium :*

- Fossil energy : only private tradeoff : marg. product of energy = marg cost + Hotelling rent

$$[e_{i,t}^f] \quad \underbrace{\mathcal{D}(\tau_{i,t})z \partial_e f(k_{i,t}, e_{i,t}) \left( \frac{e_{i,t}^f}{\omega e_{i,t}} \right)^{-\frac{1}{\sigma_e}}}_{MPe_{i,t}^f} = q_t^f = \bar{\nu} \left( \frac{E_t^f}{\mathcal{R}_t} \right)^\nu + \lambda_t^R$$

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## 2. Ramsey planner :

- Fossil energy :

$$[e_{i,t}^f] \quad \underbrace{\psi_{i,t}^k \mathcal{D}(\tau_{i,t}) z_{i,t} \partial_{ef}(k_{i,t}, e_{i,t}) \left( \frac{e_{i,t}^f}{\omega e_{i,t}} \right)^{-\frac{1}{\sigma_e}}}_{MV_{i,t}^k MPe_{i,t}^f} = \psi_{i,t}^k q_t^f + \xi SCC_t + \pi'(E_t^f) SVR_t - \partial_{ef} \widehat{\phi}_{it}^e + p_i \phi_t^{Ef} \partial_{EE} \mathcal{C}(E_t^f)$$

## Optimal energy and emissions decisions

- ▶ Planner's optimal choice for fossil energy consumption :

$$\psi_{i,t}^k MPe_{i,t}^f = \psi_{i,t}^k q_t^f + \xi SCC_t + \pi'(E_t^f) SVR_t - \partial_{ef} \hat{\phi}_{it}^e + p_i \phi_t^{Ef} \partial_{EEC}(E_t^f)$$

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- Redistribution motive : energy is more valuable in poorer countries !

$$\text{low } z_i, k_i \quad \Rightarrow \quad \text{low } c_i, \text{ high } \psi_{i,t}^k \approx \omega_i U'(c_i) p_i > \bar{\psi}_t^k = \int_{\mathbb{I}} \psi_{i,t}^k di$$

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- Climate externality :

$$SCC_t = \mathbb{C}\text{ov}_j \left( \frac{\psi_{j,t}^k}{\bar{\psi}_t^k}, LCC_{j,t} \right) + \mathbb{E}_j[LCC_{j,t}] > \mathbb{E}_j[LCC_{j,t}]$$

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- Energy rent redistribution :

$$\pi'(E_t^f) SVR_t = \text{Cov}_j \left( \frac{\psi_{j,t}^k}{\bar{\psi}_t^k}, \theta_j \pi'_j(E) \right) + \pi'(E) < \pi'(E)$$

- Curvature of the output production function :

$$\partial_{ef} \hat{\phi}_{it}^e = \partial_{ef} [MPe_t^f \phi_{j,t}^f + MPe_t^r \phi_{j,t}^r]$$

# Optimal energy and emissions decisions

- ▶ Planner's optimal choice for fossil energy consumption :

$$\psi_{i,t}^k MPe_{i,t}^f = \psi_{i,t}^k q_t^f + \xi SCC_t + \pi'(E_t^f) SVR_t - \partial_{ef} \widehat{\phi}_{it}^e + p_i \phi_t^{Ef} \partial_{EEC}(E_t^f)$$

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- Curvature of the output production function :

$$\partial_{ef} \widehat{\phi}_{it}^e = \partial_{ef} [MPe_t^f \phi_{j,t}^f + MPe_t^r \phi_{j,t}^r]$$

- Curvature of the energy supply curve :

$$\phi_t^{Ef} \partial_{EEC}(E_t^f, \mathcal{R}_t) = \phi_t^{Ef} \nu \bar{\nu} \frac{E_t^{\nu-1}}{\mathcal{R}_t^\nu}$$

## Decentralization - 1

- ▶ With inequality  $\psi_{i,t}^k \neq \psi_{j,t}^k$ , it is unclear how to decentralize
- ▶ Allowing lump sum transfer across countries solves world inequality

$$\psi_{i,t}^k = \psi_{j,t}^k = \bar{\psi}_{i,t}^k$$

- ▶ Pigouvian taxation as in the Representative agent world =  $\mathbf{t}_t^f = SCC_t = \mathbb{E}^{\mathbb{I}}(LCC_{j,t})$
- ▶ Restricting to intra-country redistribution ?
  - Allow distortive taxes and lump-sum rebate in  $i$  only
  - ⇒ Planner cares about redistribution :  $\lambda_{i,t}^k$  vs.  $\bar{\lambda}_t^k = \int_j \lambda_{j,t}^k p_j dj$
  - Combination of distortive taxes and lump-sum rebate

## Decentralization - 2

- ▶ Energy taxes :

$$MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = q_t^{e,f} + \underbrace{\frac{\bar{\psi}_t^k}{\psi_{i,t}^k}}_{=\text{redistribution}} (\mathbf{t}_t^S - \mathbf{t}_t^R - \mathbf{t}_{i,t}^e + \mathbf{t}_t^C)$$

- ▶ Combination :

- Redistributive taxes/subsidy :

$$\mathbf{t}_{i,t} = \frac{\bar{\psi}_t^k}{\psi_{i,t}^k} = \frac{\int_{\mathbb{I}} \omega_j u'(c_j) p_j dj}{\omega_i u'(c_i) p_i} \leqslant 1$$

- Pigouvian tax :  $\mathbf{t}^S$  is flat rate accounting for the climate externality

$$\mathbf{t}_t^S = - \int_{\mathbb{I}} \frac{\psi_{j,t}^k}{\bar{\psi}_t^k} \frac{\psi_{j,t}^S}{\psi_{j,t}^k} p_j dj = \text{Cov}_j\left(\frac{\psi_{j,t}^k}{\bar{\psi}_t^k}, LCC_{j,t}\right) + \overline{SCC}_t > \overline{SCC}_t$$

- Fossil rent :  $\mathbf{t}^R$  is flat rate accounting for the rent redistribution

$$\mathbf{t}_t^R = - \int_{\mathbb{I}} \frac{\psi_{j,t}^k}{\bar{\psi}_t^k} \theta_j \pi_e(E_t^f, \mathcal{R}_t) dj = \left(1 + \text{Cov}_j\left(\frac{\psi_{j,t}^k}{\bar{\psi}_t^k}, \theta_j\right)\right) \pi_e(E_t^f, \mathcal{R}_t) < \pi_e(E_t^f, \mathcal{R}_t)$$

