

The Inequality of Climate Change

WORK IN PROGRESS

Thomas Bourany

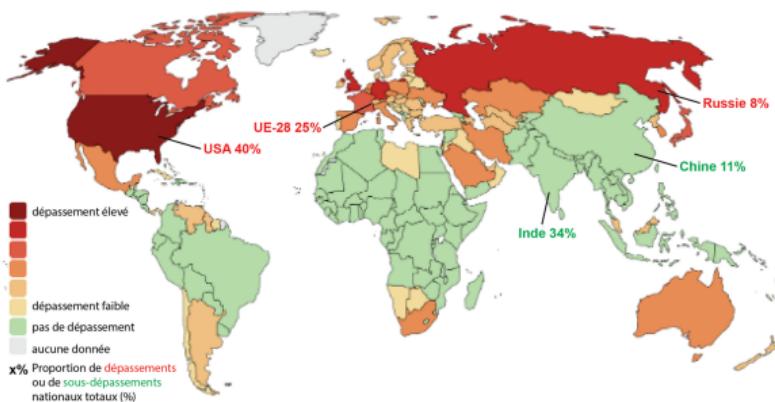
THE UNIVERSITY OF CHICAGO

Capital Theory

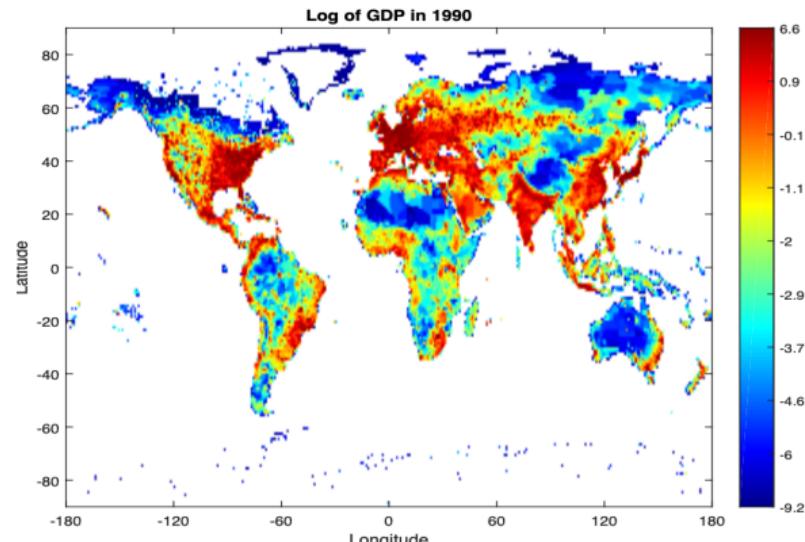
November 2022

Introduction – Motivation

- ▶ Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
 - ***Unequal causes*** : Developed economies account for over 65% of cumulative GHG emissions (~ 25% each for the EU and the US)

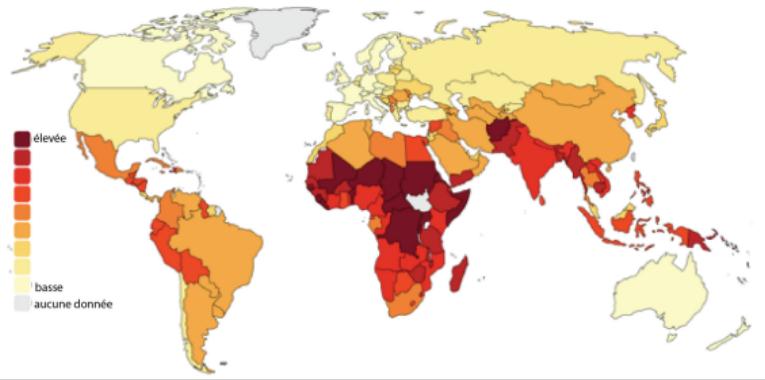


Source : Lancet planetary Health - Quantifying national responsibility for climate breakdown: an equality-based attribution approach for carbon dioxide emissions in excess of the planetary boundary - Jason Hickel 2020

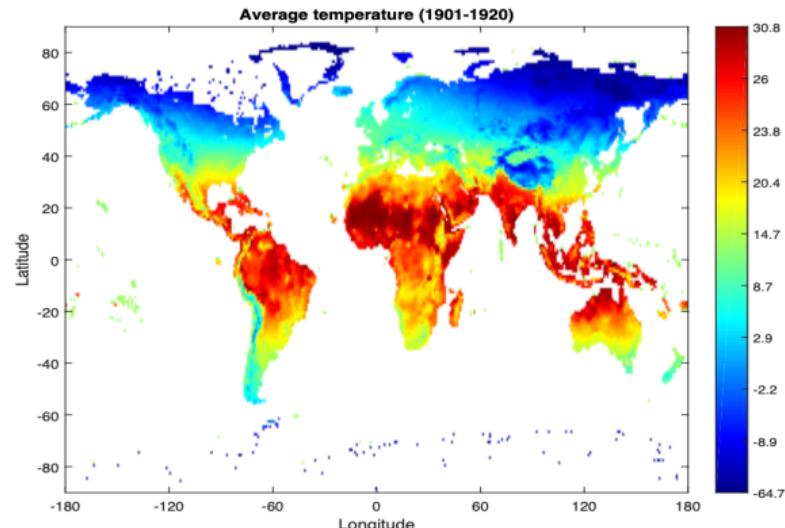


Introduction – Motivation

- ▶ Climate is warming due to greenhouse gas emissions generated by economic activity from different countries
 - ***Unequal consequences*** : Increase in temperatures will disproportionately affect developing countries where the climate is already warm



Source : Notre Dame Global Adaptation Initiative



Introduction – this project

- ▶ Which countries will be affected the most by climate change ?
 - Is the price of carbon heterogeneous across regions ? and why ?
 - What is the optimal policy in presence of externalities *and* heterogeneities ?

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- ▶ Develop a simple and flexible model of climate economics
 - Standard NCG – IAM model with heterogeneous regions
 - Every country is small relative to global GHG – no incentives to curb emissions

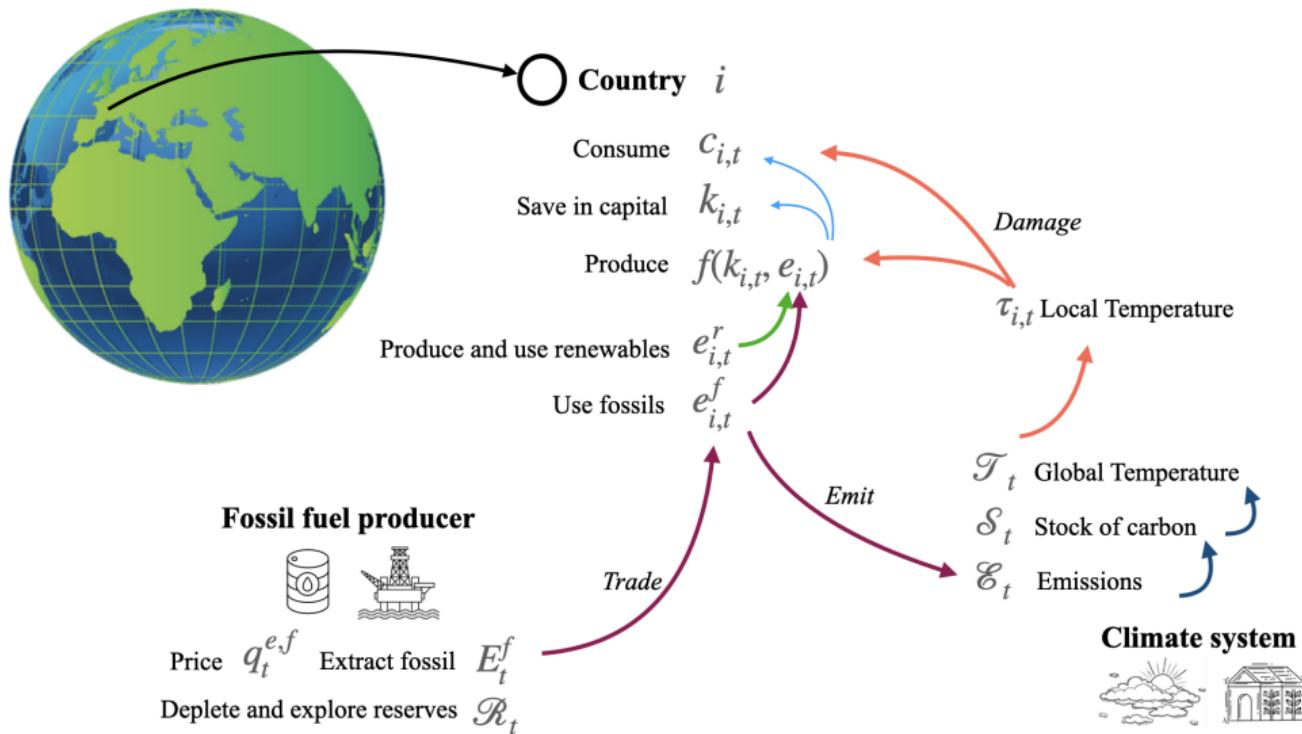
Introduction – this project

- ▶ Which countries will be affected the most by climate change ?
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- ▶ Develop a simple and flexible model of climate economics
 - Standard NCG – IAM model with heterogeneous regions
 - Every country is small relative to global GHG – no incentives to curb emissions
- Evaluate the heterogeneous welfare costs of global warming
- Provide analytical formulas and a numerical methodology to compute the cost of carbon
 - CC is linear in GDP /level of development and in temperature gaps
- Solve world optimal carbon policy with heterogeneous regions
 - Does the optimal carbon tax coincide with the social cost of carbon ?

