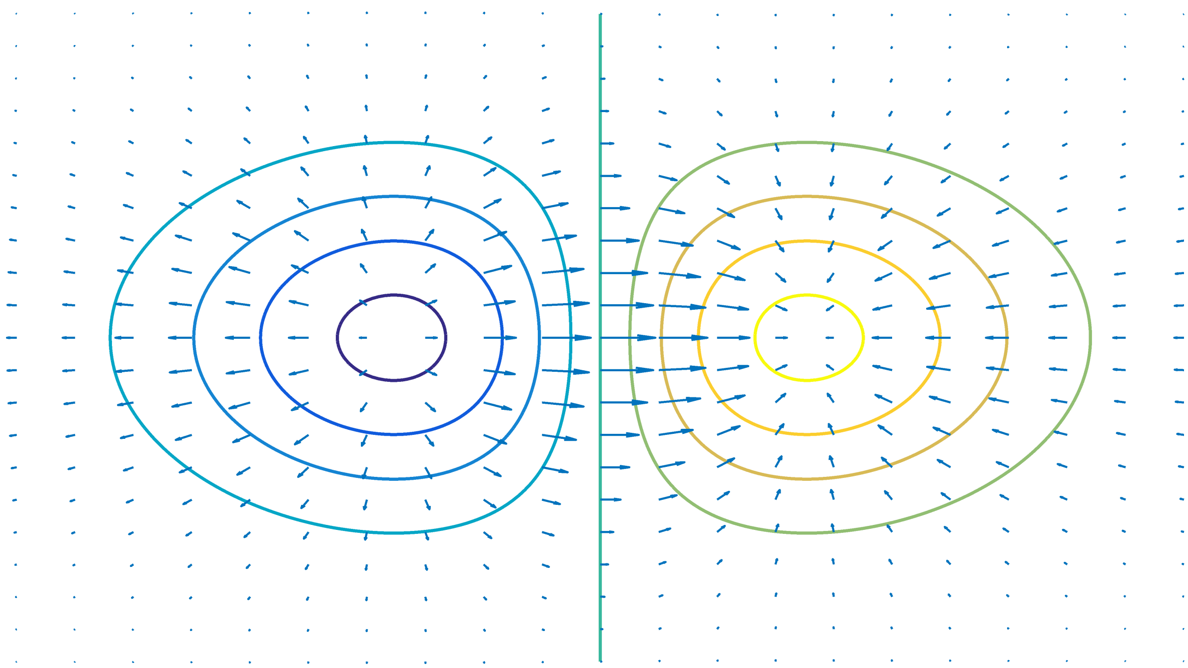
MA0232 Modelling with Differential Equations



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Contents

Chapter 1

Introduction

In this course we will be concerned with ordinary differential equations, ODEs.

Definition 1. *An ordinary differential equation (ODE) is a differential equation containing one or more functions of **exactly one** independent variable and its derivatives.*

ODEs relate the change of one variable to changes in another variable and can be used to model and understand a wide variety of phenomena, such as projectile motion, animal population interactions and the progression of chemical reactions.

In these notes we consider two important aspects in the theory of ordinary differential equations. Specifically, we seek to

1. develop methods of modelling physical phenomena;
2. understand the properties of the equations without explicitly solving them.

Point 2 may seem counter-intuitive as we have a variety of techniques that enable us to solve ODEs in closed form. Further, even if an explicit solution is not available, we can use numerical simulations to illustrate the dynamics of the ODEs. However, direct solutions are not always possible and, even when they are, they may not always enable clear interpretations and understanding of the underlying system. Equally, our analytical techniques will give us confidence in the solutions produced by numerical software.

Critically, what we gain in analytical specificity, we lose in global accuracy. Namely, we are going to learn techniques that will allow us to rigorously examine small regions of the ODE space at the expense of losing knowledge of the global dynamics. However, by the end of the course we will be able to patch together multiple parts of the local analysis in order to give us an approximate understanding of the entire dynamical system.

1.1 Preliminary definitions

We will be considering the rate of change of a variable, u, with respect to another variable, t. This dependence will be denoted

$$u(t). \tag{1.1}$$

Here, u is a scalar function (*i.e.* one-dimensional), but more generally, we will be considering systems of variables

$$\boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_k(t)) \, . \tag{1.2}$$

On the board we will usually write bold symbols with an underline[1] as it is easier to see, thus, $\boldsymbol{u} = \underline{u}$.

[1] I was once told that we use underlines to illustrate bold variables because when typesetting a document an underline would tell the printer that that symbol needed to be bold. However, if this is true, how did the writer indicate that they wanted a symbol underlined?

The values of u or $\boldsymbol{u}$ define quantities of interest. For example they could be an animal population density, a distance or a speed. Further, t can be any variable which these quantities are dependent on. Generally, however, we will take t to be time and we will be considering how these values temporally evolve.

In order to link the changes in these quantities we define a system of ODEs in the most general way possible,

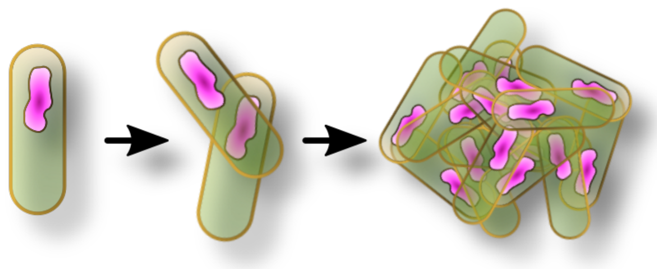
$$\boldsymbol{F}\left(t, \boldsymbol{u}, \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}, \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}, \ldots, \frac{\mathrm{d}^n\boldsymbol{u}}{\mathrm{d}t^n}\right) = 0, \tag{1.3}$$

with initial condition given by

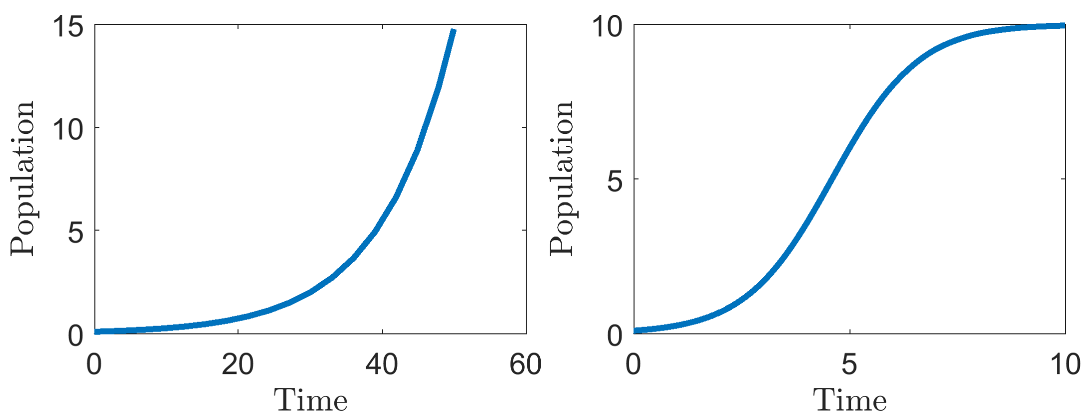
$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{1.4}$$

Note that the initial condition is kept general as we will usually be interested in how the dynamics of the system change for different starting points.

Example 1.1.1 Bacteria population growth



Example 1.1.2 Bacteria and nutrient populations.



(a) (b)

Figure 1.1: (a) Exponential growth. Parameters are $r = u_0 = 0.1$. See example 1.1.1. (b) Logistic growth. Parameters are $c = 10$, $r = u_0 = 0.1$. See example 1.1.2.

Example 1.1.3 Duffing's equations.

For our last example, consider the Duffing oscillator. The equation is simply a toy example that can be used to examine complex phenomena in a simple equation. In terms of interpretation, you can think of the equation as modelling the displacement of a beam near two magnetics. Critically, the beam and magnets are being forced to oscillate with amplitude γ and frequency ω (see Figure

1.2(a)).

$$\underbrace{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}}_{\text{Acceleration}} + \underbrace{2\delta\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{Air resistance}} + \underbrace{\beta x + \alpha x^3}_{\text{Beam's restorative force}} = \underbrace{\gamma\cos(\omega t)}_{\text{Forcing term}}. \tag{1.14}$$

We are not going to try and analytically solve or analyse Duffing's equation. Instead, we illustrate the dynamics that the equation produces as the amplitude of oscillation, γ, increases. Specifically, as γ is increased the system becomes chaotic (see Figure 1.2(b)).

Definition 2. *The **order** of a differential equation is the value of the highest derivative in the equation.*

Examples 1.1.2 and 1.1.1 are both first order equations, whilst example 1.1.3 is a second order equation. Generally (like polynomial equations of order) a differential equation of order n will have n linearly independent solutions.

Definition 3. *A system of differential equations is **autonomous** if the system does not explicitly depend on the independent variable.*

When the variable is time, they are also called time-invariant systems, this simply means that we are assuming that the defined underlying laws of the system are identical to those for any point in the past, or future.

Definition 4. *To save time we use a dot or prime mark to denote a derivative with respect to the argument, thus,*

$$\dot{\boldsymbol{u}}(t) = \boldsymbol{u}'(t) = \frac{d\boldsymbol{u}}{dt}. \tag{1.15}$$

Traditionally, dots are primarily used when the variable is time and primes are used otherwise. Note that higher orders derivatives are signified by the appropriate number of dots or primes. Namely, a second derivative would be denoted by two dots or primes, etc.

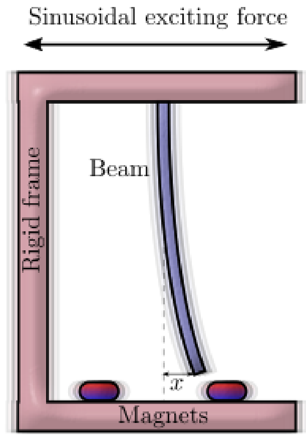
Definition 5. *A **trajectory** is a solution, $u(t)$.*

The graphs in figures 1.1 and 1.2 illustrate single trajectories of their respective systems.

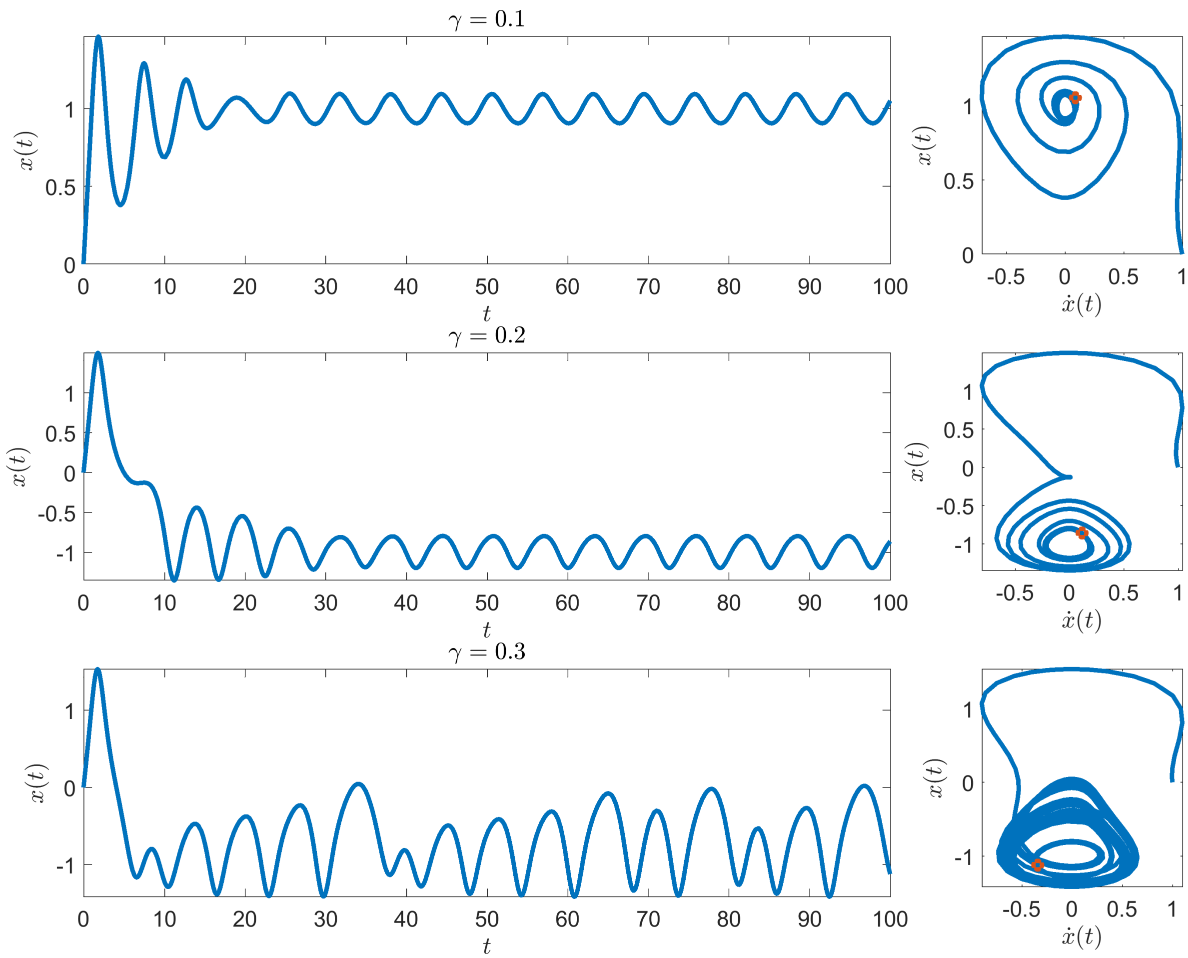
In this course we are going to occupy ourselves with systems of autonomous first order equations, of the form

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u}). \tag{1.16}$$

This may seem highly restrictive. However, systems of first order equations can have extremely complicated properties, such as oscillations and chaos, which we will try to understand.



(a)



(b)

Figure 1.2: (a) Schematic diagram of the system underlying Duffing's equation. (b) Three simulations of equation (1.14) with increasing values of γ.

Example 1.1.4 Duffing's equations without forcing.

Theorem 1.1.1. *A solution trajectory,* $\mathbf{u}(t)$*, of equation* (1.16) *cannot self-intersect (see Figure 1.3).*

Proof.

