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On multiplication in finite fields

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ABSTRACT

We present a method for multiplication in finite fields which gives multiplication algorithms with improved or best known bilinear complexities for certain finite fields. Our method generalizes some earlier methods and combines them with the recently introduced complexity notion $\hat{M}_q(\ell)$, which denotes the minimum number of multiplications needed in \mathbb{F}_q in order to obtain the coefficients of the product of two arbitrary ℓ -term polynomials modulo x^ℓ in $\mathbb{F}_q[x]$. We study our method for the finite fields \mathbb{F}_{q^n} , where $2 \leq n \leq 18$ and $q = 2, 3, 4$ and we improve or reach the currently best known bilinear complexities. We also give some applications in cryptography.

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1. Introduction

Let \mathbb{F}_q be a finite field and $n > 1$ be an integer. Let $\mathbb{F}_{q^n}^\perp$ be the dual of \mathbb{F}_{q^n} as a vector space over \mathbb{F}_q . Then the rank $R(\mathbb{F}_{q^n}/\mathbb{F}_q)$ over \mathbb{F}_q is defined to be

$$\min \left\{ \ell \in \mathbb{N} \mid \exists u_i, v_i \in \mathbb{F}_{q^n}^\perp, w_i \in \mathbb{F}_{q^n} \text{ such that } \forall a, b \in \mathbb{F}_{q^n}, ab = \sum_{i=1}^{\ell} u_i(a)v_i(b)w_i \right\}.$$

$R(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is also denoted by $\mu_q(n)$ and it is called the *bilinear complexity of multiplication in \mathbb{F}_{q^n} over \mathbb{F}_q* . It corresponds to the minimum number of \mathbb{F}_q bilinear multiplications in order to multiply two arbitrary elements of \mathbb{F}_{q^n} . Winograd [27] showed that this complexity is $\geq 2n - 1$, and it is equal to $2n - 1$ if and only if $n \leq \frac{1}{2}q + 1$. Algorithms obtaining the lower bound are based on interpolation algorithms on the rational function field [27]. D.V. Chudnovsky and G.V. Chudnovsky [14]

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generalized this idea to algebraic function fields (of one variable) over \mathbb{F}_q . Shokrollahi [22] obtained optimal algorithms for the multiplication in certain finite fields using the principle of D.V. and G.V. Chudnovsky algorithm and the elliptic curves. Shparlinski, Tsfasman and Vladut [23] gave the asymptotic bounds for multiplication in finite fields by using curves with many points. Ballet [2,3] generalized Shokrollahi's work to the algebraic function fields of genus g . Ballet and Rolland [4] gave a generalization of D.V. Chudnovsky and G.V. Chudnovsky multiplication algorithm by interpolating not only degree one places but also interpolating on degree two places. Moreover, Ballet, Rolland, Chaumine and Brigand in [4–7] have improved the asymptotic bounds given by Shparlinski, Tsfasman and Vladut in [23]. Arnaud [1] presented a method using local expansions with multiplicity 2 and places of degree one and two. In [8], new upper bounds of the bilinear complexity of multiplication in \mathbb{F}_{q^n} over \mathbb{F}_q are obtained by proving the existence of certain types of non-special divisors of degree $g - 1$ in the algebraic function fields of genus g defined over \mathbb{F}_q . Moreover, concerning the use of places of degree greater than one, Ballet and Rolland use places of degree one, two and four to improve the asymptotic bilinear complexity of multiplication in the extensions of \mathbb{F}_2 in [10].

In this paper, we use algebraic function fields of one variable with places of arbitrary degrees and moreover we use some places not only once but also many times. Here many times refers to using first $u_i > 1$ coefficients instead of the first ($u_i = 1$) coefficient in the local expansion of a place P_i (see the map φ in Section 3). The proposed method is a generalization of the methods introduced in [14,23,22,2–7,1,8,9]. In the papers cited above, mostly degree one places are used only. Among these papers, only [4–7,1,8,9] use places of degree greater than one. In these papers, [4–9] use only places of degree one and degree two but always with $u_i = 1$, i.e., using places only once. In [1], only places of degree one and degree two are used and all of such places are used at most 2 times.

In order to use places of arbitrary degree, one needs a measure to estimate the contribution of such places in the bilinear complexity. Here, we use the recently introduced complexity notion $\hat{M}_q(\ell)$ for this purpose [12]. Recall that $\hat{M}_q(\ell)$ is the minimum number of multiplications needed in \mathbb{F}_q in order to obtain coefficients of the product of two arbitrary ℓ -term polynomials modulo x^ℓ in $\mathbb{F}_q[x]$. We observe that in order to get the best linear complexities using our method, one needs to solve an optimization problem using $\hat{M}_q(\ell)$ and curves with many points over finite fields. Here, curves with many points refer to curves with many degree one and higher degree points, where the complexity notion $\hat{M}_q(\ell)$ indicates the weight of degree ℓ points of the curve in the optimization problem. Here, we would like to remark that local expansions and higher degree points of curves over finite fields have been shown to be very useful in algebraic geometry codes and low discrepancy sets and sequences (see, for example [28,29,19,20]). One of our motivations in this paper comes from these results in algebraic geometry codes and low discrepancy sets and sequences. We improve or reach the best known bilinear complexities in \mathbb{F}_{q^n} where $2 \leq n \leq 18$ and $q = 2, 3, 4$ by searching and optimizing the suitable places and multiplicities in the proposed method. Moreover, our method gives explicit multiplication formulae immediately. We also give some applications to cryptography.

The rest of the paper is organized as follows: we introduce complexity notions and a brief review of algebraic function fields in Section 2. The proposed method is presented in Section 3. In Section 4, we obtain currently best known upper bounds for the bilinear complexity $\mu_q(n)$ of multiplication for $2 \leq n \leq 18$ and $q = 2, 3, 4$ which includes our some of improvements. Using the method of Section 3, we obtain some improvements. In Section 5, we give an example of computing multiplicative complexity of finite fields with large number of elements used in cryptography. The proposed method gives explicit formulae easily. We illustrate how to obtain explicit formulae reaching the upper bounds of Section 3 with an example in Section 6.

2. Preliminaries

2.1. Some complexity notions

The notation $\mu_q(n)$ represents the *bilinear complexity of multiplication in \mathbb{F}_{q^n} over \mathbb{F}_q* . It corresponds to the minimum number of \mathbb{F}_q bilinear multiplications in order to multiply two arbitrary elements of \mathbb{F}_{q^n} . There is a related but different complexity notion. Let $M_q(n)$ denote the number of

multiplications needed in \mathbb{F}_q in order to multiply two arbitrary n -term polynomials in $\mathbb{F}_q[x]$ (cf. [13,15,16,25–27]). Here a polynomial is called an n -term polynomial in $\mathbb{F}_q[x]$ if it is of the form

$$a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in \mathbb{F}_q[x].$$

As reduction modulo an irreducible polynomial in $\mathbb{F}_q[x]$ can be performed without multiplications in \mathbb{F}_q , we have

$$\mu_q(n) \leq M_q(n). \quad (2.1)$$

However $\mu_q(n)$ and $M_q(n)$ are not necessarily equal in general. Using a polynomial basis $\{1, \xi, \xi^2, \dots, \xi^{n-1}, \dots, \xi^{2n-2}\}$ for $\mathbb{F}_{q^{2n-1}}$ over \mathbb{F}_q , it is easy to show that

$$M_q(n) \leq \mu_q(2n-1).$$

We will need another complexity notion in this paper. For a positive integer ℓ , let $\widehat{M}_q(\ell)$ denote the multiplicative complexity of computing the coefficients of the product of two ℓ -term polynomials modulo x^ℓ over \mathbb{F}_q . In other words, $\widehat{M}_q(\ell)$ is the minimum number of multiplications needed in \mathbb{F}_q in order to obtain the first ℓ coefficients of the product of two arbitrary ℓ -term polynomials in $\mathbb{F}_q[x]$. It is not difficult to obtain useful upper bounds on $\widehat{M}_q(\ell)$ for certain values ℓ . For example we have $\widehat{M}_q(2) \leq 3$, $\widehat{M}_q(3) \leq 5$, $\widehat{M}_q(4) \leq 8$ and $\widehat{M}_q(5) \leq 11$ for any prime power q (cf. [13, Proposition 1]).

