I. INTRODUCTION

Some problems regarding the elliptical cylinders can be solved by using an analytical approach like that applied to circular cylinders: one separates the variables and the exact solution is given by expansions involving angular and radial Mathieu functions. These functions have been introduced by Emile Mathieu in 1868 by investigating the vibrating modes in an elliptic membrane [1]. Details (tables or relations) concerning the Mathieu functions can be found for example in [2–12]. For circular cylinders the solutions involve readily available trigonometric and Bessel functions, while for elliptical cylinders there are still controversial and incomplete algorithms for computing the Mathieu functions. One reason for the lack of algorithms for Mathieu functions was probably the complicated and various notation existent in the literature. A main purpose for us was to simplify as much as possible the notation. With a simplified and self-contained notation, the use of Mathieu functions should be as simple as the use of Bessel functions. We largely followed the notations used by Stratton [6] and Stamnes [11, 12], but we introduced further simplifications. All formulas required to get the Mathieu functions are given explicitly. Tables of numerical values are provided. Several examples are given. Mathieu functions applied to plane wave scattering by elliptical cylinders can be found in [13, 14].

II. FUNDAMENTALS

A. Elliptical cylinder coordinates

Let consider an ellipse in the plane (x, y) defined by equation $(x/x_0)^2 + (y/y_0)^2 = 1$ with $x_0 > y_0$. The semifocal distance f is given by $f^2 = x_0^2 - y_0^2$ and the eccentricity is $e = f/x_0 < 1$. The elliptic cylindrical coordinates (u, v, z) are defined by relations

$$x = f \cosh u \, cosv, \qquad y = f \sinh u \, \sin v, \qquad z = z \tag{1}$$

with $0 \leq u < \infty$ and $0 \leq v \leq 2\pi$. In terms of (ξ, η, z), with $\xi = \cosh u$ and $\eta = \cos v$, the elliptic cylindrical coordinates are defined by relations

$$x = f\xi\,\eta, \qquad y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \qquad z = z. \tag{2}$$

The contours of constant u are confocal ellipses (of semiaxes $x_0 = f\xi$, $y_0 = f\sqrt{\xi^2 - 1}$) and those of constant v are confocal hyperbolas (see Fig. 1). The z axis coincides with the cylinder axis. The scale factors h_j, with $j = \xi, \eta, z$, are defined like as for any coordinate transformation [6],

$$h_\xi = f\frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{\xi^2 - 1}}, \qquad h_\eta = f\frac{\sqrt{\xi^2 - \eta^2}}{\sqrt{1 - \eta^2}}, \qquad h_z = 1. \tag{3}$$

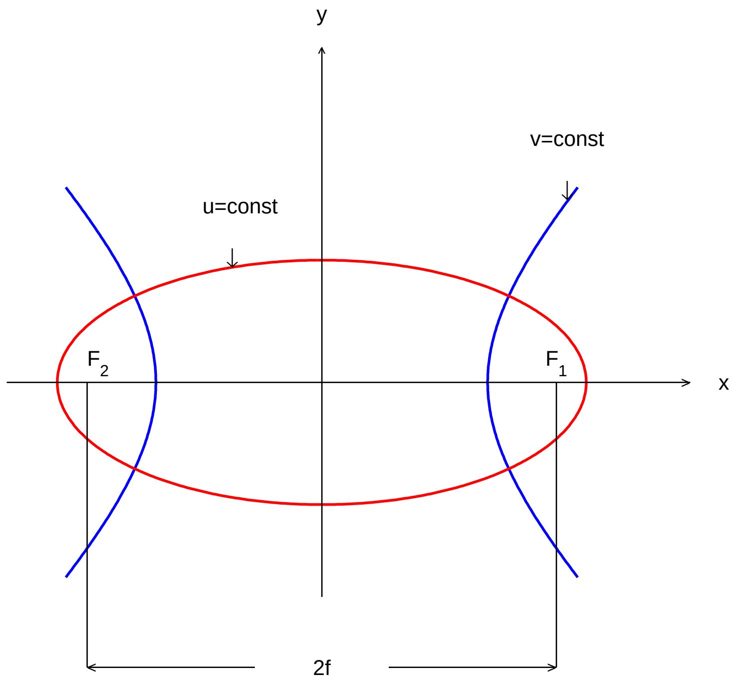

FIG. 1: Elliptic cylindrical coordinates. F_1 and F_2 are the foci of the ellipse. f is the semifocal length.

B. Wave equation in elliptic cylindrical coordinates

The scalar wave equation $(\nabla^2 + k^2)U(\mathbf{r}) = \mathbf{0}$, where $\mathbf{r}$ is the position vector, k is the wave number, $k = 2\pi\sqrt{\epsilon}/\lambda$, ϵ is the permittivity, and λ is the wavelength in vacuum, when expressed in elliptic cylindrical coordinates becomes

$$\left[\frac{2}{f^2\left(\cosh 2u - \cos 2v\right)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]U(u, v, z) = 0. \tag{4}$$

Using a solution of the form $U = Z(z)S(v)R(u)$ gives

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k_z^2\right) Z(z) = 0, \tag{5}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}v^2} + (a - 2q\cos 2v)\right] S(v) = 0, \tag{6}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}u^2} - (a - 2q\cosh 2u)\right] R(u) = 0, \tag{7}$$

where k_z is the wave vector component on z direction, $q = k_\tau^2 f^2/4$, with $k_\tau^2 = k^2 - k_z^2$, and a is separation constant. Equation (5) has solution $Z(z) = \exp(ik_z z)$. Equations (6) and (7) are known as the angular and radial Mathieu equations, respectively.

III. ANGULAR MATHIEU FUNCTIONS

In this version, only the periodic solutions of period π or 2π are considered. For a given order n, there are four categories of periodic solutions satisfying (6):

$$\begin{array}{lll} 1 & \text{even-even:} & S_{ee}(v,q,n) = \sum_{j=0}^{\infty} A_{ee}^{(2j)}(q,n)\cos(2jv), \\ 2 & \text{even-odd:} & S_{eo}(v,q,n) = \sum_{j=0}^{\infty} A_{eo}^{(2j+1)}(q,n)\cos[(2j+1)v], \\ 3 & \text{odd-even:} & S_{oe}(v,q,n) = \sum_{j=1}^{\infty} A_{oe}^{(2j)}(q,n)\sin(2jv), \\ 4 & \text{odd-odd:} & S_{oo}(v,q,n) = \sum_{j=0}^{\infty} A_{oo}^{(2j+1)}(q,n)\sin[(2j+1)v]. \end{array} \tag{8}$$

A_{pm} with $p, m = e, o$ are expansion coefficients. In the following, the angular Mathieu functions are denoted $S_{pm}(v,q,n)$, with $p, m = e, o$. Instead of two angular Mathieu functions, even S_{ep} and odd S_{op}, with $p = e, o$ [12], a single angular Mathieu function S_{pm}, with $p, m = e, o$, is considered refering to all the four categories. For a given value of q there exist four infinite sequences of characteristic values (eigenvalues) a, for either value of a corresponding an infinite sequence (eigenvector) of expansion coefficients.