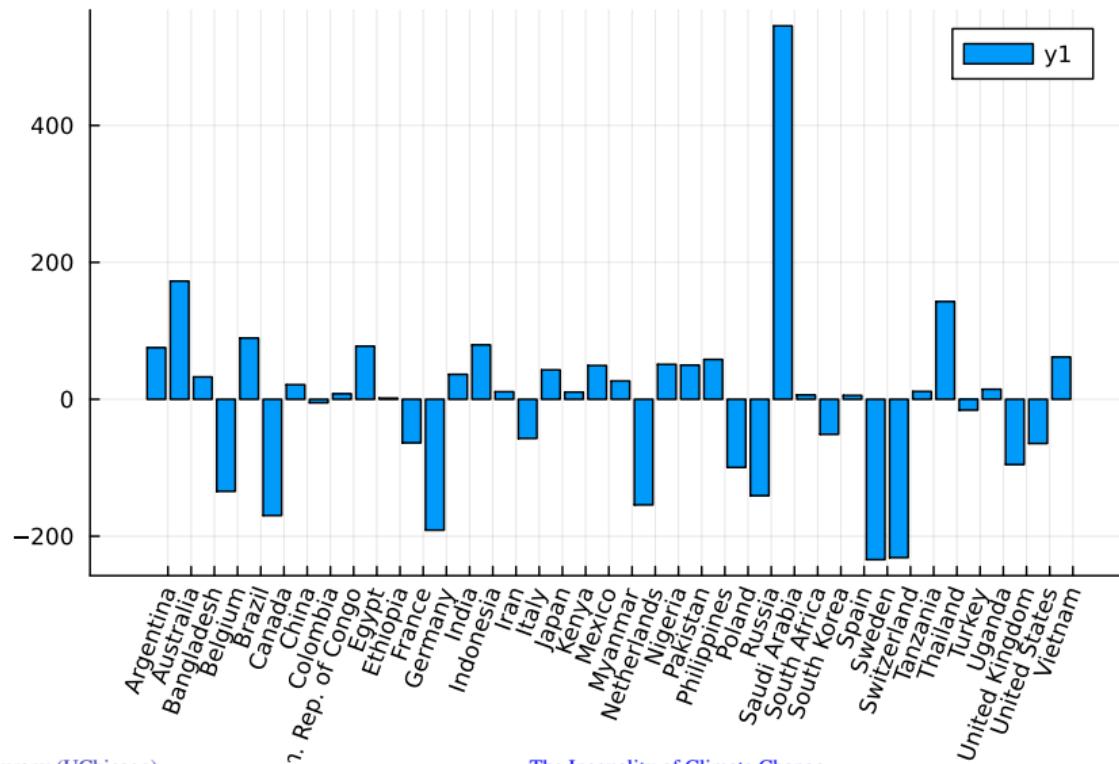
- Everything is rebated lump sum to the household in each country  $i$

## Numerical Application

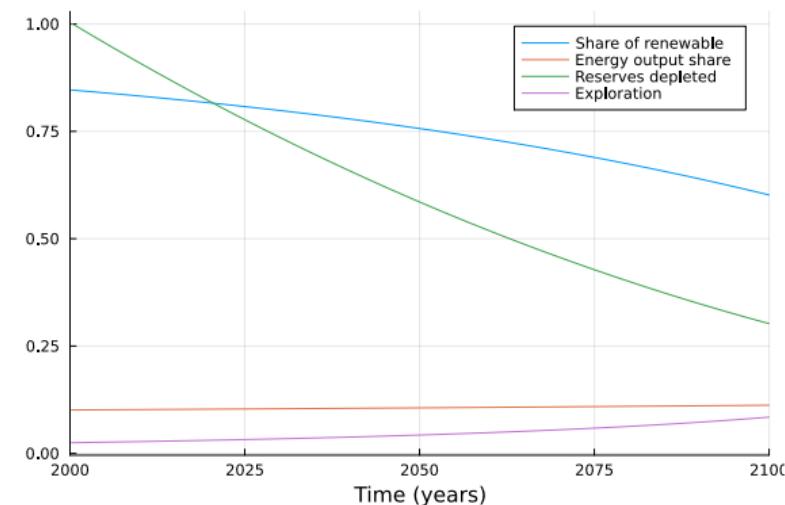
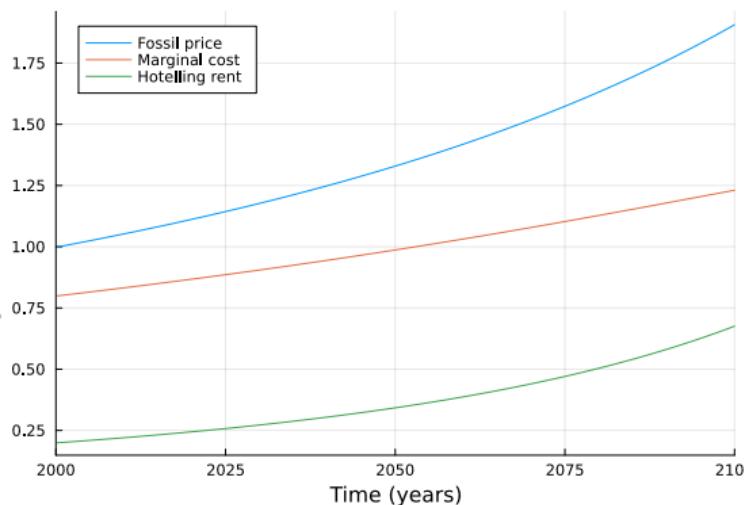
- ▶ Data : 40 countries
- ▶ Temperature (of the largest city), GDP, energy, population
- ▶ Calibrate  $z$  to match the distribution of output per capita at steady state

