

# When is aggregation enough?

## Aggregation and Projection with the Master Equation

WORK IN PROGRESS

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## Limitation of current methods for Heterogeneous Agents models

- ▶ Since Krusell-Smith (1998) a large array of methods have been developed to tackle *Heterogeneous Agent models with Aggregate Shocks*
  - Perturbation methods
    - Reiter (2009, 2010), Den Haan et al (2008)
    - Ahn, Kaplan, Moll, Wolf, Winberry (2018)
    - Bilal (2023)
    - Bhandari, Bourany, Evans, Golosov (2023)
  - Sequence space methods
    - Auclert, Bardoczy, Rognlie, Straub (2021)
  - Truncation methods
    - Legrand-Ragot (2018-)
  - Machine Learning based methods
    - Fernandez-Villaverde, Hurtado, Nuno (2023) [FVHN]
    - Han, Jentzen, E. (2018)
    - Gu, Lauriere, Merkel, Payne (2023) [GLMP]

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- ▶ Most of the recent operational methods are based on Perturbation and rely on **certainty equivalence**
- ▶ By design they can not speak about aggregate risk and decisions under **aggregate uncertainty**
- ▶ Some exceptions:
  - Second order perturbations  $\Rightarrow$  still local approximation around a stationary point
  - Machine Learning based methods  $\Rightarrow$  can be a bit opaque / case specific

## This project

- ▶ Since Krusell-Smith (1998) a large array of methods have been developed to tackle *Heterogeneous Agent models with Aggregate Shocks*
- ▶ New approaches to aggregate risk have been developed by mathematician, using the **Master equation**
  - Cardaliaguet, Delarue, Lions, Lasry (2019)
  - Also used in economics by Schaab (2021), Bilal (2023)
- ▶ This project is proposing a new method to talk about risk in H.A. models
  - Relying solely on “projection” to characterize the distribution of agents
  - Come back to the original idea by Krusell-Smith
  - Extend it to more generic models of macro-finance

## General idea with familiar notations

- Take Krusell Smith (1998) Consumption-saving model,  $c, a$ , with
  - (i) idiosyncratic income risk  $z$ , (ii) incomplete market,
  - (iii) credit constraints  $a \geq \underline{a}$
  - (iv) aggregate shock on aggregate TFP  $Z$ .

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- Firm side:

$$Y = ZK^\alpha \quad \Rightarrow \quad r = \alpha K^{\alpha-1} - \delta \quad w = (1 - \alpha)K^\alpha$$

- Distribution of households  $g(a, z)$  over wealth and income
- Household decision (KS98)

$$V(a, z, g, Z) = \max_{c, a'} u(c) + \beta \mathbb{E}^{z', Z'} [V(a', z', g', Z') \mid z, Z]$$

$$s.t. \quad c + a' = zw + (1+r)a$$

$$g' = H(g, Z, Z')$$

- Equilibrium

$$K = \int_{a, z} a dg(a, z)$$

## General idea and KS98 global solution

- ▶ Difficulty: Value function  $V(a, z, g, Z)$  depends on the whole distribution  $g$  (!)
- ▶ Need to forecast the evolution of  $g \Rightarrow$  very difficult with aggregate risk
  - Need to follow the distribution  $g_t$  on *every path* of  $\{Z_t\}_t$
  - Brute force: computationally intensive, c.f. Bourany (2018)
- ▶ Krusell-Smith solution: two assumptions related to *bounded-rationality*
  1. Assume the Household only care about aggregate capital / First-moment  $K = \int a dg(a, z)$
  2. Assume *Linear* forecasting-rule for future capital

$$K' = a_1^Z K + a_2^Z$$

- Choose parameters  $(a_1^Z, a_2^Z)$  to match the *realized* path of  $\{K_t\}_t$
- ▶ Proposal today:
  - remove assumption 2  $\Rightarrow$  bypass the linearity assumpt<sup>o</sup> (in that sense close to FVHN)
  - test robustness to 1 and 2, using methods based on the Master equation



# Primer on the Mean Field Games and the Master Equation

- Rewriting the Aiyagari model as a Mean Field Game involves a system of PDEs:

- States dynamics:

$$da_t = [z_t w_t + r_t a_t - c_t] dt \quad z_j \sim \text{Markov jump process } \lambda_j$$

1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t, a, z) + \rho v(t, a, z) = \max_c u(c) + \mathcal{L}[v](t, a, z)$$