2.2. A brief review of algebraic function fields

We start with the basics of the algebraic function fields. The details in this subsection can be found in [24].

An algebraic function field F/\mathbb{F}_q of one variable over \mathbb{F}_q is an extension field $F \supseteq \mathbb{F}_q$ such that F is a finite extension of $\mathbb{F}_q(x)$ for some element $x \in F$ which is transcendental over \mathbb{F}_q . A valuation ring of the function field F/\mathbb{F}_q is a ring $\mathcal{O} \subseteq F$ with the properties $\mathbb{F}_q \subset \mathcal{O} \subset F$ and for any $z \in F$, either $z \in \mathcal{O}$ or $z^{-1} \in \mathcal{O}$. A place of P of the function field F/\mathbb{F}_q is the maximal ideal of some valuation ring \mathcal{O} of F/\mathbb{F}_q . We will denote the set of all places of F/\mathbb{F}_q as \mathbb{P}_F . If \mathcal{O} is a valuation ring of F/\mathbb{F}_q and P is its maximal ideal, then \mathcal{O} is uniquely determined by P hence we denote \mathcal{O} by \mathcal{O}_P .

$F_P := \mathcal{O}_P/P$ is called the residue class field of P . The map $x \rightarrow x(P)$ from F to $F_P \cup \{\infty\}$ is called the residue class map with respect to P . Degree of P is $[F_P : \mathbb{F}_q] := \deg P$.

The free abelian group which is generated by the places of F/\mathbb{F}_q is called the divisor group of F/\mathbb{F}_q and it is denoted by \mathcal{D}_F . A divisor is a formal sum $D = \sum_{P \in \mathbb{P}_F} n_P P$ with $n_P \in \mathbb{Z}$, almost all $n_P = 0$. The support of D is defined by $\text{supp } D := \{P \in \mathbb{P}_F | n_P \neq 0\}$. A divisor of the form $D = P$ with $P \in \mathbb{P}_F$ is called a prime divisor. Two divisor $D = \sum n_P P$ and $D' = \sum n'_P P$ are added coefficientwise. For $Q \in \mathbb{P}_F$ and $D = \sum n_P P \in \mathcal{D}_F$ we define $v_Q(D) = n_Q$. A partial ordering on \mathcal{D}_F is defined by

$$D_1 \leq D_2 \iff v_P(D_1) \leq v_P(D_2)$$

for any $P \in \mathbb{P}_F$. A divisor $D \geq 0$ is called positive. The degree of a divisor is defined by

$$\deg D := \sum_{P \in \mathbb{P}_F} v_P(D) \cdot \deg P$$

and $\deg : \mathcal{D} \rightarrow \mathbb{Z}$ is a group homomorphism.

Let $0 \neq x \in F$ and Z (respectively N) be the set of zeros (poles) of x in \mathbb{P}_F . Then $(x)_0 := \sum_{P \in Z} v_P(x)P$ is called the zero divisor of x , $(x)_\infty := \sum_{P \in N} (-v_P(x))P$ is called the pole divisor of x and $(x) := (x)_0 - (x)_\infty$ is called the principal divisor of x .

The set $\mathcal{P}_F := \{(x) | 0 \neq x \in F\}$ is defined as the group of principal divisors of F/\mathbb{F}_q . The factor group

$$\mathcal{C} := \mathcal{D}_F / \mathcal{P}_F$$

is called the divisor class group. The divisor class of D , denoted by $[D]$, is the corresponding element in the factor group \mathcal{C}_F . For $D_1, D_2 \in \mathcal{D}_F$, we denote $D_1 \sim D_2$ if $[D_1] = [D_2]$.

For a divisor $A \in \mathcal{D}_F$ we set

$$\mathcal{L}(A) := \{x \in F \mid (x) \geq -A\} \cup \{\infty\}.$$

$\mathcal{L}(A)$ is a vector space over \mathbb{F}_q . If A' is a divisor equivalent to A then $\mathcal{L}(A) \cong \mathcal{L}(A')$. For $A \in \mathcal{D}_F$, the integer $\dim A := \dim \mathcal{L}(A)$ is called the dimension of the divisor A . The genus of F/\mathbb{F}_q is defined by

$$g := \max\{\deg A - \dim A + 1 \mid A \in \mathcal{D}_F\}.$$

For $A \in \mathcal{D}_F$,

$$i(A) := \dim A - \deg A + g - 1$$

is called the index of speciality of A . Any divisor $A \in \mathcal{D}_F$ is called non-special if $i(A) = 0$; otherwise A is called special.

3. The method

Let F/\mathbb{F}_q be an algebraic function field with full constant field \mathbb{F}_q . Let P_1, \dots, P_N be distinct places of arbitrary degrees. Assume that Q is a place of degree n . Let \mathcal{O}_Q be the valuation ring of the place Q . Note that the residue field \mathcal{O}_Q/Q is isomorphic to \mathbb{F}_{q^n} . Let D be a divisor such that $\text{supp } D \cap \{Q, P_1, P_2, \dots, P_N\} = \emptyset$. Let $\mathcal{L}(D)$ be the Riemann–Roch space of D . Assume also that the evaluation map Ev_Q from $\mathcal{L}(D)$ to the residue field \mathcal{O}_Q/Q is onto. For $1 \leq i \leq N$, let t_i be a local parameter at P_i . For $f \in \mathcal{L}(2D)$, let

$$f = \alpha_{i,0} + \alpha_{i,1}t_i + \alpha_{i,2}t_i^2 + \dots$$

be the local expansion at P_i with respect to t_i , where $\alpha_{i,0}, \alpha_{i,1}, \dots \in \mathbb{F}_{q^{\deg(P_i)}}$. Let u_i be a positive integer and consider the \mathbb{F}_q -linear map

$$\begin{aligned} \varphi_i : \mathcal{L}(2D) &\rightarrow \left(\mathbb{F}_{q^{\deg(P_i)}}\right)^{u_i} \\ f &\rightarrow (\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,u_i-1}). \end{aligned}$$

Let φ be the \mathbb{F}_q -linear map given by

$$\begin{aligned} \varphi : \mathcal{L}(2D) &\rightarrow \left(\mathbb{F}_{q^{\deg(P_1)}}\right)^{u_1} \times \left(\mathbb{F}_{q^{\deg(P_2)}}\right)^{u_2} \times \dots \times \left(\mathbb{F}_{q^{\deg(P_N)}}\right)^{u_N} \\ f &\rightarrow (\varphi_1(f), \varphi_2(f), \dots, \varphi_N(f)). \end{aligned} \quad (3.1)$$

Finally we assume that the map φ is injective.

