Discussion-answers of Old Tripos Questions

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June 3, 2022

1995

95108

Let P, Q, R be three pairwise-disjoint sets of primitive propositions; for any set S, let $\mathcal{L}(S)$ denote the set of propositional formulae whose primitive propositions are all in S. Let $s \in \mathcal{L}(P \cup Q)$ and $t \in \mathcal{L}(Q \cup R)$ be formulæ such that $(s \to t)$ is a theorem of the propositional calculus. Show that there is a formula $u \in \mathcal{L}(Q)$ such that both $(s \to u)$ and $(u \to t)$ are theorems.

Discussion

PTJ says: consider the set $U = \{u \in \mathcal{L}(Q) : \vdash (s \to u)\}$. Suppose $U \cup \{\neg t\}$ is consistent: take a suitable valuation v of $Q \cup R$, and show that the set

$$\{s\} \cup \{q : q \in Q, v(q) = 1\} \cup \{\neg q : q \in Q, v(q) = 0\}$$

is consistent. Deduce that v can be extended to a valuation of $P \cup Q \cup R$ making s true and t false, and obtain a contradiction.

Boo to that, I say. (This is Craig's Interpolation Theorem). There is a detailed constructive proof in *Logic, Induction and Sets* pp 91–92.

(Here is another proof—that i nicked from Sam Buss.

Think of the literals (and let there be k of them) from Q that occur in s, and of the valuations of those k literals which can be extended to valuations that make s true. Let $v_1 \ldots v_n$ be those valuations. Now let u be the disjunction

This might need relettering

$$\bigvee_{1 \le i \le n} ({q_1}^{(i)} \wedge {q_2}^{(i)} \ldots {q_k}^{(i)})$$

where ${q_j}^{(i)}$ is q_j or $\neg q_j$ depending on whether or not v_i believes q_j. $s \vdash u$ is immediate. To show $u \vdash t$ notice that a valuation on Q that satisfies u can be extended to a valuation on $P \cup Q$ that satisfies s. Any such valuation also satisfies t since $s \vdash t$. Hence $u \vdash t$ as desired.)

2002

2002 IIA paper 1 Q 7J

http://www.maths.cam.ac.uk/undergrad/pastpapers/2002/Part_2/index.html

The only part i'm going to answer is (ii)(b), since the earlier parts are bookwork.

I have to confess that I don't know what the subtext of this question is: usually I know this kind of thing but I'm not omniscient.[1] I'm going to have to fall back on appealing to background knowledge and sniffing around suspicious features. So what do I spot?

The first thing that catches my eye is the use of the symbol '⇒' for this operation on sets. "Might it", one wonders, "be anything to do with implication?" Can we prove the inference

$$\frac{T \subseteq B; \qquad T \subseteq (B \Rightarrow C)}{T \subseteq C} \tag{1}$$

?

You will find it easy to prove this once your suspicions have been aroused.

The second unfamiliar—and therefore eyecatching—part of this question is the condition '$(\forall t \in T)(\exists s \in S)(t \leq s)$' in clause (ii) of the definition. In the literature on quasiorders this is sometimes written '$T \leq^+ S$'. I shall use it here, as it saves space: once you spot that this '+' is useful you can prove some trivial but useful facts like:

$$T \leq^+ S \ \wedge \ S \subseteq (B \Rightarrow C) \ \rightarrow \ \ T \subseteq (B \Rightarrow C) \tag{2}$$

Now that we are armed with these I think it is safe to approach the question. A work of warning before we jump in: overloading of '→' for implication and the operation on sets defined in the question: read formulæ carefully! In fact, i think i'll doctor this file by using '⇒' for the set operation. It's obviously got something to do with Heyting algebras, after all.

To show that $B \Rightarrow C$ is R-closed whenever B and C are we have to show that if $S \subseteq (B \Rightarrow C)$ and $\langle S, a\rangle \in R$ then $a \in (B \Rightarrow C)$.

So let $S \subseteq (B \Rightarrow C)$ and a s.t. $\langle S, a\rangle \in R$ be arbitrary. We will show that $a \in (B \Rightarrow C)$.

[1] I now (2012) realise it's all about how to recover a Heyting algebra from a possible world model. (At least, that's what i tho'rt! PTJ says: "As far as I was concerned, it was actually about constructing the subobject classifier in a topos of sheaves." Well!!–he set the question and he should know)

$S \subseteq B \Rightarrow C$ is
$(\forall a \in S)(\forall b \leq a)(b \in B \to b \in C)$

We have assumed $\langle S, a\rangle \in R$. Then we have to show that $a \in (B \Rightarrow C)$. To do this we have to show

$$(\forall b \leq a)(b \in B \to B \in C)$$

$$(\forall b \leq a)(\exists T \subseteq A)(\langle T, b\rangle \in R \;\wedge\; T \leq^{+} S \;\wedge\; (\forall t \in t)(\langle \{b\}, t\rangle \in R)) \tag{3}$$

Now suppose $b \leq a$. By formula 3 we have

$$(\exists T \subseteq A)(\langle T, b\rangle \in R \;\wedge\; T \leq^{+} S \;\wedge\; (\forall t \in t)(\langle \{b\}, t\rangle \in R))$$

Assume further that $b \in B$; we want $b \in C$. We know that $b \in T$ and that $T \leq^{+} S$ from formula 3; formula 2 tells us that $T \subseteq (B \Rightarrow C)$, and then formula 1 tells us that $T \subseteq C$. But then $b \in C$ as desired. But b was arbitrary. This establishes $(\forall b \leq a)(b \in B \to B \in C)$. This is precisely the condition for a to be in $B \Rightarrow C$.

2006

Paper 4 Question

In the first part they tell you to take your wellfounded relation r and define on it by recursion the obvious rank function to the ordinals. This refines the wellfounded relation to a **prewellorder** (a wellordered partition). That was a bit cryptic, so let me be clearer. You have defined a rank function ρ from the domain of r to the ordinals. This process partitions the domain into pieces $\rho^{-1}``\{\alpha\}$ for each ordinal α. The relation $\{\langle x, y\rangle : \rho(x) < \rho(y)\}$ is wellfounded and contains every ordered pair in r. It's not a wellorder, beco's two things in the same piece are not related to each other, tho' it is what we call a *prewellorder*. AC tells us that everything can wordered, so in particular each piece of the partition can be wellordered, and we use AC to pick such a wellorder for each piece. Then we concatenate the wellorderings

That's the way the examiners expect you to do it. However there is the clever way. Consider the collection of wellorders that are compatible with r. Partially order them by **end-extension**. Then use Zorn. The point is that a union of a chain **under end-extension** of wellorderings is another wellordering. (Look at question 2009-3-16G below where end-extensions reappear.)

2007

2007-3-16G

Observe that if $\gamma \geq \omega \cdot \beta$ then $\beta + \gamma = \gamma$. Contraposing we infer that if γ does *not* "absorb β on the left" then $\neg(\gamma \geq \omega \cdot \beta)$ whence $\gamma < \omega \cdot \beta$. Analogously if β does *not* "absorb γ on the left" then $\beta < \omega \cdot \gamma$. So, if β and γ are *commensurable* (never heard this word used this way, but never mind) we have

$\gamma < \omega \cdot \beta$ and $\beta < \omega \cdot \gamma$.

Now we use the division algorithm to find the largest power of $\omega \leq \beta$.

2007-4-II-16G

We say (for the purposes of this question) that a function $f : A \to \mathcal{P}(A)$ is **recursive** if the relation $\{\langle a, b\rangle : a \in f(b)\}$ is wellfounded. Observe that any binary relation on a set A can be thought of as a map $f : A \to \mathcal{P}(A)$.

(That is the first of the two points made by this question. The other point is that not only is wellfoundedness of a relation R a sufficient condition for definitions given by recursion-over-R to have unique solutions, it is also necessary. If every definition given by recursion over R has a unique solution then R is indeed wellfounded. It is disconcerting—even if only slightly—that you are being invited to prove this converse while thinking of binary relations as functions-into-power-sets [as above].)

Suppose $g : \mathcal{P}(B) \to B$. We can attempt to define a function h recursively by:

$$h(a) := g(\{h(a') : a' \in f(a)\}).$$

Clearly we are going to be able to show (by an appeal to the recursion theorem) that this recursion has a unique solution—as long as the relation $\{\langle a, b\rangle : a \in f(b)\}$ is wellfounded. But what about a converse?

Suppose $\{\langle a, b\rangle : a \in f(b)\}$ is not wellfounded. We want to find a B and $g : \mathcal{P}(B) \to B$ such that there is more than one h satisfying

$$(\forall a \in A)(h(a) = g(\{h(a') : a' \in f(a)\})).$$

Let B be a set with at least two members, and b_1 and b_2 be two members of B, and define $g : \mathcal{P}(B) \to B$ by

$$g(B') = \texttt{if}\,(B' = \emptyset \vee B' = \{b_1\})\,\texttt{then}\,b_1\,\texttt{else}\,b_2$$

Suppose now that A' is a subset of A with no minimal member under the relation $\{\langle a, b\rangle : a \in f(b)\}$. Notice that both

$$h_1(a) := b_1$$

and

$$h_2(a) := \texttt{if}\,a \in A'\,\texttt{then}\,b_2\,\texttt{else}\,b_1$$

are solutions to

$$h(a) = g(\{h(a') : a' \in f(a)\}).$$

2008

2008-3-16G

Obviously the rank of $\mathcal{P}(x)$ has to be one greater than the rank of x. The rank function is $\subseteq$-monotone ($u \subseteq v \to \rho(u) \leq \rho(v)$) so the rank of x is maximal among ranks of members of $\mathcal{P}(x)$. So $\rho(\mathcal{P}(x))$ must be $\rho(x) + 1$, so it is successor.

The fun starts when you ask about the ordinals that can be ranks of $\mathcal{P}_{\aleph_0}(x)$ (the set of finite subsets of x) or $\mathcal{P}_{\aleph_1}(x)$ (the set of countable subsets of x). Either of those could be a fair tripos question, i think…

2009

2009-2-16G

Suppose there were an axiomatisation T of the theory of groups all of whose elements have finite order. Add, for each n, the axiom $(\forall x)(x^n = 1 \to x = 1)$. This theory has no models, but all its finite fragments do.

