# Stratification and the Implementation of Pairing-unpairing, leading eventually to a discussion of its significance for Cardinal and Ordinal Arithmetic and isomorphism types more generally *or* Three Pieces in the Form of a Pair

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## Contents

ABSTRACT

The complications brought by stratification mean that when working in NF we have to be more explicit than would otherwise be necessary about many details of implementation of cardinals, ordinals ... and relational types more generally. The difficulties have their roots in pairing functions, and that is where we start. The original goal for this paper was explaining how the stratification complexities drive the treatment of big ordinals and the Burali-Forti paradox, and that is where these discussions will conclude. *En route* we will set out lots of details that – altho' not new (they have long been well understood by the *cognoscenti*) – are generally swept under the carpet.

Some topics which could merit attention will receive little or none here: both the version of Mirimanoff's paradox for the relational types of BFEXTs (wellfounded extensional binary structures with a top element) and the circumstance that the theory of relational types is invariant will get short rations.


The purpose of this note is to examine the way in which stratification constrains the project to implement cardinal and ordinal arithmetic. There is also the more general matter of implementing abstract entities that arise from equivalence relations, but for the moment we consider only these arithmetics. Underpinning all these implementations is the machinery of pairing-with-unpairing, so we will start with that.

Do we want to consider the fact that the theory of relational types is invariant?

This document is a tutorial rather than a research paper. The material within it is known to the *cognoscenti* but is not written out anywhere, and one can see why. I remember, in about 1970, going up-country for a few days with my then girlfriend to the lake house owned by the director of the theatre troupe where we both acted, and taking with me the University of Otago's Library copy of Rosser's [7] (I don't know why they gave me borrowing rights, but they did – very nice of them) and resolving to read the chapter on ordinal arithmetic, while sitting on the lake shore amid the gorse. I remember thinking that it was *very* hard but that I damned well wasn't going to be beaten, so I took a deep breath and stuck at it. Rereading that chapter years later I formed the distinct impression that Rosser knew a lot more about this stuff than he was letting on, and that what he eventually put down on paper was a very carefully judged selection from what he actually knew, in the sense that he was very concerned not to say anything to frighten his audience who were – let us not forget – supposed to be generic mathematicians, not set theorists. Like other NFistes of my generation I have subsequently retraced Rosser's steps and come to some sort of understanding of what is afoot. The worry is that the people who once long ago read and understood this stuff will die (I have already exhausted both my Maker's patience and my threescore years and ten) – or just plain forget – before they pass on the torch... which is one reason why I am writing this down now.

*Q: What should you do in your retirement?*
*A: the things which, if* you *don't do them, won't get done.*

It will also do duty as a draught of the relevant section of the third edition of my monograph. The time has come to rethink how this material is presented.

Part of the problem in exposition is that the precise statements of the results here depend sensitively on the choice of implementation of ordered pairs, of wellorderings and ordinals. One has to find a way of narrating the Mathematics without letting the artefacts of our implementation decisions get in the way. In practice mostly (and in every case here) those spurious details tend to be the exponents on the letter '$T$'.

There may be original work here, tho' I suspect most – if not all – of the NF material (even the stuff that I worked out for myself) was known to Rosser at the time when he wrote [7].

The language of Set Theory has $\in$ and $=$. We assume that the reader is familiar with the concepts of *stratified* and *homogeneous* formulæ of that language. It doesn't have functions, pairing etc, so these things have to *implemented* in Set Theory if we are to make a start. The first of these to implement is pairing.

# 1 Prologue on Pairing-with-Unpairing Kits

**Definition 1**
*We say that a pairing function that looks inside its components is* **intrusive***; otherwise it is* **polite***.*

This terminology is not standard.

For example Quine pairs are intrusive, Wiener-Kuratowski pairs and Hausdorff pairs (see below) are polite.

We observe without proof that when our pairing is polite, $X \times X \subseteq V(X)$, the cumulative hierarchy over $X$.

Well that's not strictly true but something like it is true. Included this in thirdedition.tex

There are three features of pairing functions that we need to consider.

- *Intrusive vs polite;*
- *Do they invoke extra constants?*
- *How much do they raise types?*

We record the following observations.

- We prefer polite pairings to intrusive pairings: intrusive pairs do not work with empty atoms as in NFU. (*Quine* atoms are fine)
- We prefer pairings not to have parameters.
- We prefer pairings that do not raise types to pairings that do. For reasons that will emerge below we need '$x = \langle y, z \rangle$' to be stratified with '$y$' and '$z$' receiving the same type.

Generally theorems about cardinal multiplication can put constraints on possible pairing/unpairing kits. (See [1] for example. Specifically the well-known inequality $2^3 < 3^2$ brings us the important information that if $|X| = 3$ then $|X \times X| > |\mathcal{P}(X)|$ so you can't encode pairs in the power set unless the universe contains more than three things. To populate the universe with sufficient things to get past this roadblock we introduce constants – *parameters*.

Intrusiveness, too, compels the universe to be large: Rosser [8] showed that the existence of Quine pairs (which are intrusive) implies the axiom of infinity – in fact the existence of any type-level pair is equivalent to infinity[1]. Any pair that raises types by no more than one must either use parameters or be intrusive.

## 1.1 Pairing Kits with Parameters

Allowing parameters increases the expressive power of any apparatus of definition. In this context it can be useful to have parameters that are constants, whose effect is *inter alia* to ensure that the universe is large enough for certain constructions to work. However there can also be parameters that help by adding information, such as a total order of $V$.

For an example of use of constants, consider $\{\{x,1\},\{y,2\}\}$, the Hausdorff pair. It raises by 2, and is polite, but it needs parameters. This is not best possible: best possible is W-K pairs, which are polite, raise types by 2, and do not use parameters.

### 1.1.1 A Pairing Kit from Holmes

Holmes ([4] ch 24 p 221) exhibits an intrusive pair that raises types by 1, using eight constants:

$$\langle A, B\rangle_{\texttt{Holmes}} = \{\{a,0,1\},\ \{a,2,3\},\ \{b,4,5\},\ \{b,6,7\} : a \in A,\ b \in B\}.$$

It might be worth rephrasing this as

$$\langle A, B\rangle_{\texttt{Holmes}} = \Big\{\{x,y,z\} : \bigvee \begin{array}{l} x \in A \wedge ((y=0 \wedge z=1) \vee (y=2 \wedge z=3)) \\ x \in B \wedge ((y=4 \wedge z=5) \vee (y=6 \wedge z=7)) \end{array} \Big\}$$

Let $\mathcal{X}$ be a local universe.

If $A, B \in \mathcal{X}$ and $0 \cdots 7$ are all in $\bigcup \mathcal{X}$ then $\langle A, B\rangle_{\texttt{Holmes}}$ is in $\mathcal{P}([\bigcup \mathcal{X}]^3)$.

It gives us

$$|\mathcal{X}| \geq 8 \ \rightarrow\ |\mathcal{X}|^2 \leq 2^{|[\bigcup \mathcal{X}]^3|}$$

but that's not much use.

Notice that this does *not* imply (contrary to my first thought) that

$$\mathfrak{a} \geq 8 \rightarrow \mathfrak{a}^2 \leq 2^{\mathfrak{a}}$$

for all cardinals $\mathfrak{a}$; there's a '$\bigcup$' that gets in the way.

There is a theorem of Specker's to the effect that $2^{\mathfrak{a}} \not\leq \mathfrak{a}^2$ as long as $\mathfrak{a} > 5$. So this pair of Holmes might not be best possible; there might be a version that uses only five parameters.

[1] N.B. the so-far unpublished *aperçu* of Holmes that this implication needs full extensionality: there are models of NFU + Infinity which do not support a type-level pair.

## 1.2 Raising types by 1

### 1.2.1 Another Pairing Kit from Holmes

This pair is polite, raises types by 1, and uses parameters.