Theorem 3.1. *Under the notation and assumptions as above we have*

$$\mu_q(n) \leq \sum_{i=1}^N \mu_q(\deg(P_i)) \widehat{M}_{q^{\deg(P_i)}}(u_i). \quad (3.2)$$

Proof. Let $\{h_\ell : 1 \leq \ell \leq n\}$ be a fixed basis of $\mathcal{L}(D)$ over \mathbb{F}_q . Moreover we choose and fix h'_1, \dots, h'_m such that $\{h_\ell : 1 \leq \ell \leq n\} \cup \{h'_k : 1 \leq k \leq m\}$ is a basis of $\mathcal{L}(2D)$. We consider $\text{Ev}_Q(h_1), \dots, \text{Ev}_Q(h_n), \text{Ev}_Q(h'_1), \dots, \text{Ev}_Q(h'_m) \in \mathcal{O}_Q/Q \cong \mathbb{F}_{q^n}$ as constants since $h_1, \dots, h_n, h'_1, \dots, h'_m$ are fixed. Similarly, we consider $\varphi(h_1), \dots, \varphi(h_n), \varphi(h'_1), \dots, \varphi(h'_m) \in \left(\mathbb{F}_{q^{\deg(P_1)}}\right)^{u_1} \times \dots \times \left(\mathbb{F}_{q^{\deg(P_N)}}\right)^{u_N}$ as constants. For $f \in \mathcal{L}(2D)$, there is no cost for bilinear complexity in obtaining $\varphi(f)$. Indeed, as

$$f = \sum_{\ell=1}^n c_\ell h_\ell + \sum_{k=1}^m d_k h'_k$$

with $c_1, \dots, c_n, d_1, \dots, d_m \in \mathbb{F}_q$, we obtain $\varphi(f)$ using only multiplications with constants $\varphi(h_1), \dots, \varphi(h_n), \varphi(h'_1), \dots, \varphi(h'_m)$ and additions as in

$$\varphi(f) = c_1\varphi(h_1) + \dots + c_n\varphi(h_n) + d_1\varphi(h'_1) + \dots + d_m\varphi(h'_m).$$

Similarly for $f \in \mathcal{L}(2D)$, there is no cost for bilinear complexity in obtaining $\text{Ev}_Q(f)$. Note that the evaluation map from $\mathcal{L}(2D)$ to $\text{Ev}_Q(f)$ is surjective but not necessarily injective.

We identify $\mathcal{L}(D)$ with $\mathcal{O}_Q/Q \cong \mathbb{F}_{q^n}$ without any cost on bilinear complexity. For given $\alpha, \beta \in \mathbb{F}_{q^n} \cong \mathcal{O}_Q/Q$, let f_1, f_2 be corresponding functions in $\mathcal{L}(D)$. We obtain the coefficients $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$f_1 = a_1 h_1 + \dots + a_n h_n, \quad f_2 = b_1 h_1 + \dots + b_n h_n \quad (3.3)$$

without any cost in bilinear complexity.

Note that $f_1 f_2 \in \mathcal{L}(2D)$. The only cost on bilinear complexity stems from obtaining the coefficients $c_1, \dots, c_n, d_1, \dots, d_m \in \mathbb{F}_q$, where

$$f_1 f_2 = \sum_{\ell=1}^n c_\ell h_\ell + \sum_{k=1}^m d_k h'_k$$

using the coefficients $a_1, \dots, a_n, b_1, \dots, b_n$ given in (3.3). Indeed the product $\alpha\beta \in \mathbb{F}_{q^n}$ is obtained using $\text{Ev}_Q(f_1 f_2)$ without any extra cost in bilinear complexity provided that the coefficients $c_1, \dots, c_n, \dots, d_m \in \mathbb{F}_q$ are known.

Using our arguments above, we obtain the coefficients $c_1, \dots, c_n, d_1, \dots, d_m \in \mathbb{F}_q$ from

$$\varphi(f_1 f_2) = (\varphi_1(f_1 f_2), \varphi_2(f_1 f_2), \dots, \varphi_N(f_1 f_2)).$$

We will complete the proof by showing that the cost of obtaining $\varphi_i(f_1 f_2)$ using the coefficients $a_1, \dots, a_n, b_1, \dots, b_n$ is at most

$$\mu_q(\deg(P_i)) \widehat{M}_{q^{\deg(P_i)}}(u_i)$$

for each $1 \leq i \leq N$.

Let $1 \leq i \leq N$ be an integer and

$$\varphi_i(f_1) = (\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,u_i-1}), \quad \varphi_i(f_2) = (\beta_{i,0}, \beta_{i,1}, \dots, \beta_{i,u_i-1}).$$

Note that the coordinates $\alpha_{i,0}, \dots, \alpha_{i,u_i-1}, \beta_{i,0}, \dots, \beta_{i,u_i-1} \in \mathbb{F}_{q^{\deg(P_i)}}$ and they are obtained using the coefficients $a_1, \dots, a_n, b_1, \dots, b_n$ and the constants $\varphi_i(h_1), \dots, \varphi_i(h_n)$ without any cost.

For a transcendental x over $\mathbb{F}_{q^{\deg(P_i)}}$, we consider the polynomial ring $\mathbb{F}_{q^{\deg(P_i)}}[x]$. Let $p_1^{(i)}(x), p_2^{(i)}(x) \in \mathbb{F}_{q^{\deg(P_i)}}[x]$ be polynomials given by

$$\begin{aligned} p_1^{(i)}(x) &= \alpha_{i,0} + \alpha_{i,1}x + \dots + \alpha_{i,u_i-1}x^{u_i-1}, \\ p_2^{(i)}(x) &= \beta_{i,0} + \beta_{i,1}x + \dots + \beta_{i,u_i-1}x^{u_i-1}. \end{aligned}$$

Let $p^{(i)}(x) = p_1^{(i)}(x)p_2^{(i)}(x)$ and $\gamma_0^i, \gamma_1^i, \dots, \gamma_{u_i-1}^i \in \mathbb{F}_{q^{\deg(P_i)}}$ be the first u_i terms of $p(x)$. Namely, let $\gamma_0^i, \gamma_1^i, \dots, \gamma_{u_i-1}^i \in \mathbb{F}_{q^{\deg(P_i)}}$ such that

$$p^{(i)}(x) \equiv \gamma_0^i + \gamma_1^i x + \dots + \gamma_{u_i-1}^i x^{u_i-1} \pmod{x^{u_i}} \in \mathbb{F}_{q^{\deg(P_i)}}[x].$$

It is clear that

$$\varphi_i(f_1 f_2) = (\gamma_0^i, \gamma_1^i, \dots, \gamma_{u_i-1}^i).$$

The cost of obtaining the first u_i terms $\gamma_0^i, \gamma_1^i, \dots, \gamma_{u_i-1}^i$ of the polynomial $p^{(i)}(x)$ using the polynomials $p_1^{(i)}(x), p_2^{(i)}(x)$ is at most

$$\mu_q(\deg(P_i)) \widehat{M}_{q^{\deg(P_i)}}(u_i).$$

This completes the proof. \square

Using Theorem 3.1 we obtain explicit algorithms for multiplications in \mathbb{F}_{q^n} . The conditions of the following theorem guarantee that the assumptions of Theorem 3.1 are satisfied.

Theorem 3.2. Let F/\mathbb{F}_q be an algebraic function field with full constant field \mathbb{F}_q . Let g be the genus of F . Let P_1, P_2, \dots, P_N be distinct places of arbitrary degrees of F . Let u_1, u_2, \dots, u_N be arbitrary positive integers. Assume that

- (1) there exists a non-special divisor of degree $g - 1$,
- (2) there exists a place of degree n ,
- (3) $\sum_{i=1}^N \deg(P_i)u_i > 2n + 2g - 2$.