2009-3-16G

Let $x \subseteq y \subseteq$ the ordinals below α. x and y both inherit an ordering. Why do we know that the length of x in the inherited ordering is $\leq$ the length of y? It looks obvious, but we have to be careful. The ordering on ordinals arises from "A is iso to an an initial segment of B" rather than from "A injects into B in an order preserving fashion", and it is the second one that we want here. However these two relations are the same! This was an example sheet question in the year this question was set, so i'll provide a proof:

Remark 1 *A total order is a well order iff every subordering is iso to an initial segment.*

Proof: Let $\langle A, <_A \rangle$ be a total order where every subordering is iso to an initial segment. Consider a one-point subordering. It is iso to an initial segment so $\langle A, <_A \rangle$ has a bottom element. But now every subordering is iso to an initial segment, and so has a least element, so $\langle A, <_A \rangle$ is a well order.

For the other direction, every suborder of a well order is a well order. So, if $\langle A, <_A \rangle$ is a well order and $\langle A', <_A \rangle$ a suborder of it, then one of $\langle A, <_A \rangle$ and $\langle A', <_A \rangle$ is iso to an initial segment of the other. (The two might be iso of course). We must establish that $\langle A, <_A \rangle$ cannot be iso to a proper initial segment of $\langle A', <_A \rangle$. But this is easy: such an isomorphism would be a $<_A$-decreasing injective function $A \to A$, so that there will be sets $\{f^n(a) : n \in \mathbb{N}\} \subseteq A$ with no $<_A$-least member, contradicting wellfoundedness of $<_A$. ■

The second paragraph requires care. Observe that, at any rate, the length of the union of the x_n must be at least the sup of the lengths; that much is obvious. It's the other direction that can fail.

The operation that sends each $x \subseteq \alpha$ to an ordinal $\mu(x)$ does its work by sending each member of x to an ordinal; $\mu(x)$ is then the least ordinal not used. A member a of α can appear in lots of x's. **The point of the end-extension condition is that it**

ensures that a **gets sent to the same ordinal whatever** x **we are considering**; that ordinal is then the ordinal it is given in the computation of $\mu(\bigcup_n x_n)$. To illustrate the importance of the end-extension condition consider the following family of subsets of the ordinals below $\omega + 1$:

$$\{[0, n] \cup \{\omega\} : n < \omega\}$$

so that x_n is the ordinals below n with ω whacked on the end. Each subordering is of finite length, and the lengths are unbounded so the sup of the lengths is ω. However the union of the subsets is the set of ordinals below $\omega + 1$ which of course is of length $\omega + 1$.

Observe that in this case the ordinal that the point ω gets sent to in the calculation of $\mu(x_n)$ is not constant but depends on n.

It may be worth noting that *any countable linear order type whatever* can be obtained as a union of an ω-chain of finite total orders. Come to think of it that might make a good exercise for next year.

For the final part let x_n be $\mathbb{N} \setminus [0, n]$. Each x_n is a terminal segment of the finite ordinals and has length ω, but their intersection is empty.

2010

2010-2-16G

For the first part use the division algorithm for normal functions. See my ordinal notes if you don't know it (you should)

Show that an ordinal is indecomposible iff it is a power of ω.

One direction is easy: if α is not a power of ω then it is decomposed by applying the division algorithm to the function $\beta \mapsto \omega^\beta$, which give us the largest power of $\omega \leq \alpha$. (That was the first part of this question)

The other direction is easy if $\beta + \gamma = \omega^\lambda$ with λ limit. Then β and γ are both $< \omega^\alpha$ with $\alpha < \lambda$. But then

$$\beta + \gamma \leq \omega^\alpha \cdot 2 < \omega^{\alpha+1} < \omega^\lambda$$

We are left with the successor case. So suppose $\beta + \gamma = \omega^{\alpha+1} = \omega^\alpha \cdot \omega$.
Use the division algorithm to obtain $\beta = \omega^\alpha \cdot \beta_1 + \beta_2$ with $\beta_2 < \omega^\alpha$ and $\gamma = \omega^\alpha \cdot \gamma_1 + \gamma_2$ with $\gamma_2 < \omega^\alpha$.
Then

$$\omega^\alpha \cdot \beta_1 + \beta_2 + \omega^\alpha \cdot \gamma_1 + \gamma_2 = \omega^\alpha \cdot \omega$$

Now $\beta_2 < \omega^\alpha$ so we can simplify

$$\omega^\alpha \cdot \beta_1 + \omega^\alpha \cdot \gamma_1 + \gamma_2 = \omega^\alpha \cdot \omega$$

Clearly $\gamma_2 = 0$ so we simplify again

$$\omega^\alpha \cdot \beta_1 + \omega^\alpha \cdot \gamma_1 = \omega^\alpha \cdot \omega$$

This tells us that $\beta_1 + \gamma_1 = \omega$ so one of them is ω, from which the result follows.

2010-3-16G

I do only the hard part

Suppose x is a transitive set, of rank α. We will show that, for all $\beta < \alpha$, x has a member of rank β. That is to say, $\{\beta \in On : x \text{ has a member of rank } \beta\}$ is a proper initial segment of On. It must be a set beco's it is bounded above by the rank of x.

Observe the following:

(i) If x has a member y of rank $\beta + 1$ then y has a member y' of rank β. But x is transitive, so $y' \in x$ and x has a member of rank β.

(ii) If x has a member y of rank λ (where λ is limit) then $(\forall \beta < \lambda)(\exists \gamma < \lambda)(\beta < \gamma \wedge y$ has a member of rank $\gamma)$. But x is transitive, so the same goes for x: $(\forall \beta < \lambda)(\exists \gamma < \lambda)(\beta < \gamma \wedge x$ has a member of rank $\gamma)$.

(i) and (ii) together mean that the collection of ordinals β such that x does not have a member of rank β is closed under successor (beco's of (i)) and limits (beco's of (ii)), so it must be a terminal segment of On. So its complement must be a proper initial segment of *On*. But that is what we wanted.

The other thing one can try is to prove by $\in$-induction on x that $TC(x)$ has members of all ranks less than $\rho(x)$. Suppose that, for all $y \in x$, $TC(y)$ has members of all ranks less than $\rho(y)$. $TC(x) = x \cup \bigcup_{y \in x} TC(y)$. Now let α be an ordinal less than the rank of x. Then $\alpha \leq \rho(y)$ for some $y \in x$. If we have equality then y is the thing we are looking for that is in $TC(x)$ and has rank α. If we have $<$ then we have our witness by induction hypothesis.

2011

2011 Paper 2 Q 16H

The first two parts are bookwork. For the third part i'm assuming that each element of the space is a **finite** sum of multiples of basis-element–times–a-real. There are $\aleph_1$ basis elements and $2^{\aleph_0}$ reals, making $\aleph_1 \cdot 2^{\aleph_0}$ such pairs. How many finite sets of such pairs? The cardinality of $\mathcal{P}_{\aleph_0}(X)$ (the set of finite subsets of X) can be quite hard to compute in terms of $|X|$, whereas $|X^{<\omega}|$ (the set of finite sequences-without-repetitions of members of X) is just $|X| + |X|^2 + |X|^3 \ldots$. Evidently $X^{<\omega}$ surjects onto $\mathcal{P}_{\aleph_0}(X)$ and, since any set X of size $\aleph_1 \cdot 2^{\aleph_0}$ admits a total ordering, we can use that total order to uniformly order each finite set and thus inject $\mathcal{P}_{\aleph_0}(X)$ into $X^{<\omega}$. Evidently (whatever X is) and—certainly in this case—each of the summands in $|X^{<\omega}| = |X| + |X|^2 + |X|^3 \ldots$ is $|X|$ (which is to say $\aleph_1 \cdot 2^{\aleph_0}$) so (with the help of Cantor-Bernstein) we arrive at $\aleph_1 \cdot 2^{\aleph_0} \cdot \aleph_0 = \aleph_1 \cdot 2^{\aleph_0}$ such finite sets, and therefore $\aleph_1 \cdot 2^{\aleph_0}$ vectors.

All that was without any use of AC. If we allow ourselves even mere countable choice we can infer $\aleph_1 \leq 2^{\aleph_0}$, so there are $2^{\aleph_0}$ such pairs and $2^{\aleph_0}$ finite sets of such pairs, so there are $2^{\aleph_0}$ vectors.

That seems like a lot of work, and maybe the examiners didn't expect that much detail (you have only about half an hour per question after all) but it can't do you any harm to see it done properly. My guess is that you'd've got the brownie points simply for getting the correct answer and waving your hands artistically.

If an element of the space can be an **arbitrary** sum of multiples of basis-element–times–a-real then the sums are subtly different. There are still $2^{\aleph_0}$ pairs, but now we are interested in subsets of the collection of pairs of sizes up to $\aleph_1$, which gives us $(2^{\aleph_0})^{\aleph_1}$ elements. Now $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_0 \cdot \aleph_1}$. Now $\aleph_1 \leq \aleph_0 \cdot \aleph_1 \leq \aleph_1 \cdot \aleph_1 = \aleph_1$, so $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1}$, which won't simplify any further (in case you were wondering). The continuum hypothesis implies that $2^{\aleph_1} = 2^{2^{\aleph_0}}$, and Luzin's hypothesis [don't ask] is that $2^{\aleph_1} = 2^{\aleph_0}$. Both are consistent with ZFC—tho' of course not jointly!

2011 Paper 3 Q 16H

(i) The theory of dense linear orderings is axiomatisable, and the axioms are what you think they are.

(ii) The theory of countable dense linear orderings is not axiomatisable in the language of posets. For suppose it were, and T were an axiomatisation. We could expand the language of T by adding uncountably many constants, and add to T axioms to say that the constants all point to distinct things. That would be a theory with uncountable models—all of which would be models of T—contradicting the assumption that all models of T are countable. Note that there is a first order theory which is true in each and every countable DLO; the problem is that it has uncountable models as well.

(ii) The theory of uncountable dense linear orderings is axiomatisable *if you are allowed to add uncountably many constants*. (If the axiom of choice holds then every uncountable set has a subset of size $\aleph_1$, so we add $\aleph_1$ constants.) However, that takes you out of the language of posets, so it's not allowed; downward Skolemheim tells us that any theory of uncountable dense linear orders *in the language of posets, which is countable* will have countable models.

(iv) There cannot be a first-order theory T of wellorderings in the language of posets beco's if there were one could add constants $\langle a_i : i \in \mathbb{N} \rangle$ to the language, and axioms $a_i > a_{i+1}$ for each $i \in \mathbb{N}$. The result would be a theory all of whose models are wellorderings (since they are models of T) while not being wellorderings after all beco's of the a_i.

Here is another proof... one that you could conceivably be expected to find given the material you have been lectured, at least in a PTJ year ...

