Solutions to MATH3411 Problems 41–47

41. $H(S) = -\frac{2}{3}\log_2\frac{1}{3} - \frac{3}{9}\log_2\frac{1}{9} = \frac{2}{3}\log_2 3 + \frac{6}{9}\log_2 3 = \frac{4}{3}\log_2 3 \approx 2.113$

42.

Q29: **a)** $H(S) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{3}\log_2\frac{1}{3} - \frac{1}{6}\log_2\frac{1}{6} \approx 1.459$
The average length $L = 1.5 > H(S)$ - but pretty close.

b) $H(S) = -\frac{1}{3}\log_2\frac{1}{3} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{5}\log_2\frac{1}{5} - \frac{1}{6}\log_2\frac{1}{6} - \frac{1}{20}\log_2\frac{1}{20} \approx 2.140$
The average length $L = 2.217 > H(S)$ - but not far off.

c) $H(S) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{16}\log_2\frac{1}{16} - \frac{1}{16}\log_2\frac{1}{16} \approx 1.875$
The average length $L = 1.875 = H(S)$ - exactly the same!

d) $H(S) = -\frac{27}{40}\log_2\frac{27}{40} - \frac{9}{40}\log_2\frac{9}{40} - \frac{3}{40}\log_2\frac{3}{40} - \frac{1}{40}\log_2\frac{1}{40} \approx 1.280$
The average length $L = 1.425 > H(S)$ - not too far off.

Q33:
$$\begin{aligned} H(S) &= -0.22\log_4 0.22 - 0.2\log_4 0.2 - 0.18\log_4 0.18 - 0.15\log_4 0.15 \\ &\quad - 0.10\log_4 0.10 - 0.08\log_4 0.08 - 0.05\log_4 0.05 - 0.02\log_2 0.02 \\ &\approx 1.377 \end{aligned}$$

The average length $L = 1.47$ is greater than $H(S) \approx 1.377$ - but it's pretty close.

Q35: $H(S^1) = -\frac{2}{3}\log_2\frac{2}{3} - \frac{1}{3}\log_2\frac{1}{3} \approx 0.918$
$H(S^2) = 2H(S^1) \approx 1.837$
$H(S^3) = 3H(S^1) \approx 2.755$
The corresponding average lengths are 1, 1.889 and 2.815, respectively.
These are all greater than the corresponding entropies - but not by a lot.

Q39: $H(S) = -0.4\log_{10} 0.4 - 0.3\log_{10} 0.3 - 0.2\log_{10} 0.2 - 0.1\log_{10} 0.1 \approx 0.5558$ digits/symbol.
The 5-symbol message $s_2 s_1 s_3 s_1 \bullet$ was encoded as 0.493, so $\frac{3}{5} = 0.6$ digits per symbol were used, which is more than 0.5558 but not by a lot.

43. Measured in information per bits, the entropy of the experiment is

$$H(S) = H_2(S) = -\frac{2}{3}\log_2\frac{1}{3} - \frac{2}{9}\log_2\frac{1}{9} - \frac{3}{27}\log_2\frac{1}{27} \approx 2.2894$$

According to p.78 of the course notes, the average codeword length per symbol of the extension S^n for increasing n converges to $H(S)$, so the price for encoded long messages per bit will approximately be $2.2894 \times \$2.00 \approx \$4.56$.
Now, measured in information per ternary units, the entropy of the experiment is

$$H_3(S) = -\frac{2}{3}\log_3\frac{1}{3} - \frac{2}{9}\log_3\frac{1}{9} - \frac{3}{27}\log_3\frac{1}{27} = \frac{13}{9}$$

The price for encoded long messages per ternary unit will approximately be $\frac{13}{9} \times \$3.25 \approx \$4.69$.
We see that for sufficiently long messages, the binary encoding is (slightly) cheaper.

44.

Q29:

a)

p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{1}{2}$	2	1	0
$\frac{1}{3}$	3	2	10
$\frac{1}{6}$	6	3	110

b)

p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{1}{3}$	3	2	00
$\frac{1}{4}$	4	2	01
$\frac{1}{5}$	5	3	100
$\frac{1}{6}$	6	3	101
$\frac{1}{20}$	20	5	11000

c)

p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{1}{2}$	2	1	0
$\frac{1}{4}$	4	2	10
$\frac{1}{8}$	8	3	110
$\frac{1}{16}$	16	4	1110
$\frac{1}{32}$	32	5	11110

d)

p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{27}{40}$	$\frac{40}{27}$	1	0
$\frac{9}{40}$	$\frac{40}{9}$	3	100
$\frac{3}{40}$	$\frac{40}{3}$	4	1010
$\frac{1}{40}$	40	6	101100

Q33: **NB**: Here, we use radix 4:

p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
0.22	4.5	2	00
0.20	5	2	01
0.18	5.6	2	02
0.15	6.7	2	03
0.10	10	2	10
0.08	12.5	2	11
0.05	20	3	100
0.02	50	3	101

Q35:

S p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{2}{3}$	$\frac{3}{2}$	1	0
$\frac{1}{3}$	3	2	10

$S^{(2)}$ p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{4}{9}$	$\frac{9}{4}$	2	00
$\frac{2}{9}$	$\frac{9}{2}$	3	100
$\frac{2}{9}$	$\frac{9}{2}$	3	101
$\frac{1}{9}$	9	4	1100

