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Part I

Discrete Maths

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Lecture - Sets

📅 2024-01-23

🕒 17:00

🎓 Janka

1.1 Introduction

Sets underpin maths and Computer Science. A set is a collection of objects, which are called the elements (also known as members of the set). For example, a set of the numbers 1, 3, 8; or the collection of students in a class born in March. There are two characteristics of sets:

1. There are no repeated occurrences of elements
2. There is no particular order of the elements

1.2 Set Notation

The elements of a set are enclosed in braces with their names being denoted by a *letter*, for example:

$$A = \{1, 2, 3\}, \quad C = \{\text{Portsmouth}, \text{Brighton}, \text{London}\}$$

There are two ways that we can describe the members of a set. We can *list the elements* which is mainly used for finite sets, for example:

$$A = \{3, 6, 9, 12\}$$

We can *specify a property* that all the elements in the set have in common. The ‘|’ character is read ‘such that’, sometimes ‘:’ is used in its place. For example:

$$B = \{x | x \text{ is a multiple of 3 and } 0 < x < 15\}$$

We can also use *three dots* to informally denote a sequence of elements that we don’t wish to write down, for example:

$$C = \{1, \dots, 10\}$$

1.2.1 Sets of Numbers

There are some reserved letters to denote specific sets of numbers in maths. These are shown below:

- \mathbb{N} (or N) is used for the set of natural numbers (integers ≥ 0). $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- \mathbb{Z} (or Z) is used for the set of integers. $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$
- \mathbb{Q} (or Q) is used for the set of rational numbers (number which can be expressed as a quotient or fraction). $\mathbb{Q} = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$
- \mathbb{R} (or R) is used for the set of real numbers. $\mathbb{R} = \{\dots, -1, 0, \frac{1}{2}, \dots\}$

1.2.2 Elements of a Set

We can use the \in symbol to denote if an element is a member of a given set. For example, if x is a member of S - then we can say:

$$x \in S$$

The symbol \notin denotes an element is not a member of a given set. For example, if y is **not** a member of S - then we can say:

$$y \notin S$$

1.2.3 Many Ways to Say The Same Thing

There are several ways of describing the same set, for example for the set S of *odd integers*:

$$\begin{aligned} S &= \{\dots, -5, -3, -1, 1, 3, 5, \dots\} \\ &= \{x | x \text{ is an odd integer} \} \\ &= \{x | x = 2k + 1 \text{ for some integer } k\} \\ &= \{x | x = 2k + 1 \text{ for some } k \in \mathbb{Z}\} \\ &= \{2k + 1 | k \in \mathbb{Z}\} \end{aligned}$$

The phrase “for some [integers K]”, means “for all [integers k]”

1.2.4 Empty Sets

Where a set has *no elements*, it is called an empty set or null set. It's denoted with the \emptyset symbol, for example:

$$\emptyset = \{\}$$

1.2.5 Finite & Infinite Sets

If the number of elements in the set is fixed (for example when counting the elements at a fixed rate for a set amount of time), then the set is *finite*. If the set X is finite, then we call $|X|$ the *cardinality* of X therefore:

$$|X| = \text{number of elements in } X$$

If the counting never stops then X is an infinite set.

1.2.6 Subsets

A subset is where one set's elements are entirely present in another set. There are three conditions we need to know about:

- $A \subseteq B$: A is a subset of B therefore every element in A is also in B .
- $A \not\subseteq B$: A is not a subset of B .
- $A \subset B$: A is a proper subset of B , therefore B has at least one additional element which is not in A .

1.2.7 Equality of Sets

Two sets are *equal* if they have exactly the same elements. This is denoted by writing $A = B$. Where $A = B$, the following conditions are also true:

- $A \subseteq B$ for every a if $a \in A$, then $a \in B$
- $B \subseteq A$ for every b if $b \in B$, then $b \in A$

1.3 Operations on Sets

Sets can have *operations* performed on them - this will change something about them.

1.3.1 Intersection

The intersection of two sets A and B is defined as:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

This is the set of elements which appear in both sets only. If we take a Venn Diagram with a set on either side - its the overlapped elements which would be returned from an intersection operation. For example if $A = \{a, b, c\}$ and $B = \{c, d\}$ then $A \cap B = \{c\}$.

1.3.2 Disjoint

If an intersection returns no elements, then the two sets are *disjoint*. This is shown by:

$$A \cap B = \emptyset$$

1.3.3 Union

The *union* of the two sets A and B is defined as:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

This is the set of elements which are in either A or B , this means elements which appear in both are returned. For example, if $A = \{a, b, c\}$ and $B = \{c, d\}$ then $A \cup B = \{a, b, c, d\}$.

1.3.4 Difference

The *difference* between two sets, A and B is defined as:

$$A \setminus B = \{x | x \in A \text{ or } x \in B\}$$

This is the set of elements which are in A but not in B , so could be represented as $A - B$. Note that $A \setminus B \neq B \setminus A$.

1.3.5 Counting Elements In a Set

If we take A and B to be finite sets, we can calculate the number of elements in the union of A and B . The correct way to count this is as follows:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

We have to minus $|A \cap B|$ from the sum because otherwise it is as though we are counting it twice due to the fact that we are summing the total number of elements in A and B .

1.4 Complement

If we consider that all subsets are the subset of a particular set, U for example (the universe of discourse), then the difference $U \setminus A$ is called the *complement* of A is shown as either \overline{A} or A' . For example:

$$A' = \{x | x \in U \text{ and } x \notin A\}$$

1.5 Basic Set Properties

Sets have a number of basic properties - many of these are the same as that for Boolean Expressions

- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$
- $A \cup A = A$
- $A \cap A = A$
- Commutative
 - $A \cup B = B \cup A$
 - $A \cap B = B \cap A$
- Associative
 - $(A \cup B) \cup C = A \cup (B \cup C)$
 - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributive
 - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- de Morgan's
 - $(A \cap B)' = A' \cup B'$
 - $(A \cup B)' = A' \cap B'$

1.6 Power Set

A *power set* is the collection of all subsets of a set, S which is denoted by $P(S)$. For example, if $S = \{a, b, c\}$ then:

$$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

1.7 Partition

A *partition* of the set S is a collection of non-empty subsets of set S where every element from S belongs to exactly one member of S . This means that the sets are mutually disjoint and that the union of all the sets in the collection results in the original set, S . For example, if $S = \{a, b, c, d, e, f\}$ then $\{\{a, e\}, \{c\}, \{f, d\}, \{b\}\}$ is a partition of S .

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Lecture - Relations

📅 2024-01-30

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2.1 Ordered Pairs

An ordered pair of elements is a group of two elements which are in a specific order. They are written as (a, b) and the order matters - this means (a, b) is distinct from the pair (b, a) . Note that ordered pairs use the brackets $()$ while sets use curly braces $\{\}$

2.2 Cartesian Product

The Cartesian Product of two sets is the set of **all** ordered pairs where the first element is taken from set 1 and the second element from set 2. The formal definition is as follows:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

For example - if $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

2.3 Relations

A relation is the set of subsets from a cartesian product. For example, if we take $A = \{a, b, c, d, e\}$ and $B = 1, 2, 3$ then:

$$R_1 = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$$

$$R_2 = \{(a, 3), (a, 1), (c, 2), (c, 1), (b, 2)\}$$

R_1 and R_2 are both examples of Binary relations from A to B.

2.3.1 Describing Relations

To describe a relation, we could list all of its elements, however this can be very long and obtuse so it's better & more common practice to use "the characteristics of their elements".

2.3.2 Relations On A Set

A relation *on a set* is where both sets are equal. For example $A = B$ then a relation on A is a relation from A to A , hence a subset of $A \times A$.

For example, let R be the relation on $A = \{1, 2, 3, 4\}$ as defined by:

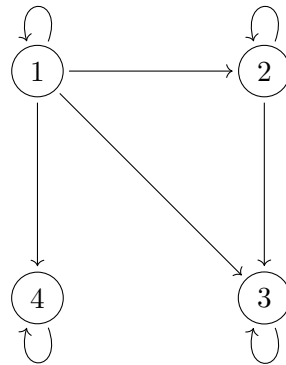
$$(x, y) \in R \text{ if and only if } x \text{ divides } y, \text{ for all } x, y \in A$$

Then we can conclude that R is:

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

2.4 A Digraph

We can use a *digraph* to picture a relation on a set. An example is shown below:



The dots (vertices) represents the elements of $A = \{1, 2, 3, 4\}$. If the element (x, y) is in the relation, an arrow (directed edge) is drawn from x to y .

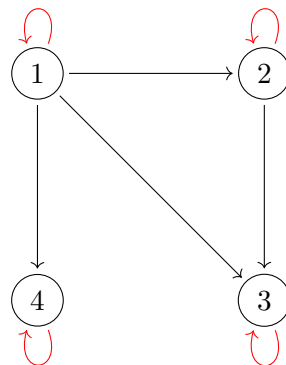
$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

2.4.1 Reflexivity

A relation is reflexive where for all elements in the set - there is an ordered pair in which both elements are the same, for example $(1, 1)$ or $(2, 2)$. In reference to the set $A = \{1, 2, 3, 4\}$, the relation:

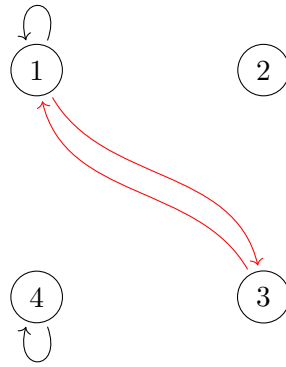
$$R = (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)$$

On the following digraph, the red arrows are the ones which display the reflexivity.