Introduction – related literature

- ▶ Classic Integrated Assessment models (IAM) :
 - Nordhaus' Multi-regions DICE (2016), Golosov, Hassler, Krusell, Tsyvinski (2014)
- ▶ Models with analytical results :
 - Dietz, van der Ploeg, Rezai, Venmans (2021)
- ▶ Climate model with risk & uncertainty :
 - Cai, Lontzek, Judd (2019), Barnett, Brock and Hansen (2022)
- ▶ Macro (IAM) model with country heterogeneity :
 - Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021)
- ▶ Quantitative spatial models :
 - Cruz, Rossi-Hansberg (2021), Bilal, Rossi-Hansberg (2023), Rudik, Lyn, Tan, Ortiz-Bobea (2022)

Summary of the model



Model

- ▶ Neoclassical economy, in continuous time
 - countries/regions $i \in \mathbb{I}$: ex-ante heterogeneous in dimensions \underline{s}
 - Here : $\underline{s}_i = \{p_i, z_i, \Delta_i\}$, relative heterogeneity doesn't change over time
 - Productivity grows at rate \bar{g} and population grow at rate n
 - regions heterogeneous ex-post \bar{s}_i
 - Here : capital and temperature $\bar{s}_i = \{k_i, \tau_i\}$
 - Future : could include z – endog. technical change – or p – migrations/demographics
 - Renormalization : all variables are values per unit of efficient labor $k_{i,t} = \frac{K_{i,t}}{p_{t_0}} e^{-(\bar{g}+n)t}$
 - Aggregate variables : global Temperature, carbon Stocks in atmosphere, fossil energy Reserves $S = \{\mathcal{T}, \mathcal{S}, \mathcal{R}\}$

Model – Household and firm

- ▶ Household problem in country i :

$$\max_{\{c_t, e_t^f, e_t^r\}} \int_{t_0}^{\infty} e^{-(\rho-n)t} u_i(c_t, \tau_t) dt$$

- ▶ Dynamics of capital in every country i :

$$\dot{k}_t = \mathcal{D}(\tau_t)f(k_t, e_t) - (n + \bar{g} + \delta)k_t - c_t - q_t^e e_t$$

- ▶ Choices :

- c_t consumption, e_t energy, with production fct :

$$f(k, e) = z \left((1 - \varepsilon)^{\frac{1}{\sigma}} k^{\alpha \frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} (z^e e)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- Damage function $\mathcal{D}_i(\tau_t)$ affect country's production
- Directed technical change z_t^e & energy mix e_t with fossil e_t^f vs. renewable e_t^r

Model – Energy markets

- ▶ Two sources of energy : fossil e_t^f and renewable e_t^r for every i

$$e_{i,t} = \left(\omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e - 1}{\sigma^e}} + (1 - \omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r)^{\frac{\sigma^e - 1}{\sigma^e}} \right)^{\frac{\sigma^e}{\sigma^e - 1}}$$

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- ▶ Fossil fuels energy producer :
 - Extended-Hotelling problem (depleting reserves with stock effects and exploration)

$$\max_{\{E_t^f, \mathcal{I}_t\}_t} \pi_t(E_t^f, \mathcal{R}_t) = \max_{\{E_t^f, \mathcal{I}_t\}_t} q_t^{e,f} E_t^f - \mathcal{C}^e(E_t^f, \mathcal{R}_t) - \mathcal{C}^i(\mathcal{I}_t, \mathcal{R}_t)$$

$$s.t. \quad E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_{i,t} di \quad \dot{\mathcal{R}}_t = -E_t^f + \delta_R \mathcal{I}_t$$

- Optimal pricing with finite-resources rents More details

$$q_t^{e,f} = \bar{\nu} \left(\frac{E_t^f}{\mathcal{R}_t} \right)^\nu + \lambda_t^R \quad \bar{\mu} \left(\frac{\mathcal{I}_t}{\mathcal{R}_t} \right)^\mu = \delta_R \lambda_t^R$$

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- ▶ Renewable energy as a substitute technology

$$e_{i,t}^r = z_{i,t}^r k_{i,t}^{r,\alpha} \quad q_{i,t}^{e,r} = r_{i,t} / (z_{i,t}^r \alpha k_{i,t}^{r,\alpha-1})$$

Fossil energy and externality

- ▶ Fossil energy input e_t^f causes climate externality
 - Change the world climate – global temperature \mathcal{T}_t and cumulative GHG in atmosphere \mathcal{S}_t :

$$\begin{aligned}\mathcal{E}_t &= \int_{\mathbb{I}} \xi e_{i,t}^f p_i di \\ \dot{\mathcal{S}}_t &= \mathcal{E}_t - \delta_s \mathcal{S}_t \quad \quad \quad \dot{\mathcal{T}}_t = \zeta(\chi \mathcal{S}_t - \mathcal{T}_t)\end{aligned}$$

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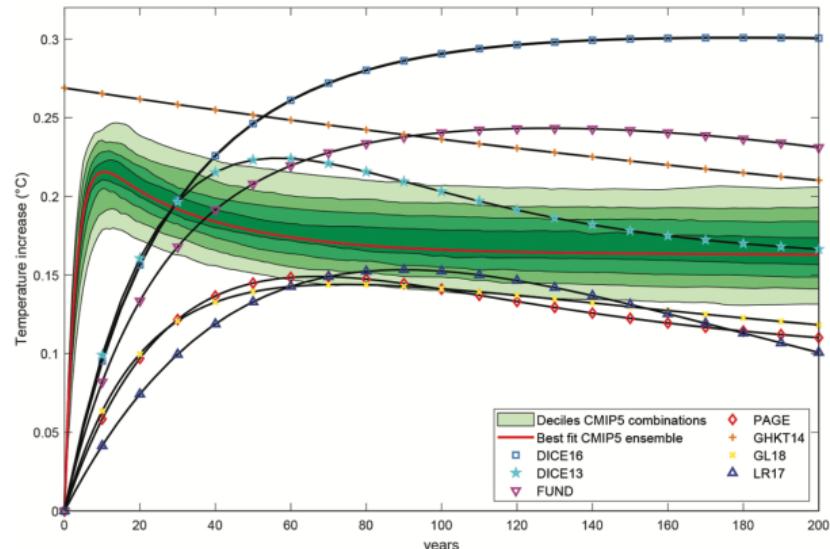
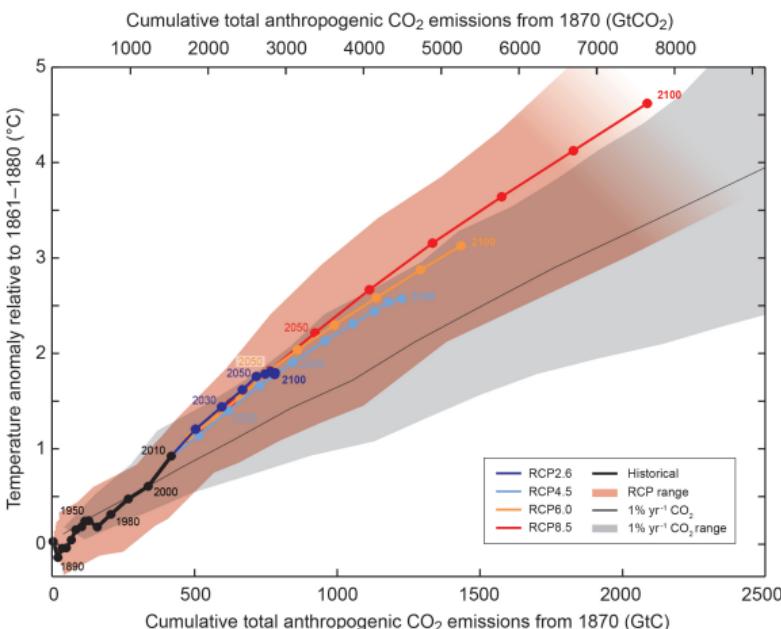
- ζ is the inverse of persistence, so if $\zeta \rightarrow \infty$, we obtain a linear model :

$$\mathcal{T}_t = \bar{\mathcal{T}} + \chi \mathcal{S}_t = \bar{\mathcal{T}} + \chi \int_{t_0}^t \int_{\mathbb{I}} \xi e_{i,s} di ds \Big|_{GIC}$$