□

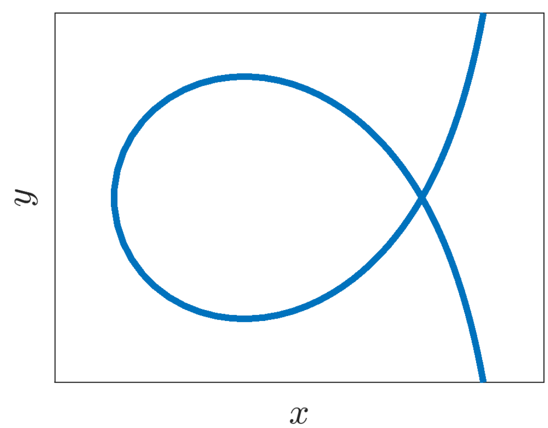


Figure 1.3: A solution of equation (1.16) cannot look like this.

1.1.1 Existence and uniqueness

With this being an applied mathematics course we are often very 'fast and loose' with our rigour. However, it is good to know that theorems have been proven regarding the existence and unique of solution to equation (1.16). Here we will quote the theorem in one dimension, but the theorem can be expanded to any number of variables.

Theorem 1.1.2. *Existence-Uniqueness theorem.*

Suppose the function $F(u)$ is differentiable and the derivative, $F'(u)$, is continuous for all values of u then there will exist some constant $c > 0$ such that

$$\dot{u} = F(u), \quad u(t_0) = u_0, \tag{1.25}$$

has a solution and it is guaranteed to exist and be unique in some finite time interval $|t - t_0| < c$.

Note that:

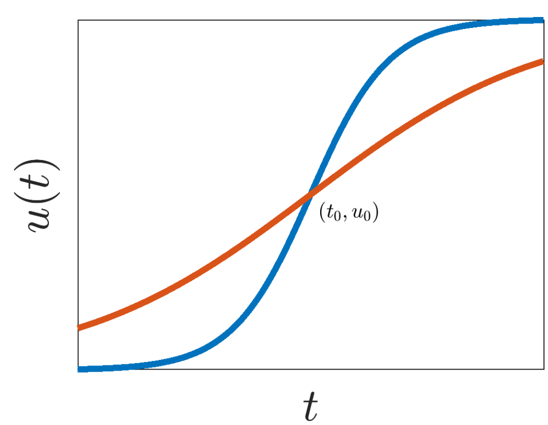
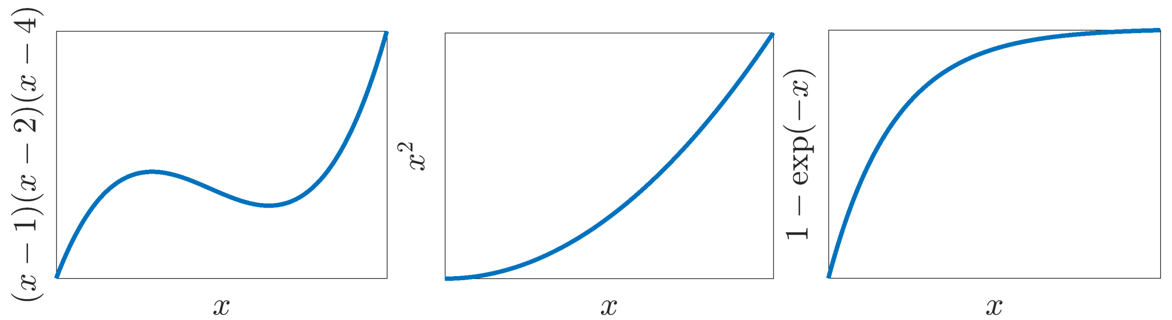


Figure 1.4: Two different solution curves of a one-dimensional ODE cannot intersect.

Definition 6. *A differentiable function is* ***monotonic*** *if its derivative never changes sign. Moreover, the function is monotonically increasing (decreasing) if the derivative is positive (negative).*

Definition 7. *A differentiable function is* ***strictly monotonic*** *if its derivative never changes sign and is never zero.*



(a) (b) (c)

Figure 1.5: (a) (b) (c)

See Figure 1.5 for examples of definitions 6 and 7.

Corollary 1.1.3. *Suppose $F(u)$ is a scalar function that is continuously differentiable. The solution, $u(t)$, of the one dimensional ODE,*

$$\dot{u} = F(u), \quad u(t_0) = u_0, \tag{1.27}$$

cannot oscillate. Specifically, $u(t)$ must either be a monotonically increasing, or decreasing function.

Proof.

□

This means that to have oscillatory phenomena in a system either we need more than one population, or the system has to be non-autonomous. See example 1.1.3 for a case where both of these factors are present and do indeed produce oscillations (and chaos).

1.2 Taylor expansions

This section is to remind you of the Taylor expansion technique. The Taylor expansion is one of the most powerful tools for an applied mathematician because very often we want to know what happens to a trajectory near some critical point. Although the kinetics maybe very non-linear and difficult to understand globally we can use the Taylor expansion to simplify the dynamics in a small region around the critical point in order to gain knowledge about the dynamics in this region.

Theorem 1.2.1. *Suppose $f(x)$ is infinitely differentiable at a point a then the Taylor series of f at a is the power series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n, \tag{1.29}$$

which is explicitly

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots, \tag{1.30}$$

where $n!$ denotes the factorial of n and $f^{(n)}(a)$ denotes the n^{th} derivative of f evaluated at the point a. The derivative of order zero of f is defined to be f itself and $(xa)^0$ and $0!$ are both defined to be 1.

Example 1.2.5 Taylor expansions.

- $\exp(x)$ at $x = 0$ (see Figure 1.6).

- $\cos(x)$ at $x = 0$.

- $1/(1 + x)$ at $x = 0$

- $\sin(x)$ at $x = \pi/2$.

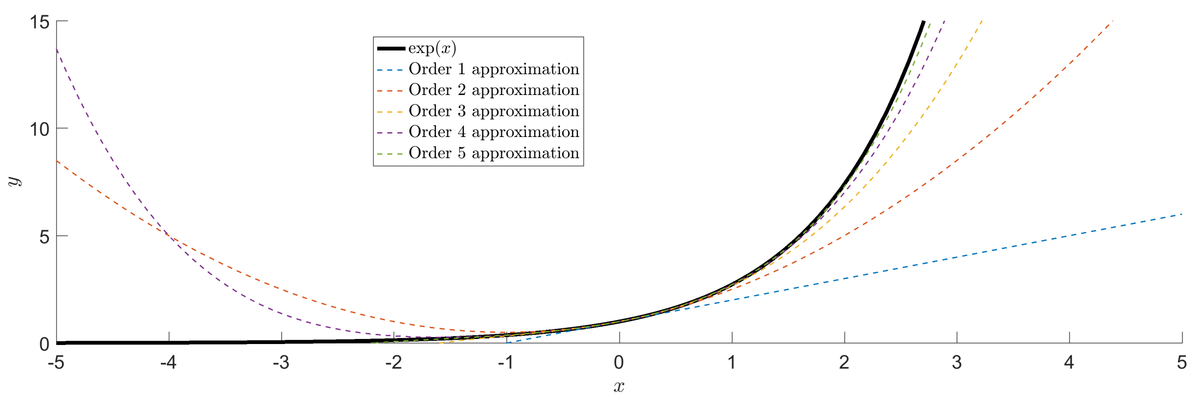


Figure 1.6: Approximating the exponential function with different orders of Taylor series.

Courses in the third year will deal with what information you get in the case that you truncate at ϵ^2, or higher. This is non-linear analysis.

1.2.1 Multivariate Taylor expansion

A similar theorem can be stated when the function f has more than one argument.

Definition 8. *If f is a function of more than one variable it is called* ***multivariate****.*

Definition 9. *For brevity we use subscripts to stand for partial derivatives,*

$$f_{x_1 x_2 \ldots x_n} = \frac{\partial^n f}{\partial x_1 \partial x_2 \ldots \partial x_n}. \tag{1.38}$$

Example 1.2.6 Multivariate Taylor expansion.

- $\sin(x + y)$ at $x = y = 0$.

- $\sin(x)\cos(y)$ at $x = y = 0$.

1.3 Polar coordinates

Many phenomena that we will model will fall under the consideration of spatial movement, for example in Chapter 2 and question sheet two we will be considering planetary movement. Critically, in many of these cases the objects tend to move in circular trajectories orbiting a single point. Thus, it is more natural to use polar coordinates (r, θ) to describe the motion, rather than Cartesian coordinates (x, y) (see Figure 1.7). However, it may be easier to model the system in Cartesian coordinates. Thus, we need to know how to convert between one set and another.

Figure 1.7 illustrates the fundamental relationships between the Cartesian and the polar coordinates, namely:

$$x = r\cos(\theta), \tag{1.41}$$

$$y = r\sin(\theta). \tag{1.42}$$

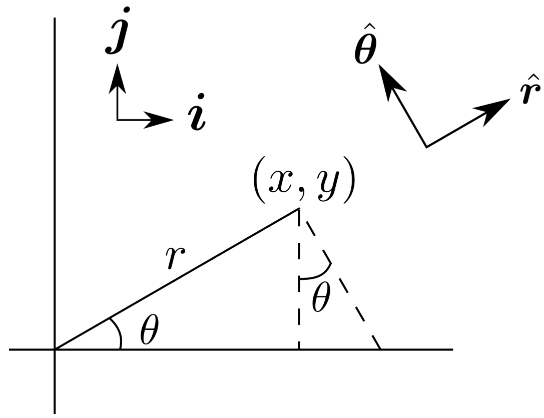


Figure 1.7: Cartesian and polar coordinates.

Critically, these specify x and y singly as functions (r, θ). These, in turn, can be used to construct equations for r and θ separately as functions of (x, y), namely,

$$r^2 = x^2 + y^2, \tag{1.43}$$

and

$$\theta = \arctan\left(\frac{y}{x}\right), \quad \text{or} \quad \theta = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right), \quad \text{or} \quad \theta = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right). \tag{1.44}$$

where the appropriate function $\theta(x, y)$ is chosen depending on which ever is easiest to use.

Example 1.3.7 Cartesian to polar conversion.

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In the above example we created $\dot{r}$ first and then used this to produce $\dot{\theta}$. In following example we show how to do the substitution all in one go.

Example 1.3.8 A quicker conversion

Generally, nonlinear equations are not solvable however, we will see in the next example the polar coordinates can convert nonlinearities in (x, r) to linearities in (r, θ)

Example 1.3.9 Solving ODEs in polar coordinates

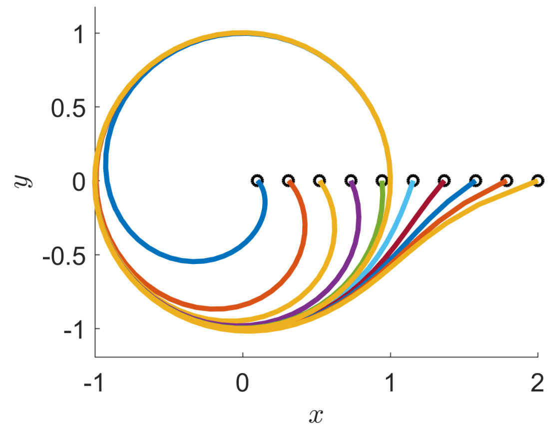

Figure 1.8: Full dynamics of equations (1.61) and (1.62).

1.4 Check list

By the end of this chapter you should be able to:

- ☐ reproduce all the definitions;
- ☐ state all theorems;
- ☐ solve simple linear ODE systems;
- ☐ prove trajectories of autonomous systems cannot cross themselves;
- ☐ prove that an ODE of one variable cannot oscillate;
- ☐ derive single variable Taylor series of any order;
- ☐ derive multivariate Taylor series up to first order;
- ☐ convert systems of ODE equations of Cartesian variables in to polar variables and back again.

Chapter 2

How to model a system

This model will be a simplification and an idealization, and consequently a falsification. It is to be hoped that the features retained for discussion are those of greatest importance in the present state of knowledge.

(The Chemical Basis of Morphogenesis. A. Turing 1952.)

All models are wrong, but some are useful.

(Empirical Model-Building and Response Surfaces. G. Box & N. Draper 1987.)

Modelling a system, whether it be physical, chemical, or biological, is, in some ways, more of art than a science. You try and strip away all extraneous information and mathematically describe that which is left. Sometimes there are physical laws to help you, *e.g.* gravity, conservation of energy and mass. Other times we only have experimental intuition, *e.g.* predator-prey interactions from population data. In either case, the central idea of modelling is that it should always form part of a cyclical process (see Figure 2.1).

You try to start with physical intuition (experiment), represent the important parts mathematically (model), hopefully reproduce reality (test) and, finally, use your mathematical model to predict unknown outcomes (predict). These prediction can then feed back into experiment and the process begins anew.

In this chapter we are going to review some of the methods that can be used to produce a mathematical interpretation of reality.

2.1 Physical laws

Definition 10. *A* ***constitutive relation*** *(or 'physical law') is a rule that the modeller adds to the system based on their experimental experience, which relates interacting components.*

Physics has many laws such as: conservation of energy, general relativity and the laws of thermodynamics. There is (as yet) no fundamental reason for these laws to hold. We just take them as laws because they fit the data that we observe.

Here are just a few examples of the laws that you might come across.

- **Newton's Law of Cooling.**

  The rate of cooling of a body is proportional to the difference between the bodies temperature and the temperature of its environment.

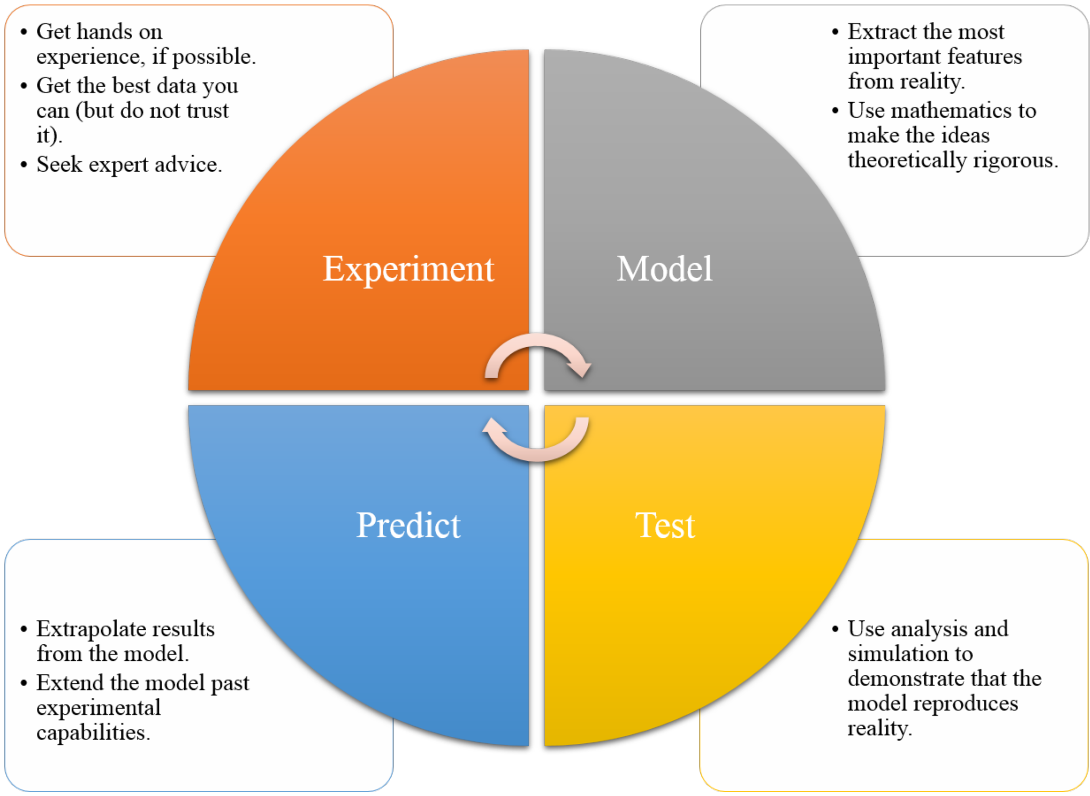

Figure 2.1: Diagram of the modelling cycle.

