# Number systems of different lengths, and a natural approach to infinitesimal analysis

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## 1 Formal System of Euclidean Arithmetic

**The language of $EA$**

- Constant: $\varnothing$ (empty set).
- Functions: P (power set); TC (transitive closure); { , } (pair set).
- Term-forming operator: $\{x \in t : A(x)\}$, whenever $A$ is bounded.
- Relations: $=$ (identity); $\in$ (membership).

**The axioms of $EA$**

- Axioms of Empty Set, Power Set, Transitive Closure, Pair Set, and Extensionality are definitions of primitive symbols.
- Instead of the Axiom of Infinity, $EA$ has the Axiom of Dedekind Finiteness: $\forall x, y(x \subsetneqq y \to x <_{\mathrm{c}} y)$
- Axiom Schema of Separation restricted to bounded formulae.

## 2 Natural Number Systems

### 2.1 Definitions

Roughly, $L$ *is generated from* $0$ *by* $\sigma$ if it has the following form:

$$[0, \sigma(0), \sigma(\sigma(0)), \cdots, a]$$

Roughly, $\mathcal{N}$ consists of the following linear orderings:

$$[\,], \quad [0_{\mathcal{N}}], \quad [0_{\mathcal{N}}, \sigma_{\mathcal{N}}(0_{\mathcal{N}})], \quad [0_{\mathcal{N}}, \sigma_{\mathcal{N}}(0_{\mathcal{N}}), \sigma_{\mathcal{N}}(\sigma_{\mathcal{N}}(0_{\mathcal{N}}))], \quad \cdots$$

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## 2.2 Examples

- $\mathcal{VN}$ is generated from $\varnothing$ by $\sigma_{\mathcal{VN}} : x \mapsto x \cup \{x\}$

$$[\,],\quad [\varnothing],\quad [\varnothing, \{\varnothing\}],\quad [\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}],\quad \cdots$$

- $\mathcal{Z}$ is generated from $\varnothing$ by $\sigma_{\mathcal{Z}} : x \mapsto \{x\}$

$$[\,],\quad [\varnothing],\quad [\varnothing, \{\varnothing\}],\quad [\varnothing, \{\varnothing\}, \{\{\varnothing\}\}],\quad \cdots$$

- $\mathcal{CH}$ is generated from $\varnothing$ by $\sigma_{\mathcal{CH}} : x \mapsto \mathrm{P}(x)$

$$[\,],\quad [\varnothing],\quad [\varnothing, \mathrm{P}(\varnothing)],\quad [\varnothing, \mathrm{P}(\varnothing), \mathrm{P}(\mathrm{P}(\varnothing))],\quad \cdots$$

## 2.3 Induction and Recusion

**Theorem 1 (Bounded induction holds)** *If $A$ is a bounded formula,*

$$EA \vdash (A(\,[\,]\,)\ \&\ (\forall L\ \mathit{in}\ \mathcal{N})[A(L) \to A(\overline{\sigma_{\mathcal{N}}}(L))]) \to (\forall L\ \mathit{in}\ \mathcal{N})A(L)$$

**Theorem 2 (Unbounded induction fails)** *If $A$ is unbounded, the following does not necessarily hold:*

$$EA \vdash (A(\,[\,]\,)\ \&\ (\forall L\ \mathit{in}\ \mathcal{N})[A(L) \to A(\overline{\sigma_{\mathcal{N}}}(L))]) \to (\forall L\ \mathit{in}\ \mathcal{N})A(L)$$

**Definition 3 (Arithmetical global functions)** *Suppose $\varphi$ is a global function. We say that $\varphi$* is arithmetical *if*

$$EA \vdash \forall x, y(x \cong y \to \varphi(x) \cong \varphi(y))$$

**Definition 4 ($\mathcal{N}$ is closed under $\varphi$)** *Suppose $\varphi$ is an arithmetical global function. Then we say that $\mathcal{N}$* is closed under $\varphi$ *if*

$$EA \vdash (\forall x\ \mathit{in}\ \mathcal{N})(\exists y\ \mathit{in}\ \mathcal{N})[\mathrm{Field}(y) \cong \varphi(\mathrm{Field}(x))].$$

**Theorem 5** *Given a natural number system $\mathcal{N}$, the family of arithmetical global functions under which $\mathcal{N}$ is closed is closed under limited recursion, but NOT under full recursion.*

- For $n = 0, 1, 2, 3$, there are natural number systems closed under all and only the arithmetical functions of Grzegorczyk's class $\mathscr{E}^n$.
- But the distinctions are more fine-grained: *e.g.*
  - There is $\mathcal{N}$ closed under $x + \log(x)$ but not under $x + x$.
  - There is $\mathcal{N}$ closed under $x\log(\log(x))$ but not under $x\log(x)$.