- The externality depends on policy $e_{i,t}^f$ as function of states $\{z, p, k, \tau\}$
 - Naturally, countries richer/more productive/with a larger population use more energy !
- Impact of climate on country's local temperature :

$$\dot{\tau}_{i,t} = \Delta_i \dot{\mathcal{T}}_t$$

Temperature dynamics



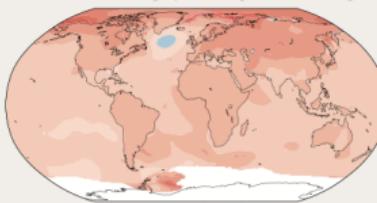
Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

Temperature dynamics

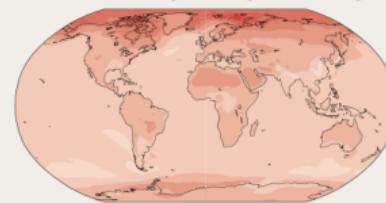
(a) Annual mean temperature change ($^{\circ}\text{C}$) at 1°C global warming

Warming at 1°C affects all continents and is generally larger over land than over the oceans in both observations and models. Across most regions, observed and simulated patterns are consistent.

Observed change per 1°C global warming



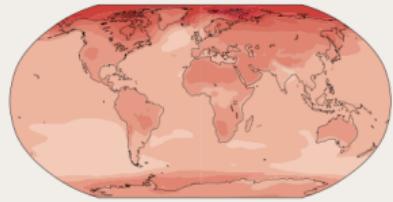
Simulated change at 1°C global warming



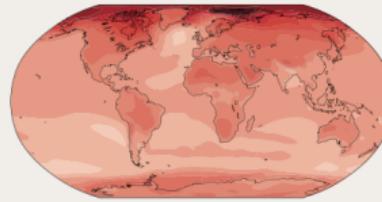
(b) Annual mean temperature change ($^{\circ}\text{C}$) relative to 1850–1900

Across warming levels, land areas warm more than ocean areas, and the Arctic and Antarctica warm more than the tropics.

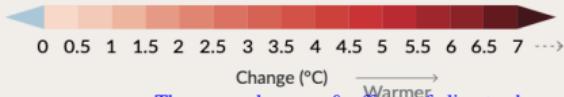
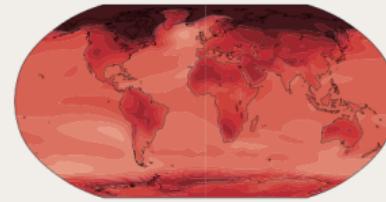
Simulated change at 1.5°C global warming



Simulated change at 2°C global warming



Simulated change at 4°C global warming



Damage functions

- ▶ Climate change has two effects :

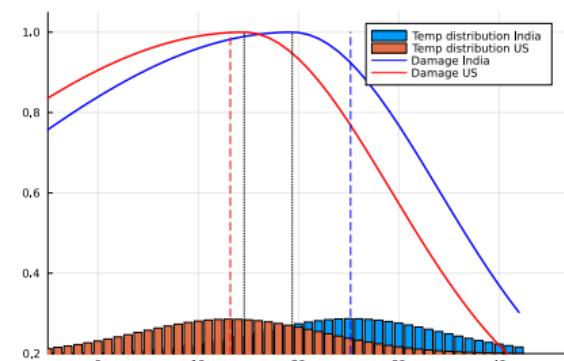
- Affect household utility function $u(c_t, \tau_t) = \mathcal{D}^u(\tau_t) \frac{c_t^{1-\eta}}{1-\eta}$

$$\mathcal{D}_u(\tau) = \begin{cases} e^{-\phi^\oplus \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau > \tau_i^* \\ e^{-\phi^\ominus \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau < \tau_i^* \end{cases}$$

- Affect firm productivity $\mathcal{D}(\tau_t)z$ as a shifter – as in Nordhaus DICE-2016

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma^\oplus \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau > \tau_i^* \\ e^{-\gamma^\ominus \frac{1}{2}(\tau - \tau_i^*)^2} & \text{if } \tau < \tau_i^* \end{cases}$$

- Deviation from "ideal" temperature
 $\tau_i^* = (1 - \alpha)\tau^* + \alpha\tau_{i,t_0}$, with $\tau^* = 15.5^\circ C$
- Damage sensitivity γ and ϕ is asymmetrical and can also be heterogeneous and uncertain



Marginal values of temperature

- ▶ Using Pontryagin Max. Principle :
 - We obtain a system of coupled ODEs [More details](#)
- ▶ Marginal values of the climate variables : λ^S and λ^T

$$\dot{\lambda}_{i,t}^\tau = \lambda_{i,t}^\tau (\tilde{\rho} + \Delta_i \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau_i^*) \mathcal{D}^y(\tau_{i,t})}^{-\partial_T \mathcal{D}^y} f(k_{i,t}, e_{i,t}) \lambda_{i,t}^k + \overbrace{\phi(\tau_{i,t} - \tau_i^*) \mathcal{D}^u(\tau_{i,t})}^{\partial_T \mathcal{D}^u} u(c_{i,t})$$

$$\dot{\lambda}_{i,t}^S = \lambda_{i,t}^S (\tilde{\rho} - \delta^s) - \Delta_i \zeta \chi \lambda_{i,t}^\tau$$

- ▶ Marg. cost for i of releasing carbon in atmosphere $\lambda_{i,t}^S$ increases with :
 - Temperature gap $\tau_{i,t} - \tau_i^*$
 - Damage sensitivity to temperature for TFP γ_i and utility ϕ
 - The development level $f(k_{i,t}, e_{i,t})$ and $u(c_{i,t})$

Social cost of carbon

- The marginal “externality damage” or “cost of carbon” can be expressed naturally :

$$CC_{i,t} := -\frac{\partial U_{i,t}/\partial \mathcal{S}_{i,t}}{\partial U_{i,t}/\partial c_{i,t}} = -\frac{\lambda_{i,t}^S}{\lambda_{i,t}^k}$$

Cost of country i of emitting one additional ton of CO_2 is

- Ratio of marg. cost of carbon vs. the marg. value of consumption/capital
- Stationary value : $t \rightarrow \infty$, with $\mathcal{E}_t = \delta_s \mathcal{S}_t$ and $\tau_t \rightarrow \tau_\infty$

$$CC_{i,t} \equiv \frac{\Delta \chi}{\tilde{\rho} - \delta^s} (\tau_{i,\infty} - \tau_i^\star) \left(\gamma_i \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} + \phi_i \mathcal{D}^u(\tau_{i,\infty}) \frac{c_{i,\infty}}{1-\eta} \right)$$

- Can integrate over time t & state-space i : $SCC_t = - \int_{\mathbb{S}} \frac{\lambda_{i,t}^S}{\lambda_{i,t}^k} di$
- Solution of the adjoint equation : [Proof](#)
- Uncertainty [SCC with uncertainty](#)

Social cost of carbon & temperature

- ▶ Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta \chi \xi \omega \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} q_t^{f-\sigma_e} \int_{j \in \mathbb{I}} (z_j z_{j,t}^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} y_{j,t} q_{j,t}^{\sigma_e - \sigma} dj dt$$

- Geographical factors determining warming Δ_i
- Climate sensitivity χ & carbon exit from atmosphere δ_s
- Growth of population n , aggregate productivity \bar{g}
- Deviation of output from trend y_i & relative TFP z_j
- Directed technical change z_t^e , elasticity of energy in output σ
- Fossil energy price $q_t^{e,f}$ and Hotelling rent $g_t^{q^f} \approx \lambda_t^R / \lambda_t^F = \rho$
- Change in energy mix, renewable share ω , price q_t^r & elasticity of source σ_e

- ▶ Approximations at $T \equiv$ Generalized Kaya (or $I = PAT$) identity More details

$$\frac{\dot{\tau}_T}{\tau_T} \propto n + \bar{g}^y - (1 - \sigma)(g^{z^e} - \tilde{\gamma}) + (\sigma_e - \sigma)(1 - \omega)g^{q^r} - (\sigma_e(1 - \omega) + \sigma\omega)g^{q^f}$$

