The Inequality of Climate Change

Heterogeneity, optimal policy and uncertainty

PRELIMINARY – WORK IN PROGRESS

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Abstract

Climate change, caused by rising global temperatures, will disproportionately affect developing economies while benefiting developed countries responsible for significant greenhouse gas emissions. Through the lens of an Integrated Assessment Model with heterogeneous countries, I analyze the costs of global warming and the trade-off of reducing emissions. I provide characterization for the carbon price, the nature of externalities, and optimal policy in this framework. The findings strongly depend on energy markets characteristics and technology path. To overcome the curse of dimensionality in this heterogeneous agents model with risk, I propose a new numerical method relying on the sequential formulation to simulate the model globally, design optimal policy and handle aggregate uncertainty.

1 Introduction

The climate is warming due to greenhouse gas emissions generated by economic activity from different countries

- ***Unequal causes:*** Developed economies account for over 65% of cumulative GHG emissions ($\sim$ 25% each for the EU and the US)

- ***Unequal consequences:*** Increase in temperatures will disproportionately affect developing countries where the climate is already warm

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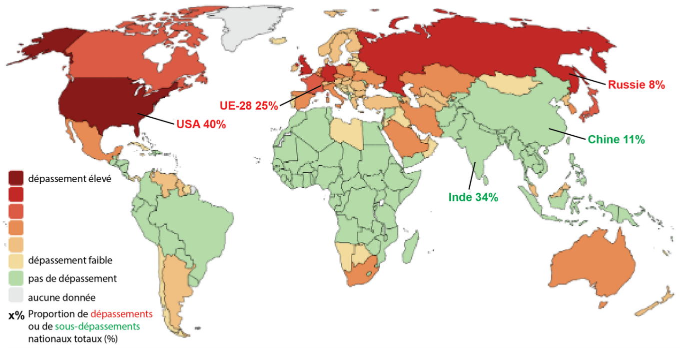
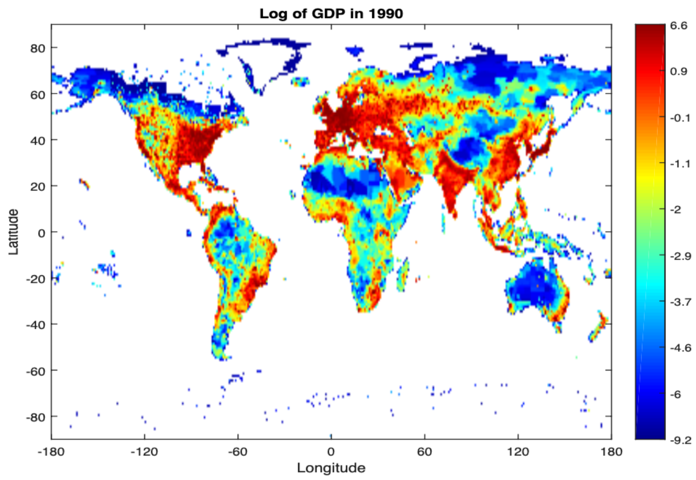
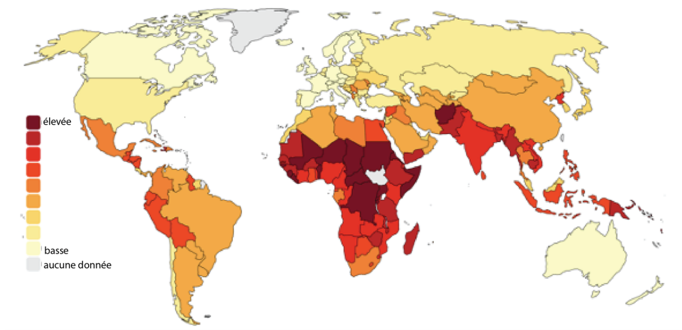
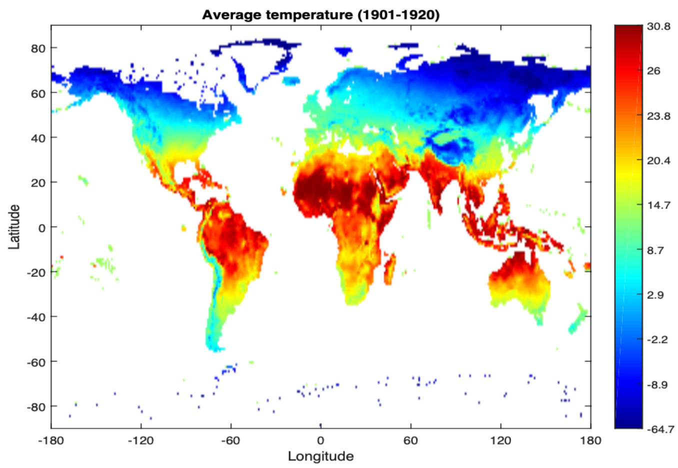


Source : Lancet planetary Health - Quantifying national responsibility for climate breakdown: an equality-based attribution approach for carbon dioxide emissions in excess of the planetary boundary - Jason Hickel 2020





Source : Notre Dame Global Adaptation Initiative



Which countries will be affected the most by climate change?

- Is the price of carbon heterogeneous across regions? and why?
- What is the optimal policy in presence of externalities *and* heterogeneities?

Develop a simple and flexible model of climate economics

- Standard NCG – IAM model with heterogeneous regions
- Every country is small relative to global GHG – no incentives to curb emissions

Results:

- Evaluate the heterogeneous welfare costs of global warming
- Provide analytical formulas and a numerical methodology to compute the cost of carbon
  - CC is linear in GDP/level of development and in temperature gaps
- Solve world optimal carbon policy with heterogeneous regions
  - Does the optimal carbon tax coincide with the social cost of carbon?

Classic Integrated Assessment models (IAM) :

- Nordhaus' Multi-regions DICE (2016), Golosov, Hassler, Krusell, Tsyvinski (2014)
- Dietz, van der Ploeg, Rezai, Venmans (2021), among others

Macro (IAM) model with country heterogeneity:

- Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021), among others
- *This paper:* Studies the optimal policy with heterogeneity and externalities

Climate model with risk & uncertainty:

- Cai, Lontzek, Judd (2019), Barnett, Brock and Hansen (2022)
- *This paper:* Includes heterogeneity and redistribution effects of climate & carbon taxation

Quantitative spatial models:

- Cruz, Rossi-Hansberg (2021), Bilal, Rossi-Hansberg (2023), Rudik et al (2022)
- *This paper:* Considers forward-looking decision of agents & optimal policy

Heterogeneous Agents models with optimal policy

- Le Grand, Ragot (2018-), Davila, Schaab (2022), Bhandari Evans Golosov Sargent (2018-)
- *This paper:* Studies climate externalities and Pigouvian taxation

2 Toy model

In this section, we develop the simplest version of the quantitative model covered in the next section. The goal is to provide intuitions on the effects of heterogeneity across countries, the source of climate externality related to energy markets, and the implementation of optimal policy.

The model is static and all the decisions are taken in one period. Consider two countries $i = N, S$, for *North* and *South*, symmetric in all regards, except for differences in productivity z_i. We consider a wide definition of z_i as productivity residuals that can accounts for technology, efficiency, and market frictions as well as institutions. A unique household in each country consumes the good c_i that is produced with energy e_i with the production function $y_i = F(e_i)$.[1] Moreover, in this world, outside of the two countries, there is an energy producer producing energy at cost. It sells this energy input at price q^e to both countries. Due to decreasing return to scale, this competitive firm still makes profit $\pi(E)$. This producer is owned by country i with share θ_i, and the profit are redistributed according to this ownership share.

The Household maximization problem is the following:

$$\max_{c_i, e_i} U(c_i)$$
$$c_i + q^e e_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) \qquad [\lambda_i^k]$$

where the production function $F(e_i)$ is increasing and concave in e_i, i.e. $F'(e) > 0$ and $F''(e) < 0$ and features Inada conditions.

Both countries are subject to climate damages $\mathcal{D}_i(\mathcal{S})$ caused by climate externalities related to energy consumption e_i:

$$\mathcal{S} = \mathcal{S}_0 + \overbrace{\xi_S e_S + \xi_N e_N}^{=\text{GHG emissions}}$$

where ξ_i is the conversion factor between energy use e_i in physical units – e.g. in Joule, Tons Oil Equivalent, kWh or Thermal units – and emissions measured in Tons of Carbon or CO_2. This obviously depends on the energy mix between fossil fuels used for energy and renewables. However, this is taken as given in the short run in our static equilibrium. The quantitative model introduces this endogenous channel of energy choice.

The global carbon emission stock is not internalized by households in their energy consumption decision leading to damage $\mathcal{D}_i(\mathcal{S})$ that affects country i's effective productivity, as in standard Integrated Assessment models, e.g. Nordhaus DICE models.

As each household consumes energy in a single international market, where the energy price

[1]Note that one could make the model slightly more general by considering continents of differing populations $p_N \neq p_S$. One could also add additional inputs in the production function, for example, capital k_i or labor ℓ_i, and make the endowment of these inputs vary across locations. As we will show in the quantitative model, these features do not change the qualitative implication of this framework.

q^e is set such as to clear the supply and demand.

$$E = e_N + e_S$$

The energy supply E – for example oil and gas extraction or nuclear, solar, and wind power – is provided by a single energy producer maximizing its profit, subject to convex cost $c(E)$, i.e. $c'(E) > 0$ and $c''(E) > 0$

$$\max_E q^e E - c(E)$$
$$\Rightarrow \qquad q^e = c'(E) \qquad \& \qquad \pi(E) := c'(E)E - c(E)$$

The Competitive Equilibrium is a system of price q^e and allocation $\{c_i, e_i\}_i$ such that (i) the Household maximizes utility, i.e. chooses c_i and e_i to maximize utility and (ii) the energy producers choose production E to maximize profit, and market clear $E = e_N + e_S$.

The competitive equilibrium results in the following optimality conditions, first for consumption :

$$\lambda_i^k = U'(c_i) \qquad \textit{with} \qquad c_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) - q^e e_i$$

where λ_i^k represents the marginal value of wealth, i.e. the marginal utility of consumption. The second optimality for energy use for production writes as follow:

$$MPe_i = q^e \qquad \textit{with} \qquad MPe_i := \mathcal{D}_i(\mathcal{S}) z_i \partial_e F(e_i)$$

This corresponds to the standard tradeoff Marginal Product of Energy = Energy Price.

For illustration purposes, we assume that the North is richer, having access to superior technology $z_N > z_S$ and higher production and consumption $c_N > c_S$ for a given price q^e of energy, and that the South is subject to higher damages $\mathcal{D}_S(\mathcal{S}) < \mathcal{D}_N(\mathcal{S})$ for all $\mathcal{S}$ the stock of carbon emissions[2].

This competitive equilibrium is inefficient for several reasons: (i) economic inequality results from the heterogeneity in productivity and climate damage: since $c_N > c_S$ we have $\lambda_S^k > \lambda_N^k$, and redistribution from the North to the South would be desirable from a utilitarian point of view. Here, an important friction we consider is trade autarky and lack of redistribution across countries: production in one country can not be exported or transferred to another country.

In addition, (ii) climate damages $\mathcal{D}_i(\mathcal{S})$ are not internalized, and energy consumption might be too high depending on the economic cost of global warming $\mathcal{D}_i(\mathcal{S})$.

Lastly (iii) the redistribution of the energy rent $\pi(E)$ is not internalized either: choosing e_i

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[2] Indeed, assuming $F(e)$ is Cobb Douglas $F(e) = \bar{k}^{1-\alpha} e^\alpha$, with $\bar{k} = 1$, we obtain $\alpha \mathcal{D}_i(\mathcal{S}) z_i e_i^{\alpha-1} = q^e$ leading to

$$e_i = \big(\alpha \mathcal{D}_i(\mathcal{S}) z_i / q^e\big)^{1/(1-\alpha)} \qquad y_i - q^e e_i = \big(\mathcal{D}_i(\mathcal{S}) z_i\big)^{1/(1-\alpha)} q^e\big)^{-\alpha/(1-\alpha)} \big[\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\big]$$

which is increasing in z_i and $\mathcal{D}_i(\mathcal{S})$.

affects the amounts of profit the energy firms make and redistributes as share θ_i to households.

We explore how the Ramsey planner would allocate consumption and energy in such an environment.

2.1 Comparison with Ramsey Problem:

Consider a Social Planner who could take the decisions instead of the agents, subject to the same frictions – climate externality and the absence of financial instruments for transfers across countries.

Moreover, it would maximize the aggregate welfare with weights ω_i for each country, choosing the allocations, subject to the optimality conditions of the Households – consumptions and energy choice – and the firm on energy production.

$$\mathbb{W} = \max_{\{c_i, e_i,\}_i, q^e} \sum_{i=N,S} \omega_i U(c_i)$$

$$\begin{aligned}
s.t \qquad c_i + q^e e_i &= \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + \theta_i \pi(E) && [\phi_i^k] \qquad \forall\, i = N, S \\
\lambda_i^k &= U'(c_i) && [\phi_i^c] \\
MPe_i &= q^e && [\phi_i^e] \\
q^e &= c'(E) && [\phi^E] \\
\mathcal{S} &:= \mathcal{S}_0 + \xi_N e_N + \xi_S e_S && E := e_N + e_S
\end{aligned}$$

The optimality conditions of the agents are internalized by the Ramsey planner in its choice of allocation: they become implementability constraints. This approach is reminiscent of the primal approach in the optimal taxation literature [Cite Atkeson Stiglitz] Moreover, the Lagrange Multipliers represent the Social Value of relaxing the budgets and optimality constraints. Optimality of the planner give us an expression of their intuitions:

First, ϕ_i^k is the social value of consumption for household in country i.

$$\phi_i^k = \omega_i U'(c_i) \qquad [c_i]$$

and represent how valuable a unit of wealth is in the country i, similar to the competitive equilibrium allocation except for the Pareto weights ω_i that the planner takes into account. Note that there is no indirect effect related to the consumption/saving decision that the planner would need to internalize since the model is static. Hence $\phi_i^c = 0$ here and $\phi_i^k \omega_i U'(c_i) \lesseqgtr U'(c_i)$.

