Solutions to MATH3411 Problems 48–59

48. Define $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$, where

a_1: student passes
a_2: student fails
b_1: student owns car
b_2: student owns no car
c_1: student lives at home
c_2: student lives away from home

We are told, and can immediately infer, that

$$\begin{aligned} P(a_1) &= 0.75 \\ P(a_2) &= 0.25 \end{aligned} \qquad \begin{aligned} P(b_1|a_1) &= 0.10 \\ P(b_2|a_1) &= 0.90 \\ P(b_1|a_2) &= 0.50 \\ P(b_2|a_2) &= 0.50 \end{aligned} \qquad \begin{aligned} P(c_1|b_1) &= 1.00 \\ P(c_2|b_1) &= 0 \end{aligned} \qquad \begin{aligned} P(c_1|b_2 \cap a_1) &= 0.40 \\ P(c_2|b_2 \cap a_1) &= 0.60 \\ P(c_1|b_2 \cap a_2) &= 0.40 \\ P(c_2|b_2 \cap a_2) &= 0.60 \end{aligned}$$

From this, we can calculate

$$\begin{aligned} P(a_1 \cap b_1) &= 0.075 \\ P(a_1 \cap b_2) &= 0.675 \\ P(a_2 \cap b_1) &= 0.125 \\ P(a_2 \cap b_2) &= 0.125 \end{aligned} \qquad \begin{aligned} P(b_1) &= 0.2 \\ P(b_2) &= 0.8 \end{aligned} \qquad \begin{aligned} P(b_1 \cap c_1) &= 0.2 \\ P(b_1 \cap c_2) &= 0 \end{aligned} \qquad \begin{aligned} P(a_1 \cap b_2 \cap c_1) &= 0.27 \\ P(a_1 \cap b_2 \cap c_2) &= 0.405 \\ P(a_2 \cap b_2 \cap c_1) &= 0.05 \\ P(a_2 \cap b_2 \cap c_2) &= 0.075 \end{aligned}$$

and thus

$$\begin{aligned} P(b_2 \cap c_1) &= 0.32 \\ P(b_2 \cap c_2) &= 0.48 \end{aligned} \qquad \begin{aligned} P(c_1) &= 0.52 \\ P(c_2) &= 0.48 \end{aligned} \qquad \begin{aligned} P(a_1 \cap b_1 \cap c_1) &= 0.075 \\ P(a_1 \cap b_1 \cap c_2) &= 0 \\ P(a_2 \cap b_1 \cap c_1) &= 0.125 \\ P(a_2 \cap b_1 \cap c_2) &= 0 \end{aligned}$$

and so

$$\begin{aligned} P(c_1|b_2) &= 0.4 \\ P(c_2|b_2) &= 0.6 \end{aligned} \qquad \begin{aligned} P(a_1 \cap c_1) &= 0.345 \\ P(a_1 \cap c_2) &= 0.405 \\ P(a_2 \cap c_1) &= 0.175 \\ P(a_2 \cap c_2) &= 0.075 \end{aligned} \qquad \begin{aligned} P(c_1|a_1) &= 0.46 \\ P(c_2|a_1) &= 0.54 \\ P(c_1|a_2) &= 0.7 \\ P(c_2|a_2) &= 0.3 \end{aligned}$$

To simplify calculations, we use the function $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$.

a. This is asking to show that $P(c_i|b_j \cap a_k) = P(c_i|b_j)$ for all $i, j, k = 1, 2$.
We could show these 8 equalities explicitly; there is however a short-cut here.
If it is assumed that the student owns a car (b_1), then that student lives at home, regardless of course performance:

$$P(c_i|b_1 \cap a_k) = P(c_i|b_1) \qquad \text{for all } i, k$$

If the student does not own a car (b_2), then that student lives at home (c_1) with 40% probability or does not (c_2) with 60% probability, regardless of course performance:

$$P(c_i|b_2 \cap a_k) = P(c_i|b_2) \qquad \text{for all } i, k$$

b. Here, we are meant to calculate $I(A, B)$:

$$\begin{aligned} I(A,B) &= H(B) - H(B|A) \\ &= H(0.2) - (P(a_1)H(B|a_1) + P(a_2)H(B|a_2) \\ &= H(0.2) - (0.75H(0.1) + 0.25H(0.5)) \\ &\approx 0.120 \end{aligned}$$

c. Here, we are meant to calculate $I(A, C)$:

$$\begin{aligned} I(A,C) &= H(C) - H(C|A) \\ &= H(0.52) - (P(a_1)H(C|a_1) + P(a_2)H(C|a_2) \\ &= H(0.52) - (0.75H(0.46) + 0.25H(0.7)) \\ &\approx 0.032 \end{aligned}$$

d. The information in the first digit is $H(A) = H(0.75) \approx 0.811$ bits.
The (new) information in the second digit is $H(B|A) = 0.75H(0.1) + 0.25H(0.5)) \approx 0.602$ bits.
The (new or extra) information in the third digit is $H(C|A \cap B)$.
By Part **a.**, this equals $H(C|B) = 0.2H(1) + 0.8H(0.4) \approx 0.777$.

49.

a. $P(a_j|b_i) = \dfrac{P(a_j \cap b_i)}{P(b_i)} = \dfrac{P(b_i|a_j)P(a_j)}{P(b_i)}$ where

$$\begin{aligned} P(b_1) &= P(b_1|a_1)P(a_1) + P(b_1|a_2)P(a_2) = 0.8 \times \frac{1}{3} + 0.4 \times \frac{2}{3} = \frac{1.6}{3} \\ P(b_2) &= P(b_2|a_1)P(a_1) + P(b_2|a_2)P(a_2) = 0.2 \times \frac{1}{3} + 0.6 \times \frac{2}{3} = \frac{1.4}{3} \end{aligned}$$

so

$$\begin{aligned} P(a_1|b_1) &= \frac{P(b_1|a_1)P(a_1)}{P(b_1)} = \frac{0.8 \times \frac{1}{3}}{\frac{1.6}{3}} = 0.5 \\ P(a_2|b_1) &= \frac{P(b_1|a_2)P(a_2)}{P(b_1)} = \frac{0.4 \times \frac{2}{3}}{\frac{1.6}{3}} = 0.5 \\ P(a_1|b_2) &= \frac{P(b_2|a_1)P(a_1)}{P(b_2)} = \frac{0.2 \times \frac{1}{3}}{\frac{1.4}{3}} \approx \frac{1}{7} \\ P(a_2|b_2) &= \frac{P(b_2|a_2)P(a_2)}{P(b_2)} = \frac{0.6 \times \frac{2}{3}}{\frac{1.4}{3}} \approx \frac{6}{7} \end{aligned}$$

b.

$$\begin{aligned} I(A,B) &= H(B) - H(B|A) \\ &= H(B) - (P(a_1)H(B|a_1) + P(a_2)H(B|a_2) \\ &= H\Big(\frac{1.6}{3}\Big) - \Big(\frac{1}{3}H(0.8) + \frac{2}{3}H(0.6)\Big) \\ &\approx 0.109 \end{aligned}$$

50. Let $P(a_1) = x$ and $P(a_2) = 1 - x$. Then

$$\begin{aligned}
P(b_1) &= P(b_1|a_1)P(a_1) + P(b_1|a_2)P(a_2) = (1-q)x + 0(1-x) = (1-q)x \\
P(b_2) &= P(b_2|a_1)P(a_1) + P(b_2|a_2)P(a_2) = 0x + (1-q)(1-x) = (1-q)(1-x) \\
P(b_3) &= P(b_3|a_1)P(a_1) + P(b_3|a_2)P(a_2) = qx + q(1-x) = q
\end{aligned}$$

so, writing $p = 1 - q$,

$$\begin{aligned}
H(B) &= -P(b_1)\log_2 P(b_1) - P(b_2)\log_2 P(b_2) - P(b_3)\log_2 P(b_3) \\
&= -px\log_2 px - p(1-x)\log_2 p(1-x) - q\log_2 q \\
&= -px\log_2 x - p(1-x)\log_2 (1-x) - p\log_2 p - q\log_2 q \\
&= pH(x) + H(q)
\end{aligned}$$