$S^{(3)}$ p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{8}{27}$	$\frac{27}{8}$	2	00
$\frac{4}{27}$	$\frac{27}{4}$	3	010
$\frac{4}{27}$	$\frac{27}{4}$	3	011
$\frac{4}{27}$	$\frac{27}{4}$	3	100
$\frac{2}{27}$	$\frac{27}{2}$	4	1010
$\frac{2}{27}$	$\frac{27}{2}$	4	1011
$\frac{2}{27}$	$\frac{27}{2}$	4	1100
$\frac{1}{27}$	27	5	11010

45. **a)** $H_2(S) = -\frac{1}{3}\log_2\frac{1}{3} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{6}\log_2\frac{1}{6} - \frac{3}{20}\log_2\frac{3}{20} - \frac{1}{10}\log_2\frac{1}{10} \approx 2.20$

b) **NB**: Here, we use radix 3:

p_i	$\frac{1}{p_i}$	SF ℓ_i	SF code
$\frac{1}{3}$	3	1	0
$\frac{1}{4}$	4	2	10
$\frac{1}{6}$	6	2	11
$\frac{3}{20}$	$\frac{20}{3}$	2	12
$\frac{1}{10}$	10	3	200

The average length $L = \frac{1}{3} \times 2 + \frac{1}{4} \times 2 + \frac{1}{6} \times 2 + \frac{3}{20} \times 2 + \frac{1}{10} \times 3 = \frac{53}{30} \approx 1.77$.
(This is bigger than the entropy found in part a) - but that was the **binary** entropy, not the **ternary** entropy, so there is no contradiction here.)

c) The smallest two probabilities for $S^{(4)}$ are $p_{624} = \frac{3}{20 \cdot 10^3}$ and $p_{625} = \frac{1}{10^4}$;
their inverses are $\frac{1}{p_{624}} = \frac{2}{3} \times 10^4 \approx 6667$ and $\frac{1}{p_{625}} = 10^4 = 10000$,
so $\ell_{624} = 13$ and $\ell_{625} = 14$ $(2^{13} = 8192$ and $2^{14} = 16384)$

46. $H(p) = -p\log_2 p - (1-p)\log_2(1-p)$.
Note that $H(0) = -0 - 1\log_2 1 = 0$ and that $H(1) = -1\log_2 1 - 0 = 0$.
Also, for $p \in (0,1)$, $-\log_2 p \geq 0$ and $-\log_2(1-p) \geq 0$, so $H(p) \geq 0$.
Hence, $H(p)$ is non-negative on the interval $[0,1]$.
Now note that $H'(p) = -\log_2 p + \log_2(1-p)$.
For $p < \frac{1}{2}$, we see that $-\log_2 p > 1$ whereas $\log_2(1-p) > -1$, so $H'(p) > 0$;
similarly for $p > \frac{1}{2}$, we see that $-\log_2 p < 1$ whereas $\log_2(1-p) < -1$, $H'(p) < 0$.
Therefore, $H(p)$ is concave down.
We also see that the maximum of $H(p)$ is at $p = \frac{1}{2}$ and equals $H(\frac{1}{2}) = 1$.

47.
Q36: Here, the equilibrium vector is $\frac{1}{11}\begin{pmatrix}3\\4\\4\end{pmatrix}$.

$$\begin{aligned}
H(S|s_1) &= -\frac{1}{3}\log_2\frac{1}{3} - \frac{1}{3}\log_2\frac{1}{3} - \frac{1}{3}\log_2\frac{1}{3} = \log_2 3 \approx 1.58\\
H(S|s_2) &= -\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} = \frac{3}{2} = 1.5\\
H(S|s_3) &= H(S|s_2) = 1.5\\
H_M &= \frac{3}{11}H(S|s_1) + \frac{4}{11}H(S|s_2) + \frac{4}{11}H(S|s_3) \approx 1.52\\
H_E &= -\frac{3}{11}\log_2\frac{3}{11} - \frac{4}{11}\log_2\frac{4}{11} - \frac{4}{11}\log_2\frac{4}{11} \approx 1.57
\end{aligned}$$

Q37: Here, the equilibrium vector is $\frac{1}{17}\begin{pmatrix}6\\7\\4\end{pmatrix}$.

$$\begin{aligned}
H(S|s_1) &= -0.7\log_2 0.7 - 0.2\log_2 0.2 - 0.1\log_2 0.1 \approx 1.16\\
H(S|s_2) &= -0.2\log_2 0.2 - 0.6\log_2 0.6 - 0.2\log_2 0.2 \approx 1.37\\
H(S|s_2) &= -0.1\log_2 0.1 - 0.4\log_2 0.4 - 0.5\log_2 0.5 \approx 1.36\\
H_M &= \frac{6}{17}H(S|s_1) + \frac{7}{17}H(S|s_2) + \frac{4}{17}H(S|s_3) \approx 1.29\\
H_E &= -\frac{6}{17}\log_2\frac{6}{17} - \frac{7}{17}\log_2\frac{7}{17} - \frac{4}{17}\log_2\frac{4}{17} \approx 1.55
\end{aligned}$$