A Model Tripos Question on Propositional Logic

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A *type* in a propositional language $\mathcal{L}$ is a set of formulæ (a countably infinite set unless otherwise specified).

For T an $\mathcal{L}$-theory a T-*valuation* is an $\mathcal{L}$-valuation that satisfies T. A valuation v *realises* a type Σ if $v(\sigma) = \texttt{true}$ for every $\sigma \in \Sigma$. Otherwise v *omits* Σ. We say a theory T *locally omits* a type Σ if, whenever ϕ is a formula such that T proves $\phi \to \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg\phi$.

Now prove the following:

(i) *Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type. If T locally omits Σ then there is a T-valuation omitting Σ.*

(ii) *Let T be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq \mathcal{L}(T)$ be a type. If T locally omits every Σ_i then there is a T-valuation omitting all of the Σ_i.*

[Hint: Show that, if n is such that you can find a family $\langle \phi_i : i \leq n \rangle$, with ϕ_i in Σ_i for every $i < n$ s.t. $T \cup \{\bigwedge_{i \leq n} \neg\phi_i\}$ is consistent, then you can extend this family to one of length $n + 1$.]

For further reading have a look at `yabloomittingtypes.pdf` linked from my home page.