- Transport/Jump-Operator

$$\mathcal{L}[v | c^*](t, a, z_j) = \partial_a v(t, a, z_j) [z_j w + r a - c^*] + \lambda_j (v(t, a, z_{-j}) - v(t, a, z_j))$$

2. Kolmogorov forward Equation:

$$\partial_t g(t, a, z) = \mathcal{L}^*[g | c^*](t, a, z)$$

- Equilibrium:

$$\iint_{z, a \geq \underline{a}} a dg(t, a, z_j) = K_t \quad r_t = \alpha K_t^{\alpha-1} - \delta$$

# Primer on the Master Equation

- The master equation combines in *one equation* both the HJB and the KFE
  - Case without aggregate risk, c.f. Cardaliaguet et al (2019), Bilal (2023)

$$\begin{aligned}
 -\partial_t v(t, a, z, g) + \rho v(t, a, z, g) = & \overbrace{\max_c u(c) + \mathcal{L}[v | c^*](t, a, z)}^{\text{standard HJB continuation value}} + \\
 & \underbrace{\iint_{z, a} \frac{dv(t, a, z, g)}{dg} [(\tilde{a}, \tilde{z})] \mathcal{L}^*[g | c^*](t, \tilde{a}, \tilde{z}) dg(t, \tilde{a}, \tilde{z})}_{\text{evolution of the distribution}}
 \end{aligned}$$

- Novelty: dependence on how the distribution  $g$  changes  
notice the forecast for all other agents  $(\tilde{a}, \tilde{z})$
- Requires to defines the derivative in the space of distribution  $\frac{dv(g)[\tilde{x}]}{dg}$ : Lions' derivative

## Primer on the Lions derivative

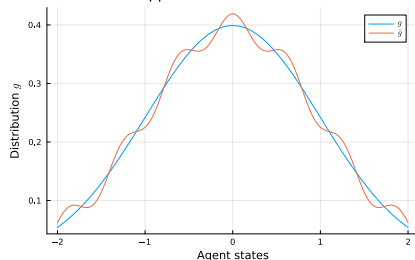
- Derivative in the space of distribution:  
how the value  $v(a, z, \mathbf{g})$  changes when the distribution of agents  $\mathbf{g}$  moves?

$$\begin{aligned} dv(a, z, \mathbf{g}) &\approx v(a, z, \tilde{\mathbf{g}}) - v(a, z, \mathbf{g}) \\ &\approx \iint_{\tilde{a}, \tilde{z}} \underbrace{\frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{a}, \tilde{z}]}_{=\text{Fréchet}} (\tilde{\mathbf{g}}(\tilde{a}, \tilde{z}) - \mathbf{g}(\tilde{a}, \tilde{z})) \end{aligned}$$

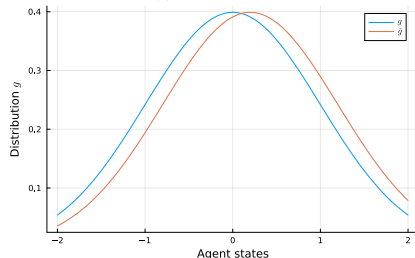
$$\approx \iint_{\tilde{a}, \tilde{z}} \underbrace{\frac{d}{d\tilde{a}} \frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{a}, \tilde{z}]}_{=\text{Lions}} \underbrace{d\tilde{a}}_{=\text{change in decision}} \mathbf{g}(\tilde{a}, \tilde{z})$$

- $\frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{x}]$  Fréchet Derivative, for a change of  $\mathbf{g}$  in  $\tilde{x}$
- $\frac{dv(a, z, \mathbf{g})}{d\mathbf{g}}[\tilde{x}] = \frac{d}{dx} \frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{x}]$  Lions Derivative, for a change of  $\tilde{x}$ , i.e. a *shift* in  $\mathbf{g}(\tilde{x})$

Point of approximation: Fréchet derivative



Point of approximation: Lions derivative



## Lions derivative and agent decision: toward aggregation?

### ► Derivative in the space of distribution

- Change in value  $v(a, z, \mathbf{g})$  with moves in the distribution of agents  $\mathbf{g}$
- Lions-derivative: what causes the change in the agents' distribution  $\mathbf{g}$ ?  
 $\Rightarrow$  change in states  $(d\tilde{a}, d\tilde{z})$
- What causes the change in states?  $\Rightarrow$  the change in agents' decisions
  - States dynamics  $(d\tilde{a}, d\tilde{z})$  change with small change in decision, i.e. consumption-saving: operator  $\mathcal{L}^*[g | c^*]$  (!)