**Definition 6 ($\varphi$ is maximally powerful in $\mathcal{N}$)** $\varphi$ is maximally powerful in $\mathcal{N}$ *if, for any arithmetical global function $\psi$, if $\mathcal{N}$ is closed under $\psi$, then there is* $\mathbf{n}$ *such that $\psi$ is eventually majorized by $\varphi^{\mathbf{n}}$.*

**Theorem 7** *Suppose there is* $\mathbf{C}$ *such that*

*(i)* $EA \vdash (\forall x)(\mathbf{C} \leq x \to x < \varphi(x))$

*(ii)* $EA \vdash (\forall x, y)(\mathbf{C} \leq x \leq y \to \varphi(x) \leq \varphi(y))$

*(iIi)* $EA \vdash (\forall x, y)(\mathbf{C} \leq x \leq y \to \varphi(x) - x \leq \mathbf{2}^y - y)$

*Then there a natural number system $\mathcal{ACK}_\varphi$ such that $\varphi$ is maximally powerful in $\mathcal{ACK}_\varphi$.*

## 2.4 Relations of length

**Definition 8** $\mathcal{M} \preceq \mathcal{N}$ *if*

$$EA \vdash (\forall x \text{ in } \mathcal{M})(\exists y \text{ in } \mathcal{N})[\mathrm{Field}(y) \cong \mathrm{Field}(x)].$$

**Theorem 9** *$\mathcal{VN}$ and $\mathcal{Z}$ are incommensurable: that is,*

$$\mathcal{VN} \not\preceq \mathcal{Z} \quad \text{and} \quad \mathcal{Z} \not\preceq \mathcal{VN}.$$

### 2.4.1 The syntactic proof

**Lemma 10 (Parikh-style Bounding Lemma)** *Suppose $A$ is a bounded formula. Then, if*

$$EA \vdash \forall x \exists! y A(x, y)$$

*then there is a classical natural number,* $\mathbf{n}$*, such that*

$$EA \vdash \forall x \exists! y (y \in \mathrm{P}^{\mathbf{n}}(\mathrm{TC}(x)) \ \&\ A(x, y))$$

*Proof of Theorem 9.* Suppose $\mathcal{VN} \preceq \mathcal{Z}$. That is,

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\exists! z \text{ in } \mathcal{Z})(\mathrm{Field}(v) \cong \mathrm{Field}(z))$$

Thus, by Parikh-style Bounding Lemma, there is $\mathbf{n}$ such that

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\exists z! \text{ in } \mathcal{Z})(z \in \mathrm{P}^{\mathbf{n}}(\mathrm{TC}(v)) \ \&\ \mathrm{Field}(v) \cong \mathrm{Field}(z))$$

But, by (meta-theoretical) induction on $\mathbf{n}$,

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\forall z \text{ in } \mathcal{Z})(z \in \mathrm{P}^{\mathbf{n}}(\mathrm{TC}(v)) \to z \in V_{\mathbf{n+4}})$$

Thus,

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\exists z! \text{ in } \mathcal{Z})(z \in V_{\mathbf{n+4}} \ \&\ \mathrm{Field}(v) \cong \mathrm{Field}(z))$$

which is false. □

### 2.4.2 The model-theoretic proof

*Proof of Theorem 9.* Let $M$ be a model of $EA$ that contains a non-standard member of $\mathcal{VN}$, $b$. Then define the following submodel of $M$:

$$C(M,b) = \bigcup_{\mathbf{n}=1}^{\infty} \{x \in M : M \models x \in \mathrm{P}^{\mathbf{n}}(b)\}$$

We call $C(M,b)$ the *cumulation model of* $EA$ *of* $b$. Then

$$C(M,b) \models EA$$

But $C(M,b)$ contains only standard members of $\mathcal{Z}$, while it contains non-standard members of $\mathcal{VN}$. Thus, it is not the case that $\mathcal{VN} \preceq \mathcal{Z}$. □

## 2.5 Measuring the universe

**Definition 11 ($\mathcal{N}$ measures the universe)** *$\mathcal{N}$ measures the universe if*

$$EA \vdash (\forall x)(\exists y \textit{ in } \mathcal{N})[x \cong \mathrm{Field}(y)]$$

**Theorem 12** *In the presence of $\Sigma_1$ induction, and thus unlimited recursion, every natural number system measures the universe.*

**Theorem 13** *In EA, no natural number system measures the universe.*

*Proof.* Suppose $\mathcal{N}$ measures the universe. If $\mathbf{k}$ is a classical natural number, let

- $v_{\mathbf{k}}$ be the $\mathbf{k}^{\text{th}}$ member of $\mathcal{VN}$,
- $z_{\mathbf{k}}$ be the $\mathbf{k}^{\text{th}}$ member of $\mathcal{Z}$, and
- $n_{\mathbf{k}}$ be the $\mathbf{k}^{\text{th}}$ member of $\mathcal{N}$.

