When is aggregation enough?
Aggregation and Projection with the Master Equation

WORK IN PROGRESS

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Limitation of current methods for Heterogeneous Agents models

- Since Krusell-Smith (1998) a large array of methods have been developed to tackle *Heterogeneous Agent models with Aggregate Shocks*
  - Perturbation methods
    - Reiter (2009, 2010), Den Haan et al (2008)
    - Ahn, Kaplan, Moll, Wolf, Winberry (2018)
    - Bilal (2023)
    - Bhandari, Bourany, Evans, Golosov (2023)
  - Sequence space methods
    - Auclert, Bardoczy, Rognlie, Straub (2021)
  - Truncation methods
    - Legrand-Ragot (2018-)
  - Machine Learning based methods
    - Fernandez-Villaverde, Hurtado, Nuno (2023) [FVHN]
    - Han, Jentzen, E. (2018)
    - Gu, Lauriere, Merkel, Payne (2023) [GLMP]

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Limitation of current methods for Heterogeneous Agents models

- Since Krusell-Smith (1998) a large array of methods have been developed to tackle *Heterogeneous Agent models with Aggregate Shocks*
- Most of the recent operational methods are based on Perturbation and rely on certainty equivalence
- By design they can not speak about aggregate risk and decisions under aggregate uncertainty
- Some exceptions:
  - Second order perturbations $\Rightarrow$ still local approximation around a stationary point
  - Machine Learning based methods $\Rightarrow$ can be a bit opaque / case specific

This project

- Since Krusell-Smith (1998) a large array of methods have been developed to tackle *Heterogeneous Agent models with Aggregate Shocks*
- New approaches to aggregate risk have been developed by mathematician, using the Master equation
  - Cardaliaguet, Delarue, Lions, Lasry (2019)
  - Also used in economics by Schaab (2021), Bilal (2023)
- This project is proposing a new method to talk about risk in H.A. models
  - Relying solely on "projection" to characterize the distribution of agents
  - Come back to the original idea by Krusell-Smith
  - Extend it to more generic models of macro-finance

General idea with familiar notations

- Take Krusell Smith (1998) Consumption-saving model, c, a, with
  (i) idiosyncratic income risk z,   (ii) incomplete market,
  (iii) credit constraints $a \geq \underline{a}$
  (iv) aggregate shock on aggregate TFP Z.

General idea with familiar notations

- Take Krusell Smith (1998) Consumption-saving model, c, a, with
  (i) idiosyncratic income risk z, (ii) incomplete market,
  (iii) credit constraints $a \geq \underline{a}$
  (iv) aggregate shock on aggregate TFP Z.
- Firm side:

$$Y = ZK^{\alpha} \qquad \Rightarrow \qquad r = \alpha K^{\alpha-1} - \delta \qquad w = (1-\alpha)K^{\alpha}$$

- Distribution of households $g(a, z)$ over wealth and income
- Household decision (KS98)

$$V(a, z, g, Z) = \max_{c,a'} \; u(c) + \beta \mathbb{E}^{z',Z'} \left[V(a', z', g', Z') \,\middle|\, z, Z \right]$$

$$s.t. \qquad c + a' = zw + (1+r)a$$

$$g' = H(g, Z, Z')$$

- Equilibrium

$$K = \int_{a,z} a \, dg(a, z)$$

General idea and KS98 global solution

- Difficulty: Value function $V(a, z, g, Z)$ depends on the whole distribution g (!)
- Need to forecast the evolution of $g \Rightarrow$ very difficult with aggregate risk
  - Need to follow the distribution g_t on *every path* of $\{Z_t\}_t$
  - Brute force: computationally intensive, c.f. Bourany (2018)
- Krusell-Smith solution: two assumptions related to *bounded-rationality*
  1. Assume the Household only care about aggregate capital / First-moment $K = \int a \, dg(a, z)$
  2. Assume *Linear* forecasting-rule for future capital

$$K' = a_1^Z K + a_2^Z$$

  - Choose parameters (a_1^Z, a_2^Z) to match the *realized* path of $\{K_t\}_t$
- Proposal today:
  - remove assumption 2 $\Rightarrow$ bypass the linearity assumpt° (in that sense close to FVHN)
  - test robustness to 1 and 2, using methods based on the Master equation