Created with mapchart.net

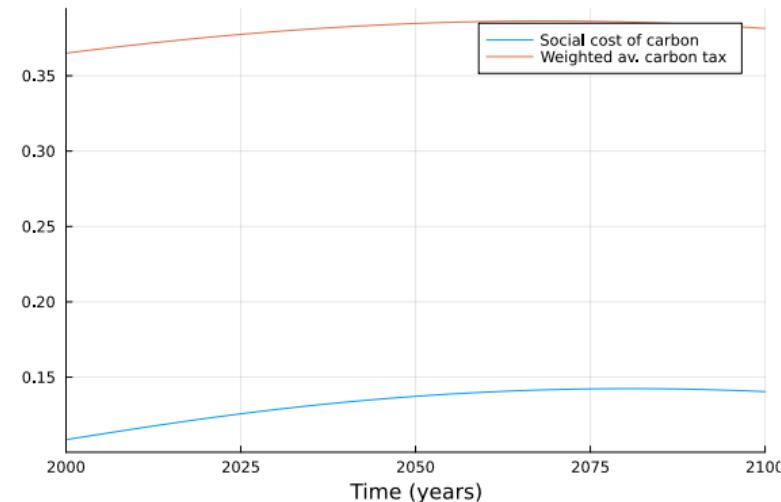
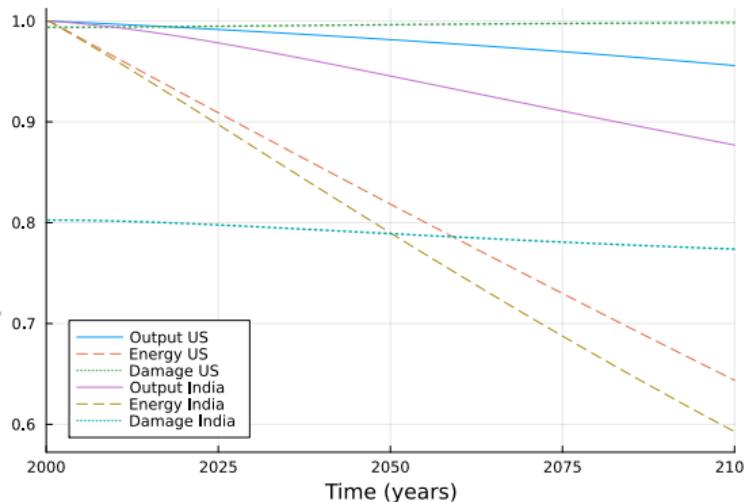
## Distribution of carbon prices :



# Energy



## Output and carbon price



## Extensions - 1 - Endogenous growth

- ▶ As of now, TFP  $z_t$  and directed technical change  $z_t^e$  are exogenous – growing at  $\bar{g}^y$  and  $g^{z^e}$
- ▶ Could easily nest an endogenous growth block in this model

- Household / firm in country  $i$  chooses an amount  $x_{i,t}$  of R&D to be allocated to increase TFP at rate  $\omega_t^z$  or energy technology (efficiency)
- Cost  $c(x_{i,t})$

$$\dot{z}_t = h^y(\omega_t^z x_t) \quad \dot{z}_t^e = h^e((1 - \omega_t^z)x_t)$$

- As a result, the marginal value of an investment in R&D is "priced" on the costates :

$$-\dot{\lambda}_t^z + \rho\lambda_t^z = \lambda_t^k \mathcal{D}(\tau_t) f(k_t, e_t) \quad \text{Recall : } y_t = z_t \mathcal{D}(\tau_t) f(k_t, e_t)$$

$$-\dot{\lambda}_t^{z^e} + \rho\lambda_t^{z^e} = \lambda_t^k z_t \mathcal{D}(\tau_t) \partial_{z^e} f(k_t, e_t)$$

- And optimal decisions depend on this shadow value ;

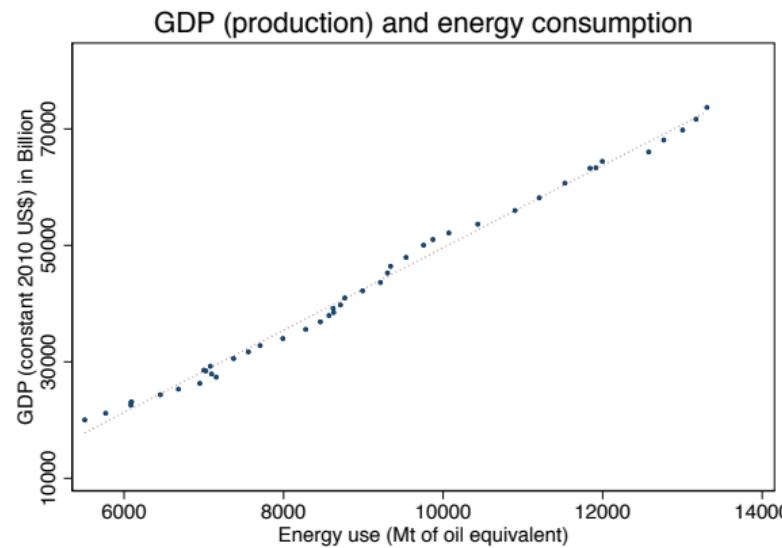
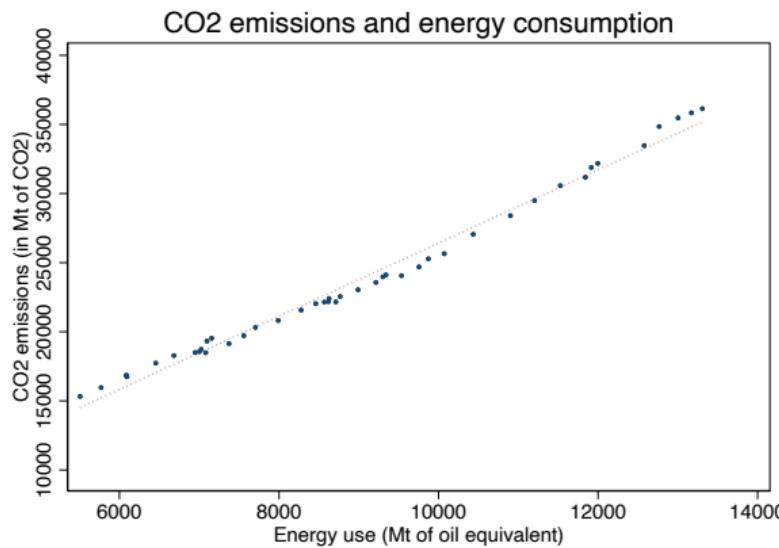
$$\omega_t^z h'^y(\omega_t^z x_t) \frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t}) \quad (1 - \omega_t^z) h'^e((1 - \omega_t^z)x_t) \frac{\lambda_t^{z^e}}{\lambda_t^k} = c'(x_{i,t})$$

