CHAPTER V: Getting One Woodin Cardinal from AD

WE NOW COME TO THE HEART OF THE ARGUMENT using *AD* that there are inner models containing many Woodin cardinals. We use the $\diamondsuit$-like object constructed in Chapter I to build a filter and then an ultrafilter having a certain normality property. This in turn will show that when the ultrafilter is used to build an ultrapower of *HOD*, the resulting elementary embedding is strong in an appropriate sense.

Our aim in this chapter is to prove the following

0·0 THEOREM *Assume that $V = L(\mathbb{R})$ and that AD holds. Then Θ is Woodin in HOD.*

According to our discussion in Chapter IV of the Woodin property, this means that in *HOD*,

$$\forall H \subseteq \Theta\, \exists\, \delta{:}{<}\Theta\, \forall \lambda{:}{\in}[\delta, \Theta)\, \exists j : V \to M\ crit(j) = \delta \ \&\ j(\delta) \geqslant \lambda \ \&\ j(H) \cap \lambda = H \cap \lambda.$$

Fix H, then, a given subset of Θ, with $H \in HOD$. To obtain δ, reconstruct $L(\mathbb{R})$ by using H as an additional predicate as we did in Chapter One: either add a one-place predicate $\dot{H}$ to the language of set theory and then repeat the definition of the $L_\zeta(\mathbb{R})$ using this additional predicate; or add a new initial function, $i_h : x \mapsto x \cap H$ and replace "rudimentary" by "rudimentary in the extra function i_h" in the definition of the $J_\zeta(\mathbb{R})$ hierarchy.

Now with (II·4·21) in mind, define δ_H to be the least ordinal δ such that $(J_\delta(\mathbb{R}; H), H \cap \omega\delta) \preceq_{\Sigma_1^{\dot{H}}} (J_\Theta(\mathbb{R}; H), H)$. As before, we write M_H for the set $J_{\delta_H}(\mathbb{R}; H)$, which by I·219 equals $J_{\delta_H}(\mathbb{R})$. We shall use the $\diamondsuit$-like object constructed in Chapter One, of which we proved the following

0·1 PROPOSITION *There is a function $\diamondsuit = \diamondsuit_H$ with domain a cofinal subset of δ_H such that for each $\eta \in dom(\diamondsuit)$, $\diamondsuit(\eta)$ is a pre-well-ordering of (a subset of) $\mathcal{N}$, and for any real a, any* OD *pre-well-ordering P of (a subset of) $\mathcal{N}$, and any Σ_1^H predicate Ψ, if $J_\Theta(\mathbb{R}; H) \models \Psi(a, P, \diamondsuit, \delta_H; H)$ then for some $\eta \in Dom(\diamondsuit)$, $J_{\delta_H}(\mathbb{R}; H) \models \Psi(a, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H)$.*

0·2 REMARK We emphasize that $\diamondsuit$ depends only on H and not on a particular P.

0·3 We shall also use the admissible coding for (partial) functions from δ_H to δ_H that was defined in §6 of Chapter III by applying the Uniform Coding Lemma to a pre-well-ordering of length δ_H. The pre-well-ordering used is the natural one, $\prec_H$, of length δ_H with field a universal set, Υ_H, for the bold-face pointclass $\mathbf{\Gamma}_H$ of those pointsets which are $\Sigma_1(J_\Theta(\mathbb{R}; H))$ always allowing a real parameter, a constant for the set of reals, and a unary symbol for the predicate H. The universal set is assumed to be good in the sense of Moschovakis; we refer the reader back to the discussion of Definition 4·17 of Chapter II for the details.

0·4 REMARK We caution the reader against confusing the above set Υ_H with the sets $\Upsilon_{H,\Lambda}$ that we are about to define, which are universal for more elaborate formulæ allowing the sequence $\diamondsuit$, the ordinal δ_H and a pre-well-ordering Λ to be named.

The stage is now set for the construction of a filter and an ultrafilter on δ_H. We shall show that δ_H will work as the δ of the Theorem.

$\lessdot$ 0·5 REMARK Since this Chapter was first drafted, a new proof of the normality of $\mathfrak{F}$ has been found which avoids induction on the length of the counterexample Q, and which thereby leads to a reduction of the hypothesis of certain results in Chapter VI. This new proof is presented in section 9 of Chapter VI, adapted for the context required there. The reader will have little difficulty in constructing a version of that proof that can replace the current sections 4 and 5 of the present chapter.

1: the filter $\mathfrak{F}$

Now let $\lambda \in [\delta_H, \Theta)$ be given: it will be convenient, and no loss of generality in proving Theorem 0·0, to assume that λ is a limit ordinal. Let Λ be an OD pre-well-ordering of length λ. We proceed to define the filter $\mathfrak{F} = \mathfrak{F}_{H,\Lambda}$. We write $\Phi_\diamondsuit(a, Q, R, \zeta; H)$ for a universal Σ_1^H statement of the form $\exists x :\in J_\Theta(\mathbb{R}; H)\ \Phi^0_\diamondsuit(a, x, Q, R, \zeta; H)$, where $\Phi^0_\diamondsuit(a, x, Q, R, \zeta; H)$ is a Δ_0^H formula of that signature, a ranges over reals, ζ over ordinals, Q will usually be an OD pre-well-ordering, R a partial function whose values are OD pre-well-orderings, and H occurs as an unary predicate: when we speak in this section of Σ_1 statements we shall always intend the existential quantifier to be restricted to $J_\Theta(\mathbb{R}; H)$.

1·0 DEFINITION $\Upsilon_{H,\Lambda} =_{\mathrm{df}} \{a \in \mathcal{N} \mid \Phi_\diamondsuit(a, \Lambda, \diamondsuit, \delta_H; H)\}$

1·1 DEFINITION $S_b =_{\mathrm{df}} \{\eta \in \mathrm{dom}(\diamondsuit) \mid \Phi_\diamondsuit(b, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H)\}$

1·2 LEMMA *(i)* $b \in \Upsilon_{H,\Lambda} \implies S_b \neq \varnothing$; *(ii) there is a recursive function π such that for all indices b, b'*

$$S_b \cap S_{b'} = S_{\pi(b,b')}.$$

Proof : (i) by the main result of I·4; (ii) by II·1·27. $\dashv$ (1·2)

1·3 DEFINITION If $\Psi(a, \Lambda, \diamondsuit, \delta_H; H)$ is a Σ_1 statement, a real b such that $\forall \eta :\in S_b\ \Psi(a, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H)$ will be said to *enforce* the statement Ψ.

If we write $\langle \ulcorner\Psi\urcorner, a\rangle$ for a real suitably coding the Gödel number of Ψ and the real a, then 'b enforces $\Psi(a, \ldots)$' means that

$$S_b \subseteq S_{\langle \ulcorner\Psi\urcorner,\, a\rangle}$$

1·4 EXERCISE Show that given a true Σ_1^H assertion $\Psi(a \ldots)$, and an ordinal $\eta < \Theta$, there is an index b which enforces Ψ with $\inf S_b > \eta$; further b can be found uniformly from a and a pair (φ, c) H-characterising an ordinal strictly between η and δ_H.

Let us revert for a moment to the comments made at the end of §I·4. There we showed that the reflection property of $\diamondsuit$ held not only for reals but also for arbitrary members of $J_{\delta_H}(\mathbb{R})$, in view of the fact that $\mathrm{Hull}_1^H(J_{\delta_H}(\mathbb{R})) = J_{\delta_H}(\mathbb{R})$. The argument shows rather more:

1·5 PROPOSITION *Let $z \in J_{\delta_H}(\mathbb{R})$ and Ψ a Σ_1^H formula such that $J_\Theta(\mathbb{R}) \models \Psi(z, \Lambda, \diamondsuit, \delta_H; H)$: then there is a real $b \in \Upsilon_{H,\Lambda}$ such that*

$$\forall \eta :\in S_b\ J_{\delta_H}(\mathbb{R}; H) \models \Psi(z, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H).$$

Hence we may speak of such b *enforcing* the statement $\Psi(z)$. We emphasize that z need not be a real.