A. Characteristic values and coefficients

By subsituting (8) in (6), the following recurrence relations among the expansion coefficients result

1 even-even:

$$aA_{ee}^{(0)} - qA_{ee}^{(2)} = 0,$$

$$(a-4)A_{ee}^{(2)} - q[2A_{ee}^{(0)} + A_{ee}^{(4)}] = 0,$$

$$[a-(2j)^2]A_{ee}^{(2j)} - q[A_{ee}^{(2j-2)} + A_{ee}^{(2j+2)}] = 0, \qquad j = 2,3,4\cdots \tag{9}$$

2 even-odd:

$$(a-1)A_{eo}^{(1)} - q[A_{eo}^{(1)} + A_{eo}^{(3)}] = 0,$$

$$[a-(2j+1)^2]A_{eo}^{(2j+1)} - q[A_{eo}^{(2j-1)} + A_{eo}^{(2j+3)}] = 0, \qquad j = 1,2,3\cdots \tag{10}$$

3 odd-even:

$$(a-4)A_{oe}^{(2)} - qA_{oe}^{(4)} = 0,$$

$$[a-(2j)^2]A_{oe}^{(2j)} - q[A_{oe}^{(2j-2)} + A_{oe}^{(2j+2)}] = 0, \qquad j = 2,3,4\cdots \tag{11}$$

4 odd-odd:

$$(a-1)A_{oo}^{(1)} + q[A_{oo}^{(1)} - A_{oo}^{(3)}] = 0,$$

$$[a-(2j+1)^2]A_{oo}^{(2j+1)} - q[A_{oo}^{(2j-1)} + A_{oo}^{(2j+3)}] = 0, \qquad j = 1,2,3\cdots. \tag{12}$$

The recurrence relations can be written in matrix form [11],

$$\text{1 even-even:} \quad \begin{pmatrix} -a & q & 0 & 0 & 0 & 0 & \cdots \\ 2q & 2^2-a & q & 0 & 0 & 0 & \cdots \\ 0 & q & 4^2-a & q & 0 & 0 & \cdots \\ 0 & 0 & q & 6^2-a & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_{ee}^{(0)} \\ A_{ee}^{(2)} \\ A_{ee}^{(4)} \\ A_{ee}^{(6)} \\ \vdots \end{pmatrix} = 0, \tag{13}$$

$$\text{2 even-odd:} \quad \begin{pmatrix} 1+q-a & q & 0 & 0 & 0 & 0 & \cdots \\ q & 3^2-a & q & 0 & 0 & 0 & \cdots \\ 0 & q & 5^2-a & q & 0 & 0 & \cdots \\ 0 & 0 & q & 7^2-a & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_{eo}^{(1)} \\ A_{eo}^{(3)} \\ A_{eo}^{(5)} \\ A_{eo}^{(7)} \\ \vdots \end{pmatrix} = 0, \tag{14}$$

$$3 \quad \text{odd-even:} \qquad \begin{pmatrix} 2^2-a & q & 0 & 0 & 0 & 0 & \cdots \\ q & 4^2-a & q & 0 & 0 & 0 & \cdots \\ 0 & q & 6^2-a & q & 0 & 0 & \cdots \\ 0 & 0 & q & 8^2-a & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_{oe}^{(2)} \\ A_{oe}^{(4)} \\ A_{oe}^{(6)} \\ A_{oe}^{(8)} \\ \vdots \end{pmatrix} = 0, \tag{15}$$

$$4 \quad \text{odd-odd:} \qquad \begin{pmatrix} 1-q-a & q & 0 & 0 & 0 & 0 & \cdots \\ q & 3^2-a & q & 0 & 0 & 0 & \cdots \\ 0 & q & 5^2-a & q & 0 & 0 & \cdots \\ 0 & 0 & q & 7^2-a & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_{oo}^{(1)} \\ A_{oo}^{(3)} \\ A_{oo}^{(5)} \\ A_{oo}^{(7)} \\ \vdots \end{pmatrix} = 0. \tag{16}$$

The matrices are real, tridiagonal, and symmetric for all categories, with the exception of the "1 even-even" category where the matrix is slightly non-symmetric. The eigenvalue problem is accurately solved in Matlab. In other computer programs it could be necessary to transform the slightly non-symmetric matrix in a symmetric one [11]. Both the eigenvalues a and the corresponding eigenvectors (A_{pm}, with $p, m = e, o$) are determined for either category at any order n. The order n takes different values for each category of Mathieu functions. For the purpose of avoiding any confusion, a distinction must be done between the n^{th} order (in the succession of all orders) and the true value of that order. Thus, let denote n the order in the succession of all orders, and t the true value of order n. The values of n and t for the four categories of Mathieu functions are

1	even-even:	$n = 0, 1, 2 \cdots$	$t = 0, 2, 4 \cdots$,
2	even-odd:	$n = 0, 1, 2 \cdots$	$t = 1, 3, 5 \cdots$,
3	odd-even:	$n = 1, 2, 3 \cdots$	$t = 2, 4, 6 \cdots$,
4	odd-odd:	$n = 0, 1, 2 \cdots$	$t = 1, 3, 5 \cdots$.

Note that, if the notation is self-contained by all routines of Mathieu functions, there is no need to determine the specific values of n and t for either category of Mathieu functions since it is done automatically.