Every substructure of a wellorder is a wellorder. So the first-order theory of wellorders (if there is one) is a universal theory. Now if T is a universal theory then—by part (iv) of sheet 3 q10 2015 (for example!)—the union of a $\subseteq$-chain of models of T is another model of T; so a union of a $\subseteq$-chain of wellorderings would have to be a wellordering. But—for example—every terminal segment of $\mathbb{Z}$ is a wellorder and the union of all of them is not. (A union of a chain of wellorderings is a wellordering if it is a chain under end-extension.)

... but that looks quite hard to me.

Paper 4 question

This is mostly bookwork. To prove that every set belongs to a V_α you are clearly going to have to use the foundation axiom somehow, since if foundation fails you can have silly things like Quine atoms that do not belong to any V_α. The usual way to use foundation is to do an $\in$-induction. This is a case in point: you prove by $\in$-induction that every set belongs to a V_α. Suppose this is true for every member y of x. To each such y associate α_y, the least α s.t. $y \in V_\alpha$. We form the set of those α_y and take the sup. Probably worth pointing out that in so doing we are using the axiom scheme of replacement.

2012

2012 paper 3 Q 16H

All bookwork. Part (v) looks scary but it's actually easy. Let α be any ordinal at all. Use the division algorithm on the function $\beta \mapsto \omega_1 \cdot \beta$ to obtain the largest multiple of ω_1 that is less than or equal to α. If that number is α itself then α is a multiple of ω_1; if it isn't then there is a remainder. This remainder is countable and so has cofinality ω (if it is limit) or is successor (in which case there is a cofinal map within the meaning of the act.).

2013

2013-1-16G

The only part that isn't bookwork is the very end bit. If you write an ordinal as a sum of a sequence of *descending* powers of ω all you get is the last term, beco's any power of ω absorbs all smaller powers of ω on the left. Multiplying by natural numbers doesn't change anything.

2013-2-16G

Explain what is meant by a chain-complete poset. State the Bourbaki–Witt fixed-point theorem for such posets.

A poset P is called directed if every finite subset of P (including the empty subset) has an upper bound in P; P is called directed-complete if every subset of P which is directed (in the induced ordering) has a least upper bound in P. Show that the set of all chains in an arbitrary poset P, ordered by inclusion, is directed-complete.

Given a poset P, let $[P \to P]$ denote the set of all order-preserving maps $P \to P$, ordered pointwise (i.e. $f \leq g$ if and only if $f(x) \leq g(x)$ for all x). Show that $[P \to P]$ is directed-complete if P is.

Now suppose P is directed-complete, and that $f : P \to P$ is order-preserving and inflationary. Show that there is a unique smallest set $C \subseteq P \to P$ satisfying

(a) $f \in C$;
(b) C is closed under composition (i.e. $g, h \in C \to g \cdot h \in C$); and
(c) C is closed under joins of directed subsets.

Show that

(i) all maps in C are inflationary;
(ii) C is directed;
(iii) if $g = \bigvee C$, then all values of g are fixed points of f ;
(iv) for every $x \in P$, there exists $y \in P$ with $x \leq y = f(y)$.

Discussion

'Directed' is an important notion. Cast your mind back to the proof that the class of models for a $\forall^*\exists^*$ ("inductive") theory is closed under unions of chains. Look carefully at that proof. You will notice that you didn't really need chains: if T is inductive then a union of a directed set of models of T is another model of T.

We have to show that the chains of an arbitrary poset form a directed-complete poset. Not a *directed* poset! That's clearly not going to happen unless the poset is a toset! If i have a directed family F of chains in a poset $\langle P, \leq_P \rangle$ then that family has a sup, namely $\bigcup F$. We have to check that $\bigcup F$ is a chain. Suppose x and y are both elements of $\bigcup F$. They belong to two chains c_x and c_y, but—by directedness—there is now a chain $\supseteq c_x \cup c_y$ to which both x and y belong. So x and y are $\leq_P$-comparable.

The set C we want is defined inductively: it's the intersection of the collection $\mathcal{C}$ of all the subsets of $P \to P$ satisfying conditions (a–c). $\mathcal{C}$ is a subset of $\mathcal{P}(P \to P)$ so it is a set, and it is nonempty, since $P \to P$ itself is such a subset. We need also to check that conditions (a–c) are *intersection-closed* (that is to say, an arbitrary intersection of sets satisfying (a–c) also satisfies (a–c)) but that is evident, beco's (a–c) are *closedness* properties. So the intersection $\bigcap \mathcal{C}$ of all such subsets satisfies (a–c) and is the set C that we want.

Clearly C is going to be the family of all iterates of f.

(i) Since C has an inductive definition it supports induction. So we should expect to be able to prove (i) by induction. If the set of all inflationary maps $P \to P$ is a member of $\mathcal{C}$ then we are home and hosed. Just check that it satisfies (a–c). It satisfies (a) beco's f is inflationary, it satisfies (b) beco's a composition of two inflationary functions is inflationary, and it satisfies (c), since the [pointwise] sup of a lot of inflationary maps is inflationary.

(ii) Suppose $D \subseteq [P \to P]$ is directed. Define $h : P \to P$ by $h(x) = \bigvee\{g(x) : g \in D\}$. For this definition to succeed we need $\{g(x) : g \in D\}$ to be directed. So let U be a finite subset of $\{g(x) : g \in D\}$. We want U to have an upper bound. Now U is $\{g(x) : g \in D'\}$ for some finite $D' \subseteq D$. But this D' has an upper bound by the assumption that D is directed.

(iii) We have to show that every value of $\bigvee C$ is a fixed point for f. For $p \in P$, $\bigvee C(p)$ is the sup of $f(p)$ for all $f \in C$. We must compute $f(\bigvee C(p))$ for arbitrary $p \in P$ and hope that it evaluates to $\bigvee C(p)$.

What is $\bigvee C$ applied to p? It's $\bigvee$ of the set $\{g(p) : g \in C\}$ of iterates of f applied to p. Now everything in C is inflationary (that was (i)) so $\bigvee\{g(p) : g \in C\} \leq_P \bigvee\{f(g(p)) : g \in C\}$. The other direction, namely $\bigvee\{f(g(p)) : g \in C\} \leq_P \bigvee\{g(p) : g \in C\}$, we get beco's $\{f(g(p)) : g \in C\} \subseteq \{g(p) : g \in C\}$, whence

$$\bigvee\{g(p) : g \in C\} = \bigvee\{f(g(p)) : g \in C\} \tag{A}$$

At this point we have to use the fact (unused so far, i think) that f is order-preserving, to infer

I don't think this works … oops

$$f(\bigvee\{g(p) : g \in C\}) \leq \bigvee\{f(g(p)) : g \in C\}.$$

Substituting using the equation (A) we obtain

$$f(\bigvee\{g(p) : g \in C\}) \leq \bigvee\{g(p) : g \in C\}.$$

The equality in the other direction

$$\bigvee\{g(p) : g \in C\} \leq f(\bigvee\{g(p) : g \in C\})$$

follows from f being inflationary, so we infer

$$\bigvee\{g(p) : g \in C\} = f(\bigvee\{g(p) : g \in C\})$$

so $\bigvee C(p)$ is a fixed point for f as desired.

(iv) This challenge reminds me very strongly of the fact that one can infer "In every chain complete poset every element has a maximal element above it" from "Every chain-complete poset has a maximal element". That hint was enough for me to know what to do, and it should work for you too. So you run the preceding constructions on $P \uparrow x$, the substructure of $\langle P, \leq_P \rangle$ that is its restriction to the elements that are $\geq_P x$: replace 'f' by '$f \restriction (P \uparrow x)$'.

2013-3-16G

Suppose P , Q and R are pairwise disjoint sets of primitive propositions, and let $\phi \in L(P \cup Q)$ and $\psi \in L(Q \cup R)$ be propositional formulae such that $\phi \to \psi$ is a theorem of the propositional calculus. Consider the set $X = \{\chi \in L(Q) : \phi \vdash \chi\}$. Show that $X \cup \{\neg\psi\}$ is inconsistent, and deduce that there exists $\chi \in L(Q)$ such that both $\phi \to \chi$ and $\chi \to \psi$ are theorems. [Hint: assuming $X \cup \{\neg\psi\}$ is consistent, take a suitable valuation v of $Q \cup R$ and show that $\{\phi\} \cup \{q \in Q | v(q) = 1\} \cup \{\neg q : q \in Q, v(q) = 0\}$ is inconsistent. The Deduction Theorem may be assumed without proof.]

Discussion

Take the hint. If $X \cup \{\neg\psi\}$ is *not* inconsistent, then there is a valuation (call it 'v') making X true and ψ false. This valuation v is defined on letters in $Q \cup R$. The plan now is to extend v to a valuation v' defined on all the letters in P as well, so that v' makes ϕ true too. Now every valuation making ϕ true makes ψ true, by assumption; so $X \cup \{\neg\psi\}$ will be inconsistent as desired. So how do we extend v to a v' making ϕ true? ϕ has propositional letters in it that come from P and from Q. Doctor ϕ by setting to $\top$ every q-letter that v makes true, and setting to $\perp$ every q-letter that v makes false, and then simplify (using rewrites such as $\alpha \wedge \perp \mapsto \perp$, $\alpha \vee \top \mapsto \top$, that sort of thing) obtaining a formula in $L(P)$. If this formula hasn't simplified to $\perp$ then it has some p-letters in it and we can find a valuation making it true, and then we are happy. But what if we can't? Can that happen? That would mean that the stuff in X prevented ϕ being true. There may be stuff that prevents ϕ being true, but the stuff in X consists of **consequences** of ϕ. So ϕ is prevented from being true by some of its consequences. But that must mean that ϕ is the negation of a truth-table tautology and that case isn't interesting.

2013-4-16G

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent in the presence of the other axioms of ZF set theory. [You may assume the existence of transitive closures.] Given a model $\langle V, \in\rangle$ for all the axioms of ZF except Foundation, show how to define a transitive class R which, with the restriction of the given relation $\in$, is a model of ZF. Given a model $\langle V, \in\rangle$ of ZF, indicate briefly how one may modify the relation $\in$ so that the resulting structure $\langle V, \in'\rangle$ fails to satisfy Foundation, but satisfies all the other axioms of ZF. [You need not verify that all the other axioms hold in $\langle V, \in'\rangle$.]

The first part is bookwork.

The second part, too, is bookwork but it's possible to get it wrong. Indeed i was contacted by a Trinmo(!) who had got into a tangle, He wanted the class of all x s.t. $(\exists y \in x)(x \cap y = \emptyset)$ (which is *not* what you want: suppose you had $x = \{x, \emptyset\}$; then x is disjoint from one of its members but is not wellfounded). The class you want is the union of all the V_α where $V_0 = \emptyset$ and thereafter $V_\alpha = \mathcal{P}(\bigcup_{\beta<\alpha} V_\beta)$.