Second, the choice of energy used e_i relates the social value of wealth ϕ_i^k of the different countries j through the impact on (i) the climate damage, (ii) the energy profit distribution and (iii) the impact on the energy market. We define the net production function $\tilde{F}(\mathcal{S}, e_i) = \mathcal{D}_j(\mathcal{S}) y_j =$

$\mathcal{D}_j(\mathcal{S})z_iF(e_i)$ and the energy profit $\pi(q^e, E) = q^eE - c(E)$, the optimality for energy writes:

$$\phi_i^k\Big(\partial_e\tilde{F}(\mathcal{S}, e_i) - q^e\Big) + \xi_i \underbrace{\sum_j \phi_j^k \mathcal{D}_j'(\mathcal{S})y_j}_{\propto-\text{social cost of carbon}} + \underbrace{\partial_E\pi(q^e, E)\sum_j \theta_j\phi_j^k}_{=\text{energy rent redistribution}}$$
$$- \underbrace{\phi^E c''(E)}_{=\text{effect on energy supply}} + \underbrace{\phi_i^e\,\partial_{ee}^2\tilde{F}(\mathcal{S}, e_i)}_{=\text{effect on energy choice}} \qquad [e_i]$$

We detail the intuition of these terms in turn. First, the climate externality is internalized in the term measuring the social cost of carbon. It accounts for the marginal change in welfare due to change in climate, rescaled by the marginal value of wealth. However, in a context where inequality are persistent, we do not have $\phi_N^k = \phi_S^k$. In such case, the planner would use the average marginal value $\partial\mathbb{W}/\partial c = \overline{\phi}^k = \frac{1}{2}\sum_j\phi_j^k = \frac{1}{2}\sum_j \omega_jU'(c_j)$. In such case, the (positive) SCC writes:

$$\begin{aligned}
SCC := -\frac{\partial\mathbb{W}/\partial\mathcal{S}}{\partial\mathbb{W}/\partial c} &= -\frac{1}{\overline{\phi}^k}\sum_j \phi_j^k\mathcal{D}_j'(\mathcal{S})y_j \\
&= -\mathbb{E}_j\left(\frac{\phi_j^k}{\overline{\phi}^k}\mathcal{D}_j'(\mathcal{S})y_j\right) \\
&= -\mathbb{C}\text{ov}_j\Big(\frac{\omega_jU'(c_j)}{\frac{1}{2}\sum_j\omega_jU'(c_j)}, \mathcal{D}_j'(\mathcal{S})y_j\Big) - \mathbb{E}_j[\mathcal{D}_j'(\mathcal{S})y_j] > -\mathbb{E}_j[\mathcal{D}_j'(\mathcal{S})y_j] = \overline{SCC}
\end{aligned}$$

where the expectation and covariances are empirical moments over the set of country $j = N, S$ and the last inequality comes from the assumption that $c_N > c_S$ as well as $\mathcal{D}_S'(\mathcal{S}) > \mathcal{D}_N'(\mathcal{S})$. In a world with heterogeneity and wealth inequality, we observe that the Social Cost of Carbon is exacerbated by the positive correlation between inequality in economic outcomes and climate damage across countries.

This term is positive and needs to be taken into consideration by the planner, reducing the energy choice. This Pigouvian taxation term is amplified by heterogeneity: the average marginal damage is greater than the damage of an "average" representative country: $SCC > -\mathbb{E}_j[\mathcal{D}_j'(\mathcal{S})y_j]$.

Second, the second term is related to the energy rent distribution, or social value of rent (SVR). Following the same logic, the planner accounts for this term by weighting it by the average marginal value of wealth:

$$\begin{aligned}
SVR &= \frac{1}{\overline{\phi}^k}\partial_E\pi(q^e, E)\sum_j \theta_j\phi_j^k \\
&= \partial_E\pi(q^e, E)\,\mathbb{C}\text{ov}_j\Big(\frac{\omega_jU'(c_j)}{\frac{1}{2}\sum_j\omega_jU'(c_j)}, \theta_j\Big) + \partial_E\pi(q^e, E)\underbrace{\mathbb{E}_j[\theta_j]}_{=1} < \partial_E\pi(q^e, E)
\end{aligned}$$

Here, we make the assumption – as is true in reality – that the energy rent correlates with income / development of the country – for example Gulf countries, United States or Russia being richer than the world average: $\theta_N > \theta_S$. That term is negative and could act as a subsidy on energy spending, but because $SVR < \partial_E \pi(q^e, E)$ its quantitative importance is smaller and can be even ambiguous in the data as we will see below.

Third, the choice of energy of country i relates to the social price of energy and its weight on the world energy markets:

$$-\phi^E c''(E) < 0$$

since the cost function of energy extraction and production is convex: the more curved the cost is, the less the planner is willing to push the production. Moreover, this term is greater with the social shadow price of energy ϕ^E that relates to the country's energy use e_i, individual valuation of energy ϕ^e_j and marginal value of wealth ϕ^k_j, becoming apparent when finding the optimality for energy price q^e in the planner's problem. As the social cost of carbon and the social value of rent, we normalize the values of energy by social value of wealth: $\widetilde{\phi}^E = \phi^E / \overline{\phi}^k$ and $\widetilde{\phi}^e = \phi^e / \overline{\phi}^k$. The social value of energy is thus:

$$\widetilde{\phi}^E = \sum_j e_j \frac{\phi^k_j}{\overline{\phi}^k} + \sum_j \widetilde{\phi}^e_j - \partial_q \pi(q^e, E) \sum_j \theta_j \frac{\phi^k_j}{\overline{\phi}^k} \qquad [q^e]$$

where here again, the covariance between energy use and marginal value of wealth ϕ^k_j increase the social value of energy. In practice this term is small, since high energy consuming countries are also the ones consuming high levels of consumption. Hence the term $-\phi^E c''(E)$ is negative, but small, and acts like a slight tax on the energy choice, reducing its use.

Fourth and last, the choice of energy relates to the individual characteristics of the production function and its curvature in terms of energy use.

$$\phi^e_i \, \partial^2_{ee} \tilde{F}(\mathcal{S}, e_i) > 0$$

The planner accounts for the fact that energy could be hard to substitute in production, generating high curvature when $e_i \to 0$.

Decentralization and First Best

The choice of decentralization of this Ramsey equilibrium depends on the number of instruments available. If the social planner has access to perfect and costly lump-sum transfers and taxes, nothing prevents it to equalize marginal value across countries and solving world's inequality:

$$\omega_i U'(c_i) = \phi^k_i = \overline{\phi}^k = \tfrac{1}{2} \textstyle\sum_j \phi^k_j = \tfrac{1}{2} \textstyle\sum_j \omega_j U'(c_j) \qquad \forall i = N, S$$

As a result, the equilibrium becomes analogous to a representative agent economy and

$$SCC = -\mathbb{E}_j[\mathcal{D}'_j(\mathcal{S}) y_j] \qquad SVR = \partial_E \pi(q^e, E)$$

as identical to standard Pigouvian taxation terms. The Ramsey planner solves the two market frictions: inequality and autarky using lump-sum transfers and the other externalities with standard Pigouvian taxes. In environment where lump-sum transfers are not available, we will se that this result does not hold.

Decentralization without Lump-sum transfers

In the absence of lump-sum taxes, there is no transfer available to the social planner, and it would implement this allocation with distortive taxes. It

$$MPe_i = \partial_e \tilde{F}(\mathcal{S}, e_i) = q^e + \underbrace{\frac{\frac{1}{2}\sum_j \omega_j U'(c_j)}{\omega_i U'(c_i)}}_{=\text{ redistribution term}} \left[\xi_i SCC - SVR + \widetilde{\phi}^E c''(E) - \widetilde{\phi}_i^e \partial_{ee}^2 \tilde{F}\right] = q^e + \underbrace{\mathbf{t}_i^e}_{=\text{energy tax}}$$

where the first and third terms inside the brackets act as a tax – potentially large – and the second and fourth terms act as a subsidy – potentially small. The net effect will be determined quantitatively in the model below.

However, we see that the two objectives – the reduction of inequalities and the correction of externalities – are merged into a single tax instrument. The redistributive wedge $\mathbb{E}_k[\omega_k U'(c_k)]/\omega_i U'(c_i)$ is small for low-income countries: the Pigouvian tax is less of a burden for them since energy is used in production to allow for high opportunity of development and consumption. In the contrary, the Pigouvian tax is amplified significantly for rich countries. We will see that this principle is general and hold in the realistic quantitative model as well. Before that, let us turn toward integrating risk in such a framework.

2.2 Effect of uncertainty

We consider risks related to both (i) economic growth, such that productivity $z_i(\epsilon_z)$ is uncertain, and (ii) temperature and climate damage $\mathcal{D}_i(\mathcal{S}|\epsilon_d)$. The probability distribution is general and writes $(\epsilon_z, \epsilon_d) =: \boldsymbol{\epsilon} \sim \varphi(\boldsymbol{\epsilon})$

Keeping the model static, the timing is as follows: the household and planners take energy decisions ex-ante before the shock is realized. Since the only channel through which production and emissions are affected is energy demand, this justifies the fact that energy choices face uncertainty. Consumption is chosen ex-post and results simply from the outcomes in temperature and productivity.

The household and planners' problems become

$$\max_{e_i} \int_{\mathcal{E}} \max_{c_i(\boldsymbol{\epsilon})} U(c_i(\boldsymbol{\epsilon})) d\varphi(\boldsymbol{\epsilon}) \qquad \text{vs.} \qquad \max_{\{e_j\}_j} \int_{\mathcal{E}} \max_{\{c_j(\boldsymbol{\epsilon})\}_j} \sum_{j=N,S} \omega_j U(c_j(\boldsymbol{\epsilon})) d\varphi(\boldsymbol{\epsilon})$$

In the competitive equilibrium, there is almost no change in behavior. The tradeoff simply balances the expected marginal product of energy with the energy price

$$\int_{\mathcal{E}} MPe_{i(\epsilon)}\ d\varphi(\epsilon) = q^e$$

The Ramsey Planner, in the contrary, faces the following trade-off for the choice of energy:

$$\begin{aligned}\int_{\mathcal{E}} \frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]} \ \big(MPe_{i(\epsilon)} - q^e\big) d\varphi(\epsilon) = \int_{\mathcal{E}} SCC(\epsilon)\ d\varphi(\epsilon) - \int_{\mathcal{E}} SVR(\epsilon)\ d\varphi(\epsilon) \\ - \widetilde{\phi}^E c''(E) + \int_{\mathcal{E}} \widetilde{\phi}_i^e(\epsilon) \partial^2_{ee} \tilde{F}(\mathcal{S}, e_i|\epsilon) d\varphi(\epsilon)\end{aligned}$$

taking into account the effect of the shock in the expected damage, social value of rent, and energy in addition to the marginal product of energy. Rewriting and using expectations formulas, we obtain:

$$\begin{aligned}\mathbb{E}_{\epsilon}\big(MPe_{i(\epsilon)}\big) = q^e + \underbrace{\frac{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}{\mathbb{E}_{\epsilon}\big(\omega_i U'(c_j(\epsilon))\big)}}_{=\text{redistributive effect}} \Bigg[\underbrace{-\mathbb{C}\text{ov}_{\epsilon}\Big(\frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, MPe_{i(\epsilon)}\Big)}_{=\text{effect of agg. risk } \epsilon \text{ on energy choice}} \\ + \mathbb{E}_{\epsilon}\big[SCC(\epsilon)\big] - \mathbb{E}_{\epsilon}\big[SVR(\epsilon)\big] - \widetilde{\phi}^E c''(E) + \mathbb{E}_{\epsilon}\big[\widetilde{\phi}_i^e(\epsilon) \partial^2_{ee} \tilde{F}(\mathcal{S}, e_i|\epsilon)\big] \Bigg]\end{aligned}$$

This formula is more involved and includes multiple effects of uncertainty. The LHS displays the expected marginal product of energy, as in the competitive equilibrium case. In the RHS, a redistributive term introduces a wedge exactly like in the deterministic case, this time taken in expectations. However, the evaluation of the marginal product of energy is also subject to risk assessment, with covariance introducing an additional wedge:

$$-\mathbb{C}\text{ov}_{\epsilon}\Big(\frac{\omega_i U'(c_i(\epsilon))}{\mathbb{E}_{k,\epsilon}[\omega_k U'(c_k)]}, MPe_{i(\epsilon)}\Big) > 0$$

This wedge is positive in the environment we consider: productivity shocks and climate damages both reduce production and consumption, hence increasing the marginal value of wealth $\omega_i U'(c_i)$. At the same time, it depreciates the marginal product of the input e_i, causing a lower benefit in investing in production. This covariance is hence negative, causing the wedge to become a positive "precautionary" tax.

The other Pigouvian terms are the same as in the deterministic case, but we now highlight the differences in the risky case only for the Social Cost of Carbon. A similar decomposition can be written for the Social Value of Rent (SVR) or the social value of energy $\widetilde{\phi}^E$.