Optimal energy and emissions decisions

- ▶ To determine the temperature paths, we need to know the decisions of energy use :
 - Different choices of emissions depending on the level of externality !
- 1. ***Business as usual*** : each country optimizes without internalization :

$$\max_{\{c_{i,t}, e_{i,t}^f, e_{i,t}^r\}_{t \geq 0}} U_{i,t_0} \quad \forall i \in \mathbb{I}$$

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- 2. ***Social planner*** : a planner chooses the decisions of all countries $i \in \mathbb{I}$:

$$\max_{\left\{ c_{i,t}, e_{i,t}^f, e_{i,t}^r \right\}_{i \in \mathbb{I}, t>0}} \int_{i \in \mathbb{I}} \omega_i U_{i,t_0} p_i di$$

- ▶ Details Details Social Planner

Optimal energy and emissions decisions

1. *Business as usual* :

- Fossil energy : only private tradeoff : marg. product of energy = marg cost + Hotelling rent

$$[e_{i,t}^f] \quad \underbrace{\mathcal{D}(\tau_{i,t})z \partial_e f(k_{i,t}, e_{i,t}) \left(\frac{e_{i,t}^f}{\omega e_{i,t}} \right)^{-\frac{1}{\sigma_e}}}_{MPe_{i,t}^f} = q_t^{e,f} = \bar{\nu} \left(\frac{E_t^f}{\mathcal{R}_t} \right)^\nu + \lambda_t^R$$

Optimal energy and emissions decisions

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2. *Social planner* :

- Fossil energy :

$$[e_{i,t}^f] \quad \underbrace{\hat{\lambda}_{i,t}^k \mathcal{D}(\tau_{i,t})z_{i,t} \partial_{ef}(k_{i,t}, e_{i,t}) \left(\frac{e_{i,t}^f}{\omega e_{i,t}} \right)^{-\frac{1}{\sigma_e}}}_{MPe_{i,t}^f} = \underbrace{\hat{\lambda}_t^{ef} + \hat{\lambda}_t^R}_{\text{=price of fossil energy}} - \underbrace{\int_{\mathbb{I}} \xi \omega_j \hat{\lambda}_{j,t}^S p_j dj}_{\hat{\lambda}_t^S = \text{externality cost}}$$

- Guess for carbon taxation ? $\mathbf{t}_t^S = -\frac{\lambda_t^S}{\lambda_{i,t}^k}$

Dynamic distortion : Competitive equilibrium vs. Social planner

- ▶ Competitive equilibrium, state : capital k and costate λ^k

$$\frac{\dot{c}_{i,t}}{c_{i,t}} = \frac{1}{\eta} (r_{i,t} - \rho)$$

- ▶ Social planner allocation :

- Impact of $\mathcal{E}_t = \int_{\mathbb{I}} e_{i,t}^f p_{i,t} di$

$$\frac{\dot{c}_{i,t}}{c_{i,t}} = \frac{1}{\eta} (r_{i,t} - \rho) - \frac{de_{i,t}^f}{dk_{i,t}} \left(\frac{\widehat{\lambda}_t^{ef} + \widehat{\lambda}_t^R - \widehat{\lambda}_t^S}{\widehat{\lambda}_{i,t}^k} \right)$$

- Marginal value of state λ_t decreases with the externality of \bar{E}
- Consume more today because more capital in the future affects the choice of energy $e(s)$ through $\partial_s e(\tilde{s})$

$$\frac{de_{i,t}^f}{dk_{i,t}} = \underbrace{\frac{\partial e_{i,t}^f}{\partial e_{i,t}} \frac{de_{i,t}}{dk_{i,t}}}_{\text{energy demand effect}} + \underbrace{\frac{\partial e_{i,t}^f}{\partial q_{i,t}^r} \frac{dq_{i,t}^r}{dr_{i,t}} \frac{dr_{i,t}}{dk_{i,t}}}_{\text{substitution effect}}$$

- Dynamic distortion : PMP for HA models

Decentralization - 1

- ▶ With inequality $\widehat{\lambda}_{i,t}^k \neq \widehat{\lambda}_{j,t}^k$, it is unclear how to decentralize
- ▶ Allowing lump sum transfer across countries solves world inequality

$$\lambda_{i,t}^k = \lambda_{j,t}^k = \bar{\lambda}_t^k$$

- ▶ Restricting to intra-country redistribution :
 - Allow distortive taxes and lump-sum rebate in i only
 - ⇒ Planner cares about redistribution : $\lambda_{i,t}^k$ vs. $\bar{\lambda}_t^k = \int_j \lambda_{j,t}^k p_j dj$
 - Combination of distortive taxes and lump-sum rebate

Decentralization - 2

- ▶ Energy taxes :

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S \quad \Leftrightarrow \quad MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \underbrace{\frac{\bar{\lambda}_t^k}{\lambda_{i,t}^k}}_{=\text{redistribution}} \left(\underbrace{q_t^{e,f}}_{= \frac{\widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R}{\bar{\lambda}_t^k}} + \underbrace{\mathbf{t}_t^S}_{= -\frac{\widehat{\lambda}_t^S}{\bar{\lambda}_t^k}} \right)$$

- ▶ Combination :

- Redistributive taxes/subsidy : $\mathbf{t}_{i,t} = \frac{\bar{\lambda}_t^k}{\lambda_{i,t}^k}$
- The Pigouvian tax : \mathbf{t}_S^f is flat rate accounting for the climate externality

$$\mathbf{t}_t^S = -\frac{\bar{\lambda}_t^S}{\bar{\lambda}_t^k} = -\int_{\mathbb{I}} \frac{\lambda_{i,t}^S}{\bar{\lambda}_t^k} p_i di \neq CC_{i,t}$$

- Capital taxes :

$$\mathbf{t}_t^k = \frac{de_{i,t}^f}{dk_{i,t}} \frac{\widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R - \widehat{\lambda}_t^S}{\bar{\lambda}_{i,t}^k}$$

- Everything is rebated lump sum to the household in each country i

Carbon taxation

- We saw two notions to prices and tax carbon emissions :

$$CC_{i,t} = -\frac{\lambda_{i,t}^S}{\lambda_{i,t}^k}$$

- The "social cost of carbon" is often expressed as :

$$SCC_t = \int_{\mathbb{I}} CC_{j,t} dj = - \int_{\mathbb{I}} \frac{\lambda_{j,t}^S}{\lambda_{j,t}^k} p_{j,t} di$$

- We noticed that Pigouvian carbon taxes on fossils $e_{i,t}^f$ for country i are :

$$\mathbf{t}_{i,t}^S = -\frac{1}{\lambda_{i,t}^k} \int_{\mathbb{I}} \omega_j \lambda_{j,t}^S p_{j,t} dj \neq - \int_{\mathbb{S}} \frac{\lambda_{j,t}^S}{\lambda_{j,t}^k} p_{j,t} di$$

- In particular, rich/developed countries would pay larger taxes (lower $\lambda_t^{k,i}$)
- Redistribution motive to correct from unequal marginal costs of emitting