B. Normalization and orthogonality

Following [6, 11], the angular Mathieu functions are normalized by requiring that

$$S_{ep}(0,q,n)=1, \qquad \left[\frac{\mathrm{d}S_{op}(v,q,n)}{\mathrm{d}v}\right]_{v=0}=1, \qquad p=e,o. \tag{17}$$

These requirements imply that,

$$\begin{array}{lll} 1 & \text{even-even:} & \displaystyle\sum_{j=0}^{\infty} A_{ee}^{(2j)}(q,n)=1, \\ 2 & \text{even-odd:} & \displaystyle\sum_{j=0}^{\infty} A_{eo}^{(2j+1)}(q,n)=1, \\ 3 & \text{odd-even:} & \displaystyle\sum_{j=1}^{\infty} 2jA_{oe}^{(2j)}(q,n)=1, \\ 4 & \text{odd-odd:} & \displaystyle\sum_{j=0}^{\infty} (2j+1)A_{oo}^{(2j+1)}(q,n)=1. \end{array} \tag{18}$$

The orthogonality relation for the angular Mathieu functions is

$$\int_0^{2\pi} S_{pm}(v,q,n)S_{pm'}(v,q,n)\,\mathrm{d}v = N_{pm}\delta_{m\,m'}, \qquad p,m,m'=e,o, \tag{19}$$

where N_{pm} is normalization factor, $\delta_{m\,m'}$ equals 1 if $m=m'$ and equals 0 otherwise. Then, the following relations for the normalization factor result,

$$\begin{array}{lll} 1 & \text{even-even:} & \displaystyle N_{ee}(q,n)=2\pi[A_{ee}^{(0)}(q,n)]^2+\pi\sum_{j=1}^{\infty}[A_{ee}^{(2j)}(q,n)]^2, \\ 2 & \text{even-odd:} & \displaystyle N_{eo}(q,n)=\pi\sum_{j=0}^{\infty}[A_{eo}^{(2j+1)}(q,n)]^2, \\ 3 & \text{odd-even:} & \displaystyle N_{oe}(q,n)=\pi\sum_{j=1}^{\infty}[A_{oe}^{(2j)}(q,n)]^2, \\ 4 & \text{odd-odd:} & \displaystyle N_{oo}(q,n)=\pi\sum_{j=0}^{\infty}[A_{oo}^{(2j+1)}(q,n)]^2. \end{array} \tag{20}$$

Since different normalization schemes have been adopted in the literature, much attention should be paid when numerical results provided by different authors are compared ones against the others.

C. Correlation factors

Let consider two regions of different permittivities, ϵ and ϵ'. The parameter q being different in the two regions, $q \neq q'$, the characteristic values and expansion coefficients are also different. Let S_{pm} and S'_{pm} be the respective angular Mathieu functions. The correlation factors $C_{pm}(q, q', n)$, with $p, m = e, o$, between the angular Mathieu functions S_{pm} and S'_{pm} are defined by relation

$$C_{pm}(q, q', n) = \delta_{m\,m'} \int_0^{2\pi} S_{pm'}(v, q, n) S'_{pm}(v, q', n)\,\mathrm{d}v, \qquad p, m, m' = e, o. \tag{21}$$

Using (8) gives

$$\begin{aligned}
&1 \quad \text{even-even:} \quad C_{ee}(q, q', n) = 2\pi A_{ee}^{(0)}(q, n) A_{ee}'^{(0)}(q', n) \\
&\qquad\qquad + \pi \sum_{j=1}^{\infty} A_{ee}^{(2j)}(q, n) A_{ee}'^{(2j)}(q', n), \\
&2 \quad \text{even-odd:} \quad C_{eo}(q, q', n) = \pi \sum_{j=0}^{\infty} A_{eo}^{(2j+1)}(q, n) A_{eo}'^{(2j+1)}(q', n), \\
&3 \quad \text{odd-even:} \quad C_{oe}(q, q', n) = \pi \sum_{j=1}^{\infty} A_{oe}^{(2j)}(q, n) A_{oe}'^{(2j)}(q', n), \\
&4 \quad \text{odd-odd:} \quad C_{oo}(q, q', n) = \pi \sum_{j=0}^{\infty} A_{oo}^{(2j+1)}(q, n) A_{oo}'^{(2j+1)}(q', n).
\end{aligned} \tag{22}$$

D. Derivatives of angular Mathieu functions

The derivatives of the angular Mathieu functions follow readily from (8),

$$\begin{aligned}
&1 \quad \text{even-even:} \quad \frac{\mathrm{d}S_{ee}(v, q, n)}{\mathrm{d}v} = -\sum_{j=1}^{\infty} 2j A_{ee}^{(2j)}(q, n) \sin(2jv), \\
&2 \quad \text{even-odd:} \quad \frac{\mathrm{d}S_{eo}(v, q, n)}{\mathrm{d}v} = -\sum_{j=0}^{\infty} (2j+1) A_{eo}^{(2j+1)}(q, n) \sin[(2j+1)v], \\
&3 \quad \text{odd-even:} \quad \frac{\mathrm{d}S_{oe}(v, q, n)}{\mathrm{d}v} = \sum_{j=1}^{\infty} 2j A_{oe}^{(2j)}(q, n) \cos(2jv), \\
&4 \quad \text{odd-odd:} \quad \frac{\mathrm{d}S_{oo}(v, q, n)}{\mathrm{d}v} = \sum_{j=0}^{\infty} (2j+1) A_{oo}^{(2j+1)}(q, n) \cos[(2j+1)v].
\end{aligned} \tag{23}$$

IV. RADIAL MATHIEU FUNCTIONS

Solutions of (7) can be obtained from (8) by replacing v by iu. Instead of $\sin v$ and $\cos v$, the terms of the series now involve $\sinh u$ and $\cosh u$. The convergence is low unless $|u|$ is small. Better convergence of series results by expressing the solutions of (7) in terms of Bessel functions associated with the same expansion coefficients that are determined once for both the angular and radial Mathieu functions. Either pair of angular and radial Mathieu functions are proportional to one another [6],

$$S_{ep}(iu,q,n) = \sqrt{2\pi}g_{ep}(q,n)J_{ep}(u,q,n), \qquad p = e, o, \tag{24}$$

where J_{ep} are even radial Mathieu functions of the first kind and g_{ep} are joining factors. When $u = 0$,

$$S_{ep}(0,q,n) = 1, \qquad J_{ep}(0,q,n) = \frac{1}{\sqrt{2\pi}g_{ep}(q,n)}, \qquad p = e, o. \tag{25}$$

Thus, one obtains,

$$\begin{aligned} &1 \quad \text{even-even:} \qquad g_{ee}(q,n) = \frac{(-1)^r}{\pi A_{ee}^{(0)}(q,n)} S_{ee}(\pi/2,q,n), \qquad r = t/2, \\ &2 \quad \text{even-odd:} \qquad g_{eo}(q,n) = \frac{-(-1)^r}{\pi\sqrt{q}A_{eo}^{(1)}(q,n)} \Big[\frac{\mathrm{d}S_{eo}(v,q,n)}{\mathrm{d}v}\Big]_{v=\pi/2}, \qquad r = (t-1)/2. \end{aligned} \tag{26}$$

