Homework 4

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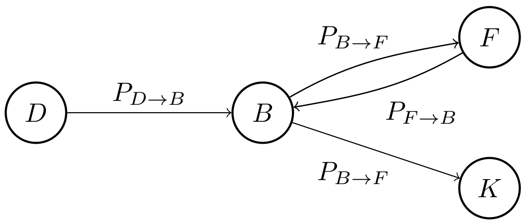
a)

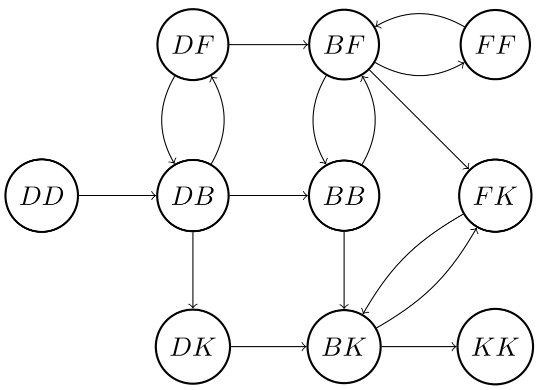


States: $Q = D, B, F, K$.
Transition Probabilities:

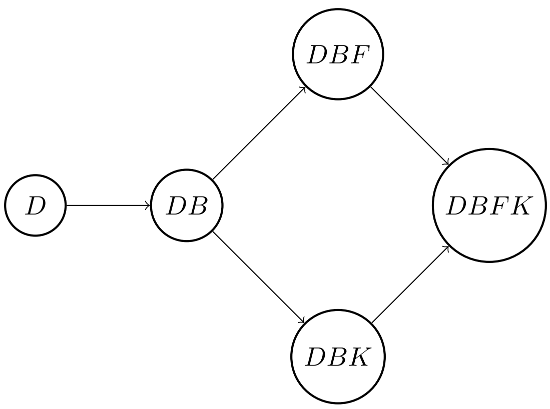
	D	B	F	K
D	$P_{D\to D}$	$P_{D\to B}$	0	0
B	0	$P_{B\to B}$	$P_{B\to F}$	$P_{B\to K}$
F	0	$P_{F\to B}$	$P_{F\to F}$	0
K	0	0	0	1

Table 1: Caption

b)