Primer on the Mean Field Games and the Master Equation

- Rewriting the Aiyagari model as a Mean Field Game involves a system of PDEs:
  - States dynamics:

$$da_t = [z_t w_t + r_t\, a_t - c_t]dt \qquad\qquad z_j \sim \text{Markov jump process } \lambda_j$$

  1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t,a,z) + \rho v(t,a,z) = \max_c \; u(c) + \mathcal{L}[v](t,a,z)$$

  - Transport/Jump-Operator

$$\mathcal{L}[\, v \,|c^\star](t,a,z_j) = \partial_a v(t,a,z_j)[z_j w + r\, a - c^\star] + \lambda_j\big(v(t,a,z_{-j}) - v(t,a,z_j)\big)$$

  2. Kolmogorov forward Equation:

$$\partial_t g(t,a,z) = \mathcal{L}^*[\, g \,|c^\star](t,a,z)$$

  - Equilibrium:

$$\iint_{z,a\geq \underline{a}} a\, dg(t,a,z_j) = K_t \qquad\qquad r_t = \alpha K_t^{\alpha-1} - \delta$$

Primer on the Master Equation

- The master equation combines in *one equation* both the HJB and the KFE
  - Case without aggregate risk, c.f. Cardaliaguet et al (2019), Bilal (2023)

$$-\partial_t v(t,a,z,\boldsymbol{g}) + \rho v(t,a,z,\boldsymbol{g}) = \overbrace{\max_c u(c) + \mathcal{L}[v\,|c^\star](t,a,z)}^{\text{standard HJB continuation value}} + \underbrace{\iint_{z,a} \frac{dv(t,a,z,\boldsymbol{g})}{dg}[(\tilde{a},\tilde{z})]\;\mathcal{L}^*[g\,|c^\star](t,\tilde{a},\tilde{z})dg(t,\tilde{a},\tilde{z})}_{\text{evolution of the distribution}}$$

  - Novelty: dependence on how the distribution g changes
    notice the forecast for all other agents $(\tilde{a},\tilde{z})$
  - Requires to defines the derivative in the space of distribution $\frac{dv(g)[\tilde{x}]}{dg}$: Lions' derivative

Primer on the Lions derivative

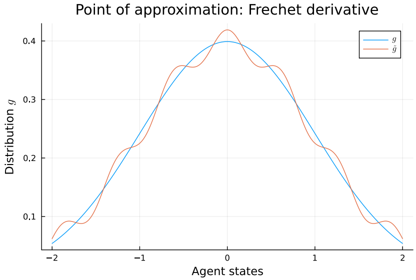
- Derivative in the space of distribution: how the value $v(a, z, \boldsymbol{g})$ changes when the distribution of agents $\boldsymbol{g}$ moves?

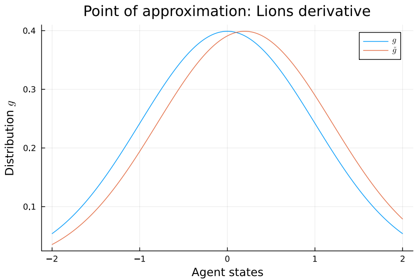
$$dv(a,z,\boldsymbol{g}) \approx v(a,z,\tilde{\boldsymbol{g}}) - v(a,z,\boldsymbol{g})$$

$$\approx \iint_{\tilde{a},\tilde{z}} \underbrace{\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{a},\tilde{z}]}_{=\text{Fréchet}} \left(\tilde{g}(\tilde{a},\tilde{z}) - g(\tilde{a},\tilde{z})\right)$$

$$\approx \iint_{\tilde{a},\tilde{z}} \underbrace{\frac{d}{d\tilde{a}} \frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{a},\tilde{z}]}_{=\text{Lions}} \underbrace{d\tilde{a}}_{=\substack{\text{change in} \\ \text{decision}}} \quad g(\tilde{a},\tilde{z})$$

- $\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{x}]$ Fréchet Derivative, for a change of g in $\tilde{x}$
- $\frac{dv(a,z,\boldsymbol{g})}{dg}[\tilde{x}] = \frac{d}{dx}\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{x}]$ Lions Derivative, for a change of $\tilde{x}$, i.e. a *shift* in $g(\tilde{x})$





Lions derivative and agent decision: toward aggregation?