### ► Can we aggregate?

- Aggregate the distribution?
  - Aggregate the change in agents' decision?
- $\Rightarrow$  Goal/method of this project!
- Before, back to the original question: aggregate risk

## Adding Aggregate Risk to the Master Equation (ARME?)

### ► Consider aggregate risk

- Agg. TFP follows a AR(1) - Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z - \bar{Z})dt + \hat{\sigma}dB_t^0$$

- The master equation doesn't change much: value  $v = v(t, a, z, g, Z)$

$$\begin{aligned}
 -\partial_t v + \rho v = & \underbrace{\max_c u(c) + \mathcal{L}[v|c](t, a, z)}_{\text{standard HJB continuation value}} \quad \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2}v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} \\
 & + \underbrace{\iint_{z,a} \frac{dv(t, a, z, g, Z)}{dg} [(\tilde{a}, \tilde{z})] \mathcal{L}^*[g|c^*](t, \tilde{a}, \tilde{z}) dg(\tilde{a}, \tilde{z})}_{\text{evolution of the distribution}}
 \end{aligned}$$

- Why?
  - Aggregate shocks don't have *direct effects* on the distribution!
  - Is that the reason why KS98 model is “boring” ?
- ⇒ linear in  $Z$  / can aggregate capital  $K$  easily / doesn't have important implication of risk  $\hat{\sigma}$  ?

## General Aggregate Risk to the Master Equation (GARME?)

- Consider aggregate risk with *direct effects* on household income, portfolio share  $\theta$

$$dR_t = \bar{\sigma} dB_t^0$$

$$da = (ra + zw - c)dt + \theta a (dR - r)$$

- The master equation now becomes ***second order***! value  $v = v(t, a, z, g, Z)$  changes a lot!

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 & + \underbrace{\frac{\theta^2 \bar{\sigma}^2}{2} \iint_{z,a} \frac{d}{d\tilde{a}} \left( \frac{dv}{dg} [(\tilde{a}, \tilde{z})] \right) dg(t, \tilde{a}, \tilde{z})}_{\text{diffusion of the distribution due to risk}} + \underbrace{\theta^2 \bar{\sigma}^2 \iint_{z,a} \frac{d}{da} \frac{dv}{dg} [(\tilde{a}, \tilde{z})] dg(t, \tilde{a}, \tilde{z})}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}}
 \end{aligned}$$

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 & + \underbrace{\theta \bar{\sigma} \hat{\sigma} \iint_{z,a} \frac{d}{dZ} \frac{dv}{dg} [(\tilde{a}, \tilde{z})] dg(t, \tilde{a}, \tilde{z})}_{\text{covariance of agg. state } Z \text{ and distribution } \tilde{a}} + \underbrace{\frac{\theta^2 \bar{\sigma}^2}{2} \iint_{(z,a)^{\otimes 2}} \frac{d^2 v}{dg^2} [(\tilde{a}, \tilde{z}, \tilde{a}', \tilde{z}')] dg(t, \tilde{a}, \tilde{z}) dg(t, \tilde{a}', \tilde{z}')}_{\text{covariance of distribution } \tilde{a} \text{ and } \tilde{a}'}
 \end{aligned}$$



## General Aggregate Risk to the Master Equation (GARME?)

- Include controlled drift, diffusion, jump on individual states  
+ mean-field interaction on drift, diffusion and jump on aggregate states
- Encompass most macro-finance models. Exception: Impulse control, fixed cost (yet!)