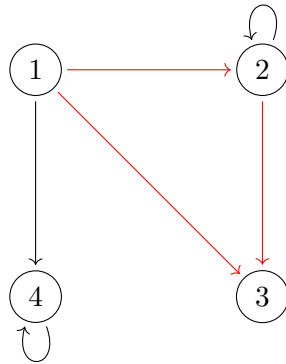
2.4.2 Symmetry

A relation is symmetrical where $(x, y) \in R$ and $(y, x) \in R$. If the condition is not true, then we do not have symmetry.



2.4.3 Transitivity

For a binary relation, R on set A ; R is transitive if and only if for all $x, y, z \in A$ if $(x, y) \in R$ and $(y, z) \in R$ and $(x, z) \in R$. We initially assume that a relation is transitive and try to disprove it; if we are unable to disprove it then the relation is transitive. In the event that there is only one element in the relation - the relation will *always* be transitive.



2.4.4 Equivalence

Where a relation is reflexive, symmetric and transitive - it is classed as an equivalence.

2.4.4.1 Equivalence Class

To be continued.

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Lecture - Functions

📅 2024-02-06

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🎓 Janka

A *function* can be described in two ways. The mathematical definition is that “a function is a special type of relation in which a single input will have at most one output”. The alternative definition of a function is that it is a mysterious black box which takes an input and returns an output. The same function, with the same input will always return the same output.

If we take A and B to be nonempty sets, then: f is a (total) function f from A to B , $f : A \rightarrow B$, is a relation from A to B such that

for all $x \in A$ there is exactly one element in B , $f(x)$

associated with x by relation f . Note that the word ‘total’ is used to describe the above function which means that every input has a defined output. The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which is defined by $f(x) = 2x$ is also an example of a total function.

It is also possible to have a ‘partial’ function, this is where some of the inputs do not have defined outputs. For example the function $f(\frac{1}{x})$ where $x = 0$, would be undefined therefore the function is classed as ‘partial’.

3.1 Describing A Function

There are a few different ways in which a function can be described.

3.1.1 By A Formula

This is the most common method used. The function f from $\mathbb{N} \rightarrow \mathbb{N}$ that maps every natural number x to its cube x^3 can be described as:

$$f(x) = x^3$$

3.1.2 By All Possible Associations

Whilst this is a valid method, it will generally not be used for efficiency reasons. The function g from $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$ would be shown as:

$$g(a) = 1, g(b) = 1, g(c) = 2$$

3.2 Domain, Co-Domain & Range

The domain of a function is the set of all input values for which there is a defined output. For example if we let $f : A \rightarrow B$ then the subset $D \subset A$ of all elements for which f is defined is the domain. In the case of a total function, $D = A$ and in the case of a partial function, $D \subsetneq A$

The co-domain of a function is the set of all possible output values; not just the ones which map to an input. For example, if we let $f : A \rightarrow B$ then the set B is the co-domain.

The range (also sometimes known as the image) of a function is the set of elements in the co-domain which map to an input. For example, if we let $f : A \rightarrow B$ then the range is denoted by $range(f)$. The range can also be expressed as:

$$range(f) = \{f(x) | x \in A\}$$

3.3 Properties of Functions

Functions have a number of properties.

3.3.1 Injective

The function $f : A \rightarrow B$ is injective (or one-to-one) if there is only one input that maps to each output. It can mathematically be defined as:

$$\text{for all } x, y \in A \text{ if } x \neq y \Rightarrow f(x) \neq f(y)$$

3.3.2 Surjective

A function $f : A \rightarrow B$ is surjective (or onto) if the $range(f)$ is the co-domain B . It can mathematically be defined as

$$\text{for all } y \in B \text{ there exists } x \in A \text{ such that } f(x) = y$$

A function which is not *onto* is *into*.

3.3.3 Bijective

A function $f : A \rightarrow B$ is bijective (or one-to-one correspondence) if it is both injective and surjective.

3.4 Composite Functions

A new function can be constructed by combining other simpler functions in some way. If we let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. The composition of g with f is the function denoted by $g \circ f : A \rightarrow C$ and defined by:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A$$

$(g \circ f)(x) = g(f(x))$ is read as g of f , which means do f first then do g .

3.5 Inverse Functions

An inverse function is where the output of function f can be fed into the input of function f^{-1} to get the original input of f . This is mathematically defined as: $f : X \rightarrow Y$ is a bijective function, then there is an inverse function $f^{-1} : Y \rightarrow X$ that is defined as:

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

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Lecture - Logic I: Introduction to Propositions & Logic

📅 2024-02-13

🕒 1700

🎓 Janka

4.1 Reasoning

Reasoning is something that we are introduced to doing from a young age. Younger children will continually ask “why?” as they attempt to make sense of the world while growing up; as they grow older they will generally only want the facts however. *Logic* is the discipline which deals with the method of reasoning:

- in mathematics to prove theorems
- in computer science to verify the correctness of programs and to prove some theorems
- in the natural and physical sciences to draw conclusions from experiments
- and in our everyday lives to solve a multitude of problems!

Over time, people come to understand that the following statement is how logic & reasoning works:

“If X then Y ” is true and “ X ” is true \Rightarrow so “ Y ” must be true

4.2 Propositions

A *proposition* is a *statement* (which is a declarative sentence) that can either be true or false; however not both. A proposition will be exact, not wishy-washy. For example - $3 - x = 5$ is not a proposition as it has an unknown value of x ; however “The earth is flat” is a proposition as it can, and only can, categorically be True or False.

4.2.1 Propositional Variables

Propositions can be quite long. Mathematicians like efficiency, therefore they apply a single letter variable to a propositional statement to make their lives easier. The letters p, q, r, \dots are used to denote propositional variables.

Statements can be combined with logical connectives to obtain compound statements. For example p and q

4.3 Logical Connectives

Logical Connectives are used to combine propositional statements together; they are very similar to Boolean Algebra¹. The truth value of a compound statement depends only on:

- the truth values of the statements being combined
- the types of connectives being used

4.3.1 Negation (not)

If p is a statement, the negation of p is the statement *not* p , denoted by $\neg p$. The truth table of negation is shown below:

p	$\neg p$
T	F
F	T

4.3.2 Conjunction (and)

If p and q are statement, the conjunction of p and q is the compound statement *p and* q , as denoted by $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

4.3.3 Disjunction (or)

If p and q are statements, the (inclusive) disjunction of p and q is the compound statement *p or* q as $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4.3.4 Conditional Proposition (implication)

If p and q are statements then the compound statement “*if* p *then* q ” dented $p \rightarrow q$ (or $p \Rightarrow q$) is called implication. The hypotheses in the statement is p and the conclusion is denoted by q .

¹Covered in A-Level Electronics, A-Level Computer Science, 1st Year ArchOS Module

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

4.3.5 Conditional Proposition (bidirectional)

If p and q are statement, the compound statement “if and only if” (abbreviated to *iff*), denoted by $p \Leftrightarrow q$, is called the biconditional of p and q .

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

4.4 Truth of Compound Properties

It is possible to combine the truth tables for single logical connectives to make *more complicated truth tables*. In similar fashion to Algebraic & standard numeric expression evaluation - there is a hierarchy of evaluation, as seen below (listed highest to lowest):

1. brackets
2. negation (\neg)
3. conjunction (\wedge)
4. disjunction (\vee)
5. implication (\rightarrow)
6. bidirectional (\leftrightarrow)

For connectives of equal priority - work from left-to-right through them.

4.5 Special Conditions

4.5.1 Tautology

A statement that is true for all possible values of its propositional variables is called a tautology. For example $p : p \vee \neg p$ is a tautology.

4.5.2 Contradiction

A statement that is false for all possible values of its propositional variables is called a contradiction. For example, any proposition $p : p \wedge \neg p$ is a contradiction. In a truth table, the final column (where the output is) will always be false.

4.5.3 Contingency

A statement that can be either true or false depending on the truth values of its propositional variables is called a contingency. For example the proposition “Murray will win the Wimbledon next year” is a contingency because the truth of the statement is dependent on the propositional variable.

4.5.4 Contrapositive

The contrapositive of a conditional statement $p \rightarrow q$ is $\neg q \rightarrow \neg p$. The conditional statement is logically equivalent to its contrapositive.

4.6 Logical Equivalence

Two statements are said to be *logically equivalent*, \equiv , if and only if they have identical truth values for each possible value of their statement variables. Logical equivalence corresponds to $=$ with numbers.

Shown below are the rules of Logical Equivalence. There is not a requirement to memorise these as it should be possible to derive them in the exam when required. Note that they are the same as Boolean algebra's laws, except with funky symbols.

