Notes of Countable Ordinals Reading Group meeting on 16/v/2014

Thomas Forster

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Under the guidance of Jeroen van der Meeren and Michael Rathjen I finally *began* to get the first glimmers of an understanding of the use of a large ordinal in describing initial segments of the countable ordinals. What follows is my notes of the discussion of this topic at the meeting of the ordinals reading group on 16/v. Present were: your humble correspondent, Professors Leader and Dawar, Arno Pauly, Philipp Kleppmann and an unidentified Ph.D. student from the Lab. We put our heads together and made some progress, and this file records my understanding of that progress.

Key word lurking in the background is **impredicativity**.

The following gadgetry goes back to Bachmann. [need a ref]

It's probably a good idea for the reader to start off by keeping in mind the Veblen picture of rows and rows of ordinals. The top row consists of powers of ω, written in increasing order left-to-right. Going down the page, each subsequent successor row consists of the fixed points in the enumeration of the row immediately above it; at limit stages the row is the intersection of all the rows above it. We assume that the reader is familiar with this picture.

For ordinals α, ζ we define a set $C(\alpha, \zeta)$ of ordinals and a function $\vartheta : On \to On$, by a *simultaneous* recursion on On^2. The thing we are really interested in is the function ϑ; the $C(\alpha, \zeta)$ are mere scaffolding, and they play no rôle in the system of notations with which the ϑ gadgetry will eventually furnish us.

To construct $C(\alpha, \zeta)$ you start with a set containing 0 and Ω, all the ordinals less than ζ, and $\vartheta(\gamma)$ for all $\gamma < \alpha$; you then close under $+$ and $\alpha \mapsto \omega^\alpha$. Our first stab at the definition of $\vartheta(\alpha)$ is: the least ζ such that $\zeta \notin C(\alpha, \zeta)$. Bear in mind that $\vartheta(\alpha)$ is **not** defined as the least thing not in $C(\alpha, \zeta)$. For one thing, it would need *two* arguments—$\vartheta(\alpha, \zeta)$—not one. It's a complex diagonalisation and you need to read the definition carefully. Bind the 'ζ' somehow, and "the least ζ such that $\zeta \notin C(\alpha, \zeta)$" sounds sensible. **However** we add a clause so that $\vartheta(\alpha)$ is not the first ζ s.t. $\zeta \notin C(\alpha, \zeta)$ but rather the first ζ s.t. $\zeta \notin C(\alpha, \zeta) \wedge \alpha \in C(\alpha, \zeta)$. ☠☠ It will become clear later what purpose is served by this extra $\alpha \in C(\alpha, \zeta)$ clause, but you should not expect it to be clear at this stage.

Here is something that threw me and it might throw you. It's pretty clear that the function $\alpha, \zeta \mapsto C(\alpha, \zeta)$ is $\subseteq$-increasing in both arguments, but you mustn't jump to the conclusion that ϑ is strictly increasing—it isn't, as we shall see. The best way to understand what is going on is to fix a small α and consider $C(\alpha, 0)$, $C(\alpha, 1)$ and so on, so let's do some of these by hand to calm our nerves. We will see that the first few values of ϑ are the first few ϵ-numbers.

$C(0, 0)$ contains 0 and Ω. We don't have to put any values of ϑ into it co's the first argument is 0. We then close under addition and $\beta \mapsto \omega^\beta$. Pretty clearly it is going to contain everything less than ϵ_0. It *won't* contain ϵ_0 itself (how could it, after all?) but it does contain a lot of stuff beyond Ω. We will see later [*much* later] what that stuff does. For the moment it does nothing.

What about $C(0, 1)$? It's just going to be the same set. $C(0, \omega)$ is going to be the same set, too. Observe that if $\zeta < \epsilon_0$ then $\zeta \in C(0, \zeta)$, so *all* the $C(0, \zeta)$ are going to be the same set all the way through all the ordinals less than ϵ_0. Indeed even $C(0, \epsilon_0)$ is the same (tho' $C(0, \epsilon_0 + 1)$ is bigger).

The first ζ such that $\zeta \notin C(0, \zeta)$ is therefore ϵ_0. The second (☠) condition on candidates for $\vartheta(\alpha)$ (the condition that requires that $\alpha \in C(\alpha, \zeta)$) is satisfied—all it requires in this case is that $0 \in C(0, 0)$—so we conclude that $\vartheta(0)$ is ϵ_0.

Notice that there is never any need for us to compute $C(0, \zeta)$ for any $\zeta > \vartheta(0)$; since the only purpose served by the $C(\alpha, \zeta)$ is to enable us to calculate $\vartheta(\alpha)$, once that is done we lose interest.

How about $C(1, 0)$? It's like $C(0, 0)$ except that we put $\vartheta(0)$ (which is ϵ_0) into it before closing under the operations. This means that we get everything less than ϵ_1 (think: Cantor Normal Forms for ordinals $< \epsilon_1$). As we run through the $\zeta < \epsilon_1$ we get nothing new in $C(1, \zeta)$ until we reach ϵ_1 itself, so we conclude that $\vartheta(1) = \epsilon_1$. As before, the (☠) condition on ζ does nothing beco's all it requires is that $C(1, \zeta)$ should contain 1, and we already know it contains everything below ϵ_1.

Similarly we conclude that $\vartheta(n) = \epsilon_n$ for $n < \omega$. A picture emerges in which, for small arguments, ϑ enumerates the ϵ numbers. In fact Jeroen tells me that ϑ is injective and all its values are ϵ-numbers.