$$\begin{aligned}
H(B|a_1) &= -P(b_1|a_1)\log_2 P(b_1|a_1) - P(b_2|a_1)\log_2 P(b_2|a_1) - P(b_3|a_1)\log_2 P(b_3|a_1) \\
&= -p\log_2 p - 0\log_2 0 - q\log_2 q \\
&= H(q)
\end{aligned}$$

$$\begin{aligned}
P(B|a_2) &= -P(b_1|a_2)\log_2 P(b_1|a_2) - P(b_2|a_2)\log_2 P(b_2|a_2) - P(b_3|a_2)\log_2 P(b_3|a_2) \\
&= -0\log_2 0 - p\log_2 p - q\log_2 q \\
&= H(q) \\
H(B|A) &= H(B|a_1)P(a_1) + H(B|a_2)P(a_2) \\
&= H(q)x + H(q)(1-x) \\
&= H(q)
\end{aligned}$$

Therefore,

$$I(A,B) = H(B) - H(B|A) = pH(x) + H(q) - H(q) = pH(x)$$

Then $\dfrac{d}{dx}I(A,B) = p\dfrac{d}{dx}H(x) = \log_2\Big(\dfrac{1-x}{x}\Big)$.
Solving $\dfrac{d}{dx}I(A,B) = 0$, we get $x = \frac{1}{2}$, and so

$$C(A,B) = \max_x I(A,B) = pH(\frac{1}{2}) = p = 1 - q$$

51. We can first calculate, regardless of the probabilities $P(a_j)$:

$$H(B|a_j) = \sum_{i=1}^{3}(-P(b_i|a_j)\log_2 P(b_i|a_j)) = -\frac{1}{2}\log_2\frac{1}{2} + 2\Big(-\frac{1}{4}\log_2\frac{1}{4}\Big) = \frac{1}{2} + \frac{1}{2}\log_2 4 = 1.5$$

$$H(B|A) = \sum_{j=1}^{3} H(B|a_j)P(a_j) = \frac{3}{2}\times\sum_{j=1}^{3}P(a_j) = 1.5$$

a.

$$\begin{aligned}
H(A) &= 3\Big(-\frac{1}{3}\log_2\frac{1}{3}\Big) = \log_2 3 \\
P(b_i) &= \sum_{j=1}^{3} P(b_i|a_j)P(a_j) = \frac{1}{2}\times\frac{1}{3} + 2\frac{1}{4}\times\frac{1}{3} = \frac{1}{3} = P(a_j) \quad \text{for all } i,j \\
H(B) &= H(A) = \log_2 3
\end{aligned}$$

Hence,

$$\begin{aligned} I(A,B) &= H(B) - H(B|A) = \log_2 3 - \frac{3}{2} \approx 0.085 \\ H(A|B) &= H(A) - I(A,B) = \frac{3}{2} = 1.5 \\ H(A,B) &= H(A) + H(B|A) = \log_2 3 + \frac{3}{2} \approx 3.085 \end{aligned}$$

b.

$$\begin{aligned} H(A) &= -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{3}\log_2\frac{1}{3} - \frac{1}{6}\log_2\frac{1}{6} \approx 1.459 \\ P(b_1) &= \sum_{j=1}^{3} P(b_1|a_j)P(a_j) = \frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{3} + \frac{1}{4}\times\frac{1}{6} = \frac{3}{8} \\ P(b_2) &= \sum_{j=1}^{3} P(b_2|a_j)P(a_j) = \frac{1}{4}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{3} + \frac{1}{4}\times\frac{1}{6} = \frac{1}{3} \\ P(b_3) &= \sum_{j=1}^{3} P(b_3|a_j)P(a_j) = \frac{1}{4}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{6} = \frac{7}{24} \\ H(B) &= -\frac{3}{8}\log_2\frac{3}{8} - \frac{1}{3}\log_2\frac{1}{3} - \frac{7}{24}\log_2\frac{7}{24} \approx 1.577 \end{aligned}$$