Then assumptions in Theorem 3.1 hold and we have

$$\mu_q(n) \leq \sum_{i=1}^N \mu_q(\deg(P_i)) \widehat{M}_{q^{\deg(P_i)}}(u_i).$$

Proof. Let G be a non-special divisor of degree $g - 1$. Let Q be a place of degree n . Let D_1 be the effective divisor given by $D_1 = G + Q$. As $D_1 \geq G$, we have that D_1 is non-special again (cf. Remark I.6.9, item (f) [24]). Hence

$$\dim \mathcal{L}(D_1) = \deg(D_1) + 1 - g = (n + g - 1) + 1 - g = n.$$

Using Strong Approximation Theorem (cf. Theorem I.6.4 [24]) we obtain a divisor D of F such that

$$D \sim D_1 \quad \text{and} \quad \text{supp } D \cap \{Q, P_1, P_2, \dots, P_N\} = \emptyset.$$

Hence D is non-special (cf. Remark I.6.9, item (c)) and the map Ev_Q from $\mathcal{L}(D)$ to the residue field \mathcal{O}_Q/Q is onto. Let φ be the \mathbb{F}_q -linear map from $\mathcal{L}(2D)$ to $\left(\mathbb{F}_{q^{\deg(P_1)}}\right)^{u_1} \times \left(\mathbb{F}_{q^{\deg(P_2)}}\right)^{u_2} \times \cdots \times \left(\mathbb{F}_{q^{\deg(P_N)}}\right)^{u_N}$ given by (3.1). It remains to prove that φ is injective. But the kernel of φ is $\mathcal{L}(2D - \sum u_i P_i)$ and as the degree of the divisor $2D - \sum u_i P_i < 0$ by the assumption (3), the kernel is $\{0\}$. So φ is injective. \square

Remark 3.3. Under the notation and assumptions of Theorem 3.2, consider the subcase that $N = N_1 + N_2$, P_i is a degree one place for $1 \leq i \leq N_1$ and P_i is a degree two place for $N_1 + 1 \leq i \leq N_1 + N_2$. Moreover let $u_i = 1$ for $1 \leq i \leq N_1 + N_2$. Note that $\mu_q(1) = 1$, $\mu_q(2) = 3$ (cf. [27]), and $\widehat{M}_{q^{\deg(P_i)}}(1) = 1$ for any $\deg(P_i)$. Therefore the condition (3) of Theorem 3.2 becomes

$$N_1 + 2N_2 > 2n + 2g - 2,$$

and the bound of Theorem 3.2 on $\mu_q(n)$ becomes

$$\mu_q(n) \leq N_1 + 3N_2.$$

These coincide with the corresponding result of Ballet and Rolland in [4].

Remark 3.4. By Theorem 3.2, in order to obtain better upper bounds on $\mu_q(n)$, we need algebraic function fields with full constant field \mathbb{F}_q , with small genus g , and with enough number of rational places of suitable degrees. It is well known that finding algebraic function fields over \mathbb{F}_q with fixed small genus g and many rational places is not easy (cf. [18, Chapter 4]). In Theorem 3.2, as $\deg(P_i)$ and u_i are further parameters to be chosen, the condition (3) is weaker than the corresponding condition in [4, Theorem 2.2].

Using $u = 2$ for degree one places and $u = 1$ for degree two places in Theorem 3.2, we obtain the following corollary.

Corollary 3.5. Let F/\mathbb{F}_q be an algebraic function field with full constant field \mathbb{F}_q . Let g be the genus of F . Assume there exist at least N_1 degree one and at least N_2 degree two places of F . If

- (1) there exists a non-special divisor of degree $g - 1$,
- (2) there exists a place of degree n ,
- (3) $2N_1 + 2N_2 > 2n + 2g - 2$,

then we have

$$\mu_q(n) \leq 3n + \frac{3g}{2}.$$

Proof. We use N_1 degree one places with $u = 2$ and N_2 degree two places with $u = 1$. Since we have $2N_1 + 2N_2 > 2n + 2g - 2$, then φ is injective with rank $2n + g - 1$. Therefore we can choose N'_1 degree one places from degree one places and N'_2 degree two places from degree two places such that $2n + g - 1 \leq 2N'_1 + 2N'_2 \leq 2n + g$. Then we get

$$\mu_q(n) \leq 3N'_1 + 3N'_2 \leq 3\left(n + \frac{g}{2}\right) = 3n + \frac{3g}{2}. \quad \square$$

We compare [Corollary 3.5](#) with the corresponding results in [\[4\]](#). The bound of [Corollary 3.5](#) is at least as good as the bounds of [\[4, Theorem 2.2\]](#) and [\[7, Theorem 2.1\]](#). The condition (3) of [Corollary 3.5](#) is weaker as the corresponding condition of [\[4\]](#) and [\[7\]](#) is $N_1 + 2N_2 > 2n + 2g - 2$. The other conditions of [Corollary 3.5](#) are the same as the ones in [\[4\]](#). Therefore [Corollary 3.5](#) gives improved bounds on $\mu_q(n)$ compared to the ones in [\[4\]](#).

For some explicit algebraic function fields, the map φ in [\(3.1\)](#) becomes injective for suitable choices of the places P_1, \dots, P_N and the divisor D even $\sum_{i=1}^N \deg(P_i)u_i = 2n + g - 1$ holds. We state such a result in the following theorem.

Theorem 3.6. *Let F/\mathbb{F}_q be an algebraic function field with full constant field \mathbb{F}_q . Let g be genus of F . Let P_1, \dots, P_N be distinct places of arbitrary degrees of F . Let u_1, u_2, \dots, u_N be arbitrary integers. Assume that*

- (1) *there exists a non-special divisor D of degree $n + g - 1$,*
- (2) *there exists a place of degree n ,*
- (3) $\sum_{i=1}^N \deg(P_i)u_i = 2n + g - 1$.

Let φ be the \mathbb{F}_q -linear map from $\mathcal{L}(2D)$ to $\left(\mathbb{F}_q^{\deg(P_1)}\right)^{u_1} \times \dots \times \left(\mathbb{F}_q^{\deg(P_N)}\right)^{u_N}$ given in [\(3.1\)](#). If φ is injective then

$$\mu_q(n) \leq \sum_{i=1}^N \mu_q(\deg(P_i)) \widehat{M}_{q^{\deg(P_i)}}(u_i).$$

Proof. As D is non-special, $\dim(\mathcal{L}(D)) = \deg D + 1 - g = n$. Moreover $\text{supp}(D) \cup \{Q\} = \emptyset$ and hence the evaluation map Ev_Q from $\mathcal{L}(D)$ to \mathcal{O}_Q/Q is bijective. Note that $\text{supp}(D) \cup \{P_1, \dots, P_N\} = \emptyset$ as well. The result follows from [Theorem 3.1](#). \square

Remark 3.7. It is enough to assume D is a non-special divisor of degree $n + g - 1$. Using Strong Approximation Theorem (cf. Theorem I.6.4 [\[24\]](#)), we can always obtain D' from such D with $D' \sim D$ and $\text{supp } D' \cap \{Q, P_1, P_2, \dots, P_N\} = \emptyset$.

Remark 3.8. In the case of places only of degree one and two and with $u = 1$, the conditions of [Theorem 3.6](#) are exactly equivalent to the conditions of Theorem 2.2 in [\[7\]](#). Moreover, The same bound was given in [\[4\]](#) under certain conditions on q and n only for degree one and degree two places with $u = 1$. The conditions on q and n in [\[4\]](#) seem to come from the choice of a non-special divisor D with extra conditions. In our case the extra conditions refer to the injectivity of the map φ , even when $\sum_{i=1}^N \deg(P_i)u_i = 2n + g - 1$. We give explicit examples of algebraic function fields satisfying this criteria in our improvements.

The following example shows that [Theorem 3.1](#) gives an improved bound for \mathbb{F}_{3^9} .

Example 3.9. Let $q = 3$ and $n = 9$. Using the results in the literature, to the best of our knowledge, the best upper bound is $\mu_3(9) \leq 27$, which can be derived by two alternative methods as follows. Using [13,16,26], we obtain the upper bounds on $M_3(9)$ as 36, 34 and 27, respectively. Hence by [13] and (2.1) we get $\mu_3(9) \leq 27$. For the method in [4], we have considered all algebraic function fields of genus 0 and 1. Let E be elliptic curve $y^2 = x^3 + x + 2$ over \mathbb{F}_3 . It has 4 degree one places, 6 degree two places and 8 degree three places. As $4 + 2 \cdot 6 < 2 \cdot 9 + 1 - 1$, the method of [4] cannot be applied directly. Using 3 degree one places, 6 degree two places, and 1 degree three places, all with $u = 1$ as in [4], we obtain that $\mu_3(9) \leq 3 \cdot 1 + 6 \cdot 3 + 6 \cdot 1 = 27$. Now we improve this to $\mu_3(9) \leq 26$ using Theorem 3.6 together with $u = 2$ for some places. We take 2 degree one places with $u = 2$, 2 degree one places with $u = 1$, and 6 degree two places with $u = 1$. Therefore we obtain that $\mu_3(9) \leq 2 \cdot 3 + 2 \cdot 1 + 6 \cdot 3 = 26$. We find an explicit formula of such an algorithm via Theorem 3.6, which can be found in Appendix. The description and details of finding explicit formula for $\mu_3(9) \leq 26$ are given in Section 6.