- $p \wedge p \equiv p$
- $p \vee p \equiv p$
- $p \wedge T \equiv p$
- $p \wedge F \equiv F$
- $p \vee T \equiv T$
- $p \vee F \equiv p$
- $\neg(\neg p) \equiv p$
- $p \vee (\neg p) \equiv T$
- $p \wedge (\neg p) \equiv F$
- $p \wedge q \equiv q \wedge p$ (commutativity)
- $p \vee q \equiv q \vee p$ (commutativity)
- $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$ (distributivity)
- $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$ (distributivity)
- $p \rightarrow q \equiv (\neg p \vee q)$
- $\neg(p \vee q) \equiv ((\neg p) \wedge (\neg q))$ (De Morgan's Law)
- $\neg(p \wedge q) \equiv ((\neg p) \vee (\neg q))$ (De Morgan's Law)

4.6.1 Example of logical equivalence proof

Prove that $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (p \vee r)$

$$\begin{aligned}
 (p \rightarrow q) \vee (p \rightarrow r) &\equiv \\
 &\equiv (\neg p \vee q) \vee (\neg p \vee r) && \text{logical equivalence law} \\
 &\equiv \neg p \vee q \vee \neg p \vee r && \text{all are `or' so remove brackets} \\
 &\equiv \neg p \vee q \vee r && \text{get rid of second } \neg p \\
 &\equiv \neg p \vee (q \vee r) && \text{add brackets in} \\
 &\equiv p \rightarrow (q \vee r) && \text{logical equivalence law again}
 \end{aligned}$$

4.7 Necessary and Sufficient Condition

A necessary condition is a condition such that statement B cannot be true without statement A being true. However it is possible for statement A to be true to even if statement B is not true.

A sufficient condition is a condition such that knowing statement A is true guarantees that statement B is true.

A statement (A) is said to be “necessary and sufficient” for the statement B when B is true if and only if A is also true.

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Lecture - Logic II: Quantified Statements

📅 2024-02-20

🕒 1700

🎓 Janka

5.1 Propositional Logic: A Recap

As we saw last week, *Propositional Logic* applies to a declarative statement where the basic propositions are either *True* or *False*. “Mr Bean is a Mathematical Major” is an example of a proposition, however “He is a Mathematical Major is not” because it depends on the value of “he”.

The sentence “All students sitting in this class in the first three rows are mathematical majors” is a more complex example however. To fully discover whether this is a proposition or not, we would need to analyse each individual atomic propositions; which would be to analyse all individuals sitting in the first three rows. For example:

- “Bella is a mathematical major”
- “Joe is a mathematical major”
- “Fred is a mathematical major”

This is obviously a lot of work and mathematicians don’t like to over-exert themselves, so there’s a method which can be used to simplify this predicament, called *Predicate Logic*.

5.2 Predicate Logic

A *predicate* (or propositional function) is a statement containing one or more variables. If values from a given set (domain) are assigned to all the variables, the resulting statement is a proposition. For example:

- $P(n) : n^2 + 2n$ is an odd integer (domain \mathbb{Z}) is a predicate because when n is substituted in for an integer, we have a proposition.
- $S(n) : \text{The student passed the exam}$ (domain of all students sitting in this class) is a predicate because when n is substituted for a student - we have a proposition.
- $\text{Test}(x, y, z) : x < y + z$ (domain of all integers) is a predicate because once x, y, z have been substituted for an integer - we have a proposition as the mathematical expression can either be True or False.
- $\text{Distance}(x, y) : \text{whether the distance between the towns } x \text{ and } y \text{ is less than 300km}$ (domain of all towns in the UK) is a predicate because once we have substituted x and y for two different Towns then we will have a proposition (as this can either be True or False).

From programming, we are familiar with If and While statements. The condition which powers these are in fact predicates (if $p(y)$ then; or while $p(y)$ do).

5.2.1 Changing a Predicate to a Proposition

1. Assign values (from the domain) to all their variables
For example: “ x is divisible by 5” (the domain \mathbb{N}) would be converted through substituting x for a number (i.e 35) then this would result in either True or False (i.e True).
2. to add *quantifiers*

5.3 Quantifiers

Quantifiers are words that refer to quantities such as “some” or “all”. Most of the statements in Maths and Computer Science use terms such as “for every” or “for some”.

For example: “For all $x \in \mathbb{N}$, x is divisible by 5” can use a different quantifier and be written as “There exists $x \in \mathbb{N}$ such that x is divisible by 5”. From this - we know that there are two quantifiers:

5.3.1 Universal Quantifiers

The symbol \forall (an upside down A) is called the universal quantifier; which has the meaning “for all” or “for each”.

The formal definition of the universal quantifier is as follows: For a predicate $p(x)$ with domain D , the statement:

“for every x from domain D , $p(x)$ ”

may be written as $\forall x \in D \ p(x)$.

For example:

- Example 1: “ All DMAFP students are happy” can be re-written as:
Let D be the set of all DMAFP students, then

$$\forall x \in D, x \text{ is happy}$$

- Example 2: Let $S = \{1, 2, 3, 4, 5, 6\}$ and consider the statement

$$\forall x \in S, x^2 \geq x$$

- Example 3: $\forall x \in \mathbb{R}, x^2 \geq x$

5.3.1.1 True Statements with \forall

We know the fact: “The statement $\forall x \in D p(x)$ is true if $p(x)$ is true for every $x \in D$ ”. To prove a quantified statement including \forall to be true, we have to show the truth of the each individual element of the domain to be true.

- Example 1: Let D be the set of all DMAFP students, then $\forall x \in D, x \text{ is happy}$, is a statement. It is true if and only if every student answers ‘Yes’.
- Example 2: Let $S = \{1, 2, 3, 4, 5, 6\}$. The statement $\forall x \in S, x^2 \geq x$ is true because $1^2 \geq 1$, $2^2 \geq 2$, ..., $6^2 \geq 6$

5.3.1.2 False Statements with \forall

We know the fact: “The statement $\forall x \in D p(x)$ is false if $p(x)$ is false for at least one $x \in D$ ”. To prove that a quantified statement including \forall to be false, we have to show at least one counterexample.

- Example 1: Let D be the set of all DMAFP students, then $\forall x \in D, x$ is happy, is a statement. It is false if and only if there is at least one unhappy student.
- Example 3: The statement $\forall x \in \mathbb{R}, x^2 \geq x$ is not true. This is because there exists $x \in \mathbb{R}$ for which $x^2 < x$ for example where $x = \frac{1}{2}$. Therefore the statement is false.

5.3.2 Existential Quantifier

The symbol \exists (a backwards E) is called the existential quantifier; which has the meaning “there exists”.

The formal definition of an existential quantifier is as follows: For a predicate $p(x)$ with the domain D :

“there exists an x from the domain D such that $p(x)$ ”

may be written as $\exists x \in D$ such that $p(x)$.

For example:

- Example 1: “There is a happy DMAFP student”. can be rewritten:
Let D be the set of all DMAFP students then:

$\exists x \in D$ such that x is happy

- Example 2: Let $S = \{1, 2, 3, 4, 5, 6\}$ and consider the statement

$\exists x \in S$ such that $x^2 \geq x$

- Example 3: $\exists x \in \mathbb{N}$ such that $x^2 > x$

5.3.2.1 True Statements with \exists

We know the fact: “The statement $\exists x \in D$ such that $p(x)$ is true if $p(x)$ is true for at least one $x \in D$ ”. To prove a quantified statement including \exists to be true, we have to find one proposition for which the predicate is true.

- Example 1: Let D be the set of all DMAFP students, the statement $\exists x \in D$ such that x is happy. To prove this is true - we only need to find one student who is happy.
- Example 3: The statement $\exists x \in \mathbb{R}$ such that $x^2 \geq x$ can be proved as true because it is true where $x = 2$.

5.3.2.2 False Statements with \exists

We know the fact that “The statement $\exists x \in D$ such that $p(x)$ is false if $p(x)$ is false for all $x \in D$ ”. To prove a quantified statement including \exists to be false, we have to prove that it can never be true. For a small, finite domain - this can be brute forced; or for a large (or infinite) domain - this has to be proved logically.

- Example n: The statement $\exists x \in \mathbb{Z}$ such that $x^2 < -2017$ is false because there are members of \mathbb{Z} which when squared, are bigger than -2017

5.4 Negation of Quantified Statements

When negating a quantified statement, it is not as simple as one might imagine. You do not simply add (or remove) the word ‘not’ and hope for the best; rather you negate components and from here a new statement will be birthed.

For example, taking the statement “There exists a fluffy cat”, to negate this - we would need to write “all cats are not fluffy”. Note how this has gone from an ‘exists’ quantifier to a ‘for all’ quantifier. Alternatively, we can take the statement “All cats are fluffy”, which negated would be “There exists a cat that is not fluffy”. Again, note that the quantifier has been changed.

Now for some mathematical examples rather than (un)fluffy cats:

The negation of a statement of the form $\forall x \in D, Q(x)$ is logically equivalent to a statement of the form $\exists x \in D$ such that $\neg Q(x)$. Symbolically:

$$\neg(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \neg Q(x)$$

The negation of a statement of the form $\exists x \in D$ such that $Q(x)$ is logically equivalent to a statement of the form $\forall x \in D, \neg Q(x)$. Symbolically:

$$\neg(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \neg Q(x)$$

5.5 Nested Quantifiers

Multiple quantifiers such as $\forall x \exists y, \exists x \forall y, \dots$ are said to be nested quantifiers.