Proof of 1·5: there is a real $\bar{a}$ and a Σ_1^H formula ϑ such that $z = \iota y\, \big[J_{\delta_H}(\mathbb{R}; H) \models \vartheta(y, \bar{a})\big]$; and hence there is an ordinal $\bar{\eta} < \delta_H$ such that for all ζ with $\bar{\eta} \leq \zeta \leq \delta_H$, $z = \iota y\, \big[J_\zeta(\mathbb{R}; H) \models \vartheta(y, \bar{a})\big]$: any $\bar{\eta}$ large enough, so that both z and a witness to its having the Σ_1 property ϑ lie in $J_{\bar{\eta}}(\mathbb{R}; H)$, will do. By II·1·27 there is a recursive function ρ such that for every Q, R, ζ, $\nu < \zeta$, and real a,

$$J_\zeta(\mathbb{R}; H) \models \Phi_\diamondsuit(\rho(a), Q, R, \nu; H) \longleftrightarrow \bigvee y\big(\vartheta(y, a) \wedge \Psi(y, Q, R, \nu; H)\big).$$

Take $b = \rho(\bar{a})$: since

$$J_\Theta(\mathbb{R}; H) \models \bigvee y\big(\vartheta(y, \bar{a}) \wedge \Psi(y, \Lambda, \diamondsuit, \delta_H; H)\big),$$

we have $J_\Theta(\mathbb{R}; H) \models \Phi_\diamondsuit(b, \Lambda, \diamondsuit, \delta_H; H)$, and so $b \in \Upsilon_{H,\Lambda}$; then for each $\eta \in S_b$, $J_{\delta_H}(\mathbb{R}; H) \models \Phi_\diamondsuit(b, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H)$ and so

$$J_{\delta_H}(\mathbb{R}; H) \models \bigvee y\big(\vartheta(y, \bar{a})\ \&\ \Psi(y, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H)\big).$$

But such y must equal z, and so

$$\forall \eta :\in S_b\ J_{\delta_H}(\mathbb{R}; H) \models \Psi(z, \diamondsuit(\eta), \diamondsuit \restriction \eta, \eta; H). \qquad \dashv (1·5)$$

1·6 DEFINITION Let $z \in M_H$. We say that a real c *establishes* z if $c \in \Upsilon_{H,\Lambda}$ and recursive in c there is a real from which z is Σ_1^H definable in M_H and such that both z and a witness to its Σ_1^H definition lie in $J_{\inf S_c}(I\!\!R; H)$.

1·7 REMARK Proposition 1·5 shows that each $z \in M_H$ is established by some real.

We may now define the filter $\mathfrak{F}_{H,\Lambda}$ and establish its fundamental character.

1·8 DEFINITION $\mathfrak{F} = \mathfrak{F}_{H,\Lambda} \ =_{\text{df}} \ \{X \subseteq \delta_H \mid \exists b{:}{\in}\Upsilon_{H,\Lambda}\ S_b \subseteq X\}$.

1·9 PROPOSITION *$\mathfrak{F}$ is a proper δ_H-complete filter extending the club filter on δ_H and containing the set $Dom(\diamondsuit)$.*

Proof : $\mathfrak{F}$ is a proper filter by the last lemma. To see that $\mathfrak{F}$ extends the club filter, we use the good coding of Chapter III. We know from (II·4·21) that there is a Σ_1^H pre-well-ordering $\leqslant$ of a (good universal) set $U^H \in \Gamma_H$ of length δ_H. Let C be a club subset of δ_H and let ε_C be a real such that

$$\forall\eta{:}{<}\delta_H \ \ \eta \in C \iff G_{\varepsilon_C}^{\leqslant\restriction\eta+1} \cap (U^H)_\eta \neq \varnothing.$$

Saying that C is cofinal in δ_H is a Σ_1 statement about δ_H, $I\!\!R$, $\leqslant$ and ε_C, namely

$$\forall\zeta{:}{<}\delta_H\ \exists\nu{:}{<}\delta_H\ (\zeta < \nu\ \&\ \varepsilon_C(\nu) \simeq 1);$$

Let b enforce that statement; then for each $\eta \in S_b$,

$$\forall\zeta{:}{<}\eta\ \exists\nu{:}{<}\eta\ (\zeta < \nu\ \&\ \varepsilon_C(\nu) \simeq 1);$$

Thus C is cofinal in η and hence η is in C as C is closed. Thus $S_b \subseteq C$ and $C \in \mathfrak{F}$.

Finally to see that $\mathfrak{F}$ is δ_H-additive, we use the Coding Lemma. Let $\xi < \delta_H$, so that ξ is the order type of a pre-well-ordering $\Xi = (\mathcal{N}, <_\Xi)$ in M_H. Suppose that $A_\nu \in \mathfrak{F}$ for $\nu < \xi$: so $\forall\nu{:}{<}\xi\ \exists b{:}{\in}\Upsilon_{H,\Lambda}\ S_b \subseteq A_\nu$. Put

$$Z = \{(a, b) \mid b \in \Upsilon_{H,\Lambda}\ \&\ S_b \subseteq A_{|a|_\Xi}\}.$$

By the Coding Lemma, there is a subset Z^* of Z, in $pos\Sigma_1^1(<_\Xi)$, such that $\forall a'\exists(a,\, b)\ (a,\, b) \in Z^*\ \&\ a{\sim_\Xi}a'$: we may say that Z^* *meets every component of* Ξ. As $\Xi \in M_H$ and M_H is admissible, $Z^* \in M_H$. Let $S^* = \{b \mid \exists a{:}{\in}\mathcal{N}\ (a,\, b) \in Z^*\}$. Then $S^* \in M_H$ since $Z^* \in M_H$. Further $S^* \subseteq \Upsilon_{H,\Lambda}$, by definition of Z and choice of Z^*: so by 1·5 there is a real d that enforces the Σ_1^H statement

$$\forall b{:}{\in}S^*\ \Phi_\diamondsuit(b, \Lambda, \diamondsuit, \delta_H\,; H).$$

Hence

$$\forall\eta{:}{\in}S_d\ \forall b{:}{\in}S^*\ \Phi_\diamondsuit(b, \diamondsuit(\eta), \diamondsuit\restriction\eta, \eta; H);$$

But then $S_d \subseteq \bigcap_{b\in S^*} S_b \subseteq \bigcap_{\nu<\xi} A_\nu$, and so that intersection is in $\mathfrak{F}$. $\dashv$ (1·9)

2: The ultrafilter $\mathfrak{U}$

In Chapter III, we studied a general method for defining ultrafilters using *AD*. We apply it now in defining the ultrafilter $\mathfrak{U}_{H,\Lambda}$.

In section one we defined the universal set $\Upsilon_{H,\Lambda}$: that is an uncountable set of reals, and therefore following III·2 there is an ultrafilter $\mathfrak{W}(\Upsilon_{H,\Lambda})$ on the set $[\Upsilon_{H,\Lambda}]^{\aleph_0}$ of countably infinite subsets of $\Upsilon_{H,\Lambda}$. This ultrafilter was defined using games, of course: for $A \subseteq [\Upsilon_{H,\Lambda}]^{\aleph_0}$, we defined a game $\mathcal{G}(\Upsilon_{H,\Lambda}, A)$: Adam and Eve each play reals a, e, bitwise: we interpreted a as a sequence of reals a^n, and e as a sequence e^n of reals according to the specific formula given in Chapter III; then the rules were that in turn a^0, e^0, a^1, e^1, were required to be in $\Upsilon_{H,\Lambda}$, failure entailing defeat for the player responsible; if all these rules were kept, we set $O(a,e) \ =_{\mathrm{df}} \ \{c \mid \exists n \ c = a^n \ \text{ or } \ c = e^n\}$, a subset then of $\Upsilon_{H,\Lambda}$; and Eve wins the run of the game if $O(a,e) \in A$. In particular, should $O(a,e)$ be finite then Adam has won.

The ultrafilter $\mathfrak{W}(\Upsilon_{H,\Lambda})$ was defined as the set of those $A \subseteq [\Upsilon_{H,\Lambda}]^{\aleph_0}$ for which Eve has a winning strategy.

The proof of the following well-known lemma is left as an exercise:

2·0 Lemma *Let U be an ultrafilter on I, and $\pi : I \to J$. Define $\pi_*(U) = \{X \subseteq J \mid \pi^{-1}[J] \in U\}$. Then $\pi_*(U)$ is an ultrafilter on J.*

Now define a map $\pi : [\Upsilon_{H,\Lambda}]^{\aleph_0} \to \delta_H$ thus: let $E \in [\Upsilon_{H,\Lambda}]^{\aleph_0}$. We know that each $b \in E$ defines a set S_b in the countably complete filter $\mathfrak{F}$: so $\bigcap_{b\in E} S_b$ is in $\mathfrak{F}$ and therefore non-empty. Let $\pi(E)$ be its least element.

We now set $\mathfrak{U}_{H,\Lambda} \ =_{\mathrm{df}} \ \pi_*\mathfrak{W}(\Upsilon_{H,\Lambda})$.

Another way of defining $\mathfrak{U}_{H,\Lambda}$, which readers may prefer, is this: for $Y \subseteq \delta_H$, define a game $\mathcal{G}^*(\Upsilon_{H,\Lambda}, Y)$ as follows: as in $\mathcal{G}$, Adam and Eve play a and e, their plays are interpreted as a sequence of reals a^n and e^n, and the first ω rules require them to ensure that each of these reals is in $\Upsilon_{H,\Lambda}$. If all those rules are kept, then Adam wins if $O(a,e)$ is finite; otherwise, if $O(a,e) \in [\Upsilon_{H,\Lambda}]^{\aleph_0}$, we set $\mathfrak{o}(a,e) = \pi(O(a,e))$ and then Eve wins if $\mathfrak{o}(a,e) \in Y$; we may refer to the ordinal $\mathfrak{o}(a,e)$ as the *deciding ordinal* of the run (a,e) of the game $\mathcal{G}^*$.

