A Simple Group Generated by Involutions Interchanging Residue Classes of the Integers

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Abstract

We find and investigate a simple group which is generated by involutions each of which interchanges two disjoint residue classes of $\mathbb{Z}$. Our group is countable, but not finitely generated. It acts highly transitively on $\mathbb{N}_0$. We show that the free group of rank 2 and free products of finite groups embed in it, and that the class of isomorphism types of its subgroups is closed under wreath products with $(\mathbb{Z}, +)$. We also show that our group has uncountably many simple subgroups, that its torsion subgroups are divisible and that it has finitely generated subgroups with unsolvable membership problem. By means of computation, we show that it contains Collatz' permutation of $\mathbb{Z}$ whose cycle structure is unknown so far. Finally we construct two simple supergroups of our group which act highly transitively on $\mathbb{Z}$, and find a locally finite simple subgroup which acts highly transitively on $\mathbb{N}_0$.

1 Introduction

Most of the general work done so far on infinite permutation groups is on groups with limited degree of transitivity. In this context, *limited degree of transitivity* means that there is a positive integer k such that the group acts k-transitively, but not $k+1$-transitively. In contrast, a transitive permutation group is called *highly transitive* if there is no such k.

A type of infinite permutation groups which are 'almost' highly transitive are the oligomorphic groups. A permutation group is called *oligomorphic* if for any k the number of orbits on the set of ordered k-tuples of distinct points is finite. Likely the best general reference for results on this type of groups is the book *Oligomorphic Permutation Groups* [8] by Peter J. Cameron.

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Already in 1976, Cameron [7] has investigated groups which are *highly homogenous*, i.e. transitive on the set of subsets of any given finite cardinality, but not highly transitive. He has shown that such a group preserves either a dense linear order, a dense linear betweenness relation, a dense cyclic order or a dense cyclic separation relation.

Continuing this work, Adeleke and Macpherson [1] have investigated infinite primitive Jordan groups which are not highly transitive and which do not preserve a relation of one of the mentioned types. They have shown that such a group preserves a dense semilinear order, a dense general betweenness relation, a Steiner k-system, a certain C- or D- relation or a certain limit of Steiner systems, general betweenness relations or D-relations.

Although some work has also been done on highly transitive groups, they are not covered by most general theorems about infinite permutation groups. Perhaps an exception to this is that nontrivial normal subgroups of highly transitive groups are highly transitive as well (cp. e.g. Corollary 7.2A in [10]). Some articles about highly transitive groups describe certain examples, like e.g. [13], or representations of certain types of groups, like e.g. [12]. It is likely almost needless to mention that at least since Reinhold Baer's article [2] the full symmetric groups in all cardinalities have attracted considerable interest.

The book [4] by Bhattacharjee, Macpherson, Möller and Neumann gives a nice survey on results concerning infinite permutation groups in general.

For finite permutation groups, a great deal of algorithms is known. For a state-of-the-art survey see Akos Seress' book [23]. In contrast, apparently so far no class of infinite permutation groups has been treated by algorithmic means.

As said, the subject of this article is the construction and investigation of certain simple groups. However in the same turn it also introduces a class of infinite permutation groups which are to a certain extent accessible to computational investigations. The author has developed a GAP [11] package RCWA [15], which provides a large variety of functionality for computing in such permutation groups and for exhibiting their structure.

Originally, this work has been inspired by Lothar Collatz' famous $3n + 1$ Conjecture. This conjecture asserts that iterated application of the mapping

$$T: \ \mathbb{Z} \longrightarrow \mathbb{Z}, \quad n \ \longmapsto \ \begin{cases} \frac{n}{2} & \text{if } \ n \text{ even}, \\ \frac{3n+1}{2} & \text{if } \ n \text{ odd} \end{cases}$$

to any positive integer eventually yields 1. The $3n + 1$ Conjecture dates back to the 1930s and is still open today. Jeffrey C. Lagarias has written a now already classical survey article [17] and a comprehensive annotated bibliography [18] on this conjecture.

The elements of the groups discussed in this article are bijective mappings of the following type:

1.1 Definition We call a mapping $f : \mathbb{Z} \to \mathbb{Z}$ *residue class-wise affine* if there is a positive integer m such that the restrictions of f to the residue classes $r(m) \in \mathbb{Z}/m\mathbb{Z}$ are all affine. This means that for any residue class $r(m)$ there are coefficients $a_{r(m)}$, $b_{r(m)}$, $c_{r(m)} \in \mathbb{Z}$ such that the restriction of the mapping f to the set $r(m) = \{r + km | k \in \mathbb{Z}\}$ is given by

$$f|_{r(m)} : \ r(m) \to \mathbb{Z}, \quad n \ \mapsto \ \frac{a_{r(m)} \cdot n + b_{r(m)}}{c_{r(m)}}.$$

We call the smallest possible m the *modulus* of f, written $\mathrm{Mod}(f)$. For reasons of uniqueness, we further assume that $\gcd(a_{r(m)}, b_{r(m)}, c_{r(m)}) = 1$ and that $c_{r(m)} > 0$. We define the *multiplier* $\mathrm{Mult}(f)$ of f by $\mathrm{lcm}_{r(m)\in\mathbb{Z}/m\mathbb{Z}}\, a_{r(m)}$ and the *divisor* $\mathrm{Div}(f)$ of f by $\mathrm{lcm}_{r(m)\in\mathbb{Z}/m\mathbb{Z}}\, c_{r(m)}$. We call f *integral* if $\mathrm{Div}(f) = 1$. We call f *class-wise order-preserving* if all $a_{r(m)}$ are positive.

It is easy to see that the permutations of this kind form a countable subgroup of $\mathrm{Sym}(\mathbb{Z})$, which acts as a group of homoeomorphisms on $\mathbb{Z}$ endowed with a topology by taking the set of all residue classes as a basis.

1.2 Definition We denote the group of all residue class-wise affine permutations of $\mathbb{Z}$ by $\mathrm{RCWA}(\mathbb{Z})$, and call its subgroups *residue class-wise affine* groups.