$$\begin{aligned}
 \mathcal{H}(x, m, \mathcal{X}, V, D_x V, D_{xx} V) = & \max_c \mathcal{L}(x, m, \mathcal{X}, c) + b(x, m, \mathcal{X}, c) \cdot D_x V + \text{Tr}([\sigma \sigma' + \overline{\sigma} \overline{\sigma}'](x, m, \mathcal{X}, c) D_{xx} V) \\
 & \sum_{n=1}^{n_J^i} \lambda^n(x, m, \mathcal{X}, c) \left( V^n(x + \gamma(x, m, \mathcal{X}, c), x, m, \mathcal{X}) - V \right) \\
 -\partial_t V + \rho V = & \mathcal{H}(x, m, \mathcal{X}, V, D_x V, D_{xx} V, c^*) \\
 & + \mu(m, \mathcal{X}) \cdot D_{\mathcal{X}} V + \text{Tr}(\widehat{\sigma} \widehat{\sigma}' D_{\mathcal{X} \mathcal{X}} V) + \sum_{n=1}^{n_J^0} \widehat{\lambda}^n(m, \mathcal{X}) \left( V \circ \widehat{\gamma}^n(m, \mathcal{X}) - V \right) \\
 & + \int_{\mathbb{X}} D_m V(x, \cdot; y) \cdot D_p \mathcal{H}(y, \cdot) m(dy) + \int_{\mathbb{X}} \sum_{n=1}^{n_J^0} \lambda^n(y, \cdot) \Delta_m V(x, \cdot; y) \circ \gamma(y, \cdot) m(dy) \\
 & + \int_{\mathbb{X}} \text{Tr}[(\sigma \sigma' + \overline{\sigma} \overline{\sigma}')(y, \cdot) D_y (D_m V(x, m, \mathcal{X}; y))] (y, m, \mathcal{X}) m(dy) \\
 & + 2 \int_{\mathbb{X}} \text{Tr}(\overline{\sigma}(x, \cdot) \overline{\sigma}(y, \cdot)' D_x D_m V(x, \cdot; y)) m(dy) + \int_{\mathbb{X}} \text{Tr}(\overline{\sigma}(y, \cdot) \widehat{\sigma}(\mathcal{X}_t)' D_m D_{\mathcal{X}} V(x, m, \mathcal{X}; y)) m(dy) \\
 & + \iint_{\mathbb{X}} \text{Tr}(\overline{\sigma}(y, \cdot) \overline{\sigma}(y', \cdot)' D_{mm}^2 V)(x, \cdot; y, y') m(dy) m(dy')
 \end{aligned}$$

## Projection and Bounded-rationality in KS98

Back to KS98. What do Households need for decisions?

- Require only changes in prices  $(r, w) \Rightarrow$  don't care of the distribution *per se*
- Neoclassical model: only need some moments, ***the mean***, of the distribution for asset prices!

$$K = \iint_{a,z} a \, dg(a, z) \qquad r = \alpha K^{\alpha-1} - \delta$$

- Bounded rationality assumption:s

$$V(a, z, \mathbf{g}, Z) \equiv \bar{V}(a, z, K^h, Z)$$

- Nice property in Lions-derivative:

$$\text{with } K^h = \int_x h(x) \, dg(x) \qquad \frac{d}{dg} V(x, g; y) \equiv \frac{d}{dK^h} \bar{V}(x, K^h) h'(y)$$

## Projection in the Master equation

- Can rewrite the Master Equation with this projection on the first-moment:

$$v = v(a, z, \mathbf{g}, Z) \equiv \bar{v}(a, z, K, Z)$$

$$\begin{aligned} \rho \bar{v} = & \underbrace{\max_c u(c) + \mathcal{L}[\bar{v} | c^*]_{(a,z)}}_{\text{standard HJB continuation value}} \underbrace{-\theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } \bar{v}} \\ & + \bar{v}_K \iint_{z,a} \underbrace{[r\tilde{a} + w\tilde{z} - c^*(\tilde{a}, \tilde{z}, K, Z)]}_{\text{change in agents } (\tilde{a}, \tilde{z}) \text{ decisions}} dg(\tilde{a}, \tilde{z}) \end{aligned}$$

- Still dependence on  $g$ , how to "get rid of it"? Not easy!
- Aggregation:

$$dK = \iint_{z,a} [r\tilde{a} + w\tilde{z} - c^*(\tilde{a}, \tilde{z}, K, Z)] dg(\tilde{a}, \tilde{z})$$

$$dK = rK + w\bar{L} - \mathcal{C}(K, Z|g)$$

with aggregate consumption function  $\mathcal{C}(K, Z|g) = \iint_{z,a} c^*(\tilde{a}, \tilde{z}, K, Z) dg(\tilde{a}, \tilde{z})$

## Back to familiar models

- The Master Equation becomes a “standard” HJB (!),  $v = v(a, z, \mathbf{g}, Z) \equiv \bar{v}(a, z, K, Z)$

$$\begin{aligned} \rho \bar{v} = \max_c & u(c) + [wz + ra - c] \bar{v}_a + \lambda(\bar{v}(a, z', \cdot) - \bar{v}(a, z, \cdot)) \\ & - \theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} \bar{v}_{ZZ} + \underbrace{[ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)]}_{=dK} \bar{v}_K \end{aligned}$$

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- Only issue:  $\mathcal{C}(t, K, Z|g)$  still depends on  $g$
- Looks exactly like the standard models (!!)