For the final part look up permutation models in my 2017 lecture notes.

2014

Paper 2 question

Ordinals less than a power of ω are closed under $+$; ordinals less than ω^{ω^α} are closed under $\times$; ordinals below an ϵ-number are closed under exponentiation.

Discussion

Let's prove that the ordinals less than a power of ω are closed under addition. Suppose not, and let $\omega^\alpha = \beta + \gamma$. Then both β and γ are below $\omega^\delta \cdot n$ for some $\delta < \alpha$. But then $\omega^\alpha = \beta + \gamma \leq \omega^\delta \cdot n + \omega^\delta \cdot n \leq \omega^\delta \cdot n \cdot 2 < \omega^{\delta+1} \leq \omega^\alpha$.

Closure under multiplication

Suppose α and β are both less than ω^{ω^γ}. Then there is a $\delta < \omega^\gamma$ such that both α and β are below ω^δ. But then $\alpha \cdot \beta$ is below $\omega^\delta \cdot \omega^\delta = \omega^{\delta+\delta}$ and now we appeal to the fact that the ordinals below ω^γ are closed under addition!

Closure under exponentiation

Suppose $\omega^\epsilon = \epsilon$, and $\alpha, \beta < \epsilon$. Then $\alpha^\beta \leq (\omega^\alpha)^\beta =^{(1)} \omega^{\alpha \cdot \beta}$.

Now $\epsilon = \omega^{\omega^\epsilon}$ so ordinals below ϵ are closed under multiplication (see above) giving $\alpha \cdot \beta < \epsilon$, whence $\omega^{\alpha\cdot\beta} <^{(2)} \omega^\epsilon = \epsilon$.

> (1) It can't do any harm to think about why this equation holds. You could prove it from the synthetic definition of exponentiation (which you were presumably not lectured) or you could prove it by fixing α ("universal generalisation") and doing an induction on β.
>
> (2) holds beco's $\gamma \mapsto \omega^\gamma$ is normal.

2015

Paper 1, Section II 13I

State and prove the Completeness Theorem for Propositional Logic.

[You do not need to give definitions of the various terms involved. You may assume the Deduction Theorem, provided that you state it precisely.]

State the Compactness Theorem and the Decidability Theorem, and deduce them from the Completeness Theorem.

Let S consist of the propositions $p_{n+1} \to p_n$ for $n = 1, 2, 3 \ldots$. Does S prove p_1? Justify your answer. [Here p_1, p_2, $p_3 \ldots$ are primitive propositions.]

Bookwork. For the last part you need the valuation that make p_1 false and all the others true. Ridiculously easy.

Paper 2, Section II 13I

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent. Give the inductive definitions of ordinal multiplication and ordinal exponentiation.

(b) Answer, with brief justification, the following:

(i) For ordinals α, β and γ with $\alpha < \beta$, must we have $\alpha + \gamma < \beta + \gamma$? Must we have $\gamma + \alpha < \gamma + \beta$?

(ii) For ordinals α and β with $\alpha < \beta$ must we have $\alpha^\omega < \beta^\omega$?

(iii) Is there an ordinal $\alpha > 1$ such that $\alpha^\omega = \alpha$?

(iv) Show that $\omega^{\omega_1} = \omega_1$. Is ω_1 the least ordinal α such that $\omega^\alpha = \alpha$?

[You may use standard facts about ordinal arithmetic.]

This is mostly bookwork. However there is a subtlety to part (iv). ω^{ω_1} is of course $\sup\{\omega^\alpha : \alpha < \omega_1\}$ and we want this to be no more than ω_1. It's clearly no *less* than ω_1 because $\alpha \leq \omega^\alpha$ always and the α we are summing over are unbounded below ω_1; for it to be no *greater* than ω_1 we need ω^α to be countable whenever α is. If we try to do this by induction on α we have no problem at successor ordinals of course, co's we're just multiplying by ω, but at limit stages we are liable to find ourselves appealing to the principle that a union of countably many countable sets is countable.

Why is ω^λ countable? Well, if λ is countable limit it is $\sup\langle\lambda_n : n \in \mathbb{N}\rangle$ and each (von Neumann ordinal) ω^{λ_n} is [a] countable [set] by induction hypothesis, so the (von Neumann ordinal) ω^λ is [a] countable [set] by countable-union-of-countable-sets-is-countable. This use of countable choice seems to be unavoidable.

However, if we use the synthetic definition of ordinal exponentiation we obtain a set (the set of all those functions from a wellordering of length α to $\mathbb{N}$ that take the value 0 at all but finitely many arguments) equipped with a natural wellordering that is of order type ω^α. This set can be shown to be countable, as follows. Each such function can be thought of as a finite set of ordered pairs of ordinals-below-α paired with naturals. There are countably many such pairs and therefore only countably many finite sets of such pairs.

You probably don't care about such things (and i bet the examiners didn't) but i didn't get where i am today by being the Old Man of Thermopylæ who never did anything properlæ. As for whether or not you can prove the analogous result for the next operation after exponentiation (namely that the countable ordinals are closed under it) without using countable choice i have to confess that i'd never thought about it. I have an awful feeling that the answer might be 'no'.

Paper 3, Section II 13I

(i) State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.]

Where in your proof have you made use of the Axiom of Choice?

(ii) Let $<$ be a partial ordering on a set X. Prove carefully that $<$ may be extended to a total ordering of X. What does it mean to say that $<$ is well-founded? If $<$ has an extension that is a well-ordering, must $<$ be well-founded? If $<$ is well-founded, must every total ordering extending it be a well-ordering? Justify your answers.

Discussion

The first part is bookwork. However they want you to say (and *I* want to hear you say it, too) that The Axiom of Choice says that every set X has a choice function blah, and that in this case the set X that has a choice function is that set that contains, for each p in your poset, the set $\{p' \in P : p \leq p'\}$.

If you discard ordered pairs from a wellfounded relation the result is still a well-founded relation. (If you remove ordered pairs you perforce remove descending sequences so you are certainly not going to *add* any new infinite descending sequences!) So: if $<$ has an extension that is a well-ordering, $<$ must be well-founded. The converse is clearly not going to be true. Consider the empty relation on $\mathbb{Q}$. It's wellfounded, but you can extend it to the usual order on $\mathbb{Q}$, and that is not wellfounded.

Paper 4, Section II 13I

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [You may assume the existence of transitive closures.]

Explain briefly how the Principle of $\in$-Induction implies that every set is a member of some V_α.

Find the ranks of the following sets:

(i) $\{\omega + 1, \omega + 2, \omega + 3\}$*;*
(ii) the Cartesian product $\omega \times \omega$*;*
(iii) the set of all functions from ω *to* ω^2*.*

[You may assume standard properties of rank.]

Discussion

The first part is bookwork. For the second we'll assume that the ordinals are von Neumann ordinals and that ordered pairs are Wiener-Kuratowski, and that will give the answers

(i) $\omega + 4$ (since the rank of the von Neumann ordinal α is α);

(ii) ω, because every member of $\omega \times \omega$ is an ordered pair of two things of finite rank, and W-K pairs lift rank by 2. . . and

(iii) $\omega^2 + 1$, beco's every function $\omega \to \omega^2$ is a subset of $\omega \times \omega^2$ and is therefore of rank at least ω (since its arguments ore of unbounded finite rank) and possibly as much as ω^2 (since its values can have any rank below ω^2) by analogy with (ii).

In connection with (iii) it might be worth reflecting that if we were considering functions $\omega \to \omega_1$ then the answer would be ω_1 not $\omega_1 + 1$. This is because a map $f : \omega \to \omega_1$ is a set of ordered pairs of a-finite-ordinal-with-a-countable-ordinal, so the ranks of all its members are countable. But this f is a countable set, so its rank is the least ordinal strictly bigger than evereything in a *countable* set of ordinals, and by countable choice this upper bound is countable.

2016

Paper 1 section II 15F

Part (b)

Part (c)

I don't know how to make it obvious that the answer is 'yes', and of course one needs the answer to be obvious if one is to know which way to jump. My thought for this one is that one should remember that $\alpha \mapsto \omega^\alpha$ is a normal function, and just give it a whirl. Low cunning would suggest that the mention of the Division algorithm at the end of the question is a hint that you should be using it. Given $\alpha = \omega \cdot \alpha$ one looks for the largest β (and there will be a largest β beco's $\alpha \mapsto \omega^\alpha$ is a normal function and we invoke the division algorithm for normal functions) s.t. $\omega^\beta \leq \alpha$.

That should crack it, so i'm leaving the details to you.

Perhaps the moral is to always remember the division algorithm for normal functions.

Actually, on second thoughts i think i owe it to you to provide a worked answer. [Thanks to Hanna Gál for pushing me.] One shows that $\alpha = \omega^n \cdot \alpha$ for every n by induction on n so certainly $\alpha \geq \omega^\omega$. One then uses the division algorithm to find β the largest multiple of $\omega^\omega \leq \alpha$. So

$$\omega^\omega \cdot \beta + \gamma = \alpha$$

where $\gamma < \omega^\omega$. What now? The only fact we have is that $\alpha = \omega \cdot \alpha$ so the obvious thing is to multiply both sides on the left by ω getting

$$\omega \cdot (\omega^\omega \cdot \beta + \gamma) = \omega \cdot \alpha = \alpha$$

But the LHS simplifies to

$$\omega^\omega \cdot \beta + \omega\gamma$$

which (by uniqueness of subtraction) gives us $\omega \cdot \gamma = \gamma$, contradicting $\gamma < \omega^\omega$.

A nice proof. Not sure whether i'd find it in the heat of the exam.

Part (e)

This is a trick question. Since $\alpha < \alpha^\omega$ always, the conditional has a false antecedent and is therefore true!

Paper 2 14F

An alternative equivalent definition of transitive set is $x \subseteq \mathcal{P}(x)$ which makes the induction a lot easier.

Paper 3, Section II 14F

This looks scary but it's actually a piece of cake. We appeal to the banal obvious fact.... If T is a universal theory then every substructure of a model of T is another model of T. So in particular the package of closed T-terms must be a model of T.

.

For the last part we of course use compactness. Suppose $T \vdash (\exists x)\psi$. Suppose further that T does not prove any finite disjunction of the indicated flavour. Then we can consistently add to T any finite collection of expressions of the form $\{\neg\psi[t_i/x] : i \in I\}$. So, by compactness, we can consistently add all negations $\neg\psi[t_i/x]$. This theory will have a model $\mathfrak{M}$. Think about the substructure of $\mathfrak{M}$ consisting of closed terms. It is a substructure of a model of T and is therefore a model of T. So it believes $\exists x\psi$ ...except that there is no term available to witness the 'x'.

