Homework 6

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Question 1

a)

Variables: $\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ w_0 \end{bmatrix}$

Where w_0 is some constant and w_j for $1 \leq j \leq n$ is the weight indicating the amount that G_j contributes to G_*

Inputs:

$$\mathbf{X} = \begin{bmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,n} & 1 \\ G_{2,1} & G_{2,2} & \dots & G_{2,n} & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ G_{m,1} & G_{m,2} & \dots & G_{m,n} & 1 \end{bmatrix}$$

where $G_{i,j} = G_j^{(i)}$ which is the steady state expression of gene j during in the i'th condition.

$$\mathbf{Y} = \begin{bmatrix} G_*^{(1)} \\ G_*^{(2)} \\ \vdots \\ G_*^{(m)} \end{bmatrix}$$

Where $G_*^{(i)}$ is the observed expression of G_* in the i'th condition.

Objective function:

$$\underset{\mathbf{W}}{argmin} \quad \|\mathbf{XW} - \mathbf{Y}\|_2^2$$

b)

Variables: $\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ w_0 \end{bmatrix}$

Where w_0 is some constant and w_j for $1 \leq j \leq n$ is the weight indicating the amount that G_j contributes to $\frac{d}{dt}G_*$

Inputs: $\mathbf{X} = \begin{bmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,n} & 1 \\ G_{2,1} & G_{2,2} & \dots & G_{2,n} & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ G_{m,1} & G_{m,2} & \dots & G_{m,n} & 1 \end{bmatrix}$

where $G_{i,j} = G_j^{(i)}$ which is the steady state expression of gene j during i'th time point.

$$\mathbf{Y} = \begin{bmatrix} G_*^{(1)} \\ G_*^{(2)} \\ \vdots \\ G_*^{(m)} \\ G_*^{(m+1)} \end{bmatrix}$$

Where $G_*^{(i)}$ is the observed expression of G_* at the i'th time point.

Model Equations:

$$\frac{d}{dt} G_*^{(i)} = w_0 + w_1 G_{i,1} + w_1 G_{i,1} + ... + w_n G_{i,n}$$

$$\begin{aligned} \frac{d}{dt} \mathbf{Y}_{1:m} &= \begin{bmatrix} \frac{d}{dt} G_*^{(1)} \\ \frac{d}{dt} G_*^{(2)} \\ \vdots \\ \frac{d}{dt} G_*^{(m)} \end{bmatrix} \\ &= X\mathbf{W} \end{aligned}$$

Objective Function:

$$\underset{\mathbf{W}}{argmin} \quad \|\hat{\mathbf{Y}}_{2:m+1} - \mathbf{Y}_{2:m+1}\|_2$$

$$\underset{\mathbf{W}}{argmin} \quad \|(\mathbf{Y}_{1:m} + \Delta t \mathbf{X}\mathbf{W}) - \mathbf{Y}_{2:m+1}\|_2^2$$

Where Δt is the time difference for each time step. If Δt is not consistent between all time points then the objective has to be rewritten as:

$$\underset{\mathbf{W}}{argmin} \quad \sum_i^m (\mathbf{Y}_i + \Delta t_{i,i+1} \mathbf{X}_{i,:} \mathbf{W} - \mathbf{Y}_{i+1})^2$$

Where $\Delta t_{i,i+1}$ is the time difference between time point i and time point $i+1$.

Question 2

$$L(w) = w_1^2 + 2w_2^2 + w_3^2 - 2w_3w_4 + w_4^2 + 2w_1 - 4w_2 + 4$$

a)

$$\nabla L(w) = [2w_1 + 2, 4w_2 - 4, 2w_3 - 2w_4, -2w_3 + 2w_4]$$

b)

$$\nabla L(w) = [-\eta(2w_1 + 2), -\eta(4w_2 - 4), -\eta(2w_3 - 2w_4), -\eta(-2w_3 + 2w_4)]$$

c)

$$\begin{aligned} 2w_1 + 2 &= 0 \quad & 4w_2 - 4 &= 0 \\ w_1 &= -1 \quad & w_2 &= 1 \end{aligned} \qquad\qquad \begin{aligned} 2w_3 - 2w_4 & \quad & -2w_3 + 2w_4 \\ w_3 = w_4 & \quad & w_3 = w_4 \end{aligned}$$