- Derivative in the space of distribution
  - Change in value $v(a, z, \boldsymbol{g})$ with moves in the distribution of agents $\boldsymbol{g}$
  - Lions-derivative: what causes the change in the agents' distribution $\boldsymbol{g}$?
    $\Rightarrow$ change in states $(d\tilde{a}, d\tilde{z})$
  - What causes the change in states? $\Rightarrow$ the change in agents' decisions
    - States dynamics $(d\tilde{a}, d\tilde{z})$ change with small change in decision, i.e. consumption-saving: operator $\mathcal{L}^\star[\,g\,|c^\star]$ (!)
- Can we aggregate?
  - Aggregate the distribution?
  - Aggregate the change in agents' decision?
  - $\Rightarrow$ Goal/method of this project!
  - Before, back to the original question: aggregate risk

Adding Aggregate Risk to the Master Equation (ARME?)

- Consider aggregate risk
  - Agg. TFP follows a AR(1) - Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z - \bar{Z})dt + \widehat{\sigma} dB_t^0$$

  - The master equation doesn't change much: value $v = v(t,a,z,\boldsymbol{g},Z)$

$$-\partial_t v + \rho v = \overbrace{\max_c \, u(c) + \mathcal{L}[\, v \,|c](t,a,z)}^{\text{standard HJB continuation value}} \quad \overbrace{-\theta(Z{-}\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v}$$
$$+ \underbrace{\iint_{z,a} \frac{dv(t,a,z,\boldsymbol{g},Z)}{dg}[(\tilde{a},\tilde{z})]\, \mathcal{L}^*[\, g \,|c^\star](t,\tilde{a},\tilde{z}) dg(t,\tilde{a},\tilde{z})}_{\text{evolution of the distribution}}$$

  - Why?
    - Aggregate shocks don't have *direct effects* on the distribution!
    - Is that the reason why KS98 model is "boring" ?
    - $\Rightarrow$ linear in Z / can aggregate capital K easily / doesn't have important implication of risk $\widehat{\sigma}$?

General Aggregate Risk to the Master Equation (GARME?)

- Consider aggregate risk with *direct effects* on household income, portfolio share θ

$$dR_t = \overline{\sigma} dB_t^0 \qquad\qquad da = (ra + zw - c)dt + \theta a\,(dR - r)$$

- The master equation now becomes ***second order***! value $v = v_{(t,a,z,\boldsymbol{g},Z)}$ changes a lot!

$$-\partial_t v + \rho v = \overbrace{\max_c \; u(c) + \mathcal{L}[v|c]_{(t,a,z)}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z-\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v} + \overbrace{\iint_{z,a} \frac{dv_{(t,a,z,\boldsymbol{g},Z)}}{dg}[(\tilde{a},\tilde{z})]\, \mathcal{L}^*[\,g\,|c^\star]_{(t,\tilde{a},\tilde{z})} dg_{(t,\tilde{a},\tilde{z})}}^{\text{deterministic evolution of the distribution}}$$

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$$+ \underbrace{\frac{\theta^2\overline{\sigma}^2}{2} \iint_{z,a} \frac{d}{d\tilde{a}}\Big(\frac{dv}{dg}[(\tilde{a},\tilde{z})]\Big) dg_{(t,\tilde{a},\tilde{z})}}_{\text{diffusion of the distribution due to risk}} + \underbrace{\theta^2\overline{\sigma}^2 \iint_{z,a} \frac{d}{da}\frac{dv}{dg}[(\tilde{a},\tilde{z})]\, dg_{(t,\tilde{a},\tilde{z})}}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}}$$

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$$+ \underbrace{\theta\overline{\sigma}\widehat{\sigma}\iint_{z,a}\frac{d}{dZ}\frac{dv}{dg}[(\tilde{a},\tilde{z})]\,dg_{(t,\tilde{a},\tilde{z})}}_{\text{covariance of agg. state } Z \text{ and distribution } \tilde{a}} + \underbrace{\frac{\theta^2\overline{\sigma}^2}{2}\iint_{(z,a)^{\otimes 2}}\frac{d^2v}{dg^2}[(\tilde{a},\tilde{z},\tilde{a}',\tilde{z}')]dg_{(t,\tilde{a},\tilde{z})}dg_{(t,\tilde{a}',\tilde{z}')}}_{\text{covariance of distribution } \tilde{a} \text{ and } \tilde{a}'}$$

General Aggregate Risk to the Master Equation (GARME?)