Fix a linear order $\leq$ on $V$. Fix two five-element sets $A$ and $B$, presumably disjoint. The pair $\langle X, Y \rangle$ is to be the symmetric difference of $\{X, Y\}$ and $A$ if $X < Y$ and the symmetric difference of $\{X, Y\}$ and $B$ if $X \geq Y$. It's polite, and it uses only 3 levels, but it does use parameters. And it uses that magical number 5, as in Specker. It might be an idea to show how to decode a pair. I suppose you take a set $P$ and consider both $P$ `XOR` $A$ and $P$ `XOR` $B$. One hopes that precisely one of them will be an unordered pair. see [3]. Write out the details!

### 1.2.2 Another Pairing Kit, from Quine

Quine ([6] ch VIII p. 110) exhibits an intrusive pair that raises types by 1:

$$\langle A, B \rangle = \iota``A \cup \{V \setminus \{b\} : b \in B\}$$

This does not use parameters but it does assume the existence of the universal set (or at the very least a set that contains every singleton and every complement of a singleton) which is not to everyone's taste.

**Remark 1** *There is no polite pairing function that raises types by 1, even using constants.*

*Proof:*

If there were such a function we would have an outright proof that $(\forall \mathfrak{a})(\mathfrak{a} \geq n \to \mathfrak{a}^2 \leq 2^{\mathfrak{a}})$ where $n$ is the number of constants we use. In particular we would be able to prove it in Fränkel's original model in [2]. But – as John Truss has pointed out to me – the Fränkel model contains a counterexample. ■

Notice that this goes beyond the observation that $2^3 < 3^2$ implies that there cannot be a pairing function that raises types by one and works globally, because that observation left open the possibility that there could be such a pairing function that worked in a sufficiently large universe.

This appears to contradict section 1.2.1 but actually it doesn't, since that pairing kit does not merely have two constants but also relies on $V$ having a total order, which the Fränkel model presumably doesn't.

There is a follow-up to this at [1]. Where do we use politeness?

In summary, any polite pairing function must either have constants or raise types by at least 2. Of course if we drop the requirement that the pairing function be polite we can have type-level pairs, but there is no possibility of a type-lowering pair, however intrusive.

Sort this out

## 1.3 Pairs cannot lower types

We have to be careful how we state that there cannot be a type-lowering pairing-and-unpairing kit. In ZF (well, GB) we can prove that there is a bijection $\pi : (V \times_{WK} V) \longleftrightarrow \iota``V$ as long as we have foundation. This gives us a pairing-with-unpairing kit: take the ordered pair of $x$ and $y$ to be $\iota^{-1}(\pi(\langle x, y\rangle_{WK}))$. However this pair is not stratified so "type-lowering" doesn't make sense. And it's not polite either.

There are at least four good reasons why there cannot be a pairing function that lowers types. One good reason should suffice of course, but we are going to cover all three of them here in order to get a more comprehensive picture of what is going on.

- First, an NF-flavoured demonstration.

There is no way of implementing in NF a stratified pairing-with-unpairing kit $x = \langle y, z\rangle$ (no assumptions beyond that it is basic in the sense of definition 2 below) that gives '$x$' a type *lower* than the other two variables. For example, if it was given a type *one* lower than the other two then the map sending $x$ to $\{\langle x, x\rangle\}$ would be homogeneous and its graph would be a set. This set would inject $V$ into $\iota``V$ which we know cannot be done. This argument is a quick and dirty fix that can be run in NF – which is our current concern.

- Here is another quick-and-dirty fix. Suppose there were a type-lowering pairing function. Work in a finitely generated model of TST. Then we would have an injection from (level $n + 1$) $\times$ (level $n + 1$) into (level $n$) But this is clearly impossible.

Admittedly this argument does not rule out the possibility of such a pairing function in a model where the axiom of infinity holds, but it is a straw in the wind.

- Here is the germ of an explanation of why there cannot be a type-lowering pair. Suppose we have a formula $\phi(x, y, z)$ whch is stratifiable with '$x$' and '$y$' both one type higher than '$z$'. By Scott's lemma we will have

$$\phi(x, y, z) \longleftrightarrow \phi((j^{n+1}\sigma)(x), (j^{n+1}\sigma)(y), (j^n\sigma)(z)),$$

for all $x$, $y$, $z$, $\sigma$, and all suff large $n$. The key observation is that $j^{n+1}\sigma$ "moves fewer things" than $j^n\sigma$ – so we should be able to find $x$, $y$, $z$, $\sigma$ and $n$ s.t. $\phi(x, y, z)$, and $x$ and $y$ are fixed by $j^{n+1}\sigma$ while $z$ is moved by $j^n\sigma$.

- The following argument is due to Randall Holmes.

Suppose $\langle x, y\rangle$ is one type lower than '$x$' and '$y$'. Now $xx$ (which is $x$ applied to $x$) is the unique $y$ s.t. $\langle x, y\rangle \in x$. So "$z = xx$" is $\langle x, z\rangle \in x$, so is stratified with '$x$' and '$z$' having the same type – in other words *homogeneous*. The function $\lambda x.(V \setminus xx)$ is a stratified set abstract, since it is $\{\langle y, x\rangle : y = V \setminus xx)\}$, and '$y = V \setminus xx$' is homogeneous. So we do the obvious self-application:

$$\lambda x.(V \setminus xx)\ \ \lambda x.(V \setminus xx)$$

This exists (it's a stratified set abstract) and becomes

$$V \setminus (\lambda x.(V \setminus xx)\, \lambda x.(V \setminus xx))$$

by $\beta$-reduction ($\in$-elimination). This gives us a set – namely

$$\lambda x.(V \setminus xx)\, \lambda x.(V \setminus xx)$$

which is equal to its own complement.

■

Notice that this is compatible with $F$ being a type-lowering pair on a domain of things that are *small* in some sense – so we can do self-application in the domain of small sets. In that setting what this argument shows is that the graph of $\lambda x_{\texttt{small}}.(V \setminus xx)$ is not a small set.

I suppose what these four observations add up to is the fact that if $F(x, y, z)$ is a stratifiable formula where the third variable is one level below the first and the second then almost any sensible theory will prove that it's not a pairing function.

## 1.4 A Note on Pairing Functions and Group Actions

Need to rewrite this section. The thing to sort out is this: A group $G$ typically starts off as a group of permutations of some set $X$, and then acts on things that arise from $X$ in natural well-motivated ways. For example it will act on $X \times X$. The way in which we think that $G$ acts on $X \times X$ doesn't depend on how we implement cartesian product.

The following fact about polite pairs seems to me to be worth recording

**REMARK 2**
*Suppose we have a group $G$ acting on a set $X$. $G$ acts on $X \times X$ in the usual obvious way (and it does so whatever pairing function we use). It also acts on $V(x)$ (the cumulative hierarchy over $x$) again in a usual obvious way.*

*Now suppose $\langle .. \rangle$ is a polite stratified pairing function raising types by $n$. Then $X \times X \subset V(x)$, so $G$'s action on $V(x)$ applies to $X \times X$.*

*These two actions on $X \times X$ are the same!*

It doesn't matter what $n$ is; it matters only that the pairing function be polite.

I think there is something one can say along these lines even if the pair is not polite, as long as "$x = \langle y, z \rangle$" is stratifiable with '$y$' and '$z$' the same level ... but it will be more complicated.

Nowadays the pairing kit used in NF studies is the Quine pair, which is type-level. The situation in the constructive fragment *i*NF is not entirely clear; the Quine pair needs the axiom of infinity to hold, and it is not yet clear whether or not *i*NF proves a version of the axiom of infinity which is suitable for supporting the Quine pair. Even if it does, it not yet entirely clear (at least not to this reader) whether or not components of Quine pairs can be extracted in a constructively safe way. Now might be a good moment to sort this out.

We hope to show that Quine pairs in a stratified set theory that interprets Heyting Arithmetic are well behaved. We have two definitions

$$\Theta_1(x) = \texttt{succ}``(x \cap \mathbb{N}) \cup (x \setminus \mathbb{N})$$
$$\Theta_2(x) = \Theta_1(x) \cup \{0\}$$

And we need everything to be either a value of $\Theta_1$ or of $\Theta_2$ but not both. Does this use excluded middle? Are we forking on '$y \in \mathbb{N}$'? We might have difficulty with things that can't make up their minds whether or not they are natural numbers.

