3A. ABSTRACT OF RESEARCH PROPOSAL

Set Theory was started about 150 years ago partly in response to what was then called *the crisis in foundations*. Its foundational rôle has made it central to modern mathematics education. I am a set-theorist, and in fact most of my teaching has been to Computer Science students, for who it is essential background knowledge.
One of the axioms from that time (called the axiom of *restriction* or *foundation*) was adopted out of caution because it closed off possibilities that seemed dangerous, that looked as if they might lead to contradiction. The thinking at the time (now largely forgotten) was that this axiom of restriction was not a core assumption so much as a temporary expedient which would be discarded later once it became clear that it was safe to do so. Over the years this policy-of-caution became the standard by default as this original motivation has been lost sight of: the axiom of restriction is now taught to students as if it had been a standard part of the kit all along.
Partly in response to this situation the American logician W.V, Quine wrote a seminal [1937] paper [8] introducing a new set of axioms for Set Theory ("New Foundations for Mathematical Logic" aka NF) which was incompatible with the axiom of restriction. For many years it languished unregarded largely because it was not known if its relaxation of restriction was safe. This was unfortunate: Quine's axioms had deep motivations which people were reluctant to explore as long as the question of their consistency remained unanswered. An additional discouraging complication came in 1953 when Specker [9] showed that NF contradicted the *Axiom of Choice* which is a core axiom of the traditional kit.
About ten years ago Randall Holmes proved that Quine's NF is consistent. It didn't cause much stir at the time, largely because the proof was very complex and very long (∼50 pages) and also because people were in any case reluctant to believe it. In the years following Holmes revised and simplified the proof, and in 2022 I instigated a Cambridge Summer project for several students to verify it using a proof assistant called *lean*. (Lean is a modern state-of-the art proof assistant and has taken the marketplace by storm). Summer came and went without the project concluding (in fairness we never expected to complete it in the summer but we did hope to break the back of it) and one of the students (Sky Wilshaw) continued collaborating with Holmes, with the result that the verification is now complete. Holmes and Wilshaw are writing it up even as we speak, and are expecting to submit it to a top journal in the coming weeks. It is of course not clear how soon actual publication will ensue but the news is already out [6]; publication is eagerly and widely awaited. (Indeed I have been asked to give a talk about it to the Oxford Logic seminar over zoom – on the day of the Marsden submission deadline!). Arguably this is the biggest development in Foundations of Mathematics for 50 years or more.

Thus the long-awaited time when the axiom of restriction is no longer needed is finally upon us. This means that the intuitions and motivations that underlay Quine's axioms can now be explored in the full confidence that the sky will not come crashing down on our heads.
The developments to which this project is a response will discommode some established researchers – they will regard it as an unwelcome and meretricious distraction from the Proper Set Theory they have become used to; however we can be certain that many people – and many *young* people – will be moved to take an interest in this area which is now suddenly no longer suspect. Now is a good time to offer some studentships to new researchers.
And there is a wealth of fruitful open problems for those students to work on. I keep a list of suitable thesis topics on [2], but we can sketch some of them here.

- Perhaps the first question must be: Is there anything new here? How can we be sure that NF isn't just the old stuff in new language? Recent work by me and Holmes [4] shows that the two approaches aren't just two different ways of saying the same thing; this raises the question of what the relationship between them is: how well do they understand each other. . . ?

- One of ways in which the axiom of restriction/foundation hampered mathematics was the way it outlawed talk about very large sets: the set of *all* sets, the set of *all* numbers, the set of all functions. The new freedom of discussion means we no longer have to chop up these classes and consider their components piecemeal; it plays to Mathematics' fundamental project – to *generalise*, to seek *universal truths*; the aim of Mathematics has always been to See Into The Mind of God.

- The emergence of Holmes' proof relied on the working out of ideas seen in embryo in Jensen's [7], identified and isolated in Holmes [5]. The idea therein of *Tangled Types* has now proved itself to be immensely fertile and is an obvious topic for a Ph.D. thesis.

NF is not only a liberating development that reopens a rich vein that had been bricked over; it also has deep connections with the type theories much used by contemporary theoretical computer scientists. The students funded by this proposal will have plenty of people to talk to. Holmes-Wilshaw is an example – probably the first – of a computer-assisted verification of a proof of something that had been genuinely uncertain hitherto. In contrast the FLYSPECK verification of Hales' proof of the Kepler conjecture was a confirmation of something that had been believed all along, and the computer-assisted proof of the Four-Colour Theorem is not a computer verification of a preëxisting humanly-derived proof but a computer checking of a raft of special cases too numerous to be done by hand. There will presumably be more proofs of this nature in the future.
Why am I the right person to supervise these forthcoming Ph.D. students? Because I am the author of the standard monograph on the subject [1] (which has had two editions so far and a third is planned in response to recent developments) and also the Stanford Encyclopædia article [3]. Because I have long experience in supervising theses on this subject: my Ph.D. students are listed at https://genealogy.math.ndsu.nodak.edu/id.php?id=77585. (It is of course true that Randall Holmes is at least as well qualified but his institution does not have a Ph.D. programme in Mathematics.) Despite my advanced age I am still very much research active – I have two articles in press with the premier Logic journal, the Journal of Symbolic Logic.

SEO codes? 3A. Expression Of Interest and 3B. References
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References

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[2] **Forster, Thomas E** “Thesis topics” www.dpmms.cam.ac.uk/ tef19/thesistopics.pdf

[3] **Forster, Thomas E** “Quine’s New Foundations” Stanford Encyclopædia of Philosophy https://plato.stanford.edu/entries/quine-nf/

[4] **Forster, Thomas E** and Randall Holmes “Synonymy Questions concerning the Quine Systems” Journal of Symbolic Logic, *in press*.

[5] Randall Holmes “The Equivalence of NF-style Set Theories with “tangled” Type Theories; the Construction of ω-models of predicative NF (and more)”. Journal of Symbolic Logic **60** (1995) pp. 178-189.

[6] Randall Holmes and Sky Wilshaw: “The Consistency of NF”

https://arxiv.org/html/1503.01406v22

[7] Jensen, R.B. [1969] “On the consistency of a slight(?) modification of Quine’s NF’; Synthese **19** pp. 250-263.

[8] Quine, W.V. “New Foundations for Mathematical Logic” (1937) most easily accessible in multiple paperback editions of *From a Logical Point of View*.

[9] Specker, E.P. [1953] “The Axiom of Choice in Quine’s New Foundations for Mathematical Logic” Proceedings of the National Academy of Sciences of the USA **39** pp. 972-975. http://www.pnas.org/cgi/reprint/39/9/972. There is a computer-verified version of this proof on https://us.metamath.org/nfeuni/nchoice.html

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