Similarly [6],

$$-iS_{op}(iu,q,n) = \sqrt{2\pi}g_{op}(q,n)J_{op}(u,q,n), \qquad p = e, o. \tag{27}$$

When $u = 0$,

$$J_{op}(0,q,n) = 0, \qquad \Big[\frac{\mathrm{d}J_{op}(u,q,n)}{\mathrm{d}u}\Big]_{u=0} = \frac{1}{\sqrt{2\pi}g_{op}(q,n)}, \qquad p = e, o. \tag{28}$$

Thus, one obtains,

$$\begin{aligned} &3 \quad \text{odd-even:} \qquad g_{oe}(q,n) = \frac{(-1)^r}{\pi q A_{oe}^{(2)}(q,n)} \Big[\frac{\mathrm{d}S_{oe}(v,q,n)}{\mathrm{d}v}\Big]_{v=\pi/2}, \qquad r = t/2, \\ &4 \quad \text{odd-odd:} \qquad g_{oo}(q,n) = \frac{(-1)^r}{\pi\sqrt{q}A_{oo}^{(1)}(q,n)} S_{oo}(\pi/2,q,n), \qquad r = (t-1)/2. \end{aligned} \tag{29}$$

Remember that t is the true value of order n.

A. Radial Mathieu functions of the first kind

Since rapidly converging series are those expressed in terms of products of Bessel functions [10, 11], in the following relations refer only to them. Similarly to the angular Mathieu functions, one may distinct four categories of radial Mathieu functions of the first kind which are denoted $J_{pm}(u,q,n)$, with $p,m=e,o$,

$$
\begin{aligned}
&1 \quad \text{even-even:} \quad J_{ee}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{ee}^{(0)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{ee}^{(2j)}(q,n)J_j(v_1)J_j(v_2),\\
&\qquad r=t/2,\\
&2 \quad \text{even-odd:} \quad J_{eo}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{eo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{eo}^{(2j+1)}(q,n)[J_j(v_1)J_{j+1}(v_2)\\
&\qquad + J_j(v_2)J_{j+1}(v_1)], \qquad r=(t-1)/2, \qquad\qquad (30)\\
&3 \quad \text{odd-even:} \quad J_{oe}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oe}^{(2)}(q,n)}\sum_{j=1}^{\infty}(-1)^j A_{oe}^{(2j)}(q,n)[J_{j-1}(v_1)J_{j+1}(v_2)\\
&\qquad - J_{j-1}(v_2)J_{j+1}(v_1)], \qquad r=t/2,\\
&4 \quad \text{odd-odd:} \quad J_{oo}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{oo}^{(2j+1)}(q,n)[J_j(v_1)J_{j+1}(v_2)\\
&\qquad - J_j(v_2)J_{j+1}(v_1)], \qquad r=(t-1)/2,
\end{aligned}
$$

where $v_1=\sqrt{q}\exp(-u)$ and $v_2=\sqrt{q}\exp(u)$. The derivatives of the radial Mathieu functions of the first kind are

1 even-even: $r=t/2$,

$$
\begin{aligned}
\frac{\mathrm{d}J_{ee}(u,q,n)}{\mathrm{d}u} =& \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{ee}^{(0)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{ee}^{(2j)}(q,n)[v_1 J_{j+1}(v_1)J_j(v_2)\\
&- v_2 J_j(v_1)J_{j+1}(v_2)],
\end{aligned}
$$

2 even-odd: $r=(t-1)/2$,

$$
\begin{aligned}
\frac{\mathrm{d}J_{eo}(u,q,n)}{\mathrm{d}u} =& \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{eo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{eo}^{(2j+1)}(q,n)\Big\{(v_2-v_1)[J_j(v_1)J_j(v_2)\\
&- J_{j+1}(v_1)J_{j+1}(v_2)] + (2j+1)[J_{j+1}(v_1)J_j(v_2) - J_j(v_1)J_{j+1}(v_2)]\Big\},
\end{aligned}
$$

3 odd-even: $r = t/2$,

$$\frac{\mathrm{d}J_{oe}(u,q,n)}{\mathrm{d}u} = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oe}^{(2)}(q,n)}\sum_{j=0}^{\infty}(-1)^{j+1}A_{oe}^{(2j+2)}(q,n)(4j+4)\Big\{J_j(v_1)J_j(v_2) \tag{31}$$
$$+\cosh 2u J_{j+1}(v_1)J_{j+1}(v_2) - (j+1)[\frac{1}{v_1}J_{j+1}(v_1)J_j(v_2) + \frac{1}{v_2}J_j(v_1)J_{j+1}(v_2)]\Big\},$$

4 odd-odd: $r = (t-1)/2$,

$$\frac{\mathrm{d}J_{oo}(u,q,n)}{\mathrm{d}u} = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^{j}A_{oo}^{(2j+1)}(q,n)\Big\{(v_1+v_2)[J_j(v_1)J_j(v_2)$$
$$+J_{j+1}(v_1)J_{j+1}(v_2)] - (2j+1)[J_{j+1}(v_1)J_j(v_2) + J_j(v_1)J_{j+1}(v_2)]\Big\}.$$

B. Radial Mathieu functions of the second kind

A second independent solution of (7) is obtained by replacing the Bessel functions of the first kind $J_n(v_2)$ in (30) by the Bessel functions of the second kind $Y_n(v_2)$, [10, 11]. This solution is denoted $Y_{pm}(u,q,n)$, with $p,m = e,o$.

$$\begin{aligned}
&1 \quad \text{even-even:} \quad Y_{ee}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{ee}^{(0)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{ee}^{(2j)}(q,n)J_j(v_1)Y_j(v_2),\\
&\qquad r = t/2,\\
&2 \quad \text{even-odd:} \quad Y_{eo}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{eo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{eo}^{(2j+1)}(q,n)[J_j(v_1)Y_{j+1}(v_2)\\
&\qquad + Y_j(v_2)J_{j+1}(v_1)], \qquad r = (t-1)/2,\\
&3 \quad \text{odd-even:} \quad Y_{oe}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oe}^{(2)}(q,n)}\sum_{j=1}^{\infty}(-1)^j A_{oe}^{(2j)}(q,n)[J_{j-1}(v_1)Y_{j+1}(v_2)\\
&\qquad - Y_{j-1}(v_2)J_{j+1}(v_1)], \qquad r = t/2,\\
&4 \quad \text{odd-odd:} \quad Y_{oo}(u,q,n) = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{oo}^{(2j+1)}(q,n)[J_j(v_1)Y_{j+1}(v_2)\\
&\qquad - Y_j(v_2)J_{j+1}(v_1)], \qquad r = (t-1)/2,
\end{aligned} \tag{32}$$