In actual English - you would see this such as something along the lines of “There is a student solving every exercise of the tutorials”. However! This sentence is ambiguous as it would have at least two different meanings:

- There is one student who solves all the exercises of the tutorials
- For any particular exercise, there is a student who solves that exercise

5.5.1 Example

Let $P(x, y)$ be the property “the student x solves the exercise y , S - the set of students, E - the set of all exercises of the tutorials.”

- $\exists x \in S \forall y \in E$ such that x and y satisfy property $P(x, y)$ is a statement.
It is true if there is (at least) one student who solves all the exercises of the tutorials.
- $\forall y \in E \exists x \in S$ such that x and y satisfy property $P(x, y)$ is a statement.
It is true if each exercise is solved by at least one student

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Lecture - Methods of Proof

📅 2024-02-27

🕒 1700

🎓 Janka

6.1 What is a Proof?

A Mathematical Proof is a deductive argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion. Mathematical Proofs are carefully reasoned and there are a number of different ways we can deduce one.

6.1.1 Formal Example

Theorem 6.1.1. *Prove that for all integers m and n , if m is odd and n is even, then $m + n$ is odd.*

An *argument* (theorem) is a finite collection of statements (p_1, p_2, \dots, p_n) called *premises* (hypothesis) followed by a statement q called the *conclusion*.

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Rightarrow q$$

In the above, the premises are “ m, n , integers, m is odd, n is even” and the conclusion is “ $m + n$ is odd”. The argument is valid (or the theorem holds) if, whenever p_1, p_2, \dots, p_n are all true, then q is also true.

6.1.2 Methods of Proof

To “prove a theorem” means to show that if all premises are true then the conclusion is also true.

m, n are integers

m is odd, n is even

...

Can we prove that $m + n$ is odd?

We have a number of different techniques which we can use. Which of the following to choose depends on the problem and experience:

1. A direct proof
2. A proof by contradiction
3. A proof by contrapositive
4. A proof by mathematical induction

6.2 A Direct Proof

In a direct proof, we start with the hypothesis of a statement (premises) and make one deduction after another until we reach the conclusion. Theorems which are able to be proved are often of the form:

$$p \text{ (hypothesis)} \Rightarrow q \text{ (conclusion)}$$

While proving the proof, we can use:

- Previously Proven Facts
- Definitions
- Known basic properties

Our task is to prove that then q is also true.

6.2.1 Our First Theorem

It's time to prove our first theorem, how exciting!

Theorem 6.2.1. *For all integers m and n , if m is odd and n is even, then $m + n$ is odd.*

As we are proving this using a direct proof, we assume that the hypothesis is true and derive the conclusion. We start doing this by realising some properties of odd and even numbers.

⌈ *Proof.* An integer r is even if and only if there exists an integer k such that $r = 2k$. (This is the basic definition of an even number, no room for disagreement here!)

Similarly, an integer r is odd if and only if there exists an integer k such that $r = 2k + 1$

To apply these derivations to our theorem:

$$m \text{ is odd} \Rightarrow \text{there exists an integer } k \text{ such that } m = 2k + 1$$

$$n \text{ is even} \Rightarrow \text{there exists an integer } l \text{ such that } n = 2l$$

We can use maths to show that the sum is:

$$\begin{aligned} m + n &= (2k + 1) + 2l \\ &= 2(k + l) + 1 \end{aligned}$$

⌋ $\therefore m + n$ is odd. ■

Note the black square (■) on the right above at the end of the proof. This stands for QED which stands for “Quod Erat Demonstrandum”, which stands for “what was to be demonstrated” which stands for “boom done, there I proved it, I’m leaving now”.

6.3 Proof by Contradiction (indirect proof)

As we know there are only, and can only, be two options for the truth value of a conclusion: *True* or *False*. If supporting that the premises are true and the conclusion is false, we are able to arrive at a contradiction (a conclusion that is contradictory to our assumptions or something obviously untrue like $1 = 0$) \Rightarrow our conclusions must be true!

While proving the proof, we can use:

- Previously known facts

- Definitions
- Known basic properties

Our task is to prove a contradiction, which is done through proving that q is true (assuming that p is true).

6.3.1 Our Second Theorem

Theorem 6.3.1. *For every $n \in \mathbb{N}$, if n^2 is even, then n is even*

We can rewrite this as:

$$n^2 \text{ is even, } n \in \mathbb{N} \Rightarrow n \text{ is even}$$

We can then take our hypothesis to be to be that n^2 is even while n is a natural number; and a false conclusion to be n is not even, hence n is odd. We want to prove any contradiction (which could be $r \wedge \neg r$ for a proposition r).

Proof. Since we take that n is odd, there exists $k \in \mathbb{N}$ such that $n = 2k + 1$. We can now derive the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

This means that n^2 is odd.

Now n^2 is even (the hypotheses) and n^2 is odd (which we have just proved). This is a *crazy* situation, an impossible contradiction! As we have found a contradiction \Rightarrow the conclusion must be true!

\therefore for every $n \in \mathbb{N}$ if n^2 is even, then n is even. ■

6.4 Proof by Contrapositive (indirect proof, again)

The contrapositive of the condition proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. Note that the conditional proposition and its contrapositive are logically equivalent

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

To prove a statement by contrapositive, we prove that the contrapositive statement is a direct proof and conclude that the original statement is true. This means that instead of the original theorem, $p \rightarrow q$, we prove by a direct proof the contrapositive theorem $\neg q \rightarrow \neg p$.

While proving the proof, we can use

- previously proven facts
- definitions
- known basic properties

Our task is to prove that $\neg p$ is true. In this way we prove that $\neg q \rightarrow \neg p$ and because of $p \rightarrow q \equiv \neg q \rightarrow \neg p$ necessarily $p \rightarrow q$ must be true as well (meaning that the theorem $p \rightarrow q$ is valid).

6.4.1 Our Second Theorem (again)

Theorem 6.4.1. *For every $n \in \mathbb{N}$, if n^2 is even then n is even.*

Our contrapositive statement is: “For every $n \in \mathbb{N}$ if n is not even, then n^2 is not even”. We can use the fact “An integer is not even, if and only if, it is odd” to derive our contrapositive:

Contrapositive. For every $n \in \mathbb{N}$, if n is odd then n^2 is odd.

We are now able to prove the contrapositive statement using a direct proof by proving that n^2 is odd.

┌ *Proof.* Since n is odd, there exists $k \in \mathbb{N}$ such that $n = 2k + 1$. We can now derive the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Therefore n^2 is odd.

This means that the contrapositive statement is true and by logical equivalence also the theorem:

“For every $n \in \mathbb{N}$, if n^2 is even, then n is even”

└ is valid ■

6.5 Mathematical Induction

Mathematical Induction is one of the most basic methods of proof. It is a useful technique to use to establish the truth of a statement about natural numbers. It is a method for proving a statement (given in the form of a proposition) $P(n)$ is true for every natural number, n , and that the infinitely many cases $P(0), P(1), P(2), P(3), \dots$ all hold. This is done in two stages.

1. Proving a simple case (the base case), which proves the statement for $n = x$ (where x is any number) without assuming any knowledges of other cases.
2. Having proven that when $n = x$, $P(n)$ is true; we assume that $P(n)$ is true for all $n \geq x$. (the inductive step) (this is *just* the case, which works through magic, don't look too deep into it.)

6.5.1 The Third Theorem

Theorem 6.5.1. *Prove that for any integer $n \geq 1$, the sum of the first n natural numbers is $S(n) = \frac{n(n+1)}{2}$*

┌ *Proof.* We will start by looking at the basic step, where we will take $n = 1$.

$$\begin{aligned} S(1) &= 1 \\ &= \frac{1(1+1)}{2} \\ &= 1 \end{aligned}$$

We will now complete the inductive step.

- (a) We can assume that $S(n)$ of the first n natural numbers is $\frac{n(n+1)}{2}$
- (b) We need to prove that the statement is true also for the first $n+1$ natural numbers, this means:

$$S(n+1) = \frac{(n+1)(n+2)}{2}$$

Which leaves us with the following assumption:

$$S(n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Now we can derive the following:

$$\begin{aligned} S(n+1) &= 1 + 2 + \dots + n + (n+1) \\ &= S(n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Therefore, assuming that the formula is true for n , we have proved the formula is true for $n+1$. ■
 └

6.6 Disproving a Universally Quantified Statement

As we saw in the previous lecture, a universally quantified statement is one which applies “for all” or “for some”.

To disprove

$$\forall x \in D \, P(x)$$

we have to find one $x \in D$ that makes $P(x)$ false. This value of x is called the ‘counterexample’.

For example, the statement:

$$\forall n \in \mathbb{N} (2^n + 1 \text{ is prime})$$

is false because, where $n = 3$, $2^3 + 1 = 9$ which is not prime.

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Lecture - Graphs: An Introduction

📅 2024-03-12

🕒 1700

🎓 Janka

7.1 History of Graphs

Euler introduced the concept of *graph theory* in 1736 when he abstracted the Bridges of Königsberg problem into a series of *vertices* and *edges*. Graphs were further developed with the Travelling Salesman Problem whereby adding *weighting* values to an edge allows the most efficient route to be calculated.