4. Multiplication in finite fields \mathbb{F}_{q^n} for $2 \leq n \leq 18$ and $q = 2, 3, 4$

In this section, for $2 \leq n \leq 18$ and $q = 2, 3, 4$, we obtain the best known (upper) bounds on $\mu_q(n)$ using the various methods in the literature and the proposed method in this paper. In particular, we indicate some improvements obtained using the proposed method on certain values of $\mu_q(n)$.

To the best of our knowledge, for this range of values of q and n , the best known (upper) bounds on $\mu_q(n)$ are obtained using the following methods:

- (i) The methods based on the idea of D.V. Chudnovsky and G.V. Chudnovsky [14], which are presented in the [2–4,7].
- (ii) The observation in (2.1) together with results presented in [12,13,15,16,26,27].
- (iii) A well known method when n is a composite number which is as follows: Let $k, \ell \geq 2$ be positive integers with $n = k \cdot \ell$. As \mathbb{F}_{q^ℓ} is a subfield of \mathbb{F}_{q^n} , it immediately follows from the definitions of $\mu_q(n)$, $\mu_{q^\ell}(k)$ and $\mu_q(\ell)$ that

$$\mu_q(n) \leq \mu_{q^\ell}(k) \cdot \mu_q(\ell). \quad (4.1)$$

- (iv) The proposed method.

Now we give some of the improvements that are obtained by using the proposed method explicitly. We start with multiplication in \mathbb{F}_{3^n} . For the cases $n = 9, 11, 13, 15, 17, 18$, we improve the best known bounds given in [13,16]. Throughout the paper we use the notation of Magma [11] for presenting the places and the divisor of algebraic function fields.

In Example 3.9, it is already explained how to obtain $\mu_3(9) \leq 26$. For the other improvements in characteristic three in this section, we again use the same elliptic curve E given in Example 3.9. Recall that E has 4 degree one places, 6 degree two places and 8 degree three places. For each choice of the places and the divisors given below, it is easy to verify, for example using Magma as in Section 6, that the corresponding map φ in (3.1) is injective and hence the proposed method applies.

In order to show that $\mu_3(11) \leq 34$, it is enough to take 2 degree one places with $u = 2$, 2 degree one places with $u = 3$ and 6 degree two places with $u = 1$ with the choice of $D = (x^{11} + 2x^9 + x^7 + x^6 + x^4 + x^3 + 2x^2 + x + 1, y + x^{10} + 2x^7 + 2x^5 + 2x^4 + 2x^3 + x + 2)$.

In order to obtain $\mu_3(13) \leq 42$, we use 4 degree one places with $u = 2$, 6 degree two places with $u = 1$ and 2 degree three places with $u = 1$ with the choice of $D = (x^{13} + 2x^{12} + x^{11} + 2x^{10} + x^9 + x^8 + x^7 + 2x^4 + 2x^3 + 1, y + x^{12} + x^{11} + 2x^{10} + 2x^9 + x^7 + x^5 + 2x^4 + 2x^3)$.

On the other hand, taking 4 degree one places with $u = 3$, 6 degree two places with $u = 1$ and 2 degree three places with $u = 1$ gives $\mu_3(15) \leq 50$ where D can be selected as $(x^{15} + 2x^{13} + 2x^{12} + 2x^{11} + x^{10} + x^8 + x^5 + 2x + 2, y + 2x^{13} + x^{12} + x^{11} + 2x^{10} + x^9 + 2x^5 + x^4 + x^3 + 2x^2 + 2x)$.

When we choose $D = (x^{17} + 2x^{16} + 2x^{15} + x^{13} + x^{10} + 2x^9 + x^8 + x^7 + 2x^6 + 2x^5 + 2x^2 + x + 1, y + 2x^{15} + x^{14} + 2x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^2 + 2)$, another improved bound $\mu_3(17) \leq 58$ is obtained by using 2 degree one places with $u = 2$, 2 degree one places with $u = 3$, 6 degree two places with $u = 1$ and 4 degree three places with $u = 1$.

Finally, $\mu_3(18) \leq 62$ is obtained by taking 3 degree one places with $u = 2$, 1 degree one places with $u = 3$, 6 degree two places with $u = 1$ and 5 degree three places with $u = 1$ where one can use $D = (x^{18} + 2x^{17} + 2x^{16} + x^{15} + x^{11} + 2x^{10} + x^4 + 2x + 2, y + 2x^{17} + x^{14} + x^{13} + 2x^{12} + 2x^8 + x^6 + 2x^5 + x^4)$.

Next we show that the proposed method improves or reach the currently best known bilinear complexities in \mathbb{F}_{4^n} for $n = 11, 13$ and 17 . In order to show that $\mu_4(11) \leq 30$, $\mu_4(13) \leq 37$ and $\mu_4(17) \leq 53$ we use the proposed method as follows. Let $\mathbb{F}_4 = \{0, 1, w, w + 1\}$ where w is a root of $x^2 + x + 1 \in \mathbb{F}_2[x]$. Let

$$\begin{aligned} E_1 : y^2 + wy &= x^3 + x^2 + wx + 1, \\ E_2 : y^2 + w^2xy + wy &= x^3 + wx + w^2, \\ E_3 : y^2 + y &= x^3 + x^2 + w^2x + w \end{aligned}$$

be elliptic curves over \mathbb{F}_4 . E_1 has 7 degree one places, 7 degree two places and 14 degree three places. E_2 has 6 degree one places, 9 degree two places and 16 degree three places. Finally, E_3 has 5 degree one places, 10 degree two places and 20 degree three places.

The bound $\mu_4(11) \leq 30$ can be obtained using the proposed method together with E_1 . When we use 1 degree one place with $u = 2$, 6 degree one places with $u = 1$ and 7 degree two places with $u = 1$, we get $\mu_4(11) \leq 30$. Note that the same bound is also obtained by the method of [4]. If we use E_2 then we obtain $\mu_4(11) \leq 30$ by using 6 degree one places and 8 degree two places.

The improved bound $\mu_4(13) \leq 37$ can be obtained by using E_2 . Let $\{P_1, \dots, P_6, Q_1, \dots, Q_9\}$ be a set of places where P_i 's are of degree one and Q_i 's are of degree two. Those are

$$\begin{aligned} P_1 &= \infty, & P_2 &= (x, y + 1), & P_3 &= (x, y + x + w^2), & P_4 &= (x + w^2, y + w), \\ P_5 &= (x + 1, y), & P_6 &= (x + 1, y + x), & Q_1 &= (x + w), & Q_2 &= (x^2 + x + w^2, y), \\ Q_3 &= (x^2 + x + w^2, y + w^2x + w), & Q_4 &= (x^2 + w^2x + 1, y + w), \\ Q_5 &= (x^2 + w^2x + 1, y + w^2x), & Q_6 &= (x^2 + w^2x + w^2, y + x), \\ Q_7 &= (x^2 + w^2x + w^2, y + wx + w), & Q_8 &= (x^2 + wx + w, y + x + w^2), \\ Q_9 &= (x^2 + wx + w, y + wx + 1). \end{aligned}$$

When we use 2 degree one places, P_1, P_2 , with $u = 2$, 4 degree one places, P_3, \dots, P_6 with $u = 1$ and 9 degree two places, Q_1, \dots, Q_9 with $u = 1$, we obtain $\mu_4(13) \leq 37$ where one can use $D = (x^{13} + w^2x^{12} + x^{11} + x^{10} + wx^9 + x^8 + wx^7 + wx^4 + x^2 + x + w, y + wx^{12} + x^{11} + w^2x^{10} + w^2x^9 + w^2x^8 + wx^7 + w^2x^6 + wx^5 + w^2x^4 + x^3 + x^2 + x + w^2)$.