The Expected Social Cost of Carbon writes as an expectation of a product and can be decomposed into covariances with respect to the cross-section of countries and with respect to aggregate risk.

$$\mathbb{E}_{\boldsymbol{\epsilon}}[SCC] = \int_{\mathcal{E}} \sum_{j=N,S} \frac{\omega_i U'(c_i(\boldsymbol{\epsilon}))}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k)]} \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z) d\varphi(\boldsymbol{\epsilon})$$

$$= -\mathbb{C}\mathrm{ov}_{j,\boldsymbol{\epsilon}}\Big(\frac{\omega_j U'(c_j(\boldsymbol{\epsilon}))}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)\Big) - \mathbb{E}_{j,\boldsymbol{\epsilon}}[\mathcal{D}'_j(\mathcal{S}) y_k]$$

This covariance can also be decomposed, with the law of total expectation between the heterogeneity across countries and the covariance due to aggregate risk:

$$\mathbb{E}_{\boldsymbol{\epsilon}}[SCC] = -\mathbb{E}_j\Bigg[\underbrace{\mathbb{C}\mathrm{ov}_{\boldsymbol{\epsilon}}\Big(\frac{\omega_j U'(c_j(\boldsymbol{\epsilon}))}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)\Big)}_{=\text{effect of agg. risk } \boldsymbol{\epsilon}}\Bigg]$$

$$\underbrace{- \mathbb{C}\mathrm{ov}_j\Bigg[\frac{\mathbb{E}_{\boldsymbol{\epsilon}}\big(\omega_j U'(c_j(\boldsymbol{\epsilon}))\big)}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k)]}, \mathbb{E}_{\boldsymbol{\epsilon}}\big(\mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)\big)\Bigg]}_{=\text{effect of heterogeneity across } j} - \underbrace{\mathbb{E}_{j,\boldsymbol{\epsilon}}[\mathcal{D}'_j(\mathcal{S}) y_k]}_{=\text{average exp. damage}}$$

In the environment we consider, the sign of the first covariance term is ambiguous:

$$-\mathbb{C}\mathrm{ov}_{\boldsymbol{\epsilon}}\Big(\frac{\omega_j U'(c_j(\boldsymbol{\epsilon}))}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)\Big) \lesseqgtr 0$$

For productivity shock, this covariance is negative, as it increases output $y_j(\epsilon_z)$ and also decreases the marginal utility of consumption $U'(c_j(\boldsymbol{\epsilon}))$. However, for climate risk, this covariance is positive: higher temperature increases marginal damage $-\mathcal{D}'(\mathcal{S}, \epsilon_d)$ – since the cost function is convex – at the same time as it reduces output and increases the marginal value of consumption $U'(c_j(\boldsymbol{\epsilon}))$. Therefore, the result depends on the joint distribution $\varphi(\epsilon_z, \epsilon_d)$.

Still, we can still argue that climate uncertainty is amplified with the presence of heterogeneity, due to the expectation term:

$$\mathbb{E}_j\Bigg[-\mathbb{C}\mathrm{ov}_{\boldsymbol{\epsilon}}\Big(\frac{\omega_j U'(c_j(\boldsymbol{\epsilon}))}{\mathbb{E}_{k,\boldsymbol{\epsilon}}[\omega_k U'(c_k)]}, \mathcal{D}'_j(\mathcal{S}, \epsilon_d) y_j(\epsilon_z)\Big)\Bigg] > \mathbb{C}\mathrm{ov}_{\boldsymbol{\epsilon}}\Big(\frac{U'(\bar{c}(\boldsymbol{\epsilon}))}{\mathbb{E}_{\boldsymbol{\epsilon}}[U'(\bar{c})]}, \bar{\mathcal{D}}'(\mathcal{S}, \epsilon_d)\bar{y}(\epsilon_z)\Big)$$

The reason lies in the fact that the positive covariance between high climate outcomes and lower marginal values of consumption is much higher in low-income and warm countries than in rich and cold economies.

$$\mathbb{E}_{\boldsymbol{\epsilon}}[SCC(\boldsymbol{\epsilon})] > \mathbb{E}_{\boldsymbol{\epsilon}}[\overline{SCC}(\boldsymbol{\epsilon})]$$

As a result, we can still conclude that climate risk exacerbates the Social Cost of Carbon both with heterogeneity and uncertainty.

$$\mathbb{E}_{\boldsymbol{\epsilon}}[SCC(\boldsymbol{\epsilon})] > SSC(\bar{\boldsymbol{\epsilon}}) > \overline{SSC}$$

We will see that this feature is robust and will also be present in our rich quantitative model.

3 Model

We develop a framework with neoclassical foundations and rich heterogeneity across regions. The time is continuous $t \in [t_0, t_T]$, where $t_0 = 2000$ and $t_T = 2100$ in the application. The countries/regions are infinitesimal and modeled as a continuum and indexed by $i \in \mathbb{I}$. They can be heterogenous in an arbitrary number of dimensions[3] s.

As of now, this model includes five states $s = \{z, p, \Delta, k, \tau\}$, respectively productivity z, population p, geographic factors Δ, capital k, and temperature τ. Moreover, the world is subject to global states which can also be changing over time $S = \{\mathcal{T}, \mathcal{S}, \mathcal{R}\}$ which are respectively world atmospheric temperature $\mathcal{T}$, world atmospheric carbon concentration $\mathcal{S}$ and reserve of fossil fuel energy sources $\mathcal{R}$. All these variables will be explained in turn below.

Countries interact with the rest of the world through their consumption of fossil-fuel energy, traded in a single world , Except for trade in energy, no trade happen between regions i. Moreover, in the baseline model no migration are allowed between countries.

3.1 Country Household and firm problem

At each instant t, each region $i \in \mathbb{I}$ is populated by a household of population size $p_{i,t}$. This population is increasing at a growth rate exogenously determined n, and $\dot{p}_{i,t} = np_{i,t}$. As a result, the population is given as $p_{i,t} = p_{i,0}e^{nt}$.

This representative household owns the representative firm that is producing output with total factor productivity $z_{i,t}$. This total factor productivity also grows with a deterministic growth rate $\bar{g}$, giving a TFP level of $z_{i,t} = z_{i,0}e^{\bar{g}t}$. In the tradition of the Neoclassical model, we normalize all the economic variables of the model by the rate of effective population $z_t p_t = e^{(n+barg)t}$, leaving only the relative difference between countries' population $p \equiv p_{i,0}$ and productivity $z \equiv z_{i,0}$. In the following, we remove the countries i indices for ease in notations and in absence of ambiguity: each country solves an independent dynamic control problem, and is subject to global variables that we shall denote with capital letters – for example, $\mathcal{T}_t$ for global temperature or $\mathcal{E}_t$ for global emissions explained below.

The household consumes the homogeneous good $c_t \equiv c_{i,t}$ and is subject to the temperature of the region $\tau_t \equiv \tau_{i,t}$. It also chooses the firms inputs in the production function[4], yielding the

[3] More precisely, state variables of heterogeneity can be split in two, $s = \{\underline{s}, \overline{s}\}$, where ex-ante heterogeneity is constant over time or relate to initial conditions and is denoted $\underline{s}$, while ex-post heterogeneity $\overline{s}$ changes over time depending on the fluctuations of the regions variables. In practice, with the method used, $\underline{s}$ can be arbitrarily large, but the size of ex-post heterogeneity $\overline{s}$ needs to be controlled, as we will explained in the next section.

[4] The original – unnormalized – production function:

$$Y_t = F(K_t, E_t, L_t) = \mathcal{D}(\tau_t) z_t \Big[(1-\varepsilon)^{\frac{1}{\sigma}} \left(K_t^{\alpha} L_t^{1-\alpha} \right)^{\frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} \left(z_t^e E_t \right)^{\frac{\sigma-1}{\sigma}} \Big]^{\frac{\sigma}{\sigma-1}}$$

We divide the output level Y_t by the growth trend in population and TFP $e^{(n+\bar{g})t}$ and by initial population $p_0 \equiv L_t$ to obtain output per effective capita.

output per capita:

$$y_t = \mathcal{D}_y(\tau_t) z f(k_t, e_t)$$

where temperature τ_t, relative productivity z, capital stock per effective capita k_t and energy input per effective capita e_t all affect production. The gross production function is a CES aggregate between the capital-labor bundle k_t and energy e_t:

$$f(k_t, e_t) = \left[(1-\varepsilon)^{\frac{1}{\sigma}} k_t^{\alpha \frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} \left(z_t^e e_t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

with $\sigma < 1$, such as energy is complementary in production[5] and where directed technical change z_t^e is exogenous and deterministic. This directed – energy augmenting – technical change allows an increase in output for a given energy consumption mix. An upward trend in such technology is sometimes argued to be behind the "relative decoupling" of developed economies: an increase in production and value-added simultaneous to a decline in energy consumption and carbon emissions. In this version of the model, this trend is taken exogenously increasing at rate $z_t^e = \bar{z}^e e^{g_e t}$, but in an extension of the model, we consider an endogenous directed technical change. Moreover, energy used in production come from two sources: either fossil e_t^f and renewable e_t^r for every country i. The production of these two sources is detailed below.

Moreover, the temperature $\tau_{i,t}$ affects the productivity through damages $\mathcal{D}_y(\tau_t)$. This is the source of climate externality as will detailed below.

The Household in the country $i \in \mathbb{I}$ owns the firms and hence solves the following intertemporal problem. They maximize present discounted utility, with discount rate ρ, by choosing consumption c_t, energy inputs e_t – bought at price q_t^e.

$$v_{i,t_0} = \max_{\{c_t, e_t^f, e_t^r\}} \int_{t_0}^{t_T} e^{-(\rho-n)t} u_i(c_t, \tau_t) dt$$

The utility that households receive from consumption is also scaled by a damage function, which represents the direct impact of temperature.

$$u_i(c_t, \tau_t) = \mathcal{D}_u(\tau) u(c_t) \qquad\qquad u(c_t) = \frac{c_t^{1-\eta}}{1-\eta}$$

To sum up, beside the cost of energy $e_t q_t^e$and expenditure in consumption c_t, the production can be invested to increase capital stock and cover depreciation $\tilde{i} = \dot{k} + \delta k_t$. Moreover, the country i receive a share of profit θ_i that the fossil sector generates $\pi(E_t^f, \mathcal{R}_t)$ and that will explained below.

Hence, with k_t being capital per effective capita – covering population n and TFP growth $\bar{g}$ rates – the dynamics of capital follows:

$$\dot{k}_{i,t} = \mathcal{D}_y(\tau_{i,t}) z f(k_{i,t}, e_{i,t}) - (n + \bar{g} + \delta) k_{i,t} + \theta_i \pi_t - q_t^e e_{i,t} - c_{i,t}$$

[5] If $\sigma = 1$ we have the Cobb Douglas : $f(k_t, e_t) = \bar{\varepsilon} z_t^{e\ \varepsilon} k_t^\alpha e_t^\varepsilon$

on $t \in [t_0, t_T]$ where the dynamics of capital starts from initial condition $k_{t_0} = k_0$ given ex-ante. This capital level constitutes the first dimension of ex-post heterogeneity.

Climate damage and externality

Change in temperatures $\tau_{i,t}$ in each country $i \in \mathbb{I}$ – given in degree Celsius, $°C$ – affects the productivity with a Damage function $\mathcal{D}_y(\tau_t)$. This scaler increase with $\tau < \tau_i^\star$ and decreases when $\tau < \tau_i^\star$, where the "optimal temperature" $\tau_i^\star$ such that $\mathcal{D}_y(\tau_i^\star) = 1$. We consider the "optimal" temperature as:

$$\tau_i^\star = \alpha^\tau \tau_{i,t_0} + (1 - \alpha^\tau)\tau^\star$$

where τ_{i,t_0} is the initial temperature in country i and $\tau^\star = 15.5°C$ is an optimal level of temperature for temperate climates. This flexible formulation allows for differing degrees of adaptability. Hot temperatures do not affect countries with histories of cold vs. hot climates in the same way, due to the presence of adaptation structures – i.e. air conditioning vs. heating infrastructures.

Productivity decays to zero when temperatures are extremely cold or hot $\lim_{\tau \to -\infty} \mathcal{D}_y(\tau) = \lim_{\tau \to \infty} \mathcal{D}_y(\tau) = 0$. We follow Nordhaus formalism and use a quadratic function for the damage function:

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma_y^{\oplus} \frac{1}{2}(\tau - \tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma_y^{\ominus} \frac{1}{2}(\tau - \tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

where $\gamma_y^{\oplus}$ and $\gamma_y^{\ominus}$ represent damage parameters on output respectively for hot v.s. cold temperatures – and they are different to allow for asymmetry on climate impact.

The utility that households receive from consumption is also scaled by a similar damage function, which represents the direct impact on population likelihood of mortality – for example, due to heatwaves or extreme weather events.

$$\mathcal{D}_u(\tau) = \begin{cases} e^{-\gamma_u^{\oplus} \frac{1}{2}(\tau - \tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma_u^{\ominus} \frac{1}{2}(\tau - \tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

where $\gamma_u^{\oplus}$ and $\gamma_u^{\ominus}$ represent also the damage parameters, but on the direct impact on utility and mortality, respectively, for hot v.s. cold temperatures.