Graphs can be a useful tool in solving a number of different mathematical problems, including those which are abstractions of a real world thing - for example the cities in which the travelling salesman needs to travel through in their problem. There are four main applications of graph theory today:

Existence Problems are problems in which we want to prove the existence (or lack thereof) of something. For example the Königsberg bridge problem.

Optimisation Problems are problems in which we are looking to find the best (most efficient) solution (this can either be a maximisation or minimisation). For example - the Travelling Salesman Problem.

Construction Problems are problems in which we are trying to prove if a solution exists and if it does - how it can be constructed. For example - finding a walk in the Königsberg bridge problem

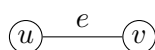
Enumeration Problems are problems in which we are looking for how many objects have a given property. For example - how many optimal routes can be found in the Travelling Salesman Problem.

7.2 Basic Terminology

A graph, G is a pair (V, E) of sets:

V is a non-empty set of vertices (nodes)

E is a set of edges. Each element of E is a set of two distinct elements of V

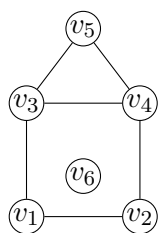


If $e \in E$, then $e = \{u, v\}$, u and v are different elements of V called the end vertices of e (where e joins vertices u and v).

Figure 7.1: Example Graph

The notation for v and u changes too; we can use uv instead of $e = \{u, v\}$ for the edge e , meaning uv and vu are the same edge. The vertices u and v are said to be *incident* with the edge uv or that they are *adjacent* because they are the end vertices of an edge.

We are only considering Finite Graphs in this module. These are graphs in which all their elements are defined and are capped in a certain place.



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$E = \{\{v_1, v_2\}, \{v_2, v_4\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$$

or $E = \{v_1v_2, v_2v_4, v_1v_3, v_3v_4, v_3v_5, v_4v_5\}$

Figure 7.2: Example of a Finite Graph

Note that although the diagrams of the graph can look nice, and help us to visualise them - they are not actually important for most of our problems.

7.2.1 Multigraph

A multigraph / pseudograph is like a graph, however it may contain loops and/or multiple edges. In this module, there is not a formal definition of one and we won't study them too much.

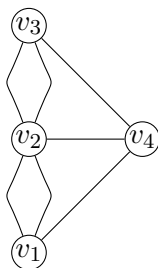


Figure 7.3: Graph with multiple edges

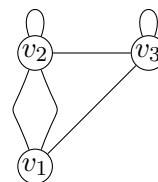
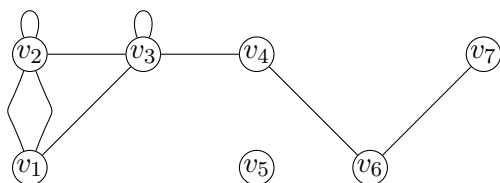


Figure 7.4: Graph with loops

7.2.2 Degree of a Vertex

The number of edges incident with a vertex, v , is called the degree of v and is denoted by $\deg v$. A vertex of degree 0 is said to be isolated.

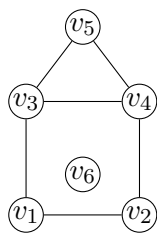


$$\begin{array}{ll} \deg v_1 = 2, & \deg v_6 = 2, \\ \deg v_7 = 1, & \deg v_5 = 0, \\ \deg v_1 = 3, & \deg v_2 = 5, \\ \deg v_3 = 5 & \end{array}$$

Figure 7.5: Degree of Vertex example graph
Note that v_5 is an isolated vertex in the above example.

7.2.3 Degree Sequence

When d_1, d_2, \dots, d_n are the degrees of the vertices of a graph (or multigraph), G , ordered so that $d_1 \leq d_2 \leq \dots \leq d_n$ Then (d_1, d_2, \dots, d_n) is called the *degree sequence* of G .



The degree sequence: $(0, 2, 2, 2, 3, 3)$

Figure 7.6: Example Graph For Degree Sequence

7.3 Euler Theorem

Euler's Theorem (or Handshaking Lemma) states that: In any graph $G = (V, E)$, the sum of all the vertex-degrees is equal to twice the number of edges,

$$\sum_{v \in V} \deg v = 2|E|$$

This is a great revelation, however it lead to two more revelations:

1. In any graph, the sum of all the vertex-degrees is an even number
2. In any graph, the number of vertices of odd degree is even

7.4 Special Types of Graphs

We will now see that there are special types of graphs that have their oen names.

7.4.1 Complete Graphs

For any positive integer n , the complete graph on n vertices, denoted K_n , is that graph with n vertices every two of which are adjacent.



Figure 7.7: K_1

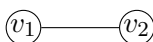


Figure 7.8: K_2

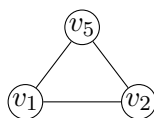


Figure 7.9: K_3

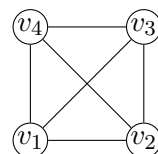


Figure 7.10: K_4

The complete graph on n has $\frac{n(n-1)}{2}$ edges because Eulers theorem.

7.4.2 Bipartite Graphs

A bipartite graph is one whose vertices can be partitioned into disjoint sets V_1 and V_2 in such a way that every edge joins in a vertex in V_1 and a vertex in V_2 (no edges within V_1 nor within V_2).

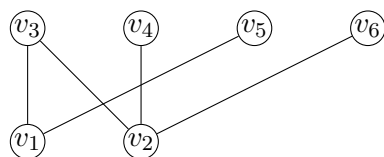


Figure 7.11: Incomplete bipartite graph

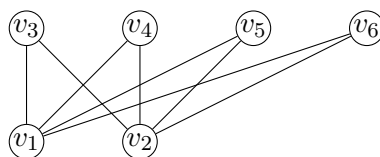
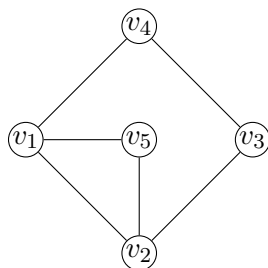
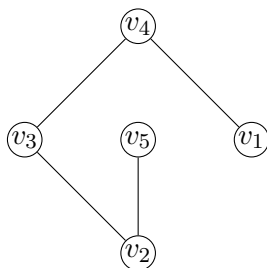
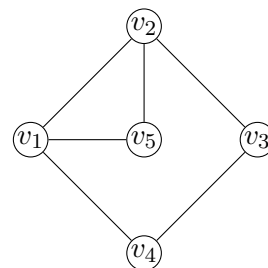


Figure 7.12: Complete bipartite graph

A complete bipartite graph is a bipartite graph in which every vertex in V_1 is joined to every vertex in V_2 .

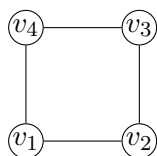
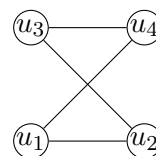
7.5 Subgraphs

A graph H is a subgraph of G if and only if the vertex and edge set of H are respectively subsets of the vertex and edge set of G .

Figure 7.13: G Figure 7.14: H_1 Figure 7.15: H_2

7.6 Isomorphic Graphs

Two graphs, G and H are said to be isomorphic if H can be obtained from G by re-labelling the vertices.

Figure 7.16: G Figure 7.17: H

In the above example, v_1 maps to u_1 , v_2 to u_2 and so on. What this means is that if the u labelling was to be replaced with the v labelling - the same graph would be obtained.

G is isomorphic to H if there is a bijective function $f : V(G) \rightarrow V(H)$ such that

- if u and v are adjacent in G then $f(u)$ and $f(v)$ are adjacent in H
- if u and v are not adjacent in G then $f(u)$ and $f(v)$ are not adjacent in H .

It is difficult to prove that two graphs are isomorphic, as we have to try all the bijections between vertex sets and check them. It can be shown, that if G and H are isomorphic graphs then G and H :

- have the same number of vertices
- have the same number of edges
- have the same degree sequence
- either both are connected or both are not connected (to be covered next week)

So it can be easier to show that graphs are not isomorphic, which is shown by proving that one of the properties above is broken.

Part II

Functional Programming

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Lecture - Introduction to Functional Programming

📅 2024-01-22

🕒 1200

🎓 Matthew

8.1 Introduction

Functional Programming is a different programming paradigm. There are all sorts of different ways we can classify a programming language, paradigm being one of them. More details on this in another module.

8.1.1 Imperative vs Functional Programming

Before we go too deep into Functional Programming, we will first look at the structure of Imperative Programming.

Imperative Programming is a paradigm where the execution of the program consists of executions of *statements*, which each impact the program's *state*. *Side Effects* can be caused by the statements; these are things that the program does where it cannot guarantee the outcome - for example get the current temperature, ask the user to enter a number or getting the system time.

Pure Functional Programming does not have a state, does not have statements and does not have side effects. However - side effects are a “necessary evil” so they get brought back in isolated from the main program. Once side effects are introduced, our functional programming becomes impure.

8.2 Functional Programming

In Functional Programming, there are three key terms - expression, evaluation and value. An expression is some text which has a value, for example $2 * 3 + 1$; a value is the thing which the expression has, for example 7; and evaluation is the process used to obtain a value from an expression.

We will start our FunProg journey looking at Mathematical Functions, which we can think of as a box which maps argument values to a result value. A Haskell program is mainly made up of Function definitions, for example

```
square :: Int -> Int
square n = n * n
```

The first line above starts with the function name, then lists the parameters (a single `Int`), then finally the result type. The second line is the actual function, starting with the name, then the names assigned to the parameters, then the value which gets returned. The double colon (`::`) is read as ‘is type of’.