The bound $\mu_4(17) \leq 53$ can be obtained by using two methods, the proposed method and method introduced in [4]. When we use the elliptic curve E_2 with 2 degree one places with $u = 2$, 4 degree one places with $u = 1$ and 9 degree two places with $u = 1$, we get $\mu_4(17) \leq 53$. On the other hand, using E_3 with 5 degree one places with $u = 1$, 10 degree two places with $u = 1$ and 3 degree three places with $u = 1$ gives the same bound.

We summarize the results of this section in Table 1. The symbol * denotes an improvement by using the proposed method compared to the best known values in the literature. In this table, we indicate the methods that achieve the bounds in the corresponding columns. These are the methods (i), (ii), (iii) or (iv) explained in the beginning of Section 4.

5. Application

Finite field multiplication is widely used in many areas such as cryptography and coding theory. For example, in elliptic curve cryptography, finite fields with large number of elements are used. Some of the suitable finite fields are proposed by NIST (National Institute of Standards and Technology) [17]. In that list, it is suggested to use the fields with 2^{163} , 2^{233} , 2^{283} , 2^{409} and 2^{571} elements. Now, we will compute the multiplicative complexity for multiplication in $\mathbb{F}_{2^{163}}$ using the proposed method. The most suitable elliptic curve over \mathbb{F}_2 (up to isomorphism) is $y^2 + y = x^3 + x + 1$ which has 1 degree one place, 2 degree two places, 4 degree three places, 5 degree four places, 8 degree five places, 8 degree six places, 16 degree seven places and 25 degree eight places. We take 1 degree one place with $u = 5$, 2 degree two places with $u = 2$, 4 degree three places with $u = 1$, 5 degree four places with $u = 1$, 8

Table 1Bounds for $\mu_q(n)$ for $2 \leq n \leq 18$ and $q = 2, 3, 4$.

n	$\mu_2(n)$	Method	$\mu_3(n)$	Method	$\mu_4(n)$	Method
2	3	(ii)	3	(ii)	3	(ii)
3	6	(ii)	6	(ii)	6	(ii)
4	9	(ii)	9	(ii)	8	(ii)
5	13	(ii)	12	(ii)	11	(ii)
6	15	(iii)	15	(ii)	14	(ii)
7	22	(ii)	19	(ii)	17	(ii)
8	24	(iii)	21	(iii)	20	(ii)
9	30	(ii)	26*	(iv)	23	(ii)
10	33	(iii)	27	(iii)	27	(ii)
11	39	(ii)	34*	(iv)	30	(i), (iv)
12	42	(iii)	36	(iii)	33	(iii)
13	48	(ii)	42*	(iv)	37*	(iv)
14	51	(iii)	45	(iii)	39	(iii)
15	54	(iii)	50*	(iv)	45	(iii)
16	60	(iii)	54	(iii)	45	(iii)
17	67	(ii)	58*	(iv)	53	(i), (iv)
18	69	(ii)	62*	(iv)	51	(iii)

degree five places with $u = 1$, 8 degree six places with $u = 1$, 15 degree seven places with $u = 1$ and 11 degree eight places with $u = 1$. Therefore we obtain

$$\mu_2(163) \leq 11 + 2 \cdot 9 + 4 \cdot 6 + 5 \cdot 9 + 8 \cdot 13 + 8 \cdot 15 + 15 \cdot 22 + 11 \cdot 24 = 916,$$

where we use Table 1 and $\widehat{M}_2(5) \leq 11$, $\widehat{M}_4(2) \leq 3$ [12]. On the other hand, the best we can expect from Karatsuba algorithm (together with (2.1)) is $\mu_2(163) \leq N$, where N is an integer with $N > 2187$, since it is given in [26] that $M_2(128) \leq 2187$.

The finite field \mathbb{F}_{397} is used in pairing based cryptography [13,21]. In order to compute $\mu_3(97)$ by using the proposed method, it would be better to use the elliptic curve $y^2 = x^3 + x^2 + 2x + 1$ which has 3 degree one places, 6 degree two places, 11 degree three places, 15 degree four places and 42 degree five places. When we use 3 degree one places with $u = 3$, 6 degree two places with $u = 1$, 11 degree three places with $u = 1$, 15 degree four places with $u = 1$ and 16 degree five places with $u = 1$, we obtain

$$\mu_3(97) \leq 3 \cdot 5 + 6 \cdot 3 + 11 \cdot 6 + 15 \cdot 9 + 16 \cdot 12 = 426$$

where we use Table 1 and $\widehat{M}_3(3) \leq 5$ [13]. Note that Karatsuba algorithm (together with (2.1)) gives $\mu_3(97) \leq 1554$ [26].

6. Multiplication in \mathbb{F}_{3^9}

In this section, we will give the details of obtaining an explicit formula for multiplication in \mathbb{F}_{3^9} by using an elliptic curve. In Example 3.9, we gave the known bounds and we showed that the proposed method provides an improved bound $\mu_3(9) \leq 26$. Now, we will give the details of how the formula for multiplication \mathbb{F}_{3^9} with $\mu_3(9) \leq 26$ is obtained explicitly.

Consider the elliptic curve $E : y^2 = x^3 + x + 2$ over \mathbb{F}_3 . Let $\{P_1, \dots, P_4, Q_1, \dots, Q_6\}$ be a set of places where P_i 's are of degree one and Q_i 's are of degree two. Those are

$$\begin{aligned} P_1 &= \infty, & P_2 &= (x+1, y), & P_3 &= (x+2, y+1), & P_4 &= (x+2, y+2), \\ Q_1 &= (x), & Q_2 &= (x^2+2x+2, y), & Q_3 &= (x^2+1, y+x), & Q_4 &= (x^2+1, y+2x), \\ Q_5 &= (x^2+x+2, y+1), & Q_6 &= (x^2+x+2, y+2). \end{aligned}$$

When we use P_1 and P_2 with $u = 1$, P_3 and P_4 with $u = 2$ and Q_1, \dots, Q_6 with $u = 1$, the map φ defined in Section 3 becomes injective. In order to find an explicit formula, we need to find the local parameters of P_3 and P_4 . The local parameters t_3 and t_4 corresponding to P_3 and P_4 respectively are

$$t_3 = \frac{y}{(x^2+x+2)} + \frac{1}{(x^2+x+2)}, \quad t_4 = \frac{y}{(x^2+x+2)} + \frac{2}{(x^2+x+2)}.$$

Let us choose

$$\mathcal{D} = (x^9 + x^8 + x^5 + 2x^3 + 2x^2 + 2x + 1, y + x^7 + x^6 + 2x^5 + x + 1).$$