We then define $\mathfrak{U}_{H,\Lambda}$ to be the set of those $Y \subseteq \delta_H$ for which Eve has a winning strategy in $\mathcal{G}^*(\Upsilon_{H,\Lambda}, Y)$.

2·1 Exercise Verify that the two definitions agree.

2·2 Exercise One of the two ways of defining $\mathfrak{U}$ uses more determinacy than the other. Which ?

2·3 Exercise Show that if $b \in \Upsilon_{H,\Lambda}$ then there are infinitely many different $c \in \Upsilon_{H,\Lambda}$ for which $S_c = S_b$.

2·4 Exercise Show that $\mathfrak{F} \subseteq \mathfrak{U}$.

2·5 Proposition *$\mathfrak{U}$ is a δ_H-complete ultrafilter extending $\mathfrak{F}$.*

Proof : that it is an ultrafilter according to the second definition may be established along the lines of the proof in Chapter III that $\mathfrak{W}(\Upsilon_{H,\Lambda})$ is an ultrafilter; that it extends $\mathfrak{F}$ has been left as an exercise; that it is complete requires the process of *seizing control of the game* which we saw in the discussion of strong partition cardinals in Chapter III. Let $\xi < \delta_H$ and let $\Xi = (\mathcal{N}, <_\Xi)$ be a prewellordering in M_H of $\mathcal{N}$ of length ξ. Suppose that for all $\nu < \xi$, $A_\nu \in \mathfrak{U}$. It will suffice to prove that $\bigcap_{\nu<\xi} A_\nu$ is non-empty. Set

$$Z = \{(a,\tau) \mid a \in \mathcal{N} \ \& \ \tau \text{ is a winning strategy for Eve in } \mathcal{G}^*(\Upsilon_{H,\Lambda}, A_{|a|_\Xi})\}.$$

By the Coding Lemma, there is a subset Z^* of Z which is in $pos\Sigma^1_1(<_\Xi)$ such that for all $\nu < \xi$ there is a pair (a,τ) in Z^* with $|a|_\Xi = \nu$; so that if we set $\mathcal{E}^* = \{\tau \mid \exists a \, (a,\tau) \in Z^*\}$, $\mathcal{E}^*$ contains at least one strategy for Eve for each of the games $\mathcal{G}(A_\nu)$. Note that Z^* and therefore also $\mathcal{E}^*$ are in M_H.

We have seen that every member of M_H is established by some real, and that such member, once established, may be used as freely as reals in Σ_1 statements that we propose to reflect. So let $a_0 \in \Upsilon_{H,\Lambda}$ establish $\mathcal{E}^*$, and consider plays a with $a^0 = a_0$. We have

$$\forall \tau\!:\in\!\mathcal{E}^* \ \forall a\!:\in\!\mathcal{N} \ \big[a^0 = a_0 \implies \Phi_\diamondsuit\big(([a]*\tau)^0, \Lambda, \diamondsuit, \delta_H; H\big)\big]$$

as Eve wishes to keep Rule 2. As $\mathcal{E}^*$ has been established by a_0, that is a Σ_1 statement [C 1] about a_0 and τ, enforced, say, by a_1 in $\Upsilon_{H,\Lambda}$.

[C 1] Note that the more general statement $\forall \tau\!:\in\!\mathcal{E}^* \ \forall a\!:\in\!\mathcal{N} \ \big[a^0 \in \Upsilon_{H,\Lambda} \implies ([a]*\tau)^0 \in \Upsilon_{H,\Lambda})\big]$, while true, would not be Σ_1.

Thus

$$\forall \eta :\in S_{a_1}\ \forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[a^0 = a_0 \implies \Phi_\diamondsuit\big(([a] * \tau)^0, \diamondsuit(\eta), \diamondsuit\restriction\eta, \eta; H\big)\big]$$

and so

$$\forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[a^0 = a_0 \implies S_{a_1} \subseteq S_{([a]*\tau)^0}\big]$$

Next,

$$\forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[a^0 = a_0\ \&\ a^1 = a_1 \implies \Phi_\diamondsuit\big(([a] * \tau)^1, \Lambda, \diamondsuit, \delta_H; H\big)\big]$$

as Eve wishes to keep Rule 4; that again is Σ_1^H, so we may choose a_2 in $\Upsilon_{H,\Lambda}$ such that

$$\forall \eta :\in S_{a_2}\ \forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[a^0 = a_0\ \&\ a^1 = a_1 \implies \Phi_\diamondsuit\big(([a] * \tau)^1, \diamondsuit(\eta), \diamondsuit\restriction\eta, \eta; H\big)\big]$$

and so

$$\forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[a^0 = a_0\ \&\ a^1 = a_1 \implies S_{a_2} \subseteq S_{([a]*\tau)^1}\big]$$

Continuing in this fashion, we may pick a_3, a_4, ... such that

$$\forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[(\forall i :\leq 2\, a^i = a_i) \implies S_{a_3} \subseteq S_{([a]*\tau)^2}\big]$$
$$\forall \tau :\in \mathcal{E}^*\ \forall a :\in \mathcal{N}\ \big[(\forall i :\leq 3\, a^i = a_i) \implies S_{a_4} \subseteq S_{([a]*\tau)^3}\big]$$

and so on. Now let $\bar a$ be such that for each k, $\bar a^k = a_k$, let $\bar\eta$ be the least element of $\bigcap_{k<\omega} S_{\bar a^k}$ and consider what happens when Adam plays $\bar a$ against a strategy in $\mathcal{E}^*$. To fix our ideas, let $\nu < \xi$ and let $\tau \in \mathcal{E}^*$ be a winning strategy for Eve in $\mathcal{G}^*(\Upsilon_{H,\Lambda}, A_\nu)$. Put $\bar e = [\bar a] * \tau$. For each k, $S_{\bar a^{k+1}} = S_{a_{k+1}} \subseteq S_{\bar e^k}$, and so $\bar\eta = \mathfrak{o}(\bar a, \bar e)$, the deciding ordinal of the run $(\bar a, \bar e)$ of the game $\mathcal{G}^*(\Upsilon_{H,\Lambda}, A_\nu)$, and therefore, since Eve has won, $\bar\eta \in A_\nu$.

As ν was arbitrary, $\bar\eta \in \bigcap_{\nu<\xi} A_\nu$, as required. ⊣ (2·5)

2·6 Exercise Show that each of a_1, a_2, … may be taken to be recursive in $\langle a_0, \tau\rangle$, and hence that the apparent use of DC in the above proof may be avoided.

3: How normality leads to strength.

We are going to prove first that $\mathfrak{F}$ and then that $\mathfrak{U}$ has a certain *normality* property. We state it first for $\mathfrak{U}$:

3·0 PROPOSITION *The ultrafilter $\mathfrak{U}_{H,\Lambda}$ on δ_H is normal in the sense that for each pre-well-ordering R that is projective in Λ, for each set $S \in \mathfrak{U}$ with $S \subseteq Field(R)$ and each function $g : S \to \delta_H$ such that $\forall \nu {:\in} S\ g(\nu) < |R(\nu)|$, there is a $T \subseteq S$ with $T \in \mathfrak{U}$ and a real a such that*

$$\forall \nu {:\in} T \ \ g(\nu) = |a|_{R(\nu)}.$$

3·1 REMARK Here $R(\nu)$ for $\nu \in \mathrm{Dom}(\diamondsuit)$ is the object defined (projectively) from $\diamondsuit(\nu)$ just as R is from Λ, as is discussed more fully at the start of section 5. Hence our $\diamondsuit$-like object is involved in this definition, so that this concept of normality is only available for ultrafilters on those ordinals, such as our δ_H, enjoying sufficiently strong reflection properties.

Before turning to the verification that $\mathfrak{F}$ and $\mathfrak{U}$ are indeed normal, we show that the normality of $\mathfrak{U}$ leads directly to the proof of the Main Theorem of this chapter. Armed with $\mathfrak{U}$, we build an ultrapower of HOD by taking all functions $f : \delta_H \to HOD$ in $V = J(\mathbb{R})$ (not just those in HOD) and factoring by $\mathfrak{U}$. This yields a well-founded model, using DC, which can then be transitised to form M. The embedding $j : HOD \to M$ then results by considering the map $x \mapsto [c_x]_{\mathfrak{U}}$ where for $x \in HOD$, c_x is the function on δ_H with constant value x.Since AC is true in HOD we get Łoś's theorem, and thus j is elementary.