- Include controlled drift, diffusion, jump on individual states
  + mean-field interaction on drift, diffusion and jump on aggregate states
- Encompass most macro-finance models. Exception: Impulse control, fixed cost (yet!)

$$\mathcal{H}(x,m,\mathcal{X},V,D_xV,D_{xx}V) = \max_c \ \mathcal{L}(x,m,\mathcal{X},c) + b(x,m,\mathcal{X},c)\cdot D_xV + \mathrm{Tr}\big([\sigma\sigma' + \overline{\sigma\sigma}'](x,m,\mathcal{X},c)\ D_{xx}V\big)$$
$$\textstyle\sum_{n=1}^{n_J^i} \lambda^n(x,m,\mathcal{X},c)\Big(V^n(x+\gamma(x,m,\mathcal{X},c),x,m,\mathcal{X}) - V\Big)$$

$$\begin{aligned}
-\partial_t V + \rho V &= \mathcal{H}\big(x,m,\mathcal{X},V,D_xV,D_{xx}V,c^\star\big)\\
&+ \mu(m,\mathcal{X})\cdot D_{\mathcal{X}}V + \mathrm{Tr}\big(\widehat{\sigma}\widehat{\sigma}' D_{\mathcal{X}\mathcal{X}}V\big) + \sum_{n=1}^{n_J^0} \widehat{\lambda}^n(m,\mathcal{X})\Big(V\circ\widehat{\gamma}^n(m,\mathcal{X}) - V\Big)\\
&+ \int_{\mathbb{X}} D_mV(x,\cdot;y)\cdot D_p\mathcal{H}(y,\cdot)m(dy) + \int_{\mathbb{X}}\sum_{n=1}^{n_J^0}\lambda^n(y,\cdot)\Delta_m V(x,\cdot;y)\circ\gamma(y,\cdot)m(dy)\\
&+ \int_{\mathbb{X}} \mathrm{Tr}\Big[(\sigma\sigma'+\overline{\sigma\sigma}')(y,\cdot)D_y\big(D_mV(x,m,\mathcal{X};y)\big)\Big](y,m,\mathcal{X})m(dy)\\
&+ 2\int_{\mathbb{X}} \mathrm{Tr}\big(\overline{\sigma}(x,\cdot)\overline{\sigma}(y,\cdot)' D_xD_mV(x,\cdot;y)\big)m(dy) + \int_{\mathbb{X}} \mathrm{Tr}\big(\overline{\sigma}(y,\cdot)\widehat{\sigma}(\mathcal{X}_t)' D_mD_{\mathcal{X}}V(x,m,\mathcal{X};y)\big)m(dy)\\
&+ \iint_{\mathbb{X}} \mathrm{Tr}\big(\overline{\sigma}(y,\cdot)\overline{\sigma}(y',\cdot)' D^2_{mm}V\big)(x,\cdot;y,y')m(dy)m(dy')
\end{aligned}$$

Projection and Bounded-rationality in KS98

Back to KS98. What do Households need for decisions?

- Require only changes in prices $(r, w) \Rightarrow$ don't care of the distribution *per se*
- Neoclassical model: only need some moments, ***the mean***, of the distribution for asset prices!

$$K = \iint_{a,z} a \, dg(a,z) \qquad\qquad r = \alpha K^{\alpha-1} - \delta$$

- Bounded rationality assumption:s

$$V(a,z,\mathbf{g},Z) \equiv \overline{V}(a,z,K^h,Z)$$

- Nice property in Lions-derivative:

$$\text{with } K^h = \int_x h(x) \, dg(x) \qquad\qquad \frac{d}{dg} V(x,g;y) \equiv \frac{d}{dK^h} \overline{V}(x,K^h) \, h'(y)$$

Projection in the Master equation

- Can rewrite the Master Equation with this projection on the first-moment: $v = v(a, z, \boldsymbol{g}, Z) \equiv \bar{v}(a, z, K, Z)$

$$\rho\bar{v} = \overbrace{\max_{c}\ u(c) + \mathcal{L}[\,\bar{v}\,|c^{\star}](a,z)}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } \bar{v}}$$
$$+ \bar{v}_K \iint_{z,a} \underbrace{[r\tilde{a} + w\tilde{z} - c^{\star}(\tilde{a}, \tilde{z}, K, Z)]}_{\text{change in agents } (\tilde{a}, \tilde{z}) \text{ decisions}} dg(\tilde{a}, \tilde{z})$$