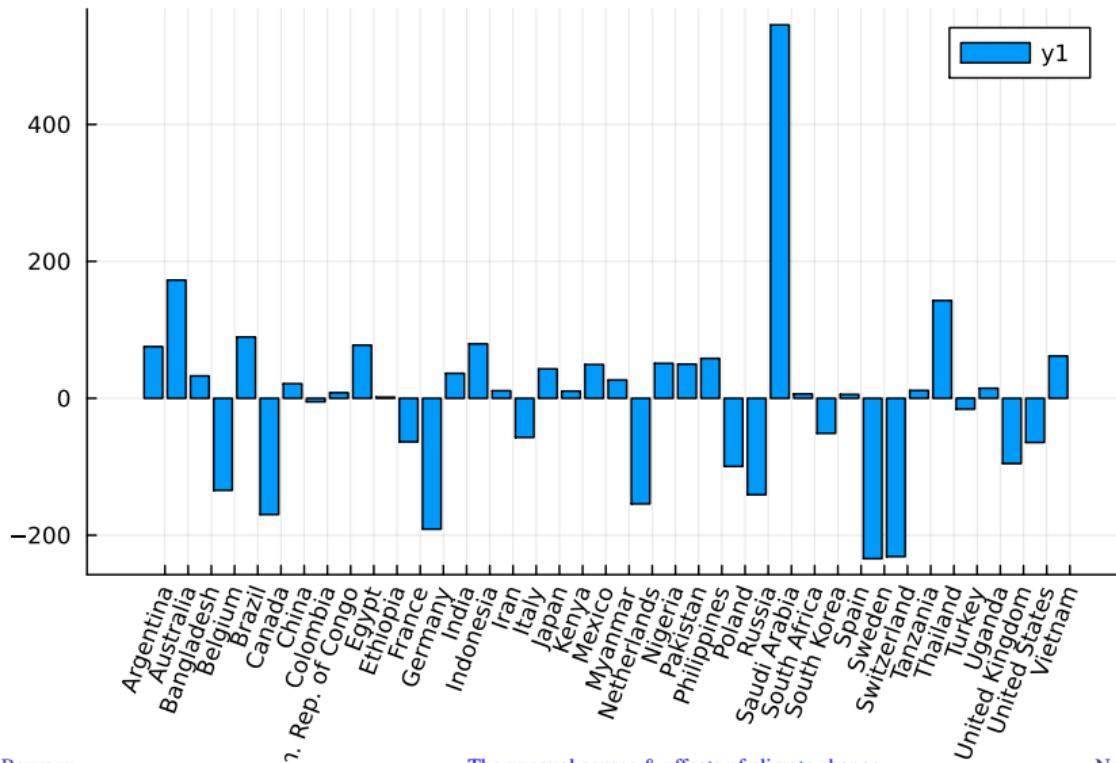
Application

- ▶ Data : 40 countries
- ▶ Temperature (of the largest city), GDP, energy, population
- ▶ Calibrate z to match the distribution of output per capita at steady state

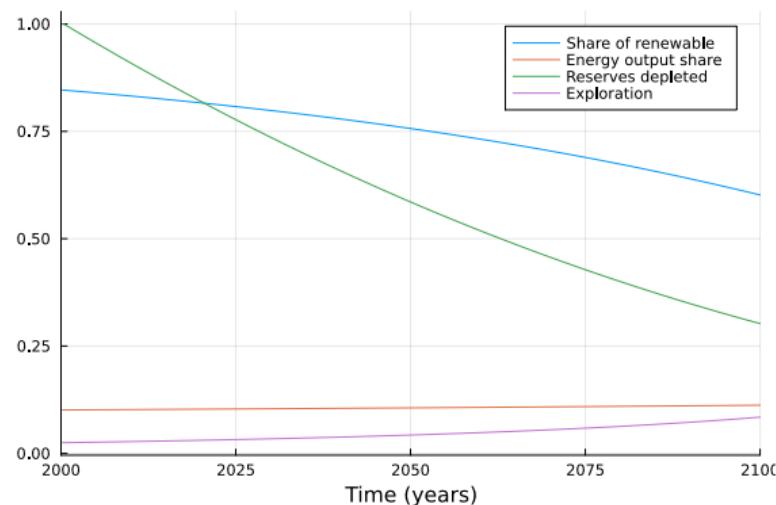
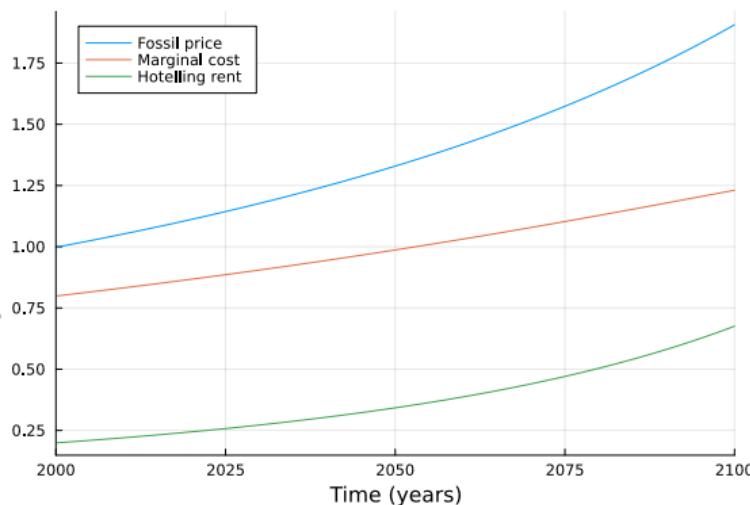


Created with mapchart.net

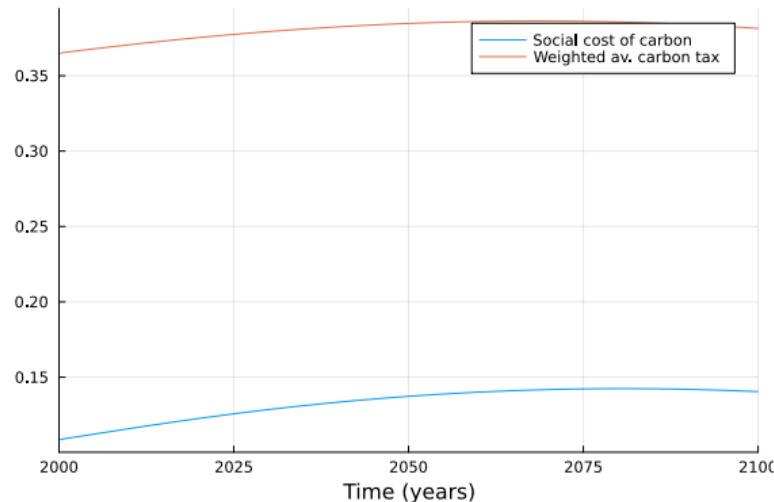
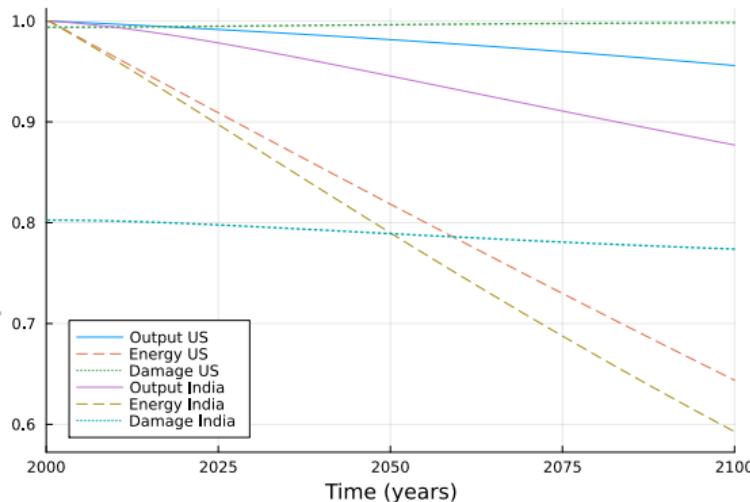
Distribution of carbon prices :



Energy



Output and carbon price



Extensions - 1 - Endogenous growth

- ▶ As of now, TFP z_t and directed technical change z_t^e are exogenous – growing at \bar{g}^y and g^{z^e}
- ▶ Could easily nest an endogenous growth block in this model

- Household / firm in country i chooses an amount $x_{i,t}$ of R&D to be allocated to increase TFP at rate ω_t^z or energy technology (efficiency)
- Cost $c(x_{i,t})$

$$\dot{z}_t = h^y(\omega^z x_t) \quad \dot{z}_t^e = h^e((1 - \omega^z)x_t)$$

- As a result, the marginal value of an investment in R&D is "priced" on the costates :

$$\begin{aligned} -\dot{\lambda}_t^z + \rho \lambda_t^z &= \lambda_t^k \mathcal{D}(\tau_t) f(k_t, e_t) && \text{Recall : } y_t = z_t \mathcal{D}(\tau_t) f(k_t, e_t) \\ -\dot{\lambda}_t^{z^e} + \rho \lambda_t^{z^e} &= \lambda_t^k z_t \mathcal{D}(\tau_t) \partial_{z^e} f(k_t, e_t) \end{aligned}$$

- And optimal decisions depend on this shadow value ;