Note that j is entirely definable from H and Λ, which are themselves OD. The ultrapower of the canonical well-ordering $<_{OD}$ of HOD yields a definable well-ordering of M which allows us to define any element of M by its place in this well-ordering. Hence $M \subseteq HOD$, and j, read as

$$\langle j(X) \mid X \in HOD \rangle$$

is amenable to HOD. The whole ultrafilter $\mathfrak{U}$ is not visible to HOD since not all subsets of δ_H are in HOD, so HOD will perceive j as an elementary embedding arising possibly from extenders but not (in view of the strength we are about to establish) arising from a single measure.

First, we examine the representation of ordinals less than λ in the model.

3·2 DEFINITION For $a \in \mathcal{N}$, define the function $f_a : \delta_H \to \delta_H$ by

$$f_a(\nu) = \begin{cases} |a|_{\Lambda(\nu)} & \text{if } \nu \in \mathrm{dom}(\diamondsuit) \text{ and } a \in \mathrm{Field}(\diamondsuit(\nu)) \\ 0 & \text{otherwise.} \end{cases}$$

3·3 PROPOSITION *If $a <_\Lambda b$ then $\{\nu \mid f_a(\nu) < f_b(\nu)\} \in \mathfrak{F}$.*

Proof : see 5·6 (iv). $\dashv$

3·4 PROPOSITION *For any $a \in \mathcal{N}$, $|a|_\Lambda = [f_a]_{\mathfrak{U}}$*

Proof : by induction on $|a|_\Lambda$, using normality for $R = \Lambda \restriction a$ to see that any h with $[h]_{\mathfrak{U}} < [f_a]_{\mathfrak{U}}$ is equal, *mod* $\mathfrak{U}$, to some f_b with $b <_\Lambda a$. $\dashv$ (3·4)

3·5 PROPOSITION *the map $\nu \mapsto |\diamondsuit(\nu)|$ represents λ.*

Proof : This map, call it h, certainly represents an ordinal, $[h]$, by Łoś. Each ordinal less than $[h]$ is of the form $[f_b]$ for some b, by the normality of $\mathfrak{U}$, *i.e.* by Proposition 3·0 with $R = \Lambda$, and each such ordinal is less than $[h]$. By 3·4, $[h] = \lambda$. $\dashv$ (3·5)

Now we verify the strength of j:

3·6 PROPOSITION $\lambda \cap H = \lambda \cap j(H)$

Proof : Let $\zeta = |a|_\Lambda < \lambda$. Then if $\zeta \in H$, $|a|_\Lambda \in H$, so by Σ_1 reflection $\{\nu \,\big|\, |a|_{\diamondsuit(\nu)} \in H\} \in \mathfrak{F} \subseteq \mathfrak{U}$, so $[f_a]_{\mathfrak{U}} \in j(H)$, and thus $\zeta \in j(H)$; similarly $\zeta \notin H \implies \zeta \notin j(H)$, and so $H \cap \lambda = j(H) \cap \lambda$. $\dashv$ (3·6)

In the light of Chapter IV, the following result is superfluous, though it is of interest to know explicit representations for subsets of λ in HOD:

3·7 PROPOSITION $\mathcal{P}(\lambda) \cap HOD \subseteq M$.

Proof : Let B in *HOD* be a subset of λ. Using the Uniform Coding Lemma we find a real b such that B is coded by G_b^{Λ}. Let B_δ be the subset of $|\diamondsuit(\delta)|$ coded by $G_b^{\diamondsuit(\delta)}$. To say that B is OD is to make the Σ_1 statement about b and Λ that some $J_\eta(\mathbb{R})$ thinks that B is OD; this statement will hold of b and $\diamondsuit(\delta)$ and thus B_δ will be in *HOD* for δ in a set Y in $\mathfrak{F}$. Hence we may define the function g_B by

$$g_B(\delta) = \begin{cases} B_\delta & \text{if } \delta \in Y; \\ \varnothing & \text{otherwise.} \end{cases}$$

This function is a map from δ_H to *HOD*: we assert that

$$[g_B]_{\mathfrak{U}} = B.$$

By 3·5, $[g_B] \subseteq \lambda$. If $|a|_\Lambda \in B$, $a \in G_b^{\Lambda}$, so $\{\nu \mid |a|_{\diamondsuit(\nu)} \in B_\nu\} \in \mathfrak{F} \subseteq \mathfrak{U}$, and so $|a|_\Lambda = [f_a] \in [g_B]$; similarly $|a|_\Lambda \notin B$ will imply that $|a|_\Lambda \notin [g_B]$.

Thus $B = [g_B]_{\mathfrak{U}}$ and so $B \in M$ as required. $\dashv$ (3·7)

3·8 Exercise Show that $j(\delta_H) > \lambda$.

3·9 Exercise Show using Proposition 3·6 that $\lambda \in H \iff \lambda \in j(H)$.

4: A non-monotonic Coding Lemma

We shall use in our proof of the normality of $\mathfrak{F}$ an observation similar in spirit, and in proof, to the Uniform Coding Lemma. But the proof of that relied on the monotonicity of the definitions we were using; here we shall have to use a non-monotone definition. Fix a (product) space $\mathcal{X}$ of type 1: for example, $\mathcal{X} = \mathcal{N}$ or $\mathcal{X} = \mathcal{N} \times \mathcal{N}$. Let κ be an ordinal less than Θ, and suppose that we have a sequence $\langle K^\delta \mid \delta < \kappa\rangle$ such that for $\delta < \kappa$ $K^\delta = (X^\delta, \leqslant_\delta)$ is a pre-well-ordering of length λ_δ, say, of a non-empty subset X^δ of $\mathcal{X}$, and suppose that the fields of these pre-well-orderings are pairwise disjoint, so that

$$\delta < \nu < \kappa \implies X^\delta \cap X^\nu = \varnothing.$$

For an element a in $\bigcup_{\delta<\kappa} X^\delta$, define $\varrho(a)$ to be the unique pair $\langle \delta, \alpha\rangle$ with $\delta < \kappa$ and $\alpha < \lambda_\delta$ such that $a \in (K^\delta)_\alpha$. We write $\leq_2$ for the lexical ordering of such pairs, so that

$$\langle \delta, \alpha\rangle <_2 \langle \gamma, \beta\rangle \iff_{\text{df}} (\delta < \gamma \text{ or } (\delta = \gamma \,\&\, \alpha < \beta)).$$

Define a pre-well-ordering $\leqslant$ on $\bigcup_{\delta<\kappa} X^\delta$ by

$$a \leqslant b \iff_{\text{df}} \varrho(a) \leq_2 \varrho(b).$$

The restriction of $\leqslant$ to each X_δ will of course be $\leqslant_\delta$. We write $X_{\delta,\alpha}$ for the set $\left(\bigcup_{\gamma<\delta} X^\gamma\right) \cup (K^\delta)_\alpha$, and $K_{\delta,\alpha}$ for the pre-well-ordering $\left(X_{\delta,\alpha}, \leqslant \cap (X_{\delta,\alpha} \times X_{\delta,\alpha})\right)$. Notice that if $\alpha < \beta < \lambda_\delta$ then neither of $X_{\delta,\alpha}$, $X_{\delta,\beta}$ contains the other, whence the non-monotonic character of the argument to come.

For T a pre-well-ordering, we shall be interested in those relations definable positively from $\leq_T$ and $\not\leq_T$. We may index $pos\Sigma^1_1(\leq_T, \not\leq_T)$ as before: but we denote the set $G_t^{\leq_T, \not\leq_T}$ for convenience by $U_t(T)$.

We now state and prove our non-monotonic Coding Lemma.

4·0 Proposition $\exists t{:}{\in}\mathcal{N}\ \forall\delta{:}{<}\kappa\ \forall\alpha{:}{<}\lambda_\delta\ \left[U_t(K_{\delta,\alpha}) \cap (K^\delta)_\alpha = \varnothing \iff \alpha = 0.\right]$

Proof : We define a recursive function $c : \mathcal{X} \times \mathcal{N} \to \mathcal{N}$ of variables $b \in \mathcal{X}$ and $t \in \mathcal{N}$ by the requirement that for any pre-well-ordering T of a subset of $\mathcal{X}$,

$$U_{c(b,t)}(T) = U_t(T) \cap \{e \mid e \leq_T e \,\&\, b \not\leq_T e\}.$$

Notice that

$$\{e \mid e \leq_T e \,\&\, b \not\leq_T e\} = \begin{cases} \text{Field}(T) & \text{if } b \notin \text{Field}(T); \\ \{e \mid e <_T b\} & \text{if } b \in \text{Field}(T). \end{cases}$$

4·1 DEFINITION The *first failure* of t, $\mathit{ff}(t)$, is the lexically first pair $(\delta,\, \alpha)$ such that $0 \le \alpha < \lambda_\delta$ and either $\alpha = 0$ & $U_t(K_{\delta,\alpha}) \cap (K^\delta)_\alpha \ne \varnothing$ or $\alpha > 0$ & $U_t(K_{\delta,\alpha}) \cap (K^\delta)_\alpha = \varnothing$. If no such pair exists, we set $\mathit{ff}(t) = \infty$.