– RBC:  $v = v(K, Z)$

$$\rho v = \max_C u(C) + [ZK^\alpha - \delta K - C]v_K - \theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}$$

– Aiyagari:  $v = v(a, z)$

$$\rho v = \max_c u(c) + [wz + ra - c]v_a + \lambda(v(a, z', \cdot) - v(a, z, \cdot))$$

## Agents' decision and global dynamical system

- ▶ With the Master equation and  $v = \bar{v}(a, z, K, Z)$  we obtain the individual decision,

$$c^*(\tilde{a}, \tilde{z}, K, Z) = u'^{-1}(\bar{v}_a(a, z, K, Z))$$

- ▶ Hence we get the dynamical system:

$$\begin{cases} da &= \left[ \overbrace{z(1-\alpha)ZK^\alpha}^{=w} + \overbrace{(\alpha ZK^{\alpha-1} - \delta)}^{=r} a - c^*(a, z, K, Z) \right] dt \\ dz &= \gamma(z)dJ_t \quad \text{intensity} \quad \lambda(z) \\ dK &= (ZK^\alpha - \delta K - \mathcal{C}(K, Z|g))dt \\ dZ &= \mu(Z)dt + \hat{\sigma}dB_t^0 \end{cases}$$

- ▶ For a guess of  $g(a, z)$  and  $\mathcal{C}(K, Z|g) = \iint_{a, z} c^*(a, z, K, Z)g(a, z)$  we have a complete characterization of the system  
 $\Rightarrow$  Can get a Kolmogorov forward equation for the system  $(a, z, K, Z)$  (!!)

## “Master-” Kolmogorov Forward for the global system

- For a guess of  $g(a, z)$  and  $\mathcal{C}(K, Z|g) = \iint_{a,z} c^*(a, z, K, Z) g(a, z)$ , the Master-KFE for states  $x = (a, z, K, Z) \in \tilde{\mathbb{X}}$  writes:

$$\begin{aligned} 0 = & -\partial_a [s(x, \bar{v}_a) \tilde{g}(x)] + \sum_n \lambda(z^n) \tilde{g}(x^n) - \lambda(z) \tilde{g}(x) \\ & - \partial_K [(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)) \tilde{g}(t, \tilde{x})] - \partial_Z [\mu(Z) \tilde{g}(x)] + \hat{\sigma} \partial_{ZZ}^2 \tilde{g}(x) \end{aligned}$$

- Easy to get from the operator of the Master-HJB! standard methods
- Consistency condition for rational-expectation equilibrium:

$$dg(a, z)|_{K,Z} = \int_{\tilde{\mathbb{X}}} \delta_{\{\tilde{K}=K, \tilde{Z}=Z\}} d\tilde{g}(a, z, \tilde{K}, \tilde{Z})$$

- Consistency for the first moment:  $\iint_{a,z} a dg(a, z^n) = \int_{\tilde{\mathbb{X}}} \delta_{\{\tilde{K}=K, \tilde{Z}=Z\}} a d\tilde{g}(a, z^n, \tilde{K}, \tilde{Z}) = K$

## Summary and numerical methods

### 1. General Master equation

- Summarize MFG systems with one equation:  $v(a, z, g, Z)$

### 2. Master HJB for “bounded-rational” agents: $v = \bar{v}(a, z, K, Z)$

- Start from guess  $g(a, z)$  and  $\mathcal{C}(K, Z|g)$
- Solve Master-HJB: standard finite difference methods
- Get individual decisions  $c^*(a, z, K, Z)$  and operator  $\mathcal{A}[\bar{v}]$  for  $(a, z, K, Z)$

### 3. Master-Kolmogorov forward for $(a, z, K, Z)$

- Obtain distribution  $\tilde{g}$  over all states  $(a, z, K, Z)$  for “free” with  $\mathcal{A}^*[\tilde{g}]$
- Update  $g$  thanks to  $\tilde{g}$  and update  $\mathcal{C}(K, Z|g)$
- Obtain Capital dynamics: potentially very non-linear!!