In contrast WK pairs are well-behaved constructively. Thanks to PTJ and Randall Holmes who both spotted it. It seems to me that this *aperçu* is probably worth writing out in some detail, and i do this in *i*NF.pdf.

We will not consider constructive scruples any further here.

A point perhaps worth recording on the fly is this: just because a typing kit is type-level doesn't mean the output has to have the same set-theoretic rank as its argument. There aren't enough things of rank 2 for all ordered pairs of them to also be of rank 2!

The Quine pair is an instance of a generic type-level pair gadget that has two parameter slots for a bijections between $V$ and two complementary moieties.. So the generic type-level pair has a parameter. However, if we are in NF then there are such bijections-and-moieties which are *definable* (since $\mathbb{N}$ is definable) so they can be massaged away.

## 1.5 What are Pairing Kits supposed to do?

There is another point of view from which to approach pairing-with-unpairing kits. It concerns itself not with the nature of the pairing function itself but rather the behaviour of the implementation that it gives rise to. It asks questions like:

*(i) Does it make the world of sets a category?*
*(ii) Is the category of sets cartesian closed?*
*(iii) Does the category of sets have coequalisers?*
*(iv) Do we get cartesian products?*

(Notice that none of these involve classifiers such as cardinal and ordinal numbers; those are for later)

Some of these things depend almost entirely on the implementation, and some depend on the axioms available in the set theory we are using. They fall into three classes:

*(i) we can control by choice of implementation;*
*(ii) is always false in NF, true in ZF;*
*(iii) depends on whether or not every partition is the same size as a set of singletons. This is true in ZF but it's a question that NF doesn't seem to have made up its mind about ... or – if it has – it's not letting on. Either way it doesn't depend much on our pairing kit: all rich kits will give the same answer.*
*(iv) doesn't depend very much on the choice of implementation but*

*rich* not defined yet.

*does depend on whether or not the theory doing the interpreting has replacement.*

**Definition 2**
*A* **basic pairing-and-unpairing kit** *is a dyadic pair-forming function with two inverse function,* `fst` *and* `snd`*, satisfying the obvious equations.*

The pair-forming function according to definition 2 doesn't have to be surjective: not everything has to be an ordered pair. The graphs of these functions are not assumed to be sets, even locally[2]. So really what we are asking for is a formula $\phi(x, y, z)$ satisfying (*inter alia*) $(\forall yz)(\exists! x)\phi(x, y, z)$ and two formulæ for the inverses.

**Definition 3**
*Let us say that a pairing-and-unpairing kit is* **rich** *if it additionally supports*

- *existence locally of the graph of the identity function, and*
- *composition of relations so that if $R$ and $S$ are sets then so is $R \circ S$;*
- *cartesian product: $A \times B$ is a set if $A$ and $B$ are.*

> We could probably add to this list the existence of projections: $R``X$, and restrictions: $R\restriction X$ (which we can get from cartesian product and $\cap$) and the other apparatus of relational algebra such as converses. (transitive closures?) Cartesian product in particular we make a fuss about beco's of the connection with the axiom scheme of replacement. There is a real question about what the next natural stopping point above basic pairing might be. One thing that we might insist on is an interpretation of a four-sorted theory with `individuals`, sets of `individuals`, pairs of `individuals` and sets of pairs of `individuals`, all set up in some (one hopes) complete theory. It will insist on composition, converses, projections and restrictions as above, that sort of thing.

But perhaps we just want to say that a pairing-with-unpairing kit is rich if it makes the universe into a category with finite limits . . . ? Or some other nice kind of category.

The reader might feel (and many readers do in fact feel) that these three bullets are an essential part of *any* implementation of pairing-with-unpairing. An implementation of widgets in set theory (or in anything else for that matter) will be an interpretation of the theory of widgets into the theory of sets. We want to define a *rich kit* as one that interprets the theory of pairing with unpairing. But what is the theory of pairing-with-unpairing? A basic pairing-and-unpairing kit captures *first-order* facts about pairing-with-unpairing; the facts about pairing-with-unpairing captured by the bullets are all *second-order*. Worries about second-order logic notwithstanding, the bullets do seem to be essential.

Let us think a bit about what this theory might be that we want rich kits to interpret. It would be good if there were a nice uncountably categorical second-order theory of pairing and unpairing lying around that we could use. Sadly

[2]See below for a discussion of what we might mean by saying that the graphs of the pairing and unpairing relations are sets!!!

there does not seem to be one. As long as we have composition and domain and codomain operators one can define what it is to be a permutation, a permutation being a relation whose domain and codomain are both the whole universe and which is $\subseteq$-minimal with this property. Thus we can express the first-order theory of infinite symmetric groups, and the trouble now is that this theory is not complete. $\mathrm{Symm}(\mathbb{Z})$ and $\mathrm{Symm}(\mathcal{P}(\mathbb{N}))$ are not elementarily equivalent, beco's $\mathrm{Symm}(\mathbb{Z})$ has an element (namely `succ`) that does not commute with any involution; however by a pigeonhole argument any permutation of an uncountable set must have two cycles of the same size and there will be an involution that conjugates them. Neither of these two observations need AC.

So there doesn't seem to be a natural property of *richness* for pairing-with-unpairing kits.

We are reconciled to the thought that if we are to have complete freedom in using Set Theory to reason about concepts that involve pairing, functions, relations etc we need our pairing-and-unpairing kit to be rich. The nonexistence of the identity function locally is not to be borne: it would prevent the world of sets from being a category. One might think that this doesn't matter if one isn't doing category theory, but it actually matters a great deal. Work in NF for the moment. Consider the pair $\langle \{x\}, y\rangle_{WK}$. This kit is basic but not rich: not only is the graph of the identity relation (on $V$) not a set (which is bad enough) but the graph of the singleton function (again, on $V$) *is* a set. This kit is trying to tell us not only that $V$ is not the same size as $V$, but also that it *is* the same size as $\iota$"$V$. There is no actual contradiction in this, but it does mean that this kit – by refusing to acknowledge that equinumerosity is an equivalence relation – does not enable us to capture cardinal arithmetic properly; it's not fit for purpose, and that's down to its inhomogeneous nature.

There is this nice *aperçu* of Mathias that, over a very basic set theory (Mac Lane set theory will do) –

**Theorem 1** *(Mathias)*
*The assertion that every basic pairing and unpairing kit is also rich is equivalent to the axiom scheme of replacement.*

*Proof:*
This is one of those pleasing deep facts that is not the kind of thing one thinks of, but is very easy to prove once spotted. Accordingly we leave it as an amusing exercise – an *amuse-gueule* – for the reader. Actually the only part of richness that Mathias uses is the existence of cartesian products. *There's* a wee **HINT**.

■

Altho' – thanks to Mathias – we know that in ZF any basic kit will perform as a rich kit (since ZF has replacement) the situation is less straightforward in NF, where we do not have full replacement[3] but only *stratified replacement*.

[3]Tho' it does have full *collection*!

One consequence of this for rich pairing-and-unpairing kits in NF is that we need "$x = \langle y, z \rangle$" to be stratified with '$y$' and '$z$' receiving the same type. This is because if this condition is not met then the restriction of the identity relation to a set $x$ will not exist (tho' if "$x = \langle y, z \rangle$" is stratified then it will exist as long as $x$ is strongly cantorian – and not otherwise). If "$x = \langle y, z \rangle$" is stratified but inhomogenous then the pairing kit will (according to NF) be basic but not rich. However, we do have the following, which we state without proof

**Theorem 2** *In NF any basic pairing kit where "$x = \langle y, z \rangle$" is stratified with '$y$' and '$z$' receiving the same type is rich.*

If "$x = \langle y, z \rangle$" is unstratified then there is not even any guarantee that ordered pairs so much as *exist* – never mind sets of them! Generally this matter of stratification of pairing functions has not attracted much attention. It's touched on briefly in [5], section 10.3 pp. 98*ff* (a discussion that bears rereading) but I know of no other treatments.

