Philosophical Skills

PHI-1A02

by Thomas Forster

Spring 2009

Contents

Foundations

We will be concentrating on the study of logic[1], the study of reason. We are interested in reason with 100% reliability, which surprisingly is easier to study than reason which gives us ‘pretty good’ reliability.

We will first look at some one important distinction:

PRESCRIPTIVE	DESCRIPTIVE
to work out how we ought to reason (with which we are concerned)	to describe how people reason[2]

Some definitions

- A **statement** or **proposition** is something that stands on its own and has a **truth value**.

- The relationship(s) between different statements are signified with **connectives** e.g. ‘and’ ‘or’ ‘not’ ‘NAND’ ‘NOR’ to make compound sentences. The connectives we are interested in are truth-functional (their truth value can also be preserved or not).

- An **argument** is made up of a **premises** and a **conclusion**.
  Conclusions are drawn from the premises e.g.

The cat sat on the mat and the dog sat in front of the fire

The cat sat on the mat
The dog sat in front of the fire

Important things to note about arguments

- If a+e I can infer a from it (and I can infer e from it)

- If a I can infer a or b from it (a simple sentence can imply a compound sentence)

[1] Not to be confused with rhetoric (using language to get people to do what you want them to do)
[2] Very important in modern science for artificial intelligence

Basic truth tables

*It is important to note the difference between **intension** and **extension** here. Think of the truth value of a statement as its extension. The meaning of a statement is its intension. For truth functional connectives we only look at extensions. (We are not interested in the content of A and B. We are only interested in whether or not the conclusion of an argument follows from the premises by pure logic.) A valid **type** of argument is one such that all **tokens** of it must have a true conclusion, when all premises are true.*

A and B

A	∧	B
I	I	I
I	0	0
0	0	I
0	0	0

(conjunction)

A or B

A	V	B
I	I	I
I	I	0
0	I	I
0	0	0

(disjunction)

Not A

¬	A
0	I
I	0

A exclusively or B

A	Δ	B
I	0	I
I	I	0
0	I	I
0	0	0

If A then B

A	→	B
I	I	I
I	0	0
0	I	I
0	I	0

(conditional)

We call the arrow in the last example a **conditional**. Anything of the form *if something then something* is a conditional. That which comes before the conditional is the **antecedent**. That which comes after the conditional is the **consequent**.

The conditional can be tricky to understand because it does not work in the same way as natural language. We must think of this as nothing to do with **semantics**. Instead it might help to think of it as analogous with validity.

1st row: the truth value of the conditional is preserved here because B is true when A is true
2nd row: the truth value of the conditional has not been preserved here as the relationship between A and B is contradicted.
3rd row: the conditional has not made a claim about the nature of B if A is false. The relationship between A and B has not been contradicted so the truth value of the conditional is actually being preserved.
4th row: the conditional has not made a claim about the nature of B if A is false. The relationship between A and B has not been contradicted so the truth value of the conditional is actually being preserved.

The same rules apply for the last connective we will look at:

.

A if and only if B

A	↔	B
I	I	I
I	0	0
0	0	I
0	I	0

Types of argument

1) Valid arguments:

- **Modus Ponens (or Affirming the Antecedent)**

-affirm the conditional, affirm the antecedent, and infer the consequent

Expressed as follows:

$$\frac{A \qquad A \rightarrow B}{B}$$

NB – this way of presenting arguments is called a 'tree proof'. It will be met later in presenting more complex proofs.

- **Modus Tollens (or Denying the Consequent)**

-affirm the conditional, deny the consequent, and deny the antecedent

Expressed as follows:

$$\frac{A \rightarrow B \qquad \neg B}{\neg A}$$

Examples of valid arguments

1. If it is sunny we will have a picnic. It is sunny. Therefore we will have a picnic.
2. If the oven is hot then the red light will go off. The red light has not gone off. Therefore the oven is not yet hot.
3. Once Annie makes a cake there are no eggs left in the fridge. Annie made a cake. Therefore there are no eggs left in the fridge.
4. All puffy food is yummy. This muffin is puffy. Therefore this muffin is yummy.

2) Fallacies

- **Fallacy of affirming the consequent**:

affirm the conditional, affirm the consequent, and infer the antecedent

It is expressed as follows:

$$\frac{A \rightarrow B \qquad B}{A}$$

NB – this is invalid because B could have been obtained another way

- **Fallacy of denying the antecedent**:

affirm the conditional, deny the antecedent, and deny the consequent

It is expressed as follows:

$$\frac{A \rightarrow B \qquad \neg A}{\neg B}$$

NB – this is invalid because B could be obtained another way

- **Fallacy of equivocation** - a word in the argument appears more than once and has different meanings.

Examples of invalid arguments

1. If George is guilty then he is reluctant to answer questions. George is reluctant to answer questions. Therefore George is guilty.

2. If Bill loves John then John is happy. John is happy. Therefore Bill loves John.

3. Emma always wears perfume if she is going clubbing. Emma is wearing perfume. Therefore Emma is going clubbing.

4. All metals are elements. Bronze is a metal. Therefore Bronze is a metal. (This argument **equivocates** on the word metal)

Complex truth tables

We can provide a truth table for any formula which is in the form of propositional logic. We fill truth tables in by:

1. Filling in the variables
2. Then fill in the connectives, starting with the connective with the smallest 'range' (brackets are used to denote where the ranges lie)

In this first example, we fill in the A and B columns first, then the Λ column because the Λ ranges between the brackets. Lastly we fill in the V because that has the largest range. The V ranges over the entire formula.

(A	Λ	B)	V	A
I	I	I	I	I
I	0	0	I	I
0	0	I	0	0
0	0	0	0	0

In this example, we fill in the A, B and C columns first[3], then the V column between the A and the B and lastly we fill in the V between the B) and the C.

(A	v	B)	v	C
I	I	I	I	I
I	I	I	I	0
I	I	0	I	I
I	I	0	I	0
0	I	I	I	I
0	I	I	I	0
0	0	0	I	I
0	0	0	0	0

In this example, we fill in the A, B and C columns first, then the V column between the B and the C and lastly we fill in the V between the A and the (B.