4·2 DEFINITION Call d *acceptable* if $\forall\delta\; U_d(K_{\delta,0}) \cap (K^\delta)_0 = \varnothing$.

For acceptable d, therefore, $\mathit{ff}(d)$ is the lexically least pair $\langle\delta,\, \alpha\rangle$ such that

$$0 < \alpha < \lambda_\delta \text{ and } U_d(K_{\delta,\alpha}) \cap (K^\delta)_\alpha = \varnothing.$$

4·3 LEMMA *Let $t \in \mathcal{N}$ and $b \in \bigcup_{\delta<\kappa} X^\delta$.*

(i) if $\varrho(b) \le_2 \mathit{ff}(t)$, then $c(b,t)$ is acceptable and $\mathit{ff}(c(b,t)) \ge_2 \varrho(b)$;

(ii) if $\varrho(b) = \mathit{ff}(t) = \langle\delta_t,\, \alpha_t\rangle$, and $\alpha_t = 0$, then $\mathit{ff}(c(b,t)) > \mathit{ff}(t)$;

(iii) if $\varrho(b) = \mathit{ff}(t) = \langle\delta_t,\, \alpha_t\rangle$, and $\alpha_t > 0$, then $\mathit{ff}(c(b,t)) = \mathit{ff}(t)$.

Proof : Put $\varrho(b) = \langle\delta_b,\, \alpha_b\rangle$. Suppose first that $\langle\gamma,\, \eta\rangle <_2 \varrho(b)$: since in this case $b \notin X_{\gamma,\eta} = \mathrm{Field}(K_{\gamma,\eta})$, $U_{c(b,t)}(K_{\gamma,\eta}) = U_t(K_{\gamma,\eta}) \cap \mathrm{Field}(K_{\gamma,\eta})$ and so hits or misses $(K^\gamma)_\eta$ correctly since $(\gamma,\, \eta) <_2 \mathit{ff}(t)$. Thus $\varrho(b) \le_2 \mathit{ff}(c(b,t))$.

Next suppose that $(\gamma,\, \eta) = \varrho(b)$: in this case $U_{c(b,t)}(K_{\gamma,\eta}) = U_t(K_{\gamma,\eta}) \cap \{e \mid e <_{K_{\gamma,\eta}} b\}$ and so misses $(K^\gamma)_\eta$. Hence $c(b,t)$ will be correct here if $\alpha_b = 0$ but will have failed if $\alpha_b > 0$. Parts (ii) and (iii) will follow from this observation once we have completed the proof of Part (i).

Finally, suppose that $(\gamma,\, \eta) >_2 \varrho(b)$. We must consider two subcases. If $\delta_b < \gamma$, then $b \in \mathrm{Field}(K_{\gamma,\eta})$, so $U_{c(b,t)}(K_{\gamma,\eta}) \subseteq \{d \mid d < b\}$ and so $U_{c(b,t)}(K_{\gamma,\eta}) \cap (K^\gamma)_\eta = \varnothing$, which is correct whenever $\eta = 0$, and so this subcase supports the acceptability of $c(b,t)$.

If on the other hand, $\delta_b = \gamma$, then $\alpha_b < \eta$, so $\eta > 0$ and this subcase does not contradict the acceptability of $c(b,t)$. ⊣ (4·3)

We play a game: Adam plays a real a, Eve a real e.

Rule 1: a must be acceptable, otherwise Adam loses.

Rule 2: If a is acceptable and e not, then Eve loses.

Rule 3: If both a and e are acceptable, then Eve wins iff $\mathit{ff}(e) > \mathit{ff}(a)$.

Let σ be a winning strategy for Adam. Let $\bar{a}$ be such that for any T,

$$U_{\bar{a}}(T) = \bigcup_{e\in\mathcal{N}} U_{\sigma\star[e]}(T).$$

Then $\bar{a}$ is acceptable, since for each e, $\sigma \star [e]$ must be acceptable and therefore $U_{\sigma\star[e]}(K_{\delta,0})$ misses $(K^\delta)_0$. Let $(\bar{\delta}, \bar{\alpha})$ be its first failure, and pick $x \in (K^{\bar{\delta}})_{\bar{\alpha}}$. Let $\widetilde{e}$ be such that for any T,

$$U_{\widetilde{e}}(T) = U_{\bar{a}}(T) \cup \{x\}.$$

By the disjointness of the component prewellorderings, $\widetilde{e}$ is acceptable and has a higher first failure than $\bar{a}$, and hence its play defeats the strategy σ, unless of course the first failure of $\bar{a}$ is ∞.

Now let τ be a winning strategy for Eve. Define a recursive function $t \mapsto t^*$ of indices by

$$U_{t^*}(T) = \big\{a \;\big|\; \exists b\big[b =_T a \;\&\; a \in U_{[c(b,t)]\star\tau}(T)\big]\big\}$$

where '$b =_T a$' means that b and a lie in the same component of T.

By the Recursion Theorem, there is a t such that for every T, $U_t(T) = U_{t^*}(T)$. We assert that the first failure of t is ∞. For if not, let $\mathit{ff}(t) = \langle\delta,\, \alpha\rangle$: we shall derive a contradiction.

(i) Suppose that $\alpha = 0$: then as t has failed here, there is some $a \in U_t(K_{\delta,0}) \cap (K^\delta)_0$. So $a \in U_{t^*}(K_{\delta,0})$ so $\exists b\; [b =_{K_{\delta,0}} a \;\&\; a \in U_{[c(b,t)]\star\tau}(K_{\delta,0})$. But that contradicts our knowledge, from the Lemma, that as $\varrho(b) = \mathit{ff}(t)$, $c(b,t)$, and therefore also $[c(b,t)] \star \tau$, is acceptable, which tells us that $U_{[c(b,t)]\star\tau}(K_{\delta,0}) \cap (K^\delta)_0$ is empty.

(ii) Now suppose that $\alpha > 0$, and let $b \in (K^\delta)_\alpha$: then we know again that $[c(b,t)] \star \tau$ is acceptable, with first failure strictly greater than $\mathit{ff}(c(b,t)) = (\delta, \alpha)$, and so $U_{[c(b,t)]\star\tau}(K_{\delta,\alpha}) \cap (K^\delta)_\alpha$ is non-empty, containing a, say. Then $a \in U_{t^*}(K_{\delta,\alpha}) \cap (K^\delta)_\alpha = U_t(K_{\delta,\alpha}) \cap (K^\delta)_\alpha$, which therefore is non-empty and so t has not failed at $\langle\delta,\, \alpha\rangle$ after all. ⊣ (4·0)

5: The normality of $\mathfrak{F}$

We shall be interested in pointsets that are projective in Λ, *i.e.* are $\Sigma^1_n(\Lambda, a)$ for some n and some real parameter a. In particular we shall be interested in pre-well-orderings that are projective in Λ.

5·0 EXAMPLE Suppose that $A \subseteq \mathcal{N} \times \mathcal{N}$ is projective in Λ: then Q is a pre-well-ordering projective in Λ, where Field$(Q) = A$, and

$$(b,\, c) \leqslant_Q (x,\, y) \Longleftrightarrow b \leq_\Lambda x.$$

5·1 EXAMPLE If Q is one such pre-well-ordering and $b \in \text{Field}(Q)$, another one will be $Q \restriction b$, where

$$Q \restriction b \ =_{\text{df}} \ \big(\{x \mid x <_Q b\}, \{(x,\, y) \mid x \leqslant_Q y <_Q b\}\big).$$

5·2 REMARK If $b \in \text{Field}(Q)$, then $|b|_Q = |Q \restriction b|$. Further, if $c <_Q b$ then $|c|_{Q \restriction b} = |c|_Q$

5·3 DEFINITION To any subset A of a product space $\mathcal{X}$ of type 1, with A projective in Λ we associate to A a function, which we call $A(\cdot)$, defined on Dom$(\diamondsuit)$: $A(\nu)$ for $\nu \in \text{Dom}(\diamondsuit)$ will be the object defined from $\diamondsuit(\nu)$ as A was from Λ. We shall speak of $A(\nu)$ as the *reflection of A to ν.*

5·4 REMARK We use this terminology for the following reason. We know that whenever a statement of the form $\exists x :\in J_\Theta(\mathbb{R}; H)\ \Psi^0(a, x, \Lambda, \diamondsuit, \delta_H; H)$ is true, where Ψ^0 is Δ_0, and a is a real or more generally a member of $J_{\delta_H}(\mathbb{R}; H)$, so is the *reflected statement* $\exists x :\in J_{\delta_H}(\mathbb{R}; H)\ \Psi^0(a, x, \diamondsuit(\nu), \diamondsuit \restriction \nu, \nu; H)$ for each ν in some set in $\mathfrak{F}$. If A is projective in Λ, then a Σ^H_1 statement $\exists x :\in J_\Theta(\mathbb{R}; H)\ \Psi(a, x, A, \Lambda, \diamondsuit, \delta_H; H)$ about A is itself a Σ^H_1 statement about a, Λ, $\diamondsuit$ and δ_H, and its truth entails the truth of the reflected statement $\exists x :\in J_{\delta_H}(\mathbb{R}; H)\ \Psi(a, x, A(\nu), \diamondsuit(\nu), \diamondsuit \restriction \nu, \nu; H)$ for each ν in a set in $\mathfrak{F}$.