Example 2.1.10 Cooling tea

Suppose I prepare two cups of tea at exactly the same time, so initially they both start at 100^{o}C. I quickly add enough milk to cup 1 to cool it to 90^{o}C. Ten minutes later I add milk to cup 2, which cools cup 2 by 10^{o}C. Which cup is hotter at that point?

- **Newton's Second Law of Motion**

  The rate of change of momentum of a body is directly proportional to the force applied to the body.

- **Newton's Law of Gravitation**[1]

  A particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.

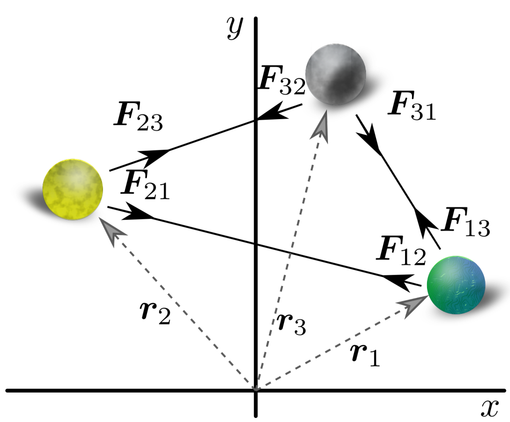

Figure 2.2: Schematic diagram of the three-body problem.

Example 2.1.11 Three body problem

We can combine the Second Law of Motion and the Law of Gravitation in order to predict the position of planets interacting through their gravitational fields (see Figure 2.2). Consider three planets with the same mass, m, and positions $\boldsymbol{r}_1(t)$, $\boldsymbol{r}_2(t)$ and $\boldsymbol{r}_3(t)$, respectively. Further, we note that since we are dealing with acceleration (a second order equation) we will need to specify two initial conditions, the position and velocity. Let the initial positions

[1]Newton devised the laws of optics, the laws of motion and invented calculus practically on a dare... then he turned 26. What have you done today?

be $\boldsymbol{r}_i(0) = \boldsymbol{r}_{i0}$ and the initial velocities $\dot{\boldsymbol{r}}_i(0) = \boldsymbol{v}_{i0}$.

The three body problem illustrates chaotic behaviour, in the sense that the outcome is extremely sensitive to the initial conditions. This can be seen in the simulations of Figure 2.3.

Simulation tip:

- when solving equations (2.8)-(2.10) numerically we could separate each equation into its Cartesian components and reduce the second order equation to two first order equations. Namely, we would introduce $(v_{ix}, v_{iy}) = (\dot{x}_i, \dot{y}_i)$. Thus, we would have a system of twelve ODEs to solve, with variables

$$(x_1, y_1, v_{1x}, v_{1y}, x_2, y_2, v_{2x}, v_{2y}, x_3, y_3, v_{3x}, v_{3y}). \tag{2.11}$$

However, since x_i and y_i are perpendicular Cartesian coordinates it turns out to be a good idea to use complex numbers. Specifically, instead of writing two ODEs for each of x_i and y_i we can simply solve one ODE in terms of the complex quantity $\boldsymbol{r}_i = x_i + Iy_i$, which can be handled by numerical solvers. Thus, we simplify the numerical solution from twelve to six equations.

- **Hooke's law**

  The force, F, needed to extend or compress a spring by some distance, x, scales linearly with respect to that distance,

2.1.0.1 Pendulums

In this section we take an extended look at pendulums depending on Hooke's law and simple Newtonian mechanics. Specifically, we consider a spring, oscillating up and down, and a bob, oscillating side to side (see Figure 2.4).

[2]Consider, for example, the ear. Small earrings to not stretch the skin very much and, thus, once the earring is removed the skin can heal. Alternatively, people who use large gauge earrings stretch their ear holes beyond the elastic limit of the skin so that they have a permanent hole.

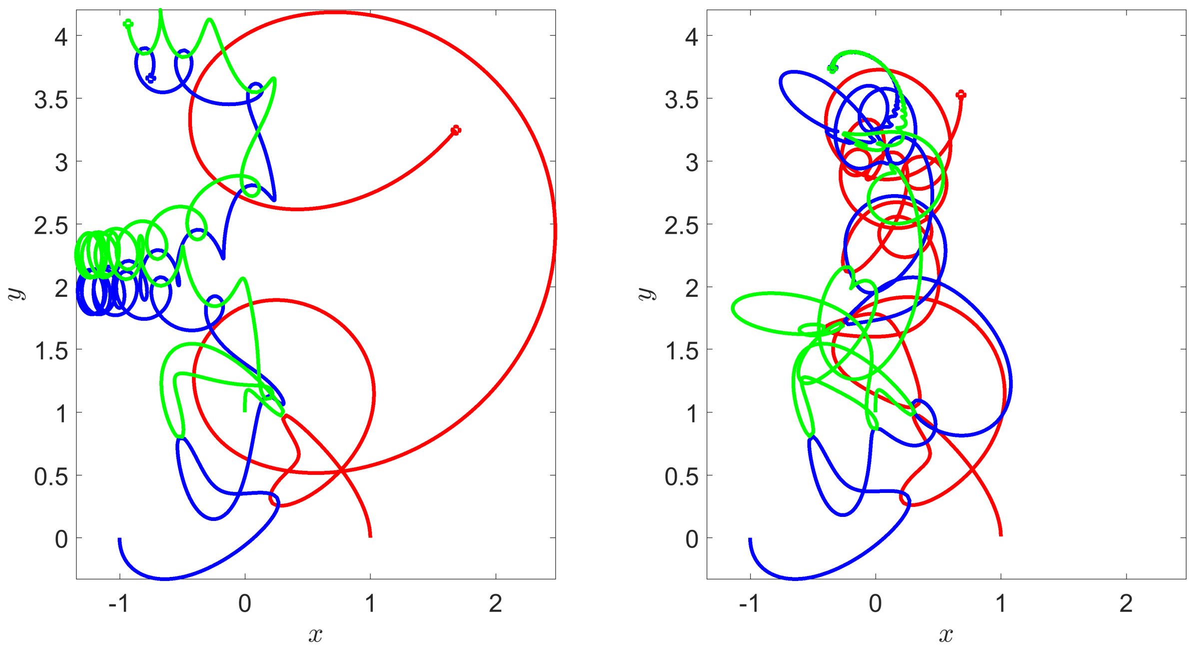

Figure 2.3: Two simulations of the three body problem. The green and red trajectories are identically initiated. The blue trajectory is initiated at $(-1, 0.01)$ in the left figure and at $(-1, 0)$ in the right figure.

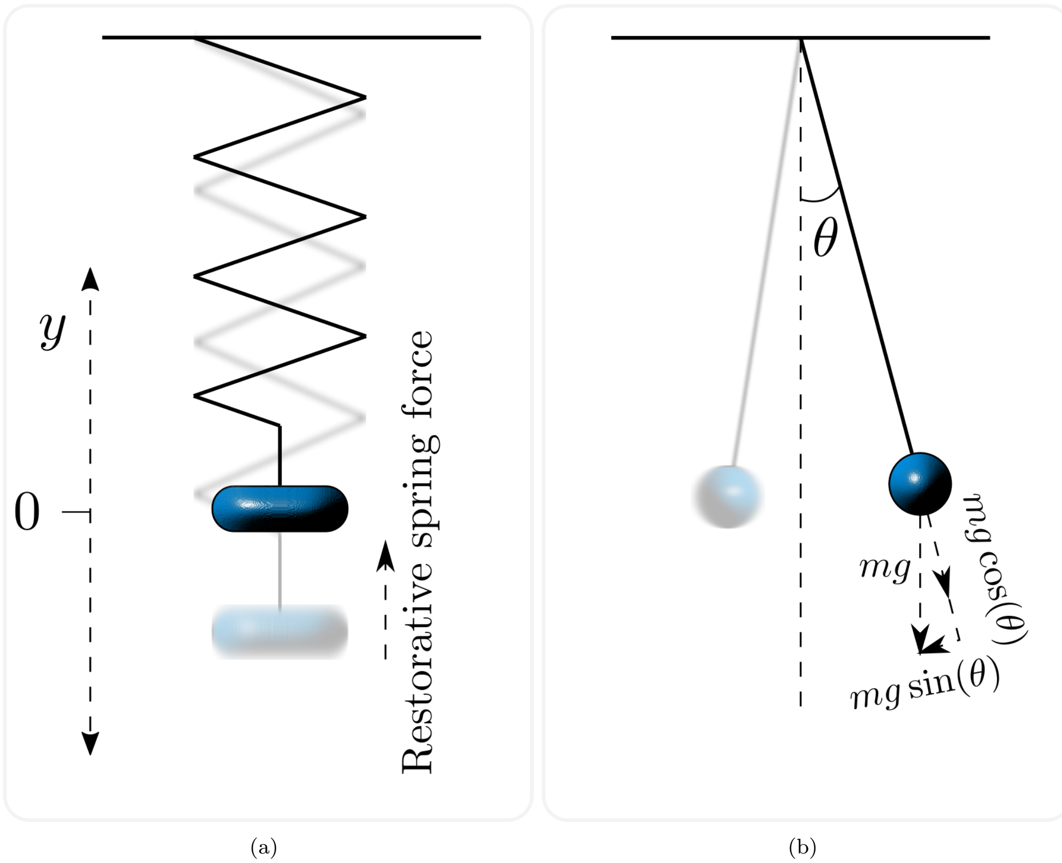

Figure 2.4: Two types of pendulums: (a) an oscillating spring. (b) a weight on a string.

Definition 11. *Any system defined by an equation of the form*

$$\ddot{u} = -k^2 u. \tag{2.16}$$

is said to under go ***simple harmonic motion****.*

Equation 2.16 can be solved to produce the solution

$$u = A\cos(kt) + B\sin(kt), \tag{2.17}$$

where A and B are specified through the initial conditions.

[3]Comparing y and θ is a little dodgy as y is a dimensional length and the θ is dimensionless, as we work in radians. However, if this bothers you we can fix this is in either of two ways. Either, we consider y normalised by its natural length (taken here to be of unit length, regardless of the dimensions involved), or, we can consider Figure 2.5 comparing equation (2.14) and its approximation in equation (2.15).

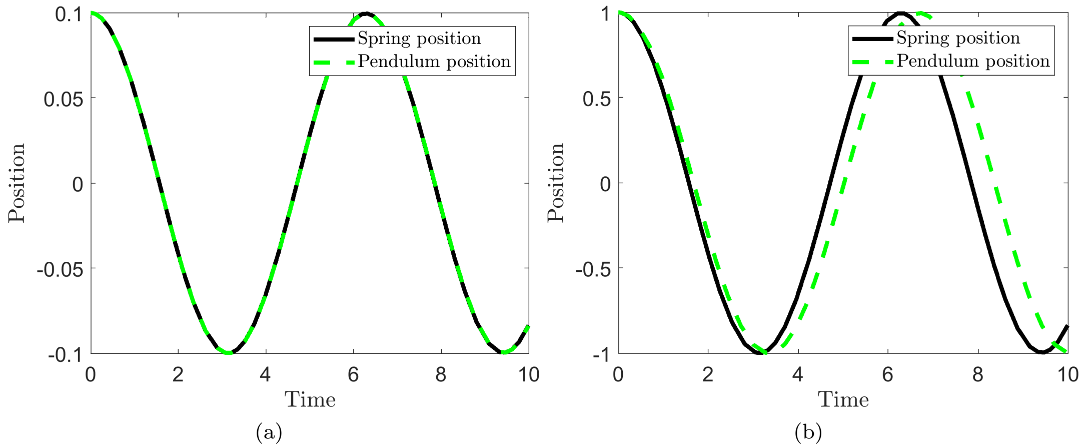

Figure 2.5: Comparing equations (2.13) and (2.14) with initial conditions (a) $y = 0.1 = \theta$ and (b) $y = 1 = \theta$. Parameter values $r = g = k = m = 1$.

2.2 Law of Mass Action

All of the above physical laws are very specific in their application. In this section we will learn about a much more general technique that will allow us to build an ODE system out of multiple interacting populations. These populations could represent chemical compounds, humans, cells or animals as well as different states within a population *i.e.* infected humans and susceptible humans. The law presented in this section is applied whenever the populations of the system are able to: (i) change identities; (ii) create more population members; or (iii) cause populations to decay. Specific examples of each of these interactions are, respectively: (i) susceptible humans becoming infected through interactions with a diseased person; (ii) animals giving birth; (iii) predators eating prey. Note that a change-of-identity interaction can itself be thought as a combination of creation and degradation operations. For example, in the above case of infection a member of the susceptible human population is removed from the system, whilst an infected human is added to the system. Thus, all interactions can be made through combining creation and degradation operations.

We use chemical reaction notation to specify the outcomes of population interactions. Consider a system composed of n different interacting populations $(u_1, \ldots, u_n)$. We assume that all interactions between the population elements lead to the creation, or destruction, of one (or more) of the n populations.

Definition 12. *A* ***rate equation*** *specifies that an interaction involves a_1 members of population u_1, a_2 members of population u_2, etc. and produces b_1 members of population u_1, b_2 members of population u_2, etc. The equation is written as*

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \xrightarrow{r} b_1 u_1 + b_2 u_2 + \cdots + b_n u_n, \tag{2.18}$$

where $r > 0$ is the ***reaction rate****.*

Note that some of the a_i and b_i values can be zero.