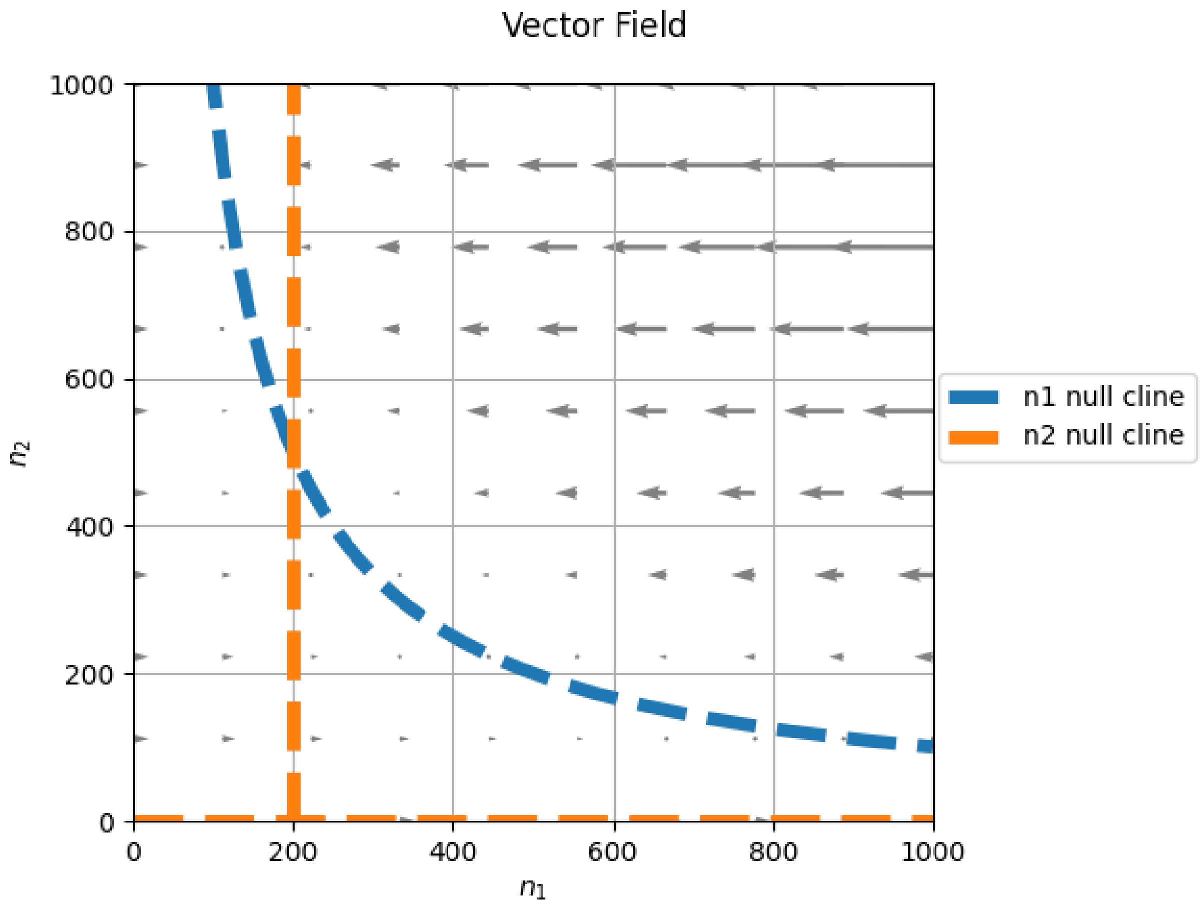


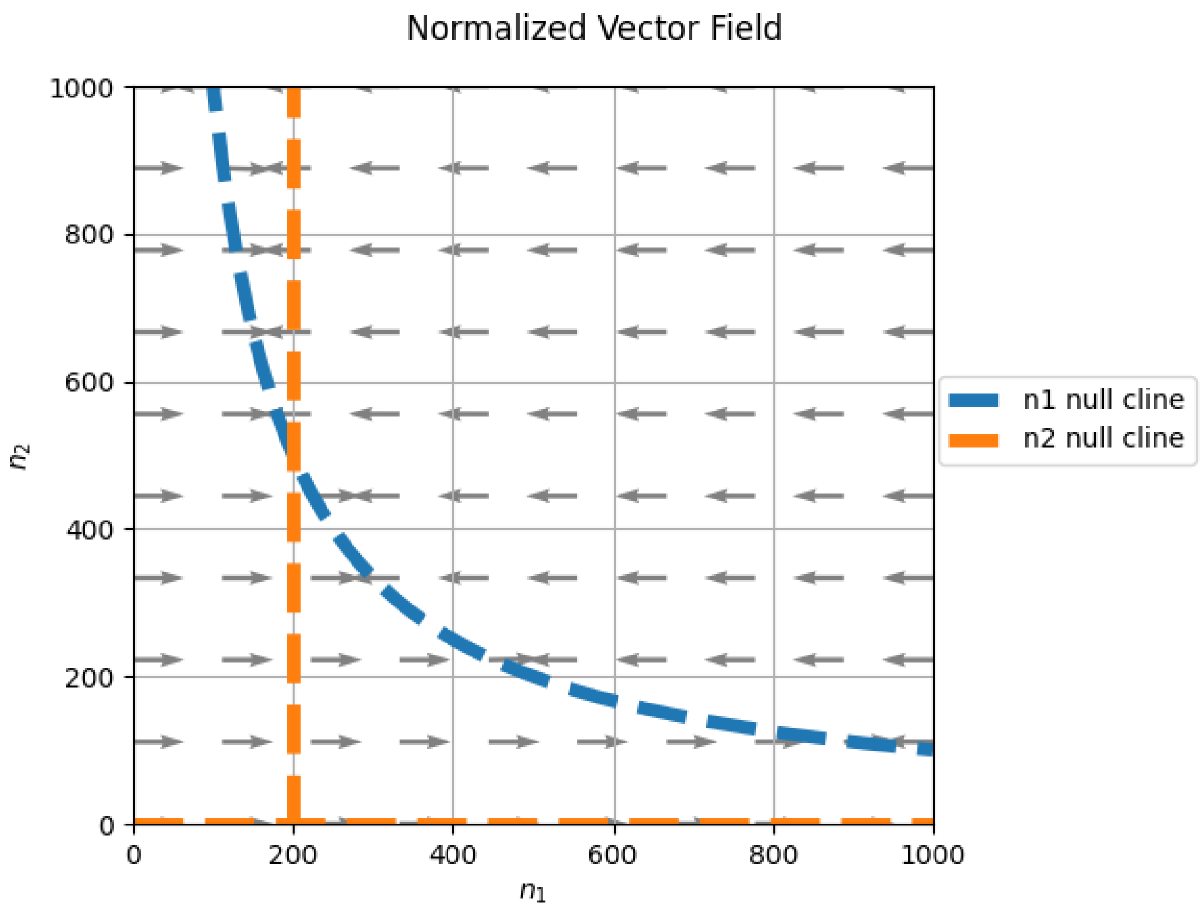
c)