## Conclusion

- ▶ Climate change is induced by externality
  - Energy/Emission choice doesn't include the impact on other countries
  - Cause strengthened by heterogeneity in wealth (capital/productivity)
  - Effect strengthened by heterogeneity in impact (temperature/damage)
- ▶ Social planner allocation correct for these different dimensions
  - Both Static correction  $\equiv$  modified Pigouvian carbon taxation
  - And dynamic : through the marginal value of states
- ▶ Future plans :
  - Simulation of the three equilibria  $CE/tax/SP$
  - Match the distribution of  $k$  using dynamics over 1960-2020
  - Social cost of carbon including heterogeneity and model uncertainty

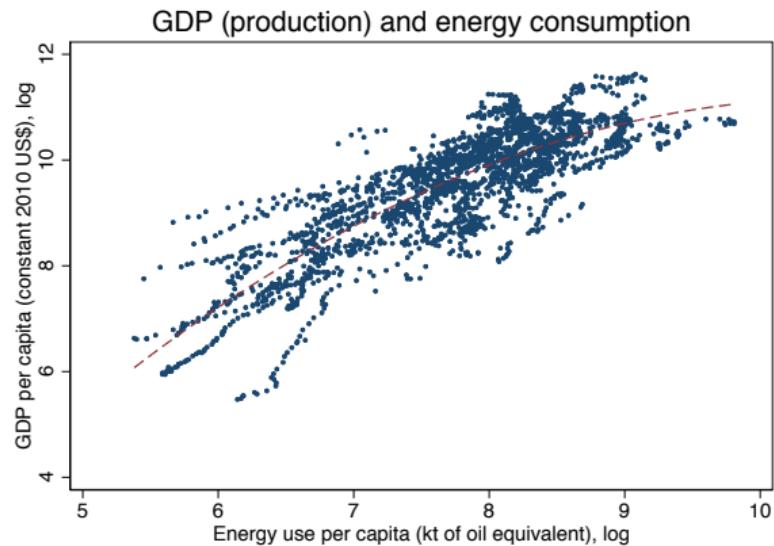
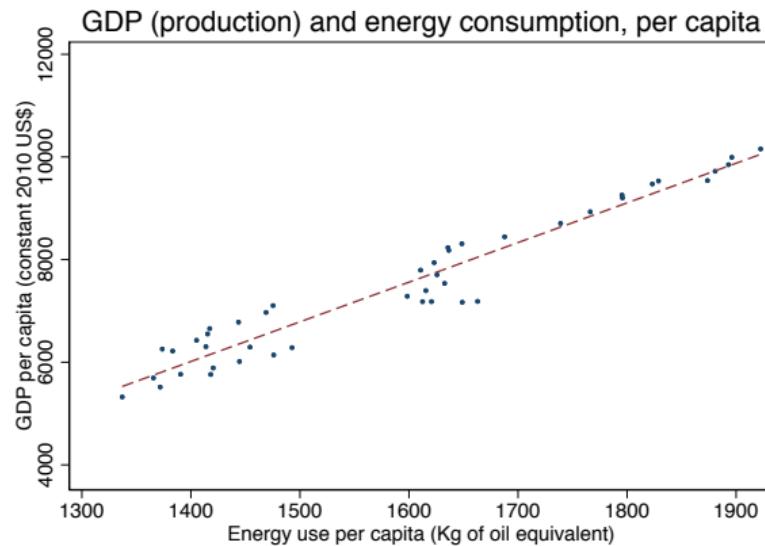
## Motivation

- ▶  $CO_2$  emissions correlate linearly with energy use
- ▶ Energy use (85% from fossils) correlates with output/growth



## Introduction – Motivation

- ▶ Also true per capita and for the trajectory of individual countries



## More details – Energy market

- Fossil fuel producer : price the Hotelling rent with the maximum principle :

$$\mathcal{H}^m(\mathcal{R}_t, \lambda_t^R, E_t, \mathcal{I}_t^e) = \pi_t(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) + \lambda_t^R (\delta^R \mathcal{I}_t^e - E_t)$$

- Rent  $\lambda_t^R$  grows with interest  $\rho$  and with the marginal gain of increasing reserves

$$\begin{aligned}\dot{\lambda}_t^R &= \rho \lambda_t^R - \partial_R \mathcal{C}(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) \\ &= \rho \lambda_t^R + \frac{\bar{\nu} \nu}{1 + \nu} \left( \frac{E_t^*}{R_t} \right)^{1+\nu} + \frac{\bar{\mu} \mu}{1 + \mu} \left( \frac{I_t^*}{R_t} \right)^{1+\mu} \\ \dot{\lambda}_t^R &= \rho \lambda_t^R + \frac{\bar{\nu}^{-1/\nu} \nu}{1 + \nu} (q^{ef} - \lambda_t^R)^{1+1/\nu} + \frac{\bar{\mu}^{-1/\mu} \mu}{1 + \mu} (\delta^R \lambda_t^R)^{1+1/\mu}\end{aligned}$$

- Because of decreasing return to scale and Hotelling rents : profits are  $> 0$

$$\pi_t(E_t^f, \mathcal{R}_t, \lambda_t^R) = \frac{1 + \nu - \bar{\nu}}{1 + \nu} \left( \frac{E_t^f}{\mathcal{R}_t} \right)^{1+\nu} \mathcal{R}_t + \lambda_t^R E_t^f - \frac{\bar{\mu}^{-1/\mu}}{1 + \mu} (\delta^r \lambda_t^R)^{1+\frac{1}{\mu}}$$

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# More details – PMP – Competitive equilibrium

- ▶ Household problem : State variables  $s_{i,t} = (k_i, \tau_i, z_i, p_i, \Delta_i)$
- ▶ Pontryagin Maximum Principle