Since the only restriction on w_3 and w_4 is that they are equal we can choose any value for them. Thus to simplify the vector we will let them equal 0.

$$\begin{aligned} 2w_1 + 2 &= 0 \quad & 4w_2 - 4 &= 0 \\ w_1 &= -1 \quad & w_2 &= 1 \end{aligned} \qquad\qquad \begin{aligned} 2w_3 - 2w_4 & \quad & -2w_3 + 2w_4 \\ w_3 = 0 & \quad & w_4 = 0 \end{aligned}$$

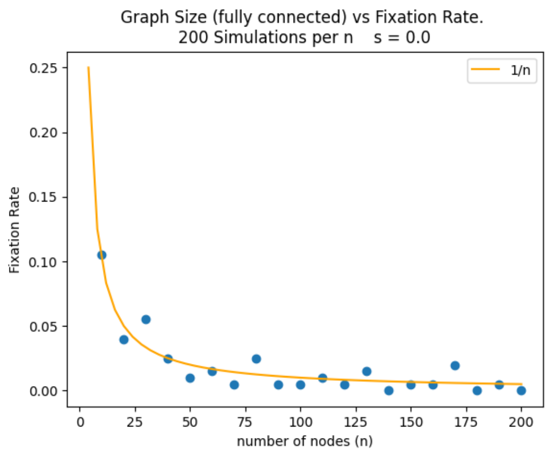
d)

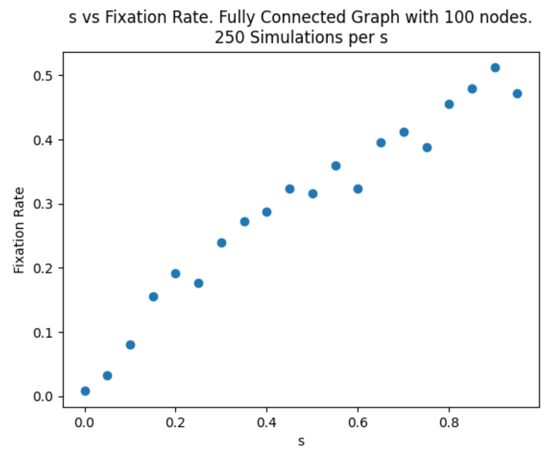
No there is more than one unique solution because w_3 and w_4 just have to be equal. Otherwise they can be any value.

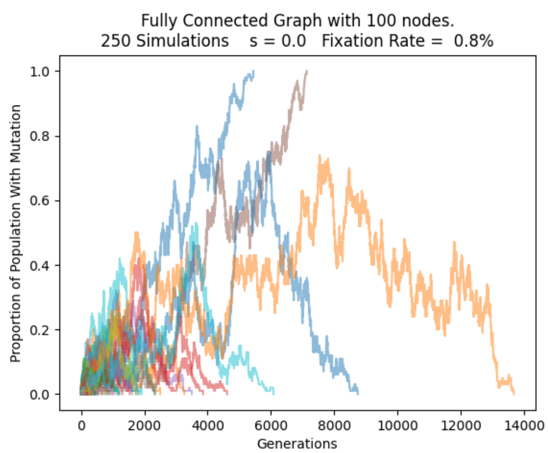
Question 3

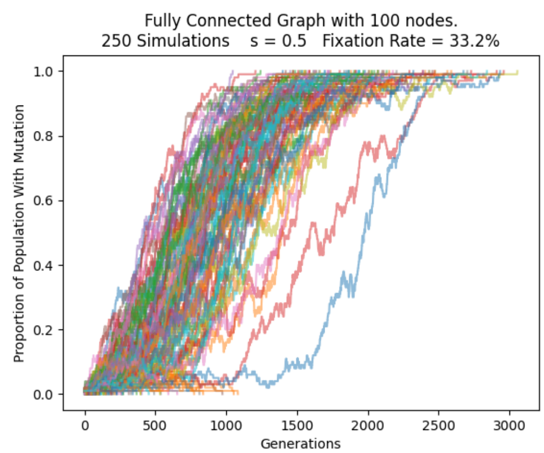
The probability of fixation is roughly the same when the graph is fully connected for the birth-death process and the death-birth process.