Paper 4, Section II 15F

Part (a) gave me a bit of a fright. Obviously you are meant to consider the chain-complete poset of partial homomorphisms $L \rightharpoonup \{0, 1\}$. But why is a maximal element of this poset a total homomorphism—one defined on the whole of L? Suppose it isn't, and that there is a maximal element f and a rogue element b on which f is not defined.

If we cannot send b (for "bad") to 1 it is beco's there is an a_0 with $f(a_0) = 0$ and $f(b \vee a_0) = 1$; and if we cannot send it to 0 it is beco's there is an a_1 with $f(a_1) = 1$ and $f(b \wedge a_1) = 0$. We have to show that these cannot both happen, and we have to use distributivity to do it.

Distributivity gives us

$$a_1 \wedge (b \vee a_0) = (a_1 \wedge b) \vee (a_1 \wedge a_0)$$

and the fact that f is a homomorphism gives us

$$f(a_1 \wedge (b \vee a_0)) = f((a_1 \wedge b) \vee (a_1 \wedge a_0))$$

Using the equations $f(a_0) = 0$, $f(b \vee a_0) = 1$, $f(a_1) = 1$ and $f(b \wedge a_1) = 0$ we can simplify the LHS to 1 and the RHS to $f(a_1 \wedge b)$ getting

$$1 = f(a_1 \wedge b)$$

Now we also have (analogously)

$$a_0 \vee (b \wedge a_1) = (a_0 \vee b) \wedge (a_0 \vee a_1)$$

and—as before—the fact that f is a homomorphism gives us

$$f(a_0) \vee f(b \wedge a_1) = f(a_0 \vee b) \wedge f(a_0 \vee a_1)$$

Similarly the LHS is 0 and the RHS is $f(a_0 \vee b)$ giving

$$f(a_0 \vee b) = 0$$

So we have both

$$f(a_0 \vee b) = 0 \text{ and } 1 = f(a_1 \wedge b)$$

But we have

$$a_1 \wedge b \leq a_0 \vee b$$

contradicting the fact that f of the LHS of this inequality is 1 and f of the RHS is 0.

There must be a simpler way of doing it, but i am an elderly wally and can't see it. O Wally wally, the water is wide and you can't get o'er.

2017

Paper 2 section II 14H

Which of the following assertions about ordinals are true? Justify your answers

(i) $\alpha + \beta = \beta + \alpha$;
(ii) $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$;
(iii) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$;
(iv) $\alpha\beta = \beta\alpha \to \alpha^2\beta^2 = \beta^2\alpha^2$;
(v) $\alpha^2 = \beta^2 \to \alpha\beta = \beta\alpha$.

Discussion Answer

I address (iv) and (v) only. (It's the least i can do, since it was i who set the questions). Observe that addition, multiplication and exponentiation preserve inequalities both ways round, and one way round they preserve *strict* inequality as well. Thus, if $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$ and $\gamma + \alpha < \gamma + \beta$. Multiplication and exponentiation similarly. We will exploit this fact in what follows.

(iv) is true. Suppose $\alpha\beta = \beta\alpha$. Multiply both sides by α on the left and β on the right to obtain $\alpha^2\beta^2 = \underline{\alpha\beta}\alpha\beta$. Now permute the underlined bits to obtain

$$\alpha^2\beta^2 = \beta\alpha\underline{\alpha\beta}$$

and permute again to obtain

$$\alpha^2\beta^2 = \beta\alpha\underline{\beta}\alpha$$

and yet *again* to obtain

$$\alpha^2\beta^2 = \beta^2\alpha^2.$$

Notice (and this has only just struck me) we haven't used the fact that α and β are ordinals. This works for any linear order types whatever—integers, rationals, reals, you name it.

(v)

How about the converse to (iv)? If $\alpha^2\beta^2 = \beta^2\alpha^2$ must we have $\alpha\beta = \beta\alpha$? (That was an old example sheet question from before the last ice age).

I am quite sobered by my student Michał Mrugała finding a mistake in my original model answer. (Sobered not just beco's it means i am fallible but also beco's it means that people haven't been reading it!!)

Assume $\alpha^2\beta^2 = \beta^2\alpha^2$ and (for a contradiction) suppose $\alpha\beta < \beta\alpha$.

You can multiply on the left by α to get $\alpha^2\beta < \alpha\beta\alpha$. (Multiplication on the left preserves strict inequality).

Now

$$\alpha\underline{\alpha\beta}\beta \leq^1 \underline{\alpha\beta}\alpha\beta \leq^2 \underline{\beta\alpha}\alpha\beta <^3 \beta\underline{\alpha\beta}\alpha \leq \beta\beta\alpha\alpha$$

(1) swap the two underlined terms

(2) swap the two underlined terms

(3) $\alpha\beta < \beta\alpha$ by assumption and multiplication on the left preserves strict inequality. giving $\alpha^2\beta^2 < \beta^2\alpha^2$ contradicting assumption.

(v) is harder than (iv) beco's you can't do it the way you did (iv)—merely by manipulating equations; you *have to* do it by contradiction. Also, in (v) you need the fact that multiplication on the left preserves strict inequality and—if you think about it—you can see that this doesn't work for arbitrary linear order types: $2 < 3$ but $\mathbb{Q} \cdot 2 = \mathbb{Q} \cdot 3$.

Paper 3 section II 14F

This seems to be the same as 2011 Paper 2, Section II 16H Logic and Set Theory

Paper 1, Section II 15H

If every valuation satisfies precisely one of A and B then (assuming neither is empty) we must have $A \vee B \models \perp$ whence $X \models \perp$ for some finite $X \subseteq A \cup B$. Then set $A' = X \cap A$ and $B' = X \cap B$. If A and B are to be finitely axiomatised one obviously looks for them to be axiomatised by A' and B' respectively. Do we have $A' \vdash A$? If not, then there is a valuation v that satisfies A' but does not satisfy A. But if v does not satisfy A then it must satisfy B, so it certainly must satisfy B'. But then it satisfies $A' \cup B'$, and this contradicts the fact that $A' \cup B' \models \perp$. So (contraposing) every valuation that satisfies A' satisfies A, so A' axiomatises A. Similarly for B and B'.

Cute. I wonder who came up with that?

Paper 4, Section II 15H

I don't know who came up with 2017 Paper 1, Section II 15H but this one was mine.

The last part is not actually difficult but it requires confidence. I suppose one should start by asking what one wants the answer to be: is there a fixed point or not? We know there is no set equal to the set of *all* its subsets, but can there be a set equal to the set of its *countable* subsets? Well, if there is, one would obtain it by iteration—as in the question, indeed. If one obtains the fixed point as a C_α then what can one say about the ordinal α? Every countable subset of C_α has to be a member of C_α, but C_α is the union of all previous C_β. So every countable subset of C_α is therefore a member of C_β for some earlier β. Now: what can you do with that? For C_α to be a fixed point it has to be that every countable set of ordinals below α is bounded below α. α cannot have cofinality ω. If you have countable choice then ω_1 is such an ordinal.

There is a delightful discussion of this question in Jech's 1980 Journal of Symbolic Logic article "On hereditarily countable sets". It has a very *very* beautiful construction. If you think you might do set theory at some point, read that paper. It's v nice, and no-one else will tell you to read it.

2018

Paper 4, Section II 16G

(i) *State and prove the* $\in$*-Recursion Theorem. [You may assume the Principle of* $\in$*-Induction.]*

(ii) *What does it mean to say that a relation* r *on a set* x *is well-founded and extensional ? State and prove Mostowski's Collapsing Theorem. [You may use any recursion theorem from the course, provided you state it precisely.]*

(iii) *For which sets* x *is it the case that every well-founded extensional relation on* x *is isomorphic to the relation* $\in$ *on some transitive subset of* V_ω *?*

Answers

(i) and (ii) are bookwork

(iii)

What is this condition on x? It says: Think of X as a bag of points. Now decorate the points with directed edges (thought of as ordered pairs from a membership relation that you are making up as you go along.). The result is a digraph that can be tho'rt of as a *set picture*. Is this picture a picture of a transitive subset of V_ω?

Now clearly this project depends only on the size of x, since any decoration of x can be copied over to any set in 1-1 correspondence with x. So what can we say about the cardinality of x? Clearly any finite set has this property. What about infinite sets? Well x is going to have to be countable, co's it has to be the same size as a subset of V_ω, and *that's* countable. But countability isn't sufficient, because i could inflict on $\mathbb{N}$ (for example) a wellordering of transfinite length, and that won't be iso to any subset of V_ω. (Any wellordering is an extensional wellfounded relation). It's going to have to be finite.

Clev-ah!

Actually Pavel Turek has pointed out something that I had overlooked (and which I am sure the examiners expected you to overlook). What happens if x cannot be decorated by a wellfounded extensional relation? Then it's vacuously true that every wellfounded extensional relation on x is isomorphic to the restriction etc etc. So, to deduce the intended answer (namely that X has to be finite) we need to assume that every set can be decorated with a wellfounded extensional relation. Now this is certainly a consequence of AC, since AC tells us that we can wellorder anything, and a wellordering is certainly a wellfounded extensional relation as noted above. Quite what the strength is of the assertion “every set supports a wellfounded extensional relation” i don’t know. And i should. But i am sure that the examiners were expecting you to assume it.

Paper 3, Section II 16G

(1) *State and prove the Compactness Theorem for first-order predicate logic.*
(2) *State and prove the Upward Löwenheim-Skolem Theorem. [You may assume the Completeness Theorem for first-order predicate logic.]*
(3) *For each of the following theories, either give axioms (in the specified language) for the theory or prove that the theory is not axiomatisable.*

(i) The theory of finite groups (in the language of groups).

(ii) The theory of groups in which every non-identity element has infinite order (in the language of groups).

(iii) The theory of total orders (in the language of posets).

(iv) The theory of well-orderings (in the language of posets).

If a theory is axiomatisable by a set S of sentences, and also by a finite set T of sentences, does it follow that the theory is axiomatisable by some finite subset of S ? Justify your answer.

Answers

(3)

(i) is clearly not axiomatisable, by compactness.

(ii) is a recent example sheet question and is discussed in my notes on that sheet

(iii) Of course

(iv) One’s first thought is that it can’t be, beco’s the theory is *prima facie* second order. However that isn’t a proof. To obtain a proper proof add infinitely many constants $c_i : i \in \mathbb{N}$ and axioms $c_i > c_{i+1}$.

The rider. Yes, of course. T must follow from S, and if S is infinite it must (by compactness) follow from some finite subset of S.