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$$\begin{aligned} \mathcal{H}^{hh}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}) &= u(c_i, \tau_i) + \lambda_{i,t}^k \left( \mathcal{D}(\tau_{it}) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - q_t^f e_{it}^f - q_{it}^r e_{it}^r - c_t \right) \\ [c_t] \quad u'(c_{it}) &= \lambda_{i,t}^k \end{aligned}$$

$$[e_t^f] \quad MPe_{it}^f = \mathcal{D}(\tau_{i,t}) z \partial_e f(k_{i,t}, e_{i,t}) \left( \frac{e_{i,t}^f}{\omega e_{i,t}} \right)^{-\frac{1}{\sigma_e}} = q_t^f$$

$$[e_t^r] \quad MPe_{it}^r = \mathcal{D}(\tau_{i,t}) z \partial_e f(k_{i,t}, e_{i,t}) \left( \frac{e_{i,t}^r}{(1-\omega)e_{i,t}} \right)^{-\frac{1}{\sigma_e}} = q_{it}^r$$

$$[k_t] \quad \dot{\lambda}_t^k = \lambda_t^k (\rho - \partial_k f(k_{i,t}, e_{i,t}))$$

- ▶ Fossil Energy Monopoly :

$$\mathcal{H}^m(\mathcal{R}_t, \lambda_t^R, E_t, \mathcal{I}_t^e) = \pi_t(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) + \lambda_t^R (\delta^R \mathcal{I}_t^e - E_t)$$

$$[\mathcal{R}_t] \quad \dot{\lambda}_t^R = \rho \lambda_t^R + \frac{\bar{\nu} \nu}{1+\nu} \left( \frac{E_t^*}{R_t} \right)^{1+\nu} + \frac{\bar{\mu} \mu}{1+\mu} \left( \frac{I_t^*}{R_t} \right)^{1+\mu}$$

$$[E_t^f] \quad q_t^{e,f} = \nu_E(E, \mathcal{R}) + \lambda_t^R = \bar{\nu} \left( \frac{E_t}{\mathcal{R}_t} \right)^\nu + \lambda_t^R$$

$$[\mathcal{I}_t] \quad \lambda_t^R \delta^R = \mu_I(I_t, R_t) = \bar{\mu} \left( \frac{I_t}{\mathcal{R}_t} \right)^\mu \quad I_t = R_t \left( \frac{\lambda_t^R \delta}{\bar{\mu}} \right)^{1/\mu}$$

## More details – PMP – Ramsey Optimal Allocation

- ▶ Hamiltonian :

$$\begin{aligned}
 \mathcal{H}^{sp}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}, \{\psi\}) = & \int_{\mathbb{I}} \omega_i \mathcal{D}^u(\tau_{it}) u(c_i) p_i di + \\
 & + \psi_{i,t}^k \left( \mathcal{D}(\tau_{it}) f(k_t, e_t) - (n + \bar{g} + \delta) k_t + \theta_i \pi(E_t^f, \mathcal{I}_t, \mathcal{R}_t) - q_t^f e_{it}^f - q_{it}^r e_{it}^r - c_t \right) \\
 & + \psi_t^s \left( \mathcal{E}_t - \delta^s \mathcal{S}_t \right) + \psi_{it}^\tau \zeta \left( \Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{i0}) \right) + \psi_{it}^{\mathcal{R}} \left( -E_t^f + \delta^R \mathcal{I}_t \right) \\
 & + \psi_{i,t}^{\lambda k} \left( \lambda_t^k (\rho - r_t) \right) + \psi_t^{\lambda R} \left( \rho \lambda_t^R + \mathcal{C}_{\mathcal{R}}^f(E_t^f, \mathcal{I}_t, \mathcal{R}_t) \right) \\
 & + \phi_{it}^c \left( \mathcal{D}^u(\tau_{it}) u'(c_i) - \lambda_{it}^k \right) + \phi_{it}^{ef} \left( MPe_{it}^f - q_t^f \right) + \phi_{it}^r \left( MPe_{it}^r - q_{it}^r \right) \\
 & + \phi_t^{Ef} \left( q_t^f - \mathcal{C}_E^f(\cdot) - \lambda_t^{\mathcal{R}} \right) + \phi_t^{\mathcal{I}f} \left( \delta \lambda_t^{\mathcal{R}} - \mathcal{C}_{\mathcal{I}}^f(\cdot) \right)
 \end{aligned}$$

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# Ramsey Optimal Allocation - FOCs

## ► FOCs

$$[c_{it}] \quad \psi_{it}^k = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{\text{=direct effect}} + \underbrace{\phi_{it}^c \mathcal{D}^u(\tau_{it}) u''(c_i)}_{\text{=effect on savings}}$$

Define :  $\widehat{\phi}_{it}^e = \phi_{it}^f MPe_t^f + \phi_{it}^r MPe_t^r$

$$[e_{it}^f] \quad \psi_{i,t}^k \left( MPe_{it}^f - q_t^f \right) + \xi_{it} p_i \psi_t^S + p_i \partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi_{jt}^k dj + \partial_{e^f} \widehat{\phi}_{it}^e - p_i \phi_t^{Ef} \partial_{EE} \mathcal{C}(\cdot) = 0$$

$$[e_{it}^r] \quad \psi_{i,t}^k \left( MPe_{it}^r - q_{it}^r \right) + \partial_{e^r} \widehat{\phi}_{it}^e = 0$$

$$[\mathcal{I}_t] \quad \delta \psi_t^{\mathcal{R}} + \partial_{\mathcal{R}\mathcal{I}}^2 \mathcal{C}(\cdot) \psi_t^{\lambda, \mathcal{R}} - \phi_t^{\mathcal{I}} \partial_{\mathcal{I}\mathcal{I}}^2 \mathcal{C}(\cdot) = 0$$

$$[q_t^f] \quad \phi_t^{Ef} = \int_{\mathbb{I}} e_{it}^f \psi_{jt}^k dj + \int_{\mathbb{I}} \phi_{jt}^f dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi_{jt}^k dj$$

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## Ramsey Optimal Allocation - FOCs