2

a, b)





c) Overtime the amount of prey increases if the current prey/predator population is below the n_1 null cline and decreases if it is above the n_1 null cline. The number of predators slowly decreases if the prey/predator popluation is to the left of the n2 null cline and increases if the prey/predator population is to the right of the n2 null cline. The steady state population of prey/predator would be 200 prey, 500 predators (intersection of the null clines for n_1 and n_2).

One limitation is that the change in predator population is very small compared to the change in prey population, so it is difficult to visualize the change in predator population from the vector field alone.

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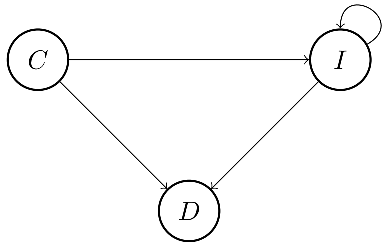
a)



$$
\begin{aligned}
C_{t+1} &= C_t + \Delta t(-\beta C - \gamma_1 C) \\
I_{t+1} &= I_t + \Delta t(\rho I + \beta C - \gamma_2 I) \\
\frac{dC}{dt} &= -\beta C - \gamma_1 C \\
\frac{dI}{dt} &= \rho I + \beta C - \gamma_2 I
\end{aligned}
$$

Null cline for C:

$$
\begin{aligned}
0 &= -\beta C - \gamma_1 C \\
0 &= C(-\beta - \gamma_1)
\end{aligned}
$$

$$
0 = C \qquad \begin{aligned} 0 &= -\beta - \gamma_1 \\ \beta &= -\gamma_1 \end{aligned}
$$

Null cline for I:

$$
\begin{aligned}
0 &= \rho I + \beta C - \gamma_2 I \\
\beta C &= \gamma_2 I - \rho I \\
C &= \frac{\gamma_2 - \rho}{\beta} I
\end{aligned}
$$

Vector Form:

$$
\begin{bmatrix} \frac{dC}{dt} \\ \frac{dI}{dt} \end{bmatrix} = \begin{bmatrix} -\beta - \gamma_1 & 0 \\ \beta & \rho - \gamma_2 \end{bmatrix} \begin{bmatrix} C \\ I \end{bmatrix} \tag{1}
$$

Finding Eigenvalues.

$$
\begin{aligned}
-\beta - \gamma_1 - \lambda &= 0 \qquad & \rho - \gamma_2 - \lambda &= 0 \\
\lambda &= -\beta - \gamma_1 \qquad & \lambda &= \rho - \gamma_2
\end{aligned}
$$

The system is stable if $\rho - \gamma_2$ is negative since both eigenvalues will be negative. If $\rho - \gamma_2$ is positive then the system will not be stable since one of the eigenvalues will be positive.

In the stable system $\rho - \gamma_2 < 0$, the stable point will be $I = 0$ and $C = 0$. C will decrease until it reaches 0. When C is 0, $\frac{dI}{dt}$ simplifies to $\frac{dI}{dt} = \rho I - \gamma_2 I$ which is negative since $\rho - \gamma_2 < 0$. Hence I will also decrease until 0. This implies the tumor won't take hold.

If $\rho - \gamma_2 = 0$ then $\frac{dI}{dt} = 0$ when $C = 0$. So the equillibrium point of I is whatever its value was when C first hits 0. This means the tumor grows up until a certain size then stays at that size.

If $\rho - \gamma_2 > 0$ then the number of cells that have invaded the organ I will grow indefinitely. And the system won't be stable. This means the tumor takes hold and continues growing.