The derivatives of the radial Mathieu functions of the second kind are

1 even-even: $r = t/2$,

$$\frac{\mathrm{d}Y_{ee}(u,q,n)}{\mathrm{d}u} = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{ee}^{(0)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{ee}^{(2j)}(q,n)[v_1 J_{j+1}(v_1)Y_j(v_2) - v_2 J_j(v_1)Y_{j+1}(v_2)],$$

2 even-odd: $r = (t-1)/2$,

$$\frac{\mathrm{d}Y_{eo}(u,q,n)}{\mathrm{d}u} = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{eo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{eo}^{(2j+1)}(q,n)\Big\{(v_2 - v_1)[J_j(v_1)Y_j(v_2) - J_{j+1}(v_1)Y_{j+1}(v_2)] + (2j+1)[J_{j+1}(v_1)Y_j(v_2) - J_j(v_1)Y_{j+1}(v_2)]\Big\},$$

3 odd-even: $r = t/2$,

$$\frac{\mathrm{d}Y_{oe}(u,q,n)}{\mathrm{d}u} = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oe}^{(2)}(q,n)}\sum_{j=0}^{\infty}(-1)^{j+1} A_{oe}^{(2j+2)}(q,n)(4j+4)\Big\{J_j(v_1)Y_j(v_2) + \cosh 2u J_{j+1}(v_1)Y_{j+1}(v_2) - (j+1)[\frac{1}{v_1}J_{j+1}(v_1)Y_j(v_2) + \frac{1}{v_2}J_j(v_1)Y_{j+1}(v_2)]\Big\}, \tag{33}$$

4 odd-odd: $r = (t-1)/2$,

$$\frac{\mathrm{d}Y_{oo}(u,q,n)}{\mathrm{d}u} = \sqrt{\frac{\pi}{2}}\frac{(-1)^r}{A_{oo}^{(1)}(q,n)}\sum_{j=0}^{\infty}(-1)^j A_{oo}^{(2j+1)}(q,n)\Big\{(v_1 + v_2)[J_j(v_1)Y_j(v_2) + J_{j+1}(v_1)Y_{j+1}(v_2)] - (2j+1)[J_{j+1}(v_1)Y_j(v_2) + J_j(v_1)Y_{j+1}(v_2)]\Big\}.$$

C. Radial Mathieu functions of the third and the fourth kinds

Radial Mathieu functions of the third kind, analogous to the Hankel functions of the first kind are defined as follows [6, 11]

$$H_{pm1}(u,q,n) = J_{pm}(u,q,n) + iY_{pm}(u,q,n), \qquad p,m = e,o. \tag{34}$$

Similarly, radial Mathieu functions of the fourth kind, analogous to the Hankel functions of the second kind are defined as follows [6, 11]

$$H_{pm2}(u,q,n) = J_{pm}(u,q,n) - iY_{pm}(u,q,n), \qquad p,m = e,o. \tag{35}$$

V. IMPLEMENTATION OF MATHIEU FUNCTIONS IN MATLAB

Following the notation of the four categories of angular Mathieu functions, the implementation in Matlab or in any other computer program is readily done by introducing a function code KF. The first step in any algorithm of Mathieu function computation is to find the characteristic values (eigenvalues) and the expansion coefficients (eigenvectors). This is done by routine "eig_Spm" which has q as input parameter (see Table I). Besides q, the function code KF should be specified. Thus, if $KF = 1$, the routine "eig_Spm" solves the eigenvalue problem for category "1 even-even" of Mathieu functions, if $KF = 2$ for category "2 even-odd", and so on. The number of expansion coefficients is the same, it is set equal to 25, for all categories of Mathieu functions. Concerning the outputs of routine "eig_Spm", va is a line vector representing the characteristic values a for all the 25 orders; mc is 25×25 matrix, where the columns represent the eigenvectors (that is, the expansion coefficients) for all orders; vt is a column vector specifying the true value t for all orders. Note that the eigenvectors in mc were processed to obey equation (18). For the purpose to save the time of computation, all the other routines have mc as input (see Table I), the routine "eig_Spm" being called once, at the beginning of the computation, for any values of coordinates u and v that intervene in that computation. Since in many cases the convergence is assured by the first several orders, all the other routines have $nmax \leq 25$ as input. It means that those routines take into account only the first $nmax$ orders, but for either order the length of the corresponding eigenvector is the same, equal to 25. The routine "extract_one_value" can be used to get a single value, and the routine "extract_one_column" to get a single eigenvector, corresponding to the order t. The derivatives of S_{pm}, with $p, m = e, o$, are computed by routine "dSpm". For both "Spm" and "dSpm", v is expressed in radians, with values in interval $(0, 2\pi)$. The normalization, correlation, and joining factors are computed by routines "Npm", "Cpm", and "gpm", respectively. The four kinds of radial Mathieu functions, $J_{pm}, Y_{pm}, H_{pm1},$ and H_{pm2}, with $p, m = e, o$, are computed by routines "Jpm","Ypm","Hpm1", and "Hpm2", respectively, and their derivatives with respect to u by routines "dJpm","dYpm","dHpm1", and "dHpm2", respectively.

Numerical values of the separation constant a, of the angular Mathieu functions S_{pm} and their derivatives S'_{pm}, with $p, m = e, o$, where the prime denotes differentiation with respect to v, are given in Tables II–IV. They can be compared with data in [2]. With the purpose to facilitate the comparison, since in [2] the normalization $N_{pm} = \pi$ is applied, the data of S_{pm} and S'_{pm} in Tables II–IV are multiplied by $\sqrt{\pi/N_{pm}}$.

Concerning the radial Mathieu functions, numerical values of $S_{ep}(iu, q, n)$ and $-iS_{op}(iu, q, n)$ are given for $u = 0.5$ in Tables V and VI. They are multiplied by $\sqrt{\pi/N_{pm}}$ and compared with data in [9]. Note that S_{ep} is correlated to the radial Mathieu function of the first kind J_{ep} by Eq. (24), whereas S_{op} is correlated to J_{op} by Eq. (27). We found that, for parameters in [9], the values of $S_{ep}(iu, q, n)$ and $-iS_{op}(iu, q, n)$ calculated with Eqs. (24) and (27) differ from those obtained with Eq. (8) by less than 7.5×10^{-12}.