- ▶ Backward equations for planner's costates

$$[k_i] \quad \dot{\psi}_{it}^k = \psi_{it}^k (\tilde{\rho} - r_{it} + \partial_k M P k_i) \psi_{it}^k - \partial_k \widehat{\phi}_{it}^e$$

$$[\mathcal{S}_i] \quad \dot{\psi}_t^{\mathcal{S}} = (\tilde{\rho} + \delta^s) \psi_t^{\mathcal{S}} - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi_{jt}^{\tau} dj$$

$$[\tau_i] \quad \dot{\psi}_t^{\tau} = (\tilde{\rho} + \zeta) \psi_t^{\tau} - \left( \omega_i \mathcal{D}'(\tau_{it}) u(c_{it}) + \psi_{it}^k \mathcal{D}'(\tau_{it}) f(k_{it}, e_{it}) + \phi_{it}^c \mathcal{D}'(\tau_{it}) u'(c_i) + \partial_{\tau} \widehat{\phi}_{it}^e \right)$$

$$[\mathcal{R}] \quad \dot{\psi}_t^{\mathcal{R}} = \psi_t^{\mathcal{R}} \left( \tilde{\rho} - \partial_{\mathcal{RR}}^2 \mathcal{C}(\cdot) \right) - \phi_t^{Ef} \partial_{\mathcal{RE}}^2 \mathcal{C}(\cdot)$$

$$[\lambda_i^k] \quad \dot{\psi}_t^{\lambda,k} = \tilde{\rho} \psi_t^{\lambda,k} - (\rho - r_{i,t}) \psi_t^k + \phi_{i,t}^c$$

$$[\lambda_i^{\mathcal{R}}] \quad \dot{\psi}_t^{\lambda,\mathcal{R}} = \psi_t^{\lambda,\mathcal{R}} (\tilde{\rho} - \rho) + \phi_t^{Ef} - \delta \phi_t^{\mathcal{If}}$$


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## Cost of carbon / Marginal value of temperature

- ▶ Solving for the cost of carbon and temperature  $\Leftrightarrow$  solving ODE

$$\begin{aligned}\dot{\lambda}_{i,t}^{\tau} &= \lambda_t^{\tau}(\tilde{\rho} + \Delta\zeta) + \gamma(\tau - \tau^*)\mathcal{D}^y(\tau)f(k, e)\lambda_i^k + \phi(\tau - \tau^*)\mathcal{D}^u(\tau)u(c) \\ \dot{\lambda}_t^S &= \lambda_t^S(\tilde{\rho} + \delta^s) - \int_{\mathbb{I}} \Delta_i \zeta \chi \lambda_{i,t}^{\tau}\end{aligned}$$

- ▶ Solving for  $\lambda_t^{\tau}$  and  $\lambda_t^S$ , in stationary equilibrium  $\dot{\lambda}_t^S = \dot{\lambda}_t^{\tau} = 0$

$$\begin{aligned}\lambda_{i,t}^{\tau} &= - \int_t^{\infty} e^{-(\tilde{\rho} + \zeta)u} (\tau_u - \tau^*) \left( \gamma \mathcal{D}^y(\tau_u) y_{\tau} \lambda_u^k + \phi \mathcal{D}^u(\tau_u) u(c_u) \right) du \\ \lambda_{i,t}^{\tau} &= - \frac{1}{\tilde{\rho} + \Delta\zeta} (\tau_{\infty} - \tau^*) \left( \gamma \mathcal{D}^y(\tau_{\infty}) y_{\infty} \lambda_{\infty}^k + \phi \mathcal{D}^u(\tau_{\infty}) u(c_{\infty}) \right) \\ \lambda_t^S &= - \int_t^{\infty} e^{-(\tilde{\rho} + \delta^s)u} \zeta \chi \int_{\mathbb{I}} \Delta_j \lambda_{j,u}^{\tau} dj du \\ &= \frac{1}{\tilde{\rho} + \delta^s} \zeta \chi \int_{\mathbb{I}} \Delta_j \lambda_{j,\infty}^{\tau} \\ &= - \frac{\chi}{\tilde{\rho} + \delta^s} \frac{\zeta}{\tilde{\rho} + \zeta} \int_{\mathbb{I}} \Delta_j (\tau_{j,\infty} - \tau^*) \left( \gamma \mathcal{D}^y(\tau_{j,\infty}) y_{j,\infty} \lambda_{j,\infty}^k + \phi \mathcal{D}^u(\tau_{j,\infty}) u(c_{j,\infty}) \right) dj \\ \lambda_t^S &\xrightarrow{\zeta \rightarrow \infty} - \frac{\chi}{\tilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_j (\tau_{j,\infty} - \tau^*) \left( \gamma \mathcal{D}^y(\tau_{j,\infty}) y_{j,\infty} \lambda_{j,\infty}^k + \mathcal{D}^u(\tau_{j,\infty}) u(c_{j,\infty}) \right) dj\end{aligned}$$

## Cost of carbon / Marginal value of temperature

► Closed form solution for CC :

- In stationary equilibrium :  $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
- Fast temperature adjustment  $\zeta \rightarrow \infty$
- no internalization of externality (business as usual)

$$LCC_{i,t} \equiv \frac{\Delta_i \chi}{\tilde{\rho} + \delta^s} (\tau_\infty - \tau^*) \left( \gamma \mathcal{D}^y(\tau_\infty) y_\infty + \phi \mathcal{D}^u(\tau_\infty) \frac{c_\infty}{1 - \eta} \right)$$

► Heterogeneity + uncertainty about models

## Uncertainty about models :

- ▶ In our model, we rely strongly on model specification :

- Parameters  $\theta$  of models :
  - Climate system and damages :  $(\xi, \chi, \zeta, \delta^s, \gamma, \phi)$
  - Economic model :  $\varepsilon, \nu, \bar{g}, n$  or extended :  $\omega, \sigma, \sigma^e, \nu, \mu$
  - Models with probability weight  $\pi(\theta)$
- Social cost of carbon, weighted for model uncertainty :

$$SCC_t(\theta) = - \int_{\mathbb{S}} \frac{\lambda_t^S(s, \theta)}{\lambda_t^k(s, \theta)} g(s) ds$$

$$S\bar{C}C_t = \int_{\Theta} SCC_t(\theta) \pi(\theta) d\theta = \int_{\Theta} \int_{\mathbb{S}} \frac{\lambda_t^S(s, \theta)}{\lambda_t^k(s, \theta)} g(s) ds \pi(\theta) d\theta$$