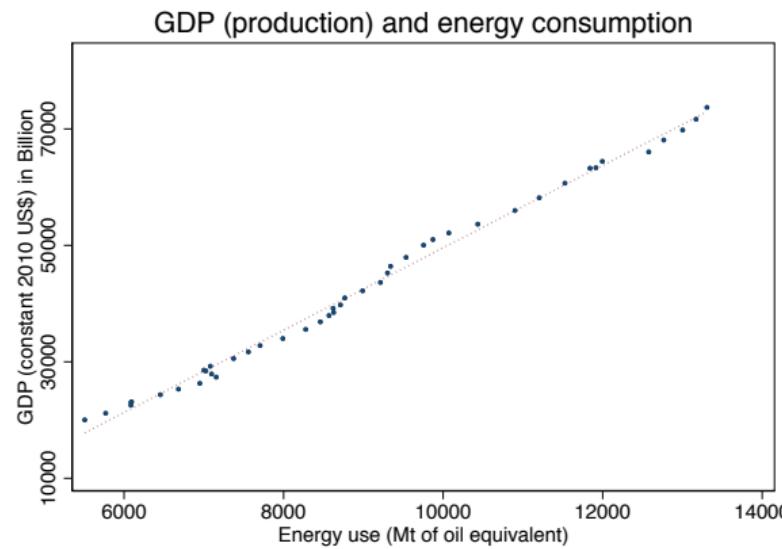
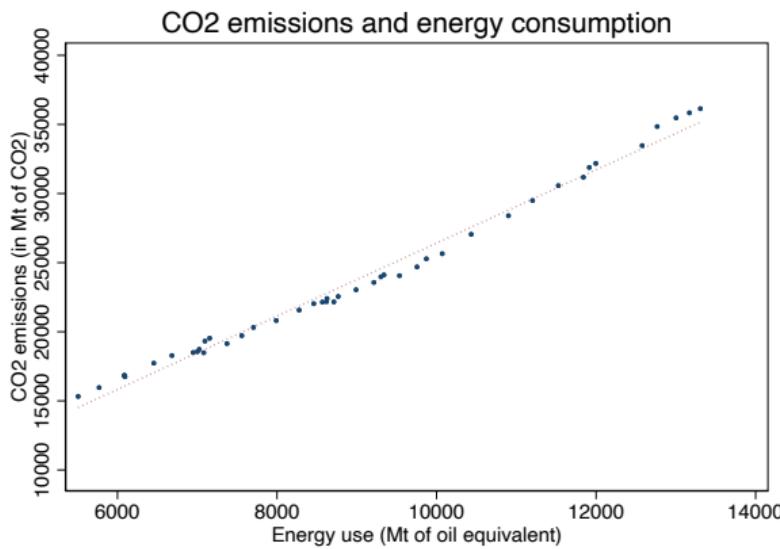
$$\omega^z h'^y(\omega^z x_t) \frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t}) \quad (1 - \omega^z) h'^e((1 - \omega^z)x_t) \frac{\lambda_t^z}{\lambda_t^k} = c'(x_{i,t})$$

Conclusion

- ▶ Climate change is induced by externality
 - Energy/Emission choice doesn't include the impact on other countries
 - Cause strengthened by heterogeneity in wealth (capital/productivity)
 - Effect strengthened by heterogeneity in impact (temperature/damage)
- ▶ Social planner allocation correct for these different dimensions
 - Both Static correction \equiv modified Pigouvian carbon taxation
 - And dynamic : through the marginal value of states
- ▶ Future plans :
 - Simulation of the three equilibria $CE/tax/SP$
 - Match the distribution of k using dynamics over 1960-2020
 - Social cost of carbon including heterogeneity and model uncertainty

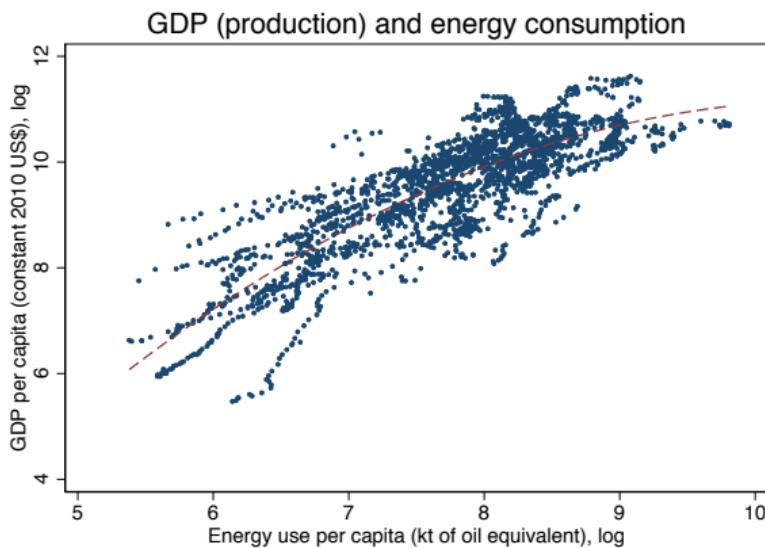
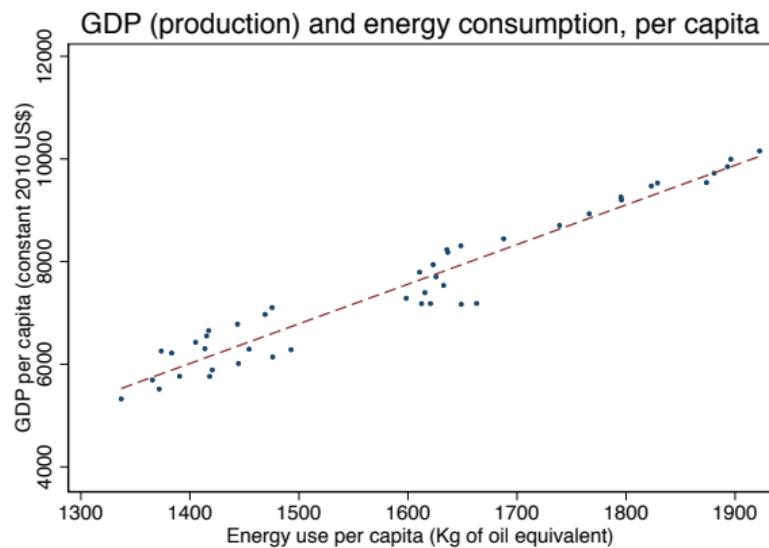
Motivation

- ▶ CO_2 emissions correlate linearly with energy use
- ▶ Energy use (85% from fossils) correlates with output/growth



Introduction – Motivation

- ▶ Also true per capita and for the trajectory of individual countries



More details – Energy market

- ▶ Fossil fuel producer : price the Hotelling rent with the maximum principle :
- ▶ Rent λ_t^R grows with interest ρ and with the marginal gain of increasing reserves

$$\dot{\lambda}_t^R = \rho\lambda_t^R + \frac{\bar{\nu}\nu}{1+\nu} \left(\frac{E_t^*}{R_t} \right)^{1+\nu} + \frac{\bar{\mu}\mu}{1+\mu} \left(\frac{I_t^*}{R_t} \right)^{1+\mu}$$

$$\dot{\lambda}_t^R = \rho\lambda_t^R + \frac{\bar{\nu}^{-1/\nu}\nu}{1+\nu} (q^{e,f} - \lambda_t^R)^{1+1/\nu} + \frac{\bar{\mu}^{-1/\mu}\mu}{1+\mu} (\delta^R \lambda_t^R)^{1+1/\mu}$$

- ▶ Alternative energy price q_t^C

$$\begin{cases} \vartheta_t = 0 & \text{if } q_t^C < q_t^e \\ \vartheta_t = 1 & \text{if } q_t^C > q_t^e \end{cases}$$

More details – PMP

- ▶ State variables $s = (p, z, \delta, k, \tau, \mathcal{T}, \mathcal{S}, \mathcal{R})$ and three controls (c, e, ϑ)

$$\dot{k}_t = \mathcal{D}(\tau_t)f(k_t, e_t) - (n + \bar{g} + \delta)k_t - c_t - q_t^e e_t - \Lambda_t(\vartheta_t)e_t$$

$$\mathcal{E}_t = e^{(n+\bar{g})t} \int_{\mathbb{S}} \xi(1 - \vartheta_t(s)) e_t^f(s) p_t(s) ds$$

$$\dot{\tau}_t = \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t) \quad \dot{\mathcal{S}}_t = \mathcal{E}_t - \delta^s \mathcal{S}_t$$

$$\dot{R}_t = -E_t + \delta_R \mathcal{I}_t$$

$$s_0 = (z_{0,i}, k_{0,i}, R_0, T_{0,i})$$

- ▶ Pontryagin Maximum Principle

$$\mathcal{H}(s, c, e, \varphi, \{\lambda\}) = u(c, \tau) + \lambda^k \dot{k} + \lambda^S \dot{S} + \lambda^T \dot{T}$$

$$\partial_c \mathcal{H}(\cdot) = 0 \quad \partial_e \mathcal{H}(\cdot) = 0 \quad -\dot{\lambda}_t^x + \tilde{\rho} \lambda_t^x = \partial_x \mathcal{H}(\cdot)$$

- Hamiltonian :

$$\begin{aligned} \mathcal{H}(s, c, \lambda) = & u(c, \tau) + \lambda^k \left(\mathcal{D}(\tau_t)f(k_t, e_t) - (n + \bar{g} + \delta)k_t - c_t - q_t^e e_t \right) \\ & + \lambda_t^S \left(\int_{\mathbb{S}} \xi e_t^f p_t ds - \delta^s \mathcal{S}_t \right) + \lambda_t^T \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t) \end{aligned}$$