Example 2.2.12 Reaction equation examples

- **Birth**

Two agents of population u come together to produce a third,

- **Death**

  An agent of population u dies (or is destroyed) due to natural causes,

- **Predation**

  A predator population, v, converts energy from eating prey, u, into offspring,

- **Infection**

  Consider a population of infected people, I, who are able to infect a susceptible population, S. Further, over time, the infected people recover and become susceptible again,

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Rate equations provide a rigorous way of defining all of the interactions a system is assumed to undergo. However, we still require a method of converting the rate equation into an ODE. This is the power of the Law of Mass Action.

Definition 13. *The* ***Law of Mass Action*** *states that production rate of a reaction is directly proportional to the product population sizes. Specifically, if*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n \xrightarrow{r} b_1u_1 + b_2u_2 + \cdots + b_nu_n$$

is the reaction of interest then the production rate is

$$ru_1^{a_1}u_2^{a_2}\ldots u_n^{a_n} \tag{2.24}$$

and the accompanying ODEs are

$$\dot{u}_1 = (b_1 - a_1)ru_1^{a_1}u_2^{a_2}\ldots u_n^{a_n}, \tag{2.25}$$
$$\dot{u}_2 = (b_2 - a_2)ru_1^{a_1}u_2^{a_2}\ldots u_n^{a_n}, \tag{2.26}$$
$$\vdots \tag{2.27}$$
$$\dot{u}_n = (b_n - a_n)ru_1^{a_1}u_2^{a_2}\ldots u_n^{a_n}. \tag{2.28}$$

Note that in converting from reaction equation to the ODE of u_i we to account for the stoichiometry, *i.e.* $(a_i - b_i)$. Further, when multiple reactions are considered, the terms arising from the Law of Mass Action are simply added together as independent terms.

Example 2.2.13 Law of Mass Action examples

- **Birth**

- **Death**

- **Predation**

- **Infection**

Example 2.2.14 Zombies

Humans, H, and zombies, Z, interact through the following three interactions (see Figure 2.6):

1. humans kill zombies at a rate a;
2. zombies kill humans at a rate b;
3. zombies infect humans at a rate c.

The reaction equations for this system are,

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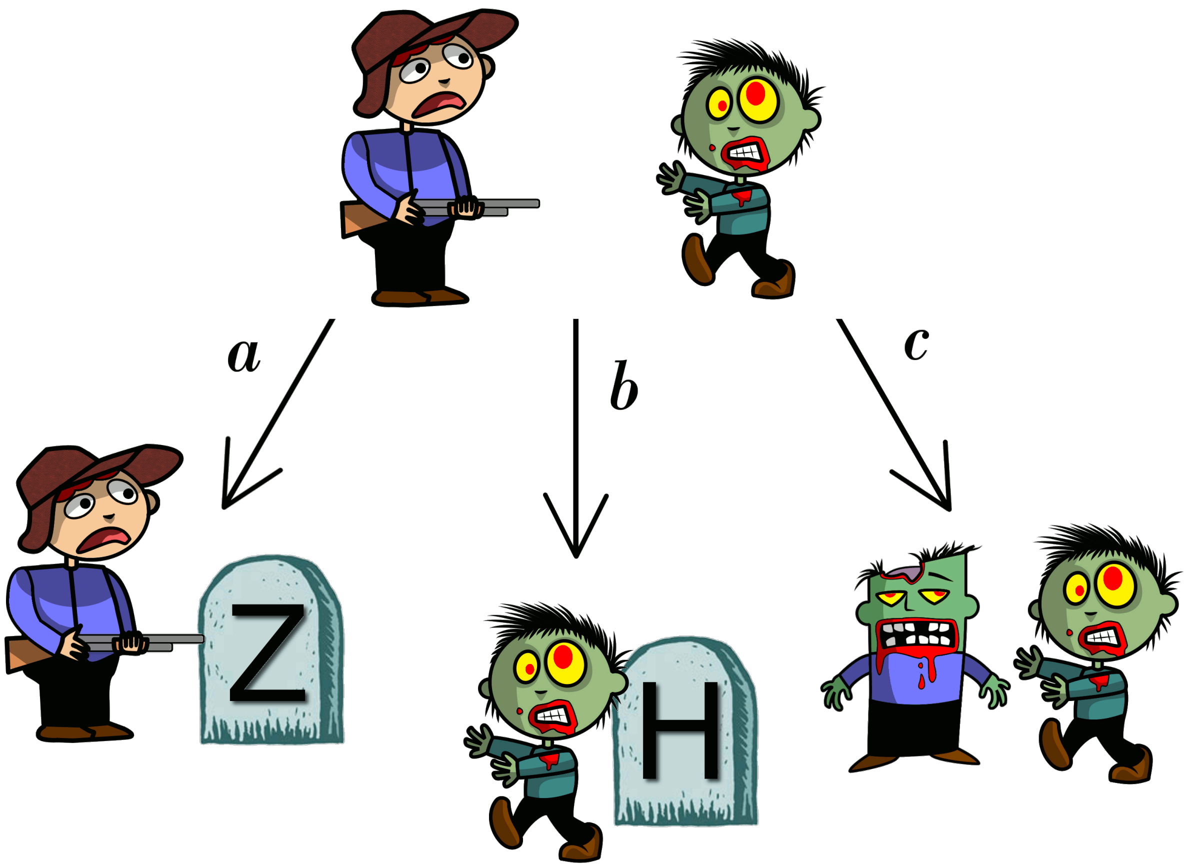

Figure 2.6: Possible outcomes of human-zombie interactions.

2.3 Check list

By the end of this chapter you should be able to:

- ☐ define all of the constitutive laws;
- ☐ solve problems involving Newton's laws, Hooke's law and simple harmonic motion;
- ☐ convert a system of population interactions into reaction equations;
- ☐ convert reaction equations into ODEs using the Law of Mass Action.

Chapter 3

Non-dimensionalisation

> *In metric, one milliliter of water occupies one cubic centimeter, weighs one gram, and requires one calorie of energy to heat up by one degree centigrade which is 1 percent of the difference between its freezing point and its boiling point. An amount of hydrogen weighing the same amount has exactly one mole of atoms in it. Whereas in the [imperial] system, the answer to "How much energy does it take to boil a room-temperature gallon of water?" is "Go fuck yourself", because you can't directly relate any of those quantities.*
>
> (Wild Thing. J. Bazell 2013.)

Thus, far we have been fairly lax about defining the quantities we have actually been measuring. Further, once we have specified what the quantity actually is, what units are we using to measure the quantity. For example, if we are measuring distance are we doing it in mm, miles or light-years? Equally, is time measured in seconds, minutes or hours? Finally, constitutive laws often introduce parameters that are not quantified accurately, or alternatively, we may be interested in understanding how a solution depends on a particular parameter as it is varied.

The Law of Mass Action, in particular, could be thought to be a troublesome law as it introduces a rate parameter for each reaction equation that is considered. For example, if a system of ODEs is defined by a set of non-linear equations it is highly unlikely to be solvable in closed form. Thus, if there are a large number of parameters in the system, it becomes very difficult to predict how varying a single parameter (or group of parameters) will influence the solution.

However, we have already seen cases in which we do not need to consider parameters individually, as specific groups of the parameters are seen to act in the same way. For example, in the case of the spring pendulum (see Section 2.1.0.1), we saw that the frequency of oscillation depended on $\sqrt{k/m}$. Thus, stiffening the spring (increasing k) has the same effect on the solution as decreasing the mass (decreasing m), *i.e.* they both increase the frequency of the oscillations. Equally, in the example of a zombie infection (see example 2.2.14), the parameters of interest were not a, b or c, but rather $\alpha = b+c$ and $\beta = c - a$.

In this chapter we introduce a technique, called non-dimensionalisation, that will benefit us in two ways. Firstly, it allow us to brush away worries about dealing with units and, secondly, it will allow us to reduce the number of effective parameters in our system. Specifically, we will be able to define parameter groupings that will influence the final result in the same way.

3.1 The central idea

To non-dimensionalise a system of equations, we have the following rules:

1. Identify all the variables;
2. Replace each variable with a quantity scaled relative to a characteristic unit of measure (to be

determined);

3. Choose the definition of the characteristic unit for each variable;

4. Rewrite the system of equations in terms of the new dimensionless quantities.

We note three particular points about these rules. Firstly, the theory behind non-dimensionalisation is straight forward. Namely, we substitute scaled variables into an equation system and massage the equations until we have rearranged the system to produce the desired outcome. However, in practice the difficulty of the technique lies in the algebraic manipulation; it is very easy for the terms to become lost during the manipulation. Thus, care must be taken during the algebraic manipulation stage.

Secondly, you will notice the word 'choose' in point 3. This means that it possible to construct many different non-dimensionalised systems from the same system of equations, *i.e.* non-dimensionalisation is non-unique. We usually choose the characteristic unit of each variable to either emphasise one of the terms in a system or to remove as many parameters as possible.

Finally, this technique is hard to demonstrate in generality. It is much better to consider a number of system and see how the technique works in action. Thus, what follows will be a select number of examples, which along with your problem sheets should give you a good basis in the theory. However, do not think that these are all the examples you could face.

It should be noted that there is little consistency in nomenclature across book when considering the separation of variables into their dimensional and non-dimensional components. Thus, always be clear in your definitions.

3.1.1 Examples of non-dimensionalisation through substitution of variables

Example 3.1.15 Substituting variables

- Consider the equation for exponential growth,

$$\dot{u} = ru, \quad u(0) = u_0. \tag{3.1}$$

- Consider the equation for logistic growth,

$$\dot{u} = ru\left(1 - \frac{u}{K}\right), \quad u(0) = u_0. \tag{3.2}$$

- Consider the following equations (the Schnakenberg kinetics)

$$\dot{u} = k_1 - k_2 u + k_3 u^2 v, \quad u(0) = u_0, \tag{3.3}$$

$$\dot{v} = k_4 - k_3 u^2 v, \quad v(0) = v_0. \tag{3.4}$$

]

3.1.2 Examples of non-dimensionalisation through the arrow method

The substitution method shown in Section 3.1.1 will always work supposing that the algebra is manipulated correctly. However, the method can be cumbersome and slow. Moreover, because it involves lots of algebraic manipulations there are many chances to make a mistake.

An alternative method rests on using arrows to identify the desired balances. This can be much quicker as the initial stages do not require laborious substitution. However, we have to be more careful because not all balances that we can 'draw' using the arrows will be valid.

The idea behind the arrow method is that you draw arrows between the quantities that are going to 'balance', which simply means they are going to have the same coefficient in the final non-dimensionalised form. The process is generally the same as the substitution method. However, we must remember that in order to specify the problem completely the number of valid arrow balances must equal the number of variables. For example, if a problem depends on u and t we would need two balances. Alternatively, if the problem depended on u, v, and t we would need three valid balances. This section is going to depend primarily on examples, again, and we will see an invalid balance at the end of the demonstrations.

Example 3.1.16 Arrow method

Consider the following equation

$$\dot{u} = k_0 + k_1 u + k_2 u^2, \quad u(0) = u_0. \tag{3.7}$$

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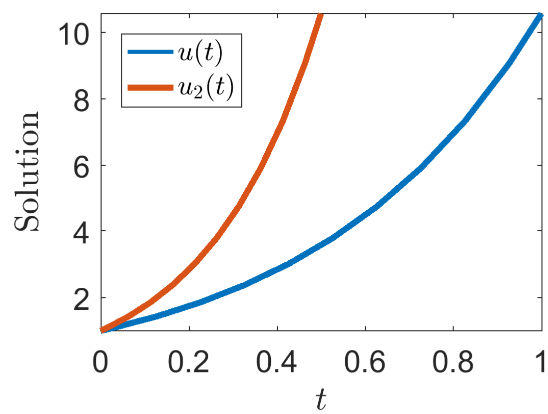

Figure 3.1: Two simulations of equation (3.7) with parameter values $k_0 = k_1 = k_2 = 1$ (blue line, $u(t)$) and $k_0 = k_1 = k_2 = 2$ (red line, $u_2(t)$). Illustrating that the evolution of the red line is the same as the blue line, except that the red line evolution occurs twice as fast, as predicted by equation (3.9).

Example 3.1.17 Non-uniqueness

To illustrate the non-uniqueness of non-dimensionalisation we rerun example 3.1.16 but this time we balance the time derivative, the constant term and the initial condition,

$$\dot{u} = k_0 + k_1 u + k_2 u^2, \quad u(0) = u_0. \tag{3.10}$$

]

Both forms of the non-dimensionalised equation, (3.8) and (3.11), are perfectly valid. The most useful form will depend on what factor dominates the equation. If k_0 is small and k_1 is big (relative to one another) then equation (3.8) would be more useful as $\alpha \approx 0$ and we would be able to manipulate the equation to provide more information. Alternatively, if k_0 was big and k_2, or k_1, was small then, equation (3.11) would be more useful as we would, again, be able to remove one of the constants based on this assumption.

Example 3.1.18 Failure

As mentioned not all balances are valid, which is what we will be seen in this example. Consider the following ODE system

$$\dot{u} = k_0 + k_1 u - k_2 uv, \quad u(0) = u_0, \tag{3.12}$$

$$\dot{v} = k_3 + k_4 v - k_2 uv, \quad v(0) = v_0. \tag{3.13}$$

One possible valid non-dimensionalisation is

$$\dot{u} = k_0 + k_1 u - k_2 uv, \quad u(0) = u_0,$$

$$\dot{v} = k_3 + k_4 v - k_2 uv, \quad v(0) = v_0.$$

See the board for details.

Although each case of non-dimensionalisation is different, the algorithm you should follow is the same in each case. The steps are:

1. write down the variables in the equations, this tells you how many balances you need;
2. specify balances and check that they are valid;
3. define non-dimensional scales that allow you to minimise the number of free parameters;
4. substitute the scales into the equations and collect together the remaining parameters into the smallest possible groups and give them a new variable name (DO NOT FORGET to do the same thing for the initial conditions. Everyone always forgets to do the initial conditions);
5. demonstrate that the scales you have derived have the correct dimension;
6. demonstrate that the new parameter groupings are dimensionless.

Although we have not completed the last two points for every example, you will be expected to do every step in an exam.

3.2 Check list

By the end of this chapter you should be able to:

- ☐ non-dimensionalise a system of equations using direct substitution, or the arrow method;
- ☐ demonstrate that the derived scales have the correct dimension;
- ☐ demonstrate that remaining parameter groupings are non-dimensional;

Chapter 4

Stationary states and stability

Now that we are able to model and simplify a physical system, we want to predict what the equations will do without having to simulate the system each time. Specifically, we are not interested in the transient initial behaviour of the equations, we want to understand what the trajectories will like far into the future. In one dimension we have proven that the equations must either monotonically increase, decrease, or tend to a fixed value. In two-dimensions we have the additional complications of persistent oscillatory dynamics. In higher dimensions we have the further option of chaotic systems, which are outside the scope of this course. However, even by restricting ourselves to one and two dimensions, how do we know what will happen? To enable us to generate these insights we first need two important definitions.