The notation '$\mathrm{RCWA}(\mathbb{Z})$' reflects that generalizations to suitable rings other than $\mathbb{Z}$ make perfect sense. Basic thoughts in this direction can be found in [16]. For the sake of simplicity, in this article we restrict ourselves to residue class-wise affine groups over $\mathbb{Z}$.

The main group which we investigate is the one which is generated by all involutions of the following type:

1.3 Definition Let $r_1(m_1)$ and $r_2(m_2)$ be disjoint residue classes. We define the *class transposition* $\tau_{r_1(m_1),r_2(m_2)} \in \mathrm{RCWA}(\mathbb{Z})$ by

$$\tau_{r_1(m_1),r_2(m_2)} : \quad n \ \mapsto \ \begin{cases} \dfrac{m_2 n + (m_1 r_2 - m_2 r_1)}{m_1} & \text{if } \ n \in r_1(m_1), \\ \dfrac{m_1 n + (m_2 r_1 - m_1 r_2)}{m_2} & \text{if } \ n \in r_2(m_2), \\ n & \text{otherwise,} \end{cases}$$

where we assume $0 \leqslant r_1 < m_1$ and $0 \leqslant r_2 < m_2$. For convenience, we set $\tau := \tau_{0(2),1(2)} : n \mapsto n + (-1)^n$. We denote the group generated by all class transpositions by $\mathrm{CT}(\mathbb{Z})$.

Any finite symmetric group S_m has a monomorphic image in $\mathrm{CT}(\mathbb{Z})$: Let $m \geqslant 2$, and let S_m act naturally on the set $\{0, 1, \dots, m-1\}$. Then an example of a corresponding monomorphism is

$$\varphi_m: \quad \mathrm{S}_m \hookrightarrow \mathrm{CT}(\mathbb{Z}), \quad \sigma \mapsto (\sigma^{\varphi_m}: \ n \mapsto n + (n \mod m)^\sigma - n \mod m).$$

Just like the group S_m itself, its image under φ_m acts m - transitively on the set $\{0, 1, \dots, m-1\}$. Since m can be chosen arbitrary large and as class transpositions map nonnegative integers to nonnegative integers, this means that the group $\mathrm{CT}(\mathbb{Z})$ acts highly transitively on $\mathbb{N}_0$.

It is easy to see that the multiplier resp. divisor of the product of two residue class-wise affine permutations divides the product of the multipliers resp. divisors of the factors. Further it is obvious that inversion interchanges multiplier and divisor. Therefore as there are infinitely many primes and as for any positive integer n there is a class transposition $\tau_{1(2),0(2n)}$ with multiplier and divisor n, the group $\mathrm{CT}(\mathbb{Z})$ is not finitely generated.

The free group of rank 2 embeds in $\mathrm{CT}(\mathbb{Z})$. An example of an embedding is given by

$$\varphi_{\mathrm{F}_2}: \quad \mathrm{F}_2 = \langle a, b \rangle \ \hookrightarrow \ \mathrm{CT}(\mathbb{Z}), \quad a \ \mapsto \ (\tau \cdot \tau_{0(2),1(4)})^2, \quad b \ \mapsto \ (\tau \cdot \tau_{0(2),3(4)})^2.$$

This can be seen by applying the Table-Tennis Lemma (see e.g. [9], Section II.B.) to the infinite cyclic groups generated by the images of a and b under φ_{F_2} and the sets $0(4) \cup 1(4)$ and $2(4) \cup 3(4)$. Likewise, the Table-Tennis Lemma can be used to show that

$$\begin{aligned} \varphi_{\mathrm{PSL}(2,\mathbb{Z})}: \quad & \mathrm{PSL}(2,\mathbb{Z}) \ \cong \ \mathrm{C}_3 \star \mathrm{C}_2 \ \cong \ \langle a, b \mid a^3 = b^2 = 1 \rangle \ \hookrightarrow \ \mathrm{CT}(\mathbb{Z}), \\ & a \ \mapsto \ \tau_{0(4),2(4)} \cdot \tau_{1(2),0(4)}, \quad b \ \mapsto \ \tau \end{aligned}$$

is indeed an embedding of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. In this case one can use the sets $0(2)$ and $1(2)$ in place of $0(4) \cup 1(4)$ and $2(4) \cup 3(4)$.

The classes of isomorphism types of subgroups of $\mathrm{RCWA}(\mathbb{Z})$ resp. $\mathrm{CT}(\mathbb{Z})$ are closed under forming direct products. To see this, we look at a class of monomorphisms from the group $\mathrm{RCWA}(\mathbb{Z})$ into itself:

1.4 Definition Let f be an injective residue class-wise affine mapping. Further let $\pi_f: \mathrm{RCWA}(\mathbb{Z}) \hookrightarrow \mathrm{RCWA}(\mathbb{Z})$, $\sigma \mapsto \sigma_f$ be the monomorphism defined by the properties $\forall \sigma \in \mathrm{RCWA}(\mathbb{Z}) \ f\sigma_f = \sigma f$ and $\mathrm{supp}(\mathrm{im}\,\pi_f) \subseteq \mathrm{im}\, f$. Then we call π_f the *restriction monomorphism* associated to f.

Now given any two subgroups $G, H \leqslant \mathrm{RCWA}(\mathbb{Z})$, the group generated by $G^{\pi_{n \mapsto 2n}}$ and $H^{\pi_{n \mapsto 2n+1}}$ is clearly isomorphic to $G \times H$. As the image of a class transposition $\tau_{r_1(m_1),r_2(m_2)}$ under a restriction monomorphism $\pi_{n \mapsto mn+r}$ is $\tau_{mr_1+r(mm_1),mr_2+r(mm_2)}$, the same argument can be used for $\mathrm{CT}(\mathbb{Z})$ as well.