There are two ways to write this down. Either argue that for each $t \in T$ there is a finite subset S_t of S which entails it (by compactness). Then take the union of the (finitely many) S_t to obtain a finite axiomatisation $\subseteq S$. Or consider the conjunction of all the members of T and argue that that must follow from a finite subset of S—again, by compactness.

Paper 2, Section II 16G

(i) State and prove the Knaster-Tarski Fixed-Point Theorem.

(ii) Deduce the Schröder-Bernstein [sic] Theorem.

(iii) Show that the poset P of all countable subsets of $\mathbb{R}$ (ordered by inclusion) is not complete.

(iv) *Find an order-preserving function $f : P \to P$ that does not have a fixed point. [Hint: Start by well-ordering the reals.]*

Answers

(iii) Well of course it's not complete: it hasn't got a top element. Duh!

Annoying question: is it *countably* complete? Is a union of countably many countable sets of reals a countable set of reals? Normally one needs countable choice to show that a countable family of countable sets has a countable sumset, but what if you are given the additional information that each countable set in the family is a set of reals? Does that help? I think it is known that if AC fails badly enuff then $\mathbb{R}$ can be a union of countably many countable sets—so the extra information *doesn't* help... as it happens, and we *do* need countable choice to show that P is countably complete. But it's good to get into the habit of asking questions like that.

Another annoying question: is it chain-complete? No, it isn't even chain-complete. It shouldn't be hard to come up with a nested sequence of countable subsets of $\mathbb{R}$ whose union is uncountable.

And it isn't *directed-complete* either. But now we are straying too far into poset stuff ... which you have been told not to worry about.

(iv) I had to think a bit about this last one. If you wellorder $\mathbb{R}$ by some wellordering $<<$ then you can extend any countable subset of $\mathbb{R}$ by inserting the $<<$-first real not in it. That is to say: $F(A)$ is to be $A \cup \{$the $<<$-first thing in $X \setminus A\}$. (Notice that all we need is a wellordered uncountable subset X of $\mathbb{R}$, beco's no countable subset of $\mathbb{R}$ can contain all of X so we just add to A the first thing in $X \setminus A$).

However, in a discussion with my supervisee John Dawson he brought home to me that—although it's obvious that this function is inflationary—it's not *quite* so obvious that it is order-preserving. But it is, nevertheless. Suppose A is a proper subset of B. We want the $<<$-first thing in $\mathbb{R} \setminus A$ to be in $F(B)$. If it is in B then it is certainly in $F(B)$ so we're happy. If it is not in B then it must be the first thing not in B. Thank you jad200!

There's is actually an easier way that some of you found. Wellorder $\mathbb{R}$ and use this to get a "successor" function $s : \mathbb{R} \to \mathbb{R}$. Then send each $A \subseteq \mathbb{R}$ to $s``A$.

Can one find such a function without any use of choice? Not the way we have done it, beco's we need at least *some* AC to show that $\mathbb{R}$ has an uncountable wellordered subset. More formally: $\aleph_1 \leq 2^{\aleph_0}$ is not a theorem of ZF without AC. I would expect that we can, but i can't see how to do it offhand. We do know, even without AC, that there is a surjection from $\mathbb{R}$ to the set of countable sets of reals, but i can't see how to make good use of it. Dunno! Pester me if you need an answer.

Paper 1, Section II 16G

(i) Give the inductive definition of ordinal exponentiation. Use it to show that $\alpha^\beta \leq \alpha^\gamma$ whenever $\beta \leq \gamma$ (for $\alpha \geq 1$), and also that $\alpha^\beta < \alpha^\gamma$ whenever $\beta < \gamma$ (for $\alpha \geq 2$).

(ii) Give an example of ordinals α and β with $\omega < \alpha < \beta$ such that $\alpha^\omega = \beta^\omega$.

(iii) Show that $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$, for any ordinals $\alpha\ \beta\ \gamma$,

(iv) Give an example to show that we need not have
$(\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$.

(v) For which ordinals α do we have $\alpha^{\omega_1} \geq \omega_1$?

And

(vi) for which α do we have $\alpha^{\omega_1} \geq \omega_2$?

Justify your answers. [You may assume any standard results not concerning ordinal exponentiation.]

Answers

(i) Bookwork. Pester me if you really want an answer.

[Actually it may be pædogogically worth spelling out that in order to prove that $(\forall\alpha\beta\gamma)(\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma)$ one fixes α and β and proves by induction on γ that $(\forall\gamma)(\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma)$. One does not prove by induction on γ that $(\forall\alpha\beta\gamma)(\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma)$. This second induction is much stronger than the one we are doing: it has $\forall\alpha\beta$ at the front.]

(ii) Try $\alpha = \omega^2$ and $\beta = \omega^3$. $\alpha^\omega = (\omega^2)^\omega = \omega^{2\cdot\omega} = \omega^\omega$;
β similarly.

(iv) For this last bit let α and β be finite and let $\gamma = \omega$.

(v) All of them, or at least all of them bigger than 1. You prove by induction that $\alpha^\beta \geq \beta$. Fix $\alpha > 1$ and do an induction on β. (Do not induct on α!)

(vi) This is a bit tricky, but since it is the rider that makes the difference between an α and a β that is to be expected. That said, i do think it is really quite difficult if you are not allowed to wave your arms.

The way in is to think about the cardinals associated with these ordinals. (Thinking about cardinals being associated to ordinals is a good idea anyway since it helps you to remember that cardinals and ordinals are related things rather than identical things!) If α is an ordinal it is the order type of some wellordering $\langle A, <_A \rangle$, and the set A has a cardinal—all sets do, after all. Also the cardinal we get doesn't depend on our choice of A but only on α. This cardinal-of function is well behaved:

• cardinal-of($\alpha + \beta$) is clearly cardinal-of(α)+ cardinal-of(β), and

• cardinal-of($\alpha \cdot \beta$) is clearly cardinal-of(α)· cardinal-of(β), and[2] in both cases the answer is just the bigger of the two cardinals. (Sum and product of alephs is just max). Much less obviously

• cardinal-of(α^β) is max(cardinal-of(α), cardinal-of(β)) which can be seen by considering the definition of ordinal exponentiation, which was an example sheet question. Recall that α^β is the order type of the set of functions of finite support from B to A, where $\langle A, <_A \rangle$ is a worder of otype α and $\langle B, <_B \rangle$ is a worder of otype β. We order the functions colex, but for present purposes all that matters is how many of the buggers there are. And that is going to be the number of finite subsets of $A \times B$ which is going to be just $\max(|A|, |B|)$, because all the cardinals involved are alephs.

[2] Note the overloading of '+' and '·' to mean both ordinal and cardinal multiplication: these are **not** the same operations!

Notice that the answer is **not** (cardinal-of(α))$^{(\text{cardinal-of})(\beta))}$; this time the ordinal operation doesn't get sent to the cardinal operation.

Back to the question.

What is cardinal-of(α^{ω_1})? It's (cardinal-of(α)) $\cdot$ (cardinal-of(ω_1)). Since these cardinals are alephs, this product is simply the bigger of the two. So we want the bigger of (cardinal-of(α) and (cardinal-of(ω_1) to turn out to be cardinal-of(ω_2)—which is of course $\aleph_2$. So cardinal-of(α) had better be $\aleph_2$, whence we deduce $\alpha \geq \omega_2$.

As i say, i think this last bit is quite hard.

2019

Paper 4, Section II 16 I

(i) Define the cardinals $\aleph_\alpha$, and explain briefly why every infinite set has cardinality an $\aleph$.

(ii) Show that if κ is an infinite cardinal then $\kappa^2 = \kappa$.

(iii) Let $X_1, X_2, ..., X_n$ be infinite sets. Show that $X_1 \cup X_2 \cup \ldots \cup X_n$ must have the same cardinality as X_i for some i.

(iv) Let $X_1, X_2, ...$ be infinite sets, no two of the same cardinality. Is it possible that $X_1 \cup X_2 \cup ...$ has the same cardinality as some X_i? Justify your answer.

Answers

(i) An aleph is simply the cardinal of a(n infinite) wellordered set. The alephs are wellordered by the obvious order relation on cardinals (this is a nontrivial fact, which the examiners may or may not be expecting you to prove) so you can enumerate them in increasing order using the ordinals, starting with 0 (obviously!) so the first infinite aleph is $\aleph_0$.

The stream of cardinals of wellordered sets is always there, and it's always wellordered by the natural order relation on cardinals and we always use the letter '$\aleph$' (with subscripts) to denote its members—whether we have AC or not. The Axiom of Choice says that every set can be wellordered, so it tells us that every infinite cardinal is an aleph.

This is a question that reveals the extent to which the examinee is confused about cardinals. People learning set theory are very liable to confuse two things:

- Q: What are cardinals? A: cardinals are those things such that two things have the same cardinal iff there is a bijection between them.
- (In set theory) Q: What is a cardinal? This is a question about how to concretise/implement these abstract mathematical objects *as sets*—given that you engaged in the fantasy of thinking that everything is a set. You can use Scott's trick cardinals as long as you have foundation. If you have replacement you can concretise every ordinal as a von Neumann ordinal. If you have AC then every set is wellordered and you can concretise its

> cardinal as the least ordinal (length) to which it can be wellordered. However this tells you how to concretise cardinals *if you have replacement and choice*. It doesn't tell you what cardinals *are* (You knew *that* all along). Do **not** go away with the idea that if AC fails then certain sets do not have cardinals. *Every* set has a bloody cardinal. It's just that if you don't have AC then you can concretise those cardinals in the way you have been led to expect.

(ii) This is bookwork... bloody hard bookwork but bookwork. (Look at the discussion in my lecture notes for 2017, or my tutorial `www.dpmms.cam.ac.uk/~tf/ordinalsforwelly.pdf`)
For (iii) you exploit the fact that the sum of finitely many alephs is the biggest of them. We are assuming AC, so every cardinal is an aleph.
(iv)

The question is vague about the size of the family of Xs, perhaps deliberately so. Is it finite? Might it be infinite? My guess is that they are expecting you to think it is countable. If it's finite, then the size of the union is simply the biggest, beco's ... assuming AC (as one does) all cardinals are alephs, and the sum of finitely many alephs is simply the biggest of the summands. OTOH, suppose the family is infinite: alephs are wellordered by magnitude, so the countable many cardinals of these countably many sets will form a chain, and that chain will be wellordered by magnitude. Might it have a top element? Nothing to say it can't. What happens if i take the union of countably many things, all but one of them smaller than α? Can the union be the size of the biggest? Again, nothing to say that it can't.