A	v	(B	v	C)
I	I	I	I	I
I	I	I	I	0
I	I	0	I	I
I	I	0	0	0
0	I	I	I	I
0	I	I	I	0
0	I	0	I	I
0	0	0	0	0

[3] The relationship between variables and rows is number of $r = 2^n$, where r is the number of rows and n is the number of variables. As we have added a third variable there will be not 8 rows, but 16.

Tautologies

For a given formula with n number of premises e.g. $(P_1 \wedge P_2 \wedge P_3 \wedge \dots P_n) \rightarrow C$

If this comes out true, no matter what the premises, it is a tautology and a VALID ARGUMENT.

We can easily discover if a formula is tautologous using truth tables. Each row of a truth table represents a possible assignment of truth values. If all assignments of the principal connective column come out true then the formula is a **tautology**.

A	→	(B	→	A)
I	I	I	I	I
I	I	0	I	I
0	I	I	0	0
0	I	0	I	0

((A	→	B)	→	A)	→	A
I	I	I	I	I	I	I
I	0	0	0	I	I	I
0	I	I	0	0	I	0
0	I	0	0	0	I	0

The following tautologous formula is called Peirce's Law. Peirce's gave us the type/token distinction.

((A	→	B)	→	A)	→	A)
I	I	I	I	I	I	I
I	0	0	I	I	I	I
0	I	I	0	0	I	0
0	I	0	0	0	I	0

Type/token distinction

Example 1

'I am writing a book' — this book is a **TYPE** *abstract object*
'I bought a book yesterday' — this book is a **TOKEN** *physical object*

Example 2

'Rose is a rose is a rose is a rose' — Here there are 3 **types** of words – 'rose', 'is' and 'a' and 10 **tokens** of words – 10 words in the sentence.

We can use the type/token distinction to define validity – a valid argument [TYPE] is one such that all [tokens] of it have a true conclusion.

Properties of connectives

ASSOCIATIVITY

A	v	(B	v	C)
I	I	I	I	I
I	I	I	I	0
I	I	0	I	I
I	I	0	0	0
0	I	I	I	I
0	I	I	I	0
0	I	0	I	I
0	0	0	0	0

(A	v	B)	v	C
I	I	I	I	I
I	I	I	I	0
I	I	0	I	I
I	I	0	I	0
0	I	I	I	I
0	I	I	I	0
0	0	0	I	I
0	0	0	0	0

The principal connective column comes out the same which tells us that it doesn't matter where the brackets go! We say that V has the property of associativity. Λ also has the property of associativity.

DISTRIBUTION OVER

A	Λ	(B	v	C)
I	I	I	I	I
I	I	I	I	0
I	I	0	I	I
I	0	0	0	0
0	0	I	I	I
0	0	I	I	0
0	0	0	I	I
0	0	0	0	0

These are logically equivalent

(A	Λ	B)	v	(A	Λ	C)
I	I	I	I	I	I	I
I	I	I	I	I	0	0
I	0	0	I	I	I	I
I	0	0	0	I	0	0
0	0	I	0	0	0	I
0	0	I	0	0	0	0
0	0	0	0	0	0	I
0	0	0	0	0	0	0

Again the principal connective column comes out the same. We say that 'Λ distributes over V'.

This is just the same as multiplication distributes over addition – $x(y+z) = xy + xz$

But just as addition does not distribute over multiplication…

A	V	(B	Λ	C)
I	I	I	I	I
I	I	I	0	0
I	I	0	0	I
I	I	0	0	0
0	I	I	I	I
0	0	I	0	0
0	0	0	0	I
0	0	0	0	0

These are logically equivalent

(A	v	B)	Λ	(A	V	C)
I	I	I	I	I	I	I
I	I	I	I	I	I	0
I	I	0	I	I	I	I
I	I	0	I	I	I	0
0	I	I	I	0	I	I
0	I	I	0	0	0	0
0	0	0	0	0	I	I
0	0	0	0	0	0	0

V ACTUALLY DOES DISTRIBUTE OVER Λ!

V and Λ are also both IDEMPOTENT- A V A provides the same truth table as A, and A Λ A provides that same truth table as A.

Some examples of logically equivalent formulae

1)

¬	(A	Λ	B)
0	I	I	I
I	I	0	0
I	0	0	I
I	0	0	0

¬	A	V	¬	B
0	I	0	0	I
0	I	I	I	0
I	0	I	0	I
I	0	I	I	0

2)

¬	(A	V	B)
0	I	I	I
I	I	0	0
I	0	0	I
I	0	0	0

¬	A	Λ	¬	B
0	I	0	0	I
0	I	I	I	0
I	0	I	0	I
I	0	I	I	0

3)

(A	Λ	B)	→	C
0	I	0	0	I
0	I	I	I	0
I	0	I	0	I
I	0	I	I	0

A	→	(B	→	C)
0	I	0	0	I
0	I	I	I	0
I	0	I	0	I
I	0	I	I	0

Assessing validity with truth tables

If there is no case in which the conclusion is false and both premises are true we have a valid argument.

Examples

1)

Premises	(A	→	B),	¬B	Conclusion	¬A
	I	I	I	0		0
	I	0	0	I		0
	0	I	I	0		I
	0	I	0	I		I

Here the conclusion is only false in the first two instances. In the first instance ¬B is also false. In the second instance A → B is false. So there is no instance where all premises are true and the conclusion is false so the argument is valid.

2)

Premises	A	→	B,	B	Conclusion	A
	I	I	I	I		I
	I	0	0	0		I
	0	I	I	I		0
	0	I	0	0		0

Here the conclusion is only false in the last two columns. In the third column both premises are true so there is an instance where all premises are true and the conclusion is false. The argument is therefore invalid.

3)

Premises	A	A	→	B,	B	→	C	Conclusion	C
	I	I	I	I	I	I	I		I
	I	I	I	I	I	0	0		0
	I	I	0	0	0	I	I		I
	I	I	0	0	0	I	0		0
	0	0	I	I	I	I	I		I
	0	0	I	I	I	0	0		0
	0	0	I	0	0	I	I		I
	0	0	I	0	0	I	0		0

Again, here the conclusion is only ever false if one or more of the premises is false. The argument is therefore valid.