Hence,

$$\begin{aligned} I(A,B) &= H(B) - H(B|A) \approx 1.577 - 1.5 = 0.077 \\ H(A|B) &= H(A) - I(A,B) \approx 1.459 - 0.077 = 1.382 \\ H(A,B) &= H(A) + H(B|A) \approx 1.459 + 1.5 = 2.959 \end{aligned}$$

c. Your guess is your guess :)

d. Write $P(a_1) = x_1$ and $P(a_2) = x_2$.

$$\begin{aligned} P(b_1) &= \sum_{j=1}^{3} P(b_1|a_j)P(a_j) = \frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{4}(1 - x_1 - x_2) = \frac{1}{4}(1 + x_1) \\ P(b_2) &= \sum_{j=1}^{3} P(b_2|a_j)P(a_j) = \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}(1 - x_1 - x_2) = \frac{1}{4}(1 + x_2) \\ P(b_3) &= \sum_{j=1}^{3} P(b_3|a_j)P(a_j) = \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{2}(1 - x_1 - x_2) = \frac{1}{4}(2 - x_1 - x_2) \end{aligned}$$

$$\begin{aligned} H(B) &= -\frac{1}{4}(1+x_1)\log_2\frac{1}{4}(1+x_1) - \frac{1}{4}(1+x_2)\log_2\frac{1}{4}(1+x_2) - \frac{1}{4}(2-x_1-x_2)\log_2\frac{1}{4}(2-x_1-x_2) \\ &= I\Big(\frac{1+x_1}{4}\Big) + I\Big(\frac{1+x_2}{4}\Big) + I\Big(\frac{2-x_1-x_2}{4}\Big) \end{aligned}$$

where $I(x) = -x\log_2 x$. Hence,

$$I(A,B) = H(B) - H(B|A) = I\Big(\frac{1+x_1}{4}\Big) + I\Big(\frac{1+x_2}{4}\Big) + I\Big(\frac{2-x_1-x_2}{4}\Big) - \frac{3}{2}$$

To find the maximum of this function, we solve the equations $\frac{d}{dx_1}I(A,B) = 0$ and $\frac{d}{dx_y}I(A,B) = 0$. Let us first solve the first equation:

$$0 = \frac{d}{dx_1}I(A,B) = \frac{1}{4}\log_2\Big(\frac{2-x_1-x_2}{1+x_1}\Big)$$

$$\text{so} \quad 2 - x_1 - x_2 = 1 + x_1$$

and so $x_1 = \frac{1}{2}(1-x_2)$. Since $I(A,B)$ is symmetric with respect to x_1 and x_2, the second equation will imply that $x_2 = \frac{1}{2}(1-x_1)$, so $x_1 = \frac{1}{2}(1-\frac{1}{2}(1-x_1)) = \frac{1}{4}+\frac{1}{4}x_1$ and so $x_1 = \frac{1}{3}$. Hence, $x_2 = 1 - x_1 - x_2 = \frac{1}{3}$, so we find that $I(A,B)$ is maximal for the probabilities $P(a_j) = \frac{1}{3}$ from part **a.**, and that the capacity is

$$C(A,B) = \max_{x_1,x_2} I(A,B) = H(B) - H(B|A) = \log_2 3 - \frac{3}{2} \approx 0.085$$

52.

$$P(a_j|b_i) = \frac{P(a_j \cap b_i)}{P(b_i)} = \frac{P(b_i|a_j)P(a_j)}{P(b_i)}$$

where

$$\begin{aligned}
P(b_1) &= P(b_1|a_1)P(a_1) + P(b_1|a_2)P(a_2) = \frac{5}{7}x \\
P(b_2) &= P(b_2|a_1)P(a_1) + P(b_2|a_2)P(a_2) = \frac{2}{7}x \\
P(b_3) &= P(b_3|a_1)P(a_1) + P(b_3|a_2)P(a_2) = \frac{1}{10}(1-x) \\
P(b_4) &= P(b_4|a_1)P(a_1) + P(b_4|a_2)P(a_2) = \frac{9}{10}(1-x)
\end{aligned}$$