$$dK = ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)$$

### ► Procedure standard and general

- No need for deep-learning/splines/polynomials: use standard finite difference methods
- Method robust to higher-order moments (in the paper!)

$$K_2 = \iint_{a,z} (a-K)^2 dg(a, z) \dots \Rightarrow \text{imply additional terms in HJB (+ larger state-space)}$$



## Master-Equation with higher moments:

► HJB with 2nd-order moments:

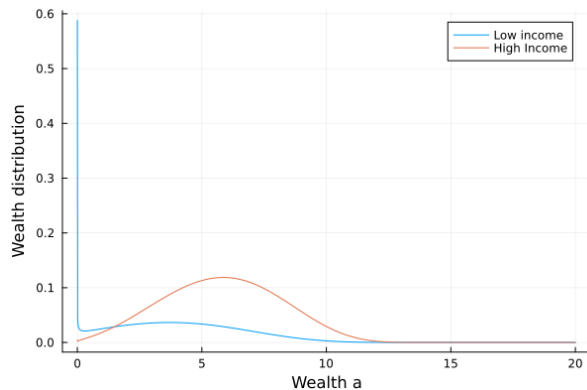
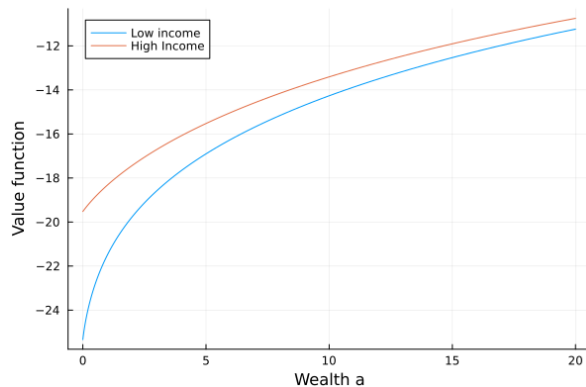
$$v = v(a, z, \mathbf{g}, Z) \equiv \bar{v}(a, z, K, K_2, L_2, KL, Z) = \bar{v}(a, z, K, K_2, Z)$$

- $K_2 = \mathbb{V}\text{ar}(a)$ ,  $L_2 = \mathbb{V}\text{ar}(z)$ ,  $KL = \mathbb{C}\text{ov}(a, z)$
- In KS98, you don't need all of them!

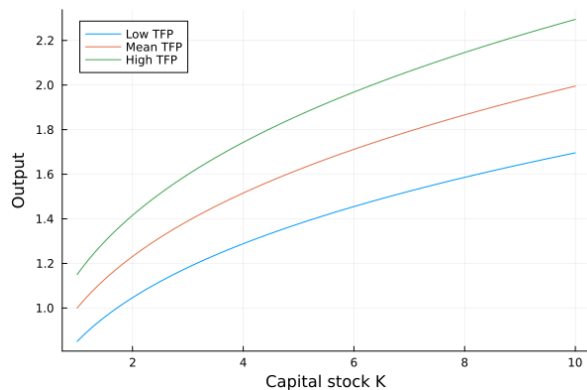
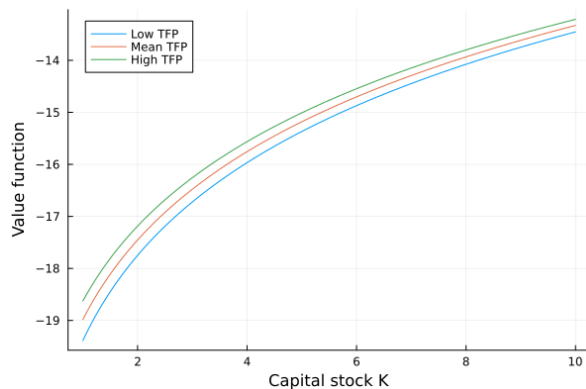
$$\begin{aligned} \rho \bar{v} = \max_c u(c) + [wz + ra - c] \bar{v}_a + \lambda(\bar{v}(a, z', \cdot) - \bar{v}(a, z, \cdot)) - \theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} \bar{v}_{ZZ} \\ + \underbrace{[ZK^\alpha - \delta K - \mathbb{E}^g[c^*]]}_{=dK} \bar{v}_K + \underbrace{[-\mathbb{C}\text{ov}^g(a, c^*)]}_{dK_2} \bar{v}_{K_2} \end{aligned}$$