Definition 14. *A state, $\boldsymbol{u}_s$, is a **steady state** or **stationary state** of the ODE system*

$$\dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u}) \tag{4.1}$$

if is satisfies $\boldsymbol{F}(\boldsymbol{u}_s) = 0$.

This definition simply states that if the ODE system ever reaches $\boldsymbol{u}_s$ then the system will not evolve further because all of the dynamics are in equilibrium. This is a useful concept, but currently incomplete.

For example, you can (theoretically) stand a pencil on its tip and it would remain stationary, if it were not perturbed (see Figure 4.1). Hence, this is a stationary state orientation of the pencil. However, it would require only a very small perturbation to cause the coin to fall over and, thus, transition from the state of being on its point to being on its side (see Figure 4.1). Given a large enough perturbation (*i.e.* picking the pencil up) you could reset the pencil to the previous state of standing on its point. However, it requires a larger perturbation to reset the pencil than it does to knock it over and, so, we see that although these state are both stationary states they are somehow fundamentally different. This difference comes down to the intuitive concept of 'stability'.

Definition 15. *A steady state, $\boldsymbol{u}_s$, of the ODE system*

$$\dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u}) \tag{4.2}$$

*is **stable** if for all $\epsilon > 0$, there exists a $\delta > 0$ and a $t_0 > 0$ such that whenever $|\boldsymbol{u}(t) - \boldsymbol{u}_s| < \delta$ then $|\boldsymbol{u}(t) - \boldsymbol{u}_s| < \epsilon$ for all $t \geq t_0$. Otherwise the steady state is unstable*

Simply put, this means that a state, $\boldsymbol{u}_s$, is stable if whenever a solution $\boldsymbol{u}(t)$ comes close enough to it then the solution tends to the state *i.e.* $\boldsymbol{u}(t) \to \boldsymbol{u}_s$. In the example of the pencil, both the vertical and horizontal orientations of the pencil are stationary state. However, only the horizontal orientation is stable.

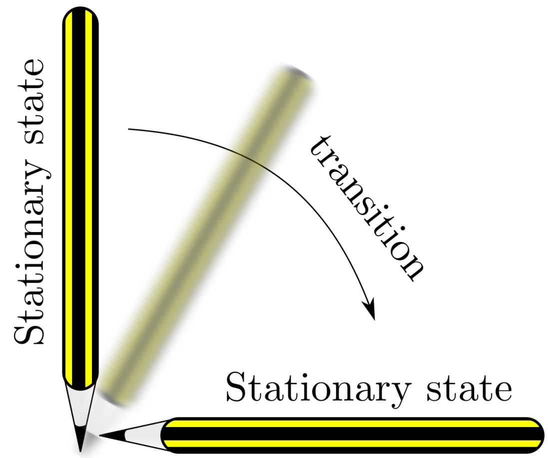

Figure 4.1: Stationary states of a pencil.

Example 4.0.19 Balls on surfaces

Consider Figure 4.2, using your intuition, state which of the balls are stationary and which are stable, assuming that the surface that they are moving on has a small amount of friction.

Moving beyond the case of categorising drawings let us consider the specific mathematical example of the logistic equation.

Example 4.0.20 Stationary states and stability of the logistic equation

The non-dimensionalised logistic equation is (as we have seen before)

$$\dot{u} = u(1 - u). \tag{4.3}$$

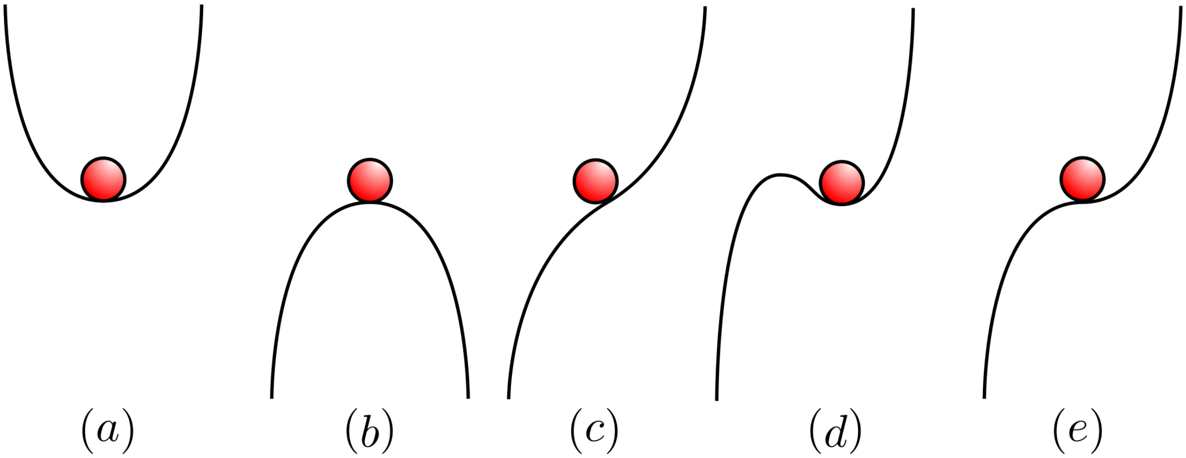

Figure 4.2: Which balls are stationary and stable?

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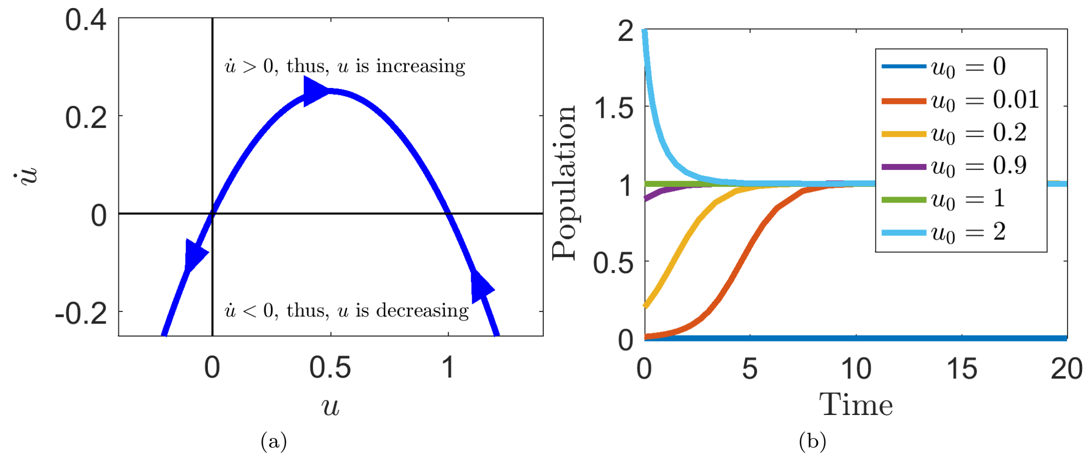

Figure 4.3: Illustrating the stationary states and stability characteristics of the logistic equation (4.3). (a) A plot of the curve in $(u, \dot{u})$ coordinates. (b) Multiple simulations of equation (4.3) with different initial conditions, u_0, noted in the legend.

As we saw in example 4.0.20, when working with a single dependent variable, u, all of the stationary and stability information can be gained from plotting the 'phase plane', *i.e.* the $(u, \dot{u})$ coordinate system. Specifically, the stationary states are given where the curve crosses the 'x-axis' (*i.e.* $\dot{u} = 0$) and the stability of the states can be given by considering where the the system is increasing or decreasing in the vicinity of the stationary point (*i.e.* the sign of $\dot{u}$).

However, drawings can be misleading and may not be possible if the function on the right-hand side of the differential equation is too complicated. Equally, drawing the curve will not work if the system has more than one variable. Thus, we need an analytical method to characterise the stability of a system, which can be generalised to higher order systems.

4.1 Linear stability

The crux of this method is to consider the dynamics of an ODE system near its stationary points. To do this we substitute a solution into the equations that is a perturbation about the steady state. Using Taylor series we expand the system in terms of the perturbation and keep only the linear terms as we are assuming that the perturbation is small. Since the system is now linear we can solve the approximate equations completely and, thus, they will tell us what dynamics to expect close to the steady states.

Theorem 4.1.1. *Suppose u_s is a steady state of the one dimensional ODE,*

$$\dot{u} = F(u), \tag{4.4}$$

then u_s is linearly stable if $dF(u_s)/\,du < 0$ and linearly unstable if $dF(u_s)/\,du > 0$.

Proof.

□

We make a number of remarks about the theorem's statement and proof:

Example 4.1.21 Linearising around multiple steady states

Consider the following equation, which can be used to model harvesting, with constant effort, $E > 0$, of a population, u,

$$\dot{u} = f(u) = \underbrace{\frac{2u^2}{u^2+1}}_{\text{Population growth with saturating rate}} - \underbrace{Eu}_{\text{Constant harvesting effort}}. \tag{4.10}$$

]

]

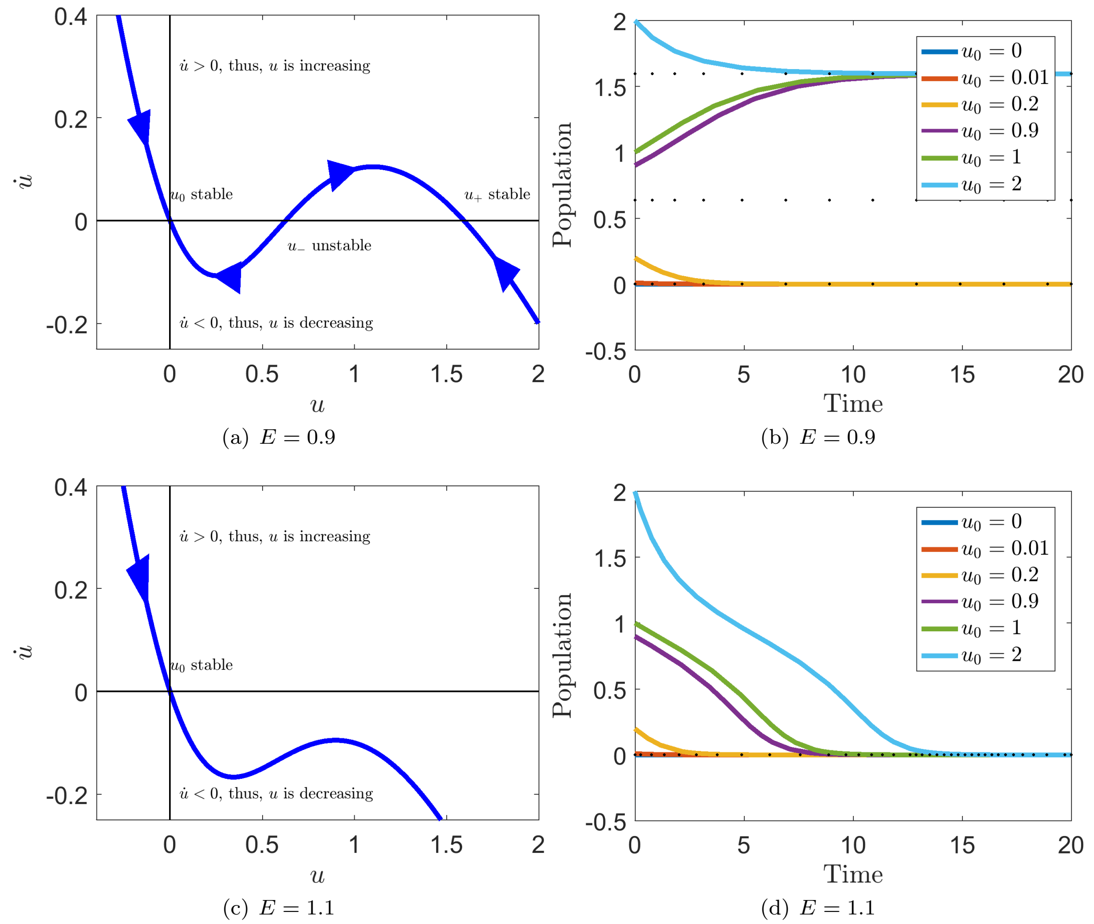

Figure 4.4: Illustrating the stationary states and stability characteristics of equation (4.10). Top row: case when $E = 0.9$ and three steady states exist. (a) A plot of the curve in $(u, \dot{u})$ coordinates. (b) Multiple simulations of equation (4.10) with different initial conditions, u_0, noted in the legend. Bottom row: case when $E = 1.1$ and only $u_0 = 0$ is a stationary state. (a) A plot of the curve in $(u, \dot{u})$ coordinates. (b) Multiple simulations of equation (4.10) with different initial conditions, u_0, noted in the legend.

4.2 Bifurcations and hysteresis

As seen in example 4.1.21 the existence and stability of steady states can depend on model parameters, here E.

Definition 16. *A* ***bifurcation point*** *of a system is a point at which the characteristics of the steady states change. This can be either in number of steady states, or their stability.*

In example 4.1.21, $E = 1$ is a bifurcation point of the system.

The amount of information gained in example 4.1.21 can be quite overwhelming. Thus, we use a bifurcation diagram to illustrate the complexity in a simple way. Specifically, Figure 4.5 shows equation (4.12) as a function of E and captures the following features:

- u_0 always exists;
- $u_\pm$ exists whenever $E < 1$;
- u_0 and u_+ are stable when they exist;
- u_- is always unstable when it exists.

Figure 4.5 can also be used to tell us what happens in the case when we think about varying E and how it can have unexpected impacts on the system. Consider the case where we are happily fishing in a lake, which can be modelled by equation (4.10), such that $E = 0.7$ and, so the level of fish in the lake is stable at around $u_+(0.7) \approx 2$ (taken from Figure 4.5[1]).

Suppose we become greedy and increase our effort, thus pushing E to 1.1. The population begins to die out rapidly, due to over fishing. Critically, we notice the huge reduction in population size and reduce our effort to the previous stable case, $E = 0$. Unfortunately, we have left it too late and the population has reduced past $u_-(0.7)$, thus, even though a fish population still exists and our harvesting rate $E < 1$, the population will still die. This is an example of hysteresis.