The rules of natural deduction

Introduction		Elimination /Use	
Rule	**Explanation**	**Rule**	**Explanation**
or-introduction v-int $\dfrac{A}{A \vee B}$ or $\dfrac{B}{A \vee B}$	If A is true then A V B is also true. So, if we have A, we can infer A V B	or-elimination v-elim $\dfrac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$	*This will be explained further on*
and-introduction Λ-int $\dfrac{A \quad B}{A \wedge B}$	If A is true, and B is true then A Λ B is true. So if we have A and we have B we can infer A Λ B	and-elimination Λ-elim $\dfrac{A \wedge B}{A}$ or $\dfrac{A \wedge B}{B}$	If A Λ B is true then A is true, and B is true. So if we have A Λ B we can infer A an we can infer B
arrow-introduction →-int $\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}$	*This will be explained further on*	arrow-elimination →-elim $\dfrac{A \quad (A \to B)}{B}$	If you have a conditional expression and its antecedent you necessarily have the consequent.

The false and it's uses

$\dfrac{\perp}{A}$

⊥ means 'the false'

If I have the false I can infer A.

ex falso sequitur quodlibet -
from the false, anything follows

The 'do nothing' rule

If I have two assumptions A and B, I can ignore one of them in order to derive certain conclusions e.g.

$$\dfrac{\dfrac{\dfrac{[A] \quad [B]}{A}}{B \to A}}{A \to (B \to A)}$$

or present it with Λ – int:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B}}{A}}{B \to A}}{A \to (B \to A)}$$

Further explanation of some rules

Arrow introduction rule explained

I start with A

get to B

therefore I can infer:

$$\frac{A}{A \to B}$$

We look at this like borrowing A, using A to infer B and then giving A back when we no longer need it.

Or elimination rule explained

$$\dfrac{A \vee B \qquad \begin{matrix} A \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} B \\ \vdots \\ C \end{matrix}}{C}$$

If I have proof of C from A and I have proof of C from B, then if I have A V B I have C

‘From the false anything follows’ explained

$\neg$ A is to say that ($A \to \perp$); in other words A entails a falsehood, a *logical contradiction*, and so cannot be true. Therefore the rule works as follows:

$$\begin{array}{c} [A] \\ \downarrow \\ \perp \\ \hline A \to \perp \\ \neg A \end{array}$$

If we can derive the false from an assumption, we use the rule of -int to express that negated expression as a conditional expression.

[We use square brackets to denote an assumption which can be discharged at the end of the rule.]

Examples of compound arguments

Using these rules we can now 'stick together' some compound arguments.

Example 1:

Always notate proofs in this way!

$$\dfrac{\dfrac{A \wedge B}{A}\,{\scriptstyle \wedge\text{-elim}} \qquad A \to C}{C}\,{\scriptstyle \to\text{-elim}}$$

Example 2:

$$\dfrac{\dfrac{\dfrac{A \wedge B}{A}\,{\scriptstyle \wedge\text{-elim}} \qquad A \to C}{C}\,{\scriptstyle \to\text{-elim}} \qquad \dfrac{A \wedge B}{B}\,{\scriptstyle \wedge\text{-elim}}}{C \wedge B}\,{\scriptstyle \wedge\text{-int}}$$

Example 3:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \wedge B}{B}\,{\scriptstyle \wedge\text{-elim}} \qquad B \to C}{C}\,{\scriptstyle \to\text{-elim}} \qquad C \to D}{D}\,{\scriptstyle \to\text{-elim}}}{D \vee E}\,{\scriptstyle \vee\text{-int}}}{(A \wedge B) \to (D \vee E)}\,{\scriptstyle \to\text{-int}}$$

Example 4:

$$\dfrac{\dfrac{P \qquad P \to \bot}{\bot}\,{\scriptstyle \to\text{-elim}}}{Q}$$

(here I can infer anything I want)

Example 5:

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\,\wedge\text{-int} \qquad (A \wedge B) \to C}{C}\,\to\text{-elim}$$

Example 6:

NB – it is ok that A appears twice here!

$$\dfrac{\dfrac{A \qquad A \to (B \to C)}{B \to C}\,\to\text{-elim} \qquad \dfrac{A \qquad A \to B}{B}\,\to\text{-elim}}{C}\,\to\text{-elim}$$

Example 7:

$$\dfrac{\dfrac{\dfrac{A \wedge B}{A}\,\wedge\text{-elim} \qquad A \to (B \to C)}{B \to C}\,\to\text{-elim} \qquad \dfrac{A \wedge B}{B}\,\wedge\text{-elim}}{C}\,\to\text{-elim}$$

Example 8:

$$\dfrac{\dfrac{[A \vee (A \to \bot)) \to \bot]^2 \qquad \dfrac{\dfrac{\dfrac{[A \vee (A \to \bot)) \to \bot]^2 \qquad \dfrac{[A]^1}{A \vee (A \to \bot)}}{\bot}}{A \to \bot}}{A \vee (A \to \bot)}}{\bot}}{A \vee (A \to \bot)}$$

-Try this one **$(A \to (B \vee C)) \to ((A \to B) \vee (A \to C))$** (hint: must be reductio)

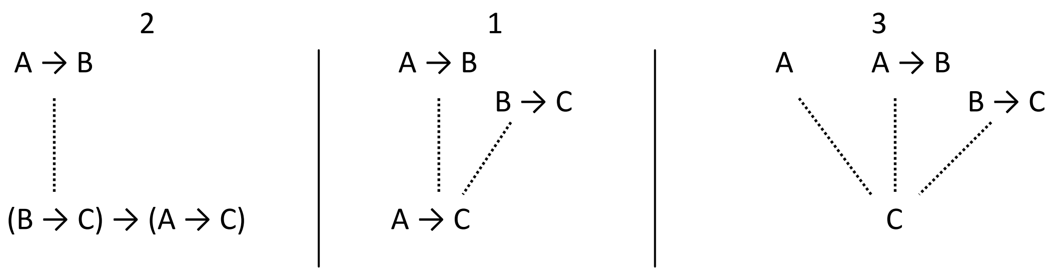
Other methods of presentation

Framing

-used for presenting proofs

Example Prove $(A \to B) \to ((B \to C) \to (A \to C))^3$

to prove this let's assume $A \to B$
In this example we will use framing:



The frame numbers match up to the flag numbers when we solve the formula by natural deduction

$\underline{[A]^1 \; [A \to B]^3}$

$\underline{B \qquad\qquad\qquad [B \to C]^2}$

$\underline{C}$

$\underline{A \to C}$

$\underline{(B \to C) \to (A \to C)}$

$(A \to B) \to ((B \to C) \to (A \to C))$

Disjunctive normal forms

-used for presenting formula

Something is in disjunctive normal form if it is of the form:

( ) ∨ ( ) ∨ ( )

where the brackets contain only ∧s and/or ¬s

e.g. $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p) = (p \wedge q) \vee (\neg p \wedge \neg q)$

Disjunctive normal form corresponds to rows in truth tables, like so:

p	↔	q
I	I	I
I	0	0
0	0	I
0	I	0

Any formula at all can be put into disjunctive normal forms (Similarly for conjunctive normal form[4]):

$A \wedge (B \vee C)$	=	$(A \wedge B) \vee (A \wedge C)$
$\neg(A \wedge B)$	=	$\neg A \wedge \neg B$

Parse trees

- used for presenting formula

Example Peirce's law $((A \rightarrow B) \rightarrow A) \rightarrow A$ would be presented as such:

```
            →
           /  \
         →      A
        /  \
      →      A
     /  \
    A    B
```

4 This is similar to DNF except the ∨s are inside the brackets and the ∧s are outside the brackets.

Predicate logic

With the following Lewis Carroll riddle:

Barry takes salt	**P**	**Barry takes mustard**	**U**
Mill takes salt	**Q**	**Mill takes mustard**	**V**
Cole takes salt	**R**	**Cole takes mustard**	**W**
Lang takes salt	**S**	**Lang takes mustard**	**X**
Dix takes salt	**T**	**Dix takes mustard**	**Y**

If we try to express it as

$$P \leftrightarrow ((\neg R \lor \neg W) \lor (\neg S \lor \neg X))$$

the letters do not properly denote the associations

To solve problems like these it is easier to use predicate logic, also known as 1^{st} order logic. Predicate logic denotes properties e.g.

green	**is bigger than**	**between**
unary	*binary*	*ternary*

We express predicate logic as such:

Romeo loves Juliet

$$L\,(r, j)$$

Relation symbol (always a capital)

Romeo

Juliet

Relatives (lower case)

We cannot write predicate logic as say rLj because if we have more than 2 relatives where would the relation symbol go?

So now the Lewis Carroll riddle becomes:

Barry	b	T (b, s)	T (b, u)
Mill	m	T (m, s)	T (m, u)
Cole	c	T (c, s)	T (c, u)
Lang	l	T (l, s)	T (l, u)
Dix	d	T (d, s)	T (d, u)
salt	s		
mustard	u		
takes	T		

now we have sufficient internal structure in the sentences to explain the relation between sentences

Examples of turning language into predicate logic

Romeo loves Juliet and Juliet loves Romeo:

$L(r, j) \wedge L(j, r)$

this is much more informative than $p \wedge q$!
-it captures the fact that the two statements have common grounds

Balbus loves Julia and Julia does not love Balbus, what a pity:

$L(b, j) \wedge \neg L(j, b) \wedge p$

Fido sits on the sofa, Herbert sits on the chair:

$S(f, s) \wedge S(h, c)$

Fido sits on Herbert:

$S(f, h)$

The chair sits on Herbert:

$S(c, h)$

Here we need a more complicated formula to explain that $S(c, h)$ is incorrect while $S(f, h)$ is fine!

Alfred drinks more whiskey than Herbert. Herbert drinks more whiskey than Mary:

It is not necessary here to have a symbol for whiskey and a symbol for drinks more than because the only drinking happening is drinking of whiskey.

$D(a, h) \wedge D(h, m)$

If we have
Alfred drinks more whiskey than Herbert. Herbert drinks more whiskey than Mary. Mary drinks more fruit juice than Alfred. we would need separate symbols for 'drinks more' 'whiskey' and 'fruit juice'.

Variables and quantifiers

Variables

Up until now our examples of predicate logic have only used constants such as Juliet, Romeo, Herbert etc. We can also express predicate logic using variables, which are traditionally denoted by letters w, x, y and z.

So instead of **Romeo loves Juliet:** L (r, j) we can have **x loves Juliet:** L(x, j)
where x is a variable

Quantifiers

Using quantifiers we can acquire more information about the nature of x. The two quantifiers we will look at are:

$\exists$	and	$\forall$
Existential quantifier 'there exists an'		Universal quantifier 'all'

So instead of…	**x loves Juliet:**	L(x,j)
we can now say that…	**there is an such that x loves Juliet:**	($\exists$x)(L(x,j))
and we can now say that…	**all xs love Juliet:**	($\forall$x)(L(x,j)

We say that quantifiers **BIND** variables.
A variable that is not bound by a quantifier is said to be **FREE**.

A quantifier binds every given variable to the right of it as illustrated below.

$\exists$ binds these xs
$\forall$ binds these ys

$$(\exists x)\overbrace{(\forall y)\overbrace{(L(x,y)}}$$

Examples of turning language into predicate logic (with variables and quantifiers)

1. **Socrates is a man**
<u>All men are mortal</u>
Socrates is mortal

becomes:

Socrates	s
is a man	H
Is mortal	M

$$\begin{array}{l} H(s) \\ \underline{(\forall x)(H(x) \rightarrow M(x))} \\ M(s) \end{array}$$

2. **No fossil can be crossed in love.**
<u>An oyster may be crossed in love.</u>
Oysters are not fossils.

becomes:

is a fossil	F
is an oyster	O
Can be crossed in love	C

$$\begin{array}{l} (\forall x)(F(x) \rightarrow \neg C(x)) \\ \underline{(\forall x)(O(x) \rightarrow C(x))} \\ (\forall x)(\neg F(x) \wedge O(x)) \end{array}$$

Think of this as if you were an oyster – you can be crossed in love so ANY oyster may be crossed in love.