Looking at the monomorphisms π_{mn+r} and φ_m, it is immediate to see that the classes of isomorphism types of subgroups of $\mathrm{RCWA}(\mathbb{Z})$ resp. $\mathrm{CT}(\mathbb{Z})$ are closed under forming wreath products with finite permutation groups as well. Laurent Bartholdi has furthermore pointed out that they are also closed under forming wreath products with $(\mathbb{Z}, +)$: Given a subgroup G, a subgroup isomorphic to $G \wr (\mathbb{Z}, +)$ is generated by $G^{\pi_{n \mapsto 4n+3}}$ and $\tau \cdot \tau_{0(2),1(4)}$. This is seen best by checking that the orbit of the residue class $3(4)$ under the action of the cyclic group $\langle \tau \cdot \tau_{0(2),1(4)} \rangle$ consists of disjoint residue classes, thus that the conjugates of $G^{\pi_{n \mapsto 4n+3}}$ under powers of $\tau \cdot \tau_{0(2),1(4)}$ have pairwise disjoint supports.

Free products of finite groups $G_0, \dots, G_{m-1}$ embed in $\mathrm{CT}(\mathbb{Z})$ as well. To see this, first consider regular permutation representations φ_r of the groups G_r on the residue classes (mod $|G_r|$). Then take conjugates $H_r := (\mathrm{im}\,\varphi_r)^{\sigma_r}$ of the images of these representations under mappings $\sigma_r \in \mathrm{CT}(\mathbb{Z})$ which map $0(|G_r|)$ to $\mathbb{Z} \setminus r(m)$. Finally use that point stabilizers in regular permutation groups are trivial and apply the Table-Tennis Lemma to the groups H_r and the residue classes $r(m)$ to see that the group generated by the H_r's is isomorphic to their free product.

Torsion subgroups of $\mathrm{CT}(\mathbb{Z})$ are divisible: Given an element $g \in \mathrm{CT}(\mathbb{Z})$ of finite order and a positive integer k, there is always an $h \in \mathrm{CT}(\mathbb{Z})$ such that $h^k = g$. Since the element g has finite order, it permutes a partition $\mathcal{P}$ of $\mathbb{Z}$ into finitely many residue classes on all of which it is affine. A k-th root h can be constructed from g by 'slicing' cycles $\prod_{i=2}^{l} \tau_{r_1(m_1),r_i(m_i)}$ on $\mathcal{P}$ into cycles $\prod_{i=1}^{l} \prod_{j=\max(2-i,0)}^{k-1} \tau_{r_1(km_1),r_i+jm_i(km_i)}$ of the k-fold length on the refined partition which one gets from $\mathcal{P}$ by decomposing any $r_i(m_i) \in \mathcal{P}$ into residue classes (mod km_i).

Words in generators of residue class-wise affine groups can always be evaluated and compared. Thus as there are finitely presented groups with unsolvable word problem (cp. [22], [5], [6]), not all finitely presented groups embed in $\mathrm{RCWA}(\mathbb{Z})$. We have seen above that $\mathrm{CT}(\mathbb{Z})$ contains nontrivial wreath products $G \wr (\mathbb{Z}, +)$. Therefore by the main theorem in [3], on the other hand not all finitely generated subgroups of $\mathrm{CT}(\mathbb{Z})$ are finitely presented.

Let $\mathrm{F}_2 = \langle a, b \rangle$ be the free group of rank 2, and let $r_1, \dots, r_k \in \mathrm{F}_2$ be the relators of a finitely presented group with unsolvable word problem. Then the membership problem for the group $\langle (a,a), (b,b), (1,r_1), \dots, (1,r_k) \rangle < \mathrm{F}_2 \times \mathrm{F}_2$ is unsolvable (cp. [20], [21]; see also [19], Chapter IV.4). Thus as $\mathrm{F}_2 \times \mathrm{F}_2$ embeds in $\mathrm{CT}(\mathbb{Z})$, there are finitely generated subgroups $G < \mathrm{CT}(\mathbb{Z})$ for which the membership problem is unsolvable.

Now we already know the following:

1.5 Theorem The group $\mathrm{CT}(\mathbb{Z})$ is not finitely generated, and its torsion subgroups are divisible. It acts highly transitively on $\mathbb{N}_0$. The class of isomorphism types of its subgroups is closed under forming direct products and under forming wreath products with finite groups and with $(\mathbb{Z}, +)$. It includes the free group of rank 2 and free products of finitely many finite groups. The group $\mathrm{CT}(\mathbb{Z})$ has finitely generated subgroups which are not finitely presented. It also has finitely generated subgroups with unsolvable membership problem.

2 The Group $\mathrm{CT}(\mathbb{Z})$ is Simple

The aim of this section is to show that the group $\mathrm{CT}(\mathbb{Z})$ is simple. For this we need some lemmata:

2.1 Lemma *Given two class transpositions $\tau_{r_1(m_1),r_2(m_2)}$ and $\tau_{r_3(m_3),r_4(m_4)}$ which are both not equal to τ, there is always a product π of 6 class transpositions such that $\tau_{r_1(m_1),r_2(m_2)}^{\pi} = \tau_{r_3(m_3),r_4(m_4)}$.*

Proof: Let $r_5(m_5), r_6(m_6) \subset \mathbb{Z} \setminus (r_1(m_1) \cup r_2(m_2))$ be disjoint residue classes such that $\cup_{i=3}^{6} r_i(m_i) \neq \mathbb{Z}$. Further let $r_7(m_7), r_8(m_8) \subset \mathbb{Z} \setminus \cup_{i=3}^{6} r_i(m_i)$ be disjoint residue classes. Then the following hold:

1. $\tau_{r_1(m_1),r_2(m_2)}{}^{\tau_{r_1(m_1),r_5(m_5)} \cdot \tau_{r_2(m_2),r_6(m_6)}} = \tau_{r_5(m_5),r_6(m_6)}$.
2. $\tau_{r_5(m_5),r_6(m_6)}{}^{\tau_{r_5(m_5),r_7(m_7)} \cdot \tau_{r_6(m_6),r_8(m_8)}} = \tau_{r_7(m_7),r_8(m_8)}$.
3. $\tau_{r_7(m_7),r_8(m_8)}{}^{\tau_{r_3(m_3),r_7(m_7)} \cdot \tau_{r_4(m_4),r_8(m_8)}} = \tau_{r_3(m_3),r_4(m_4)}$.