- Counterfactual computation of  $SCC$ 
  - Representative country / no uncertainty  $\frac{\lambda_t^S}{\lambda_t^k}$
  - With heterogeneous regions / no uncertainty  $SCC_t(\bar{\theta})$
  - No heterogeneity / model uncertainty  $\int_{\Theta} \frac{\lambda_t^S(\bar{s}, \theta)}{\lambda_t^k(\bar{s}, \theta)} \pi(\theta) d\theta$
  - With heterogeneous regions / with model uncertainty  $S\bar{C}C_t$

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## Long term temperature

- ▶ Integrating temperature dynamics :

$$\begin{aligned}\tau_{i,T} - \tau_{i,t_0} &= \Delta_i(\mathcal{T}_T - \mathcal{T}_{t_0}) = \Delta_i \chi \int_{t_0}^T e^{-\delta_s(T-t)} \mathcal{E}_t dt \\ &= \Delta_i \chi \xi \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e_{j,t}^f (1 - \vartheta_{j,t}) dj dt\end{aligned}$$

- Use CES demands :  $e_{j,t}^f = \omega e_{j,t} q_t^{-\sigma_e} q_t^{\sigma_e}$  for energy and  $e_t = (z z_t^e \mathcal{D}(\tau_{j,t})^{\sigma-1} q_t^{-\sigma})$
- Moreover, CES price index  $q_t = (\omega q_t^{f,1-\sigma_e} + (1-\omega)q_t^{r,1-\sigma_e})^{1/(1-\sigma_e)}$ , so first order approximation :  $g^q = \omega g^{q^f} + (1-\omega)g^{q^r}$  with growth for  $q^f$  and  $q^r$  as well as  $z_t^e = e^{g^e t}$
- Gives :

$$e_{j,t}^f = \omega q_t^{-\sigma_e} q_{j,t}^{\sigma_e} (z z_t^e \mathcal{D}(\tau_{j,t})^{\sigma-1} q_{j,t}^{-\sigma})$$

# Temperature dynamics

- ▶ Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e_{j,t}^f (1 - \vartheta_{j,t}) dj dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} q_t^{-\sigma_e} \int_{j \in \mathbb{I}} \omega(z z^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} e^{g^e t} q_{j,t}^{\sigma_e - \sigma} (1 - \vartheta_{j,t}) dj dt$$

$$\begin{aligned} \tau_{i,T} - \tau_{i,t_0} &= \Delta_i \chi \xi \omega \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} e^{(-\sigma_e + (\sigma_e - \sigma)\omega) g^f t} e^{(\sigma_e - \sigma)(1-\omega) g^r t} \\ &\quad \times \int_{j \in \mathbb{I}} (z z_t^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} y_{j,t} (1 - \vartheta_{j,t}) dj dt \end{aligned}$$

▶ back

## Social Planner allocation

- ▶ Solving the social planner allocation : Hamiltonian

$$\begin{aligned} \mathcal{H}^{sp}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}) = & \int_{\mathbb{I}} \omega_i u(c_i, \tau_i) p_i di - w L_t^f + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^k \left( \mathcal{D}(\tau_t) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - c_t \right) p_i di \\ & + \widehat{\lambda}_t^S \left( \int_{\mathbb{I}} \xi^f e_t^f p_i di - \delta^S \mathcal{S}_t \right) + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^\tau \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t) p_i di \\ & + \widehat{\lambda}_t^R \left( -E_t^f + \delta^R \mathcal{I}_t \right) + \widehat{\lambda}_t^{e^f} \left( \widetilde{\mathcal{F}}(L_t^f, \mathcal{R}_t) - E_t^f \right) + \int_{\mathbb{I}} \widehat{\lambda}_t^{e^r} \left( z_{i,t}^r k_{i,t}^{r,\alpha} - e_t^r \right) p_i di \end{aligned}$$

with  $E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_i di$  and  $e_t = \left( \omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e-1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r)^{\frac{\sigma^e-1}{\sigma^e}} \right)^{\frac{\sigma^e}{\sigma^e-1}}$

- ▶ Results :

$$\omega_i u_c(c_i, \tau_i) = \widehat{\lambda}_{i,t}^k$$

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S$$

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^r} = \widehat{\lambda}_t^{e^r}$$



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## Social Planner allocation

- ▶ Moreover, using Pontryagin principle, accounting for the distribution, we should have an adjustment for the state dynamics :
  - Valuation of the state changes when the planner knows if after the externality

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## Decentralization

- ▶ With inequality  $\widehat{\lambda}_{i,t}^k \neq \widehat{\lambda}_{j,t}^k$ , it's unclear how to decentralize
- ▶ Allowing lump sum transfer across countries solves world inequality  $\lambda_{i,t}^k = \lambda_{j,t}^k = \bar{\lambda}_t^k$ , as a result :

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{ef} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S \quad \Leftrightarrow \quad MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \left( \underbrace{\mathcal{C}'(E_t^f) + \lambda_t^R}_{=q_t^{e,f} = \frac{\widehat{\lambda}_t^{ef} + \widehat{\lambda}_t^R}{\bar{\lambda}_t^k}} + \underbrace{T_t^f}_{= \frac{\widehat{\lambda}_t^S}{\bar{\lambda}_t^k}} \right)$$

- ▶ The climate  $\mathbf{t}_t^f$  is flat rate accounting for the climate externality

$$\mathbf{t}_t^f = \frac{\widehat{\lambda}_t^S}{\bar{\lambda}_t^k} = \int_{\mathbb{I}} \frac{\lambda_{i,t}^S}{\bar{\lambda}_t^k} p_i di \neq CC_{i,t}$$

# Optimal abatement of emissions decisions

## 1. *Business as usual* :

- Abatement :

$$[\vartheta_t] \quad \partial_\vartheta \Lambda(\vartheta_{i,t}) e_{i,t}^f = 0 \quad \Rightarrow \vartheta_{i,t} = 0$$

# Optimal abatement of emissions decisions

## 1. *Business as usual :*

- Abatement :

$$[\vartheta_t] \quad \partial_\vartheta \Lambda(\vartheta_{i,t}) e_{i,t}^f = 0 \quad \Rightarrow \vartheta_{i,t} = 0$$

## 2. *Social planner :*

- Abatement :

$$[\vartheta_t] \quad \partial_\vartheta \Lambda(\vartheta_t^i) = \bar{\theta} (\vartheta_{i,t})^\theta = - \underbrace{\frac{1}{\lambda_{i,t}^k} \int_{\mathbb{I}} \omega_j \lambda_{j,t}^S p_j dj}_{=\text{carbon tax for } i}$$