Cost of carbon / Marginal value of temperature

- ▶ Solving for the cost of carbon and temperature \Leftrightarrow solving ODE

$$\begin{aligned}\dot{\lambda}_t^T &= \lambda_t^T(\tilde{\rho} + \Delta\zeta) + \gamma(T - T^*)\mathcal{D}^y(T)f(k, e)\lambda_t^k + \phi(T - T^*)\mathcal{D}^u(T)u(c) \\ \dot{\lambda}_t^S &= \lambda_t^S(\tilde{\rho} - \delta^s) - \Delta\zeta\chi\lambda_t^T\end{aligned}$$

- ▶ Solving for λ_t^T and λ_t^S , in stationary equilibrium $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$

$$\begin{aligned}\lambda_t^S &= - \int_t^\infty e^{-(\tilde{\rho} - \delta^s)\tau} \Delta\zeta\chi\lambda_\tau^T d\tau \\ \lambda_t^T &= - \int_t^\infty e^{-(\tilde{\rho} + \Delta\zeta)\tau} (T_\tau - T^*) \left(\gamma\mathcal{D}^y(T_\tau)y_\tau\lambda_\tau^k + \phi\mathcal{D}^u(T_\tau)u(c_\tau) \right) d\tau \\ \lambda_t^T &= - \frac{1}{\tilde{\rho} + \Delta\zeta} (T - T^*) \left(\gamma\mathcal{D}^y(T_\infty)y_\infty\lambda_\infty^k + \phi\mathcal{D}^u(T_\infty)u(c_\infty) \right) \\ \lambda_t^S &= \frac{1}{\tilde{\rho} - \delta^s} \Delta\zeta\chi\lambda_\infty^T \\ &= - \frac{\Delta\chi}{\tilde{\rho} - \delta^s} \frac{\zeta}{\tilde{\rho} + \Delta\zeta} (T - T^*) \left(\gamma\mathcal{D}^y(T_\infty)y_\infty\lambda_\infty^k + \phi\mathcal{D}^u(T_\infty)u(c_\infty) \right) \\ \lambda_t^S &\xrightarrow{\zeta \rightarrow \infty} - \frac{\Delta\chi}{\tilde{\rho} - \delta^s} (T - T^*) \left(\gamma\mathcal{D}^y(T_\infty)y_\infty\lambda_\infty^k + \mathcal{D}^u(T_\infty)u(c_\infty) \right)\end{aligned}$$

Cost of carbon / Marginal value of temperature

► Closed form solution for CC :

- In stationary equilibrium : $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
- Fast temperature adjustment $\zeta \rightarrow \infty$
- no internalization of externality (business as usual)

$$CC_t(s) \equiv \frac{\Delta\chi}{\tilde{\rho} - \delta^s} (T_\infty - T^*) \left(\gamma \mathcal{D}^y(T_\infty) y_\infty + \phi \mathcal{D}^u(T_\infty) \frac{c_\infty}{1 - \eta} \right)$$

► Heterogeneity + uncertainty about models

Uncertainty about models :

- ▶ In our model, we rely strongly on model specification :

- Parameters θ of models :
 - Climate system and damages : $(\xi, \chi, \zeta, \delta^s, \gamma, \phi)$
 - Economic model : $\varepsilon, \nu, \bar{g}, n$ or extended : $\omega, \sigma, \sigma^e, \nu, \mu$
 - Models with probability weight $\pi(\theta)$
- Social cost of carbon, weighted for model uncertainty :

$$SCC_t(\theta) = - \int_{\mathbb{S}} \frac{\lambda_t^S(s, \theta)}{\lambda_t^k(s, \theta)} g(s) ds$$

$$S\bar{C}C_t = \int_{\Theta} SCC_t(\theta) \pi(\theta) d\theta = \int_{\Theta} \int_{\mathbb{S}} \frac{\lambda_t^S(s, \theta)}{\lambda_t^k(s, \theta)} g(s) ds \pi(\theta) d\theta$$

- Counterfactual computation of SCC
 - Representative country / no uncertainty $\frac{\lambda_t^S}{\lambda_t^k}$
 - With heterogeneous regions / no uncertainty $SCC_t(\bar{\theta})$
 - No heterogeneity / model uncertainty $\int_{\Theta} \frac{\lambda_t^S(\bar{s}, \theta)}{\lambda_t^k(\bar{s}, \theta)} \pi(\theta) d\theta$
 - With heterogeneous regions / with model uncertainty $S\bar{C}C_t$
- back

Long term temperature

- ▶ Integrating temperature dynamics :

$$\begin{aligned}\tau_{i,T} - \tau_{i,t_0} &= \Delta_i(\mathcal{T}_T - \mathcal{T}_{t_0}) = \Delta_i \chi \int_{t_0}^T e^{-\delta_s(T-t)} \mathcal{E}_t dt \\ &= \Delta_i \chi \xi \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e_{j,t}^f (1 - \vartheta_{j,t}) dj dt\end{aligned}$$

- Use CES demands : $e_{j,t}^f = \omega e_{j,t} q_t^{-\sigma_e} q_t^{\sigma_e}$ for energy and $e_t = (z z_t^e \mathcal{D}(\tau_{j,t})^{\sigma-1} q_t^{-\sigma})$
- Moreover, CES price index $q_t = (\omega q_t^{f,1-\sigma_e} + (1-\omega)q_t^{r,1-\sigma_e})^{1/(1-\sigma_e)}$, so first order approximation : $g^q = \omega g^{q^f} + (1-\omega)g^{q^r}$ with growth for q^f and q^r as well as $z_t^e = e^{g^e t}$
- Gives :

$$e_{j,t}^f = \omega q_t^{-\sigma_e} q_{j,t}^{\sigma_e} (z z_t^e \mathcal{D}(\tau_{j,t})^{\sigma-1} q_{j,t}^{-\sigma})$$

Temperature dynamics

- ▶ Integrating temperature dynamics :

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} \int_{j \in \mathbb{I}} e_{j,t}^f (1 - \vartheta_{j,t}) dj dt$$

$$\tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \int_{t_0}^T e^{(n+\bar{g})t - \delta_s(T-t)} q_t^{-\sigma_e} \int_{j \in \mathbb{I}} \omega(z z^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} e^{g^e t} q_{j,t}^{\sigma_e - \sigma} (1 - \vartheta_{j,t}) dj dt$$

$$\begin{aligned} \tau_{i,T} - \tau_{i,t_0} = \Delta_i \chi \xi \omega \int_{t_0}^T & e^{(n+\bar{g})t - \delta_s(T-t)} e^{(-\sigma_e + (\sigma_e - \sigma)\omega) g^f t} e^{(\sigma_e - \sigma)(1-\omega) g^r t} \\ & \times \int_{j \in \mathbb{I}} (z z_t^e \mathcal{D}(\tau_{j,t}))^{\sigma-1} y_{j,t} (1 - \vartheta_{j,t}) dj dt \end{aligned}$$

- ▶ back

Social Planner allocation

- ▶ Solving the social planner allocation : Hamiltonian

$$\begin{aligned} \mathcal{H}^{sp}(s, \{c\}, \{e^f\}, \{e^r\}, \{\lambda\}) = & \int_{\mathbb{I}} \omega_i u(c_i, \tau_i) p_i di - w L_t^f + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^k \left(\mathcal{D}(\tau_t) f(k_t, e_t) - (n + \bar{g} + \delta) k_t - c_t \right) p_i di \\ & + \widehat{\lambda}_t^S \left(\int_{\mathbb{I}} \xi^f e_t^f p_i di - \delta^S \mathcal{S}_t \right) + \int_{\mathbb{I}} \widehat{\lambda}_{i,t}^\tau \Delta_i \zeta (\chi \mathcal{S}_t - \mathcal{T}_t) p_i di \\ & + \widehat{\lambda}_t^R \left(-E_t^f + \delta^R \mathcal{I}_t \right) + \widehat{\lambda}_t^{e^f} \left(\widetilde{\mathcal{F}}(L_t^f, \mathcal{R}_t) - E_t^f \right) + \int_{\mathbb{I}} \widehat{\lambda}_t^{e^r} \left(z_{i,t}^r k_{i,t}^{r,\alpha} - e_t^r \right) p_i di \end{aligned}$$

with $E_t^f = \int_{\mathbb{I}} e_{i,t}^f p_i di$ and $e_t = \left(\omega^{\frac{1}{\sigma^e}} (e_{i,t}^f)^{\frac{\sigma^e-1}{\sigma^e}} + (1-\omega)^{\frac{1}{\sigma^e}} (e_{i,t}^r)^{\frac{\sigma^e-1}{\sigma^e}} \right)^{\frac{\sigma^e}{\sigma^e-1}}$