The assertion follows. $\square$

2.2 Lemma *Let $\sigma, \upsilon \in \mathrm{RCWA}(\mathbb{Z})$, and put $m := \mathrm{Mod}(\sigma)$. If υ is integral and fixes all residue classes (mod m) setwisely, then $[\sigma, \upsilon]$ is integral as well.*

Proof: Since υ fixes residue classes (mod m), an affine partial mapping α of $[\sigma, \upsilon]$ is a product $\alpha_\sigma^{-1}\alpha_{\upsilon^{-1}}\alpha_\sigma\alpha_\upsilon$ of affine partial mappings α_σ, α_υ and $\alpha_{\upsilon^{-1}}$ of σ, υ resp. υ^{-1}. The assertion follows since the subgroup generated by translations and reflections is normal in the affine group of the rationals. $\square$

2.3 Lemma *Let G be a subgroup of $\mathrm{RCWA}(\mathbb{Z})$ which contains $\mathrm{CT}(\mathbb{Z})$. Then a nontrivial normal subgroup $N \trianglelefteq G$ must have an integral element $\iota \neq 1$.*

Proof: Let $\sigma \in N \setminus \{1\}$, and let $m := \mathrm{Mod}(\sigma)$. Without loss of generality we can assume that there is a residue class $r(m)$ such that $r(m)^\sigma \neq r(m)$. Put $\iota := [\sigma, \tau_{r(2m),r+m(2m)}] \in N \setminus \{1\}$. By Lemma 2.2, ι is integral. □

Now we can prove our theorem:

2.4 Theorem *The group* $\mathrm{CT}(\mathbb{Z})$ *is simple.*

Proof: We have to show that any nontrivial normal subgroup $N \trianglelefteq \mathrm{CT}(\mathbb{Z})$ contains any class transposition. By Lemma 2.1 all class transpositions $\neq \tau$ are conjugate in $\mathrm{CT}(\mathbb{Z})$. Further we have $\tau = \tau_{0(4),1(4)} \cdot \tau_{2(4),3(4)}$. Thus it is already sufficient to show that one class transposition $\neq \tau$ lies in N.

By Lemma 2.3 the normal subgroup N has an integral element $\iota_1 \neq 1$. Let $m \geqslant 3$ be a large enough multiple of the modulus of ι_1 such that there is a residue class $r(m) \in \mathbb{Z}/m\mathbb{Z}$ which ι_1 does not fix setwisely. Then put $\iota_2 := \tau_{r(2m),r+m(2m)} \cdot \tau_{r(2m)^{\iota_1},r+m(2m)^{\iota_1}} = [\tau_{r(2m),r+m(2m)}, \iota_1] \in N$.

By the choice of m we can now choose two distinct residue classes $r_1(2m)$, $r_2(2m) \notin \{r(2m), r+m(2m), r(2m)^{\iota_1}, r+m(2m)^{\iota_1}\}$. Then we have

$$\begin{aligned}\tau_{r_1(2m),r_2(2m)} &= \iota_2^{\tau_{r(2m),r_1(4m)} \cdot \tau_{r+m(2m),r_2(4m)}} \\ &\quad \cdot \iota_2^{\tau_{r(2m),r_1+2m(4m)} \cdot \tau_{r+m(2m),r_2+2m(4m)}} \in N,\end{aligned}$$

which completes the proof of the theorem. □

2.5 Remark Assume $\mathrm{CT}(\mathbb{Z}) \leqslant G \leqslant \mathrm{RCWA}(\mathbb{Z})$, and let N be a nontrivial normal subgroup of G. Then the proof of Theorem 2.4 shows in fact that N contains $\mathrm{CT}(\mathbb{Z})$.

2.6 Definition Given a set of primes $\mathbb{P}$, let $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z}) < \mathrm{CT}(\mathbb{Z})$ denote the subgroup which is generated by all class transpositions whose moduli have only prime factors in $\mathbb{P}$.

2.7 Corollary *If the set* $\mathbb{P}$ *contains 2, then* $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ *is simple as well. Hence the group* $\mathrm{CT}(\mathbb{Z})$ *has uncountably many simple subgroups.*

Proof: In case $2 \in \mathbb{P}$, all of our arguments in this section apply to $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ as well: In the proof of Lemma 2.1, we can choose the four residue classes $r_5(m_5), \dots, r_8(m_8)$ in such a way that all prime factors of their moduli already divide $m_1m_2m_3m_4$. The proofs of Lemma 2.2, Lemma 2.3 and Theorem 2.4 likewise do not require the presence of class transpositions whose moduli have certain odd factors. □

3 Collatz' Permutation Lies in $\mathrm{CT}(\mathbb{Z})$

In 1932, Lothar Collatz investigated the permutation

$$\alpha \in \mathrm{RCWA}(\mathbb{Z}): \quad n \ \mapsto \ \begin{cases} \frac{2n}{3} & \text{if } n \in 0(3), \\ \frac{4n-1}{3} & \text{if } n \in 1(3), \\ \frac{4n+1}{3} & \text{if } n \in 2(3) \end{cases}$$

of the integers (cp. Keller [14], Wirsching [24] and Lagarias [17]).

The permutation α commutes with the involution $n \mapsto -n$, thus the cycle structures of its restrictions to the positive resp. negative integers are the same. The fixed points of α are -1, 0 and 1. In Keller [14] it is shown that α has at most finitely many cycles of any given finite length. It further looks likely that the only finite cycles are the transpositions $\pm(2\ 3)$, the 5-cycles $\pm(4\ 5\ 7\ 9\ 6)$ and the 12-cycles $\pm(44\ 59\ 79\ 105\ 70\ 93\ 62\ 83\ 111\ 74\ 99\ 66)$. However according to Wirsching [24] the latter has still not been proven, and in particular it is not yet known whether the cycle

$$(\ \dots\ 32\ 43\ 57\ 38\ 51\ 34\ 45\ 30\ 20\ 27\ 18\ 12\ 8\ 11\ 15\ 10\ 13\ 17\ 23\ 31\ 41\ \dots\ )$$

of α is finite or infinite.