- Still dependence on g, how to "get rid of it"? Not easy!
- Aggregation:

$$dK = \iint_{z,a} [r\tilde{a} + w\tilde{z} - c^{\star}(\tilde{a}, \tilde{z}, K, Z)] dg(\tilde{a}, \tilde{z})$$
$$dK = rK + w\bar{L} - \mathcal{C}(K, Z|g)$$

with aggregate consumption function $\mathcal{C}(K, Z|g) = \iint_{z,a} c^{\star}(\tilde{a}, \tilde{z}, K, Z) dg(\tilde{a}, \tilde{z})$

Back to familiar models

- The Master Equation becomes a "standard" HJB (!), $v = v(a, z, \boldsymbol{g}, Z) \equiv \bar{v}(a, z, K, Z)$

$$\rho\,\bar{v} = \max_c\ u(c) + \left[wz + ra - c\right]\bar{v}_a + \lambda\big(\bar{v}(a,z',\cdot) - \bar{v}(a,z,\cdot)\big)$$

$$- \theta(Z-\bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{\left[ZK^{\alpha} - \delta K - \mathcal{C}(K,Z|g)\right]}_{=dK}\,\bar{v}_K$$

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$$\rho \, \bar{v} = \max_c \; u(c) + [wz + ra - c] \, \bar{v}_a + \lambda\big(\bar{v}(a,z',\cdot) - \bar{v}(a,z,\cdot)\big)$$

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  - Only issue: $\mathcal{C}(t, K, Z|g)$ still depends on g
  - Looks exactly like the standard models (!!)
    - RBC: $v = v(K, Z)$

$$\rho \, v = \max_C \; u(C) + [ZK^\alpha - \delta K - C] v_K - \theta(Z - \bar{Z}) v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}$$

    - Aiyagari: $v = v(a, z)$

$$\rho \, v = \max_c \; u(c) + [wz + ra - c] v_a + \lambda\big(v(a,z',\cdot) - v(a,z,\cdot)\big)$$

Agents' decision and global dynamical system

- With the Master equation and $v = \bar{v}(a, z, K, Z)$ we obtain the individual decision,

$$c^{\star}(\tilde{a}, \tilde{z}, K, Z) = u'^{-1}\big(\bar{v}_a(a, z, K, Z)\big)$$

- Hence we get the dynamical system:

$$\begin{cases} da & = \Big[z\,\overbrace{(1-\alpha)Z\,K^{\alpha}}^{=w} + \overbrace{(\alpha Z\,K^{\alpha-1} - \delta)}^{=r}\,a - c^{\star}(a, z, K, Z)\Big]dt \\ dz & = \gamma(z)dJ_t \qquad\qquad\qquad \text{intensity} \qquad \lambda(z) \\ dK & = \big(ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)\big)dt \\ dZ & = \mu(Z)dt + \widehat{\sigma}dB_t^0 \end{cases}$$

- For a guess of $g_{(a,z)}$ and $\mathcal{C}(K, Z|g) = \iint_{a,z} c^{\star}(a,z,K,Z)g(a,z)$ we have a complete characterization of the system
  - $\Rightarrow$ Can get a Kolmogorov forward equation for the system (a, z, K, Z) (!!)

"Master-" Kolmogorov Forward for the global system

- For a guess of $g(a,z)$ and $\mathcal{C}(K,Z|g) = \iint_{a,z} c^\star(a,z,K,Z)g(a,z)$, the Master-KFE for states $x = (a,z,K,Z) \in \widetilde{\mathbb{X}}$ writes:

$$0 = -\partial_a\left[s(x,\bar{v}_a)\widetilde{g}(x)\right] + \sum_n \lambda(z^n)\widetilde{g}(x^n) - \lambda(z)\widetilde{g}(x)$$
$$- \partial_K\left[\left(ZK^\alpha - \delta K - \mathcal{C}(K,Z|g)\right)\widetilde{g}(t,\widetilde{x})\right] - \partial_Z[\mu(Z)\widetilde{g}(x)] + \widehat{\sigma}\partial^2_{ZZ}\widetilde{g}(x)$$