8.3 Data Types: A Brief Introduction

Haskell includes a number of basic data types which we can make use of.

8.3.1 Boolean

The `Bool` data type has the values `True` and `False`. As well as having the boolean operators `&&` (and); `||` (or); and `not` (not). These can be seen implemented in the following function which implements the *exclusive or* operator (which gives `True` when exactly one of its arguments is `True`):

```
exOr :: Bool -> Bool -> Bool
exOr x y = (x || y) && not (x && y)
```

8.3.2 Int & Float

Haskell includes a number of different numerical data types - we'll start of using the `Int` and `Float` types. `Int` is a fixed-space integer data type; and `Float` is a floating point data type. Operators for these data types include:

- arithmetic operators: `+`, `-` and `*`
- The `Float` data type includes floating point division `/`
- The `Int` has integer division and remainder functions `div` and `mod`
- relation operators: `==`, `/=` (not equals), `<`, `>`, `<=` and `>=`

The operators use the standard precedence rules (as experienced in other languages).

8.4 Conditional Expressions

Haskell includes a conditional expression which takes the form:

```
if condition then m else n
```

where `condition` is a boolean expression and `m` & `n` are expressions of the same type Where `condition` is true - the expression evaluates to `m`; and where `condition` is false - the expression evaluates to `n`.

In the next lecture, we'll see an alternative to conditional expressions.

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Lecture - Introduction To Functional Programming II

📅 2024-01-29

🕒 1200

🎓 Matthew

9.1 Evaluation and Calculation

In a similar way to that of tracing an imperative program to understand the effect of executing their statements on the program state; we can evaluate expressions of a functional program step-by-step to understand the operation of it. This is also known as calculation.

The process of calculation will be explained with the example function:

```
twiceSum x y = 2 * (x + y)
```

which we can evaluate using the example inputs 4 and (2 + 6). The complete calculation of the expression is as follows:

<u>twiceSum 4 (2 + 6)</u>	
↪ 2 * (4 + (2 + 6))	def of twiceSum
↪ 2 * (4 + 8)	arithmetic
↪ 2 * 12	arithmetic
↪ 24	arithmetic

9.2 Guards

Guards are Boolean expressions used in function definitions to give alternative results dependent on the parameter values. The following function gives the largest of the two `Int` values:

```
maxVal :: Int -> Int -> Int
maxVal x y
  | x >= y    = x
  | otherwise = y
```

We saw the `if ... then ... else ...` construct last week, which could be used in place of guards. However guards are the preferential thing to use where there are more than one case - as they allow for this easier. Shown below is a function, `maxThree`, which returns the largest of three `Int` values passed in:

```
maxThree :: Int -> Int -> Int -> Int
maxThree x y z
  | x >= y && x >= z = x
  | y >= z           = y
  | otherwise       = z
```

Calculations involving guards are represented slightly differently, note the ?? denoting when the function is inside the guard. Shown below is the calculation of `maxThree`:

```

maxThree 3 2 5
?? 3 >= 2 && 3 >= 5      first guard
?? ~> True && 3 >= 5      def of >=
?? ~> True && False       def of >=
?? ~> False              def of &&
?? 2 >= 5                second guard
?? ~> False              def of >=
?? otherwise             third guard
?? ~> True               def of otherwise
~> 5

```

9.3 Local Definitions

Single line definitions in Haskell can be slightly unwieldy to read, write and understand. For this reason - we may want to breakdown the definition's expression to make it easier to read. For example the function:

```

distance :: Float -> Float -> Float -> Float -> Float
distance x1 y1 x2 y2 = sqrt ((x1-x2)^2 + (y1-y2)^2)

```

can be broken down to the following:

```

distance x1 y1 x2 y2 = sqrt (dxSq + dySq)
where
    dxSq = (x1 - x2) ^ 2
    dySq = (y1 - y2) ^ 2

```

From the above example, we see that the main expression uses the local definitions and the local definitions use the function's parameters. The local definitions can only be used within the functions that they are defined within; they are "hidden" from the rest of the program. The local definitions can appear in any order within the `where`.

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Lecture - Pattern Matching & Recursion

📅 2024-02-12

🕒 1200

🎓 Matthew

10.1 Modules

As with Python and other popular programming languages, Haskell supports modules (libraries) which provide pre-written and tested code to do certain things. As with other languages, when using a module in Haskell - we have to import it with the `import` command. The first line below shows importing the entire module and the second line shows importing only two functions:

```
import Data.Char
import Data.Char (toUpper, toLower)
```

Haskell's standard library which is auto-imported into every other module & the interpreter is called the *standard prelude*.

10.2 Functions & Operators

Haskell includes both functions and operators. Functions (`sqrt`, `mod`, etc) are used in prefix notation (i.e. `mod n 2`). Operators (`+`, `-`, `**`, etc) are used in infix notation (i.e. `1 + x`); apart from the unary minus operator which is a prefix operator.

It is possible to use any binary (two-argument) function as an operator by surrounding it with backquotes (```). For example `n `mod` 2` is equivalent to `mod n 2`. Similarly, we can use an operator as a function by encasing the operator in parentheses (`()`). For example `(+) 1 x` is equivalent to `1 + x`.

10.3 Pattern Matching

So far, we have seen two ways of defining functions:

- using single equations
- using guards

We will now add a third function definition mechanism to our options: pattern matching.

Pattern matching consists of a sequence of equations; each *pattern* (on the left hand side) is associated with a different result (on the right hand side). An example of this is seen below, defining the `not` function which is included in the Prelude.

```
not :: Bool -> Bool
not True  = False
not False = True
```

It is also possible to use a wildcard to simplify pattern matching. This can be seen below where the Boolean *or* operator is redefined.

```
(||) :: Bool -> Bool -> Bool
False || False = False
_ | _          = True
```

Alternatively, we can also use *named parameters* which can take a value from the pattern and use it in the output. An example of this is shown below

```
(||) :: Bool -> Bool -> Bool
True || _   = True
False || p  = p
```

10.4 Recursion

As with other programming languages, Haskell supports recursion. A recursive definition is one that is defined in terms of itself.

Recursion is a critical component in Haskell as pure functional programming does not support loops as these are imperative constructs (as they operate on a program's state). We shall recuse lots especially when using lists.

10.4.1 Recursive Definition of Factorial

To illustrate recursion in Haskell, we will examine an example of the Factorial function. $fact(n)$ is the product of the integer n and all the integers below it $n - 1$, $n - 2$, For example:

$$fact(3) = 3 \times 2 \times 1 = 6$$

Note that the factorial of a number $n > 0$ can be defined in terms of the factorial of $n - 1$, for example

$$fact(4) = 4 \times fact(3)$$

This leads us to the following recursive function definition

```
fact :: Int -> Int
fact n
  | n > 0      = n * fact (n - 1)
  | n == 0     = 1
  | otherwise  = error "undefined for negative ints"
```

10.4.2 Primitive vs General Recursion

Primitive recursion is recursion where

- the base case considers the parameter value 0
- the recursive case considers how to get from value $n - 1$ to n

General Recursion is a recursive function where there is not a precise halting case, but there is a halting condition. For example, in the `divide` function below - the halting case is where $n < m$ - not when either of them are at exact values.

```
divide :: Int -> Int
divide n m
  | n < m      = 0
  | otherwise  = 1 + divide (n - m) m
```

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Lecture - Tuples, Strings & Lists

📅 2024-02-12

🕒 12:00

🎓 Matthew

11.1 Characters & Strings

Haskell comes with both the `String` and `Char` types. In Haskell, a single quote (`'`) is used to denote characters and double quotes (`"`) for strings. For example:

```
ghci> :type 'a'
'a' :: Char

ghci> :type "Sam"
"Sam" :: String
```

The `Char` module defines some useful functions on characters. Such as converting to uppercase or checking if a provided character is a digit.

```
ghci> import Data.Char

ghci> toUpper 'a'
'A'

ghci> isDigit 'a'
False
```

11.2 Tuples

As seen in other languages, we can combine multiple pieces of data into one data type. For example, we may want to combine a name (represented as a string) and a mark (represented as an integer) for example `("Dave", 74)` which would have the tuple type `(String,Int)`. This example can be furthered to a small program which takes two Tuples representing a student and outputs the name of the student with the higher mark:

```
betterStu :: (String,Int) -> (String,Int) -> String
betterStu (s1,m1) (s2,m2)
  | m1 >= m2 = s1
  | otherwise = s2
```

We can define a *type synonym* which can be used in the place of the raw definition of the tuple. For example, if we define the type synonym on the first line below, we can then re-define `betterStu` to use that.

```
type StudentMark = (String, Int)
betterStu :: StudentMark -> StudentMark -> String
```

```

betterStu (s1,m1) (s2,m2)
  | m1 >= m2  = s1
  | otherwise = s2

```

Tuples can also be used to enable a function to return more than one value.

11.3 Lists

Lists are the main data structure in Haskell. They are used to store any number of data values *of the same type*. For example the first list below is a list of integers and the second example is a list of strings.

```

[12, 64, -92, 85, 12]
["This", "is", "a", "list"]

```

The type of a list (l) is denoted by [l]. For example:

```

ghci> :type [True, False, False]
[True, False, False] :: [Bool]

```

The empty list ([]) is an element of any list type.