In spite of this we can show that the permutation α lies in $\mathrm{CT}(\mathbb{Z})$ by determining an explicit factorization into generators.

The major obstacle we are confronted with when trying to obtain such a factorization is the fact that multiplier and divisor of α are coprime, whereas multiplier and divisor of a class transposition are always the same. We have already observed that the multiplier resp. divisor of a product of two residue class-wise affine mappings always divides the product of the multipliers resp. divisors of the factors. In the given case we need to form a product of class transpositions in such a way that one prime divisor gets eliminated from the multiplier of the product, but appears in the denominators of all of its affine partial mappings.

As a first step towards a solution of the factorization problem, we hence attempt to determine some product of class transpositions which has coprime multiplier and divisor. It turns out that 6 class transpositions are sufficient to form such a product: Given an odd prime p, the permutation

$$\begin{aligned} \sigma_p \ := \ & \tau_{0(8),1(2p)} \cdot \tau_{4(8),2p-1(2p)} \\ & \cdot \tau_{0(4),1(2p)} \cdot \tau_{2(4),2p-1(2p)} \\ & \cdot \tau_{2(2p),1(4p)} \cdot \tau_{4(2p),2p+1(4p)} \end{aligned}$$

has multiplier p and divisor 2.

Indeed, evaluating this product yields

$$\sigma_p \in \mathrm{CT}(\mathbb{Z}): \quad n \ \mapsto \ \begin{cases} (pn+2p-2)/2 & \text{if } n \in 2(4), \\ n/2 & \text{if } n \in 0(4) \setminus 4(4p) \cup 8(4p), \\ n+2p-7 & \text{if } n \in 8(4p), \\ n-2p+5 & \text{if } n \in 2p-1(2p), \\ n+1 & \text{if } n \in 1(2p), \\ n-3 & \text{if } n \in 4(4p), \\ n & \text{if } n \in 1(2) \setminus 1(2p) \cup 2p-1(2p). \end{cases}$$

The GAP [11] package RCWA [15] provides a factorization routine for residue class-wise affine permutations, which uses a kind of elaborate heuristics. The permutations σ_p and their images under restriction monomorphisms $\pi_{n \mapsto mn+r}$ play a key role in this routine. It has been used to obtain the following factorization of α:

$$\begin{aligned} \alpha \ = \ & \tau_{0(6),4(6)} \cdot \tau_{0(6),5(6)} \cdot \tau_{0(6),3(6)} \cdot \tau_{0(6),1(6)} \cdot \tau_{0(6),2(6)} \\ & \cdot \tau_{2(3),4(6)} \cdot \tau_{0(3),4(6)} \cdot \tau_{2(3),1(6)} \cdot \tau_{0(3),1(6)} \cdot \tau_{0(36),35(36)} \\ & \cdot \tau_{0(36),22(36)} \cdot \tau_{0(36),18(36)} \cdot \tau_{0(36),17(36)} \cdot \tau_{0(36),14(36)} \cdot \tau_{0(36),20(36)} \\ & \cdot \tau_{0(36),4(36)} \cdot \tau_{2(36),8(36)} \cdot \tau_{2(36),16(36)} \cdot \tau_{2(36),13(36)} \cdot \tau_{2(36),9(36)} \\ & \cdot \tau_{2(36),7(36)} \cdot \tau_{2(36),6(36)} \cdot \tau_{2(36),3(36)} \cdot \tau_{2(36),10(36)} \cdot \tau_{2(36),15(36)} \\ & \cdot \tau_{2(36),12(36)} \cdot \tau_{2(36),5(36)} \cdot \tau_{21(36),28(36)} \cdot \tau_{21(36),33(36)} \cdot \tau_{21(36),30(36)} \\ & \cdot \tau_{21(36),23(36)} \cdot \tau_{21(36),34(36)} \cdot \tau_{21(36),31(36)} \cdot \tau_{21(36),27(36)} \cdot \tau_{21(36),25(36)} \\ & \cdot \tau_{21(36),24(36)} \cdot \tau_{26(36),32(36)} \cdot \tau_{26(36),29(36)} \cdot \tau_{10(18),35(36)} \cdot \tau_{5(18),35(36)} \\ & \cdot \tau_{10(18),17(36)} \cdot \tau_{5(18),17(36)} \cdot \tau_{8(12),14(24)} \cdot \tau_{6(9),17(18)} \cdot \tau_{3(9),17(18)} \\ & \cdot \tau_{0(9),17(18)} \cdot \tau_{6(9),16(18)} \cdot \tau_{3(9),16(18)} \cdot \tau_{0(9),16(18)} \cdot \tau_{6(9),11(18)} \\ & \cdot \tau_{3(9),11(18)} \cdot \tau_{0(9),11(18)} \cdot \tau_{6(9),4(18)} \cdot \tau_{3(9),4(18)} \cdot \tau_{0(9),4(18)} \\ & \cdot \tau_{0(6),14(24)} \cdot \tau_{0(6),2(24)} \cdot \tau_{8(12),17(18)} \cdot \tau_{7(12),17(18)} \cdot \tau_{8(12),11(18)} \\ & \cdot \tau_{7(12),11(18)} \cdot \sigma_3^{-1} \cdot \tau_{7(12),17(18)} \cdot \tau_{2(6),17(18)} \cdot \tau_{0(3),17(18)} \cdot \sigma_3^{-3}. \end{aligned}$$

This shows constructively that $\alpha \in \mathrm{CT}(\mathbb{Z})$.

