1) a) Let $u, v, w \in \mathbb{R}^n$ be linearly independent and let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix.

We define U as the subspace spanned by u, v, w:

$$U = \text{span}\{u, v, w\}$$

where $\dim(U) = 3$ because U is spanned by 3 linearly independent vectors (basis of U) (see Definition 5.5)

By Theorem 2.5.2 a square matrix such as A is invertible if and only if it is a product of elementary matrices therefore:

$$A = E_K \cdot E_{K-1} \cdot \ldots \cdot E_2 \cdot E_1 \cdot I$$

$$A \cdot u = E_K \cdot E_{K-1} \ldots E_2 E_1 \cdot u$$

$$A \cdot v = E_K \cdot E_{K-1} \ldots E_2 \cdot E_1 \cdot v$$

$$A \cdot w = E_K \cdot E_{K-1} \ldots E_2 \cdot E_1 \cdot w$$

By theorem 2.2.5 and 2.3.2 we can say that the product of the matrices A and

$$B = \{u \quad v \quad w\}$$

is $AB = \{Au \quad Av \quad Aw\} = C$

Therefore $E_K \cdot E_{K-1} \cdot \ldots E_2 \cdot E_1 \cdot B = C$ so C can be obtained from B by elementary row operations.

By Lemma 5.4.1, $\text{row}(B) = \text{row}(C)$

1) a) Since we know that the columns of B are linearly independent by Theorem 5.4.3 the rows of B span $\mathbb{R}^3$.

Therefore we also know that the columns of C are linearly independent by Theorem 5.4.3 because the rows of C span $\mathbb{R}^3$ and the two conditions are equivalent.

It has been therefore proven that: if $u, v, w \in \mathbb{R}^n$ are linearly independent and $A \in \mathbb{R}^{n \times n}$ then $\{Au, Av, Aw\}$ are also linearly independent.

Q.E.D.

1) b) Let u, v, w be linearly independent.

Now, if we assume that

$$\text{span}\{u, v\} = \text{span}\{u, v, w\}$$

then w must be a linear combination of u and v

because by Theorem 5.1.1

$$U = \text{span}\{u, v\} = \text{span}\{u, v, w\}$$

contains the vectors u, v, w and so if $\text{span}\{u, v\}$ contains w then by Definition 5.2 w is a linear combination of u and v.

1) b) However u, v and w are linearly independent by assumptions so by Definition 5.3

$$t_1 u + t_2 v + t_3 w = 0 \quad \text{only when} \quad t_1 = t_2 = t_3 = 0$$

If we rewrite w as $w = a \cdot u + b \cdot v$

we get

$$t_1 \cdot u + t_2 \cdot v + t_3 \cdot a \cdot u + t_3 \cdot b \cdot v = 0$$

If $t_1 = -t_3 \cdot a \neq 0$ and $t_2 = -t_3 \cdot b \neq 0$ then the result of the above expression would still be 0 therefore u, v and w would not be linearly independent.

Therefore assuming $\text{span}\{u, v\} = \text{span}\{u, v, w\}$ led to a contradiddiction so it has been proven that

$$\text{span}\{u, v\} \neq \text{span}\{u, v, w\}$$

Q.E.D.

2) a) Let $v, w \in \mathbb{R}^n$ be normalised vectors then

$$v \cdot w = \|v\| \cdot \|w\| \cdot \cos\theta$$

by Theorem 4.2.2.

By definition $\|v\| = 1$, $\|w\| = 1$, $-1 \leq \cos\theta \leq 1$ so

$$v \cdot w \geq -1$$

2) b) Let $u, v, w \in \mathbb{R}^n$, $n \geq 3$ be three normalised vectors

Supposing and

$$u \cdot v = -\frac{1}{2} \qquad u \cdot w = -\frac{1}{3}$$

we can say that, by Theorem 4.2.2

$$\|u\| \cdot \|v\| \cdot \cos \alpha = -\frac{1}{2}$$

$$1 \cdot 1 \cdot \cos \alpha = -\frac{1}{2}$$

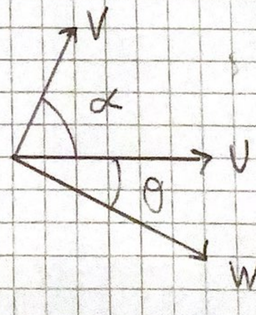
$$\cos \alpha = -\frac{1}{2} \quad \text{so} \quad \alpha = \frac{2}{3}\pi$$

and

$$\|u\| \cdot \|w\| \cdot \cos \theta = -\frac{1}{3}$$

$$1 \cdot 1 \cdot \cos \theta = -\frac{1}{3}$$

$$\cos \theta = -\frac{1}{3} \quad \text{so} \quad \theta = 1{,}04719755$$



The maximum possible angle between v and w is the sum of the angles between v and u and u and w however it could be less than that in R3 and higher dimensions. This is due to the fact that the vectors could form the edges of a tetrahedron and an angle at a vertex will always be less than the sum of the neighbouringn angles.

Therefore $v \cdot w \leq \|v\| \cdot \|w\| \cdot \cos \gamma$ where

$\gamma = \alpha + \theta$ by Theorem 4.2.2.

We can write this as:

$$v \cdot w \leq 1 \cdot 1 \cdot \cos \gamma \leq \cos \gamma$$

2b) where $\cos\gamma = \cos\alpha\cos\theta - \sin\alpha\sin\theta$

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) - \sin\alpha \cdot \sin\theta$$

we can write $\sin\alpha$ as $\sqrt{1-\cos^2\alpha}$

and $\sin\theta$ as $\sqrt{1-\cos^2\theta}$

to get $\sin\alpha = \frac{\sqrt{3}}{2}$

$\sin\theta = \frac{\sqrt{8}}{3}$

so $\cos\gamma = \frac{1}{6} - \frac{\sqrt{3}\cdot\sqrt{8}}{6}$

Since $\sqrt{24} \approx 4{,}898979485 57\ldots$

$$-\frac{\sqrt{24}}{6} > -\frac{5}{6}$$

so $\cos\gamma > \frac{1}{6} - \frac{5}{6}$

so $v \cdot w \geq -\frac{2}{3}$

Q.E.D.

2c) Let $u, v, w \in \mathbb{R}^n$, $n \geq 3$ be three normalised vectors with $u \cdot v = -\frac{1}{2}$, $u \cdot w = -\frac{1}{3}$, $v \cdot w = 0$, $v = e_1$.

We know that if u is of the form $u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ and $w = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$: $u_x \cdot 1 = -\frac{1}{2}$, $w_x = 0$ and $-\frac{1}{3} = u_x \cdot w_x + u_y \cdot w_y + u_z \cdot w_z$

2) c) So: $v = \begin{pmatrix} -\frac{1}{2} \\ v_y \\ v_z \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ w_y \\ w_z \end{pmatrix}$

$$\|v\| = 1 = \sqrt{\frac{1}{4} + v_y^2 + v_z^2} \rightarrow \frac{3}{4} = v_y^2 + v_z^2$$

$$\|w\| = 1 = \sqrt{w_y^2 + w_z^2} \rightarrow 1 = w_y^2 + w_z^2$$

$$-\frac{1}{3} = v_y w_y + v_z w_z$$

To find an admissable value for the triple $\{v, v = e_1, w\}$ we can choose a value for a component, for example $v_z = 0$

This would give three vectors:

$$\left\{ \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{2\sqrt{3}}{9} \\ \frac{\sqrt{69}}{9} \end{pmatrix} \right\}$$

that satisfy the conditions.

Q.E.D.