5·5 EXAMPLE Let Q be a pre-well-ordering projective in Λ, and $b \in \text{Field}(Q)$. Then for $\nu \in \text{Dom}(Q(\cdot)) = \text{Dom}(\Lambda)$,

$$\begin{aligned}(Q \restriction b)(\nu) &= \big(\{x \mid x <_{Q(\nu)} b\}, \{(x, y) \mid x \leqslant_{Q(\nu)} y <_{Q(\nu)} b\}\big) \\ &= Q(\nu) \restriction b\end{aligned}$$

Of course a given Q will be definable in more than one way from Λ. The following lemma covers that and related points.

5·6 LEMMA *Let Q and R be pre-well-orderings projective in Λ.*

(i) If $R = Q$ then $\{\nu \in Dom(\diamondsuit) \mid R(\nu) = Q(\nu)\} \in \mathfrak{F}$;

(ii) if $b \in Field(Q)$, then $C(b) \ =_{\text{df}} \ \{\nu \mid b \in Field(Q(\nu))\} \in \mathfrak{F}$;

(iii) for $\nu \in C(b)$, $|(Q \restriction b)(\nu)| = |b|_{Q(\nu)}$;

(iv) if $c <_Q b$, $F(c, b) \ =_{\text{df}} \ \{\nu \mid c <_{Q(\nu)} b\} \in \mathfrak{F}$;

(v) for $\nu \in C(b) \cap F(c, b)$, $|c|_{(Q \restriction b)(\nu)} = |c|_{Q(\nu)}$.

Proof : (i), (ii) and (iv) since the statements $R = Q$, $b \in \text{Field}(Q)$, $c <_Q b$ are all Σ^H_1 statements about Λ, c and b, and other reals used in defining R and Q from Λ. (iii) and (iv) follow from those and Remark 5·2. $\dashv$ (5·6)

5·7 DEFINITION If $\mathcal{F}$ is a filter on a set I, we set $\tilde{\mathcal{F}} \ =_{\text{df}} \ \{X \subseteq I \mid I \setminus X \in \mathcal{F}\}$ and $\mathcal{F}^+ \ =_{\text{df}} \ \{X \subseteq I \mid X \notin \tilde{\mathcal{F}}\}$.

Thus if $\mathcal{F}$ were the sets of measure one with respect to some $[0, 1]$-valued measure, $\tilde{\mathcal{F}}$ would be the sets of measure 0 and $\mathcal{F}^+$ the sets of positive measure.

5·8 THEOREM *Let Q be a pre-well-ordering projective in Λ. Suppose that $X \subseteq \text{dom}(\diamondsuit)$, $X \in \mathfrak{F}^+$ and $g : X \to ON$ is such that $\forall \nu :\in X\ \ g(\nu) < |Q(\nu)|$. Then there is a real $a \in Field(Q)$ such that*

$$\{\nu \mid \nu \in X \ \&\ a \in \textit{Field}(Q(\nu)) \ \&\ g(\nu) = |a|_{Q(\nu)}\} \in \mathfrak{F}^+.$$

The proof of 5·8 falls into five steps.

Step 1: We suppose that Q is a counterexample of minimal length. That tells us that for $b \in \text{Field}(Q)$ the theorem is true of $Q \restriction b$. For $b \in \text{Field}(Q)$, define

$$D'(b) \ =_{\text{df}} \ C(b) \cap \{\nu \in X \mid g(\nu) < |(Q \restriction b)(\nu)|\,\}.$$

If some $D'(b)$ is in $\mathfrak{F}^+$, then, by the truth of the theorem for $Q\restriction b$ there is an $a\in\mathrm{Field}(Q\restriction b)$ such that

$$\big\{\nu\in D'(b)\mid a\in\mathrm{Field}((Q\restriction b)(\nu))\ \&\ g(\nu)=|a|_{(Q\restriction b)(\nu)}\big\}\in\mathfrak{F}^+.$$

But then by part (v) of the Lemma, $\{\nu\mid g(\nu)=|a|_{Q(\nu)}\}$ contains $D'(b)\cap C(b)\cap F(a,b)$ and so is in $\mathfrak{F}^+$.

Hence we may assume that each $D'(b)$ is in $\widetilde{\mathfrak{F}}$. We may also assume that for each $b\in\mathrm{Field}(Q)$, $D''(b)\ =_{\mathrm{df}}\ \{\nu\mid\nu\in X\,\&\,b\in\mathrm{Field}(Q(\nu))\ \&\ g(\nu)=|b|_{Q(\nu)}\}\in\widetilde{\mathfrak{F}}$, since otherwise we would have reached the conclusion of the theorem; so if we set $D(b)=D'(b)\cup D''(b)=\{\nu\in X\cap C(b)\mid g(\nu)\le|b|_{Q(\nu)}\}$, we shall have that for all $b\in\mathrm{Field}(Q)$,

$$D(b)\in\widetilde{\mathfrak{F}}. \tag{5·8·0}$$

Step 2: Define

$$Z\ =_{\mathrm{df}}\ \{(b,\,c)\mid b\in\mathrm{Field}(Q)\ \&\ c\in\Upsilon_{H,\Lambda}\ \&\ S_c\cap D(b)=\varnothing\}.$$

By the Coding Lemma, there is a $Z^*\subseteq Z$, in $pos\boldsymbol{\Sigma}^{\mathbf{1}}_{\mathbf{1}}(<_Q)$, and hence projective in Λ, which meets every component of Q in the sense that

$$\forall b'\,{:}{\in}\,\mathrm{Field}(Q)\ \exists(b,\,c)\,{:}{\in}\,Z^*\ b=_Q b'.$$

Note that we can turn Z^* into the field of a prewellordering Q^* by setting

$$(b,d)\le_{Q^*}(b',d')\iff b\le_Q b'.$$

Then $|Q^*|=|Q|$, and $|(b,d)|_{Q^*}=|b|_Q$. Since Z^* is of the form $G^{<_Q}_{a^*}$, we may set, for $\delta\in\mathrm{Dom}(\diamondsuit)$, $Z^*(\delta)=G^{<_{Q(\delta)}}_{a^*}$ and reflect Q^* correspondingly to $Q^*(\delta)$, which for each δ in a set in $\mathfrak{F}$ will be a pre-well-ordering.

Reflection tells us, of course, that since Z^* meets every component of Q, for almost all[C 2] δ, $Z^*(\delta)$ will meet every component of $Q(\delta)$; our aim though is to show that Z^* itself meets every component of $Q(\delta)$, for many δ.

Now $\qquad \forall(b,\,c)\,{:}{\in}\,Z^*\ \Phi_\diamondsuit(c,\Lambda,\diamondsuit,\delta_H;H);$

so for δ in some $X_2\in\mathfrak{F}$, $\qquad \forall(b,\,c)\,{:}{\in}\,Z^*(\delta)\ \Phi_\diamondsuit(c,\diamondsuit(\delta),\diamondsuit\restriction\delta,\delta;H);$

in other words,

$$\delta\in X_2\ \&\ (b,\,c)\in Z^*(\delta)\implies\delta\in S_c. \tag{5·8·1}$$

Suppose that $(b,\,c)\in Z^*\cap Z^*(\delta)$; then $(b,\,c)\in Z$, so $S_c\cap D(b)=\varnothing$; but also $\delta\in S_c$; so $\delta\notin D(b)$; and so we have

$$\delta\in X\cap X_2\ \&\ (b,\,c)\in Z^*\cap Z^*(\delta)\implies g(\delta)>|(Q\restriction b)(\delta)|=|b|_{Q(\delta)};$$

or, put another way,

$$\delta\in X_2\cap X\ \&\ g(\delta)\le\alpha<|Q^*(\delta)|\implies(Q^*(\delta))_\alpha\cap Z^*=\varnothing \tag{5·8·2}$$

But (5·8·2) is a Σ_1 statement about Λ; or at least it would be if we could code $X_2\cap X$ and g appropriately. An admissible coding is to hand, as we saw in §6 of Chapter III, starting from the pre-wellordering $\prec_H$ of length δ_H of the good universal set Υ_H for Γ_H. So let ε_X be an admissible code of the characteristic function of $X_2\cap X$, and let ε_g be an admissible code of the function g. Then we may re-write (5·8·2) as the following Σ^H_1 statement about Λ, ε_X, ε_g and δ_H:

$$\forall\delta{:}{<}\,\delta_H\ \Big(\varepsilon_X(\delta)\simeq 1\implies\forall\alpha{:}{<}\,|Q^*(\delta)|\ \big(\varepsilon_g(\delta){\downarrow}\le\alpha\implies(Q^*(\delta))_\alpha\cap Z^*=\varnothing\big)\Big)$$

[C 2] by which we mean ‘for all δ in a set in $\mathfrak{F}$’.