Four files are attached as examples of Mathieu functions computation: example1_Spm.m, example2_Spm.m, example1_Jpm.m, and example2_Jpm.m. Results are shown in Figs. 2–5.

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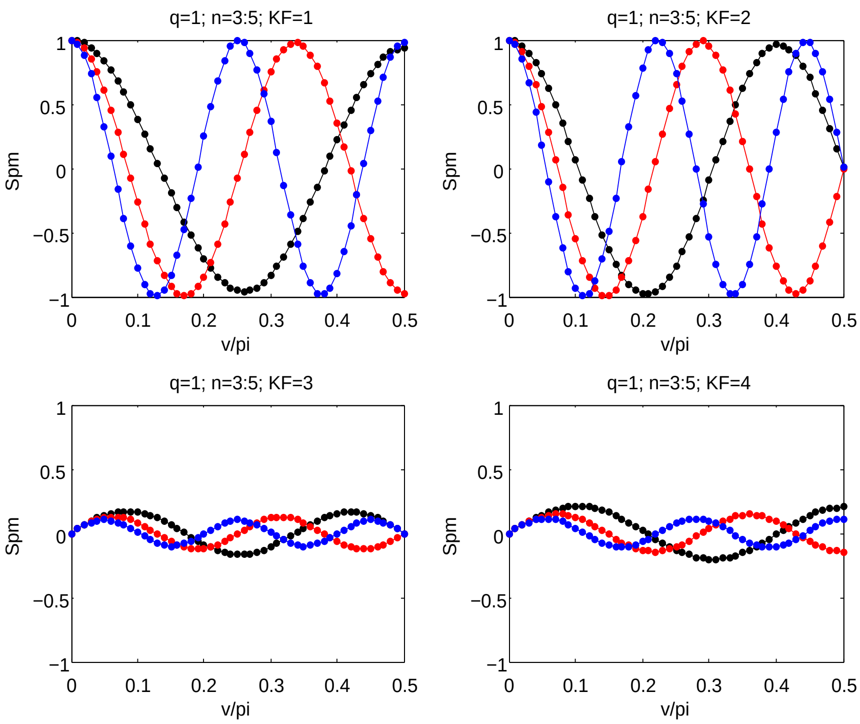

FIG. 2: Examples of angular Mathieu functions at $q = 1$ and orders $n = 3$ (black), $n = 4$ (red), and $n = 5$ (blue). The angular coordinate v is varied from 0 to $\pi/2$.

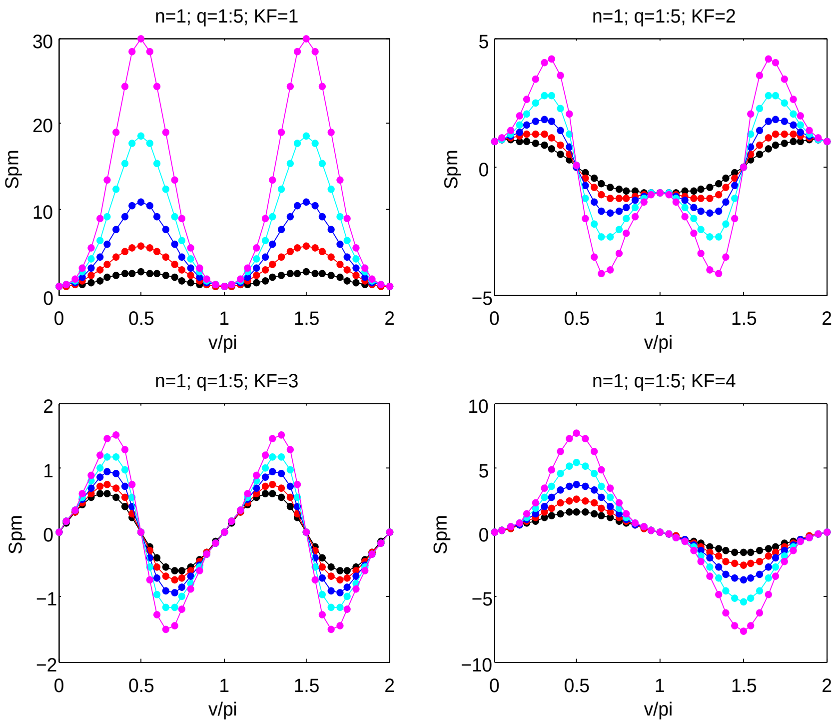

FIG. 3: Examples of angular Mathieu functions at $n = 1$ and different values of q: $q = 1$ (black), $q = 2$ (red), $q = 3$ (blue), $q = 4$ (cyan), and $q = 5$ (magenta). The angular coordinate v is varied from 0 to $\pi/2$.

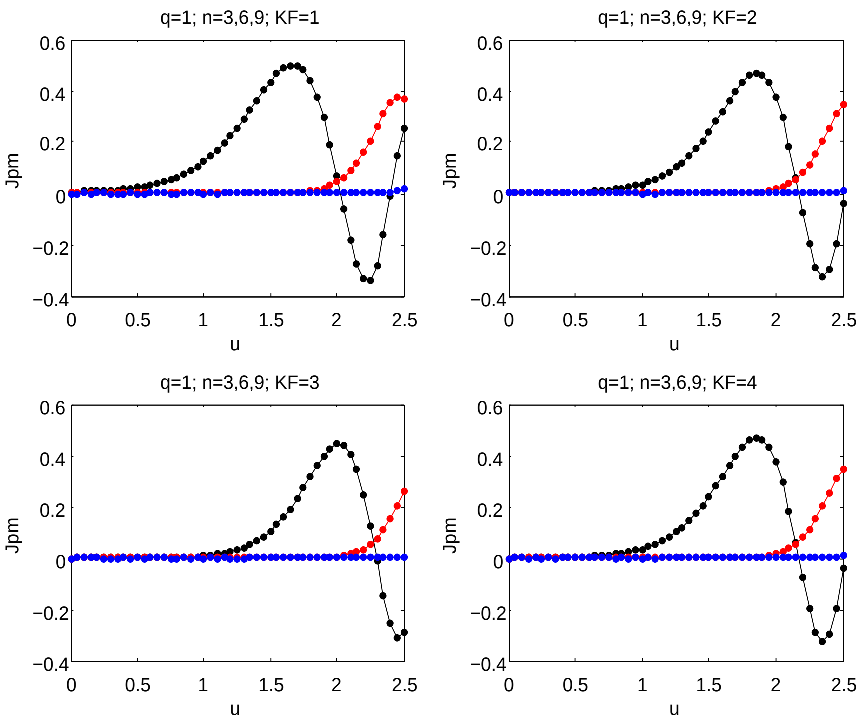

FIG. 4: Examples of radial Mathieu functions of the first kind at $q = 1$ and orders $n = 3$ (black), $n = 6$ (red), and $n = 9$ (blue).