I think this question is ridiculously hard. Granted, the correct answer is 'yes', and it is plausible that it should be 'yes', but a rigorous proof is beyond the resources of this course. Showing that the sum of infinitely many alephs is the largest of them (if there is a largest, indeed) is quite a lot of work... at least if you do it properly. For a start, you need the axiom of choice if the concept of infinite sums of cardinals is to even make sense. (You have to pick a set of each size and take the union...)

So: i think the answer they are expecting is that $|X_1| = \aleph_\omega$ and thereafter $|X_n| = \aleph_n$. Proving rigorously that that works is more work than there is space for under exam conditions, so you are expected to wave your arms artistically.

Paper 3, Section II 16 I

Define the von Neumann hierarchy of sets V_α. Show that each V_α is transitive, and explain why $V_\alpha \subseteq V_\beta$ whenever $\alpha \leq \beta$.

Prove that every set x is a member of some V_α.

Which of the following are true and which are false? Give proofs or counterexamples as appropriate. [You may assume standard properties of rank.]

(i) If the rank of a set x is a (non-zero) limit then x is infinite.
(ii) If the rank of a set x is countable then x is countable.
(iii) If every finite subset of a set x has rank at most α then x has rank at most α.
(iv) For every ordinal α there exists a set of rank α.

Answers

To prove that every x belongs to a V_α you do an $\in$-induction.

(i) Yes. If x has infinite rank then the set of ranks of its members has no maximal element, so x must have infinitely many members.

(ii) *Obviously* not: $V_{\omega+1}$ has countable rank but its cardinality is $2^{\aleph_0}$

(iii) I had to think about this. Hold on to your hat . . . the rank of a set is the least ordinal bigger than all the ranks of its members. If the rank of x is successor—$\alpha + 1$ for some α—then it has at least one member of rank α so it has finite subsets of rank $\alpha + 1$. So: yes. What happens if the rank of x is some limit ordinal λ? Then the rank of any finite subset of x is the sup of some finite set of ordinals below λ and accordingly is below λ. But the set of such ordinals is not bounded below λ. So the least α s.t. every finite subset of x has rank at most α is λ.

Another take:

The rank of a finite subset of x is the sup of some ordinals of the form rank(y) + 1 for various $y \in x$, and rank(x) is the sup of all such ordinals, so rank(x) is at least the sup of all the ranks of the finite subsets. Can it be greater? No: every member of x appears in at least one such finite set.

(iv) Not hard to prove by induction on the ordinals that the von Neumann ordinal α has rank α.

A tangential pædogogical point at this stage. . . . It is possible to worry about the legitimacy of the construction of the cumulative hierarchy of the V_αs by recursion over the ordinals. I mean, dammit, the ordinals are inhabitants of the cumulative hierarchy so they aren't initially there for us to do the recursion on! This isn't a *real* worry, but it does create an opening for one to make the point that the morally correct way to think of the recursive construction of the cumulative hierarchy is being run on ordinals-as-abstract-mathematical-objects, which exist in the wider world of mathematics quite independently of any set theoretic clothing they may or may not have. You don't have to think of 17 or π as sets, and you don't have to think of ordinals as sets either, and part (iv) is a good time to exercise that right. I would answer (iv) with something like "for every ordinal α its von Neumann concretisation is a set of rank (birthday!) α."

You can think of this construction as a way in to Set Theory. The narrative is: you start off not knowing about sets (tho' you do know about numbers, incl ordinals, which are the kind of number that measures the lengths of transfinite processes). You then execute the Indian Rope Trick that is the cumulative hierarchy, and now you have got your hands on some sets!

Paper 2, Section II 16I

Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false?

Give proofs or counterexamples as appropriate.

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(ii) If α and β are uncountable then $\alpha + \beta = \beta + \alpha$.

(iii) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

(iv) If α and β are infinite and $\alpha + \beta = \beta + \alpha$ then $\alpha\beta = \beta\alpha$.

Answers

(i) and (iii) are true for *all* order types, not just ordinals, so use the synthetic definition.

(ii) is obviously false: try ω_1 and ω_2;

(iv) brought me up short. I had to think for a bit. $\alpha + \beta = \beta + \alpha$ happens only if the two ordinals are "close": $\alpha < \beta \cdot \omega$ and $\beta < \alpha \cdot \omega$. If they are further apart than that the bigger one will absorb the smaller one on the left. So you might expect the answer to be 'yes', but it ain't: try $\alpha = \omega$ and $\beta = \omega \cdot 2$.

So "Close for the purposes of addition" is weaker than "close for the purposes of multiplication". I haven't tho'rt about closeness for the purposes of exponentiation; when do you have $\alpha^\beta = \beta^\alpha$?

Paper 1, Section II 16I

(i) State the completeness theorem for Propositional Logic. Explain briefly how the proof of this theorem changes from the usual proof in the case when the set of primitive propositions may be uncountable.
(ii) State the compactness theorem and the decidability theorem, and deduce them from the completeness theorem.
(iii) A poset $(X, <)$ is called *two-dimensional* if there exist total orders $<_1$ and $<_2$ on X such that $x < y$ if and only if $x <_1 y$ and $x <_2 y$. By applying the compactness theorem for Propositional Logic, show that if every finite subset of a poset is two-dimensional then so is the poset itself.

[Hint: Take primitive propositions $p_{x,y}$ and $q_{x,y}$, for each distinct $x, y \in X$, with the intended interpretation that $p_{x,y}$ is true if and only if $x <_1 y$ and $q_{x,y}$ is true if and only if $x <_2 y$.]

Answers

(i) and (ii) are bookwork.

(iii) This is a really *really* nice question. I could bore for Britain on all the cute points one can extract from it.

We start with a warning: 'two dimensional' is a nonce notation, dreamt up specifically for this question. You will not see it anywhere else.

The question setter doesn't say whether the relation $<$ is a partial ordering or a strict partial ordering. The use of '$<$' rather than '$\leq$' suggests that it is the strict version that they have in mind, but i suspect they mean the nonstrict (reflexive) version. That's the assumption i would make, anyway.

This is a cute illustration of how a Zorn proof of the existence of a maximal widget can sometimes be coded up inside propositional logic. There are some standard examples, but this one is new to me.

The intersection of (the graphs of) two partial ordering relations on a given set is another partial ordering of that same set. (You don't need the partial orderings to be total orderings, as here)

> The *logician* in me wants to emphasise that this last fact is because of the very simple syntax of 'R is a partial order'. In contrast the intersection of two total orderings is not a total ordering, for similar syntactic reasons.
>
> The *amateur combinatorist* in me wants to say that there is a lot to be said about how the structure of the antichains in a partial ordering $R_1 \cap R_2$ is controlled by the structure of the antichains[3] in R_1 and R_2. You measure the complexity of the antichains with an ordinal, and the ordinal for the antichains in $R_1 \cap R_2$ is the Hessenberg sum of the ordinals for R_1 and R_2[4].

For another thing, assuming AC, every partial ordering is two-dimensional. (The finite case of this is obvious). This is a sexed-up version of OEP, the order extension principle, that says that every partial ordering can be extended (by adding ordered pairs) to a total ordering. The sexed-up version will say that any partial order can be extended in two incompatible ways whose intersection is the partial order you started with.

It's a standard result (it's probably an old example sheet question or old tripos question) that every *wellfounded* partial order can be extended to a *wellfounded* total order. Can this be sexed-up too? No: if a wellfounded partial order is an intersection of two wellorderings then it has no infinite antichains: the empty relation on an infinite set (for example) is not the intersection of two wellorderings. [Why not!?]

With the hint and with the suggestion that you use propositional compactness this is really a piece of cake. Contact me if you get stuck. I promise not to tell your friends.

Oh, Ok, i suppose i should say *something*. The hint is good: "Take primitive propositions $p_{x,y}$ and $q_{x,y}$, for each distinct $x, y \in X$, with the intended interpretation that $p_{x,y}$ is true if and only if $x <_1 y$ and $q_{x,y}$ is true if and only if $x <_2 y$."

Then the theory that says that $<_X$ is two-dimensional (in virtue of $<_1$ and $<_2$) has the following axiom (schemes):

- $p_{x,y} \wedge q_{x,y}$ for all pairs $x <_X y$;
- $p_{x,y} \wedge p_{y,z} \to p_{x,z}$ for all x, y, z in X;
- $q_{x,y} \wedge q_{y,z} \to q_{x,z}$ for all x, y, z in X;
- $p_{x,y} \vee p_{y,x}$ for all x, y in X;
- $q_{x,y} \vee q_{y,x}$ for all x, y in X.

$<_X$ is two-dimensional as long as this theory has a valuation. By compactness it has a valuation iff every finite subset has a valuation. But that is going to happen iff

[3] R_1 and R_2 are two partial orderings of one and the same set... so we mean the intersection of the two graphs, of the two relations...

[4] Look at `www.dpmms.cam.ac.uk/~tf/ordinalsforwelly.pdf` if you want to know about Hessenberg sum

every restriction of $<_X$ to a finite subset of X is two-dimensional (in virtue of $<_1$ and $<_2$). And we are told that every finite substructure of $\langle X, <_X\rangle$ is two-dimensional.

I'm pretty sure that every finite poset is two-dimensional.

While we are about it, here are some standard examples of Zorn-like things you can prove using propositional compactness:

- A group is orderable if all its finitely generated subgroups are.
- A graph is k-colourable if all its finite subgraphs are.
- Every partial ordering can be extended to a total ordering on the same carrier set.
- Every filter on a set can be extended to an ultrafilter[5].

[5]If you don't yet know what an ultrafilter is you should look at the question on them on p. 52.

2020

Paper 2 16H

Parts (a) and (b), (i) are bookwork. The third part gave me a bad fright. It makes me realise i am becoming old and slow; time to retire.

let X be an infinite set. For each $x \in X$, let L_x be a subset of X. Suppose that, for any finite $Y \subseteq X$, there is a function $f_Y : X \to \{1, \dots 100\}$ such that, for all $x \in Y$ and all $y \in Y \cap L_X$, $f_Y(x) \neq f_Y(y)$.

Show that there is a function $F : X \to \{1, \dots 100\}$ such that, for all $x \in X$ and all $y \in L_x$, $F(x) \neq F(y)$.

Answer:

This is obviously a compactness question. In questions of this kind the plan is to set up a propositional language $\mathcal{L}$ with a propositional theory T included in it in such way that a valuation making T true contains a description of the object you are being invited to build. Best bet is that $\mathcal{L}$ has a propositional letter for every pair in $X \times [0, 100]$ whose meaning is that $x \in X$ is joined to a number less than 100. For each $x \in X$, T contains a disjunction of 100 propositional letters that say that x is connected to a number ≤ 100, plus axioms to say that it isn't additionally connected to any others. Also, for each $x \in X$ you have (finitely many)—negated—letters saying that x is not joined to the same number as any member of L_x. Any finite set of these axioms is consistent in virtue of the existence of the f_Y for finite $Y \subset X$.