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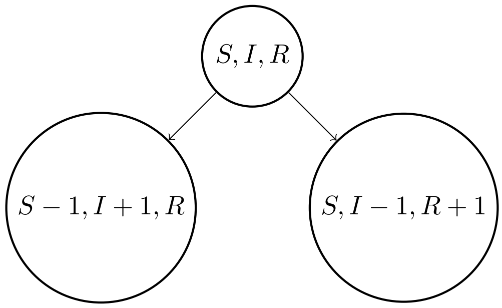
a) Markov Model for System as a whole



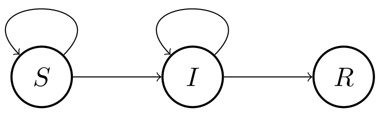
States: $Q = \{(S, I, R), (S-1, I+1, R), (S, I-1, R+1)\}$
Transitions:

$$S, I, R \rightarrow S-1, I+1, R$$
$$S, I, R \rightarrow S, I-1, R+1$$

	S, I, R	$S-1, I+1, R$	$S, I-1, R+1$
SIR	0	$\lambda_1 I \frac{S}{N}$	$\lambda_2 I$

Table 2: Transition table

Markov Model for individuals.



States: $Q = \{S, I, R\}$
Transitions:

$$S \rightarrow I$$
$$S \rightarrow S$$
$$I \rightarrow R$$
$$I \rightarrow I$$

	S	I	R
S	$1 - \lambda_1 I \frac{S}{N}$	$\lambda_1 I \frac{S}{N}$	0
I	0	$1 - \lambda_2 I$	$\lambda_2 I$

Table 3: Transition table

b)

Data: λ_1, λ_2, N, m
Result: Final S, I, R
$I \leftarrow m$;
$S \leftarrow N - m$;
while $I > 0$ *and* $I \neq N$ **do**
  $t_1 = Exp(\lambda_1 I \frac{S}{N})$;
  $t_2 = Exp(\lambda_2 I)$;
  if $t_1 < t_2$ **then**
    $I \leftarrow I + 1$;
    $S \leftarrow S - 1$;
    $t \leftarrow t + t_1$
  else
    $I \leftarrow I - 1$;
    $R \leftarrow R + 1$;
    $t \leftarrow t + t_2$;
  end
end

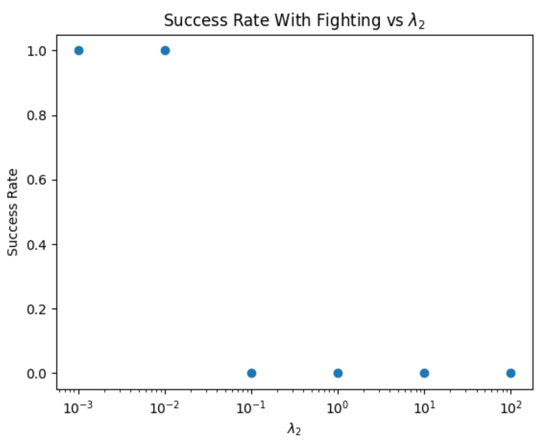
Algorithm 1: CTMM Simulation

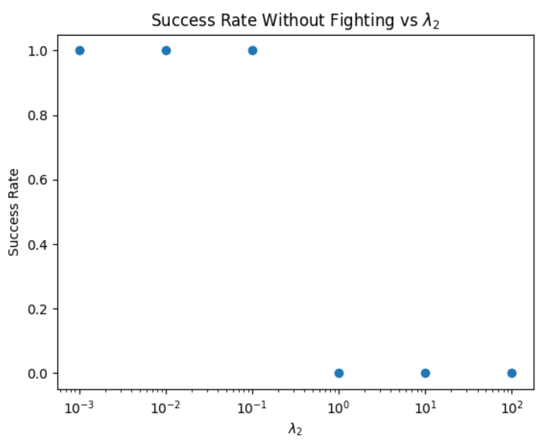
c)

Data: λ_1, λ_2, N, m
Result: Final S, I, R
$I \leftarrow m$;
$S \leftarrow N - m$;
while $I > 0$ *and* $I \neq N$ **do**
  $t_1 = Exp(\lambda_1 I \frac{S}{N})$;
  $t_2 = Exp(\lambda_2 I)$;
  $t_3 = Exp(\frac{lambda_3 IS}{S+I+R})$;
  if $min(t_1, t_2, t_3) == t_1$ **then**
    $I \leftarrow I + 1$;
    $S \leftarrow S - 1$;
    $t \leftarrow t + t_1$
  else
    $I \leftarrow I - 1$;
    $R \leftarrow R + 1$;
    $t \leftarrow t + min(t_2, t_3)$;
  end
end

Algorithm 2: CTMM Simulation with zombie death

e)





f) Fighting back is sometimes helpful. This can be seen between the two plots. When not fighting back the success rate at $\lambda_2 = 0.1$ is 0, but if the people do fight back then the success rate is 1. Fighting back helps because it increase the rate at which zombies die. However this only helps if the rate at which zombies die exceeds a certain threshold. In cases where λ_2 is very small $(0.001, 0.01)$ fighting back doesn't change the end result because the rate at which zombies die is not sufficiently increased by people fighting back.