In the previous graph, we present an example of such damage function for two countries, USA and India, with the distribution of temperature (approximated by a normal distribution), their average yearly temperature (respectively $13.5°C$ and $25°C$) in dashed lines and their optimal temperature in dotted black lines (respectively $15°C$ and $20°C$)

3.2 Energy sector

Given the demand for energy inputs e_t in each country, the firm has the choice among two sources of energy: one fossil-fuel source in finite supply e_t^f and one renewable source e_t^r. We consider that these two sources are substitutable, and total energy inputs quantity e_t is given by the CES

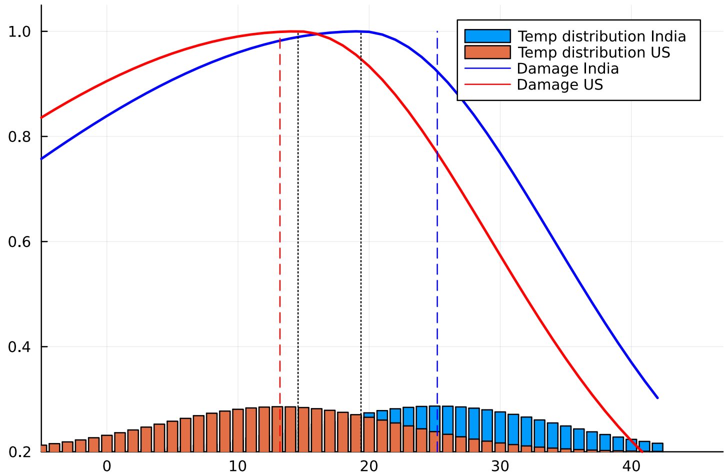


Figure 1: Damage function for two example countries, US and India

aggregator, where σ_e represents the elasticity of substitution.

$$e_t = \left(\omega_f^{\frac{1}{\sigma_e}} (e_t^f)^{\frac{\sigma_e - 1}{\sigma_e}} + (1-\omega_f)^{\frac{1}{\sigma_e}} (e_t^r)^{\frac{\sigma_e - 1}{\sigma_e}}\right)^{\frac{\sigma_e}{\sigma_e - 1}} \qquad \text{if} \qquad \sigma_e \in (1, \infty)$$
$$e_t = e_t^f + e_t^r \qquad \text{if} \qquad \sigma_e \to \infty$$

subject to the budget for energy expenditures:

$$e_t^f q_t^{e,f} + e_t^r q_t^{e,r} = q_t^e e_t$$

As a result, the demand curve for both fossil and renewable energies are given by usual demands:

$$\frac{e_t^f}{e_t} = \omega^f \left(\frac{q_t^{e,f}}{q^e}\right)^{-\sigma_e} \qquad \& \qquad \frac{e_t^r}{e_t} = (1-\omega^f)\left(\frac{q_t^{e,r}}{q_t^e}\right)^{-\sigma_e}$$
$$q_t^e = \left(\omega_f (q_t^{e,r})^{1-\sigma_e} + (1-\omega_f)(e_t^r)^{1-\sigma_e}\right)^{\frac{1}{1-\sigma_e}} \qquad \text{if} \qquad \sigma_e \in (1, \infty)$$
$$q_t^e = \min\{q_t^{e,f}, q_t^{e,r}\} \qquad \text{if} \qquad \sigma_e \to \infty$$

where the price of the energy bundle q_t is some weighted sum of the energy price of fossil fuel $q_t^{e,f}$ and renewable $q_t^{e,r}$.

Fossil fuel extraction and exploration

Fossil energy is produced and sold in a centralized market at the world level. The single competitive producer is extracting the fuel quantity E_t from a single pool of resources $\mathcal{R}_t$, with production cost $\nu(E_t^f, \mathcal{R}_t)$. Fossil energy can be shipped costlessly around the world, where the global market in energy clears:

$$E_t^f = \int_{\mathbb{I}} p_{i,0}\, e^{(n+\bar{g})t} e_{i,t}^f di$$

where the demand comes from the aggregation of individual energy per capita inputs in each country

$i \in \mathbb{I}$ and energy input is rescaled by the population and technology exponential trends $e^{(n+\bar{g})t}$.

Moreover, the fossil-fuel reserves $\mathcal{R}_t$ are depleted with extraction E_t^f, but can be regenerated by exploration, which require investment $\mathcal{I}_t^e$ to obtain $\delta^R \mathcal{I}_t^e$ additional reserves for an exploration cost $\mu(\mathcal{I}_t^e, \mathcal{R}_t)$

$$\dot{\mathcal{R}}_t = -E_t^f + \delta^R \mathcal{I}_t^e$$

The parameter δ^R can be interpreted in two ways: first, it can represent the probability intensity $\delta^R \mathcal{I}_t^e$ of finding developable reserves among possible reserves $\mathcal{I}_t^e$ in a continuum of fossil fuel fields and mines. Second, it can also represent the fraction of individual producers discovering developable reserves, aggregating up a representative producer. This stylized model is a simplified version of the rich framework developed in **?**.

Moreover, the fossil-fuel producer hence faces a modified Hotelling finite-resources problem – c.f. Heal and Schlenker – allowing for exploration of additional reserves. As a result, its dynamic problem is given by :

$$v^e(\mathcal{R}_t) = \max_{\{E_t^f, \mathcal{I}_t^e\}_{t \geq t_0}} \int_0^\infty e^{-\rho t} \pi(\mathcal{R}_t, E_t^f, \mathcal{I}_t^e) dt$$

$$\text{with} \qquad \pi_t(\mathcal{R}_t, E_t^f, \mathcal{I}_t^e) = q_t^{e,f} E_t^f - \nu(E_t^f, \mathcal{R}_t) - \mu(\mathcal{I}_t^e, \mathcal{R}_t)$$

$$s.t. \qquad \dot{\mathcal{R}}_t = -E_t^f + \delta^R \mathcal{I}_t^e \qquad\qquad E_t^f = \int_{\mathbb{I}} p_{i,0} e^{(n+\bar{g})t} e_{i,t}^f di$$

This can be solved using the Pontryagin maximum principle, where we denote λ_t^R the Hotelling rent, which is the costate of the resource depletion dynamics. The price of the fossil energy supplied and the optimal exploration are given by optimality conditions:

$$[E_t^\star] \qquad\qquad q_t^{e,f} = \nu_E(E_t^\star, \mathcal{R}_t) + \lambda_t^R$$

$$[\mathcal{I}_t^\star] \qquad\qquad \delta^R \lambda_t^R = \mu_E(\mathcal{I}_t^{e\ \star}, \mathcal{R}_t)$$

Price is hence the sum of marginal cost, plus an additional rent meant to price the finiteness of the resource. Moreover, the dynamics of that Hotelling rent are given by the equation:

$$\dot{\lambda}_t^R = \rho \lambda_t^R + \partial_R \nu(E_t^\star, \mathcal{R}_t) + \partial_R \mu(\mathcal{I}_t^\star, \mathcal{R}_t)$$

In standard Hotelling model without stock effects – i.e. where $\partial_R \nu(E^\star, \mathcal{R}) = 0$ and no exploration $\mu(\mathcal{I}^\star, \mathcal{R}) = 0$ – we have the standard expression for the finite resource rent $\dot{\lambda}_t^R = \rho \lambda_t^R$ and $\lambda_t^R = e^{\rho t} \lambda_{t_0}^R$, and $R_t \to 0$ as $t \to \infty$. In our context, the rent grows less fast because (i) the producer anticipate that the depletion of reserves will increase marginal cost in the future $\partial_R \nu(E^\star, \mathcal{R}) < 0$ and (ii) it can invest in exploration, increasing future reserves which can lower even further the future cost of exploring $\partial_R \mu(\mathcal{I}^\star, \mathcal{R}) < 0$.

As a result, even with simple functional forms that yield isoelastic supply curves for fossil energy extraction and exploration, we can solve the dynamics of the rent price.[6]

Note that this centralized market for fossil fuels is in equilibrium: the supply curve $(q^{e,f}, E_t^f)$ determined by the fossil-fuel producers meets the demand coming from the aggregation of all individual countries $(q^{e,f}, e_t^f)$. Moreover, fossil fuels emit CO_2 and other GHG emissions, as we will see in the next section.

Renewable energy production

Renewable energy is not subject to the finiteness of the stock of reserves and is produced with capital k_t^e.

$$e_t^r = z_t^r f(k_t^r)$$

We assume that capital k_t^r is fungible with the capital k_t that produces the homogeneous good and is hence subject to the same interest r_t on the common capital market of the country.

$$q_t^r z_t^r f'(k_t^r) = r_t$$

where q_t^r is the price of that renewable energy demanded. We make these stylized assumptions to keep the model tractable.

For now, renewable energy production is assumed constant return to scale, i.e. $f^r(k_t^r) = k_t^r$, and the two sources of energy are perfectly substitutable, i.e. $\sigma_e \to \infty$, then we obtain that renewables act as a perfect "backstop" technology to fossil fuel. If $q_t^{e,f}$ grows up to $q_t^{e,r}$ which is given exogenously by:

$$q_t^{e,r} = \frac{r_t}{z_t^r}$$

then all the energy is produced using renewable $e_t = e_t^r$ and the emissions collapse to zeros.

Moreover, the carbon emissions associated with renewable energy are null, minimizing the externality on the climate when the energy transition is complete.

3.3 Climate system, emissions and externality

Economic activity are emitting carbon and other greenhouse gas emissions, which change the climate and increase the temperature of the atmosphere. Due to these activities coming from the energy sector, each country is emitting the amount:

$$\epsilon_{i,t} = \xi^f e_{i,t}^f p_{i,t}$$

[6] Details of the fossil energy producers can be found in appendix .

where ξ^f denote the carbon content of fossil fuels[7]. As a result, since the energy use is normalized by growth of TFP and population, the absolute amount of global emissions aggregates to:

$$\mathcal{E}_t = \int_{i \in \mathbb{I}} \epsilon_{i,t} di = e^{(n+\bar{g})t} \int_{\mathbb{I}} \epsilon_{i,t} di$$

These emissions are released in the atmosphere, adding up to the cumulative stock of greenhouse gas $\mathcal{S}_t$.

$$\dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s \mathcal{S}_t$$

However, a part of these emissions exit the atmosphere and can be stored in oceans or the biosphere, discounting the current stocks by an amount δ_s. Moreover, these cumulative emissions push the global atmospheric temperature $\mathcal{T}_t$ upward linearly with parameter χ with some inertia and delay represented by parameter ζ

$$\dot{\mathcal{T}}_t = \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)$$

This simple two-equations climate system is a good approximation of large-scale climate models[8] with a small set of parameters $\xi^f, \delta_s, \zeta, \chi$.

More particularly, ζ is the inverse of persistence, and modern calibrations set $\zeta \approx 0.1$ is such that the pick of emissions happens after 10 years. Dietz et al (2021) show that classical IAM models such at Nordhaus' DICE tend to set ζ to low generating too large inertia, as shown in the figure below. Moreover, if $\zeta \to \infty$, temperature reacts immediately and we obtain a linear model – which is a good long-run approximation:

$$\mathcal{T}_t = \bar{\mathcal{T}}_{t_0} + \chi \mathcal{S}_t = \bar{\mathcal{T}}_{t_0} + \chi \int_{t_0}^{t} \int_{\mathbb{I}} \epsilon_{i,t} \, di \; ds \Big|_{GtC}$$

As we see, the global externality depends on the path of individual policies $\epsilon_{i,t} \propto e^f_{i,t}$ as of function of states of the country $\{z_i, p_i, k_i, \tau_i\}$

[7] We can consider an alternative, like in Nordhaus' DICE model, with

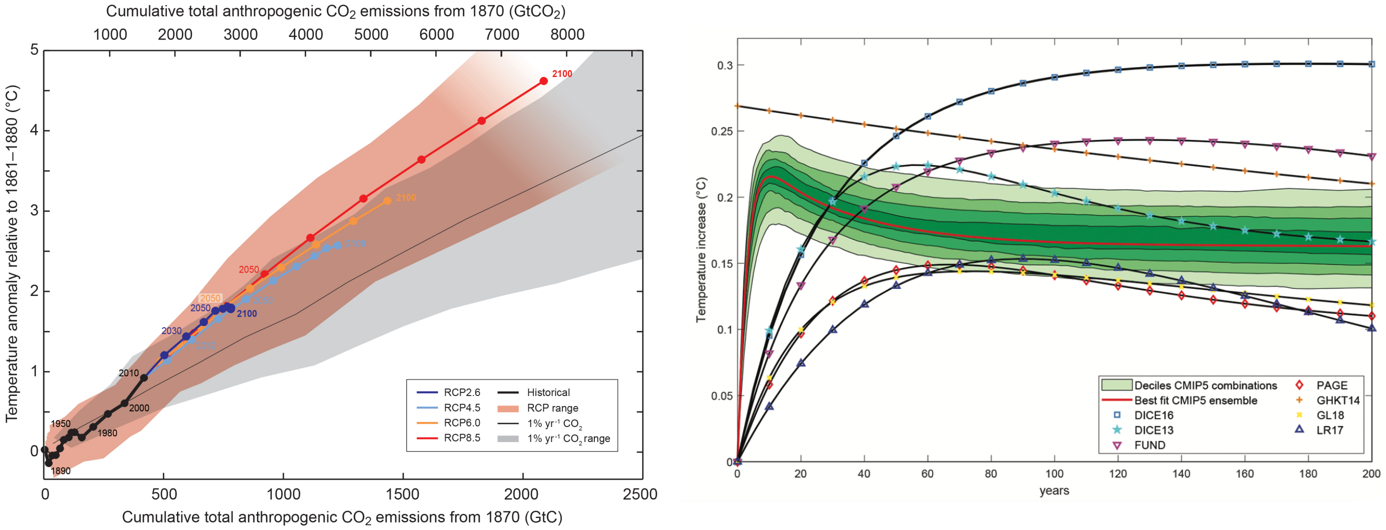
$$\epsilon_{i,t} = \xi^f (1 - \vartheta_{i,t}) e^f_{i,t} p_{i,t} \qquad \& \qquad \mathcal{E}_t = e^{(n+\bar{g})t} \int_{\mathbb{I}} \xi^f (1 - \vartheta_{i,t}) e^f_{i,t} p_i di$$

ϑ_t represents the abatement policy taken in country i. It represents all the policies that allow reducing the emissions for a given choice of the energy mix – for example, additional environmental regulations or investment in carbon capture technology – and its optimal choice will be determined in appendix.