so

$$\begin{aligned}
P(a_1|b_1) &= \frac{P(b_1|a_1)P(a_1)}{P(b_1)} = 1 \\
P(a_1|b_2) &= \frac{P(b_2|a_1)P(a_1)}{P(b_2)} = 1 \\
P(a_1|b_3) &= \frac{P(b_3|a_1)P(a_1)}{P(b_3)} = 0 \\
P(a_1|b_4) &= \frac{P(b_4|a_1)P(a_1)}{P(b_4)} = 0 \\
P(a_2|b_1) &= \frac{P(b_1|a_2)P(a_2)}{P(b_1)} = 0 \\
P(a_2|b_2) &= \frac{P(b_2|a_2)P(a_2)}{P(b_2)} = 0 \\
P(a_2|b_3) &= \frac{P(b_3|a_2)P(a_2)}{P(b_3)} = 1 \\
P(a_2|b_4) &= \frac{P(b_4|a_2)P(a_2)}{P(b_4)} = 1
\end{aligned}$$

a. By the above, $H(a_j|b_i) = 0$ for all i,j, so $H(A|B) = 0$.
This reflects that the elements of B imply which elements of A are given; in other words, there is no

information in A that is not determined by B.
In particular, if b_1 or b_2 are received, then a_1 has been sent; otherwise if b_3 or b_4 are received, then a_2 has been sent.

b. $I(A,B) = H(A) - H(A|B) = H(A) = H(x)$, which has maximum

$$C(A,B) = \max_x I(A,B) = H(\frac{1}{2}) = 1$$

53. The block code $\mathbb{Z}_2^{15}$ contains $2^{15} = 32768$ which is enough to encode the students, for instance by a binary decision tree.

54.

a) First use the Euclidean Algorithm forwards:

$$\begin{aligned} 3876 &= 11 \times 324 + 312 \\ 324 &= 1 \times 312 + 12 \\ 312 &= 26 \times 12 + 0\,, \end{aligned}$$

so $d = \gcd(312, 3876) = 12$. Now use the Euclidean Algorithm backwards:

$$\begin{aligned} 12 &= 324 - 312 \\ &= 324 - (3876 - 11 \times 324) \\ &= 12 \times 324 - 3876 \end{aligned}$$

Hence, $12 = \gcd(324, 3876) = 324x + 3876$ for $x = 12$ and $y = -1$.

b) First use the Euclidean algorithm forwards:

$$\begin{aligned} 7412 &= 4 \times 1513 + 1360 \\ 1513 &= 1 \times 1360 + 153 \\ 1360 &= 8 \times 153 + 136 \\ 153 &= 1 \times 136 + 17 \\ 136 &= 8 \times 17 + 0\,, \end{aligned}$$

so $d = \gcd(7412, 1513) = 17$. Now use the Euclidean algorithm backwards:

$$\begin{aligned} 17 &= 153 - 136 \\ &= 153 - (1360 - 8 \times 153) \\ &= 9 \times 153 - 1360 \\ &= 9 \times (1513 - 1360) - 1360 \\ &= 9 \times 1513 - 10 \times 1360 \\ &= 9 \times 1513 - 10 \times (7412 - 4 \times 1513) \\ &= 49 \times 1513 - 10 \times 7412 \end{aligned}$$

Hence, $d = \gcd(7412, 1513) = 7412x + 1513y$ for $x = -10$ and $y = 49$.

c) First use the Euclidean algorithm forwards:

$$\begin{aligned}
2187 &= 2 \times 1024 + 139 \\
1024 &= 7 \times 139 + 51 \\
139 &= 2 \times 51 + 37 \\
51 &= 1 \times 37 + 14 \\
37 &= 2 \times 14 + 9 \\
14 &= 1 \times 9 + 5 \\
9 &= 1 \times 5 + 4 \\
5 &= 1 \times 4 + 1
\end{aligned}$$

so $\gcd(1024, 2187) = 1$. Now use the Euclidean algorithm backwards:

$$\begin{aligned}
1 &= 5 - 4 \\
&= 5 - (9 - 5) \\
&= 2 \times 5 - 9 \\
&= 2 \times (14 - 9) - 9 \\
&= 2 \times 14 - 3 \times 9 \\
&= 2 \times 14 - 3 \times (37 - 2 \times 14) \\
&= 8 \times 14 - 3 \times 37 \\
&= 8 \times (51 - 37) - 3 \times 37 \\
&= 8 \times 51 - 11 \times 37 \\
&= 8 \times 51 - 11 \times (139 - 2 \times 51) \\
&= 30 \times 51 - 11 \times 139 \\
&= 30 \times (1024 - 7 \times 139) - 11 \times 139 \\
&= 30 \times 1024 - 221 \times 139 \\
&= 30 \times 1024 - 221 \times (2187 - 2 \times 1024) \\
&= 472 \times 1024 - 221 \times 2187
\end{aligned}$$

Hence, $1 = \gcd(1024, 2187) = 1024x + 2187y$ for $x = 472$ and $y = -221$.

55.

$$\begin{array}{lllll}
\mathbb{U}_{24} & = \{1,5,7,11,13,17,19,23\} & & \varphi(24) & = |\mathbb{U}_{24}| = 8 \\
\mathbb{U}_{36} & = \{1,5,7,11,13,17,19,23,25,29,31,35\} & \text{so} & \varphi(36) & = |\mathbb{U}_{24}| = 12 \\
\mathbb{U}_{17} & = \{1, \dots 16\} & & \varphi(17) & = |\mathbb{U}_{17}| = 16
\end{array}$$

56.

$$\begin{array}{rllrl}
72 & = 2^3 \cdot 3^2 & & \phi(72) & = \varphi(2^3)\varphi(3^2) = (2^3 - 2^2)(3^2 - 3^1) = 4 \times 6 = 24 \\
1224 & = 2^3 \cdot 3^2 \cdot 17 = 72 \cdot 17 & \text{so} & \phi(1224) & = \varphi(72)\varphi(17) = 24 \times 16 = 384 \\
561561 & = 3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 & & \phi(561561) & = \varphi(3)\varphi(7)\varphi(11^2)\varphi(13)\varphi(17) \\
& & & & = 2 \cdot 6 \cdot (11^2 - 11^1) \cdot 12 \cdot 16 = 253440
\end{array}$$

57.

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	3	2
5	0	5	4	3	2	1

Every non-zero element in $\mathbb{Z}_5$ has an inverse and is therefore a unit,
in contrast to the elements 3, 4, and 5 in $\mathbb{Z}_6$.
Therefore, $\mathbb{Z}_5$ is a field and $\mathbb{Z}_6$ is not.

58.

a)
$$\begin{aligned} & 6x \equiv 7 \pmod{17} \\ \Leftrightarrow\quad & 6x \equiv 24 \pmod{17} \\ \Leftrightarrow\quad & x \equiv 4 \pmod{17} \quad (\text{since } \gcd(6,17)=1) \end{aligned}$$

b)
$$\begin{aligned} & 6x \equiv 8 \pmod{11} \\ \Leftrightarrow\quad & 6x \equiv 30 \pmod{11} \\ \Leftrightarrow\quad & x \equiv 5 \pmod{11} \quad (\text{since } \gcd(6,11)=1) \end{aligned}$$

c)
$$\begin{aligned} & 6x \equiv 9 \pmod{13} \\ \Leftrightarrow\quad & 2x \equiv 3 \pmod{13} \quad (\text{since } \gcd(3,13)=1) \\ \Leftrightarrow\quad & 2x \equiv 16 \pmod{13} \\ \Leftrightarrow\quad & x \equiv 8 \pmod{13} \quad (\text{since } \gcd(2,13)=1) \end{aligned}$$

59.

a) In $\mathbb{Z}_{11}$, $6 \times 2 = 12 = 1$, so $6^{-1} = 2$.

b) $\gcd(6, 10) = 2 \neq 1$, so 6 has no inverse in $\mathbb{Z}_{10}$.

c) In $\mathbb{Z}_{23}$, $6 \times 4 = 24 = 1$, so $6^{-1} = 4$.