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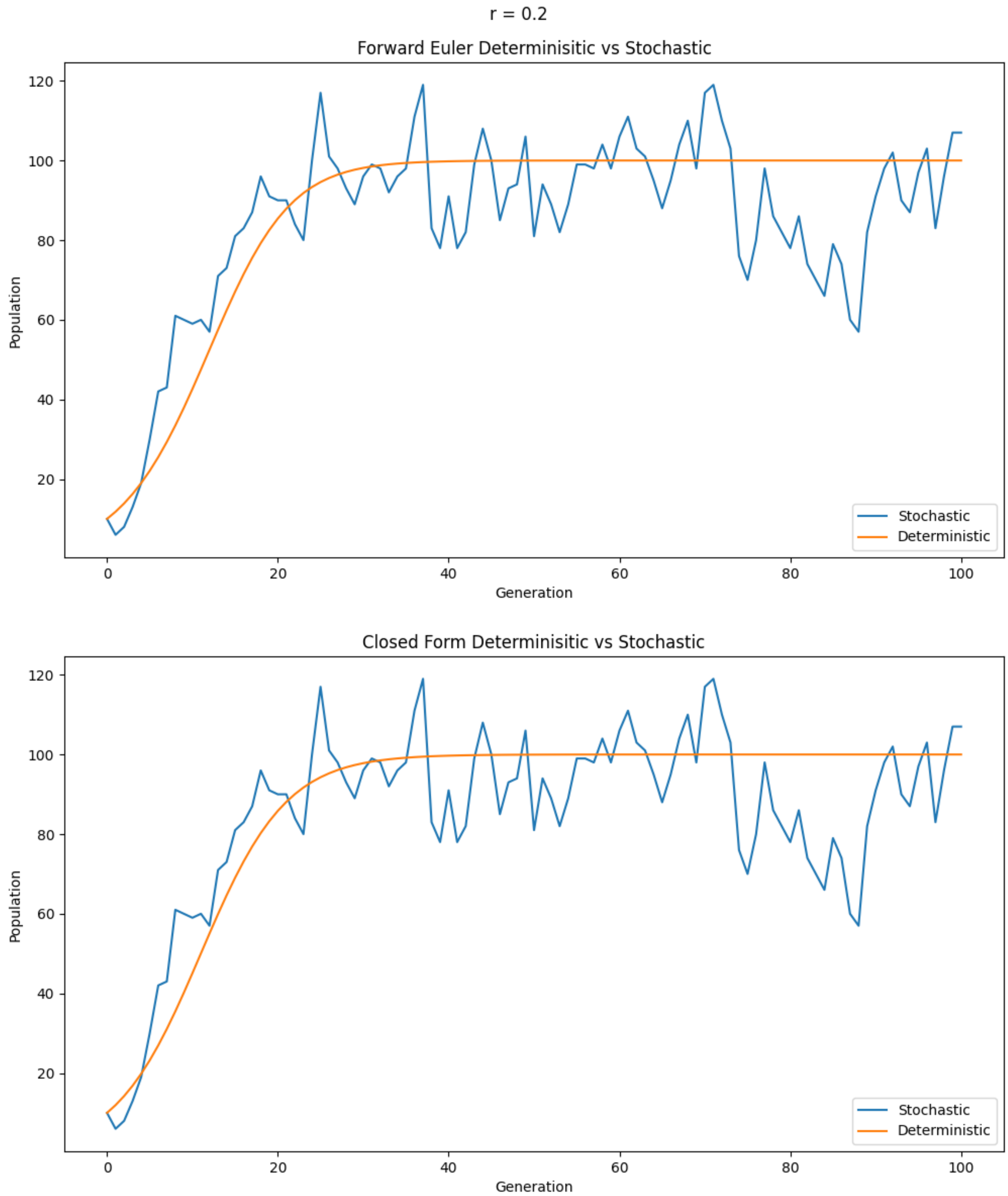
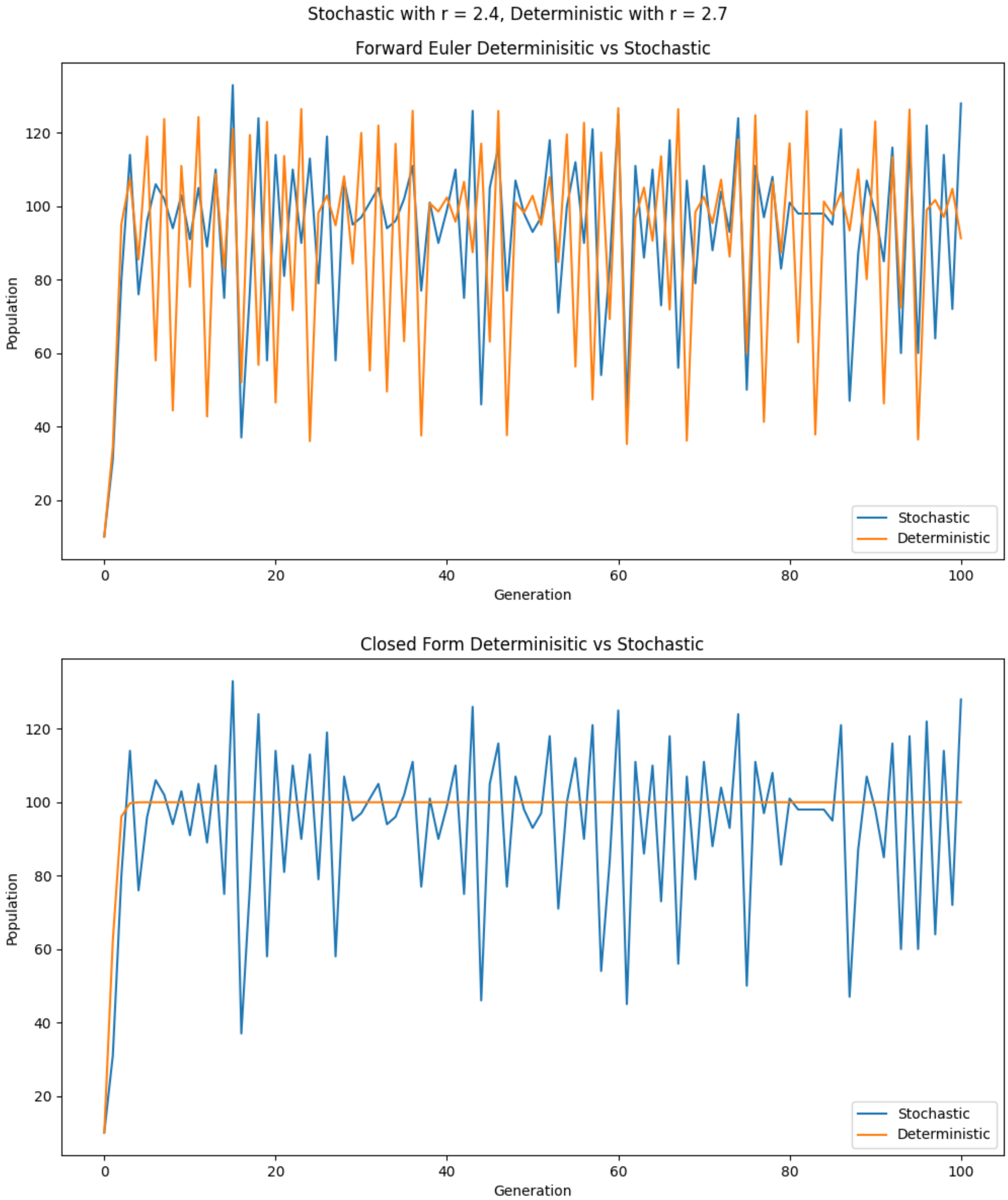


r = 0.2
Forward Euler Determinisitic vs Stochastic
Population
Generation
Stochastic
Deterministic
0
20
40
60
80
100
120
Closed Form Determinisitic vs Stochastic
Population
Generation
Stochastic
Deterministic



Looking at the plots the stochastic model looks random, while the forward euler deterministic model seems to have "controlled" chaos in that it looks chaotic but the values repeat. The closed form deterministic looks like a well behaved function.

From the plots, it appears that the closed form deterministic model can be used to get a general sense for the trend of the stochastic model, while the forward euler version of the deterministic can somewhat represent the random behavior of a stochastic model.

As r increases the stability of the stochastic and forward euler deterministic model decreases. We are able to infer that both the forward euler and closed form determinisitic model can be used to model a stochastic logistic growth processes relatively well when r is small.

However as r increases, the closed form determinisitic model represents the stochastic behavior less while the forward euler form of the determinisitic model still represents the stochastic process moderately well.