3. **All lions are fierce.**
<u>Some lions do not drink coffee.</u>
Some creatures that drink coffee are not fierce.

becomes:

is a lion	L
drinks coffee	D
Is fierce	F

$$\begin{array}{l} (\forall x)(L(x) \rightarrow C(x)) \\ \underline{(\exists x)(L(x) \wedge \neg D(x))} \\ (\exists x)(D(x) \wedge \neg F(x)) \end{array}$$

4. **No muffins are wholesome.**
<u>All puffy food is unwholesome.</u>
All muffins are puffy.

becomes:

is a muffin	M
is puffy	P
is wholesome	W

$$\begin{array}{ll} (\forall x)(M(x) \rightarrow \neg W(x)) & \textit{or } \neg(\exists x)(M(x) \wedge W(x)) \\ \underline{(\forall x)(P(x) \rightarrow \neg W(x))} & \\ (\forall x)(M(x) \rightarrow P(x)) & \end{array}$$

note that this is not actually a valid argument

5.	**Babies are illogical.**	*becomes:*	is a Baby	B
	Nobody is despised who		is illogical	I
	can manage a crocodile.		is despised	D
	Illogical persons are despised.		can manage a crocodile	M

$(\forall x)(B(x) \rightarrow I(x))$
$\neg(\exists x)(D(x) \wedge M(x))$ *or* $(\forall x)(M(x) \rightarrow \neg D(x))$
$(\forall x)(I(x) \rightarrow D(x))$

NB - In this example we could actually think of 'manage a crocodile' as a two place relation:

Instead of $M(x)$ have $M(x,y)$ and $C(x)$

x can manage y is a crocodile

6.	**My saucepans are the only things**	*becomes:*	is made of tin	T
	that I have that are made of tin.		is a saucepan	S
	I find all your presents very useful.		is useful	I
	<u>None of my saucepans are of any use.</u>		is a present	D
	Your presents are not made of tin.		(from you to me)	

$(\forall x)(T(x) \rightarrow S(x))$
$(\forall x)(P(x) \rightarrow U(x))$
$\underline{\neg(\exists x)(S(x) \wedge U(x))}$ *or* $(\forall x)(S(x) \rightarrow \neg U(x))$
$\neg(\exists x)(P(x) \wedge T(x))$

7.	**No potatoes of mine that are new**	*becomes:*	is new	N
	have been boiled.		is fit to eat	F
	All of my potatoes in this dish are fit to eat.		is in this dish	I
	No unboiled potatoes of mine are fit to eat .		has been boiled	B

NB – x ranges over all potatoes which are mine

$(\forall x)(N(x) \rightarrow \neg B(x))$
$(\forall x)(D(x) \rightarrow F(x))$
$(\forall x)(\neg B(x) \rightarrow \neg F(x))$ *or* $\neg(\exists x))\neg B(x) \wedge F(x))$ *or* $(\forall x)(F(x) \rightarrow B(x))$

8. **No ducks can waltz.** *becomes:*
No officers ever decline to waltz.
<u>All my poultry are ducks.</u>
My poultry are not officers.

is a duck	T
waltzes	W
is an officer	O
is my poultry	P

$(\forall x)(T(x) \rightarrow S(x))$
$(\forall x)(P(x) \rightarrow U(x))$
$\underline{\neg(\exists x)(S(x) \wedge U(x))}$ *or* $(\forall x)(S(x) \rightarrow \neg U(x))$
$\neg(\exists x)(P(x) \wedge T(x))$

9. **Everyone who is sane can do logic.** *becomes:*
No lunatics are fit to serve on a jury.
<u>None of your sons can do logic.</u>
None of your sons are fit to serve on a jury

is sane	S
can do logic	L
is your son	O
is fit to serve on a jury	J

$(\forall x)(S(x) \rightarrow L(x))$
$(\forall x)(\neg S(x) \rightarrow J(x))$
$\underline{(\forall x)(O(x) \rightarrow \neg L(x))}$ *or* $(\forall x)(J(x) \rightarrow S(x))$
$(\forall x)(O(x) \rightarrow \neg J(x))$

NB – this alternative formula is its CONTRAPOSITIVE, that is, it is of the form:

$A \rightarrow B = \neg B \rightarrow \neg A$

10. **There are no pencils of mine In this box.** *becomes:*
No sugar plums of mine are cigars.
The whole of my property that is
<u>Not in this box consists of cigars.</u>
No pencils of mine are sugar plums

is in this box	B
is a sugar plum	S
is a cigar	C
is a pencil	P

$(\forall x)(P(x) \rightarrow \neg B(x))$ *or* $(\forall x)(B(x) \rightarrow \neg P(x))$
$(\forall x)(S(x) \rightarrow \neg C(x))$ *or* $\neg(\exists x)(S(x) \wedge C(x))$
$\underline{(\forall x)(\neg B(x) \rightarrow C(x))}$ *or* $(\forall x)(B(x) \vee C(x))$
$(\forall x)(P(x) \rightarrow \neg S(x))$ *or* $\neg(\exists x)(P(x) \wedge S(x))$

NB – here it is tempting to say that the V should be an XOR. But this is only because we can intuitively infer it. The language does not actually imply that it is the case.

All formulas so far have had one quantifier. Consider formulae with two quantifiers…

Examples with more than one quantifier

George is the best student:	George	g
	is a student	S
$(\forall x)(S(x) \rightarrow B(g,x))$	is better than	B

Everyone loves someone:

$\forall x \exists y\ L(x,y)$

There is something such that everyone loves it:

$\exists y \forall x\ L(x,y)$

these two may be be very similar but they are NOT! the same

There is no best student (for any x, if it is a student there is somebody it isn't better than):

$(\forall x)(S(x) \rightarrow (\exists y)(\neg B(x,y)))$

I know a pop star.
I know two pop stars.