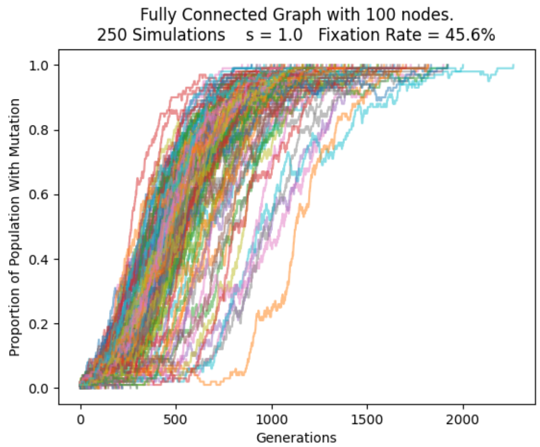
It appears that if the graph isn't fully connected then the chance of fixation decreases depending on s. When s is 0, there doesn't seem to be much correlation between fixation rate and the number of edges each node has. When s is larger the connectivity of the graph seems to have slightly more impact. When s is large, graphs with lower connectivity have slightly lower chance of reaching fixation.

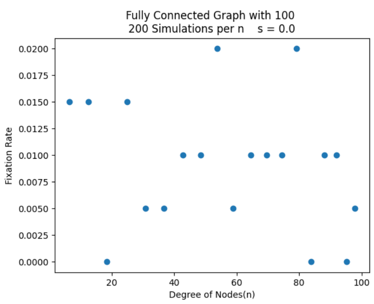


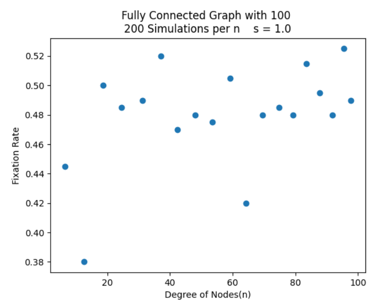


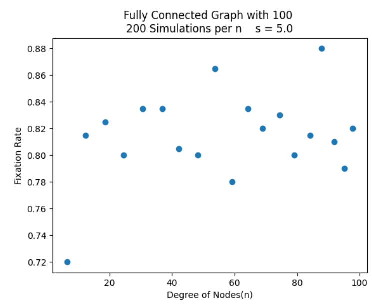


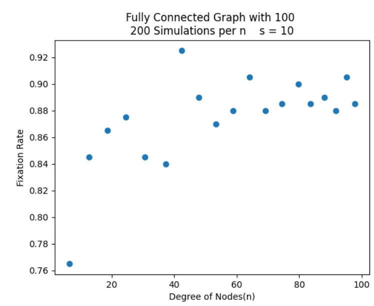


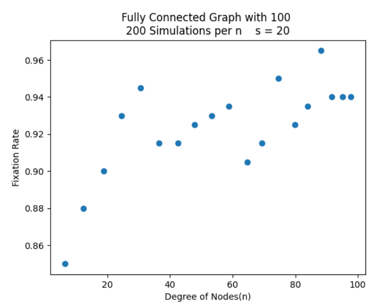

Fully Connected Graph with 100 nodes.
250 Simulations s = 1.0 Fixation Rate = 45.6%
Proportion of Population With Mutation
Generations

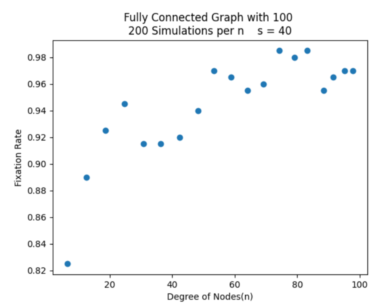

Fully Connected Graph with 100
200 Simulations per n s = 0.0
Fixation Rate
Degree of Nodes(n)


Fully Connected Graph with 100
200 Simulations per n s = 1.0
Fixation Rate
Degree of Nodes(n)