> One wants to say that if "$x = \langle y, z \rangle$" is homogeneous then its graph is a set. But the concept of a graph of a relation involves ordered tuples, so the situation is not straightforward! What is true is that if we use a homogeneous pairing relation (or rather, the homogeneous tripling relation that it brings with it) then the graph of any homogeneous three-place relation is a set.

Altho' (as we have seen) it is essential in NF that '$y$' and '$z$' receive the same type if the kit is to be rich there is no need for '$x$', too, to receive that same type.

Quine ordered pairs are available in NF, and they are a rich kit, and they give all three variables the same type. This makes life very easy. For example the graphs of the two unpairing functions are actually sets. However they are not polite!

## 1.6 Appendix to section 1: work still to do

There should be a theorem that says something like: any two rich kits are conjugated by a setlike map. There is always the map that sends $\langle x, y \rangle_1$ to $\langle x, y \rangle_2$. Is this map setlike when the two kits are rich…?

Might have to say something about acyclic pairing functions!!!

No basic CO model admits a rich pairing kit. This is because a cartesian product of two co-low sets is intermediate, and no basic CO model contains any intermediate sets. And that is so whatever the pairing kit is! The sets of a basic CO model do not form a category – the identity relation on a co-low set is intermediate, whatever kit you are using.

Incidentally ETCS is a stronger theory than just the assertion that sets form a category. It says *inter alia* that the world of sets is a topos. It has separation but not foundation. Thank you Clive Newstead for explaining this to me.

Is there a rich kit in $\mathrm{NF}_3$? What happens?

# 2 Cardinals and Ordinals

Now that we have got a definition of rich pairing-with-unpairing kit, and resolved to use only rich kits, we can move onto material that builds on that consensus. We can start thinking about classifiers for relations that need ordered pairs, in particular equipollence and order isomorphism. Classifiers for these equivalence relations will give us implementations of cardinals and ordinals.

## 2.1 Definitions and Notation

We use lower-case Greek letters for variables over ordinals, and lower-case $\mathfrak{fraktur}$ characters to range over cardinals. This use of $\mathfrak{fraktur}$ characters has a respectably long history going back at least as far as Sierpiński. For example, '$\mathfrak{c}$' has long been used to denote the cardinality of the continuum.

**Definition 4**

- *We have already been writing '$|x|$' for the cardinal of $x$, so our announcement here is purely* pro forma*.*
- *An* **aleph** *is the cardinal of a wellorderable set.*
- *We write '$\mathbf{card}(\alpha)$' for the aleph associated with the ordinal $\alpha$, in the sense that if $\alpha$ is the order type of a wellordering $\langle A, <_A \rangle$, then $\mathbf{card}(\alpha)$ is $|A|$.*
- *An (infinite) ordinal $\alpha$ is* **initial** *iff $(\forall \beta < \alpha)(\mathbf{card}(\beta) < \mathbf{card}(\alpha))$.*
- **Hartogs' aleph function***: $\aleph(\mathfrak{a})$ is the least aleph $\not\leq$ the cardinal $\mathfrak{a}$ when such a thing exists.*
- **Hartogs' Lemma** *states that Hartogs' aleph function is total.*

For example $\Omega$ is an initial ordinal. This is because (as we shall see, lemma 2), for all $\beta < \Omega$, $\beta$ is $T^2$ of something, but $\Omega$ itself isn't. It will turn out that $\mathbf{card}(T^{-1}\Omega) = \aleph(T|V|)$.

$\Omega$ not defined until definition 6

Notice that these definitions just gazetted make sense even before we have decided on what classifiers we want to use for equipollence and isomorphisms.

$T^n|V| = |\iota^n``V|$. It might be an idea to decide which notation to use, or perhaps to have some system about when you use one and when you use the other.

In the following sections we consider the variety of possible implementations of classifiers for equinumerosity, equinumerosity of finite sets, and other equivalence relations. We start with classifiers for the equivalence relation on finite sets of equinumerosity... in other words, implementations of the arithmetic of natural numbers.

## 2.2 The Finite Case

We have more freedom in how we implement natural numbers than we have in how we implement ordinals or arbitrary cardinals, so we shall deal with them first.

There are many ways of implementing `natural-number-of` with a stratified formula. To each such implementation we can associate a concrete integer $k$ which is the difference (`type-of` '$y$') − (`type-of` '$x$') in '$y = |x|$'. In fact:

**REMARK 3**
*For every concrete integer $k$ there is an implementation of* `natural-number-of` *making '$y = |x|$' stratified with the type difference (*`type-of` *'$y$') − (*`type-of` *'$x$') equal to $k$.*

*Proof:*
For $k = 1$ there is the natural and obvious implementation that declares $|x|$ to be $[x]_\sim$, the equipollence class of $x$ – the set of all things that are the same size as $x$. This object is one type higher than $x$ whatever pairing kit we are using, of course. For $k \geq 1$ we take $|x|$ to be $\iota^{k-1}([x]_\sim)$. (This works for all cardinals, not just natural numbers).

For $k < 1$ we have to do a bit of work and, although the steps we take will not work for arbitrary cardinals, they do work for naturals. We need the fact that there is a closed stratified set abstract without parameters that points to a wellordering of length precisely[4] $\omega$. The obvious example of such a wellordering is the usual Frege-Russell implementation of $\mathbb{N}$ as equipollence classes (with the usual ordering) which we have just used above with $k \geq 1$, but it is probably worth emphasising that we don't *have* to use the Frege-Russell $\mathbb{N}$ here; whenever we have a definable injective total function $f$ where $V \setminus f``V$ is nonempty, with a definable $a \notin f``V$, then

$$\bigcap\{A : a \in A \wedge f``A \subseteq A\}$$

will do just as well[5]. The wellordering relation will be the ancestral of $f$. The usual definition of $\mathbb{N}$ as a set abstract is merely a case in point. (It may or may not be worth noting that there is no such set abstract in Zermelo or ZF!) Let's use the usual $\mathbb{N}$-as-the-set-of-equipollence-classes.

Consider $\{\iota^k(n) : n \in \mathbb{N}\}$. It is denoted by a closed set abstract and has an obvious wellordering of length $\omega$. For every inductively finite set $x$ there is a unique initial segment $i$ of this wellordering equipollent to it, and the (graph of the) function that assigns $x$ to that initial segment is a set. We conclude that the function $x \mapsto \bigcup^k i$ is an implementation of `natural-number-of` that lowers types by $k$. ■

Here is another proof. We can take $|x|$ to be $[y]_\sim$ for any $y$ such that $\iota^k``y \sim x$. (We can prove by induction that for any finite $x$ we can find such a $y$). This gives us a natural-number-of $x$ that is $k-1$ types lower than $x$. For a natural-number-of $x$ that is $k+1$ types *higher* than $x$ take $|x|$ to be $[\iota^k``x]_\sim$.
■

[4]The ultra-cautious might not be happy using this Greek letter at this stage. It's a shorthand for *a wellordering with no last element and no limit points.* Any such wellordering is unique up to canonical isomorphism – even in NF!

[5]It has to be admitted that proving that NF exhibits such an $f$ involves a nontrivial amount of work!

We mustn't forget that the graph of the `natural-number-of` function is guaranteed to be a set only if it's captured by a homogeneous expression. Admittedly it might be a set even if this condition is not met, but that's not to be relied on.

## 2.3 The Axiom of Counting

Rosser's Axiom of Counting does two things.

One thing it does (the *primary* thing, from which it takes its name) is say that each natural number counts the set of natural numbers below it. In this capacity it is much used in the philosophical and psychological literature on numerosity: to ascertain the cardinality of a finite set has size $n$ you *count* it, telling off its members with natural numbers. If the last number you use is $n$ then you know that it has $n$ members. This strategy of course relies on the fact that the set $[0, n]$ of natural numbers below $n$ is of size precisely $n$ – and *that* fact is Rosser's Axiom of Counting.