4 A Larger Simple Group

In this section we would like to construct a simple subgroup of $\mathrm{RCWA}(\mathbb{Z})$ which properly contains $\mathrm{CT}(\mathbb{Z})$ and which acts highly transitively on $\mathbb{Z}$. For this purpose we need to take a slightly closer look at the group $\mathrm{RCWA}(\mathbb{Z})$, its subgroups and its elements.

It turns out that there are two entirely different classes of residue class-wise affine groups and -permutations. One of these classes comprises what could be called the 'trivial cases'. The members of the other have typically a quite complicate structure and are often very difficult to investigate:

4.1 Definition We call an element $\sigma \in \mathrm{RCWA}(\mathbb{Z})$ *tame* if it permutes a partition $\mathcal{P}$ of $\mathbb{Z}$ into finitely many residue classes on all of which it is affine, and *wild* otherwise. We call a group $G < \mathrm{RCWA}(\mathbb{Z})$ *tame* if there is a common such partition for all elements of G, and *wild* otherwise. We call partitions with the described properties *respected partitions* of σ resp. G.

Obviously, finite residue class-wise affine groups and integral residue class-wise affine permutations are tame. Tameness is invariant under conjugation: If $\alpha \in \mathrm{RCWA}(\mathbb{Z})$ respects a partition $\mathcal{P}$, then a conjugate α^β respects the partition consisting of the images of the intersections of the residue classes in $\mathcal{P}$ with the sources of the affine partial mappings of β under β. Of course the product of two tame permutations is in general not tame. Further, tameness of products does not induce an equivalence relation on the set of tame permutations: Let for example $a := \tau_{1(6),4(6)}$, $b := \tau_{0(5),2(5)}$ and $c := \tau_{3(4),4(6)}$. Then ab and bc are tame, but ac is not. If a tame group does not act faithfully on a respected partition, the kernel of the action clearly does not act on $\mathbb{N}_0$. Thus as the group $\mathrm{CT}(\mathbb{Z})$ acts on $\mathbb{N}_0$, its tame subgroups are finite. It can be shown that a residue class-wise affine group is tame if and only if the set of moduli of its elements is bounded (see Theorem 2.5.8 in [16]).

It is easy to see that all tame elements of $\mathrm{RCWA}(\mathbb{Z})$ can be written as products of class transpositions and members of the following two series:

4.2 Definition Let $r(m) \subset \mathbb{Z}$ be a residue class.

1. We define the *class shift* $\nu_{r(m)} \in \mathrm{RCWA}(\mathbb{Z})$ by
$$\nu_{r(m)}: \quad n \ \mapsto \ \begin{cases} n+m & \text{if} \ \ n \in r(m), \\ n & \text{otherwise.} \end{cases}$$

2. We define the *class reflection* $\varsigma_{r(m)} \in \mathrm{RCWA}(\mathbb{Z})$ by
$$\varsigma_{r(m)}: \quad n \ \mapsto \ \begin{cases} -n+2r & \text{if} \ \ n \in r(m), \\ n & \text{otherwise,} \end{cases}$$
where we assume $0 \leqslant r < m$.

For convenience, we set $\nu := \nu_{\mathbb{Z}} : n \mapsto n+1$ and $\varsigma := \varsigma_{\mathbb{Z}} : n \mapsto -n$.

In contrast, it is an open problem whether the tame elements generate the group RCWA($\mathbb{Z}$) or a proper normal subgroup thereof (cp. Conjecture 2.9.8 in [16] and the corresponding factorization routine in [15]). In the former case we would have $\mathrm{RCWA}(\mathbb{Z}) = \langle \mathrm{CT}(\mathbb{Z}), \nu, \varsigma_{0(2)} \rangle$:

1. It is $\nu_{0(2)} = \tau\nu$, $\nu_{1(2)} = \nu_{0(2)}^{\tau}$, $\varsigma_{1(2)} = \varsigma_{0(2)}^{\tau}$ and $\varsigma = \varsigma_{0(2)} \cdot \nu_{1(2)} \cdot \varsigma_{1(2)}$. Therefore we know that $\{\nu_{0(2)}, \nu_{1(2)}, \varsigma_{1(2)}, \varsigma\} \subset \langle \mathrm{CT}(\mathbb{Z}), \nu, \varsigma_{0(2)} \rangle$.

2. Let $r(m) \subsetneq \mathbb{Z}$ be a residue class $\neq 1(2)$. We choose a residue class $\tilde{r}(\tilde{m}) \subset \mathbb{Z} \setminus (0(2) \cup r(m))$, and put $\vartheta := \tau_{0(2),\tilde{r}(\tilde{m})} \cdot \tau_{\tilde{r}(\tilde{m}),r(m)} \in \mathrm{CT}(\mathbb{Z})$. Then we have $\{\nu_{r(m)}, \varsigma_{r(m)}\} = \{\nu_{0(2)}^{\vartheta}, \varsigma_{0(2)}^{\vartheta}\} \subset \langle \mathrm{CT}(\mathbb{Z}), \nu, \varsigma_{0(2)} \rangle$.

There is an epimorphism from RCWA($\mathbb{Z}$) to $\mathbb{Z}^{\times} \cong \mathrm{C}_2$. Using the notation for the coefficients from Definition 1.1, this sign epimorphism is given by

$$\sigma \longmapsto (-1)^{\frac{1}{m}\left(\sum\limits_{r(m) \in \mathbb{Z}/m\mathbb{Z}} \frac{b_{r(m)}}{a_{r(m)}} + \sum\limits_{r(m):\ a_{r(m)}<0} (m-2r)\right)}$$

(see Section 2.12 in [16]).

4.3 Definition We denote the kernel of the sign epimorphism by K.

Obviously it is $\mathrm{CT}(\mathbb{Z}) < K$. Likewise it is easy to see that class shifts and class reflections do not lie in K.