Fully Connected Graph with 100
200 Simulations per n s = 5.0
Fixation Rate
Degree of Nodes(n)

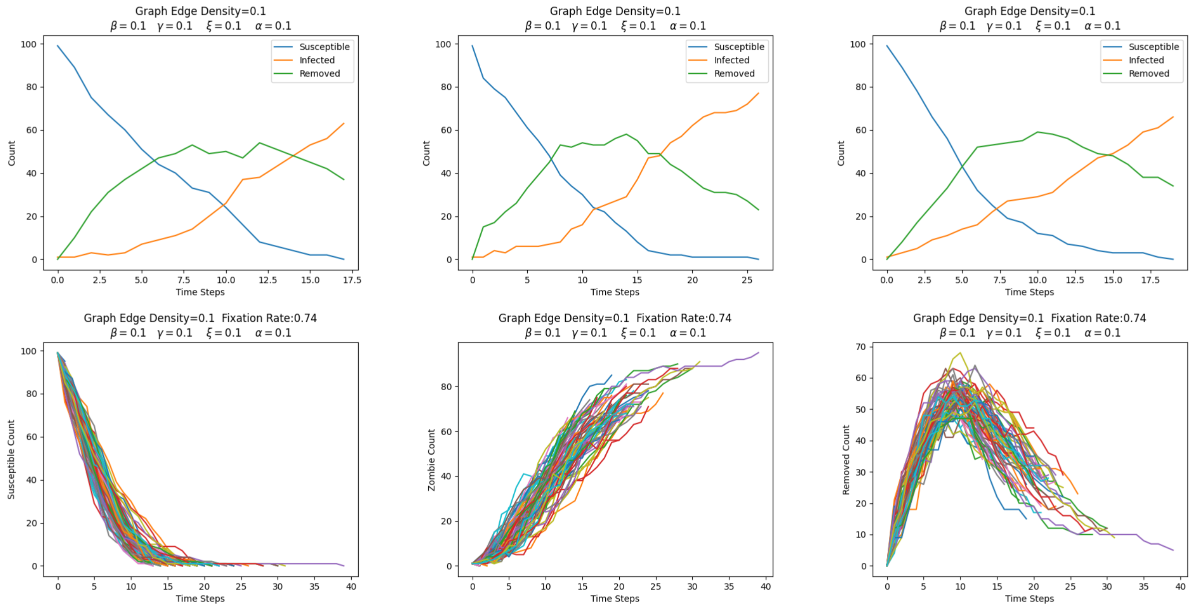

Fully Connected Graph with 100
200 Simulations per n s = 10
Fixation Rate
Degree of Nodes(n)


Fully Connected Graph with 100
200 Simulations per n s = 20
Fixation Rate
Degree of Nodes(n)


Fully Connected Graph with 100
200 Simulations per n s = 40
Fixation Rate
Degree of Nodes(n)

Question 4

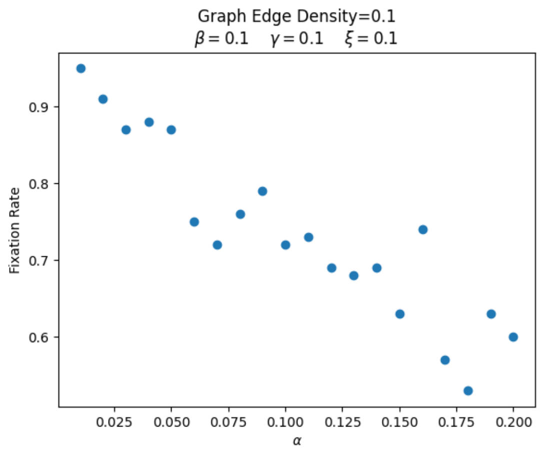
Given all of the data below. It appears that the probability of zombies growing exponentially at the start is most affected by β and the edge density of the graph. When these two values are high relative to α then the probability of short term exponential growth is higher.