The other thing is the bundle of assertions:

*(a) Every inductively finite set is cantorian;*
*(b) Every inductively finite set is strongly cantorian;*
*(c)* $\mathbb{N}$ *is strongly cantorian.*

Ad (c) we mention *en passant* that if we want to prove merely that $\mathbb{N}$ is *cantorian* we can be do it outright.

An important difference between the Axiom of Counting and the bundle (a)–(c) is that the truth-value of the Axiom of Counting (but not (a)–(c)) is contingent on a choice of implementation of cardinal. (a) and (b) do not mention cardinals at all; (c) does, admittedly, but altho' changing the implementation might in principle change the size of a set of cardinals it affects it only by applications of $T$ or $T^{-1}$ and this won't alter whether or not a given set of cardinals is strongly cantorian.

(a)–(c) can be proved equivalent *without* proving that they are equivalent to the axiom of Counting. They make use of the concept of a bijection but not of any classifier for equinumerosity. Proving their equivalence relies only on the pairing kit being rich.

In contrast whether or not counting appears to be true depends on our choice of cardinal classifier, specifically on the type difference between $x$ and the `natural number of` $x$. If our pairing kit is rich then the Axiom of Counting is either a theorem of NF (if the type difference is $-1$) or is equivalent to everything in the bundle (all other values of the type difference).

This of course means that whether or not (a)–(c) turn out to be equivalent to The Axiom of Counting depends on our choice of implementation of `natural-number-of`.

First we prove that (a)–(c) are equivalent.

**Lemma 1** *(a)–(c) are all equivalent.*

*Proof:*

Clearly (b) implies (a), so it will suffice to prove (a) $\to$ (c) and (c) $\to$ (b).

(a) $\to$ (c) (a) implies $(\forall n \in \mathbb{N})(n = Tn)$. $Tn$ is always one type higher than $n$ whatever `natural-number-of` classifier we are using, so the map $Tn \mapsto \{n\}$ is homogeneous and its graph is a set; the map $n \mapsto Tn$ is the identity map by assumption so its graph is a set. The composition of these two maps is $\iota \restriction \mathbb{N}$, in virtue of which $\mathbb{N}$ is strongly cantorian.

(c) $\to$ (b) Observe that every subset of a strongly cantorian set is strongly cantorian, and every inductively finite set injects into $\mathbb{N}$.

■

If we were to prove the Axiom of Counting by induction we would need "$m$ is the cardinality of the set of natural numbers less than $n$" to be homogeneous. That is in order that the collection of putative counterexamples be a set. For this to be the case we need our `cardinal-of` function to lower types by 1. If we have such a `cardinal-of` function we can prove that every natural number counts the numbers below it as follows. Let $X$ be any finite set. It has a wellordering $<_X$. Send each $x \in X$ to the cardinal of the $<_X$-initial segment bounded by $x$. This map bijects $X$ with the set of natural numbers $< |X|$. It's homogeneous so its graph is a set, so $|X| = |\{n : n < |X|\}|$.

**REMARK 4**

- *If the type-difference in "$y$ is the natural number of $x$" is $-1$ then the Axiom of Counting is a theorem of NF.*
- *Otherwise it is equivalent to each of (a) – (c).*

*Proof:*

First bullet.

If the natural number of $n$ is one type lower than $n$ then "$|\{m : m <_{\mathbb{N}} n\}| = n$" is stratified and can be proved by induction.

Second bullet.

Let '$y = |x|$' be stratified with '$y$' $k$ types higher than '$x$'. Send $Tn$ to $\{m : m < n\}$ and then send *that* to $T^{-k}|\{m : m < n\}|$. That way the map will be homogeneous.

By assuming The Axiom of Counting we get $|\{m : m < n\}| = n$, so $T^{-k}|\{m : m < n\}| = T^{-k}n$. So we have the graph of a function sending each natural number $Tn$ to $T^{-k}n$. More simply put, we have the graph of a function sending each natural number $n$ to $T^{k-1}n$. So we can prove by induction that this map is the identity.

So $(\forall n \in \mathbb{N})(n = T^{k-1}n)$. If $k = 1$ this is trivial, but if $k$ is anything else it is informative. Not only that, but all assertions $(\forall n \in \mathbb{N})(n = T^{k-1}n)$ for $k \neq 1$ are equivalent. What we want to show is that, for any concrete natural $k$, and all $n \in \mathbb{N}$, if $n = T^j n$ then $n = Tn$. To do this we fix $n$ and consider the sequence $\{n, Tn, T^2 n \cdots T^j n\}$. Since $T$ is an automorphism of $<_{\mathbb{N}}$ this sequence must be either $<_{\mathbb{N}}$-increasing or decreasing – or, of course, constant. However we are told that $n = T^j n$, so the two ends of the sequence are equal, which means that it must be constant, whence $n = Tn$. This is (a).

■

Notice that we have used not only that the singleton function gives rise to an automorphism of $T$ of $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$ but also that $<_{\mathbb{N}}$ is a total order. (Tho' we don't use this last fact when $k = 0$ or 2, when the sequence has length 2). For arbitrary infinite $\mathfrak{a}$ we do not seem to be able to infer $\mathfrak{a} = T\mathfrak{a}$ from $\mathfrak{a} = T^j \mathfrak{a}$.

## 2.4 The Infinite Case

There are other details that merit attention. At present it seems to be an open problem whether or not NF is consistent with a type-lowering implementation of `cardinal-of` for *all* sets. The trick used to obtain a type-lowering implementation of `cardinal-of` for finite sets will work also for wellordered sets whose sizes are sufficiently small alephs. If there are only finitely many different sizes of infinite sets then NC (the set of all cardinals) is countable and is therefore the size of a set of singletons$^k$ for concrete $k$ as big as you please. So in those circumstances we would have, for each concrete $k$, an implementation of `cardinal-of` making the type difference between '$|x|$' and '$x$' precisely equal to $k$. In contrast there cannot be a type-lowering implementation of `ordinal-of` – because of Burali-Forti. (We will discuss this below)

Wellorderings can also be represented as ordernestings. This turns out not to be much use to us; it is true that it enables us to conduct at least some of the discussion of wellorderings without using pairing-and-unpairing, but the ordernesting of a wellordering of a set $X$ is one type higher than $X$ whereas its representation as a set of ordered pairs can be the same type. It turns out that there is nothing clever we can do with ordernestings so we won't consider them any further here.

Ordinals are the entities that arise from the order-isomorphism relation on wellorderings. Two wellorderings partake of the same ordinal iff they are isomorphic. We implement a wellordering as a set decorated with a set of ordered pairs. How next are we to implement ordinals? We look for a *classifier* for order-isomorphism; that is to say a function $f$ defined on the set of all wellorderings with the property that, for wellorderings $R$ and $S$, $f(R) = f(S)$ iff $R$ and $S$ are order-isomorphic. The first function that comes to mind is that which sends a wellordering to its equivalence class under order-isomorphism. This function is definable by a stratified expression, but its graph isn't a set beco's its definition is not homogeneous: the value is one level higher than the argument. This (minor) blemish cannot be remedied by using a different implementation of pairing since it would require a type-lowering pairing function, and we saw in section

1.3 that this is not to be had. The blemish is, as I say, minor, and it is this function that we will use as our classifier; and we will notate it '$\mathtt{No}(\langle X, <_X\rangle)$' (as in Rosser [7]). Even tho' it's not homogeneous, its definition is at least stratified, so the image of any set in it is also a set (*i.e.*, it obeys replacement); in particular its range – which of course will be the set of our ordinals – is a set. We will **not** write it '*On*'; that string is too strongly associated with the class of von Neumann ordinals in ZF for it to be safe for us to use it here. We will notate it '*NO*' as in [7]; *NO* means *NO*!!

Let us set this definition up in lights.