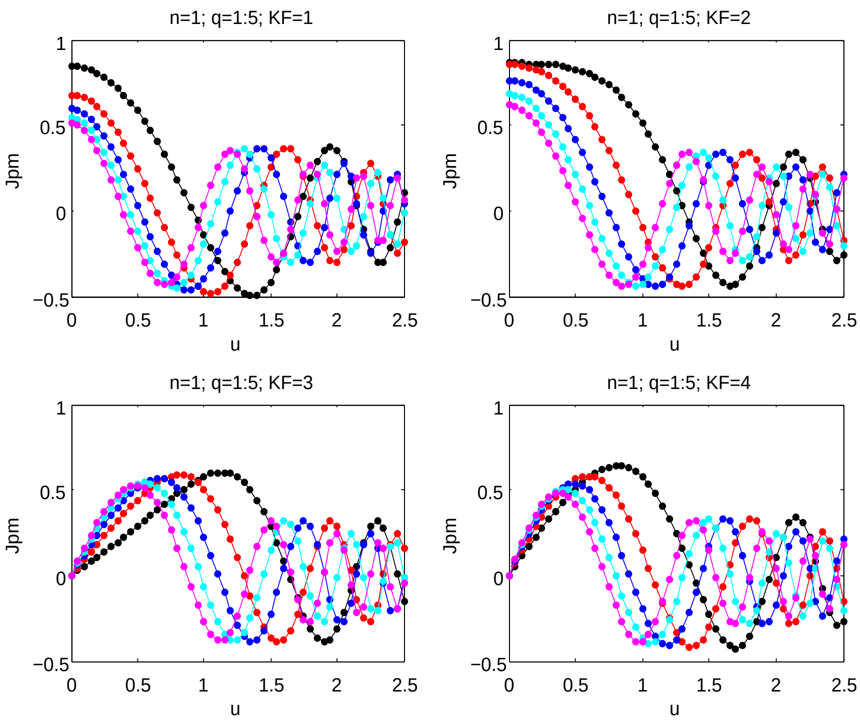

FIG. 5: Examples of radial Mathieu functions of the first kind at $n = 1$ and different values of q: $q = 1$ (black), $q = 2$ (red), $q = 3$ (blue), $q = 4$ (cyan), and $q = 5$ (magenta).

TABLE I: Routines comprised in the toolbox

Name of routine	Routine call	What the routine computes
eig_Spm	$[va, mc, vt]$=eig_Spm(KF, q)	Vector of characteristic values va, matrix of coefficients mc, and vector of orders vt, at given function code KF and elliptical parameter $q \geq 0$.
Spm	y=Spm$(KF, v, mc, nmax)$	Angular Mathieu functions S_{pm}, [Eq. (8)].
dSpm	y=dSpm$(KF, v, mc, nmax)$	Derivatives with respect to v of S_{pm}, [Eq. (23)].
Npm	y=Npm$(KF, mc, nmax)$	Normalizing factors of angular Mathieu functions S_{pm}, [Eqs. (19) and (20)].
Cpm	y=Cpm$(KF, mc, mc', nmax)$	Correlation factors of S_{pm} and S'_{pm}, having matrices of coefficients mc and mc', [Eqs. (21) and (22)].
Jpm	y=Jpm$(KF, u, q, mc, nmax)$	Radial Mathieu functions of the first kind J_{pm}, [Eq. (30)].
dJpm	y=dJpm$(KF, u, q, mc, nmax)$	Derivatives with respect to u of J_{pm}, [Eq. (31)].
gpm	y=gpm$(KF, q, mc, nmax)$	Joining factors for pairs of angular, S_{pm} and radial, J_{pm} Mathieu functions, [Eqs. (24)–(29)].
Ypm	y=Ypm$(KF, u, q, mc, nmax)$	Radial Mathieu functions of the second kind Y_{pm}, [Eq. (32)].
dYpm	y=dYpm$(KF, u, q, mc, nmax)$	Derivatives with respect to u of Y_{pm}, [Eq. (33)].
Hpm1	y=Hpm1$(KF, u, q, mc, nmax)$	Radial Mathieu functions of the third kind H_{pm1}, [Eq. (34)].
dHpm1	y=dHpm1$(KF, u, q, mc, namax)$	Derivatives with respect to u of H_{pm1}.
Hpm2	y=Hpm2$(KF, u, q, mc, nmax)$	Radial Mathieu functions of the fourth kind H_{pm2}, [Eq. (35)].
dHpm2	y=dHpm2$(KF, u, q, mc, namax)$	Derivatives with respect to u of H_{pm2}.
extract_one_column	y=extract_one_column(KF, t, mc)	Extracts one column from mc at given t.
extract_one_value	y=extract_one_value(KF, t, vec)	Extracts one value from vec at given t.

TABLE II: Values of S_{ee} multiplied by $\gamma_{ee} = \sqrt{\pi/N_{ee}}$ to be compared with data in [2]

t	q	a	$\gamma_{ee}S_{ee}(0,q,n)$	$\gamma_{ee}S_{ee}(\pi/2,q,n)$
0	0	0	0.7071067811865	0.7071067811865
	5	-5.8000460208515	0.0448001816519	1.3348486746980
	10	-13.9369799566589	0.0076265175709	1.4686604707129
	15	-22.5130377608640	0.0019325083152	1.5501081466866
	20	-31.3133900703364	0.0006037438292	1.6098908573959
	25	-40.2567795465667	0.0002158630184	1.6575102983235
2	0	4.0000000000000	1.0000000000000	-1.0000000000000
	5	7.4491097395292	0.7352943084007	-0.7244881519677
	10	7.7173698497796	0.2458883492913	-0.9267592641263
	15	5.0779831975435	0.0787928278464	-1.0199662260303
	20	1.1542828852468	0.0286489431471	-1.0752932287797
	25	-3.5221647271583	0.0115128663309	-1.1162789532953
10	0	100.0000000000000	1.0000000000000	-1.0000000000000
	5	100.1263692161636	1.0259950270894	-0.9753474872360
	10	100.5067700246816	1.0538159921009	-0.9516453181790
	15	101.1452034473016	1.0841063118392	-0.9285480638845
	20	102.0489160244372	1.1177886312594	-0.9057107845941
	25	103.2302048044949	1.1562399186322	-0.8826919105637