4.4 Definition We denote the subgroup of K which is generated by the tame elements with sign 1 by $\tilde{K}$.

The groups RCWA($\mathbb{Z}$), K and $\tilde{K}$ are not finitely generated for the same reasons as CT($\mathbb{Z}$). Let $\varphi_m : \mathrm{S}_m \hookrightarrow \mathrm{CT}(\mathbb{Z})$ denote the monomorphism given in the introduction. Then the conjugate of the image of φ_m under the mapping $\nu^{-2\lfloor m/4 \rfloor} \in \tilde{K}$ acts m - transitively on the set $\{-2\lfloor m/4\rfloor, \dots, m-2\lfloor m/4\rfloor-1\}$. Since m can be chosen arbitrary large, $\tilde{K}$ acts highly transitively on $\mathbb{Z}$.

4.5 Theorem *The group $\tilde{K}$ is simple.*

Proof: Looking at respected partitions, we check that $\tilde{K}$ is generated by

1. all products of a class reflection and a class shift with the same support which has a nontrivial complement in $\mathbb{Z}$,

2. all class transpositions and

3. all products of two class shifts with disjoint supports whose union has a nontrivial complement in $\mathbb{Z}$.

This is easiest to see by looking at the process of factoring a given tame permutation ϑ with sign 1 into these elements:

1. Let $m \geqslant 2$ be a multiple of $\mathrm{Mod}(\vartheta)$. Make the permutation ϑ class-wise order-preserving by dividing it from the left by elements $\varsigma_{r(m)} \cdot \nu_{r(m)}$, where $r(m)$ runs over all residue classes (mod m) on which ϑ is order-reversing.

2. Let $\mathcal{P}$ be a respected partition of ϑ of length at least 3. Divide ϑ by a product of class transpositions which respects $\mathcal{P}$ as well and which induces on $\mathcal{P}$ the same permutation as ϑ does.

3. Factor the remaining quotient which is integral and fixes $\mathcal{P}$ into products of two class shifts with disjoint supports and inverses of such products. This works since for $n \geqslant 3$ the lattice in $\mathbb{Z}^n$ consisting of all vectors with even coordinate sum is spanned by the sums of two distinct canonical basis vectors. The attribute *even coordinate sum* reflects the assumption that the sign of ϑ is 1.

Now let N be a nontrivial normal subgroup of $\tilde{K}$. From Remark 2.5 we know that $\mathrm{CT}(\mathbb{Z}) \leqslant N$, thus N contains series (2.) of generators of $\tilde{K}$.

Given two disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ whose union is not $\mathbb{Z}$, for any residue class $r_3(m_3) \subset \mathbb{Z} \setminus (r_1(m_1) \cup r_2(m_2))$ we have

$$\begin{aligned}\nu_{r_1(m_1)} \cdot \nu_{r_2(m_2)} \ &= \left[\tau_{r_1(m_1),r_2(m_2)}, \nu_{r_3(m_3)} \cdot \varsigma_{r_1(m_1)}\right] \\ &\quad \cdot \left[\tau_{r_1(m_1),r_2(m_2)}, \varsigma_{r_1(m_1)} \cdot \nu_{r_1(m_1)}\right] \ \in \ N.\end{aligned}$$

This shows that N contains series (3.) of generators of $\tilde{K}$.

Let $r(m) \subsetneq \mathbb{Z}$ be a residue class. Then for an arbitrary residue class $\tilde{r}(\tilde{m}) \subset \mathbb{Z} \setminus r(m)$ we have

$$\begin{aligned}\varsigma_{r(m)} \cdot \nu_{r(m)} \ &= \left[\tau_{r(m),\tilde{r}(\tilde{m})}, \varsigma_{r(m)} \cdot \nu_{\tilde{r}(\tilde{m})}\right] \cdot \left[\tau_{\tilde{r}(2\tilde{m}),\tilde{r}+\tilde{m}(2\tilde{m})}, \varsigma_{\tilde{r}(2\tilde{m})} \cdot \nu_{\tilde{r}(2\tilde{m})}\right] \\ &\quad \cdot \left(\nu_{\tilde{r}(2\tilde{m})} \cdot \nu_{\tilde{r}+\tilde{m}(2\tilde{m})} \cdot \tau_{\tilde{r}(2\tilde{m}),\tilde{r}+\tilde{m}(2\tilde{m})}\right)^{-1} \ \in \ N.\end{aligned}$$

This shows that N contains also series (1.) of generators of $\tilde{K}$, and therefore completes the proof of our theorem. $\square$

It is not known so far whether $\tilde{K} = K$ or whether $\tilde{K}$ is a proper normal subgroup of K.

5 A Simple Group 'In Between'

There is an epimorphism from the subgroup of RCWA($\mathbb{Z}$) consisting of all class-wise order-preserving permutations to $(\mathbb{Z}, +)$. Using the notation for the coefficients from Definition 1.1, this epimorphism is given by

$$\sigma \ \longmapsto \ \frac{1}{m} \sum_{r(m) \in \mathbb{Z}/m\mathbb{Z}} \frac{b_{r(m)}}{a_{r(m)}}$$

(see Section 2.11 in [16]). We denote the kernel of this epimorphism by K^+. Obviously class transpositions lie in K^+, but class shifts do not. While it is so far not known whether the group K^+ is generated by its tame elements or not, we can show the following:

5.1 Theorem *The subgroup $\tilde{K}^+ \trianglelefteq K^+$ generated by the tame elements is simple, and it acts highly transitively on $\mathbb{Z}$.*

Proof: Looking at respected partitions, we check that $\tilde{K}^+$ is generated by

1. all class transpositions and
2. all quotients of two class shifts with disjoint supports whose union has a nontrivial complement in $\mathbb{Z}$.

For this we look at the process of factoring a given tame $\vartheta \in K^+$ in the into these elements:

1. Let $\mathcal{P}$ be a respected partition of ϑ. Divide ϑ by a product of class transpositions which respects $\mathcal{P}$ as well and which induces on $\mathcal{P}$ the same permutation as ϑ does.
2. Factor the remaining quotient which is integral and fixes $\mathcal{P}$ into quotients of two class shifts with disjoint supports which respect $\mathcal{P}$. This works since the lattice in $\mathbb{Z}^n$ consisting of all vectors with zero coordinate sum is spanned by the differences of two distinct canonical basis vectors.