**DEFINITION 5**
*The ordinal of a wellordering $\langle X, <_X\rangle$ is the isomorphism class of $\langle X, <_X\rangle$ and we write it* $\mathtt{No}(\langle X, <_X\rangle)$.
*The collection of all ordinals is a set and is notated 'NO'.*

Both the notations in definition 5 are in [7]. We assume that the reader is familiar with fact that the order relation on ordinals is a wellorder. We will write this wellorder $<_{NO}$. It is important that the existence of this wellordering is a fact about abstract ordinals, not about ordinals as implemented in a theory of sets. There are textbooks that say that Von Neumann ordinals are wellfounded because their order relation is $\in$ and we have an axiom that says that $\in$ is wellfounded. This is a distraction! (And it's another reason for writing '*NO*' rather than '*On*' for the set of ordinals.)

**DEFINITION 6** *We write*[6] *'$\Omega$' for the order type of $\langle NO, <_{NO}\rangle$.*

Quite what ordinal this notation picks out depends on our choice of implementation of ordinal number which (since we have decided to implement ordinals as equivalence classes of binary relations) means it depends on our choice of implementation of ordered pair.

Corollary 1 below will declare that there is also a *homogeneous* classifier for order-isomorphism, and therefore an implementation of ordinals such that "$y$ is the ordinal of $R$" is a homogeneous expression (albeit one with a parameter). Its graph will be a set. However we know nothing about its values so it is not much use to us and we shall not use it.

It is perhaps illuminating to compare and contrast the finite and infinite versions of the Counting Principle: the Axiom of Counting and the Extended Axiom of Counting. The Axiom of Counting (which is the finite case) can be made true by a judicious choice of implementation of `natural-number-of` and, for any concrete natural $k$, an implementation can be found whose type difference is $k$. In the infinite case cardinals and ordinals part company. Corresponding to the axiom of counting we have in the infinite caase what one might call the *Extended Axiom of Counting* that says that every ordinal is the order type of its predecessors in the obvious order. The Burali-Forti paradox will tell us that there is no way of choosing an implementation of `ordinal-of` that

[6] This notation is not in [7], tho' it is now commonplace among *NFistes*.

makes the Extended Axiom of Counting true. As regards cardinals it is still open whether or not there can be a type-lowering implementation of "$y$ is the cardinal of $x$".

The existence of a homogeneous classifier for `ordinal-of` is far from obvious, and we will prepare the ground with some factoids and definitions, some of which may be familiar but which I will include anyway because the context is nonstandard and readers may be suspicious.

**Lemma 2** *The order type of the ordinals below $\alpha$ is $T^2\alpha$.*

We prove this by induction on the ordinals. ■

This is the formulation in NF-speak of an early insight of Cantor's, to the effect that every ordinal counts the set of its predecessors. Rosser [7] considered the restriction of this principle to $\mathbb{N}$, which he called the *Axiom of Counting*, and which we considered in section 2.3. We could call the full principle (for all ordinals) the *Extended Axiom of Counting* – as we have just done. It would be good for it to have a name, and currently it doesn't seem to have one.

Von Neumann exploited this extended axiom of counting to implement ordinals in ZF by simply declaring each ordinal to simply *be* the set of its predecessors. This implementation is very cute (*very* cute) but it "disappears" this important fact about ordinals by turning it into a notational trivality. (Quine's point about how our notation controls which truths appear to be logical truths is a good one that needs to be borne in mind in Mathematics not merely in Philosophy of Science). So we should put von Neumann ordinals to one side and instead consider the challenge of proving the extended axiom of counting in ZF using *Scott's-trick* ordinals. Then there is an actual induction to perform. Exactly the same induction can be proved in NF, as long as we stratify it by inserting a suitably placed '$T^2$'. (The reader is assumed to understand by this stage why it is only stratified inductions that we can perform on $<_{NO}$). Here we have an encounter – of the kind that I warned about earlier – with the exponent on '$T$'. The exponent here is 2, and that is because we are using *Quine* pairs, which are type-level. We could also have used *Wiener-Kuratowski* ordered pairs – which raise types by 2. Had we been using them the exponent would have been 4. Were we to use a pairing function that raised types by $n$ then the exponent would be $n + 2$.

To summarise: when there are no stratification worries to slow us down we embrace the extended axiom of counting and say that the order type of an initial segment of the ordinals is the least ordinal not in it. The extended axiom of counting plus the assumption that $NO$ is a set gives us Burali-Forti, so in NF we have to modify it to the assertion that the order type of an initial segment of the ordinals is $T^2$ of the least ordinal not in it.

But wait! Might not lemma 2 look different if we were to use a different classifier, such as the one whose existence we prove in corollary 1)? Yes: the exponent 2 will be replaced by 1. The set of ordinals below $\alpha$ is one type higher than $\alpha$ and the order relation on it will also be one type higher, as will the ordinal of that order relation. So we get a different exponent, but we never get an exponent of zero. The exponent on $T$ in lemma 2 can be any nonzero natural number.

# 3 Big Ordinals in NF

The word 'Big' is a deliberate choice. We are concerned here not with measurable/compact/strongly compact etc cardinals, but with cardinals whose magnitude is of an altogether different nature: sizes of things that in ZF wouldn't be sets, such as the universe or the set of all ordinals. The cardinal of the set of ordinals according to NF could – for all we know – be $\aleph_n$ where $n$ is a nonstandard natural number, and not be *large* at all. Hence 'big' not 'large'.

## 3.1 Hartogs' Lemma in an NF Context

Hartogs' Lemma asserts that for every set there is a wellorderable set that is too big to be injected into it. Hartogs' lemma obviously fails in NF, because every set can be injected into $V$. Let's try executing the proof and see what happens.

The way to prove Hartogs' lemma for a set $X$ is to consider the set of wellorderings of subsets of $X$, and the set $\mathcal{X}$ of ordinals of those wellorderings. This is an initial segment of the ordinals, and it's a set. Being an initial segment its order type is $T^2$ of the least ordinal it doesn't contain, which of course is the least ordinal *not* the length of a wellordering of a subset of $X$. (Lemma 2 tells us as much). We want its cardinality to *not* be $\leq |X|$. Unfortunately it's two types too high. We're in luck as long as $\mathcal{X}$ is the same size as a set $\iota^2$"$\mathfrak{X}$ of double singletons, beco's then $\mathfrak{X}$ is a wellorderable set admitting a wellordering longer than any wellordering of any subset of $X$, and is the set we desire.

Thus we can prove Hartogs' lemma for sets that are the same size as a set of double singletons:

**Lemma 3** *In NF, $\aleph(\mathfrak{a})$ is defined for all cardinals $\mathfrak{a} \leq T^2|V|$.*

Actually we can do better than that; we can prove Hartogs' lemma for $|\iota$"$V|$ too!

**Remark 5** *There is a wellorderable set not the size of any set of singletons.*

*Proof:*

Suppose *per contra* that every wellorderable set is the same size as a set of singletons. Then every wellorderable set is the same size as a set of *double* singletons. This is because if a wellorderable set $X$ is the same size as a set $\iota$"$X'$ of singletons, then $X'$, too, is wellorderable and therefore will be the same size as a wellorderable set $\iota$"$X''$ of singletons, making $X$ the same size as $\iota^2$"$X''$. So – by assumption – we can peel off the curly brackets as desired. We consider the set of ordinals of wellorderings of subsets of $\iota$"$V$. If this set were the size of a set of double singletons we would be able to remove two layers of curly brackets to obtain a set bigger than any wellordering of singletons. Contraposing, not every wellorderable set is the size of a set of singletons. ■

The statement of remark 5 does not mention ordinals, and the suspicious reader (good on 'em!) may want to satisfy themselves that this proof does not rely on our choosing any particular classifier for wellorderings (i.e., implementation of ordinals). It does so depend, but only to the extent of varying the exponent on '$T$' in the proof. If we were using a homogeneous classifier the exponent would be 1 not 2.

Remark 5 tells us that Hartogs' Lemma works for $|\iota``V|$: there is a smallest wellordered set not the size of any set of singletons. A feature of this situation of some interest is that we know of no definable set of that size.

I am going to continue to use isomorphism classes as ordinals and will henceforth no longer flag the fact that exponents will be different if we use different classifiers.