[8] These climate models have typically much more complex climate block, adding 3 to 4 more state variables, with $\mathbf{J}$ the vector of carbon "boxes": layers of the atmosphere and sinks such as layers of oceans:

$$\dot{\mathbf{J}}_t = \Phi^J \mathbf{J}_t + \rho^e \int_{\mathbb{I}} \xi^f e^f_i p_i di$$
$$F_t = \mathcal{F}(\mathbf{J}_t) \qquad \dot{\mathcal{T}}_t = \Phi^T \mathcal{T} + \eta F_t$$

with F_t Carbon forcing and ρ^e, vector of parameters, Φ^J and Φ^T Markovian transition matrices and $\mathcal{F}(\cdot)$ a non-linear function.



Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

The temperature in country i is affected by global warming of the atmosphere $\mathcal{T}_t$ with sensitivity Δ_i

$$\dot{\tau}_{i,t} = \Delta_i \, \dot{\mathcal{T}}_t$$

In general, the temperature scaler Δ_i depends on the geographic properties of country i – like temperature, latitude, longitude, elevation, distance from coasts and water bodies, vegetation, and albedo (sunlight reflexivity due to ice, vegetation and soil properties). Moreover, a simple linear equation is a good first-order approximation:

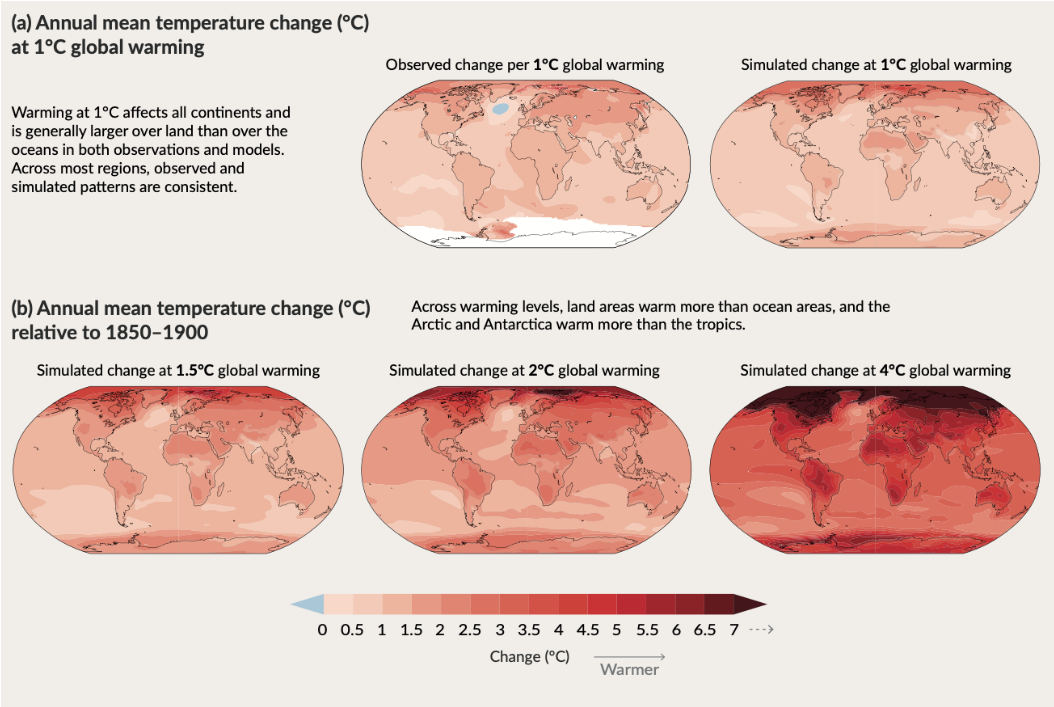
$$\Delta_i = 1.537 - 0.0288 \times \tau_{t_0,i}$$

4 Competitive equilibrium and Business as usual

To solve for the competitive equilibrium, we use the Pontryagin Max. Principle, we obtain a system of coupled ODEs. The equilibrium boils down to the standard neoclassical model dynamics, for each country $i \in \mathbb{I}$.

$$\begin{cases} \dot{c}_{i,t} &= c_{it} \, \frac{1}{\eta} \, (r_{it} - \rho) \\ \dot{k}_{i,t} &= \mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e^f_{i,t}, e^r_{i,t}) + \theta^R_i \pi^R_t - (n + \bar{g} + \delta) k_{i,t} - q^f_t \, e^f_{i,t} - q^r_{i,t} e^r_{i,t} - c_{i,t} \\ q^f_t &= MPe^f_{it} \qquad\qquad\qquad q^r_{it} = MPe^r_{it} \end{cases}$$

with, in addition, the climate block for carbon stock $\mathcal{S}_t$ and temperature $\tau_{i,t}$ and the dynamics of Hotelling rents λ^R_t for the fossil energy price q^f_t. Details of the system can be found in appendix **??**. The specificities of this system are that the ODEs are coupled through two mechanisms: first, energy markets clear such that the energy demand from all the individual countries impact the fossil fuel price that countries face. Second, the emissions from each country affect the global climate



and local temperatures.

Despite the infinite dimensionality of this system, this problem is well-posed, as it is the solution of Forward Backward McKean Vlasov system of equations.

This Business as Usual scenario features unrestricted use of fossil energy until its price increase when resources are depleted. In particular, temperature increase to high levels, and climate damages are large. We will analyze the result in the quantitative section below. We now turn to the Ramsey policy to take into account the climate externalities.

5 Ramsey problem and optimal policy

We consider the optimal policy of a social planner that maximize the weighted sum of the Household utility, subject to the optimality conditions of the agents. In this context, it would not only internalize all the dynamics of economic variables, the climate and energy markets, but also the decisions that Household and firms takes.

The Ramsey planner choses, consumption/saving $c_{i,t}$, energy mix $e^f_{i,t}$, the energy price $q^{e,f}_t$ and extraction $\mathcal{I}_t$, the trajectories of dynamic states $(k, \tau, \mathcal{S}, \mathcal{R})$

$$\mathcal{W}_{t_0} = \max_{\{c, e^f, e^r, k, \tau, \mathcal{S}, \mathcal{R}, \mathcal{I}\}} \int_{t_0}^{\infty} \int_{\mathbb{I}} e^{-(\widetilde{\rho}+n)t} \; \omega_i \; \mathcal{D}(\tau_{i,t}) u(c_{i,t}) p_i \; di \; dt$$

subject to (i) the optimality conditions of households, for c_i, e^f_i, e^r_i and k_i, (ii) the optimality conditions of the Fossil fuel producers for E^f, $\mathcal{I}$ and $\mathcal{R}$ and (iii) the Climate and temperature dynamics τ_i and $\mathcal{S}$. Note that the planner has discount factor $\widetilde{\rho}$ might be different than the agent discount parameter ρ, and notably smaller, if we believe the planner could be more patient.

We apply again the Pontryagin Maximum Principle in infinite dimension. The resulting system of McKean Vlasov differential equations is very large, and details of the entire system can be found in appendix **??**. The Lagrange multipliers corresponding to states dynamics equations are denoted ψ's and the ones corresponding to agents optimality conditions are named with ϕ's.

We provide some intuitions of the most important results and the ones that have connections with the rest of the literature.

First, the optimality for consumption yields the marginal value of capital, i.e. wealth ψ^k_{it}. This multiplier informs on the value of consumption in country i and measures directly the extent of inequality across countries. This is directly related to the marginal utility of consumption and the distortion of the saving decisions:

$$[c_{it}] \qquad \psi^k_{it} = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{=\text{direct effect}} + \underbrace{\phi^c_{it} \mathcal{D}^u(\tau_{it}) u''(c_i)}_{=\text{effect on savings}}$$

This expression for the social shadow value of wealth is analogous to the "marginal value of liquidity" in heterogenous agents analysis like **?** and **?**. To have a measure of inequality, we compare this with the average marginal value:

$$\frac{\psi^k_{i,t}}{\overline{\psi}^k_t} \lessgtr 1 \qquad \text{with} \qquad \overline{\psi}^k_t = \int_{\mathbb{I}} \psi^k_{jt} dj$$

If the ratio is higher than 1, we can argue that the country is relatively poorer, with a lower welfare than average.

Second, let us turn toward the social value of energy. We denote it by $\widehat{\phi}^e_{it}$, and it is a local value and is a weighted sum of the social values of fossil energy ϕ^f_{it} and non-carbon energy ϕ^r_{it}, and weights are marginal products of each energy sources:

$$\widehat{\phi}^e_{it} = \phi^f_{it} MPe^f_t + \phi^r_{it} MPe^r_t$$

As in the Toy model of **??**, the optimality condition for the fossil energy combines multiple terms we detail below.

$$[e^f_{it}] \qquad \underbrace{\frac{\psi^k_{i,t}}{\overline{\psi}^k_t}}_{=\substack{\text{redistribution}\\ \text{term}}} \left(MPe^f_{it} - q^f_t\right) + \xi_i \, p_i \underbrace{\frac{\psi^S_t}{\overline{\psi}^k_t}}_{=-SCC} + \; p_i \; \underbrace{\partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \frac{\psi^k_{jt}}{\overline{\psi}^k_t} dj}_{=SVR}$$

$$+ \underbrace{\frac{\partial_{e^f} \widehat{\phi}^e_{it}}{\overline{\psi}^k_t}}_{=\substack{\text{effect on}\\ \text{energy choice}}} - \; p_i \underbrace{\frac{\phi^{Ef}_t}{\overline{\psi}^k_t} \, \partial_{EE} \mathcal{C}(\cdot)}_{=\substack{\text{effect on}\\ \text{fossil market}}} = 0$$

where ψ_t^S is the social shadow value of carbon stock $\mathcal{S}$ in the atmosphere, and ϕ_t^{Ef} the social value of aggregate fossil energy supply. Moreover, $\partial_{EE}\mathcal{C}(\cdot)$ is the curvature of the extraction cost function of fossils and $\partial_E \pi^f(\cdot)$ is the marginal profit for an additional unit of fossil fuel extracted. We denote SCC the social value of carbon and SVR the social value of rent. As in the Toy model of section 1, these terms account for both externality and wealth distribution, and we detail them in turn.

Social cost of carbon

In this model, the social cost of carbon writes very simply:

$$SCC_t := \frac{\partial \mathcal{W}_t / \partial \mathcal{S}_t}{\partial \mathcal{W}_t / \partial c_t} = -\frac{\psi_t^S}{\overline{\psi}_t^k}$$

The costate for the stock of carbon $\mathcal{S}_t$ measures the marginal – social – shadow value of an additional ton of GHG emitted in the atmosphere. To convert this welfare measure, one to translate it into monetary units using the marginal value of wealth or capital. As the cost of climate is a global measure, the standard naive intuition "representative agent" is to use the average marginal value $\overline{\psi}_t^k$. This allows to consider an average value, but we will see that redistribution terms need to be accounted for in the optimal taxation results.

To measure the welfare cost of climate damage, one can follow the dynamics of ψ_t^S along the trajectories of climate and aggregate temperatures.

Applying the Pontryagin Max Principle in this Ramsey problem – or using integration by part as in the proof of the PMP – we can follow the costate for carbon $\mathcal{S}$ that depends on the costate for local temperatures.

$$\dot{\psi}_{i,t}^{\tau} = \psi_{i,t}^{\tau}(\widetilde{\rho} + \zeta) + \underbrace{\gamma_i(\tau_{i,t} - \tau_i^{\star})\mathcal{D}^y(\tau_{i,t})}_{-\partial_\tau \mathcal{D}^y} f(k_{i,t}, e_{i,t})\psi_{i,t}^k + \underbrace{\phi_i(\tau_{i,t} - \tau_i^{\star})\mathcal{D}^u(\tau_{i,t})}_{\partial_\tau \mathcal{D}^u} u(c_{i,t})$$

$$\dot{\psi}_t^S = \psi_t^S(\widetilde{\rho} + \delta^s) - \zeta\,\chi \int_{\mathbb{I}} \Delta_i\, \psi_{i,t}^{\tau}\, di$$

The marginal cost for country i of releasing carbon in atmosphere ψ_t^S depends on the shadow value of temperatures, through the climate parameters: ζ the climate inverse persistence (e.g. lags), χ the climate sensitivity and Δ_i the catching up effect" of temperature at the cold location.

The marginal value for country i of being subject to an increase in local temperature is measured by ψ_t^{τ}. It increases with different terms: the temperature gap $\tau_{i,t} - \tau_i^{\star}$, due to the convexity of the damage function, the damage sensitivity to temperature for TFP γ_i and utility/mortality ϕ_i. It depends on the "catching up" effect. Finally, it is proportional to the development level $f(k_{i,t}, e_{i,t})$ and $u(c_{i,t})$, due to the multiplicative nature of the Climate Damage.

Solving for the SCC can be found in appendix **??**. Moreover, since the marginal damage affects all the countries locally and symmetrically through a value $\psi_{i,t}^{\tau}$, and these gain/costs are added as a sum additively, we can perform this (exact) decomposition:

$$\psi_t^S = \int_{\mathbb{I}} \psi_{i,t}^S di$$

More particularly, the Social Cost of Carbon can hence be reexpressed as:

$$SCC_t = -\frac{\psi_t^S}{\overline{\psi}_t^k} = -\int_{\mathbb{I}} \underbrace{\frac{\psi_{i,t}^k}{\overline{\psi}_t^k}}_{=\text{redistribution term}} \underbrace{\frac{\psi_{i,t}^S}{\psi_{i,t}^k}}_{=LCC_{i,t}} di$$

where we consider the *Local Cost of Carbon* (LCC_i) as country i specific. When we convert the local impact on welfare using its own measure of the marginal value of wealth/income $\psi_{i,t}^k$. Note that this notion is exactly analogous to the concept of Local Cost of Carbon in **?**.