$(\exists x)(P(x) \wedge K(x))$
$(\exists x)(\exists y)((P(x) \wedge P(y) \wedge K(x) \wedge K(y)) \wedge \neg (x=y))$

There is always someone worse off than you

$(\forall y)(\exists x)(W(x,y))$

If anyone can do it, Jon can.
If Jon can do it, anyone can.

$(\forall x)(D(x) \rightarrow D(j))$
$(\forall x)(D(j) \rightarrow D(x))$

NB – sometimes you will be asked to put formulae into Prenex normal form. This means simply quantifiers must be pulled to the front e.g. I know 3 pop stars:

$(\exists x)(\exists y)(\exists z)(x \neq y \wedge y \neq z \wedge x \neq z)$

Everybody loves a lover:

$(\forall x)(\forall y)((\exists z)(L(y,z)) \rightarrow L(x,y))$ is not in PNF, $(\forall x)(\forall y)(\forall z)(L(y,z) \rightarrow L(x,y))$ is!

Syllogistic validity and relations

$(\forall x)(x=y)$
$\underline{(\exists z)(z=x)}$

The calculation follows from the structure of the argument – not in virtue of the content

Socrates is a man		**All cows are mad**
<u>All men are mortal</u>	*or*	**<u>Daisy is a cow</u>**
Socrates is mortal		**Daisy is mad**

BUT consider this example:

George is taller than Bill

<u>Bill is taller than Harry</u>

George is taller than Harry

Consider this with a different relation; say 'is the first cousin of'. It doesn't work. So this form of argument may be <u>truth preserving</u> but it is not <u>valid</u>.

In propositional logic we can assess the argument by devising truth tables. We need the equivalent tool for predicate logic.

'is better than' has a special quality – it is TRANSITIVE

We say R is transitive if:

$(\forall x)(\forall y)(\forall z)((R(x,y) \land (R\ (y,z)) \rightarrow R(x,z)$

We say a relation R is reflexive if it relates everything to itself:

$(\forall x)(R(x,x))$ $\quad$ $(x=y)(\forall R)((\forall X)(R(x,x)) \rightarrow R(x,y))$

We say R is symmetrical if the converse relation is the same:

$(\forall xy)(R(x,y) \rightarrow R(y,x))$ e.g. siblings

Converse relations:

If x is related to y by R, then y is related to x by S, the converse of R:

$R(x,y)$ x is the parent of y and $S(y,x)$ y is the child of x

Is taller than	T	$T(g,b)$	this may be valid but it doesn't give us enough information to determine whether T(g, b) could be true while T(b,g) is true.
George	g	$\underline{T(b,h)}$	
Henry	h		
Bill	b	$T(g,h)$	

$(\forall x,y,z)(T(x,y) \land T(y,z) \rightarrow T(x,z))$ - this does give us all the information about 'taller than'

We say R is in Equivalence Relation if R is transitive, reflexive and symmetrical.

R is irreflexive if:

$(\forall x,y,z)(R(x,y) \rightarrow x \neq y)$ e.g. 'is bigger than' 'eats'

this is not the same as (¬ reflexive) because a relation can be not reflexive if only one thing is not related back to it, whereas irreflexive says all things are not related back.

R is asymmetrical if:

$(\forall x,y)(R(x,y) \rightarrow \neg R(y,x))$

again this is not the same as (¬ symmetric).

If we have $(\forall x,y)(R(x,y) \rightarrow R(y,x))$

it is the same as $(\forall x,y)(R(x,y) \leftrightarrow R(y,x))$

Formula, meaning and tautology

$\neg F(x)$
$\exists x\, \neg F(x)$
$\neg\exists x\, \neg F(x) \quad = \quad \forall x\, F(x)$

$\neg F(x)$
$\forall x\, \neg F(x)$
$\neg\forall x\, \neg F(x) \quad = \quad \exists x\, F(x)$

de Morgan Laws
$\neg(\neg A \vee \neg B \vee \neg C) = A \wedge B \wedge C$
$\neg(\neg A \wedge \neg B) = A \vee B$

$\exists$ is a kind of $\vee$
$\forall$ is a kind of $\wedge$

$(\neg(\forall x)\, \neg F(x)) \leftrightarrow \exists x\, F(x)$	TAUTOLOGOUS
$(\forall x)(F(x) \vee \neg F(x))$	TAUTOLOGOUS
$(\forall x)(F(x)) \vee \forall x\, (\neg F(x))$	NOT TAUTOLOGOUS
$(\exists x)(F(x) \vee \neg F(x))$	this is true as long as there is a thing in the Universe. Does this make it logically true? Our school of thought says it is logically true. The others say that it is NOT a logically true statement
of the form $P \vee \neg P$	that the Universe is not empty.

Therefore it is better to accept that it is logically true.
As such,
$\exists x\, F(x) \vee \exists x\, \neg F(x)$ is logically true too.

$(\exists y)(\forall x)(L(x,y))$
There is a y such that for all x, x loves y or there is somebody that everyone loves

$(\forall x)(\exists y)(L(x,y))$
For all x there is a y such that x loves y or everyone loves someone

therefore $(\exists y)(\forall x)(L(x,y)) \quad \neq \quad (\forall x)(\exists y)(L(x,y))$

$(\exists y)(\forall x)(L(y,x))$
There is a y such that for all x, y loves x or everyone loves someone

$(\forall x)(\exists y)(L(y,x))$
For all x there is a y such that y loves x or is loved by someone

If a is a person such that
$(\forall x)(L(x,a))$
'a' is witness to the '$\exists y$' if $(\exists y)(\forall x)(L(y,x))$

$-(\exists y)(\forall x)(L(y,x)) \rightarrow (\forall x)(\exists y)(L(y,x))$ is logically true
$-(\exists y)(\forall x)(L(x,y)) \rightarrow (\forall x)(\exists y)(L(x,y))$ is logically true

1. $(\exists x)(\forall y)(F(y) \rightarrow F(x))$ there is an x such that for all y, if y is an F then x is an F
2. $(\exists x)((\exists y)(F(y) \rightarrow F(x)))$ there is an x such that if there is an F, a will be an F

This must be logically true. There is a thing that if there are any frogs, it will be a frog. If there are no frogs, anything can be a witness to the second. If there are frogs then select one of them to be a witness. 1. says the same as 2. and therefore is also a logical truth.

$$\frac{(\exists x)F(x) \qquad (\forall y)(F(y) \rightarrow G(x))}{(\exists x)G(x)}$$

this is a valid argument

'is better than' example (reliant on definition)
Only tautologous when $(\forall xyz)$ x taller then y
Y taller than x etc

$$\frac{x=y \qquad y=z}{x=z}$$

this is tautologous

= has a fixed meaning. It is treated as part of the logical vocabulary.