0.1 Paper 3: 16H

Let $\langle V, \in \rangle$ be a model of ZF.

(i) Give the definition of a class and a function class in V.

(ii) Use the concept of function class to give a short, informal statement of the Axiom of Replacement.

(iii) Let $z_0 = \omega$ and, for each $n \in \omega$, let $z_{n+1} = \mathcal{P}(z_n)$.

(iv) Show that $y = \{z_n : n \in \omega\}$ is a set.

We say that a set x is small if there is an injection from x to z_n for some $n \in \omega$. Let HS be the class of sets x such that every member of $TC(x)$ is small, where $TC(x)$ is the transitive closure of $\{x\}$.

(v) Show that $n \in HS$ for all $n \in \omega$ and deduce that $\omega \in HS$. Show further that $z_n \in HS$ for all $n \in \omega$.

(vi) Deduce that $y \in HS$. Is $\langle HS, \in\rangle$ a model of ZF?

(vii) Justify your answer. [Recall that $0 = \emptyset$ and that $n + 1 = n \cup \{n\}$ for all $n \in \omega$.]

Answers

I can't think of anything particularly helpful to say about this. There are seven parts to this question, so the examiners can't be asking for anything very detailed.

(ii) "The image of a set in a function is a set".

(v) Of course you use replacement. I don't think he's expecting you to explain attempts.

(vi) $\bigcup y$ is not in HS. So HS is witness to the independence of the axiom of sumsets from ZF. It's the standard proof of this fact.

Paper 4: 16H

(a) State Zorn's lemma.

[Throughout the remainder of this question, assume Zorn's lemma.]

(b) Let P be a poset in which every non-empty chain has an upper bound and let $x \in P$.

By considering the poset $P_x = \{y \in P : x \leq y\}$, show that P has a maximal element σ with $x \leq \sigma$.

(c) A filter is a non-empty subset $F \subseteq \mathcal{P}(\mathbb{N})$ satisfying the following three conditions:

- *if $A, B \in F$ then $A \cap B \in F$;*
- *if $A \in F$ and $A \subseteq B$ then $B \in F$;*
- *$\emptyset \notin F$.*

An ultrafilter is a filter $\mathcal{U}$ such that, for all $A \subseteq \mathbb{N}$, we have either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

(i) For each $n \in \mathbb{N}$, show that $\mathcal{U}_n = \{A \subseteq \mathbb{N} : n \in A\}$ is an ultrafilter.

(ii) Show that $F = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$ is a filter but not an ultrafilter, and that, for all $n \in \mathbb{N}$, we have $F \not\subseteq \mathcal{U}_n$.

(iii) Does there exist an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \neq \mathcal{U}_n$ for any $n \in \mathbb{N}$? Justify your answer.

This is all pretty routine. The only part that involves any work is (iii). It's an application of Zorn's lemma. Fix any filter, and consider the set of filters that are supersets of it. Clearly this is a chain-complete poset, so we get an ultrafilter extending the filter we started with. If we start with a nonprincipal filter (such as the "cofinite" (or *Fréchet*) filter consisting of the elements with finite complement—called 'F' in (ii) above) then the filter we get cannot be a $\mathcal{U}_n$. '$\mathcal{U}_n$' is not a standard notation BTW.

Ultrafilters are dead cool; i use to lecture them but they aren't actually in the syllabus. They should be. They lead to ultraproducts which are cooler still.

0.2 Paper 1, Section II 16H

Throughout this question, assume the axiom of choice.]

Let κ, λ and μ be cardinals. Define $\kappa + \lambda, \kappa \cdot \lambda$ and κ^λ.

What does it mean to say $\kappa \le \lambda$?

Show that $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$. Show also that $2^\kappa > \kappa$.

Assume now that κ and λ are infinite. Show that $\kappa \cdot \kappa = \kappa$. Deduce that $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.

Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) $\kappa^\lambda = 2^\lambda$;
(ii) $\kappa \le \lambda \to \kappa^\lambda = 2^\lambda$;
(iii) $\kappa^\lambda = \lambda^\kappa$.

(i) is obviously false c's κ can be as big as you like; (iii) is obviously false—take the two cardinals to be $\aleph_0$ and $2^{\aleph_0}$. (ii), on the other hand, is true:

$$2^\lambda \le \kappa^\lambda \le (2^\kappa)^\lambda \le 2^{\kappa \cdot \lambda} \le 2^{\kappa^2} = 2^\kappa$$

Quite a nice question...

2021

Paper 1, Section II 16G

"Let S and T be sets of propositional formulæ.

(a) What does it mean to say that S is deductively closed ?

What does it mean to say that S is consistent? Explain briefly why if S is inconsistent then some finite subset of S is inconsistent.

(b) We write $S \vdash T$ to mean $S \vdash t$ for all $t \in T$. If $S \vdash T$ and $T \vdash S$ we say S and T are equivalent. If S is equivalent to a finite set F of formulæ we say that S is *finitary*.

Show that if S is finitary then there is a finite set $R \subset S$ with $R \vdash S$.

(c) Now let $T_0, T_1, T_2, ...$ be deductively closed sets of formulæ with $T_0 \subset T_1 \subset T_2 \cdots$ (the inclusions are strict)

Show that each T_i is consistent.

Let $T = \bigcup_{i \in \mathbb{N}} T_i$

Show that T is consistent and deductively closed, but that it is not finitary."

This is a rather cute question. Since T_i is a strict subset of T_{i+1} it must be consistent, since everything follows from an inconsistent set of axioms. Then T must be consistent by compactness. If T were finitary it would follow from finitely many of the T_i, as it might be all the Ts up to 17. Now there are consequences of T that don't appear until T_{18}—and we are told that T_{17} doesn't imply T_{18}.

The student was asked me for a discussion of this question has been given name suppression.

Paper 3, Section II 16G

The first four lines of part (b) are stuff you should have done in 1a. Part II students shouldn't be asked to write out proofs of this sort of thing. Get a 1a student to do it for you. Does your college still have fagging? Probably not. I don't know what the world is coming to.

The last part is an idiomatic piece of set theory, and recalls the proof of Hartogs' Lemma. On being given a relation on (the von Neuman ordinal) ω_α—which is a set of size $\aleph_\alpha$. send it to 0 if it is not a wellordering. If it is a wellordering send it to (the von Neumann ordinal) of its order type. This maps the set of wellorderings of (the von

Neuman ordinal) ω_α onto the set of (von Neumann) ordinals $< \omega_{\alpha+1}$, which last set is of course the (von Neumann) ordinal $\omega_{\alpha+1}$ itself, as desired.

Notice that—in particular—there is a map from $\mathbb{R}$ onto the second number class, the set of all countable ordinals, *and we know what that map is*. Observe, in contrast, that there is no obvious injection from the second number class into the reals. There was an example sheet question about this, the question that asked you to embed every countable ordinal in $\mathbb{R}$.

Paper 4, Section II 16G

The third part of this question has an interesting history. Years ago i set it as an exercise on $\in$-induction in the form "Show that any $\in$-automorphism of V must be the identity". The proof is pretty routine, the only subtlety being the point that one might fix a function-class f and then prove by $\in$-induction that f is the identity, or prove it simultaneously for all function-classes. This raises the question of whether or not the thing one is proving by means of $\in$-induction is allowed to contain quantifiers over function-classes. But perhaps it's best not to think about that!

Anyway, subsequent versions of this question were formulated without using the word 'automorphism'; apparently it's scary. So the question became something like: suppose f is a function-class s.t., for all x and y, $x \in y \longleftrightarrow f(x) \in f(y)$; prove that f is the identity. The problem is that to establish that it is the identity one needs the extra condition that f be surjective. If we drop that requirement then we can declare a counterexample recursively by

$f(\emptyset) =: \{\emptyset\}$; thereafter $f(x) =: f``x$.

My student August (he's a star) then says: it's obvious that $x \in y$ implies $f(x) \in f(y)$, but what about the other direction?? Suppose $f(x) \in f(y) = \{f(z) : z \in y\}$. That doesn't tell us that $x \in y$ unless f is injective! After all we might have $x \notin y$ but $f(x) = f(z)$ for some $z \in y$. To my shame this possibility had never occurred to me!! We'd better prove that f is injective. Presumably we do this by induction.

Suppose, for all $x \in X$, that $(\forall u)(f(x) = f(u) \to x = u)$. We want to show $(\forall Y)(f(Y) = f(X) \to Y = X)$. Accordingly suppose that $f(X) = f(Y)$. Then $\{f(x) : x \in X\} = \{f(y) : y \in Y\}$ so every f of a member of X is an f of a member of Y and *vice versa*. Perhaps a little detail wouldn't go amiss. Suppose $u \in X$. Then $f(u) \in f(X))$ and *vice versa* beco's of the induction hypothesis on members of X. But $f(X) = f(Y)$ so that's iff $f(u) \in f(Y) = f``Y$. But then $u \in X$ beco's the only z st $f(z) = f(u)$ is u, by induction hypothesis. Whence $X = Y$ by extensionality. Whew.

(For one ghastly moment i feared i was going to have to do it by a wellfounded induction on $V \times V$ using the relation $\langle x, y\rangle R \langle x', y'\rangle$ iff $x \in x' \wedge y \in y'$).

Izaak Mammadov had an interesting take on this question, even if only beco's he misread it. He read the condition as

$$(\forall xy)(x \in y \to f(x) \in f(y))$$

i.e., as a conditional not a biconditional. And he had difficulty showing that f must be the identity—even with the assumption that f is onto. Well, he's not the only one. I can't prove it either! And my current guess is that it's not true.

One thought that i had at the back of my mind is that thinking about fs like this—that satisfy the biconditional and not merely the conditional and are injective but not required to be surjective—can be connected to fs that satisfy an extra "elementarity" condition. This condition says that

$$(\forall \vec{x})(\phi(\vec{x}) \longleftrightarrow \phi(\vec{f}(x)))$$

Such an f must send the empty set to the empty set. And the singleton of the empty set to the singleton of the empty set. In fact it's quite hard to see how f can avoid being the identity. Indeed the assertion that there is a nonidentity f of this kind is incredibly strong. The least ordinal α s.t. $\alpha \neq f(\alpha)$ is a monster. Study of functions of this kind is absolutely central to postmodern set theory with large cardinals.