**Remark 6** *There is a set of singletons the same size as $NO$.*

*Proof:*
By lemma 2 the order type of an initial segment of the ordinals is $T^2$ of the least ordinal not in it, so $\Omega$ wants to be $T^2$ of the least ordinal not in $NO$, but of course there is no such ordinal. So $\Omega$ is not $T^2$ of any ordinal. Now any proper initial segment of $NO$ has an order that really is $T^2$ of something, namely the least ordinal not in it. So $\Omega$ is the least ordinal that is not $T^2$ of anything. Now we must have

$$\Omega \leq T\Omega \ \vee \ T\Omega < \Omega.$$

We can't have the first disjunct because that would imply $\Omega \leq T^2\Omega$ – in which case $\Omega$ would be $T^2$ of something after all. So we must have $T\Omega < \Omega$. But $\Omega$ was the least ordinal that isn't $T^2$ of anything, so $T\Omega$ is $T^2$ of something, so $T^{-1}\Omega$ is defined. So $NO$ is the size of a set of singletons. ■

**Corollary 1** *There is a homogeneous classifier for the equivalence relation of order-isomorphism of wellorderings.*

*Proof:*
By remark 6 there is an injection $f : NO \hookrightarrow \iota``V$. The function

$$\langle X, <_X\rangle \ \mapsto \ \iota^{-1}(f(\mathtt{No}(\langle X, <_X\rangle)))$$

is a homogeneous classifier.

■

Notice however that this argument gives us no reason to suppose that the homogeneous classifier thus obtained picks one wellordering from each isomorphism-class of wellorderings. Indeed there is no reason to suppose even that its values are wellorderings. In fact we know nothing about its values at all beyond the fact that they are all distinct. If there were a homogeneous classifier picking a wellordering from each equivalence class then we would be able to show that there was a last aleph, as follows. We concatenate all the wellorderings that are values of the classifier, with some modification forced on us by the fact that the carrier sets of these wellorderings might not be disjoint. We order elements of the union of all these carrier sets by first appearance. The resulting wellordering will not necessarily be a longest wellordering (that would be nice!) but it will be a wellordering of maximal cardinality. (Think what happens if, for each countable ordinal, you have a wellordering of $\mathbb{N}$ of that length, and you execute this construction. Since all the wellorderings have the same carrier set, the later wellorderings make no contribution to the ordering of the concatenation, so you get the naturals in their usual order.) This is just a manifestation in NF guise of the fact that wellordered choice enables us to prove that every value of the Hartogs' function is a successor aleph.

## 3.2 Prewellorderings and Prewellordinals

It is reasonable to suspect that our ability to understand the cardinalities and cardinal inequalities of large alephs might be improved by considering the weak versions of AC that crop up in this connection: the partition principle, that sort of thing. Is every wellordered partition the same size as a set of singletons? This line of thought leads us to take an interest in big prewellordeings.

Prewellordings are familiar (for example to people used to Descriptive Set Theory) but it can do no harm to spell things out, particularly since we need to be clear what our notion of isomorphism of prewellorderings is. This will enable us to give a unified treatment of both ordinals and the isomorphism classes of prewellorderings.

**Definition 7**

*A* **Prewellordering** *is a wellfounded strict poset $\langle X, <_X \rangle$ satisfying the extra condition that every element belongs to a unique maximal antichain*[7]*.*
*This is equivalent to the first-order condition that the (reflexive closure of the) relation $u \not<_X v \not<_X u$ is an equivalence relation. Let us write it $\sim_X$.*

*A* **Prewellorder Morphism** $I : \langle X, <_X \rangle$ *to* $\langle Y, <_Y \rangle$ *is a subset of* $X \times Y$ *satisfying*

- $I(x, y) \wedge x \sim_X x' \wedge y \sim_Y y' \rightarrow I(x', y')$ *and*
- $I(x, y) \wedge I(x', y') \wedge x <_X x' \rightarrow y <_Y y'$.

*A morphism whose converse is a morphism is an* **isomorphism.**

*The isomorphism classes of prewellorderings are* **prewellordinals**.

- *A prewellordinal that contains a wellordering is an* **ordinal**. *(we could use this to define ordinals should we wish to start from prewellorderings instead of wellorderings.)*

[7] This is an exercise for the reader. Go on, stir yer stumps, whaddarya!

- *When $\alpha$ is the prewellordinal of $\langle A, <_A \rangle$ we define $T\alpha$ to be the prewellordinal of $\langle \iota``A, (<_A)^\iota \rangle$ (aka $\langle \iota``A, \texttt{RUSC}(<_A) \rangle$) which of course is also a prewellordering.*
- *If $\alpha$ is the prewellordinal of $\langle A, <_A \rangle$ and $\beta$ is the prewellordinal of $\langle B, <_B \rangle$ and there is a morphism $\langle A, <_A \rangle \to \langle B, <_B \rangle$ but not in the other direction we shall write $\alpha <_{NO} \beta$. It is permissible to use the same subscript because of the previous bullet.*

We record the following observations, mostly without proof.

- A prewellorder $\langle X, <_X \rangle$ is an actual wellorder iff the $\sim_X$-equivalence classes are singletons; ordinals are a special kind of prewellordinal: the definition was formulated with that in mind.
- All sufficiently small prewellordinals are ordinals (a prewellordering with an ordinal above it is another ordinal).
- Prewellordinals behave exactly like ordinals: the obvious order relation on them is a prewellordering – in fact a wellordering. This is beco's of the following

**Remark 7** *$T$ of a prewellordinal is always an ordinal.*

*Proof:*

If the prewellordinal $\alpha$ is the relational type of an ordering $\langle X, <_X \rangle$ then $T\alpha$ is the relational type of $\langle \iota``X, \texttt{RUSC}(<_X) \rangle$. But notice that this structure is isomorphic-as-a-prewellordering to the prewellordering induced on the quotient $X/\sim_X$, and *this* prewellorder is clearly a wellordering, so $T\alpha$ is an ordinal. ■

The order type of the prewellordinals below a prewellordinal $\alpha$ is $T^2\alpha$, and this prewellordinal $T^2\alpha$ is a proper ordinal since it contains an actual wellordering, namely the prewellordinals below $\alpha$ equipped with $<_{NO}$.

It might help to have a proper name for the order type of the wellordering of all prewellordinals (it is a kosher ordinal not just a mere prewellordinal). Let's call it $pw\Omega$. It's a pun!

This prewellordering lark is probably worth taking seriously. Maybe the correct notion is not $\aleph^*(\alpha)$ but something like: Find the least prewellordinal $\beta$ that does *not* have a representative that is a prewellordering of a set of size $\alpha$. We've got to call it *something* but i can't think of a satisfactory notation. For the moment let's write it '$\mathcal{H}^*(\alpha)$'. The $\mathcal{H}$' reminds us of $H$artogs and the asterisk means 'surjections'.

So $\mathcal{H}^*(|X|)$ is the least prewellordinal that does not contain a prewellordering of $X$.

But we have to be careful ... the use of the symbol '$\aleph$' in this notation might suggest (by analogy with the Hartogs aleph function) that the function takes values in the cardinals, so that $\aleph^{**}(|X|)$ is **card** of the least prewellordinal that does not contain a prewellordering of $X$. But we cannot do that, beco's **card** cannot be defined on prewellordinals, since we can have a pair of pwo-isomorphic prewellorderings of different sizes. Perhaps this is telling us that morally the Hartogs aleph function takes values in the ordinals rather than in the cardinals. I'd always pushed back against the people who declare it as a function taking values in the ordinals beco's it seemed to me that this was merely another manifestation of the error of conflating cardinals and ordinals, but perhaps i was being too jumpy.

There is in principle the possibility that the prewellordinal $\mathcal{H}^*(\alpha)$ that we obtain for $\alpha$ might be a prewellordinal that is not an ordinal, so $\mathcal{H}^*$ looks like a genuine addition to our toolkit.

The next thing is to prove a version of Sierpinski-Hartogs for $\mathcal{H}^*$.