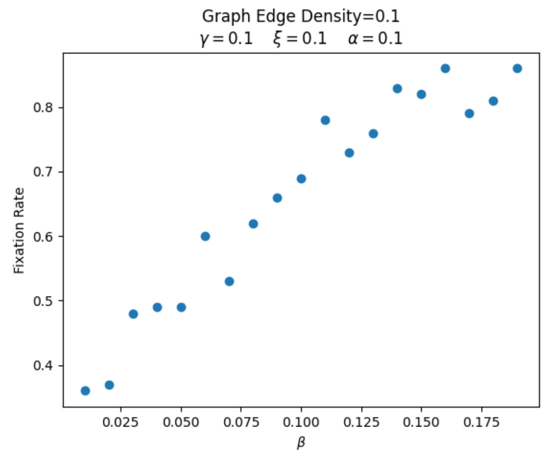


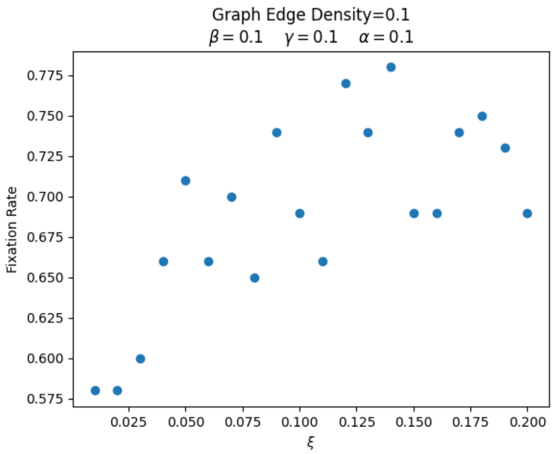
Given:
$\alpha =$ Rate at which susceptible individuals defeat zombies
$\beta =$ Rate at which susceptible individuals are infected when they come into contact with a zombie.
$\gamma =$Rate at which susceptible individuals die due to non-zombie related reasons.
$\xi =$Rate at which the removed are resurrected into zombies.

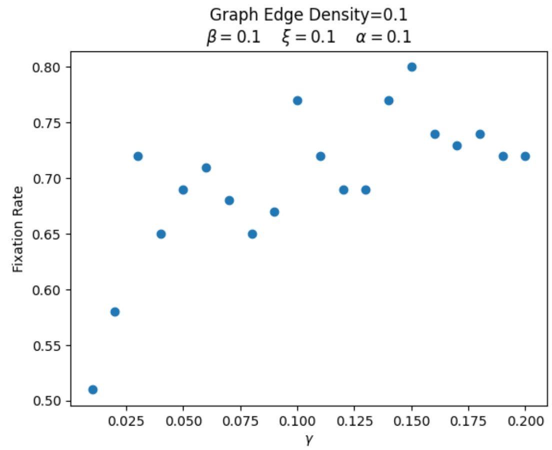
Fixation rate (the chance that zombies infect all susceptible) decreases when α increases. Fixation rate increases when β, γ, ξ or the edge density of the graph increases.

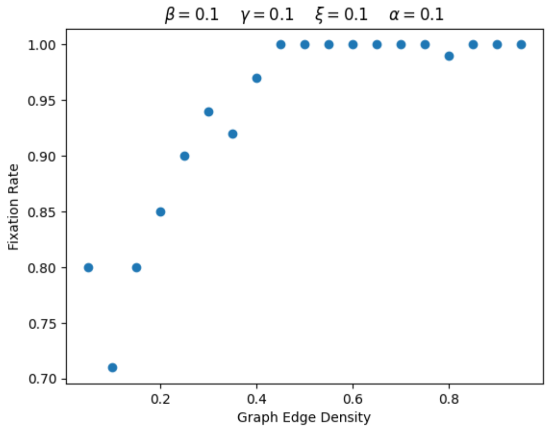
If exponential growth of zombies is the goal, then having small α, large β, γ, ξ and high edge density of the graph is ideal.






Graph Edge Density=0.1
$\beta = 0.1$ $\gamma = 0.1$ $\alpha = 0.1$
Fixation Rate
ξ


Graph Edge Density=0.1
$\beta = 0.1$ $\xi = 0.1$ $\alpha = 0.1$
Fixation Rate
γ


$\beta = 0.1$ $\gamma = 0.1$ $\xi = 0.1$ $\alpha = 0.1$
Fixation Rate
Graph Edge Density