Now let N be a nontrivial normal subgroup of $\tilde{K}$. From Remark 2.5 we know that $\mathrm{CT}(\mathbb{Z}) \leqslant N$, thus N contains series (1.) of generators of $\tilde{K}^+$.

Let $r_1(m_1)$ and $r_2(m_2)$ be two disjoint residue classes whose union is not $\mathbb{Z}$. Then for any residue class $r_3(m_3) \subset \mathbb{Z} \setminus (r_1(m_1) \cup r_2(m_2))$ we have

$$\nu_{r_1(m_1)} \cdot \nu_{r_2(m_2)}^{-1} \ = \ [\tau_{r_1(m_1),r_2(m_2)}, \nu_{r_3(m_3)} \cdot \nu_{r_2(m_2)}^{-1}] \ \in \ N,$$

thus N contains also series (2.) of generators of $\tilde{K}^+$. This shows that the group $\tilde{K}^+$ is simple.

Now let k be a positive integer, and let $(n_1, \dots, n_k)$ and $(\tilde{n}_1, \dots, \tilde{n}_k)$ be two k-tuples of pairwise distinct integers. We have to show that there is a $\sigma \in \tilde{K}^+$ such that $(n_1^\sigma, \dots, n_k^\sigma) = (\tilde{n}_1, \dots, \tilde{n}_k)$.

Let $m := 2k + 1$, and choose a residue class $r(m)$ which does not contain one of the points n_i or $\tilde{n}_i$. Define $\sigma_1, \tilde{\sigma}_1 \in \tilde{K}^+$ by

$$\sigma_1 \ := \prod_{i:n_i<0} (\nu_{r(m)} \cdot \nu_{n_i(m)}^{-1})^{\lfloor \frac{n_i}{m} \rfloor} \quad \text{resp.} \quad \tilde{\sigma}_1 \ := \prod_{i:\tilde{n}_i<0} (\nu_{r(m)} \cdot \nu_{\tilde{n}_i(m)}^{-1})^{\lfloor \frac{\tilde{n}_i}{m} \rfloor}.$$

Then the images of all points n_i resp. $\tilde{n}_i$ under σ_1 resp. $\tilde{\sigma}_1$ are nonnegative. Since we know that $\mathrm{CT}(\mathbb{Z})$ acts highly transitively on $\mathbb{N}_0$, we can choose a $\sigma_2 \in \mathrm{CT}(\mathbb{Z}) < \tilde{K}^+$ which maps the images of the n_i under σ_1 to the images of the $\tilde{n}_i$ under $\tilde{\sigma}_1$. Now the permutation $\sigma := \sigma_1 \cdot \sigma_2 \cdot \tilde{\sigma}_1^{-1}$ serves our purposes. □

6 A Locally Finite Simple Subgroup of $\mathrm{CT}(\mathbb{Z})$

The class transpositions which interchange residue classes with the same modulus generate a proper subgroup of $\mathrm{CT}(\mathbb{Z})$:

6.1 Definition Let $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ denote the subgroup of $\mathrm{CT}(\mathbb{Z})$ which is generated by all integral class transpositions.

The group $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ acts highly transitively on $\mathbb{N}_0$ for the same reasons as $\mathrm{CT}(\mathbb{Z})$ does so. Finitely generated subgroups of $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ are obviously tame. Further we have seen above that tame subgroups of $\mathrm{CT}(\mathbb{Z})$ are even finite. Hence the group $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ is locally finite. We still have to show the second property announced in the section title:

6.2 Theorem *The group* $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ *is simple.*

Proof: We have to show that any nontrivial normal subgroup $N \trianglelefteq \mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ contains any class transposition of the form $\tau_{r_1(m),r_2(m)}$.

Let $\iota_1 \in N \setminus \{1\}$. Further let m be a large enough multiple of the modulus of ι_1 such that there is a residue class $r(m) \in \mathbb{Z}/m\mathbb{Z}$ which ι_1 does not fix setwisely. Then $\iota_2 := [\iota_1, \tau_{r(2m),r+m(2m)}] \in N$ is a product of two class transpositions with disjoint supports.

Given any two class transpositions $\tau_1 = \tau_{r_1(m),r_2(m)}$ and $\tau_2 = \tau_{r_3(2m),r_4(2m)}$ with disjoint supports, we have

$$\begin{aligned} \tau_1 \ &= \ \tau_2 \cdot \tau_{r_1(2m),r_2(2m)} \\ &\quad \cdot \tau_2 \cdot \tau_{r_1+m(2m),r_2+m(2m)}. \end{aligned}$$

Therefore any integral class transposition $\neq \tau$ can be written as a product of two suitable products of two integral class transpositions with disjoint supports.

Integral class transpositions of same modulus are conjugate in $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$: For any $\tau_{r_1(m),r_2(m)}, \tau_{r_3(m),r_4(m)} \in \mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ we have

$$\tau_{r_1(m),r_2(m)}{}^{\tau_{r_1(m),r_3(m)} \cdot \tau_{r_2(m),r_4(m)}} \quad = \quad \tau_{r_3(m),r_4(m)},$$

where we read $\tau_{r_1(m),r_3(m)}$ resp. $\tau_{r_2(m),r_4(m)}$ as the identity if the interchanged residue classes are equal. Likewise we observe that given $m \geqslant 4$, all products of two integral class transpositions with modulus m and disjoint supports are conjugate in $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$.

Given $k, m \in \mathbb{N}$, any integral class transposition $\tau_{r_1(m),r_2(m)}$ of modulus m can be written as a product of k integral class transpositions of modulus km. This can be seen from the equality $\tau_{r_1(m),r_2(m)} = \prod_{i=0}^{k-1} \tau_{r_1+im(km),r_2+im(km)}$.

In the second paragraph of the proof we can choose m to be a multiple of any given positive integer. Therefore from all observations made since then together, we finally can conclude that N contains indeed any integral class transposition. This is what had to be shown. $\square$

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