1) Let A be an $n \times n$ matrix, ~~Show~~ by definition λ

a) is an eigenvalue of A if and only if

$$A\vec{x} = \lambda\vec{x}$$ where $\vec{x}$ is a vector of $\mathbb{R}^n$ (the corresponding eigenvector)

Assuming A to be invertible

$$A^{-1}\cdot A\cdot\vec{x} = A^{-1}\cdot\lambda\vec{x}$$

$$1\cdot\vec{x} = A^{-1}\cdot\lambda\cdot\vec{x}$$ by commutativity of scalar multiplication

$$\Updownarrow$$

$$1\cdot\vec{x} = \lambda\cdot(A^{-1}\cdot\vec{x})$$ multiplying by $\frac{1}{\lambda}$ on both sides gives:

$$\Updownarrow$$

$$\frac{1}{\lambda}\cdot\vec{x} = A^{-1}\cdot\vec{x}$$ so $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

Therefore we can ~~write~~ show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} if λ is an eigenvalue of A.

~~By working backwards we get~~ If

$$A^{-1}\cdot\vec{x} = \frac{1}{\lambda}\cdot\vec{x}$$ then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Therefore we can claim that

$\lambda = \dfrac{1}{\frac{1}{\lambda}}$ is an eigenvalue of $(A^{-1})^{-1} = A$ by the lines above

and so λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Q.E.D.

1) Let A be an $n \times n$, the eigenvalues of A are the roots of characteristic polynomial $c_A(x)$ of A by theorem 3.3.2, where

$$c_A(x) = \det(xI - A) = x^n + ax^{n-1} + bx^{n-1} + \dots yx + z$$

This polynomial can be rewritten by the Fundamental Theorem of Algebra as:

$$c_A(x) = \prod_{i=1}^{n} (x - k_i)$$

So by the zero product property the eigenvalues of A are the values of k_i for every $1 \leq i \leq n$ because

$$\prod_{i=1}^{n} (x - k_i) = 0 \quad \text{when} \quad x = k_i \text{ for every } 1 \leq i \leq n.$$

Therefore if we wanted to write the product of the eigenvalues of A we could express this as:

$$\prod_{i=1}^{n} (k_i)$$

We also know that * For proof see p. 4

$$\det A = (-1)^n \det(0 \cdot I - A) = (-1)^n \det(-A)$$

so, substituting the characteristic polynomial of A with $x = 0$ we get

$$\det A = (-1)^n \cdot c_A(0)$$

$$\det(A) = (-1)^n \cdot \left(\prod_{i=1}^{n} (0 - k_i) \right)$$

$$\det(A) = (-1)^n \cdot \left(\prod_{i=1}^{n} (-k_i) \right)$$

$$\det(A) = \prod_{i=1}^{n} ((-1)(-k_i))$$

$$\det(A) = \prod_{i=1}^{n} (k_i)$$

So therefore it has been proven that the determinant of A is the product of it's eigenvalues.

Q.E.D.

* We know that by Theorem 3.1.2 if a ~~column~~ row of the matrix ~~det~~ A is multiplied by a constant then the determinant of the new matrix is:

$$\det A' = k \cdot \det A$$

where $A' = E^i_k A$

and $E^i_k = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & k & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 1 \end{pmatrix}$ = One row(i) is multiplied by k

~~We use define mul~~

By definition 2.2, $-A$ is the same as the matrix A' where every row of A is multiplied by -1

Therefore if A has n rows

$$-A = (E^i_{-1})^n A$$

every row of A is multiplied by -1

so the determinant of $-A$ is:

$$\det(-A) = \det((-1)^n A) = \det A \cdot (-1)^n$$

Therefore

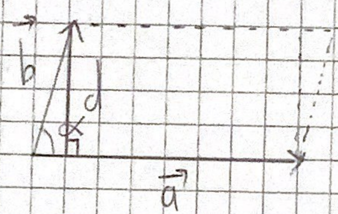
$$\det A = (-1)^n \cdot \det(-A)$$

2) Given a parallellepiped V whose sides are given by the linearly independent vectors a, b, c and assuming the base of the parallellepiped to be the parallellogram built on a, b, we can determine the volume of V in the following way:

For any parallellepiped the volume is the area of the base × the ~~area~~ height of the parallellepiped

$$V = A_{ab} \cdot h$$

The area of the parallellogram is



$d \cdot \|a\|$ where d is the height of the parallellogram

By the defintion of sin

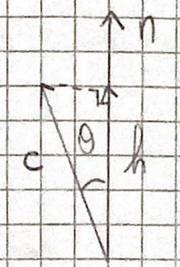
$$\sin\alpha = \frac{d}{\|b\|} \quad \text{so} \quad d = \|b\| \cdot \sin\alpha$$

Therefore $A_{ab} = \|a\| \cdot \|b\| \cdot \sin\alpha$ which is the value of the crossproduct of a and b:

$$A_{ab} = \|a \times b\| \quad \text{(See Theorem 4.3.4)}$$

The height of the parallellepiped is the projection of c on the normal vector to the plane ab

Therefore it is given by



$$h = \cos\theta \cdot \|c\|$$

Since we have determined the length of the normal vector is the same as the area of the base we can rewrite the formula for the volume as:

$$V = \|(a \times b)\| \cdot \|c\| \cdot \cos\theta$$

where θ is the angle between $(a \times b)$ and $\vec{c}$.
We can, therefore, recognise this is the formula for the dot product. ~~magnitude~~ So

$$V = (a \times b) \cdot c$$ (See Theorem 4.2.2)

However since V is a volume it must be always positive so for completeness:

$$V = |(a \times b) \cdot c|$$

Q.E.D.