$(\forall x)(\forall y)(F(x) \wedge F(y)) \rightarrow x=y$
If I have two things which are F they are the same thing
i.e. there exists more than one F

To say, there is precisely one F:

$(\exists x)(F(x) \wedge (\forall y)F(y) \rightarrow y=x))$ which is abbreviated to $\exists!xF(x)$

$(\exists!x)(F(x) \wedge G(x))$

The present King of France is bald (B. Russell)
$(\exists x)(F(x) \wedge (\forall y)(F(y) \rightarrow y=x) \wedge G(x))$

F(x) x is the King of France
G(x) x is bald

This is simply false because there is no King of France.

Quantifier rules

∃ introduction:

$$\frac{F(a)}{\exists x F(x)}\ {\scriptstyle \exists\text{-int}}$$

∃ elimination:

$$\frac{\exists x F(x) \qquad \begin{array}{c} F(a) \\ \vdots \\ P \end{array}}{P}\ {\scriptstyle \exists\text{-int}}$$

∀ introduction:

a is arbitrary

$$\frac{\dfrac{\begin{array}{c} [P(a)] \\ \vdots \\ O(a) \end{array}}{P(a) \rightarrow O(a)}}{(\forall x)(P(x) \rightarrow O(x))}\ {\scriptstyle \forall\text{-int}}$$

∀ elimination:

$$\frac{\forall x\, F(x)}{F(a)}\ {\scriptstyle \forall\text{-elim}}$$

Language/metalanguage distinction

$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

This symbol is a turnstile. It means 'there exists a proof'.
$(A\rightarrow (B \rightarrow C)) \;\vdash\; (A \rightarrow B) \rightarrow (A \rightarrow C)$
This means there is a proof of $(A \rightarrow B) \rightarrow (A \rightarrow C)$ from $(A\rightarrow (B \rightarrow C))$

A and B etc. talk about the world – they are part of the language we are using
Γ and $\vdash$ etc. are part of a metalanguage which talks about that language.

It is important to note that $A \rightarrow B$ is not the same as $A \vdash B$. The first is the language of propositional logic and the second is its metalanguage.

As such, A and B etc. should be in a different font to show they are not symbols but symbols of symbols.

A worked example

$\vdash A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$

implies a tautology

$$\underline{B \vdash B}^{13} \qquad\qquad\qquad \underline{C \vdash C}^{14}$$
$$\underline{A \vdash A}^{12} \qquad \underline{B \wedge C \vdash B}^{11} \qquad\qquad \underline{B \wedge C \vdash C}^{10} \qquad \underline{A \vdash A}^{9}$$
$$\underline{A \vdash A \vee B}^{7} \quad \underline{B \wedge C \vdash A \vee B}^{8} \qquad \underline{B \wedge C \vdash A \vee C^{5} \qquad A \vdash A \vee C}^{6}$$
$$\underline{A \vee (B \wedge C) \vdash A \vee B}^{3} \qquad \underline{A \vee (B \wedge C) \vdash A \vee C}^{4}$$
$$\underline{A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)}^{2}$$
$$\vdash A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)^{1}$$

line 1 follows from line 2 because of the rule of $\rightarrow$-int
line 2 follows from line 3 and 4 because of the rule of $\wedge$-int
line 4 follows from line 5 and 6 because of the rule of $\vee$-elim
line 3 follows from line 7 and 8 because of the rule of $\vee$-elim
line 6 follows from line 9 because of the rule of $\vee$-int
line 5 follows from line 10 because of the rule of $\vee$-int
line 8 follows from line 11 because of the rule of $\vee$-int
line 7 follows from line 12 because of the rule of $\vee$-int
line 11 follows from line 13 because of the rule of $\wedge$-elim
line 10 follows from line 14 because of the rule of $\wedge$-elim

A sequent proof like this is the project log of the search for a proof (tree/natural deduction proof) of the formula to the right of the turnstile.

- Try this one $\qquad \vdash \mathbf{A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)}$

Metalanguage rules

L Elimination	R Introduction	
∧-int $\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$	∧-elim $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \vee B}$	$\Gamma \vdash A$ – there is a proof of A using assumptions in Γ
V-int $\dfrac{\Gamma, A \vdash B \quad \Gamma C \vdash B}{\Gamma, A \vee C \vdash B}$	V-elim $\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$	Sequent – Formula$_1$ $\vdash$ Formula$_2$
→-elim $\dfrac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}$	→-int $\dfrac{\Gamma, A \vdash B}{\Gamma\ A \to B}$	

Examples using these rules

$$\frac{\Gamma \vdash \Phi(a)}{\Gamma \vdash \exists x\, \Phi(x)} \qquad \exists\text{-R}$$

$$\frac{\Gamma, \Phi(a)}{\Gamma, \exists x\, \Phi(x) \vdash B} \qquad \exists\text{-L}$$

$$\frac{\Gamma \vdash \Phi(a)}{\Gamma \vdash \forall x\, \Phi(x)} \qquad \forall\text{-R}$$

$$\frac{\Gamma, \Phi(a) \vdash B}{\Gamma, \forall x\, \Phi(x) \vdash B} \qquad \forall\text{-L}$$

$$\left.\frac{\Gamma \vdash \Phi(x)}{\Gamma \vdash \forall x\, \Phi(x)}\right\} \quad \text{as long as x is not in } \Gamma$$

Now relax the rules to allow multiple formulae on the right or possibly no formulae at all! –

$\Gamma \vdash A, B$ or $\Gamma \vdash$

$\Gamma \vdash \Delta$ now means if everything in Γ is true, at least one of the things in Δ is true.