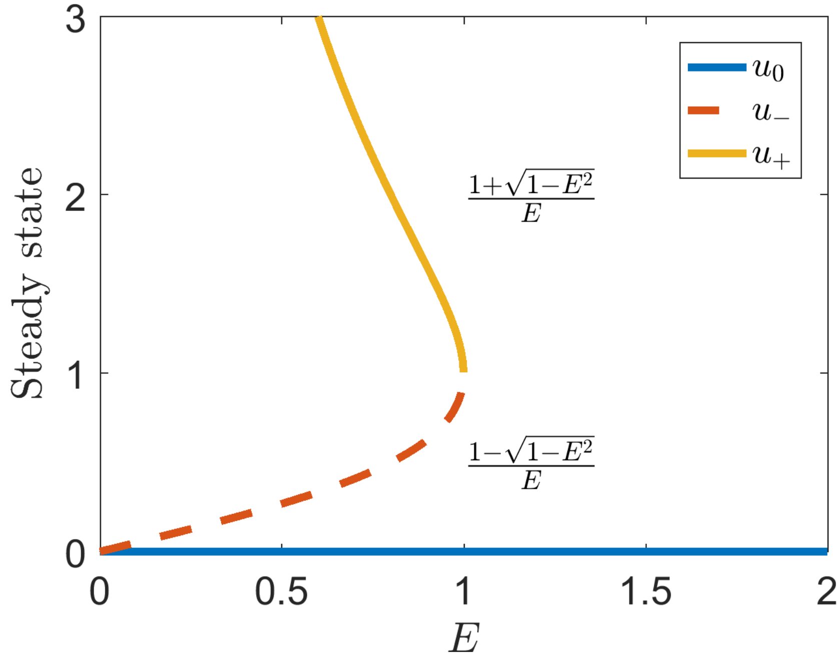

Figure 4.5: Bifurcation plot of equation (4.10). The dependence of the existence and stability of u_0, u_- and u_+ on E is plotted. The u_- is dashed to illustrate that the steady states are always unstable, whilst u_0 and u_+ are stable wherever they exist. However, the steady states, $u_\pm$, disappear for $E > 1$.

Definition 17. *A system exhibits* ***hysteresis*** *if, when a parameter of the system is altered and subsequently returned to the initial value, the system does not return to its original state.*

Example 4.1.21 and Figure 4.5 demonstrate a simple way showing a system exhibits hysteresis. Specifically, you should:

[1] We are assuming that the system is non-dimensionalised, so I am not saying this is two fish, or two tons, just a measure of two times some scale.

1. derive the steady states and their dependence on any given parameters;
2. derive the stability of the steady states and their dependence on any given parameters;
3. note any bifurcation points of the parameters;
4. define and illustrate the characteristics of the system before and after the bifurcation point;
5. consider the system before the bifurcation point;
6. identify what happens to the system as the bifurcation increases passes its bifurcation point;
7. identify what happens to the system as the bifurcation is reduced to its initial value;
8. if the system state in point 5 is the same as 7 then the system does not exhibit hysteresis. Otherwise hysteresis is present in the system.

4.3 Check list

By the end of this chapter you should be able to:

- ☐ derive the steady states of a system;
- ☐ categorise the stability of the steady states using graphical means;
- ☐ prove that the stability of a steady state depends on the sign of the first derivative (with respect the system variable) evaluated at the steady state;
- ☐ analytically specify the parameter dependencies of the steady states and stability criteria;
- ☐ identify bifurcation points;
- ☐ plot the steady state curves in a bifurcation diagram;
- ☐ identify whether a system could exhibit hysteresis.

Chapter 5

Stability of ODE systems

In the last chapter we focused on systems of single variables. We now extend our stability theory to account for any number of variables.

First, we note that the definition of a steady state immediately generalises to any number of variables. Specifically, if we have n variables, $\boldsymbol{u} = (u_1, \dots, u_n)$ then there must be n ODEs, $\boldsymbol{F}(\boldsymbol{u}) = (F_1(u_1, \dots, u_n), \dots, F_n(u_1, \dots, u_n))$, one for each variable, in order for the system to be uniquely defined. Thus, the steady states, $\boldsymbol{u}_s$, are found from solving $\boldsymbol{F}(\boldsymbol{u}_s) = 0$. The derivation of linear stability also extends to higher similarly, however, we need to first define the Jacobian.

Definition 18. *The Jacobian, $\boldsymbol{J}$, of an ODE system,*

$$\dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u}), \tag{5.1}$$

is the matrix of partial derivatives of each function, with respect to each argument,

$$\boldsymbol{J} = \left[\frac{\partial F_i}{\partial u_j}\right]_{i,j=1,\dots,n} = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial F_n}{\partial u_1} & \frac{\partial F_n}{\partial u_2} & \cdots & \frac{\partial F_n}{\partial u_n} \end{bmatrix}. \tag{5.2}$$

For brevity, it is common practice to write a partial derivative as a subscript, *i.e.*

$$\frac{\partial F}{\partial u} = F_u. \tag{5.3}$$

Equally, unless otherwise specified, we assume that the Jacobian is evaluated at the steady state.

Theorem 5.0.1. *Suppose $\boldsymbol{u}_s$ is a steady state of the ODE system*

$$\dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u}). \tag{5.4}$$

The linear stability of $\boldsymbol{u}_s$ will depend on the eigenvalues of the Jacobian.

Proof.

□

Critically, now we are in higher dimensions, the eigenvalues can have complex values. If we let $\lambda_i = \alpha_i + \beta_i I$ then

$$\exp(\lambda_i t) = \exp(\alpha t)\left(\cos(\beta_i t) + I\sin(\beta_i t)\right). \tag{5.11}$$

Thus, real part of the eigenvalues determines the growth rate, whilst the imaginary part determines the frequency of oscillation in time. Namely, if all eigenvalues have negative real parts the small perturbations die out. However, if there is at least one eigenvalue with positive real part then the perturbations will grow and the steady state is not stable.

5.1 Steady state classification of two-dimensional systems

In the last section we demonstrated that the stability of the steady states depends on the eigenvalues of the Jacobian. In this section, we restrict ourselves to considering two-dimensional systems only and illustrate that all steady states can be defined to fit a small number of categories.

The following derivation is going to be an explicit form of the proof shown in the last section. The reason for this is that the condensed vector form of proof is less transparent and it is always good to see a full sprawling derivation to illustrate the subtleties. Critically, although you may be specifically be required to reproduce the proof, in a specific case you can generally just calculate the Jacobian straight away and not bother with the initial linearisation steps.

Consider the general two-dimensional system

$$\dot{u} = f(u, v), \tag{5.12}$$

$$\dot{v} = g(u, v). \tag{5.13}$$

Let (u_s, v_s), be a steady state, *i.e.* $f(u_s, v_s) = g(u_s, v_s) = 0$. Linearising around the steady state with $u = u_s + \epsilon_1$ and $v = v_s + \epsilon_2$ produces

$$\begin{aligned} \dot{\epsilon_1} &= f(u_s + \epsilon_1, v_s + \epsilon_2), \\ &\approx \underbrace{f(u_s, v_s)}_{=0} + f_u(u_s, v_s)\epsilon_1 + f_v(u_s, v_s)\epsilon_2. \end{aligned} \tag{5.14}$$

and, similarly,

$$\dot{\epsilon_2} = g_u(u_s, v_s)\epsilon_1 + g_v(u_s, v_s)\epsilon_2. \tag{5.15}$$

The eigenvalues will, thus, depend on the four parameters (f_u, f_v, g_u, g_v). Note that we have not restricted the signs of these parameters. Thus, any of them could be positive or negative. Due to not knowing the signs of the derivatives we are unable to non-dimensionalise them out. However, in a specific example, this maybe be possible, thus, reducing down the number of free parameter groups in the steady state and stability conditions.

Combining equations (5.14) and (5.15) we derive

$$\begin{pmatrix} \dot{\epsilon_1} \\ \dot{\epsilon_2} \end{pmatrix} = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}. \tag{5.16}$$

Thus, we are left to find the eigenvalues of

$$\boldsymbol{J} = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}, \tag{5.17}$$

5.1.1 $D < 0$

If $D < 0$ then $\lambda_\pm$ are both real. Moreover $T^2 - 4D > T^2$, thus $\lambda_- < 0 < \lambda_+$. Since one of the eigenvalues has positive real part the steady state is unstable. More specifically, it is called a 'saddle point'.

Definition 19. *A steady state is a* ***saddle point*** *if not all of the real parts of the eigenvalues have the same sign.*

For a more intuitive understanding such a steady state is called a saddle point because the trajectories want to converge along one direction and diverge along another (see Figure 5.1), *i.e.* the energy surface around the steady state is shaped like a saddle.

Example 5.1.22 Saddle point

Consider the system

$$\dot{u} = u/(v+2), \tag{5.21}$$

$$\dot{v} = -v/(u+1). \tag{5.22}$$

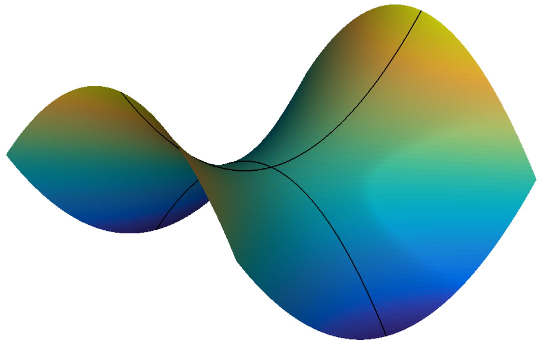

Figure 5.1: A Saddle shaped surface. If a marble is placed at the top of the surface its trajectory will initially tend to the centre, before diverging to infinity.

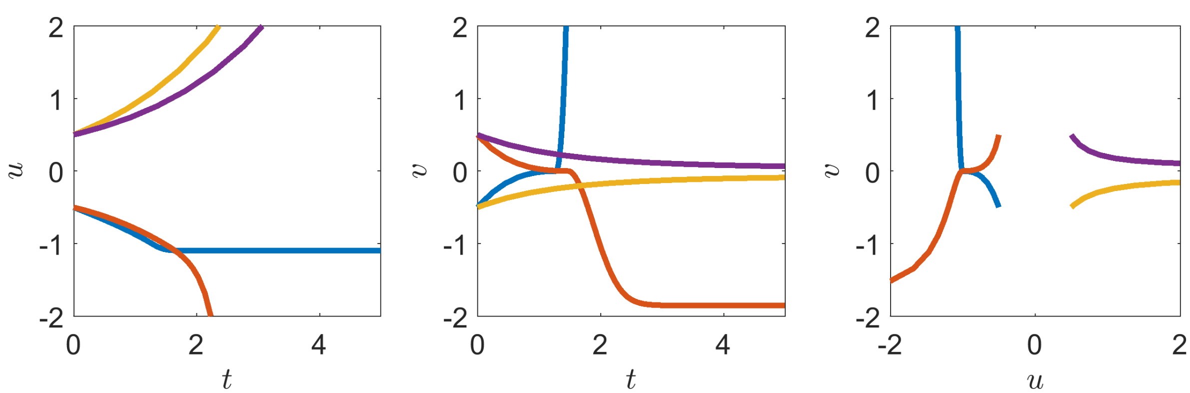

Figure 5.2: Saddle point system trajectories, solutions of equations (5.21) and (5.22). Left: plot of (u, t) for different initial conditions. Middle: plot of (v, t) for different initial conditions. Right: plot of (u, v) combining the solutions from the left and middle plots. Trajectories from the same initial conditions have the line colour across all three figures. All trajectories have a least one coordinate that grows without bound.

5.1.2 $D > 0$

If $D > 0$ then the eigenvalues may be real or imaginary. However, what is certain is $T^2 - 4D < T^2$. Thus the sign of the real part of the eigenvalue depends on the sign of T. Hence, we break this subsection up into to further cases.

5.1.2.1 $T = 0$

If $T = 0$ then the eigenvalues are purely imaginary, $\mathrm{Re}(\lambda_-) = \mathrm{Re}(\lambda_-) = 0$. This means that the linear analysis suggests that the trajectories neither growing, nor shrinking, the trajectories, simply oscillate around the steady state. Such points are called centre points.

Note that this is a marginal case and higher order terms may still cause the system to converge or diverge, but slowly, thus, although the linear analysis says that the trajectory simply oscillates we should go to higher orders to check, but this is outside the scope of this course.

Example 5.1.23 Centre point

Consider the system

$$\dot{u} = -v - u^2, \tag{5.24}$$
$$\dot{v} = -u + v^2. \tag{5.25}$$

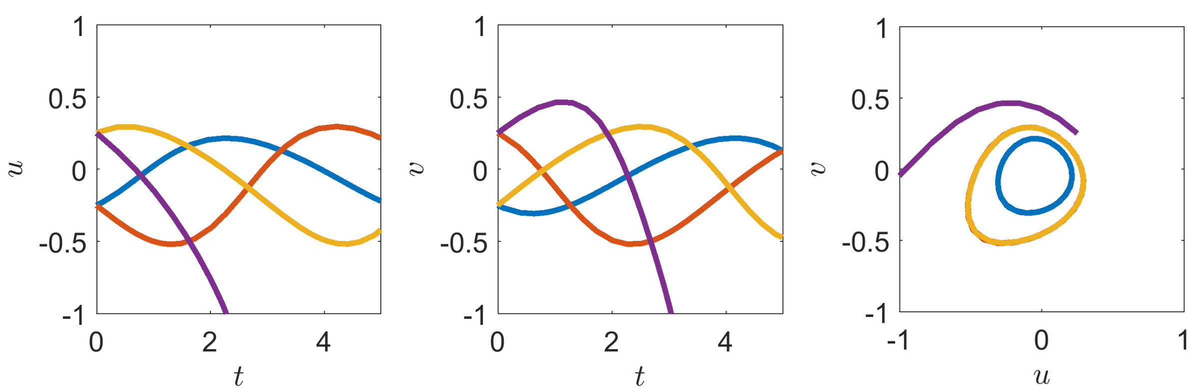

Figure 5.3: Stable system trajectories, solutions of equations (5.24) and (5.25). Left: plot of (u, t) for different initial conditions. Middle: plot of (v, t) for different initial conditions. Right: plot of (u, v) combining the solutions from the left and middle plots. Trajectories from the same initial conditions have the line colour across all three figures.

Example 5.1.23 demonstrates well that our analysis is only valid near the steady state. Namely, three out of the four initial conditions appear to form closed loops that oscillate around (0,0) (see the right image of Figure 5.3). However, one of the initial conditions diverges away.

5.1.2.2 $T < 0$

If $T < 0$ then $\mathrm{Re}(\lambda_-) \leq \mathrm{Re}(\lambda_-) < 0$ and, so, all eigenvalues have negative real part, meaning that the steady state is stable. This case can further be sub-divided depending on the sign of $T^2 - 4D$. Namely, if $T^2 - 4D > 0$ the steady state is a stable node whilst if $T^2 - 4D < 0$ the steady state is a stable spiral.