## FBSDE for MFG systems – general formulation

- ▶ State  $X_t \equiv (a_t, z_t) \in \mathbb{X} \subset \mathbb{R}^d$  (possibly with state-constraints), and  $X$  diffusion process with control  $\alpha^\star(t, X, P_X, Y) \equiv c_t^\star$

$$dX_t = b(X_t, P_{X_t}, \alpha_t^\star) dt + \sigma dB_t$$

- ▶ Set up the Hamiltonian :

$$\mathcal{H}(t, x, P_X, y) = \max_{\alpha \in \mathbb{A}} (u(\alpha) + b(x, P_X, \alpha) \cdot y)$$

- ▶ Optimal control  $c^\star \in \operatorname{argmax}_{\alpha \in \mathbb{A}} (u(\alpha) + b(x, P_X, \alpha) \cdot y)$
- ▶ Using the Pontryagin maximum principle :

$$dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \tilde{Z}_t dB_t$$

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## FBSDE system for MFG

- ▶ Coupled FBSDE system for each agent

$$\begin{cases} dX_t &= D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t \\ dY_t &= -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \tilde{Z}_t dB_t \end{cases}$$

- ▶ Question : What else do we need ?

## FBSDE system for MFG

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- ▶ Question : What else do we need ?

- The risk loading in the costate  $\tilde{Z}_t$  :
  - ▶ Intuitions : expectation error in the law of motion of  $Y_t$

$$\tilde{Z}_t(x) = \mathbb{E} \left[ \frac{dY_t + D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt}{dB_t^0} \right]$$

- ▶ BSDE theory : keep the co-state measurable w.r.t.  $dB_t$ , despite running backward)
- ▶ Can compute that by Monte Carlo

## FBSDE system for MFG

- ▶ Coupled FBSDE system for each agent

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- The initial condition  $Y_0$  as a function of  $X_0$ 
  - ▶ BSDE theory : keep the co-state measurable w.r.t.  $dB_t$ , despite running backward)
  - ▶ Can compute that by Monte Carlo

## FBSDE system for MFG

- ▶ Coupled FBSDE system for each agent

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- ▶ Question : What else do we need ?

- The risk loading in the costate  $\tilde{Z}_t$  :
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$$\tilde{Z}_t(x) = \mathbb{E} \left[ \frac{dY_t + D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt}{dB_t^0} \right]$$

- BSDE theory : keep the co-state measurable w.r.t.  $dB_t$ , despite running backward)
  - ▶ Can compute that by Monte Carlo
- The initial condition  $Y_0$  as a function of  $X_0$
- A boundary condition of  $Y_T$  or transversality  $\lim_{t \rightarrow \infty} e^{-\rho t} X_t Y_t = 0$

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## Social planner : Mean Field Control/McKean Vlasov

- ▶ Suppose now that the social planner controls the dynamic of each agents ... *accounting for its effect on the distribution*
- ▶ Get an additional term :

$$\begin{cases} dX_t = D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t \\ dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + Z_t dB_t - \tilde{\mathbb{E}}[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)] \end{cases}$$

- ▶ Effect on the distribution :

## Social planner : Mean Field Control/McKean Vlasov

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- ▶ Effect on the distribution :
  - Social planner internalize the externality coming from heterogeneity
    - ▶  $D_\mu H$  is the L-derivative w.r.t the measure  $\mu \equiv \mathbb{P}_{X_t}$
    - ▶ Idea : lifting of the function  $H(x, \mu) = \hat{H}(x, \hat{X})$  where  $\hat{X} \sim \mu$  and hence  $D_\mu H(x, \mu)(\hat{X}) = D_{\hat{x}} \hat{H}(x, \hat{X})$
    - ▶ Intuition : shift the distribution of states  $\hat{X}$  for all agents
    - ▶ Probabilistic approach : easy to compute  $\tilde{\mathbb{E}}[D_\mu H(\tilde{X}_t, \mu)] = \tilde{\mathbb{E}}[D_{\hat{x}} \hat{H}(\tilde{X}_t, \hat{X})]$
  - Here : the effect is homogeneous for all agents : the interaction with the measure is non-local !

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## Sequential approach FBSDE w/ Agg. shocks

- ▶ Coupled FBSDE system for each agent

$$\begin{cases} dX_t &= D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t + \tilde{\sigma} dB_t^0 \\ dY_t &= -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + Z_t dB_t + \tilde{Z}_t^0 dB_t^0 \end{cases}$$

- ▶ Question : What else do we need ?

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- ▶ Question : What else do we need ?
  - The risk loading on idiosyncratic shocks  $\tilde{Z}_t$  :
  - The risk loading on aggregate shocks  $\tilde{Z}_t^0$  :

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- ▶ Question : What else do we need ?

- The risk loading on idiosyncratic shocks  $\tilde{Z}_t$  :
- The risk loading on aggregate shocks  $\tilde{Z}_t^0$  :
- The initial conditions  $Y_0(X_0)$  and boundary condition on  $Y_T$  or transversality  
 $\lim_{t \rightarrow \infty} e^{-\rho t} X_t Y_t = 0$

## MFG system : Recursive approach w/ Agg. shocks

- ▶ Here : recursive w.r.t. idiosyncratic shocks, but sequential w.r.t. aggregate shocks.
- ▶ System for  $v$  and  $g$  :

$$\begin{aligned} -\partial_t v + \rho v &= \max_{\alpha} u(\alpha) + \mathcal{A}(v)v + Z_t^0 dB_t^0 \\ \partial_t g &= \mathcal{A}^*(v)g + \partial_x[\sigma g]dB_t^0 \end{aligned}$$

- ▶ Solve the PDE system :
  - Finite difference, upwinding scheme
  - View that as a non-linear system : use Quasi Newton methods
  - New part : forcing terms  $\partial_x[\sigma g]dB_t^0$  and  $Z_t^0 dB_t^0$
  - Initial and terminal conditions

$$v_T = v^\infty \quad g_0 = g^\infty$$

- ▶ Direct effect of uncertainty on measure
- ▶ Indirect effect through agent expectations : shadow price of aggregate risk  $Z_t^0$