There will be an $M_1 \in \mathfrak{F}$ with the corresponding reflected statement holding for all $\zeta \in M_1$. In other words,

$$\forall\delta\,{:<}\,\zeta\ \Big(\varepsilon_X(\delta) \simeq 1 \implies \forall\alpha\,{:<}\,|Q^*(\delta)|\ \big(\varepsilon_g(\delta)\!\downarrow\, \leq \alpha \implies (Q^*(\delta))_\alpha \cap Z^*(\zeta) = \varnothing\big)\Big)$$

Let us check that that is indeed the correct reflected statement. First, note that

$$\forall\delta\,{:<}\,\delta_H\ Q^*(\delta) \in J_{\delta_H}(\mathbb{R}; H)\ \&\ |Q^*(\delta)| < \delta_H,$$

because the pre-well-orderings $Q^*(\delta)$ are all members of M_H; Q^* is projective in Λ, and so $Q^*(\delta)$ is projective in $\diamondsuit(\delta)$; the $\diamondsuit$ function is very local in its Σ_1^H recursive definition, and thus the definition of $Q^*(\delta)$ will retain its meaning in $J_\zeta(\mathbb{R}; H)$; hence we may assume that for all $\zeta \in M_1$,

$$\forall\delta\,{:<}\,\zeta\ Q^*(\delta) \in J_\zeta(\mathbb{R}; H)\ \&\ |Q^*(\delta)| < \zeta.$$

Secondly, every H-safe ordinal is a limit of ordinals closed under the Gödel pairing function for ordinals that we used in establishing the admissible coding, and the initial segments of the canonical pre-well-ordering of Υ_H used in computing the admissible coding can be recovered in a Δ_1 way, so the Δ_1 statements $\varepsilon_X \simeq 1$ and $\varepsilon_g(\delta)\!\downarrow\, \leq \alpha$ are going to preserve their meaning on reflection.

Thus taking $\bar{X} = M_1 \cap X_2 \cap X$ and translating back to the original terms, we obtain this statement:

(5·8·3) $$\exists\bar{X}\,{:\in}\,\mathfrak{F}^+\ \forall\zeta\,{:\in}\,\bar{X}\ \forall\delta\,{:\in}\,\zeta\cap\bar{X}\ \big[g(\delta) \leq \alpha < |Q^*(\delta)| \implies (Q^*(\delta))_\alpha \cap Z^*(\zeta) = \varnothing\big]$$

Step 3: Now we build pre-well-orderings K^δ with a view to applying our non-monotonic coding lemma. Write $\mu_\delta = |Q^*(\delta)|$.

If we write

$$X^\delta = \bigcup_{g(\delta)\leq\alpha<\mu_\delta} (Q^*(\delta))_\alpha$$

then (5·8·3) tells us that

$$\delta \in \bar{X}\ \&\ \zeta \in \bar{X}\ \&\ \delta < \zeta \implies X^\delta \cap X^\zeta = \varnothing.$$

As $X^\delta \subseteq Z^*(\delta)$, we may make X^δ into a pre-well-ordering K^δ by giving it the ordering inherited from $Q^*(\delta)$. We set $\lambda_\delta = |K^\delta|$.

We may now, as in §4, define $X_{\delta,\alpha}$ and $K_{\delta,\alpha}$: it will be convenient, though, to think of α as ranging not over the interval $[0, \lambda_\delta)$ but over the interval $[g(\delta), \mu_\delta)$. Of course, $\mu_\delta = g(\delta) + \lambda_\delta$.

The way we set up these definitions has been guided by our wish that our definition of $Q_{\delta,\alpha}$ shall make sense when the first parameter is δ_H. We have in particular to avoid speaking of $g(\delta_H)$, which is of course undefined. In this limit case, instead of using α as an index we shall instead use a real a with $|a|_{Q(\delta)} = \alpha$.

So for $a \in \mathrm{Field}(Q^*)$ define

$$X_{\delta_H,a} = \Big(\bigcup_{\gamma<\delta_H} X^\gamma\Big) \cup \big(Q^*\big)_{|a|_{Q^*}}$$

This reflects correctly to

$$X_{\delta,a} = \Big(\bigcup_{\gamma<\delta} X^\gamma\Big) \cup \big(Q^*(\delta)\big)_{|a|_{Q^*(\delta)}}$$

for $a \in \mathrm{Field}(Q^*(\delta))$, in the sense that $X_{\delta,a} = X_{\delta,\alpha}$ for $\alpha = |a|_{Q^*(\delta)}$. We may turn $X_{\delta_H,a}$ into a pre-well-ordering $K_{\tau_H,a}$ with a top component $(Q^*)_{|a|_{Q^*}}$ and the earlier portion with the pre-well-ordering dictated by the lexical ordering of pairs.

Step 4: We invoke our non-monotonic Coding Lemma. In our context, in which we have altered the range of the index α, it will read

(5·8·4) $$\exists t\,{:\in}\,\mathcal{N}\ \forall\delta\,{:\in}\,\bar{X}\ \forall\alpha\,{:\in}\,[g(\delta), \mu_\delta)\ \big[U_t(K_{\delta,\alpha}) \cap (K^\delta)_\alpha = \varnothing \iff \alpha = g(\delta).\big]$$

Step 5: Now we complete the proof of the Normality Theorem. Fix $\bar{t}$ as promised in the assertion of (5·8·4). Then on a set of positive $\mathfrak{F}$ measure, namely $\bar{X}$,

$$\exists\gamma{:<}\mu_\delta\ \forall\alpha{:\in}[\gamma,\mu_\delta)\ \big[U_{\bar{t}}(K_{\delta,\alpha})\cap(Q^*(\delta))_\alpha=\varnothing\iff\alpha=\gamma\big]$$

Write γ_δ for this γ, which is necessarily unique for given $\delta\in\bar{X}$. We assert that therefore

$$\exists\gamma{:<}|Q^*|\ \forall\alpha{:\in}[\gamma,|Q^*|)\ \big[U_{\bar{t}}(K_{\delta_H,\alpha})\cap(Q^*)_\alpha=\varnothing\iff\alpha=\gamma\big]$$

for its denial, when we replace the quantification over ordinals less than $|Q^*|$ by a quantification over members of Z^*, which is of course projective in Λ, would be a Σ^H_1 statement that could not reflect to a set in $\mathfrak{F}$, since any such necessarily meets $\bar{X}$. Fix such a γ, and let $(\bar{b},\bar{d})$ be in$(Q^*)_\gamma$. Since γ is definable as the largest α for which $U_{\bar{t}}(K_{\delta_H,\alpha})\cap(Q^*)_\alpha$ is empty, and since this definition, when reflected, defines γ_δ, we see that there is a set $M_2\in\mathfrak{F}$ such that for $\delta\in M_2\cap\bar{X}$,

$$(\bar{b},\bar{d})\in(Q^*(\delta))_{\gamma_\delta}.$$

But it is plain from (5·8·4) that $\gamma_\delta=g(\delta)$ for $\delta\in M_2\cap\bar{X}$, and for such δ, $(\bar{b},\bar{d})\in Z^*\cap Z^*(\delta)$, contradicting (5·8·3). ⊣ (5·8)

The Normality theorem has a corollary:

5·9 Corollary *Let R be a pre-well-ordering projective in Λ, and let A be a set that meets each component of R: then*

$$\big\{\nu\ \big|\ \forall\alpha :< |R(\nu)|\ A\cap(R(\nu))_\alpha\neq\varnothing\big\}\in\mathfrak{F}.$$

Proof : let $g(\nu)=$ the least $\alpha<|R(\nu)|$ such that $(R(\nu))_\alpha\cap A=\varnothing$. If the corollary is false, g will be defined on a set X in $\mathfrak{F}^+$, and therefore there is an $b\in Field(R)$ such that on a set $Y\subseteq X$ in $\mathfrak{F}^+$, $g(\nu)=|b|_{R(\nu)}$. By assumption, $\exists a{:\in}A\ a=_R b$, so $T\ =_{\rm df}\ \{\nu\mid a=_{R(\nu)} b\}$ is in $\mathfrak{F}(\Lambda)$ and so meets Y. But then for $\nu\in Y\cap T$, $a\in(R(\nu))_{g(\nu)}\cap A$, a contradiction. ⊣

5·10 Remark If A were (say) in $pos\boldsymbol{\Sigma}^1_1(\Lambda)$, so that we had reflected versions A_ν to hand, then of course A_ν would meet each component of $R(\nu)$, for almost all ν. However the corollary states that A itself meets all components of $R(\nu)$, for many ν, and A itself is not required to be reflectible. In the next section we apply this corollary to establish the normality of $\mathfrak{U}_{\Lambda,H}$: in our application there will be a set, projective in A, of strategies for certain games; the corresponding sets, projective in $A(\nu)$, will certainly be of strategies for something, but we would not know that they were strategies for the same games. Hence the importance of the corollary as stated.

