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Second Year

Discrete Mathematics and Functional Programming (`DMAFP`)

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Lecture - Sets

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1.1 Introduction

Sets underpin maths and Computer Science. A set is a collection of objects, which are called the elements (also known as members of the set). For example, a set of the numbers 1, 3, 8; or the collection of students in a class born in March. There are two characteristics of sets:

1. There are no repeated occurrences of elements
2. There is no particular order of the elements

1.2 Set Notation

The elements of a set are enclosed in braces with their names being denoted by a *letter*, for example:

$$A = \{1, 2, 3\}, \quad C = \{Portsmouth, Brighton, London\}$$

There are two ways that we can describe the members of a set. We can *list the elements* which is mainly used for finite sets, for example:

$$A = \{3, 6, 9, 12\}$$

We can *specify a property* that all the elements in the set have in common. The '|' character is read 'such that', sometimes ':' is used in it's place. For example:

$$B = \{x | x \text{ is a multiple of 3 and } 0 < x < 15\}$$

We can also use *three dots* to informally denote a sequence of elements that we don't wish to write down, for example:

$$C = \{1, \ldots, 10\}$$

1.2.1 Sets of Numbers

There are some reserved letters to denote specific sets of numbers in maths. These are shown below:

- $\mathbb{N}$ (or N) is used for the set of natural numbers (integers >= 0). $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- $\mathbb{Z}$ (or Z) is used for the set of integers. $\mathbb{N} = \{\ldots, -1, 0, 1, \ldots\}$
- $\mathbb{Q}$ (or Q) is used for the set of rational numbers (number which can be expressed as a quotient or fraction). $\mathbb{Q} = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$
- $\mathbb{R}$ (or R) is used for the set of real numbers. $\mathbb{R} = \{\ldots, -1, 0, \frac{1}{2}, \ldots\}$

1.2.2 Elements of a Set

We can use the $\in$ symbol to denote if an element is a member of a given set. For example, if x is a member of S - then we can say:

$$x \in S$$

The symbol $\notin$ denotes an element is not a member of a given set. For example, if y is **not** a member of S - then we can say:

$$y \notin S$$

1.2.3 Many Ways to Say The Same Thing

There are several ways of describing the same set, for example for the set S of *odd integers*:

$$\begin{aligned} S &= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \\ &= \{x | x \text{ is an odd integer }\} \\ &= \{x | x = 2k + 1 \text{ for some integer } k\} \\ &= \{x | x = 2k + 1 \text{ for some } k \in \mathbb{Z}\} \\ &= \{2k + 1 | k \in \mathbb{Z}\} \end{aligned}$$

The phrase “for some [integers K]”, means “for all [integers k]”

1.2.4 Empty Sets

Where a set has *no elements*, it is called an empty set or null set. It’s denoted with the $\emptyset$ symbol, for example:

$$\emptyset = \{\}$$

1.2.5 Finite & Infinite Sets

If the number of elements in the set is fixed (for example when counting the elements at a fixed rate for a set amount of time), then the set is *finite*. If the set X is finite, then we call $|X|$ the *cardinality* of X therefore:

$$|X| = \text{number of elements in } X$$

If the counting never stops then X is an infinite set.

1.2.6 Subsets

A subset is where one set’s elements are entirely present in another set. There are three conditions we need to know about:

- $A \subseteq B$: A is a subset of B therefore every element in A is also in B.
- $A \nsubseteq B$: A is not a subset of B.
- $A \subset B$: A is a proper subset of B, therefore B has at least one additional element which is not in A.

1.2.7 Equality of Sets

Two sets are *equal* if they have exactly the same elements. This is denoted by writing $A = B$. Where $A = B$, the following conditions are also true:

- $A \subseteq B$ for every a if $a \in A$, then $a \in B$
- $B \subseteq A$ for every b if $b \in B$, then $b \in A$

1.3 Operations on Sets

Sets can have *operations* performed on them - this will change something about them.

1.3.1 Intersection

The intersection of two sets A and B is defined as:

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

This is the set of elements which appear in both sets only. If we take a Venn Diagram with a set on either side - its the overlapped elements which would be returned from an intersection operation. For example if $A = \{a, b, c\}$ and $B = \{c, d\}$ then $A \cap B = \{c\}$.

1.3.2 Disjoint

If an intersection returns no elements, then the two sets are *disjoint*. This is shown by:

$$A \cap B = \emptyset$$

1.3.3 Union

The *union* of the two sets A and B is defined as:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

This is the set of elements which are in either A or B, this means elements which appear in both are returned. For example, if $A = \{a, b, c\}$ and $B = \{c, d\}$ then $A \cup B = \{a, b, c, d\}$.

1.3.4 Difference

The *difference* between two sets, A and B is defined as:

$$A \setminus B = \{x | x \in A \text{ or } x \in B\}$$

This is the set of elements which are in A but not in B, so could be represented as $A - B$. Note that $A \setminus B \neq B \setminus A$.

1.3.5 Counting Elements In a Set

If we take A and B to be finite sets, we can calculate the number of elements in the union of A and B. The correct way to count this is as follows:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

We have to minus $|A \cap B|$ from the sum because otherwise it is as though we are counting it twice due to the fact that we are summing the total number of elements in A and B.

1.4 Complement

If we consider that all subsets are the subset of a particular set, U for example (the universe of discourse), then the difference $U \setminus A$ is called the *complement* of A is shown as either $\overline{A}$ or A'. For example:

$$A' = \{X | x \in U \text{ and } x \notin A\}$$

1.5 Basic Set Properties

Sets have a number of basic properties - many of these are the same as that for Boolean Expressions

- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$
- $A \cup A = A$
- $A \cap A = A$
- Commutative
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Associative
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributive
  - $A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cup C) = (A \cup B) \cap (A \cup C)$
- de Morgan's
  - $(A \cap B)' = A' \cup B'$
  - $(A \cup B)' = A' \cap B'$

1.6 Power Set

A *power set* is the collection of all subsets of a set, S which is denoted by $P(S)$. For example, if $S = \{a, b, c\}$ then:

$$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

1.7 Partition

A *partition* of the set S is a collection of non-empty subsets of set S where every element form S belongs to exactly one member of S. This means that the sets are mutually disjoint and that the union of all the sets in the collection results in the original set, S. For example, if $S = \{a, b, c, d, e, f\}$ then $\{\{a, e\}, \{c\}, \{f, d\}, \{b\}\}$ is a partition of S.

Part II

Functional Programming