- Similarly, solve for dynamical system  $(a, z, K, K_2, Z)$ , the “master” KFE and then plug  $g$  back into  $\mathbb{E}^g[c^*]$  and  $\mathbb{C}\text{ov}^g(a, c^*)$

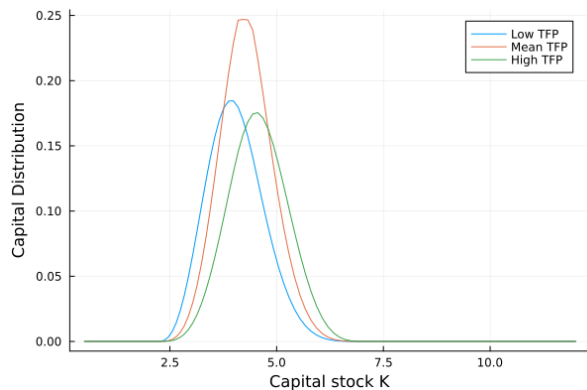
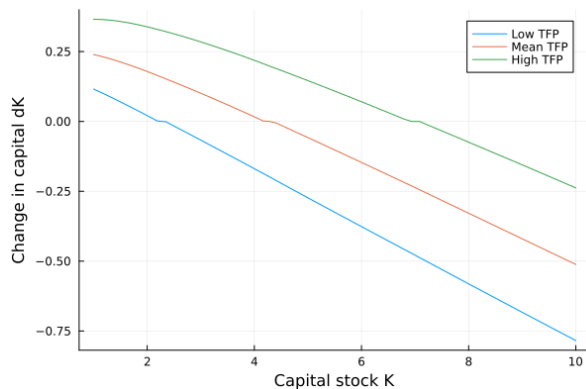
## Numerical experiment - Aiyagari model



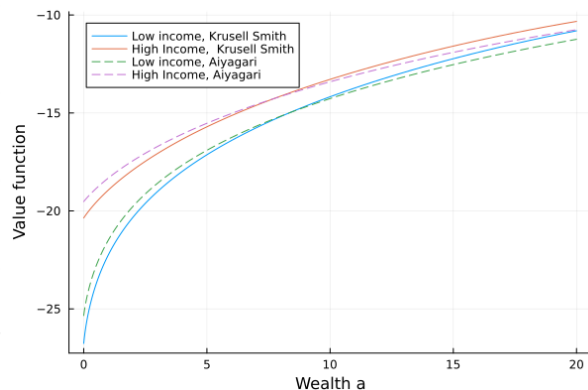
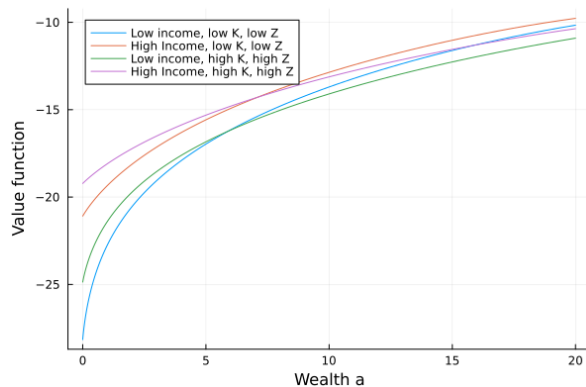
## Numerical experiment - Brock-Mirman / RBC



## Numerical experiment - Brock-Mirman / RBC



# Numerical experiment - Master equation, Krusell-Smith



## Conclusion

- ▶ In this project, I propose a new method to solve Heterogeneous Agent Models with aggregate risk
- ▶ Next steps:
  - Properties of KS98: is the model Markovian in capital? i.e. is the consumption function  $\mathcal{C}(K, Z|g)$  robust to change in  $g$  (e.g. to change in  $K_2 = \mathbb{V}\text{ar}(a)$ ).
  - Comparison with Krusell-Smith's linearity in capital flow
  - Overidentification test for SMM: do agents need second-order (or higher-order) moments when making their decision?
  - Solving a “more interesting” macro-finance model:  
Model with meaningful distribution of portfolios, exposure and impact of aggregate risk