5·11 Remark The conclusion of the theorem may be derived from the corollary, which therefore may be considered as an alternative formulation of normality.

6: The normality of $\mathfrak{U}$

Again we use the notion of *seizing control of the game.*

6·0 PROPOSITION *Let R be a pre-well-ordering projective in Λ. Let $g : \delta_H \to \delta_H$ be a function such that $\forall\nu < \delta_H\, g(\nu) < |R(\nu)|$. Then there is a real $b \in Field(R)$ such that $T_b =_{\rm df} \{\nu \mid b \in Fld(R(\nu))\ \&\ g(\nu) = |b|_{R(\nu)}\} \in \mathfrak{U}$.*

Proof : If not, every such T_b is in $\widetilde{\mathfrak{U}}$: then Adam has a winning strategy for every game $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_b)$ with $b \in Field(R)$. Let $Z = \{(b,\sigma) \mid b \in \text{Field}(R)\ \&\ \sigma \text{ is a winning strategy for Adam in } \mathcal{G}^*(\Upsilon_{H,\Lambda}, T_b)\}$. By now familiar arguments there is a subset Z^* of Z which is $pos\Sigma^1_1(<_R)$ and such that, setting $\mathcal{A}^* = \{\sigma \mid \exists b\, (b,\sigma) \in Z^*\}$, for each b there is a $b' =_R b$ and a $\sigma \in \mathcal{A}^*$ with σ a winning strategy for Adam in $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_{b'})$. Z^* is projective in Λ and so for each $\delta \in \text{Dom}(\diamondsuit)$, we may define $Z^*(\delta)$ from $\diamondsuit(\delta)$ as Z^* is from Λ: as before, if we start from two possible definitions of Z^*, the reflected definitions will agree almost everywhere.

$\forall(b,\, \sigma){:\in} Z^*\ b \in \text{Field}(R)$, and so

$$X_0 =_{\rm df} \{\delta \mid \forall(b,\, \sigma){:\in} Z^*(\delta)\ b \in \text{Field}(R(\delta))\} \in \mathfrak{F}.$$

We may treat the elements of Z as pre-well-ordered by their first co-ordinate, and this pre-well-ordering is inherited by Z^*. Since Z^* meets every component of Z^*, we may apply Corollary 5·9 to conclude that

$$X_1 =_{\rm df} \{\delta \mid Z^* \text{ meets every component of } Z^*(\delta)\} \in \mathfrak{F}.$$

Since Z^* meets every component of R , we know that for every ordinal $\eta < |R|$,

$$(Z^*)_\eta = \big\{(b,\sigma) \in Z^* \,\big|\, |b|_R = \eta\big\},$$

or, quantifying over reals instead of ordinals,

$$\forall a{:\in}\text{Field}(R)\ \big[(Z^*)_{|a|_R} = \big\{(b,\sigma) \in Z^* \,\big|\, b \in \text{Field}(R)\ \&\ |b|_R = |a|_R\big\}\big]; \qquad \text{hence}$$

$$X_2 =_{\rm df} \Big\{\delta \,\Big|\, \forall a{:\in}\text{Field}(R(\delta))\ \big[(Z^*(\delta))_{|a|_{R(\delta)}} = \big\{(b,\sigma) \in Z^*(\delta) \,\big|\, b \in \text{Field}(R(\delta))\ \&\ |b|_{R(\delta)} = |a|_{R(\delta)}\big\}\big]\Big\} \in \mathfrak{F}.$$

Since everything in $\mathcal{A}^*$ is an Adam strategy,

$$\forall\sigma{:\in}\mathcal{A}^*\ \forall e{:\in}\mathcal{N}\ \big[\Phi_\diamondsuit\big((\sigma * [e])^0, \Lambda, \diamondsuit, \delta_H; H\big)\big] :$$

that is a Σ^H_1 statement about Λ, $\diamondsuit$ and δ_H; hence we may choose $e_0 \in \Upsilon_{H,\Lambda}$ such that

$$S_{e_0} \subseteq X_0 \cap X_1 \cap X_2 \cap \big\{\delta \,\big|\, \forall\sigma{:\in}\mathcal{A}^*(\delta)\ \forall e{:\in}\mathcal{N}\ \big[\Phi_\diamondsuit\big((\sigma * [e])^0, \diamondsuit(\delta), \diamondsuit{\restriction}\delta, \delta; H\big)\big]\big\}.$$

That last requirement implies that

$$\delta \in S_{e_0}\ \&\ \sigma \in \mathcal{A}^*(\delta) \implies \forall e{:\in}\mathcal{N}\ \ \delta \in S_{(\sigma*[e])^0}.$$

$$\text{Again,} \quad \forall\sigma{:\in}\mathcal{A}^*\ \forall e{:\in}\mathcal{N}\ \big[e^0 = e_0 \implies \Phi_\diamondsuit\big((\sigma * [e])^1, \Lambda, \diamondsuit, \delta_H; H\big)\big] :$$

that is a Σ^H_1 statement; hence we may choose $e_1 \in \Upsilon_{H,\Lambda}$ with

$$S_{e_1} \subseteq \big\{\delta \,\big|\, \forall\sigma{:\in}\mathcal{A}^*(\delta)\ \forall e{:\in}\mathcal{N}\ \big[e^0 = e_0 \implies \Phi_\diamondsuit\big((\sigma * [e])^1, \delta, \diamondsuit(\delta), \diamondsuit{\restriction}\delta, \delta; H\big)\big]\big\}.$$

In other words,

$$\delta \in S_{e_1}\ \&\ \sigma \in \mathcal{A}^*(\delta) \implies \forall e{:\in}\mathcal{N}\ (e^0 = e_0 \implies \delta \in S_{(\sigma*[e])^1}).$$

Continue for ω steps, to build a sequence e_k of elements of $\Upsilon_{H,\Lambda}$ such that

$$\delta \in S_{e_k}\ \&\ \sigma \in \mathcal{A}^*(\delta) \implies \forall e{:\in}\mathcal{N}\ \big((\forall i{:<}k\ e^i = e_i) \implies \delta \in S_{(\sigma*[e])^k}\big).$$

Let $\bar{e}$ be the real with $\forall k\, \bar{e}^k = e_k$, let $\bar{\delta}$ be the least element of $\bigcap_{k<\omega} S_{\bar{e}^k}$, and let $(\bar{b},\, \bar{\sigma}) \in Z^* \cap (Z^*(\bar{\delta}))_{g(\bar{\delta})}$: this last is possible as $\bar{\delta} \in X_1$. As $\bar{\delta} \in X_0$, and $(\bar{b},\, \bar{\sigma}) \in Z^*(\bar{\delta})$, we know that $\bar{b} \in \text{Field}(R(\bar{\delta}))$. As $(\bar{b},\, \bar{\sigma}) \in Z^* \subseteq Z$, we know that $\bar{b} \in \text{Field}(R)$ and that $\bar{\sigma}$ is a winning strategy for Adam in $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_{\bar{b}})$. Further, as $(\bar{b},\, \bar{\sigma}) \in Z^*(\bar{\delta})$, we know that $\bar{\sigma} \in \mathcal{A}^*(\bar{\delta})$. So for each k, $\bar{\delta} \in S_{(\bar{\sigma}*[\bar{e}])^k}$, and hence $\bar{\delta} = \mathfrak{o}(\bar{\sigma} * [\bar{e}], \bar{e})$. Adam has won $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_{\bar{b}})$; his victory must be under Rule ω since each $\bar{e}^k = e_k \in \Upsilon_{H,\Lambda}$, and so $\bar{\delta} \notin T_{\bar{b}}$; but $\bar{b} \in \text{Field}(R(\bar{\delta}))$, and as $\bar{\delta} \in X_2$ $g(\bar{\delta}) = |(\bar{b},\, \bar{\sigma})|_{Z^*(\bar{\delta})} = |\bar{b}|_{R(\bar{\delta})}$. Contradiction ! $\dashv$ (6·0)

With that, the proof of the main theorem of the chapter is complete.