As a result, we can express the social cost of carbon emissions as:

$$\begin{aligned} SCC_t &= \int_{\mathbb{I}} \frac{\psi_{i,t}^k}{\overline{\psi}_t^k} LCC_{i,t}\, di \\ &= \mathbb{E}^{\mathbb{I}}\big[LCC_{i,t}\big] + \mathbb{C}\text{ov}^{\mathbb{I}}\Big(\frac{\psi_{i,t}^k}{\overline{\psi}_t^k}, LCC_{i,t}\Big) \\ &> \mathbb{E}^{\mathbb{I}}\big[LCC_{i,t}\big] =: \overline{SCC}_t \end{aligned}$$

where the last inequality follows the empirical observation that marginal damage – i.er. high local temperature $\tau_{i,t}$ – tend to be negatively correlated with development levels y_i, i.e. lower production, consumption and hence a higher marginal utility of consumption.

To conclude, the presence of heterogeneity and the correlation between local damage and poverty increases the Social Cost of Carbon from the Social Planner perspective.

6 Long-run analysis

In this section, we provide analytical results of the Ramsey problem on the cost of carbon, the path of emissions, and temperature in the asymptotic stationary equilibrium.

6.1 The Social Cost of Carbon

Given the path for the costate that informs on the social value of carbon emission, we canÂ find a balance-growth path that keeps the SCC stationary. We consider the long-run equilibrium where the terminal time horizon $T \to \infty$. In this context, only a stable temperature makes the system stationary, such that the emissions entering the atmosphere $\mathcal{E}_t$ are exactly offset by the one rejected outside the climate system δ_i

$$\mathcal{E}_t = \delta_s \mathcal{S}_t \qquad \text{and} \qquad \tau_t \to \tau_\infty$$

Solving the stationary differential equations, we find an analytical characterization for the Social Cost of Carbon:

Proposition:
In the stationary equilibrium, the Social Cost of Carbon can be expressed as:

$$SCC_t \equiv \frac{1}{\bar{\psi}_t^k} \frac{\chi}{\widetilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_i(\tau_{i,\infty} - \tau_i^\star)\Big(\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} \psi_{i,\infty}^k + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \omega_i u(c_{i,\infty}) p_i\Big) di$$

This formula is analogous to the Social Cost of Carbon expressed in **?**. Considering a linear instead of quadratic damage function – and only applied to TFP, without direct effects on mortality, would yield an exactly identical expression. We rely on a different set of assumptions – stationarity and continuous time – while the analysis in **?** relies on a representative agent, discrete-time and the log-assumption such income and substitution forces offset each other.

In particular, the noticeable feature is the proportionality of the SCC with $y_{i,\infty}$ and the temperature gap $(\tau_{i,\infty} - \tau_i^\star)$.

One could also consider the *"Local cost of carbon"* as the marginal damage for the region $i \in \mathbb{I}$:

$$LCC_{i,t} = \frac{\chi}{\widetilde{\rho} + \delta^s} \Delta_i(\tau_{i,\infty} - \tau_i^\star)\Big(\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \frac{u(c_{i,\infty})}{u'(c_{i,\infty})}\Big) v(tau_{i,\infty}$$

Following the same logic as above, we observe that:

$$\begin{aligned} SCC_t &= \int_{\mathbb{I}} \frac{\psi_{i,t}^k}{\overline{\psi}_t^k} LCC_{i,t}\, di \\ &= \mathbb{E}^{\mathbb{I}}\big[LCC_{i,t}\big] + \mathbb{C}\mathrm{ov}^{\mathbb{I}}\Big(\frac{\psi_{i,t}^k}{\overline{\psi}_t^k}, LCC_{i,t}\Big) \\ &> \mathbb{E}^{\mathbb{I}}\big[LCC_{i,t}\big] =: \overline{SCC}_t \end{aligned}$$

6.2 Green Growth and decoupling from energy

Empirically, energy use has correlated strongly with GDP levels and industrial production in the last century, as seen in figures in **??**. However, lowering GHG emissions tend to go hand in hand with reducing energy consumption. This asks the question of the possibility of decoupling between economic growth and energy supply, and fossils in particular.

To examine this in our framework, let us study the optimality conditions for energy and express the energy share in the final output.

$$\begin{cases} MPe_i = {z_i}^{1-\frac{1}{\sigma}} {y_{i,t}}^{\frac{1}{\sigma}} \varepsilon^{\frac{1}{\sigma}} (z_{i,t}^e)^{1-\frac{1}{\sigma}} e_{i,t}^{-\frac{1}{\sigma}} & = q_t^e \\ MPe_i\big(\frac{e_t^f}{\omega e_t}\big)^{-\frac{1}{\sigma_e}} & = q_t^{e,f} \\ MPe_i\big(\frac{e_t^r}{(1-\omega)e_t}\big)^{-\frac{1}{\sigma_e}} & = q_t^{e,r} \end{cases}$$

As a result, the total energy share writes:

$$s_{e,t} := \frac{e_{i,t} q_t^e}{y_{i,t}} = (q_t^e)^{1-\sigma} z_i^{\sigma-1} (z_t^e)^{\sigma-1} \varepsilon$$

Since all the variable are already expressed in efficient unit per capita, accounting for the trend in population n and TFP growth $\bar{g}$, we have z_i constant and all the variables growth in absolute value. However, all the other variables can feature additional long-run trends, such as energy price $\dot{q}_t^e / q_t^e = g_q$ or directed technical change $\dot{z}_t^e / z_t^e = g_e$.

We consider two case: (i) the cost share of energy stays stable in output and (ii) this share falls to zeros.

$$\begin{aligned} &(i) \qquad s_{e,t} \rightarrow_{t\rightarrow\infty} \bar{s}_e \qquad &&\Leftrightarrow \qquad g_q(1-\sigma) + g_e(\sigma-1) = 0 \\ &(ii) \qquad s_{e,t} \rightarrow_{t\rightarrow\infty} 0 \qquad &&\Leftrightarrow \qquad g_q - g_e < 0 \end{aligned}$$

In our quantitative exercise, following empirical evidence that energy share $s_{e,t}$ tends to comove strongly with energy price q_t^e, we assume that $\sigma < 1$ and energy is a complementary factor in production. As result, $g_e = g_q$ for (i) and $g_e > g_q$ for (ii). For the energy share to stay stable or decline, directed technical change should at least compensate for the increase in price.

To determine the path of price in our context, recall the supply side of the energy market, we have:

$$\frac{\dot{q}_t^e}{q_t^e} = s_{ef,t} \frac{\dot{q}_t^{e,f}}{q_t^{e,f}} + s_{er,t} \frac{\dot{q}_t^{e,r}}{q_t^{e,r}}$$

where $s_{ef,t} = \frac{e_t^f q_t^{e,f}}{e_t q_t}$ is the expenditure share in fossil and $s_{er,t} = 1 - s_{ef,t}$ the share in renewable. Recall that in our context,

$$q_t^f = \Big(\frac{E_t^f}{\mathcal{R}_t}\Big)^\nu + \lambda_t^R \qquad \Rightarrow \qquad \frac{\dot{q}_t^f}{q_t^f} = s_{\mathcal{C}} \nu \Big(\frac{\dot{E}_t^f}{E_t^f} - \frac{\dot{\mathcal{R}}_t}{\mathcal{R}_t}\Big) + (1 - s_{\mathcal{C}}) \frac{\dot{\lambda}_t^R}{\lambda_t^R}$$

where $s_{\mathcal{C}} = \frac{\mathcal{C}_E(\cdot)}{q_t^f}$ is the share of marginal in the fossil price, and $\frac{\dot{\lambda}_t^R}{\lambda_t^R}$ is the growth of the Hotelling rent, which is ρ at the first order. Obviously if extraction rate is faster than exploration of new reserves, the price will grow to infinity. Moreover, the rent of the monopolist will at least grow at the speed ρ in the first order,

Similarly, to get decoupling from fossils in the energy mix, we must have $g_r = \frac{\dot{q}_t^{e,r}}{q_t^{e,r}} < \frac{\dot{q}_t^{e,f}}{q_t^{e,f}} = g_f$. In this case, $g_q \rightarrow g_r$.

To conclude, to obtain a balance green growth equilibrium in our context, we need: (i) fossil prices to grow sufficiently fast due to extraction or rise in Hotelling rents, (ii) the price of renewables to grow less fast than fossils and (iii) that the directed technical change grows at a rate at least faster than the growth in the relative price of the resulting energy.

6.3 Path of emissions and temperature

The cost of carbon depends only on final temperatures and path of emissions:

$$\tau_T - \tau_{t_0} = \Delta\,\chi\,\xi\,\omega\int_{t_0}^{T} e^{(n+\bar{g})t-\delta_s(T-t)} q_t^{f\ -\sigma_e} \int_{j\in\mathbb{I}} \left(z_j z_{j,t}^e \mathcal{D}(\tau_{j,t})\right)^{\sigma-1} y_{j,t}\, q_{j,t}^{\sigma_e-\sigma} dj\ dt$$

Geographical factors determining warming Δ_i, the climate sensitivity parameter χ and carbon exit from atmosphere δ_s, the growth of population n and aggregate productivity $\bar{g}$, the deviation of output from trend y_i & relative TFP z_j, the directed technical change z_t^e, elasticity of energy in output σ, the Fossil energy price $q^{e,f}$ and Hotelling rent $g^{q^f} \approx \dot{\lambda}_t^R/\lambda_t^R = \rho$ and finally the change in energy mix, renewable share ω, price q_t^r & elasticity of source σ_e

Approximations at terminal time $T \equiv$ Generalized Kaya (or $I = PAT$) identity

$$\frac{\dot{\tau}_T}{\tau_T} \propto\ n + \bar{g}^y - (1-\sigma)\left(g^{z^e} - \widetilde{\gamma}\right) + (\sigma_e - \sigma)(1-\omega)g^{q^r} - (\sigma_e(1-\omega) + \sigma\omega)g^{q^f}$$

7 Calibration

This section is forthcoming

8 Numerical Experiments

This section is forthcoming

Rewritten for intuitions

$$\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\},\{\psi\}) = \int_{\mathbb{I}} \omega_i \mathcal{D}^u(\tau_{it}) u(c_i) p_i di + \psi^k_{i,t}\Big(\mathcal{D}(\tau_{it}) f(k_t, e_t) - (n+\bar{g}+\delta)k_t + \theta_i \pi(E^f_t, \mathcal{I}_t, \mathcal{R}_t) - q^f_t e^f_{it} - q^r_{it} e$$

$$+ \psi^S_t\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big) + \psi^\tau_{it}\zeta\Big(\Delta_i\,\chi\,\mathcal{S}_t - (\tau_{it}-\tau_{i0})\Big) + \psi^{\mathcal{R}}_{it}\Big(-E^f_t + \delta^R \mathcal{I}_t\Big)$$

$$+ \psi^{\lambda k}_{i,t}\Big(\lambda^k_t(\rho - r_t)\Big) + \psi^{\lambda R}_t\Big(\rho\lambda^R_t + \mathcal{C}^f_{\mathcal{R}}(E^f_t, \mathcal{I}_t, \mathcal{R}_t)\Big)$$

$$+ \phi^c_{it}\big(\mathcal{D}^u(\tau_{it})u'(c_i) - \lambda^k_{it}\big) + \phi^{ef}_{it}\Big(MPe^f_{it} - q^f_t\Big) + \phi^r_{it}\Big(MPe^r_{it} - q^r_{it}\Big)$$

$$+ \phi^{Ef}_t\Big(q^f_t - \mathcal{C}^f_E(\cdot) - \lambda^{\mathcal{R}}_t\Big) + \phi^{\mathcal{I}f}_t\big(\delta\lambda^{\mathcal{R}}_t - \mathcal{C}^f_{\mathcal{I}}(\cdot)\big)$$

Rewritten for intuitions:

FOCs

$$[c_{it}] \qquad \psi^k_{it} = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{\text{=direct effect}} + \underbrace{\phi^c_{it}\mathcal{D}^u(\tau_{it}) u''(c_i)}_{\text{=effect on savings}}$$

$$\text{Define :} \qquad \widehat{\phi}^e_{it} = \phi^f_{it} MPe^f_t + \phi^r_{it} MPe^r_t$$

$$[e^f_{it}] \qquad \psi^k_{i,t}\Big(MPe^f_{it} - q^f_t\Big) + \xi_{it} p_i \psi^S_t + p_i \partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj + \partial_{e^f} \widehat{\phi}^e_{it} - p_i \phi^{Ef}_t \partial_{EE}\mathcal{C}(\cdot) = 0$$

$$[e^r_{it}] \qquad \psi^k_{i,t}\Big(MPe^r_{it} - q^r_{it}\Big) + \partial_{e^r}\widehat{\phi}^e_{it} = 0$$

$$[\mathcal{I}_t] \qquad \delta\,\psi^{\mathcal{R}}_t + \partial^2_{\mathcal{R}\mathcal{I}}\mathcal{C}(\cdot)\,\psi^{\lambda,\mathcal{R}}_t - \phi^{\mathcal{I}}_t \partial^2_{\mathcal{I}\mathcal{I}}\mathcal{C}(\cdot) = 0$$

$$[q^f_t] \qquad \phi^{Ef}_t = \int_{\mathbb{I}} e^f_{it}\psi^k_{jt} dj + \int_{\mathbb{I}} \phi^f_{jt} dj - \partial_{q^f}\pi^f(\cdot)\int_{\mathbb{I}} \theta_j \psi^k_{jt} dj$$