Example 5.1.24 Stable node

Consider the system

$$\dot{u} = -u + v, \tag{5.27}$$

$$\dot{v} = -v/(u+1). \tag{5.28}$$

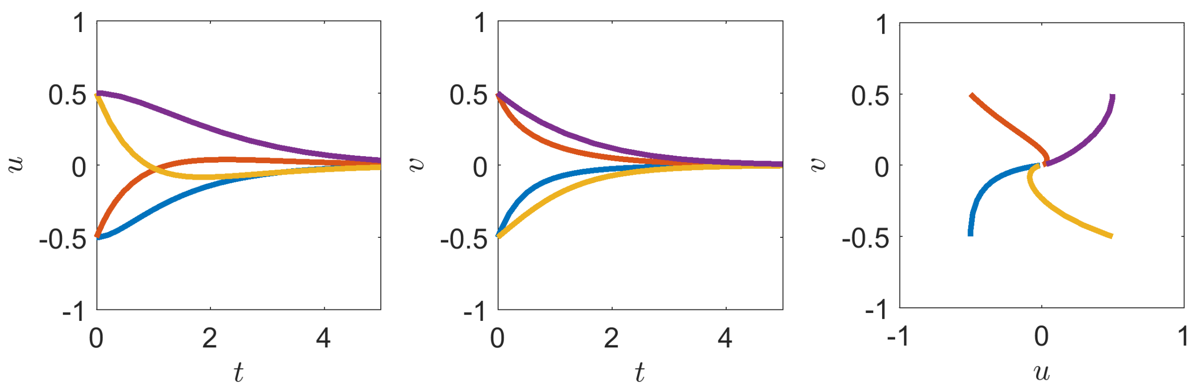

Figure 5.4: Stable system trajectories, solutions of equations (5.27) and (5.28). Left: plot of (u, t) for different initial conditions. Middle: plot of (v, t) for different initial conditions. Right: plot of (u, v) combining the solutions from the left and middle plots. Trajectories from the same initial conditions have the line colour across all three figures. All trajectories converge to (0,0).

Example 5.1.25 Stable spiral

Consider the system

$$\dot{u} = -\frac{u}{1+v} + v, \tag{5.30}$$

$$\dot{v} = -u + \frac{v}{v+2}. \tag{5.31}$$

The unique steady state is $(u, v) = (0, 0)$. The Jacobian is

$$\boldsymbol{J} = \begin{bmatrix} -\frac{1}{1+v} & \frac{v^2+u+2v+1}{(1+v)^2} \\ -1 & \frac{2}{(2+v)^2} \end{bmatrix} \Longrightarrow \boldsymbol{J}(0,0) = \begin{bmatrix} -1 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}. \tag{5.32}$$

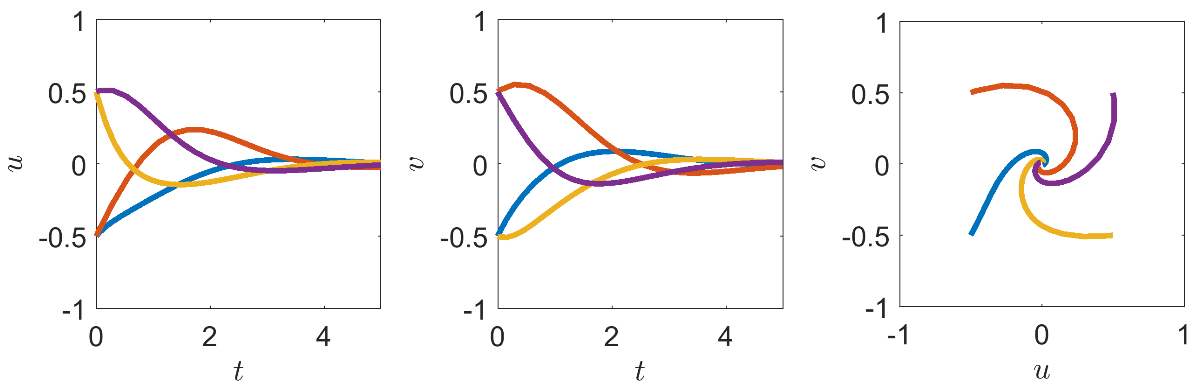

Figure 5.5: Saddle point system trajectories, solutions of equations (5.27) and (5.28). Left: plot of (u, t) for different initial conditions. Middle: plot of (v, t) for different initial conditions. Right: plot of (u, v) combining the solutions from the left and middle plots. Trajectories from the same initial conditions have the line colour across all three figures. All trajectories converge to (0,0).

5.1.2.3 $T > 0$

Opposite to the previous case $0 < \mathrm{Re}(\lambda_-) \le \mathrm{Re}(\lambda_-)$ and, so, all eigenvalues have positive real part, meaning that the steady state is unstable. Similar to the previous naming convention, if $T^2 - 4D > 0$ the steady state is an unstable node whilst if $T^2 - 4D < 0$ the steady state is a unstable spiral.

Example 5.1.26 Unstable node

Consider the system

$$\dot{u} = \frac{u}{1+v^2}, \tag{5.33}$$
$$\dot{v} = u + v/(2+v^2). \tag{5.34}$$

The unique steady state is $(u, v) = (0, 0)$. The Jacobian is

$$\boldsymbol{J} = \begin{bmatrix} \frac{1}{1+v^2} & -\frac{2uv}{(1+v^2)^2} \\ 1 & -\frac{v^2-2}{(2+v^2)^2} \end{bmatrix} \Longrightarrow \boldsymbol{J}(0,0) = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}. \tag{5.35}$$

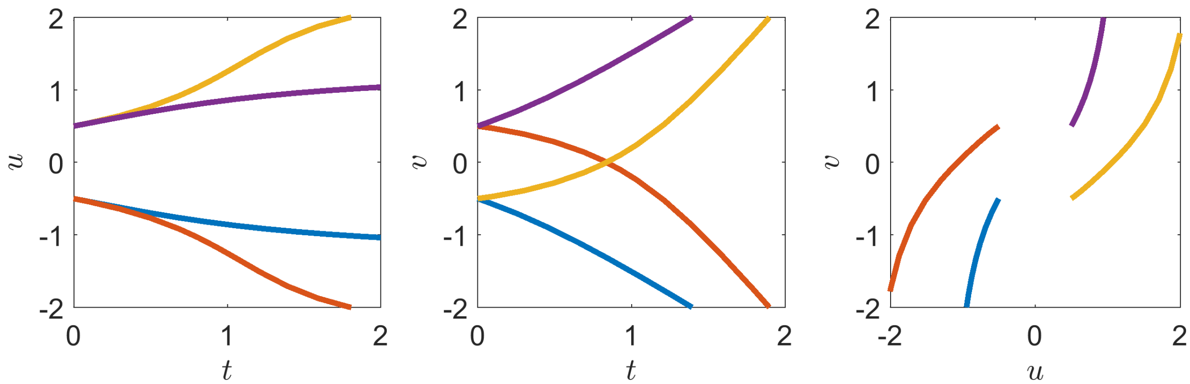

Figure 5.6: Unstable system trajectories, solutions of equations (5.33) and (5.34). Left: plot of (u, t) for different initial conditions. Middle: plot of (v, t) for different initial conditions. Right: plot of (u, v) combining the solutions from the left and middle plots. Trajectories from the same initial conditions have the line colour across all three figures. All trajectories diverge away from (0,0).

Example 5.1.27 Unstable spiral

Consider the system

$$\dot{u} = \frac{u}{1+v^2} + v, \tag{5.36}$$

$$\dot{v} = -u + \frac{v}{v^2+2}. \tag{5.37}$$

5.2 Comments

Note that we do not consider the marginal cases $D = 0$ or $T^2 = 4D$. This is because these cases need to be approached on a case by case basis, because it is the non-linear terms which may dominate the kinetics. Even in the case $T = 0$, where we generate centre points, we have seen that the analysis breaks down when the initial condition is too far away from the steady state.

All of the above definitions can be encompassed in a single diagram of the (T, D) plane (see Figure 5.8). Critically, although Figure 5.8 is useful, it is suggested that instead of calculating the trace and determinant of the Jacobian and figuring out where in the stability diagram that you lie, you calculate the eigenvalues of any system explicitly.

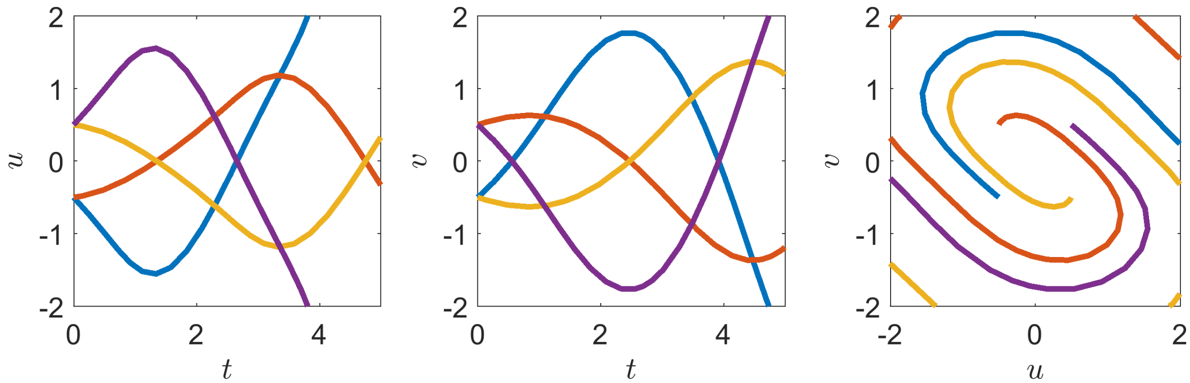

Figure 5.7: Unstable spiral system trajectories, solutions of equations (5.36) and (5.37). Left: plot of (u, t) for different initial conditions. Middle: plot of (v, t) for different initial conditions. Right: plot of (u, v) combining the solutions from the left and middle plots. Trajectories from the same initial conditions have the line colour across all three figures. All trajectories diverge away from (0,0).

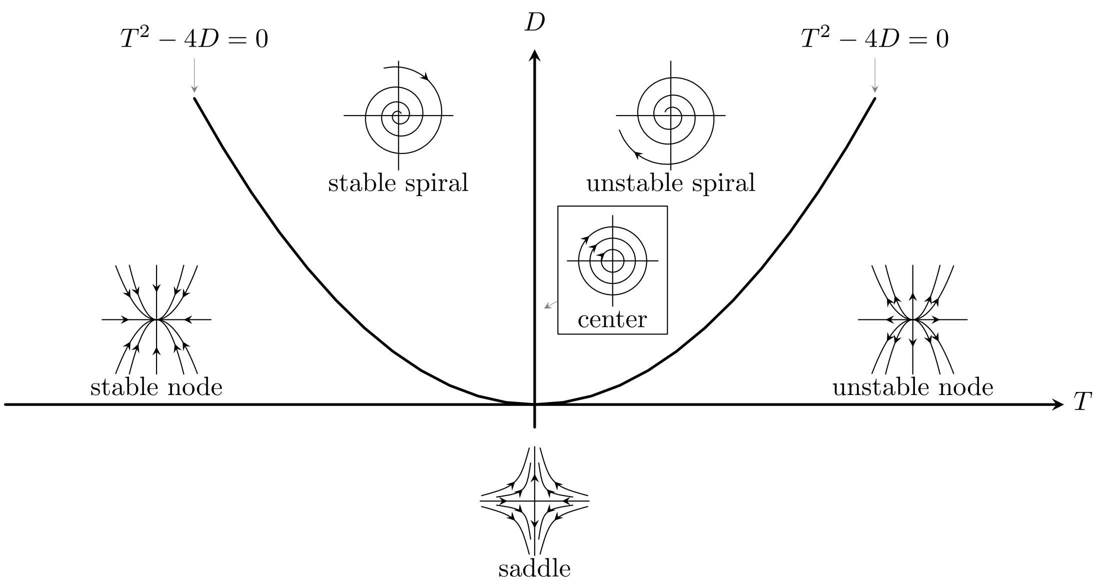

Figure 5.8: Stability diagram in terms of the trace and determinant of the Jacobian.

5.3 Check list

By the end of this chapter you should be able to:

- ☐ derive the steady states of an ODE system;
- ☐ prove that the stability of a steady state depends on the eigenvalues of the Jacobian of a system;
- ☐ explicitly derive the eigenvalues of the Jacobian of a two-species system;
- ☐ use the eigenvalues to characterise steady states in terms of whether they are centres, (un)stable nodes, (un)stable spirals or saddle points.

Chapter 6

Phase plane analysis

In the last chapter we considered ODE systems with only a single steady state. Even though we are going to restrict ourselves to a two-dimensional ODE systems, such systems can have many non-trivial steady states. We need to be able to combine such information to give an idea of what the global dynamics will be, even though we only have local analysis.

This will be a graphical method and in some ways provides a two-dimensional extension to the methods seen in Chapter 4. Specifically, in Chapter 4 we could understand the entire dynamics of the system in the $(u, \dot{u})$ plane, when we have two variables, we consider the (u, v) plane instead, which is known as a 'phase plane'. To construct a phase plane, instead of considering a single trajectory as in the (t, u) simulation, we consider the motion of a trajectory across all points in the (u, v) space. To aid in our understanding we introduce a new concept.

Definition 20. *Consider an ODE system*

$$\dot{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{u}), \tag{6.1}$$

where $\boldsymbol{F}(\boldsymbol{u}) = (F_1(u_1, \dots, u_n), \dots, F_n(u_1, \dots, u_n))$. *The nullclines are the curves defined by*

$$F_i(u_1, \dots, u_n) = 0, \tag{6.2}$$

for all $i = 1, \dots, n$.

Nullclines are a useful concept because on each separate curve the dynamics of at least one variable is stationary, thus, the direction across a nullcline is simplified. Moreover, if all nullclines meet at a given point all dynamics must be stationary, *i.e.* by definition all nullclines meet at steady states.

Example 6.0.28 Nullclines

Consider the system

$$\dot{u} = v - (u - 2)(u - 3), \tag{6.3}$$
$$\dot{v} = v - \ln(u), \tag{6.4}$$

in the half plane $u > 0$.

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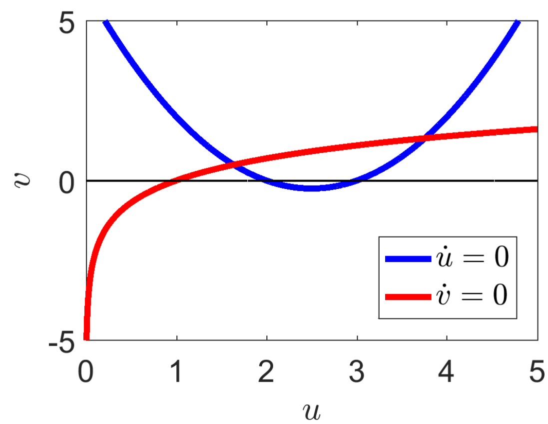

Figure 6.1: Plot of the nullclines of equations (6.3) and (6.4).