Fixed point ϵ numbers are sometimes called *κ-numbers*, so that κ_0 is the least solution to $\kappa = \epsilon_\kappa$. Let us think a bit about what $\vartheta(\kappa_0)$ might be. We start with $C(\kappa_0, 0)$. This set contains Ω and all the ϵ-numbers below κ_0, and is closed under $+$ and $\zeta \mapsto \omega^\zeta$. Now, recalling what we know about Cantor Normal Forms, we can see that this act of closure will put into $C(\kappa_0, 0)$ every ordinal below κ_0 (plus a lot of big rubbish beyond Ω). This immediately tells us that the sets $C(\kappa_0, \zeta)$ for $\zeta \leq \kappa_0$ are all going to be the same set as $C(\kappa_0, 0)$. We observe that $\kappa_0 \notin C(\kappa_0, \kappa_0)$ so we might expect that we then declare $\vartheta(\kappa_0)$ to be κ_0. However note that κ_0 is not only the *second* argument at this stage, but also the *first*, so we look at the ☠ condition—"$\alpha \in C(\alpha, \zeta)$"—and we see that it is not satisfied! So we have to look at a few more $C(\kappa_0, \zeta)$ before we can say we have reached $\vartheta(\kappa_0)$. In fact we have to go as far as $C(\kappa_0, \epsilon_{\kappa_0+1})$.

The picture i now have is that, for $\alpha < \Omega$, ϑ enumerates the ϵ-numbers less than Ω—except that it misses out the fixed points (that is what the ☠ condition is doing). Another way of putting this is that it enumerates those ordinals in the first row that do not appear in the second row; *yet* another way of putting it is to say that the purpose of the ☠ clause is to prevent ϑ from having fixed points.

That was what one might call the *first pass*. I am assured by Jeroen that $\vartheta(\Omega)$ is the first fixed-point ϵ-number (the first κ-number)—aka $\phi(2, 0)$—and that $\vartheta(\Omega + 1)$ is the second fixed-point ϵ-number.

OK, so: thus emboldened, let us check these allegation for ourselves and start by thinking about what $\vartheta(\Omega)$ might be. We obtain $C(\Omega, \zeta)$ by starting with $\{\vartheta(\alpha) : \alpha < \Omega\}$ and all the ordinals less than ζ and closing under $\beta \mapsto \omega^\beta$ and $+$. If i was right earlier, then we have all the ϵ numbers less than the first κ number. So $C(\Omega, \kappa_0)$ contains Ω but does not contain κ_0 so $\vartheta(\Omega)$ is going to be κ_0 as foretold. Observe that we have now reached a stage where all the stuff $\geq \Omega$ that we always put into the $C(\alpha, \zeta)$s starts doing something.

This is consonant with what the preceding paragraph is telling us, namely us that, in the *second* pass, ϑ goes back and enumerates those ordinals in the *second* row that do not appear in the *third* row. Indeed one has the impression that in the αth pass ϑ enumerates in increasing order those ordinals in the αth row of the Veblen table that do not appear in the $\alpha + 1$th row. Jeroen and Michael tell me that $\vartheta(\Omega^2) = \Gamma_0$. This would appear to confirm what i have just been saying, beco's, after all, once one has made Ω passes (and thereby reached $\vartheta(\Omega^2)$) one should have hit every power of ω below Γ_0.

Stuff to sort out

There now follow some observations from Jeroen and Michael that i am reassured to find plausible but which i can't at this stage actually prove.

Jeroen also sez $\alpha < \Omega \to \alpha < \vartheta(\alpha)$.

All values of ϑ are less than Ω.

ϑ is injective.

[These last two observations cannot bothe be true! What did he mean?]

The values of ϑ do not depend on the choice of Ω. You can even take Ω to be ω_1^{CK}.

Every ϵ-number below the Bachmann-Howard ordinal is a value of ϑ.

$C(\epsilon_{\Omega+1}, 0) \restriction \Omega$ is the ordinals below the Bachmann-Howard ordinal.

If $\alpha < \epsilon_{\Omega+1}$ then α has a CNF with base Ω. That much is obvious. Let $K(\alpha)$ be the set of ordinals that appear in the CNF for α, and let $\alpha^* = \max(K(\alpha))$. Then we can say

$$\begin{array}{lll} \vartheta(\alpha) < \vartheta(\beta) \text{ iff either} & \alpha < \beta \wedge \alpha^* < \vartheta(\beta) \\ \qquad\qquad\qquad\quad \text{or} & \alpha > \beta \wedge \vartheta(\alpha) \leq \beta^* \end{array}$$

Then

$$\vartheta(\alpha) = \min\{\zeta \in E : \alpha^* < \zeta \wedge (\forall \beta < \alpha)(\beta^* < \zeta \to \vartheta(\beta) < \zeta)\}$$

where $\zeta \in E$ means that ζ is an ϵ-number.

All this machinery presumably supports a notational system. There is a binary ϕ(-,-) function that we can use to denote ordinals in sufficiently early levels of the Veblen table. I would like to understand that properly.

Should say something about why all these ordinals described by this Bachmann gadgetry are recursive. Anuj says that the ordering on the ordinals denoted by these notations is decidable. So, for any of these ordinals—α,say—the set of [gnumbers of] notations for ordinals below α gives a wellordering of $\mathbb{N}$. [but why is this set of notations for ordinals below α a decidable set? Why isn't it merely r.e.. . . ?]

Apparently it's straightforward to show that $C(\Omega, \beta)$ never exhausts all the ordinals, so that $\vartheta(\alpha)$ is well-defined. Should find something to say about this.