**THEOREM 3** $\mathcal{H}^*(\alpha) < \mathcal{H}^*(2^{\alpha^2})$

(I think this is a novel theorem. Ask Asaf and JKT)

*Proof:*

Let $A$ be a set of cardinality $\alpha$. Quine ordered pairs are type-level, and if $2^{\alpha^2}$ is defined then $\mathcal{P}(A \times A)$ is the same size as some set $\iota``X$ of singletons. That means that we can think of $X$ as $\mathcal{P}(A \times A)$. We define a function on $\mathcal{P}(A \times A)$ by sending an argument to its prewellordinal if it is a prewellordering and to 0 if it isn't. We now have a set of prewellorderings that is a surjective image of $\mathcal{P}(A \times A)$. It is of course a prewellordering; we want to show that this prewellordering is not isomorphic to any prewellordering of $A$.

I think it's going to happen that two things are going to turn out to be the same:

- The least ordinal not containing a wellordering of a surjective image of $X$
- The least prewellordinal not containing a prewellordering of a subset of $X$.

Or at least they are equal if they are both defined. The first one has to be an ordinal; the second might be a prewellordinal while not being an ordinal. The second might be defined even if the first one isn't.

My brain hurts. Sort this out

The next claim is that

**THEOREM 4**

$$\aleph^*(\alpha) \leq^* 2^{\alpha^2}.$$

*Proof:*

This is a classical result in the mould of Sierpinski-Hartogs. God knows who proved it first; even John Truss doesn't know. It's probably on a granite stele from Sumeria. Notice that it can be captured by a stratifiable expression. This fact tells us that everything is going to be all right in the end, even tho' *en route* we have problems occasioned by things cropping up at the wrong type/level.

We start by recalling the proof in ZF.

$2^{\alpha^2}$ is of course the cardinality of $\mathcal{P}(A \times A)$ when $|A| = \alpha$. $\mathcal{P}(A \times A)$ is of course the set of binary relations on $A$. We are looking for a surjection. Let $f$ be the following function. On being given a binary relation $R \in \mathcal{P}(A \times A)$ it sends it to 0 if it is not a prewellordering. If it is a prewellordering send it to its equivalence class. Clearly any equivalence class of a prewellordering of $A$ gets hit. The idea is that the range of this function is of size $\aleph^*(\alpha)$ but on the face of it it is at the wrong level. ■

Now we modify the proof to accommodate the stratification requirements of a stratified theory such as NF. Notice that ‘ $\aleph^*(\alpha) \leq^* 2^{\alpha^2}$ ’ is stratifiable, so we have grounds for hope that there will be a proof in NF. We have a type-level pair in NF.

If $2^{\alpha^2}$ is defined then there is a set $B$ and a bijection $f : \iota``B \longleftrightarrow (A \times A)$. We want to map $\mathcal{P}(B)$ onto a smallest wellordered set that is not a surjective image of $A$. The obvious way to obtain such a set is to start with the set of ordertypes of wellorderings that *are* surjective images of $A$.

Finish this off

The point i want to make at this stage is that the first two paragraphs of Holmes’ remark, quoted above, hold also for prewellordinals. Recall that we write ‘$pw\Omega$’ for the otype of the set of prewellorderings. It’s clearly an ordinal not merely a prewellordinal.

$$\begin{array}{lllllll}
\Omega & \leq & pw\Omega & \leq & T^{-1}\Omega & \leq & T^{-1}pw\Omega \\
 & & pw\Omega & & < & & T^{-1}pw\Omega \\
\Omega & & < & & T^{-1}\Omega & &
\end{array}$$

So $pw\Omega$ might be $\Omega$ and it might be $T^{-1}\Omega$ but it cannot be both, so there are three possibilities:

(i) $pw\Omega = \Omega$:

The ordinals and the prewellordinals coincide. That means that every wellorderable partition is the same size as a set of singletons. This says:

$$\aleph \leq^* T|V| \to \aleph \leq T|V|.$$

(ii) $pw\Omega = T^{-1}\Omega$:

The prewellordinals are as long as they can be. This means that every wellorderable set is the same size as a partition, or every ordinal is $T$ of a prewellordinal; in other words

$$(\forall \aleph)(\aleph \leq^* T|V|).$$

(iii) There is a first ordinal that is not $T$ of a prewellordinal.

This says that there are alephs $\kappa < \lambda$ with $\kappa \leq^* T|V| \wedge \kappa \not\leq T|V|$, and $\lambda \not\leq^* T|V|$.

I think (ii) might be a profitable possibility to follow up.

Suppose we are given a wellordering. It would be nice to obtain from it a wellordering of a partition. How do you do that? Well, by recursion on the wellordering, you remove from each point any element that appears in an earlier point. Of course if your bottom element is $V$ then this process doesn't get you very far! Your chances look better if all the points in the wellordering are small sets. We are looking for a notion of small set s.t. there are enuff small sets around for every aleph to contain a set of small sets. So (for example) how many wellorderings are there? Are there enough for every wellordering to be the same length as a wellordering whose carrier set/domain is a set of wellorderings?

One thing that this development does is to drive a wedge between the interesting fact that ordinals in TST can be relied on to increase every *two* upwards steps, and the equally interesting fact that the order type of the ordinals is two levels higher than the ordinals. The appearance of the number 2 in both these settings is coincidence. Exactly the same increase-of-two happens with the prewellordinals too.

### 3.3 Afterthoughts

Much of the above is in [7], certainly in Rosser's head even if not on paper: have a look at Theorem XII.3.15 and Exercise XII.3.5.

May be worth pointing out that $\aleph(T^2|V|) = |NO|$, and (which is the same thing) $\aleph(T|V|) = T^{-1}|NO|$.

The worrying thing about the proof of remark 6 is that it seems to be thoroughly nonconstructive: We prove that there must be a set of singletons that is the same size as $NO$, but our proof doesn't tell what that set might be. To put it another way, $T^{-1}\Omega$ is a perfectly respectable definable set and is known to be nonempty, but we do not seem to be able to point to any definable member of it. If Fate Decrees That NF Is Not To Have A Term Model this may be the way to prove it. Again, it is important to emphasise that that phenomenon – there being a definable provably nonempty set with no obvious definable member – happens however we implement ordered pairs. The *really* bizarre feature is that this happens simply because there is no type-lowering ordered pair. (see section 1.3.) How can the nonexistence of a type-lowering ordered pair result in an existence theorem?

Might every wellorderable set be a surjective image of a set of singletons? Let's try tweaking the proof that not every wellorderable set is the size of a set of singletons.

The analogue of Hartogs' for a set $X$ starts by considering the set of ordinals of wellorderings of surjective images of $X$. The idea is that this set cannot be a surjective image of $X$, but we have the same old problem with the set being two types too high. This time however we are hampered. If every wellorderable set is the same size as a set of singletons, then every wellorderable set is the same size as a set of double singletons. However the analogous inference from "every wellorderable set is a surjective image of $\iota$"$V$" to "every wellorderable set is a surjective image of $\iota^2$"$V$" seems not to work: just beco's a set is a surjective image of a set of singletons isn't going to tell us that it's a surjective image of a *wellorderable* set of singletons.

That said, I know of no proof that there are wellorderable sets that are not surjective images of $\iota$"$V$.

However we can prove that

**REMARK 8**
*Not every wellorderable set is a surjective image of $\iota^2$"$V$.*

*Proof:*

This is a consequence of the fact that any surjective image of $\iota^2$"$V$ (wellordered or not) injects into $\iota$"$V$:

$$\text{If } f : \iota^2\text{“}V \twoheadrightarrow X \text{ then } x \mapsto \{\bigcup\bigcup f^{-1}\text{“}\{x\}\} \text{ injects } X \text{ into } \iota\text{“}V.$$

■

It may not look like it, but remark 8 is a manifestation in NF guise of the standard fact that every surjective image of $X$ can be injected into $\mathcal{P}(X)$. After all, $2^{T^2|V|} = T|V|$.

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