TABLE III: Values of S_{eo} and S'_{eo} multiplied by $\gamma_{eo} = \sqrt{\pi/N_{eo}}$ to be compared with data in [2]

t	q	a	$\gamma_{eo}S_{eo}(0,q,n)$	$\gamma_{eo}S'_{eo}(\pi/2,q,n)$
1	0	1.0000000000000	1.0000000000000	-1.0000000000000
	5	1.8581875415478	0.2565428793224	-3.4690420034057
	10	-2.3991424000363	0.0535987477472	-4.8504383044964
	15	-8.1011051316418	0.0150400664538	-5.7642064390510
	20	-14.4913014251748	0.0050518137647	-6.4905657825800
	25	-21.3148996906657	0.0019110515067	-7.1067412352901
5	0	25.0000000000000	1.0000000000000	-5.0000000000000
	5	25.5499717499816	1.1248072506385	-5.3924861549882
	10	27.7037687339393	1.2580199413083	-5.3212765411609
	15	31.9578212521729	1.1934322304131	-5.1191498884064
	20	36.6449897341328	0.9365755314226	-5.7786752500644
	25	40.0501909858077	0.6106943100507	-7.0598842916553
15	0	225.0000000000000	1.0000000000000	15.0000000000000
	5	225.0558124767096	1.0112937325296	15.1636574720602
	10	225.2233569749644	1.0228782824382	15.3198803056623
	15	225.5029562446541	1.0347936522369	15.4687435032830
	20	225.8951534162079	1.0470843441629	15.6102785232380
	25	226.4007200447481	1.0598004418139	15.7444725050679

TABLE IV: Values of S_{op} and S'_{op} multiplied by $\gamma_{op} = \sqrt{\pi/N_{op}}$, where $p = e, o$, to be compared with data in [2]

t	q	a	$\gamma_{oe}S'_{oe}(0,q,n)$	$\gamma_{oe}S'_{oe}(\pi/2,q,n)$
2	0	4.0000000000000	2.0000000000000	-2.0000000000000
	5	2.0994604454867	0.7331661960372	-3.6405178524082
	10	-2.3821582359570	0.2488228403985	-4.8634220691653
	15	-8.0993467988959	0.0918197143696	-5.7655737717278
	20	-14.4910632559807	0.0370277776852	-6.4907522240373
	25	-21.3148606222498	0.0160562170491	-7.1067719073739
10	0	100.0000000000000	10.0000000000000	-10.0000000000000
	5	100.1263692156019	9.7341731518695	-10.2396462566908
	10	100.5067694628784	9.4404054347686	-10.4539475316485
	15	101.1451722929092	9.1157513395126	-10.6428998776563
	20	102.0483928609361	8.7555450801360	-10.8057241781325
	25	103.2256800423735	8.3526783655914	-10.9413538308191
t	q	a	$\gamma_{oo}S'_{oo}(0,q,n)$	$\gamma_{oo}S_{oo}(\pi/2,q,n)$
1	0	1.0000000000000	1.0000000000000	1.0000000000000
	5	-5.7900805986378	0.1746754006198	1.3374338870223
	10	-13.9365524792501	0.0440225659111	1.4687556641029
	15	-22.5130034974235	0.0139251347875	1.5501150743576
	20	-31.3133861669129	0.0050778849001	1.6098915926038
	25	-40.2567789846842	0.0020443593656	1.6575103983745
5	0	25.0000000000000	5.0000000000000	1.0000000000000
	5	25.5108160463032	4.3395700104946	0.9060779302024
	10	26.7664263604801	3.4072267604013	0.8460384335355
	15	27.9678805967175	2.4116664728002	0.8379493400125
	20	28.4682213251027	1.5688968684857	0.8635431218534
	25	28.0627658994543	0.9640716219024	0.8992683245108
15	0	225.0000000000000	15.0000000000000	-1.0000000000000
	5	225.0558124767096	14.8287889732852	-0.9889607027406
	10	225.2233569749643	14.6498600449581	-0.9781423471832
	15	225.5029562446537	14.4630006940372	-0.9675137031855
	20	225.8951534161767	14.2679460909928	-0.9570452540613
	25	226.4007200438825	14.0643732956172	-0.9467086958781

TABLE V: Values of $S_{ep}(iu, q, n)$ for $u = 0.5$ multiplied by $\sqrt{\pi/N_{ep}}$, where $p = e, o$, compared with data in [9]

t	q	Values at $p = e$	Data in [9]	t	q	Values at $p = o$	Data in [9]
0	5	-0.019325304910071	-0.01932	1	5	0.021440743185527	0.02144
	10	-0.007055239716193	-0.00705		10	-0.038634237458525	-0.03863
	20	-0.000169411415735	-0.00016		20	-0.003373888309642	-0.00337
2	5	0.446937465741068	0.44693	3	5	1.205528267066838	1.2055
	10	-0.063855921612085	-0.06385		10	0.235940782144547	0.23594
	20	-0.024916657795101	-0.02491		20	-0.097385461808731	-0.09738
4	5	2.234088244534832	2.2341	5	5	3.864089377116713	3.8641
	10	1.039103163573830	1.0391		10	2.285610444240526	2.2856
	20	-0.143991090269732	-0.14399		20	0.274270780278172	0.27427

TABLE VI: Values of $-iS_{op}(iu, q, n)$ for $u = 0.5$ multiplied by $\sqrt{\pi/N_{op}}$, with $p = e, o$, compared with data in [9]

t	q	Values at $p = e$	Data in [9]	t	q	Values at $p = o$	Data in [9]
2	5	0.238342768735937	0.23834	1	5	0.036613617783886	0.03661
	10	0.028675814044625	0.02867		10	0.000750806874015	0.00075
	20	-0.003176296415956	-0.00317		20	-0.000538258353937	-0.00053
4	5	1.883560277440876	1.8836	3	5	0.806555153528872	0.80655
	10	0.769679129538722	0.76968		10	0.204495885546638	0.20449
	20	0.040515136278697	0.04051		20	-0.005279473480675	-0.00527
6	5	6.6066602369876	6.6067	5	5	3.667530204538722	3.6675
	10	4.1161420952367	4.1161		10	1.972361938552091	1.9724
	20	1.1805904286267	1.1806		20	0.320398855944192	0.32040