Then a basis $\{f_1, f_2, \dots, f_{18}\}$ of $\mathcal{L}(2\mathcal{D})$ containing the basis of $\mathcal{L}(\mathcal{D})$ is

$$\begin{aligned} f_1 &= \frac{x^7y}{f} + \frac{(2x^8 + 2x^7 + x^6 + 2x^4 + x^3 + x^2 + 2x + 2)}{f}, \\ f_2 &= \frac{x^6y}{f} + \frac{(x^8 + 2x^6 + x^5 + x^4 + 2x^3 + 1)}{f}, \\ f_3 &= \frac{x^5y}{f} + \frac{(2x^8 + 2x^5 + x^3 + x + 1)}{f}, \quad f_4 = \frac{x^4y}{f} + \frac{(2x^8 + x^7 + x^4 + 2x^2 + x + 2)}{f}, \\ f_5 &= \frac{x^3y}{f} + \frac{(x^8 + x^6 + x^4 + x^3 + 2x^2 + x)}{f}, \quad f_6 = \frac{x^2y}{f} + \frac{(x^7 + x^5 + x^3 + x^2 + 2x + 1)}{f}, \\ f_7 &= \frac{xy}{f} + \frac{(2x^8 + 2x^7 + x^6 + 2x^2 + 2x)}{f}, \quad f_8 = \frac{y}{f} + \frac{(2x^7 + 2x^6 + x^5 + 2x + 2)}{f}, \quad f_9 = 1 \\ f_{10} &= \frac{(x^{14} + x^{13} + 2x^{12} + x^{10} + 2x^8 + x^7 + x^5 + x^3 + 2x^2)y}{f^2} \\ &\quad + \frac{(x^{18} + 2x^{17} + 2x^{16} + 2x^{15} + 2x^{13} + 2x^{12} + 2x^{10} + x^9 + x^8 + 2x^7 + 2x^4 + 2x)}{f^2}, \\ f_{11} &= \frac{(x^{13} + x^{12} + 2x^{11} + x^9 + 2x^7 + x^6 + x^4 + x^2 + 2x)y}{f^2} \\ &\quad + \frac{(x^{17} + 2x^{16} + 2x^{15} + 2x^{14} + 2x^{12} + 2x^{11} + 2x^9 + x^8 + x^7 + 2x^6 + 2x^3 + 2)}{f^2}, \\ f_{12} &= \frac{(x^{12} + x^{11} + 2x^{10} + x^8 + 2x^6 + 2x^5 + 2x^4 + 2x^3 + x)y}{f^2} \\ &\quad + \frac{(x^{16} + 2x^{15} + 2x^{14} + 2x^{13} + 2x^{11} + 2x^{10} + x^9 + 2x^8 + x^6 + x^4 + x^3 + x + 1)}{f^2}, \\ f_{13} &= \frac{(x^{11} + x^{10} + 2x^9 + x^7 + x^5 + x^3 + 2x^2)y}{f^2} \\ &\quad + \frac{(x^{15} + 2x^{14} + 2x^{13} + 2x^{12} + 2x^{10} + x^9 + x^8 + 2x^4 + 2x)}{f^2}, \\ f_{14} &= \frac{(x^{10} + x^9 + 2x^8 + x^6 + x^4 + x^2 + 2x)y}{f^2} \\ &\quad + \frac{(x^{14} + 2x^{13} + 2x^{12} + 2x^{11} + 2x^9 + x^8 + x^7 + 2x^3 + 2)}{f^2}, \\ f_{15} &= \frac{(x^9 + x^8 + 2x^7 + 2x^5 + 2x^4 + 2x^3 + x)y}{f^2} \\ &\quad + \frac{(x^{13} + 2x^{12} + 2x^{11} + 2x^{10} + x^9 + 2x^8 + x^6 + x^5 + x^4 + x^3 + x + 1)}{f^2}, \\ f_{16} &= \frac{(x^8 + x^7 + 2x^6 + 2x^5 + x^3 + 2x^2)y}{f^2} + \frac{(x^{12} + 2x^{11} + 2x^{10} + x^9 + x^8 + 2x)}{f^2}, \\ f_{17} &= \frac{(x^7 + x^6 + 2x^5 + 2x^4 + x^2 + 2x)y}{f^2} + \frac{(x^{11} + 2x^{10} + 2x^9 + x^8 + x^7 + 2)}{f^2}, \\ f_{18} &= \frac{(x^6 + 2x^5 + x^4 + x)y}{f^2} + \frac{(x^{10} + 2x^8 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)}{f^2} \end{aligned}$$

where $\{f_1, f_2, \dots, f_9\}$ is a basis of $\mathcal{L}(\mathcal{D})$ and $f = x^9 + x^8 + x^5 + 2x^3 + 2x^2 + 2x + 1$.

Now consider the elements $a = \sum_{i=1}^9 a_i f_i \in \mathcal{L}(\mathcal{D})$ and $b = \sum_{i=1}^9 b_i f_i \in \mathcal{L}(\mathcal{D})$. Let $c = \sum_{i=1}^{18} c_i f_i$ be the product of a and b given by

$$\left(\sum_{i=1}^9 a_i f_i \right) \cdot \left(\sum_{i=1}^9 b_i f_i \right) = \sum_{i=1}^{18} c_i f_i. \quad (6.1)$$

Then we get the following system of linear equations

$$\underbrace{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 - m_3 - m_5 \\ m_6 \\ m_7 - m_6 - m_8 \\ m_9 - m_{11} \\ m_{10} - m_9 - m_{11} \\ m_{12} + m_{13} \\ m_{14} - m_{12} \\ m_{15} - m_{16} \\ m_{17} - m_{15} - m_{16} \\ m_{18} - m_{19} \\ m_{20} - m_{18} - m_{19} \\ m_{21} + m_{22} \\ m_{23} - m_{21} - m_{22} \\ m_{24} + m_{25} \\ m_{26} - m_{24} + m_{25} \end{bmatrix}}_M = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 \\ 2 & 0 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 2 \end{bmatrix}}_G \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \\ c_{15} \\ c_{16} \\ c_{17} \\ c_{18} \end{bmatrix}}_C$$

where multiplications m_i for $1 \leq i \leq 26$, are given in [Appendix](#).

Since G is invertible, we have $C = G^{-1} \cdot M$. Then we can find the multiplication in \mathbb{F}_{3^9} by using $\text{Ev}_Q(c)$ where we choose

$$Q = (x^9 + 2x^8 + x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2, y + x^8 + 2x^6 + 2x^4 + x^3 + 1).$$

The explicit formula is given in the [Appendix](#).

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Appendix

We give an explicit formula for multiplication in \mathbb{F}_{3^9} . We represent \mathbb{F}_{3^9} as the field $\mathbb{F}_3(w) = \mathbb{F}_3[x]/(p(x))$ where w is the root of the irreducible polynomial $p(x) = x^9 + 2x^8 + x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2$. Let $\alpha = \sum_{i=1}^9 a_i \xi_i$, $\beta = \sum_{i=1}^9 b_i \xi_i$, and $\gamma = \sum_{i=1}^9 c_i \xi_i \in \mathbb{F}_{3^9}$ with

$$\left(\sum_{i=1}^9 a_i \xi_i \right) \cdot \left(\sum_{i=1}^9 b_i \xi_i \right) = \sum_{i=1}^9 c_i \xi_i,$$

where $\{\xi_1, \xi_2, \dots, \xi_9\}$ is a basis of \mathbb{F}_{3^9} over \mathbb{F}_3 such that

$$\begin{aligned}\xi_1 &= w^8 + 2w^7 + w^6 + w^4 + 2w^3 + 2w^2 + 2, \\ \xi_2 &= w^7 + 2w^6 + w^5 + w^3 + 2w^2 + 2w, \\ \xi_3 &= 2w^8 + w^7 + w^6 + w^5 + 2w^4 + w^3 + 2w^2 + 2, \\ \xi_4 &= w^8 + w^7 + w^6 + 2w^5 + w^3 + w, \\ \xi_5 &= w^8 + w^6 + 2w^5 + w^4 + 2w^3 + 2w + 1, \\ \xi_6 &= w^8 + 2w^5 + w^4 + w^2 + 2w + 2, \\ \xi_7 &= w^8 + w^5 + w^4 + 2w^2 + 2, \\ \xi_8 &= 2w^8 + 2w^7 + 2w^5 + 2w^4 + 2w^3 + w^2, \\ \xi_9 &= 1.\end{aligned}$$

The following explicit formula consisting of the 26 multiplications in \mathbb{F}_3 gives γ from α and β . We first define the multiplications m_i , for $1 \leq i \leq 26$ and then we give the formula for obtaining the coefficients of γ using these multiplications.