- ▶ Results :

$$\omega_i u_c(c_i, \tau_i) = \widehat{\lambda}_{i,t}^k$$

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{e^f} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S$$

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^r} = \widehat{\lambda}_t^{e^r}$$

▶ Back

Social Planner allocation

- ▶ Moreover, using Pontryagin principle, accounting for the distribution, we should have an adjustment for the state dynamics :
 - Valuation of the state changes when the planner knows if after the externality

▶ [Back](#)

Decentralization

- ▶ With inequality $\widehat{\lambda}_{i,t}^k \neq \widehat{\lambda}_{j,t}^k$, it's unclear how to decentralize
- ▶ Allowing lump sum transfer across countries solves world inequality $\lambda_{i,t}^k = \lambda_{j,t}^k = \bar{\lambda}_t^k$, as a result :

$$\widehat{\lambda}_{i,t}^k MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \widehat{\lambda}_t^{ef} + \widehat{\lambda}_t^R + \widehat{\lambda}_t^S \quad \Leftrightarrow \quad MPe_{i,t} \frac{\partial e_{i,t}}{\partial e_{i,t}^f} = \left(\underbrace{\mathcal{C}'(E_t^f) + \lambda_t^R}_{=q_t^{e,f} = \frac{\widehat{\lambda}_t^{ef} + \widehat{\lambda}_t^R}{\bar{\lambda}_t^k}} + \underbrace{T_t^f}_{= \frac{\widehat{\lambda}_t^S}{\bar{\lambda}_t^k}} \right)$$

- ▶ The climate \mathbf{t}_t^f is flat rate accounting for the climate externality

$$\mathbf{t}_t^f = \frac{\widehat{\lambda}_t^S}{\bar{\lambda}_t^k} = \int_{\mathbb{I}} \frac{\lambda_{i,t}^S}{\bar{\lambda}_t^k} p_i di = CC_{i,t}$$

Optimal abatement of emissions decisions

1. ***Business as usual :***

- **Abatement :**

$$[\vartheta_t] \quad \partial_{\vartheta} \Lambda(\vartheta_{i,t}) e_{i,t}^f = 0 \quad \Rightarrow \vartheta_{i,t} = 0$$

Optimal abatement of emissions decisions

1. ***Business as usual :***

- Abatement :

$$[\vartheta_t] \quad \partial_\vartheta \Lambda(\vartheta_{i,t}) e_{i,t}^f = 0 \quad \Rightarrow \vartheta_{i,t} = 0$$

2. ***Social planner :***

- Abatement :

$$[\vartheta_t] \quad \partial_\vartheta \Lambda(\vartheta_t^i) = \bar{\theta} (\vartheta_{i,t})^\theta = - \underbrace{\frac{1}{\lambda_{i,t}^k} \int_{\mathbb{I}} \omega_j \lambda_{j,t}^S p_j dj}_{=\text{carbon tax for } i}$$

FBSDE for MFG systems – general formulation

- ▶ State $X_t \equiv (a_t, z_t) \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), and X diffusion process with control $\alpha^\star(t, X, P_X, Y) \equiv c_t^\star$

$$dX_t = b(X_t, P_{X_t}, \alpha_t^\star) dt + \sigma dB_t$$

- ▶ Set up the Hamiltonian :

$$\mathcal{H}(t, x, P_X, y) = \max_{\alpha \in \mathbb{A}} (u(\alpha) + b(x, P_X, \alpha) \cdot y)$$

- ▶ Optimal control $c^\star \in \operatorname{argmax}_{\alpha \in \mathbb{A}} (u(\alpha) + b(x, P_X, \alpha) \cdot y)$
- ▶ Using the Pontryagin maximum principle :

$$dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \tilde{Z}_t dB_t$$

[back](#)

FBSDE system for MFG

- ▶ Coupled FBSDE system for each agent

$$\begin{cases} dX_t = D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t \\ dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \tilde{Z}_t dB_t \end{cases}$$

- ▶ Question : What else do we need ?

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- ▶ Question : What else do we need ?

- The risk loading in the costate \tilde{Z}_t :
 - ▶ Intuitions : expectation error in the law of motion of Y_t

$$\tilde{Z}_t(x) = \mathbb{E} \left[\frac{dY_t + D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt}{dB_t^0} \right]$$

- ▶ BSDE theory : keep the co-state measurable w.r.t. dB_t , despite running backward)
- ▶ Can compute that by Monte Carlo

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$$\tilde{Z}_t(x) = \mathbb{E} \left[\frac{dY_t + D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt}{dB_t^0} \right]$$

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 - ▶ Can compute that by Monte Carlo
- The initial condition Y_0 as a function of X_0
- A boundary condition of Y_T or transversality $\lim_{t \rightarrow \infty} e^{-\rho t} X_t Y_t = 0$

[back](#)

Social planner : Mean Field Control/McKean Vlasov

- ▶ Suppose now that the social planner controls the dynamic of each agents ... *accounting for its effect on the distribution*
- ▶ Get an additional term :

$$\begin{cases} dX_t = D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t \\ dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + Z_t dB_t - \widetilde{\mathbb{E}}[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t)(X_t)] \end{cases}$$

- ▶ Effect on the distribution :

Social planner : Mean Field Control/McKean Vlasov

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- ▶ Effect on the distribution :
 - Social planner internalize the externality coming from heterogeneity
 - ▶ $D_\mu H$ is the L-derivative w.r.t the measure $\mu \equiv \mathbb{P}_{X_t}$
 - ▶ Idea : lifting of the function $H(x, \mu) = \hat{H}(x, \hat{X})$ where $\hat{X} \sim \mu$ and hence $D_\mu H(x, \mu)(\hat{X}) = D_{\hat{x}} \hat{H}(x, \hat{X})$
 - ▶ Intuition : shift the distribution of states \hat{X} for all agents
 - ▶ Probabilistic approach : easy to compute $\tilde{\mathbb{E}}[D_\mu H(\tilde{X}_t, \mu)] = \tilde{\mathbb{E}}[D_{\hat{x}} \hat{H}(\tilde{X}_t, \hat{X})]$
 - Here : the effect is homogeneous for all agents : the interaction with the measure is non-local !

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Sequential approach FBSDE w/ Agg. shocks

- ▶ Coupled FBSDE system for each agent

$$\begin{cases} dX_t = D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t + \tilde{\sigma} dB_t^0 \\ dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + Z_t dB_t + \tilde{Z}_t^0 dB_t^0 \end{cases}$$

- ▶ Question : What else do we need ?

Sequential approach FBSDE w/ Agg. shocks

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- ▶ Question : What else do we need ?
 - The risk loading on idiosyncratic shocks \tilde{Z}_t :
 - The risk loading on aggregate shocks \tilde{Z}_t^0 :

Sequential approach FBSDE w/ Agg. shocks

- ▶ Coupled FBSDE system for each agent

$$\begin{cases} dX_t = D_y \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + \sigma dB_t + \tilde{\sigma} dB_t^0 \\ dY_t = -D_x \mathcal{H}(t, X_t, P_{X_t}, Y_t) dt + Z_t dB_t + \tilde{Z}_t^0 dB_t^0 \end{cases}$$

- ▶ Question : What else do we need ?

- The risk loading on idiosyncratic shocks \tilde{Z}_t :
- The risk loading on aggregate shocks \tilde{Z}_t^0 :
- The initial conditions $Y_0(X_0)$ and boundary condition on Y_T or transversality
 $\lim_{t \rightarrow \infty} e^{-\rho t} X_t Y_t = 0$

MFG system : Recursive approach w/ Agg. shocks

- ▶ Here : recursive w.r.t. idiosyncratic shocks, but sequential w.r.t. aggregate shocks.
- ▶ System for v and g :

$$\begin{aligned} -\partial_t v + \rho v &= \max_{\alpha} u(\alpha) + \mathcal{A}(v)v + Z_t^0 dB_t^0 \\ \partial_t g &= \mathcal{A}^*(v)g + \partial_x[\sigma g]dB_t^0 \end{aligned}$$

- ▶ Solve the PDE system :
 - Finite difference, upwinding scheme
 - View that as a non-linear system : use Quasi Newton methods
 - New part : forcing terms $\partial_x[\sigma g]dB_t^0$ and $Z_t^0 dB_t^0$
 - Initial and terminal conditions

$$v_T = v^\infty \quad g_0 = g^\infty$$

- ▶ Direct effect of uncertainty on measure
- ▶ Indirect effect through agent expectations : shadow price of aggregate risk Z_t^0