$\Gamma \vdash$ (nothing here means Γ is FALSE)

The false is true.
The disjunction of the empty set of formulae is the false.

Worked example 1.

Using the above rules prove $\exists y \forall x Lxy \vdash \forall x \exists y Lxy$:

$$\begin{array}{lcll}
Lba & \vdash & L(ba) & _{\forall\text{-L}} \\ \hline
\forall x Lxa & \vdash & L(ba) & _{\exists\text{-R}} \\ \hline
\forall x Lxa & \vdash & \exists y L(by) & _{\exists\text{-L}} \\ \hline
\exists y \forall x Lxy & \vdash & \exists y L(by) & _{\forall\text{-R}} \\ \hline
\exists y \forall x Lxy & \vdash & \forall x \exists y Lxy &
\end{array}$$

Notice how we have managed to prove a fact about b from something which didn't even mention b!

This is obvious: *If everybody loves Alfie then obviously Bert loves Alfie.*
If Bert loves Alfie then Bert loves somebody.

Try proving this the other way around:

$$\begin{array}{lcl}
 & \mathbf{X} & \\
L(ba) & \vdash & L(ca) \\ \hline
\exists y Lby & \vdash & \forall x Lxa \\ \hline
\forall x \exists y Lxy & \vdash & \forall x Lxa \\ \hline
\forall x \exists y Lxy & \vdash & \exists y \forall x Lxy
\end{array}$$

It doesn't work!

Worked example 2.

$$\frac{X \vdash Y \quad Z \vdash W}{X, Y \to Z \vdash W}$$

Use rule:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ {\to}\text{-R}$$

BECAUSE OF →- int

$$\frac{P(a) \vdash P(a)}{P(a) \vdash \exists x)(P(x)}\ \exists\text{-R}$$

$$\frac{P(a) \vdash \exists x)(P(x) \qquad A \vdash A}{(\exists x)(P(x)) \to A, P(a) \vdash A}\ {\to}\text{-L}$$

$$\frac{(\exists x)(P(x)) \to A, P(a) \vdash A}{(\exists xP(x)) \to A \vdash P(a) \to A}\ {\to}\text{-R}$$

$$\frac{(\exists xP(x)) \to A \vdash P(a) \to A}{(\exists xP(x)) \to A \vdash (\forall x)(P(x) \to A)}\ \forall\text{-R}$$

Check that these special side conditions are met! No 'a' on the LHS

Worked example 3.

Use rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\text{-int}$$

$$\frac{\dfrac{\Gamma \vdash A, B}{\Gamma \vdash A \vee B, B}}{\underbrace{\Gamma \vdash A \vee B, A \vee B}}$$

occurrence of the same formula twice

NB – this justifies $\dfrac{\Gamma \vdash A, B}{\Gamma \vdash A \vee B}$

We looked at this formula **(A → (B V C)) → ((A → B) V (A → C))** when we learned natural deduction and it could only be solved with reductio. We can now solve it with a different method, using what we have just learned:

$$\frac{B, A \vdash B, C}{B \vee C, A \vdash B, C}\ \vee\text{-L}$$

$$\frac{A \vdash A \qquad B \vee C, A \vdash B, C}{A \to (B \vee C), A \vdash B, C}\ {\to}\text{-L}$$

$$\frac{A \to (B \vee C), A \vdash B, C}{A \to (B \vee C), A \vdash B, A \to C}\ {\to}\text{-R}$$

$$\frac{A \to (B \vee C), A \vdash B, A \to C}{A \to (B \vee C) \vdash A \to B, A \to C}\ {\to}\text{-R}$$

$$\frac{A \to (B \vee C) \vdash A \to B, A \to C}{A \to (B \vee C) \vdash (A \to B) \vee (A \to C)}\ \vee\text{-R (new style)}$$

$$\frac{A \to (B \vee C) \vdash (A \to B) \vee (A \to C)}{\vdash (A \to (B \vee C)) \to ((A \to B) \vee (A \to C))}\ {\to}\text{-R}$$

Glossary

Alphabetic variants – formulae that differ only in the letters/words used

Antecedent – that which comes before the conditional

Conclusion – a proposition which has been logically deduced

Conditional – a connective which denotes a relationship between variables of the form 'if then'. It is indicated by →

Connective – denotes the relationship held by variables. They, like variables, are truth functional

Consequent – that which comes after the conditional

Equivocation – using a word in more than one sense e.g. something can be light in colour or light in weight

Extension – the synthetic truth value of a proposition

Fallacy - an invalid argument

Heterological - a word is said to be heterological if it doesn't describe itself e.g. 'long', 'German'

Inference – the act of getting the conclusions from the premises

Intension – the analytic truth value of a proposition

Law of excluded middle – holds that P V ¬P must be logically true by its form.

Modus ponens – a valid argument of the form if A then B, A, therefore B.

Modus tollens – a valid argument of the form if A then B, ¬B, therefore ¬A.

Parse tree – a method of displaying formula in propositional logic

Premises – propositions which can logically support a conclusion

Prinicple connective – the connective with the largest range in any propositional formula

Proposition – an informative statement

Propositional logic – a branch of logic which formalises natural language into symbols which represent variables and their connectives

Semantics – the meaning of a proposition

Sound – a sound argument is one which is valid and has premises which are true of their intension.

Syllogism – a three part argument with 2 premises and 1 conclusion of the form all x is y, all y is z, therefore all x is z

Tautology – a formula which is true for all possible truth value assignments of the propositions which make it up

Token – a physical example of an abstract concept e.g. that book on my shelf is a token of Pride and Prejudice

Truth functional – logically dependent on truth value

Truth value

Type – an abstract concept e.g. Pride and Prejudice

Valid – a valid type of argument is one such that all tokens of it must have a true conclusion, when all premises are true.