$$\begin{aligned}m_1 &= a_9 b_9 \\ m_2 &= (2a_2 + 2a_3 + 2a_4 + a_7 + 2a_8 + a_9)(2b_2 + 2b_3 + 2b_4 + b_7 + 2b_8 + b_9) \\ m_3 &= (a_2 + 2a_8 + 2a_4 + 2a_7 + a_9)(b_2 + 2b_8 + 2b_4 + 2b_7 + b_9) \\ m_4 &= (a_8 + 2a_4 + a_9 + 2a_3)(b_8 + 2b_4 + b_9 + 2b_3) \\ m_5 &= (2a_2 + a_7 + 2a_3 + 2a_8)(2b_2 + b_7 + 2b_3 + 2b_8) \\ m_6 &= (2a_1 + a_9 + a_7 + 2a_3 + a_8 + a_4 + 2a_5 + 2a_6) \\ &\quad \times (2b_1 + b_9 + b_7 + 2b_3 + b_8 + b_4 + 2b_5 + 2b_6) \\ m_7 &= (a_1 + a_9 + 2a_7 + 2a_6 + 2a_2)(b_1 + b_9 + 2b_7 + 2b_6 + 2b_2) \\ m_8 &= (2a_2 + a_3 + 2a_4 + a_7 + a_5 + 2a_8 + 2a_1)(2b_2 + b_3 + 2b_4 + b_7 + b_5 + 2b_8 + 2b_1) \\ m_9 &= (2a_1 + a_2 + a_3 + 2a_4 + a_6 + 2a_8 + a_9)(2b_1 + b_2 + b_3 + 2b_4 + b_6 + 2b_8 + b_9) \\ m_{10} &= (2a_1 + a_2 + a_3 + 2a_4 + a_6 + a_9)(2b_1 + b_2 + b_3 + 2b_4 + b_6 + b_9) \\ m_{11} &= a_8 b_8 \\ m_{12} &= (a_1 + a_5 + a_9 + 2a_7)(b_1 + b_5 + b_9 + 2b_7) \\ m_{13} &= (2a_2 + 2a_1 + a_5 + 2a_3 + 2a_4 + 2a_7 + 2a_8 + a_6) \\ &\quad \times (2b_2 + 2b_1 + b_5 + 2b_3 + 2b_4 + 2b_7 + 2b_8 + b_6) \\ m_{14} &= (2a_5 + a_9 + a_7 + 2a_2 + 2a_3 + 2a_4 + 2a_8 + a_6) \\ &\quad \times (2b_5 + b_9 + b_7 + 2b_2 + 2b_3 + 2b_4 + 2b_8 + b_6) \\ m_{15} &= (a_1 + a_3 + a_9 + a_4 + a_6)(b_1 + b_3 + b_9 + b_4 + b_6) \\ m_{16} &= (a_2 + a_5 + a_3)(b_2 + b_5 + b_3) \\ m_{17} &= (a_1 + 2a_3 + a_9 + a_4 + a_6 + a_2 + a_5)(b_1 + 2b_3 + b_9 + b_4 + b_6 + b_2 + b_5) \\ m_{18} &= (a_1 + a_9 + 2a_2 + a_3 + 2a_4 + a_8)(b_1 + b_9 + 2b_2 + b_3 + 2b_4 + b_8) \\ m_{19} &= (a_2 + 2a_1 + a_7 + 2a_3)(b_2 + 2b_1 + b_7 + 2b_3) \\ m_{20} &= (a_9 + 2a_4 + a_8 + a_7)(b_9 + 2b_4 + b_8 + b_7) \\ m_{21} &= (a_5 + a_9 + a_4 + 2a_8)(b_5 + b_9 + b_4 + 2b_8) \\ m_{22} &= (2a_1 + 2a_4 + 2a_3 + a_6 + 2a_7 + a_2 + 2a_5)(2b_1 + 2b_4 + 2b_3 + b_6 + 2b_7 + b_2 + 2b_5) \\ m_{23} &= (a_9 + 2a_8 + 2a_1 + 2a_3 + a_6 + 2a_7 + a_2)(b_9 + 2b_8 + 2b_1 + 2b_3 + b_6 + 2b_7 + b_2) \\ m_{24} &= (a_9 + 2a_2 + a_7 + 2a_3 + a_6 + 2a_4 + a_5 + a_8) \\ &\quad \times (b_9 + 2b_2 + b_7 + 2b_3 + b_6 + 2b_4 + b_5 + b_8) \\ m_{25} &= (a_2 + a_3 + a_6 + a_4 + a_8 + a_1)(b_2 + b_3 + b_6 + b_4 + b_8 + b_1) \\ m_{26} &= (a_9 + a_7 + 2a_6 + a_5 + 2a_8 + a_1)(b_9 + b_7 + 2b_6 + b_5 + 2b_8 + b_1).\end{aligned}$$

The coefficients of $\gamma \in \mathbb{F}_{3^9}$ are found by using the following equations.

$$\begin{aligned}
 c_1 &= (2m_6 + m_{11} + m_{10} + m_{13} + m_{14} + 2m_{16} + 2m_{17} + 2m_{19} + m_{25} + 2m_{26} \\
 &\quad + 2m_{20} + 2m_{21} + 2m_{22} + 2m_2 + m_1) \\
 c_2 &= (m_6 + 2m_9 + 2m_{10} + m_{15} + m_{16} + 2m_{17} + 2m_{18} + 2m_{19} + 2m_{25} + m_{26} \\
 &\quad + m_{20} + m_{21} + m_{23} + 2m_2 + m_3 + 2m_5 + m_4) \\
 c_3 &= (m_6 + 2m_9 + 2m_{10} + m_{13} + m_{14} + m_{15} + 2m_{16} + m_{19} + 2m_{24} + 2m_{25} \\
 &\quad + m_{20} + m_{22} + 2m_{23} + 2m_2 + 2m_3 + m_5 + 2m_4) \\
 c_4 &= (m_7 + 2m_8 + m_9 + m_{11} + 2m_{10} + m_{13} + m_{14} + m_{15} + m_{17} + m_{18} + 2m_{19} \\
 &\quad + m_{25} + 2m_{26} + m_{21} + m_{22} + m_2 + m_1 + m_5 + 2m_4) \\
 c_5 &= (2m_6 + m_7 + 2m_8 + 2m_9 + 2m_{11} + m_{10} + 2m_{13} + 2m_{14} + 2m_{18} + m_{19} \\
 &\quad + m_{25} + 2m_{26} + m_2 + 2m_1) \\
 c_6 &= (2m_6 + 2m_9 + 2m_{10} + 2m_{12} + 2m_{13} + 2m_{15} + 2m_{17} + m_{18} + 2m_{19} \\
 &\quad + 2m_{25} + m_{26} + m_{22} + 2m_{23} + m_2 + m_1) \\
 c_7 &= (m_6 + 2m_7 + m_8 + m_{12} + m_{13} + 2m_{15} + m_{16} + 2m_{18} + 2m_{19} + m_{24} \\
 &\quad + m_{25} + m_{20} + 2m_{21} + 2m_{22} + 2m_1 + 2m_3 + m_5 + 2m_4) \\
 c_8 &= (m_6 + 2m_{11} + 2m_{10} + 2m_{12} + m_{13} + 2m_{14} + m_{16} + m_{17} + 2m_{19} \\
 &\quad + 2m_{24} + 2m_{26} + 2m_{20} + 2m_{21} + 2m_{23} + m_1 + 2m_3) \\
 c_9 &= (2m_6 + 2m_9 + m_{11} + 2m_{13} + 2m_{14} + 2m_{15} + 2m_{17} + 2m_{18} + m_{19} + m_{24} \\
 &\quad + 2m_{25} + 2m_{26} + 2m_{21} + m_{22} + m_{23} + 2m_5 + m_4).
 \end{aligned}$$

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