11.3.1 Strings as a List of Chars

Strings in Haskell are simply lists of characters, therefore the type `String` is declared as:

```

type String = [Char]

```

This means that the two expressions below are the same

```

['h', 'e', 'l', 'l', 'o']
"hello"

```

This also means that the operations that we will cover later in this lecture (for example concatenation of lists) therefore also apply to strings.

11.3.2 Lists from Ranges

It is possible to use the operator `..` to generate lists. For example:

```

[3 .. 9]      = [3, 4, 5, 6, 7, 8, 9]
[3.1 .. 9]    = [3.1, 4.1, 5.1, 6.1, 7.1, 8.1, 9.1]
['a' .. 'z'] = "abcdefghijklmnopqrstuvwxyz"

```

We can also add an argument to give steps different from 1 as seen below:

```

[3, 5 .. 15]    = [3, 5, 7, 9, 11, 13, 15]
[0, 0.1 .. 0.5] = [0.0, 0.1, 0.2, 0.3, 0.4, 0.5]

```

11.3.3 List Comprehension

We can use list comprehension to build one list from another list. For example, if we define:

```

aList = [1, 2, 3, 4, 5]

```

Then the list comprehension shown on the first line below will have the value on the second line below:

```

[2*i | i <- aList]

[2, 4, 6, 8, 10]

```

We read this as “take all $2*i$ where i comes from `aList`”. The `<-` symbol represents the set member symbol (\in) from Maths.

List comprehension is extremely powerful as can be seen in the following example:

```
bList = [2, 3, 6, 9, 4, 7]

ghci> [mod i 2 == 0 | i <- bList]
[True, False, True, False, True, False]
```

We can take this another step further where we can add a test at the end of the generator. The following example has given all the values that are less than 5 from `cList`

```
cList = [2, 3, 6, 9, 4, 8]

ghci> [i * 2 | i <- cList, i < 5]
[4, 6, 8]
```

We can use list comprehension and a pattern on the left hand side of the `<-` as seen below:

```
addPairs :: [(Int, Int)] -> [Int]
addPairs pairList = [i + j | (i, j) <- pairList]

ghci> addPairs [(1, 2), (4, 8), (6, 3)]
[3, 12, 9]
```

11.4 Polymorphic Functions

The *Prelude* comes with a number of functions built in, we will now examine some of these. While doing so, we will examine *polymorphism*.

If we consider the `length` function, which gives the number of elements in a list (returning an `Int`), which works for any type of lists. For this reason, we may think of it to have the following types:

```
length :: [String] -> Int
length :: [Bool] -> Int
```

However! The actual type of `length` is given as:

```
length :: [a] -> Int
```

Here, we are using a *type variable* which stands for an *arbitrary type*. By convention `a`, `b`, `c`, ...are used as type variables.

The expressions `[a] -> Int` is known as the *most general type* for `length`

11.4.1 List Functions

We can define polymorphic functions, by not giving a type declaration. Haskell will try and infer the most general type by looking at the structure of the function.

11.4.1.1 Adding Elements To Front Of List

For lists in Haskell, one of the most used operations is `:. This adds an element to the front of the list and is of type a -> [a] -> [a]. For example:`

```
ghci> 3:[5, 7, 2]
[3, 5, 7, 2]
```


11.4.1.2 List Concatenation

The ++ operator can be used to join two lists together:

```
ghci> "hello" ++ "world"
"helloworld"
```

11.4.1.3 Return Element From Specific Position

The !! operator returns an element at a given position. For example:

```
ghci> ["fish", "ham", "cheese", "spam"] !! 2
"cheese"
```

Note that eventhough indexing a list like this is common in other languages, we will not tend to use this operator that often.

11.4.1.4 Checking If List Is Empty

The null function tests whether a list is empty. For example:

```
ghci> null [1, 2]
False
```

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Lecture - List Patterns and Recursion

📅 2024-02-19

🕒 1200

🎓 Matthew

12.1 Patterns, a Recap

Many of the functions we have been using so far have been defined using patterns. These can include literal values, variables, and wildcards. All of these can be seen below in the redefinition of `or` from lecture 3.

```
or :: Bool -> Bool -> Bool
or True _    = True
or False a   = a
```

Patterns can also include Tuples. For example the `fst` and `snd` functions from the Prelude:

```
fst (x,_) = x
snd (_,y) = y
```

Note that these projection functions are polymorphic - they work for tuples of data of any kind.

12.2 Lists and List Patterns

As we saw in the last lecture, the `:` operator will append whatever precedes it to the list which succeeds it. For example:

```
ghci> 4:[7,3]
[4,7,3]
```

This means that every list can be built from `[]` and `:`, which are constructors for lists.

Note that the `:` operator is right-associative meaning that it works from right to left adding the right-most to the second right-most, and so on. For example:

```
ghci> 1:2:[]
[1,2]
```

We can use these constructors in patterns. Where we use `:` in list patterns when we want to deal with the first element and the rest of the list separately. For example, the following example (from the Prelude) returns the first element of a list:

```
head :: [a] -> a
head (x:xs) = x
```

Note that the naming convention `x:xs` is standard; where we say “x, exes”.

It is also possible to use wildcard matching:

```
head :: [a] -> a
head (x:_) = x
```

12.3 Recursion over Lists

As we already know from recursion in the previous weeks, we require a base case and a recursive case. When recursing over lists - the base case considers the empty list `[]` and the recursive case gives the result for any non-empty list (one matched by `x:xs`) from the result of the tail of the list `xs`.

If we take the example of the implementation of a Prelude defined function `sum`.

```
sum :: [Int] -> Int
```

The function is defined to return a sum of a list of integers.

The base case of our function will be when the list is empty, as the sum of `[]` is obviously 0. The recursive case will be where `xs` is greater than 0. Therefore, we can define the function as

```
sum []      = 0
sum (x:xs)  = x + sum xs
```

12.3.1 General Recursion over Lists

The above recursive function we have explored is of the type ‘primitive recursion’. It is also possible to use general recursion over lists.

For example, we take the Prelude function `zip` which has the following definition:

```
zip :: [a] -> [b] -> [(a,b)]
```

This function joins two lists into a single list of tuples:

```
ghci> zip ['r', 'h', 'a'] [4, 7, 2]
[('r',4),('h',7),('a',2)]
ghci> zip [5, 7, 1, 5] ['a', 'b']
[(5,'a'),(7,'b')]
```

Note that the lists don't have to be of the same length, and that if the lists are different lengths then the last few elements of the longer elements are dropped.

The easiest way to define `zip` is by beginning with the recursive case of two non-empty lists `x:xs` and `y:ys`. In this case we can place `x` and `y` into tuple, and zip up the tails `xs` and `ys` which give us:

```
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

Now we have one part of the program left - handling when one argument is `[]`. There are three possibilities for this:

- `y` is empty
- `x` is empty
- `x` and `y` are empty

These can all be written together with a wildcard:

```
zip _ _ = []
```

which gives us the entire function:

```
zip :: [x] -> [y] -> [(x,y)]
zip (x:xs) (y:ys) = (x,y) : zip xs ys
zip _ _ = []
```

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Lecture - Functions as Values

📅 2024-02-26

🕒 12:00

🎓 Matthew

13.1 Functions as Arguments

In Functional Programming, functions can be treated as data. This means that they can be passed as arguments to other functions; and that they can be returned as results from functions. A function that either takes another function as an argument; or returns a function is known as a *Higher-Order Function*. Higher-Order programming can be very expressive.

The following function definition applies a function to a value and then applies the same function again to the result.

```
twice :: (Int -> Int) -> Int -> Int
twice f x = f (f x)
```

We would also need to define a function to be used in the above function:

```
succ :: Int -> Int
succ n = n + 1
```

Now that we have two functions, we can look at an example calculation:

```
twice succ 5
~> succ (succ 5)    def of twice
~> succ (5 + 1)     def of succ
~> (succ 6)         arithmetic
~> (6 + 1)          def of succ
~> 7                arithmetic
```

13.1.1 Function Composition

Haskell includes a function composition operator “.”. For two functions, *f* and *g*, the expression

$$(f \ . \ g) \ x$$

means apply *g* to *x* then apply *f* to the result. Note that this is the same as the mathematical operation $f \circ g$. The below code snippet shows the order in which the operator works:

$$(f \ . \ g) \ x = f \ (g \ x)$$

The output type of the first function must be of the same type as the input type of the second function.

13.1.2 Function-Level Definitions

The composition operator, `.`, allows us to easily define functions in terms of other functions. For example, we can replace the definition:

```
twice f x = f (f x)
```

with:

```
twice f = f . f
```

A function defined just in terms of other functions (without reference to other arguments) is known as a function-level definition. This notation can also be known as “point-free style”

13.2 Partial Application

A powerful feature shared by many functional languages is that functions can be partially applied. If we consider the following simple function definition which multiplies two numbers together:

```
multiply :: Int -> Int -> Int
multiply x y = x * y
```

We would typically consider the only thing that this function does is to multiply two parameters together and return the result. However - it would be more correct to consider that we can apply the function to one argument and are left with a function of one argument.

Returning to our example, we may use it as follows:

```
double = multiply 2
```

which would then mean that:

```
ghci> double 5
10
```

As we can see - `double` is taking a single argument which is being passed to `multiply` along with it's defined argument. This allows us to understand more about how Haskell works, in that every function in Haskell takes exactly one argument. This means that `multiply`, with two input parameters, is actually taking a single parameter (`Int`) then returning another function, `Int -> Int` which is used to produce the final result.

We can see that:

```
multiply :: Int -> Int -> Int
```

is shorthand for:

```
multiply :: Int -> (Int -> Int)
```

13.2.1 Operator Sections

It is not only the functions which can be partially applied - it is also possible to partially apply an operator. The following are examples of operator sections:

- `(2*)` - a function that multiplies its arguments by 2
- `(/2)` - a function that divides its argument by 2
- `(2/)` - a function which gives 2 divided by its argument
- `(>3)` - a function which tests if its argument is greater than 3

13.3 Patterns of Computation

When designing algorithms, there are a number of things that we often want to do to a list:

- transform every element of a list in some way
- remove those elements of a list that don't possess a given property
- combine all the elements with a particular operation

Haskell includes a number of higher-order functions that implement each of these patterns.

13.3.1 Mapping

If we take the example function, `doubleAll` which doubles every element in the list. A recursive example of this is below:

```
doubleAll :: [Int] -> [Int]
doubleAll [] = []
doubleAll (x:xs) = 2*x : doubleAll xs
```

This function has a pattern in it, where an operator (could also be a function) is applied to every element of the list.

The Prelude defines a function `map` which takes the operation to be applied as a parameter. Using `map`, the definition of the function `doubleAll` become (note the use of an operator section):

```
doubleAll xs = map (2*) xs
```

However, we can make this simpler. We do not need to specify `xs` because that is added complexity. The simplest form of this function can be seen below:

```
doubleAll = map (2*)
```

13.3.2 Filtering

Filtering doesn't change any of the elements of the list - rather it checks if the elements pass the test and keeps them if so. For example:

```
keepPositive [] = []
keepPositive (x:xs)
  | x > 0      = x : keepPositive xs
  | otherwise = keepPositive xs
```

Using the `filter` function, we are able to condense this down to the following:

```
keepPositive = filter (>0)
```

Note here that we have used the simplest form of the function as we do not need to specify the list we want to filter in the function definition.

13.3.3 Folding

The Prelude contains the function `foldr` which applies the function specified on the provided list. It is used to combine the elements of the list in some way. For example, the following function which adds every element in the list:

```
addUp :: [Int] -> [Int]
addUp [] = 0
addUp (x:xs) = x + addUp xs
```

can be rewritten using `foldr` as follows

```
addUp = foldr (+) 0
```

Note that you still have to specify the type for the function when using `foldr`, for example for `addUp` this would be:

```
addUp :: [Int] -> Int
```

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Lecture - Algebraic Types

📅 2024-03-11

🕒 12:00

🎓 Matthew

“There’s nothing in the blob, it’s a blob”

14.1 Types in Haskell: A Recap

As we have already seen, there are a number of built-in types in Haskell:

- `Int`, `Float`, `Bool` and `Char` (the basic types)
- `(Int, Int, Char)` (tuple types)
- `[Int]` or `[(Int, Char)]` (the list type)

It is also possible to give convenient names to types using synonyms, for example:

- `type HouseNumber = Int`
- `type StreetName = String`
- `type Address = (HouseName, StreetName)`

These types are only so well and good, as they do not provide a convenient way to model more complex structures (such as Binary Trees). Within Haskell, we are able to use *Algebraic Types* to define arbitrarily complex types.

14.2 Algebraic Types

We declare the algebraic type using the keyword `data`, followed by:

- the name of the type being defined
- a list of constructors

Note that both have to begin with a capital letter. For example:

```
data Day = Mon | Tue | Wed | Thur | Fri | Sat | Sun
```

As we can see above, the simplest algebraic types are those where the constructors don’t take any arguments. Here, the constructors are the data values (or members of the type). We call a type which has been defined in this way an *enumerated type*.

14.2.1 Enumerated Type

Another example of an enumerated type is Haskell's Boolean type:

```
data Bool = False | True
```

The simplest way to define a function on an Enumerated Type is by pattern matching. Using the `Day` data type an example is:

```
isWeekend :: Day -> Bool
isWeekend Sat = True
isWeekend Sun = True
isWeekend _   = False
```

This might look rather obtuse, and that would be a correct observation. This is because our data type (`Day`) doesn't include any operators such as `==`. It is possible to define `==` ourselves, however this would be tedious.

14.2.2 Algebraic Types & Type Classes

We can get Haskell to provide a `==` operator by declaring that we want our type (`Day`) to be a member of the `Eq` type class. (Note that `Eq` includes all types that include `==` and `/=`). To do this, we would need to define `Day` as follows:

```
data Day = Mon | Tue | Wed | Thur | Fri | Sat | Sun
         deriving (Eq)
```

Now that we have the `==` operator implemented for `Day`, we can redefine `isWeekend` as:

```
isWeekend :: Day -> Bool
isWeekend day = day == Sat || day == Sun
```

However, `Eq` only includes *Equality* operators - what happens if we want to compare two values and see if one is greater than the other.

Fortunately, Haskell includes more type classes which can provide us with the functionality that we require for this:

- `Ord` provides us with `<`, `<=`, `>` and `>=`
- `Show` to provide a function

```
show :: Day -> String
```

which converts `Day` values into strings

- `Read` which provides a function

```
read :: String -> Day
```

which converts strings (for example "Mon") into `Day` values.

When using all these additional type classes, we may end up with the following definition of `Day`:

```
data Day = Mon | Tue | Wed | Thur | Fri | Sat | Sun
         deriving (Eq,Ord,Show,Read)
```

This therefore means we can redefine `isWeekend` as follows:

```
isWeekend :: Day -> Bool
isWeekend day = day >= Sat
```

14.2.3 Product Types

As we saw in an earlier lecture, we can define student records (their name and a mark) as a type synonym giving a name to a tuple:

```
type StudentMark = (String, Int)
```

However, it is also possible for us to use an algebraic type with a constructor that has two arguments - one for the name and one for the mark:

```
data StudentMark = Student String Int
```

Here, the constructor `Student` is followed by its argument types. For example:

```
Student "Sam" 44
Student "Jill" 64
```

As could be expected, it is also possible to define functions which use the product data types:

```
betterStu :: StudentMark -> StudentMark -> String
betterStu (Student s1 m1) (Student s2 m2)
  | m1 >= m2 = s1
  | otherwise = s2
```

Which would be executed with:

```
ghci> betterStu (Student "Sam" 44) (Student "Jo" 73)
"Jo"
```

The main advantage of using an algebraic type is that every data value has an explicit label of its purpose (for example, `Student`)

14.2.4 Sum Types

It is possible for us to combine the ideas of enumerated types and product types, this gives us types whose elements can be built in different ways. For example, a shape might be specified by its radius or a rectangle specified by its height and width. It is possible to represent this using a *sum* type:

```
data Shape = Circle Float |
           Rectangle Float Float
```

Which would be declared as:

```
Circle 9.0
Rectangle 4.5 6.0
```

This is a bit like an Abstract Class, where the parent class has no types of its own and that the child classes are Concrete classes. `Shape` would be the parent, abstract class, and `Circle` & `Rectangle` would be the child, concrete classes.

As expected, it is possible to write a single function for a sum type (i.e `Shape`) which runs different code depending on the type of `Shape` used. This is done through Pattern Matching:

```
area :: Shape -> Float
area (Circle r)      = pi * r * r
area (Rectangle h w) = h * w
```

This can be extremely useful, for example in the problem of representing addresses. Some buildings have numbers, and some buildings have names. We can define a data type for representing the first line of an address as:

```
data Address = Address Building String
data Buildings = Number Int | Name String
```

Which we can use as:

```
Address (Number 42) "High Street"
Address (Name "Seaview") "Uplands Road"
```

14.2.5 Recursive Types

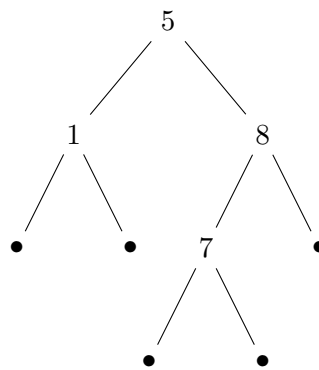


Figure 14.1: Binary Tree Example

Types can be described in terms of themselves. If we take the example of a binary tree, which we know is defined recursively as:

- a null node; or
- a node with a value, left sub-tree and a right-sub-tree

We can define a data type directly from this definition:

```
data Tree = Null |
          Node Int Tree Tree
```

Which we can see in-use below (note that the final value is represented by the diagram on the previous page):

```
Null
Node 7 Null Null
Node 5 (Node 1 Null Null)
      (Node 8 (Node 7 Null Null) Null)
```

It is possible to define functions which interact with recursive types. Generally, functions on trees will mirror the recursive structure of the type (meaning they use `Null` as the base case & `Node` for the recursion). For example, the following function returns the height of a binary tree:

```
height :: Tree -> Int
height Null = 0
height (Node _ st1 st2) = 1 + max (height st1) (height st2)
```