PMP rewritten for intuitions

$$[k_i] \qquad \dot{\psi}^k_{it} = \psi^k_{it}(\tilde{\rho} - r_{it} + \partial_k MPk_i)\psi^k_{it} - \partial_k \widehat{\phi}^e_{it}$$

$$[\mathcal{S}_i] \qquad \dot{\psi}^{\mathcal{S}}_t = \big(\tilde{\rho} + \delta^s\big)\psi^{\mathcal{S}}_t - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi^\tau_{jt} dj$$

$$[\tau_i] \qquad \dot{\psi}^\tau_t = \big(\tilde{\rho} + \zeta\big)\psi^\tau_t - \Big(\omega_i \mathcal{D}'(\tau_{it}) u(c_{it}) + \psi^k_{it}\mathcal{D}'(\tau_{it}) f(k_{it}, e_{it}) + \phi^c_{it}\mathcal{D}'(\tau_{it}) u'(c_i) + \partial_\tau \widehat{\phi}^e_{it}\Big)$$

$$[\mathcal{R}] \qquad \dot{\psi}^{\mathcal{R}}_t = \psi^{\mathcal{R}}_t\Big(\tilde{\rho} - \partial^2_{\mathcal{R}\mathcal{R}}\mathcal{C}(\cdot)\Big) - \phi^{Ef}_t \partial^2_{\mathcal{R}E}\mathcal{C}(\cdot)$$

$$[\lambda^k_i] \qquad \dot{\psi}^{\lambda,k}_t = \tilde{\rho}\psi^{\lambda,k}_t - (\rho - r_{i,t})\psi^k_t + \phi^c_{i,t}$$

$$[\lambda^{\mathcal{R}}_i] \qquad \dot{\psi}^{\lambda,\mathcal{R}}_t = \psi^{\lambda,\mathcal{R}}_t(\tilde{\rho} - \rho) + \phi^{Ef}_t - \delta\phi^{\mathcal{I}f}_t$$

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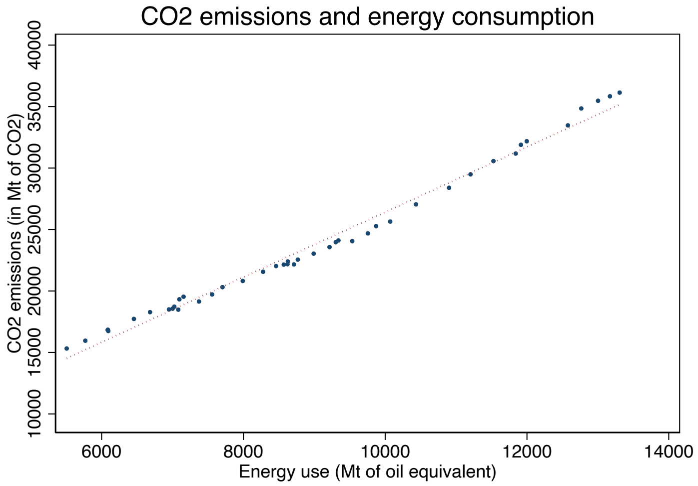
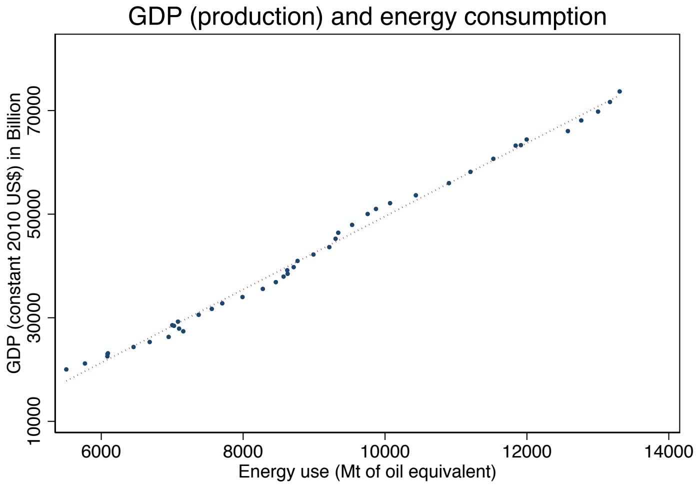
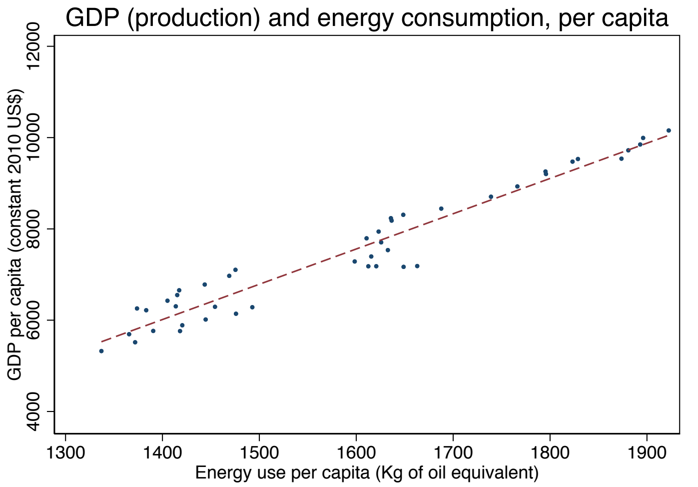
A Coupling GDP, Energy and Emissions

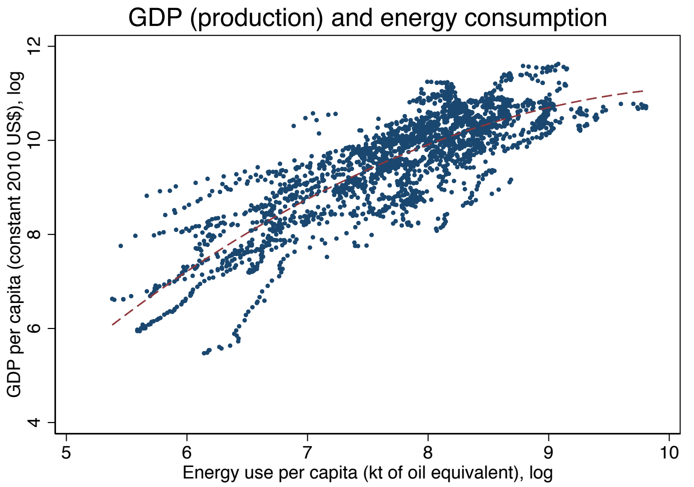
CO_2 emissions correlate linearly with energy use. Energy use (including 85% from fossil fuels sources) correlates with output/growth





This trend is also true per capita and for the trajectory of individual countries





B Toy model – More details on the Ramsey problem

The Lagrangian of the Ramsey writes as follow:

$$\mathcal{L}(\{c_i, e_i, \lambda_i^k\}_i, q^e) = \sum_{i=N,S} \omega_i U(c_i)p_i + \phi_i^k \big(\overbrace{\mathcal{D}_i(\mathcal{S}) z_i F(k_i, e_i)}^{=\tilde{F}(\mathcal{S},k_i,e_i)} + \theta_i \pi(q^e, E) - q^e e_i - c_i \big)$$
$$+ \phi_i^e \big(\partial_e \tilde{F}(\mathcal{S}, k_i, e_i) - q^e\big)$$
$$+ \phi_i^c \big(U'(c_i)p_i - \lambda_i^k\big)$$
$$+ \phi^E \big(q^e - c'(E)\big)$$
$$\text{with} \qquad E := e_N + e_S \qquad \& \qquad \mathcal{S} := \mathcal{S}_0 + \xi_N e_N + \xi_S e_S$$

FOC:

$$[c_i] \qquad \phi_i^k = \underbrace{\omega_i U'(c_i)}_{=\text{direct effect}} + \underbrace{\phi_i^c U''(c_i)}_{=\text{effect on consumption/saving choice}}$$

$$[\lambda_i^k] \qquad \phi_i^c = 0$$

$$[e_i] \qquad \phi_{i,t}^k \Big(\partial_e \tilde{F}(S, k_i, e_i) - q^e\Big) + \xi_i \underbrace{\sum_j \phi_j^k \mathcal{D}_j'(\mathcal{S}) y_j}_{=-\text{social cost of carbon}} + \underbrace{\partial_E \pi(q^e, E) \sum_j \theta_j \phi_j}_{=\text{energy rent redistribution}}$$
$$+ \underbrace{\phi_i^e \partial_{ee}^2 \tilde{F}(S, k_i, e_i)}_{=\text{effect on energy choice}} - \underbrace{\phi^E c''(E)}_{=\text{effet on energy supply}}$$

$$[q^e] \qquad \phi^E = \underbrace{\sum_j e_j \phi_j^k}_{=\text{expenditure impact}} + \underbrace{\sum_j \phi_j^e}_{=\text{individual energy choice}} - \underbrace{\partial_q \pi(q^e, E) \sum_j \theta_j \phi_i^k}_{=\text{aggregate supply \& rent}}$$

C Energy producers – fossil fuel company

We consider the simplest functional forms, yielding isoelastic supply curves for fossil energy extraction and exploration:

$$\nu(E, \mathcal{R}) = \frac{\bar{\nu}}{1+\nu} \Big(\frac{E}{\mathcal{R}}\Big)^{1+\nu} \mathcal{R} \qquad \mu(\mathcal{I}^e, \mathcal{R}) = \frac{\bar{\mu}}{1+\mu} \Big(\frac{\mathcal{I}^e}{\mathcal{R}}\Big)^{1+\mu} \mathcal{R}$$

Setting up the Hamiltonian,

$$\mathcal{H}(\mathcal{R}_t, \lambda_t^R, E_t, \mathcal{I}_t^e) = \pi_t(\mathcal{R}_t, E_t, \mathcal{I}_t^e) + \lambda_t^R (\delta^R \mathcal{I}_t^e - E_t)$$

The optimal decisions are given by:

$$[E_t] \qquad q_t^{e,f} = \nu_E(E,R) + \lambda_t^R = \bar{\nu}\Big(\frac{E_t}{\mathcal{R}_t}\Big)^{\nu} + \lambda_t^R$$

$$[\mathcal{I}_t] \qquad \lambda_t^R \delta^R = \mu_I(\mathcal{I}_t, \mathcal{R}_t) = \bar{\mu}\Big(\frac{\mathcal{I}_t}{\mathcal{R}_t}\Big)^{\mu} \qquad \mathcal{I}_t = \mathcal{R}_t\Big(\frac{\lambda_t^R \delta}{\bar{\mu}}\Big)^{1/\mu}$$

The Pontryagin Maximum Principle yields the dynamics of the costate :

$$\begin{aligned}
-\dot{\lambda}_t^R + \rho\lambda_t^R &= \partial_R \mathcal{H}(R, E^\star, I^\star) \\
\dot{\lambda}_t^R &= \rho\lambda_t^R + \partial_R \nu(E_t^\star, \mathcal{R}_t) + \partial_R \mu(\mathcal{I}_t^\star, \mathcal{R}_t) \\
\dot{\lambda}_t^R &= \rho\lambda_t^R - \frac{\bar{\nu}\nu}{1+\nu}\Big(\frac{E_t^\star}{\mathcal{R}_t}\Big)^{1+\nu} - \frac{\bar{\mu}\mu}{1+\mu}\Big(\frac{I_t^\star}{\mathcal{R}_t}\Big)^{1+\mu} \\
\dot{\lambda}_t^R &= \rho\lambda_t^R - \frac{\bar{\nu}\nu}{1+\nu}\Big(\frac{E_t^\star}{\mathcal{R}_t}\Big)^{1+\nu} - \frac{\bar{\mu}\mu}{1+\mu}\Big(\frac{I_t^\star}{\mathcal{R}_t}\Big)^{1+\mu}
\end{aligned}$$

Replacing it with the optimal decisions, we obtain a non-linear equation for the Hotelling rent:

$$\dot{\lambda}_t^R = \rho\lambda_t^R - \frac{\bar{\nu}^{-1/\nu}\nu}{1+\nu}\big(q^{e,f} - \lambda_t^R\big)^{1+1/\nu} - \frac{\bar{\mu}^{-1/\mu}\mu}{1+\mu}\big(\delta^R\lambda_t^R\big)^{1+1/\mu}$$

Moreover, we should add the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t^R \mathcal{R}_t = 0$$

and since we know that λ_t^R grows less fast than $e^{\rho t}$, we have the transversality respected even if $\mathcal{R}_t \not\to 0$ when $t \to \infty$.

This implies a (highly!) non-linear ODE for the Hotelling rent λ_t^R, where λ_0^R is chosen such that $\mathcal{R}_t = 0$ by terminal time $t = \bar{t}$. We can "simplify" the ODE, in the case where the cost are quadratic $\mu = \nu = 1$ and

$$\dot{\lambda}_t^R = \rho\lambda_t^R + \frac{1}{2\bar{\nu}}\big(q_t^{e,f} - \lambda_t^R\big)^2 + \frac{1}{2\bar{\mu}}\big(\delta^R\lambda_t^R\big)^2$$

We see that the Hotelling rent account for the extraction cost (scaled by $\bar{\nu}$) and the exploration cost (scaling in $\bar{\mu}$) and depend on the price/inverse demand for determining the quantity produced in equilibrium.