Consider a general nullcline, for example $\dot{u} = 0$. This line must delineate the regions where the derivative is positive and negative. Namely, on one side of the line $\dot{u} > 0$, whilst on the other $\dot{u} < 0$. The same can be said of the $\dot{v} = 0$. Thus, the nullclines segment the (u, v) into regions of different dynamics. We return to example 6.0.28 with this knowledge and specify the signs of the derivatives in each region.

Example 6.0.29 Derivative signs

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From this example we have seen that phase planes are helpful diagrams, which encapsulate lots of stability information. However, as illustrated in comparing the diagram with the actual analytical values of the eigenvalues it can be difficult to tell the difference between (un)stable nodes and (un)stable spirals. Equally, as we saw in the last chapter, sketches only provide the correct insight if you draw the system correctly. If there had been a parameter in this system that we could vary then there may have been a stability case, dependent on the parameter, that we would miss if we had only drawn one diagram. Thus, a phase plane should always be backed up with linear analysis. The linear analysis provides the local information, whilst the phase plane allows us to approximately see how all the dynamics fit together.

6.1 Check list

By the end of this chapter you should be able to:

- ☐ define what a nullcline is;
- ☐ understand the relationship between steady states and the points at which nullclines cross;
- ☐ plot nullclines;
- ☐ sketch arrows showing general trajectory directions on the phase plane;
- ☐ interpret the stability of the steady states from the information plotted on a phase plane.

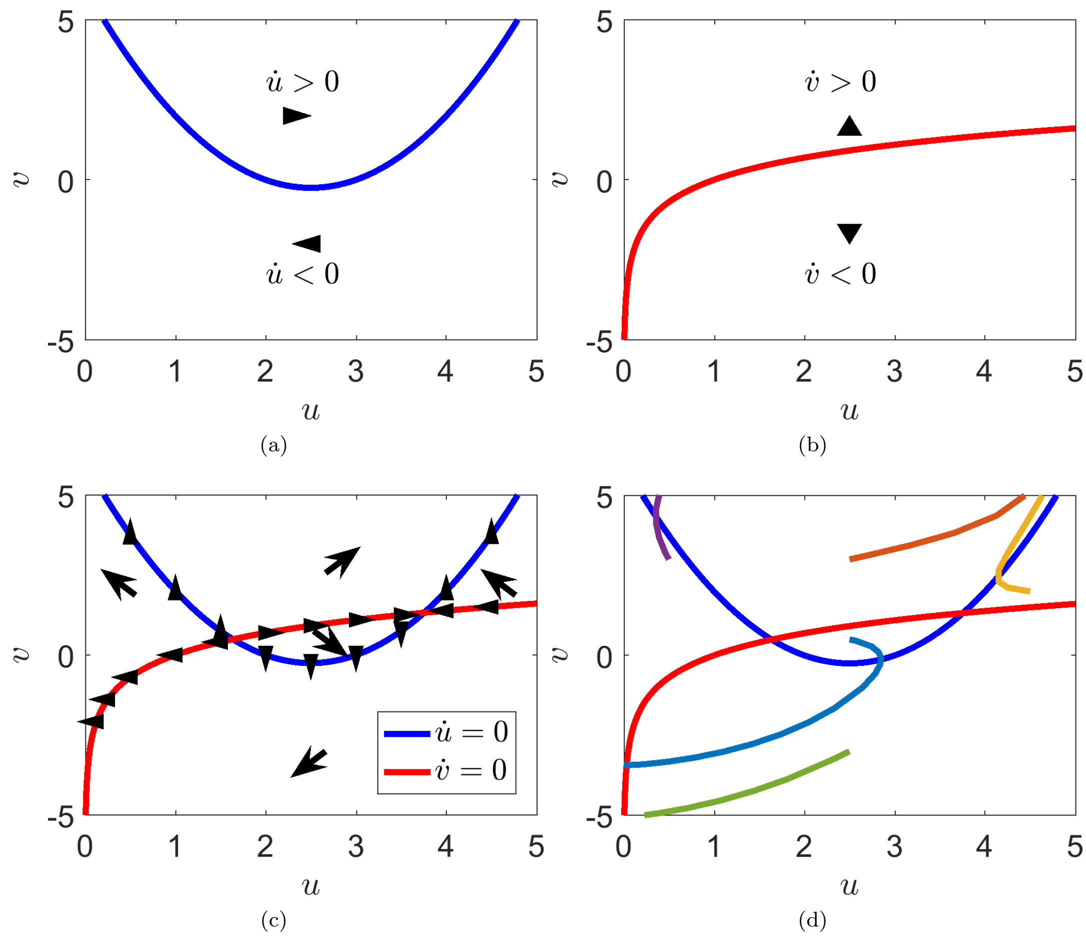

Figure 6.2: Specifying the signs of the derivatives on either side of the (a) $\dot{u}$ and (b) $\dot{v}$ nullcline. The arrowheads indicate the general direction that a trajectory will be heading. These results can then be combined into the direction plots seen in (c). Finally, in (d), we simulate a number of trajectories, which demonstrate that the arrows in (c) provide the correct general idea.

Chapter 7

Putting it all together

Throughout this course we have learned how to construct an ODE system from an intuitive understanding of the dynamics (Chapter 2). From this point we simplify the system using non-dimensionalisation, which reduces the number of free parameters that we need to consider (Chapter 3). Having combined the system parameters into smaller groupings we are able derive how the systems steady states and stability rest on these parameters (chapters 4 and 5). Finally, we saw how to illustrate these local dependencies using a phase plane, in order to better understand the global phenomena (Chapter 6). In this chapter we combine all of these techniques and completely analyse a number of examples.

7.1 Fish example

Example 7.1.30 Fishing

Consider a lake with fish that attract fishermen. We wish to model the fish-fishermen interaction under the following assumptions:

- in the absence of fishing the fish population growth is proportional to the current population, but is suppressed by binary competition;
- the presence of fishermen suppresses the fish growth rate at a rate jointly proportional to the size of the fish and fisherman populations;
- fishermen are attracted to the lake at a rate directly proportional to the number of fish in the lake;
- binary competition between fishermen discourages fishermen.

7.1.1 Model the system

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7.1.2 Non-dimensionalise

$$\dot{F} = k_1 F - k_{-1} F^2 - k_2 F M, \quad F(0) = F_0, \tag{7.7}$$

$$\dot{M} = k_3 F - k_4 M^2, \quad M(0) = M_0. \tag{7.8}$$

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7.1.3 Identify steady states

7.1.4 Calculate stability

]

7.1.5 Plot the phase-plane

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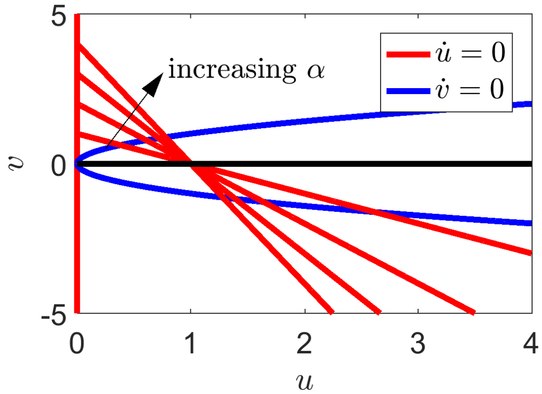
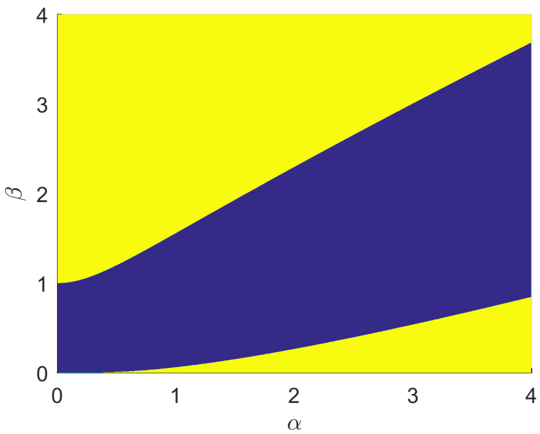
7.1.6 What does it mean?

7.2 Pendulum example

The last example was explicitly described throughout and verbose. This example of the pendulum equation will be more terse.



(a)



(b)

Figure 7.1: (a) Nullclines of equations (7.11) and (7.12). (b) Plotting the surface defined by equation (7.20). The yellow region illustrates the region where $\mathrm{Tr}(\boldsymbol{J})^2 - 4\mathrm{Det}(\boldsymbol{J}) > 0$ making the steady state a stable node, whilst the blue region is where $\mathrm{Tr}(\boldsymbol{J})^2 - 4\mathrm{Det}(\boldsymbol{J}) < 0$ and the steady is a stable spiral.

Example 7.2.31 Pendulum

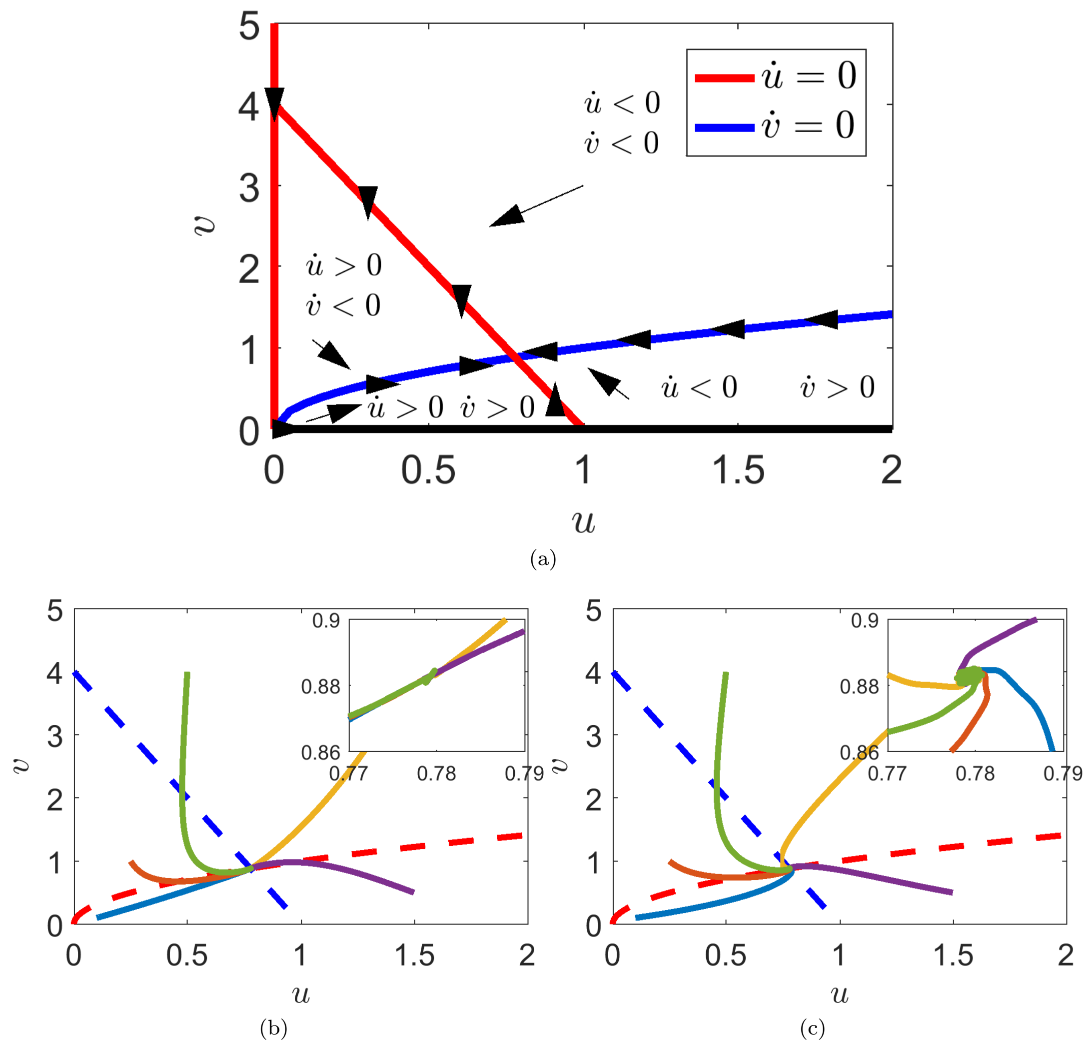


Figure 7.2: (a) Dynamics of equations (7.11) and (7.12) in all regions and on the nullclines. (b), (c) Multiple simulations of equations (7.11) and (7.12) with different initial conditions. In (b) the parameters are $\alpha = 4$, $\beta = 4$, making the steady state a stable node (see Figure 7.1(b)). In (c) the parameters are $\alpha = 4$, $\beta = 1$, making the steady state a stable spiral (see Figure 7.1(b)). The insets of each image demonstrate the dynamics very close to the steady state.

small perturbation away from the downward vertical then the dynamics cycles back and forth as
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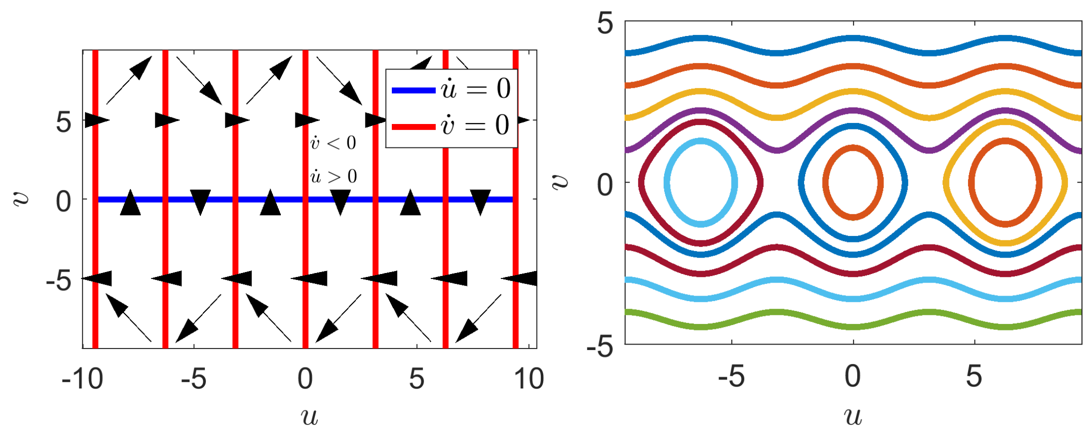

(a) (b)

Figure 7.3: (a) Dynamics of equations (7.25) and (7.26) in all regions and on the nullclines. (b) Multiple simulations of equations (7.25) and (7.26) with different initial conditions.

7.3 Check list

By the end of this chapter you should be able to:

- ☐ use all the tools developed throughout these notes to completely analyse a system of first order ordinary differential equations in terms of the steady states available and their stability .