A Discrete Maths Exercise

Thomas Forster

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First some preliminaries:

Products

You can make products of more than two things. For example we can product $\{\top, \bot\}$ with itself to get

$$\{\top, \bot\} \times \{\top, \bot\}$$

and we can do it again to get

$$\{\top, \bot\} \times \{\top, \bot\} \times \{\top, \bot\}$$

Why have I left out the brackets? Altho'

$$(\{\top, \bot\} \times \{\bot, \top\}) \times \{\bot, \top\}$$

and

$$\{\bot, \top\} \times (\{\bot, \top\} \times \{\bot, \top\})$$

are not *literally identical* they are obviously isomorphic. Indeed they are not only *obviously* isomorphic they are (to use a term of art) **naturally** isomorphic. The meaning of this term of art doesn't need to be spelled out here and now. Suffice for the moment is to say that there is a definable bijection (which you should try writing in `ML`) between them. For many purposes we do not really need to distinguish between isomorphic structures.

You will need this exercise [hint, hint!!!]

Prove that $\mathcal{P}(\{0, 1, 2\})$ and $\{\bot, \top\}^3$ are isomorphic.

$\mathcal{P}(\{0, 1, 2\})$ is the power set of $\{0, 1, 2\}$ ordered by $\subseteq$ (set inclusion). Clearly we are ordering $\{\bot, \top\}$ by $\bot < \top$.

Equivalence relations

(An equivalence relation $\sim$ is a **congruence relation for** an n-ary function f if, whenever $x_i \sim y_i$ for $i \leq n$, then $f(x_1 \ldots x_n) \sim f(y_1 \ldots y_n)$.

An equivalence relation $\sim$ is a **congruence relation for** an n-ary **relation** R if, whenever $x_i \sim y_i$ for $i \leq n$, then $R(x_1 \ldots x_n) \longleftrightarrow R(y_1 \ldots y_n)$.)

The Exercise

Consider the relation between humans "It is safe for x to receive a transfusion of blood from y." Ignoring the blood-borne diseases like HIV, CJD, Hep C and so on, we find that if x can safely receive a transfusion of blood from y, and y' belongs to the same blood group as y, then x can safely receive a transfusion of blood from y'. That is to say, the equivalence relation of having-the-same-blood-group is a congruence relation for the binary relation "x can safely receive a transfusion of blood from y".

That way we can think of the relation "x can safely receive a transfusion of blood from y" as really a relation between the blood groups, and summarise it in the following matrix.

Columns are donors, rows are recipients.

	$O-$	$O+$	$B-$	$B+$	$A-$	$A+$	$AB-$	$AB+$
$O-$	$\times$							
$O+$	$\times$	$\times$						
$B-$	$\times$		$\times$					
$B+$	$\times$	$\times$	$\times$	$\times$				
$A-$	$\times$				$\times$			
$A+$	$\times$	$\times$			$\times$	$\times$		
$AB-$	$\times$		$\times$		$\times$		$\times$	
$AB+$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

So the table is telling you that O- people can donate blood to everyone, and so on.

The blood groups themselves (O-, O+, B-, B+, A-, A+, AB- and AB+) are the equivalence classes under this equivalence relation.

Look at the compatibility table for blood groups above. You may be struck by the pleasing pattern made by the $\times$s: I certainly was. In every column the number of '$\times$'s is a power of two! The same goes for rows. What I want you to do is follow the same path I trod when I spotted this pattern, namely:

1. Does this pattern tell you anything significant about the properties of this compatibility relation? I'm thinking of reflexivity, transitivity, symmetry, antisymmetry etc. etc. When you've got that sorted out, procede to part 2.

2. We have already seen other ways of representing binary relations. Try some of them on this data and see if you get any pretty pictures. I tried it and I got a three-dimensional shape. I want you to find that

three-dimensional shape. The fact that it is a *three*-dimensional shape is important.

3. One moral I want to draw from this coursework is that Discrete Mathematics and mere thought can be useful in developing profitable hypotheses in the sciences. You probably know enough biology to know that characters are inherited by genes, and that we each have two copies of each gene. Each gene can come in several forms called *alleles*. Some alleles are *dominant* in that you express them even if you have only one copy (brown eyes); some are *recessive* in that you do not express them unless you have two copies (blue eyes, green eyes). The picture you have developed will suggest to you a hypothesis about how many genes there are that control your blood group, and how many alleles there are at each gene, and which are dominant. Formulate this hypothesis.

Do not forget that this is a discrete mathematics question, not a biology question. There is nothing to be gained—and time to be lost—in surfing the web for information about blood groups.

This exercise was originally designed as a group coursework, and will benefit from brainstorming.