A stationary solution can be found in the case where $\dot{\lambda}_t^R = 0$

$$\begin{aligned}
\rho\lambda_t^R + \frac{1}{2\bar{\nu}}\big(q_t^{e,f} - \lambda_t^R\big)^2 + \frac{1}{2\bar{\mu}}\big(\delta^R\lambda_t^R\big)^2 &= 0 \\
\rho\lambda_t^R - \frac{1}{\bar{\nu}}q_t^{e,f}\lambda_t^R + \frac{1}{2\bar{\nu}}(\lambda_t^R)^2 + \frac{1}{2\bar{\nu}}(q_t^{e,f})^2 + \frac{1}{2\bar{\mu}}(\delta^R)^2(\lambda_t^R)^2 &= 0 \\
\lambda_\infty^R = \frac{\frac{q_t^{e,f}}{\bar{\nu}} - \rho \pm \sqrt{(\frac{q_t^{e,f}}{\bar{\nu}} - \rho)^2 - (\frac{1}{\bar{\nu}} + \frac{\delta^2}{\bar{\mu}})\frac{1}{\bar{\nu}}(q_t^{e,f})^2}}{\frac{1}{\bar{\nu}} + \frac{\delta^2}{\bar{\mu}}}
\end{aligned}$$

We obtain two stationary positive solutions: for a given energy price (demanded) $q^{e,f}$, in one equilibrium, the rent is very high, incentiving a lot of exploration as a share of reserve ($\mathcal{I}/\mathcal{R}$ is high) but the production is relatively low ($q^{e,f} - \lambda^R$ is low and so is the marginal cost and quantity $E/\mathcal{R}$). In a second stationary equilibrium, the rent is lower and the marginal cost is higher since the extraction is larger as a share of reserves. Note, that this stationary equilibrium is not consistent with state $\mathcal{R}_t$ dynamics since the reserves are depleting at different rates: only the first case is consistent with a sustainable level of extraction and exploration.

D Competitive equilibrium

Dynamics of the individual state variables $s_{i,t} = (k_{i,t}, \tau_{i,t}, z_i, p_i, \Delta_i)$ and aggregate ones $(\mathcal{S}_t, \mathcal{T}_t, \mathcal{R}_t)$:

$$\begin{aligned}
\dot{k}_t &= \mathcal{D}(\tau_t) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - c_t - q^e_t e_t - \Lambda_t(\vartheta_t) e^f_t \\
\mathcal{E}_t &= e^{(n+\bar{g})t} \int_{\mathbb{S}} \xi (1 - \vartheta_{i,t}) e^f_{i,t} p_{i,t} ds \\
\dot{\tau}_{i,t} &= \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t) \qquad \dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s \mathcal{S}_t \\
\dot{\mathcal{R}}_t &= -E^f_t + \delta_R \mathcal{I}_t \qquad q^{e,f}_t = \bar{\nu} (E^f_t / \mathcal{R}_t)^{\nu}
\end{aligned}$$

Household problem: Pontryagin Maximum Principle

$$\mathcal{H}^{hh}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}) = u(c_i, \tau_i) + \lambda^k_{i,t} \Big(\mathcal{D}(\tau_{it}) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t \Big)$$

$$\begin{aligned}
&[c_t] && u'(c_{it}) = \lambda^k_{i,t} \\
&[e^f_t] && MPe^f_{it} = \mathcal{D}(\tau_{i,t}) z \, \partial_e f(k_{i,t}, e_{i,t}) \Big(\frac{e^f_{i,t}}{\omega e_{i,t}} \Big)^{-\frac{1}{\sigma_e}} = q^f_t \\
&[e^r_t] && MPe^r_{it} = \mathcal{D}(\tau_{i,t}) z \, \partial_e f(k_{i,t}, e_{i,t}) \Big(\frac{e^r_{i,t}}{(1-\omega) e_{i,t}} \Big)^{-\frac{1}{\sigma_e}} = q^r_{it} \\
&[k_t] && \dot{\lambda}^k_t = \lambda^k_t \big(\rho - \partial_k f(k_{i,t}, e_{i,t}) \big)
\end{aligned}$$

Fossil Energy Monopoly problem:

$$\mathcal{H}^m(\mathcal{R}_t, \lambda^R_t, E_t, \mathcal{I}^e_t) = \pi_t(E^f_t, \mathcal{I}^f_t, \mathcal{R}_t) + \lambda^R_t (\delta^R \mathcal{I}^e_t - E_t)$$

$$\begin{aligned}
&[\mathcal{R}_t] && \dot{\lambda}^R_t = \rho \lambda^R_t + \frac{\bar{\nu}\nu}{1+\nu} \Big(\frac{E^\star_t}{R_t} \Big)^{1+\nu} + \frac{\bar{\mu}\mu}{1+\mu} \Big(\frac{I^\star_t}{R_t} \Big)^{1+\mu} \\
&[E^f_t] && q^{e,f}_t = \nu_E(E, \mathcal{R}) + \lambda^R_t = \bar{\nu} \Big(\frac{E_t}{\mathcal{R}_t} \Big)^{\nu} + \lambda^R_t \\
&[\mathcal{I}_t] && \lambda^R_t \delta^R = \mu_I(I_t, R_t) = \bar{\mu} \Big(\frac{I_t}{\mathcal{R}_t} \Big)^{\mu} \qquad I_t = R_t \Big(\frac{\lambda^R_t \delta}{\bar{\mu}} \Big)^{1/\mu}
\end{aligned}$$

E Optimal policy and Ramsey problem

The dynamic optimization problem of the Ramsey planner can be summarized by the Hamiltonian of the system, for the state $s_{i,t} = (k_{i,t}, \tau_{i,t}, z_i, p_i, \Delta_i)$.

$$\begin{aligned}\mathcal{H}^{sp}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}, \{\psi\}) = \int_{\mathbb{I}} \omega_i \mathcal{D}^u(\tau_{it}) u(c_i) p_i di + \psi^k_{i,t}\Big(\mathcal{D}(\tau_{it}) f(k_t, e_t) - (n + \bar{g} + \delta) k_t + \theta_i \pi(E^f_t, \mathcal{I}_t, \mathcal{R}_t) - q^f_t e^f_{it} - q^r_{it} e\\
+ \psi^S_t\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big) + \psi^\tau_{it}\zeta\Big(\Delta_i\, \chi\, \mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \psi^{\mathcal{R}}_{it}\Big(- E^f_t + \delta^R \mathcal{I}_t\Big)\\
+ \psi^{\lambda k}_{i,t}\Big(\lambda^k_t(\rho - r_t)\Big) + \psi^{\lambda R}_t\Big(\rho \lambda^R_t + \mathcal{C}^f_{\mathcal{R}}(E^f_t, \mathcal{I}_t, \mathcal{R}_t)\Big)\\
+ \phi^c_{it}\big(\mathcal{D}^u(\tau_{it}) u'(c_i) - \lambda^k_{it}\big) + \phi^{ef}_{it}\Big(MPe^f_{it} - q^f_t\Big) + \phi^r_{it}\Big(MPe^r_{it} - q^r_{it}\Big)\\
+ \phi^{Ef}_t\big(q^f_t - \mathcal{C}^f_E(\cdot) - \lambda^{\mathcal{R}}_t\Big) + \phi^{\mathcal{I}f}_t\big(\delta \lambda^{\mathcal{R}}_t - \mathcal{C}^f_{\mathcal{I}}(\cdot)\big)\end{aligned}$$

The FOCs of the Planners with respect to all the controls.

$$[c_{it}] \qquad \psi^k_{it} = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{\text{=direct effect}} + \underbrace{\phi^c_{it} \mathcal{D}^u(\tau_{it}) u''(c_i)}_{\text{=effect on savings}}$$

$$\text{Define :} \qquad \widehat{\phi}^e_{it} = \phi^f_{it} MPe^f_t + \phi^r_{it} MPe^r_t$$

$$[e^f_{it}] \qquad \psi^k_{i,t}\Big(MPe^f_{it} - q^f_t\Big) + \xi_{it} p_i \psi^S_t + p_i \partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj + \partial_{e^f} \widehat{\phi}^e_{it} - p_i \phi^{Ef}_t \partial_{EE} \mathcal{C}(\cdot) = 0$$

$$[e^r_{it}] \qquad \psi^k_{i,t}\Big(MPe^r_{it} - q^r_{it}\Big) + \partial_{e^r} \widehat{\phi}^e_{it} = 0$$

$$[\mathcal{I}_t] \qquad \delta\, \psi^{\mathcal{R}}_t + \partial^2_{\mathcal{R}\mathcal{I}} \mathcal{C}(\cdot)\, \psi^{\lambda,\mathcal{R}}_t - \phi^{\mathcal{I}}_t \partial^2_{\mathcal{I}\mathcal{I}} \mathcal{C}(\cdot) = 0$$

$$[q^f_t] \qquad \phi^{Ef}_t = \int_{\mathbb{I}} e^f_{it} \psi^k_{jt} dj + \int_{\mathbb{I}} \phi^f_{jt} dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj$$

Applying the Pontryagin Maximum Principle, we obtain the dynamics of the costate / Lagrange multipliers for state dynamics of the system.

$$[k_i] \qquad \dot{\psi}^k_{it} = \psi^k_{it}(\tilde{\rho} - r_{it} + \partial_k MPk_i)\psi^k_{it} - \partial_k \widehat{\phi}^e_{it}$$

$$[\mathcal{S}_i] \qquad \dot{\psi}^{\mathcal{S}}_t = \big(\tilde{\rho} + \delta^s\big)\psi^{\mathcal{S}}_t - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi^\tau_{jt} dj$$

$$[\tau_i] \qquad \dot{\psi}^\tau_t = \big(\tilde{\rho} + \zeta\big)\psi^\tau_t - \Big(\omega_i \mathcal{D}'(\tau_{it}) u(c_{it}) + \psi^k_{it} \mathcal{D}'(\tau_{it}) f(k_{it}, e_{it}) + \phi^c_{it} \mathcal{D}'(\tau_{it}) u'(c_i) + \partial_\tau \widehat{\phi}^e_{it}\Big)$$

$$[\mathcal{R}] \qquad \dot{\psi}^{\mathcal{R}}_t = \psi^{\mathcal{R}}_t\Big(\tilde{\rho} - \partial^2_{\mathcal{R}\mathcal{R}} \mathcal{C}(\cdot)\Big) - \phi^{Ef}_t \partial^2_{\mathcal{R}E} \mathcal{C}(\cdot)$$

$$[\lambda^k_i] \qquad \dot{\psi}^{\lambda,k}_t = \tilde{\rho} \psi^{\lambda,k}_t - (\rho - r_{i,t}) \psi^k_t + \phi^c_{i,t}$$

$$[\lambda^{\mathcal{R}}_i] \qquad \dot{\psi}^{\lambda,\mathcal{R}}_t = \psi^{\lambda,\mathcal{R}}_t(\tilde{\rho} - \rho) + \phi^{Ef}_t - \delta \phi^{\mathcal{I}f}_t$$

F Closed form solution for the Social Cost of Carbon

Solving for the shadow cost of carbon and temperature $\Leftrightarrow$ solving ODE

$$
\begin{aligned}
\dot{\psi}_{i,t}^{\tau} &= \psi_t^{\tau}(\widetilde{\rho} + \Delta\zeta) + \gamma(\tau - \tau^{\star})\mathcal{D}^y(\tau) f(k,e)\psi_t^k + \phi(\tau - \tau^{\star})\mathcal{D}^u(\tau)u(c) \\
\dot{\psi}_t^S &= {\psi^S}_t(\widetilde{\rho} + \delta^s) - \int_{\mathbb{I}} \Delta_i \zeta \chi \psi_{i,t}^{\tau}
\end{aligned}
$$

We need to solve for ψ_t^{τ} and ψ_t^S. In stationary equilibrium $\dot{\psi}_t^S = \dot{\psi}_t^{\tau} = 0$. As a result, we obtain:

$$
\begin{aligned}
\psi_{i,t}^{\tau} &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\zeta)u}(\tau_u - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_u) y_{\tau}\psi_u^k + \phi\mathcal{D}^u(\tau_u)u(c_u)\Big)du \\
\psi_{i,t}^{\tau} &= -\frac{1}{\widetilde{\rho} + \Delta\zeta}(\tau_{\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{\infty})y_{\infty}\psi_{\infty}^k + \phi\mathcal{D}^u(\tau_{\infty})u(c_{\infty})\Big) \\
\psi_t^S &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\int_{\mathbb{I}}\Delta_j\psi_{j,u}^{\tau}dj\ du \\
&= \frac{1}{\widetilde{\rho}+\delta^s}\zeta\chi\int_{\mathbb{I}}\Delta_j\psi_{j,\infty}^{\tau} \\
&= -\frac{\chi}{\widetilde{\rho}+\delta^s}\frac{\zeta}{\widetilde{\rho}+\zeta}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_{\infty}\psi_{j,\infty}^k + \phi\mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)dj \\
\psi_t^S &\xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho}+\delta^s}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_{j,\infty}\psi_{j,\infty}^k + \mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)dj
\end{aligned}
$$

which proves the analytical formula in the main text.

Moreover, observing that we obtained an expression for the Social Cost, we can rewrite it as the integral of Local Cost, invoking Fubini's theorem:

$$
\begin{aligned}
\psi_t^S &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\Delta_j\psi_{j,u}^{\tau}dj\ du \\
&= -\int_{\mathbb{I}}\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\Delta_j\psi_{j,u}^{\tau}\,du\,dj \\
&= \int_{\mathbb{I}}\psi_{j,t}^S dj
\end{aligned}
$$

with

$$
\begin{aligned}
\psi_{j,t}^S &= \int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\Delta_j\psi_{j,u}^{\tau}\,du \\
&\xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho}+\delta^s}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^y(\tau_{j,\infty})y_{j,\infty}\psi_{j,\infty}^k + \mathcal{D}^u(\tau_{j,\infty})u(c_{j,\infty})\Big)
\end{aligned}
$$