- Easy to get from the operator of the Master-HJB! standard methods
- Consistency condition for rational-expectation equilibrium:

$$dg(a,z)\big|_{K,Z} = \int_{\widetilde{\mathbb{X}}} \delta_{\{\widetilde{K}=K,\widetilde{Z}=Z\}} d\widetilde{g}(a,z,\widetilde{K},\widetilde{Z})$$

  - Consistency for the first moment: $\iint_{a,z} adg(a,z^n) = \int_{\widetilde{\mathbb{X}}} \delta_{\{\widetilde{K}=K,\widetilde{Z}=Z\}} ad\widetilde{g}(a,z^n,\widetilde{K},\widetilde{Z}) = K$

Summary and numerical methods

1. General Master equation
   - Summarize MFG systems with one equation: $v(a,z,g,Z)$
2. Master HJB for "bounded-rational" agents: $v = \bar{v}(a,z,K,Z)$
   - Start from guess $g(a,z)$ and $\mathcal{C}(K,Z|g)$
   - Solve Master-HJB: standard finite difference methods
   - Get individual decisions $c^{\star}{(a,z,K,Z)}$ and operator $\mathcal{A}[\bar{v}]$ for (a,z,K,Z)
3. Master-Kolmogorov forward for (a,z,K,Z)
   - Obtain distribution $\widetilde{g}$ over all states (a,z,K,Z) for "free" with $\mathcal{A}^*[\widetilde{g}]$
   - Update g thanks to $\widetilde{g}$ and update $\mathcal{C}(K,Z|g)$
   - Obtain Capital dynamics: potentially very non-linear!!

$$dK = ZK^{\alpha} - \delta K - \mathcal{C}(K,Z|g)$$

- Procedure standard and general
   - No need for deep-learning/splines/polynomials: use standard finite difference methods
   - Method robust to higher-order moments (in the paper!)
     $K_2 = \iint_{a,z}(a-K)^2 dg(a,z)...$ $\Rightarrow$ imply additional terms in HJB (+ larger state-space)

Master-Equation with higher moments:

- HJB with 2nd-order moments:
  $v = v(a, z, \mathbf{g}, Z) \equiv \bar{v}(a, z, K, K_2, L_2, KL, Z) = \bar{v}(a, z, K, K_2, Z)$
  - $K_2 = \mathbb{V}\mathrm{ar}(a)$, $L_2 = \mathbb{V}\mathrm{ar}(z)$, $KL = \mathbb{C}\mathrm{ov}(a, z)$
  - In KS98, you don't need all of them!

$$\rho\,\bar{v} = \max_c\, u(c) + \left[wz + ra - c\right]\bar{v}_a + \lambda\left(\bar{v}_{(a,z',\cdot)} - \bar{v}_{(a,z,\cdot)}\right) - \theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ}$$

$$+ \underbrace{\left[ZK^\alpha - \delta K - \mathbb{E}^g[c^\star]\right]}_{=dK}\bar{v}_K + \underbrace{\left[-\mathbb{C}\mathrm{ov}^g(a, c^\star)\right]}_{dK_2}\bar{v}_{K_2}$$

  - Similarly, solve for dynamical system (a, z, K, K_2, Z), the "master" KFE and then plug g back into $\mathbb{E}^g[c^\star]$ and $\mathbb{C}\mathrm{ov}^g(a, c^\star)$

Numerical experiment - Aiyagari

- Standard Aiyagari value/consumption
  - Graphs forthcoming for Monday

Numerical experiment - Brock-Mirman / RBC

- Standard Brock-Mirman / RBC, value / consumption
  - Graphs forthcoming for Monday

Numerical experiment - Master equation, Krusell-Smith

- Graphs forthcoming for Monday

Conclusion

- In this project, I propose a new method to solve Heterogeneous Agent Models with aggregate risk
- Next steps:
  - Properties of KS98: is the model Markovian in capital? i.e. is the consumption function $\mathcal{C}(K, Z|g)$ robust to change in g (e.g. to change in $K_2 = \mathbb{V}\mathrm{ar}(a)$).
  - Comparison with Krusell-Smith's linearity in capital flow
  - Overidentification test for SMM: do agents need second-order (or higher-order) moments when making their decision?
  - Solving a "more interesting" macro-finance model:
    Model with meaningful distribution of portfolios, exposure and impact of aggregate risk