Since $\mathcal{N}$ measures the universe,

$$EA \vdash (\forall x)(\exists y! \text{ in } \mathcal{N})[x \cong y]$$

Thus, by the Parikh-style Bounding Lemma, there is $\mathbf{n}$ such that

$$EA \vdash (\forall x)(\exists y! \text{ in } \mathcal{N})[y \in \mathrm{P}^{\mathbf{n}}(x) \;\&\; x \cong y]$$

Thus, for all classical natural numbers, $\mathbf{k}$,

$$n_{\mathbf{k}} \in \mathrm{P}^{\mathbf{n}}(v_{\mathbf{k}}) \quad \text{and} \quad n_{\mathbf{k}} \in \mathrm{P}^{\mathbf{n}}(z_{\mathbf{k}})$$

Thus,

$$n_{\mathbf{k}} \in \mathrm{P}^{\mathbf{n}}(v_{\mathbf{k}}) \cap \mathrm{P}^{\mathbf{n}}(z_{\mathbf{k}})$$

Thus,

$$n_{\mathbf{k}} \in V_{\mathbf{n+4}}$$

But this gives a contradiction, since $V_{\mathbf{n+4}}$ cannot contain sufficiently many members of $\mathcal{N}$ to measure all standard members of $\mathcal{VN}$ and $\mathcal{Z}$. □

# 3 Infinitesimal Analysis

## 3.1 Extending $EA$

**Definition 14 ($\mathcal{N}$-small and $\mathcal{N}$-large)** *Suppose $\mathcal{N}$ is a natural number system.*

- $x$ *is* $\mathcal{N}$*-small* $\leftrightarrow (\exists y \textit{ in } \mathcal{N})[x < \text{Field}(y)]$
- $x$ *is* $\mathcal{N}$*-large* $\leftrightarrow (\forall y \textit{ in } \mathcal{N})[\text{Field}(y) < x]$

**Definition 15** *$EA^+$ is obtained from $EA$ by adding the following axiom:*

$$(\exists x)[x \textit{ is } \mathcal{ACK}\textit{-large}]$$

**Theorem 16** *If $EA$ is consistent, then $EA^+$ is consistent.*

## 3.2 Infinitesimal analysis in $EA^+$

**Definition 17 (Integers in $EA^+$)** *An integer is an ordered pair $(a, b)$ where $a$ and $b$ are sets. (Intuitively, $(a, b)$ is $a - b$.)*

$$(a, b) =_Z (c, d) \leftrightarrow a + d \cong b + c$$

**Definition 18 (Rationals in $EA^+$)** *A rational is an ordered pair $(a, b)$ where $a$ and $b$ are integers, and $b \neq_Z 0$. (Intuitively, $(a, b)$ is $\frac{a}{b}$.)*

$$(a, b) =_Q (c, d) \leftrightarrow a \times_Z d \cong b \times_Z c$$

**Definition 19 (Reals in $EA^+$)**

$$r \textit{ in } R \leftrightarrow (\exists x)[x \textit{ is } \mathcal{ACK}\textit{-small} \ \& \ |r| < x]$$

**Definition 20 (Infinitesimal in $EA^+$)**

$$r \textit{ in } I \leftrightarrow (\forall x)\left[x \textit{ is } \mathcal{ACK}\textit{-small} \rightarrow |r| < \frac{1}{x}\right]$$

**Definition 21 ($x \simeq y$)** *If $x$ and $y$ are in $R$, then $x \simeq y \leftrightarrow x - y$ in $I$*

**Theorem 22** *$R$ is 'almost' real closed.*

## 3.3 Continuous functions in $EA^+$

**Definition 23 ($f$ is continuous)** *If $f : J \rightarrow R$, then $f$* is continuous *if*

$$(\forall x, y \textit{ in } J)[x \simeq y \rightarrow f(x) \simeq f(y)]$$

### 3.4 Differential and integral calculus in $EA^+$

**Definition 24 ($f$ is differentiable)** *Suppose $f : J \to R$, $x$ is in $J$, and $\alpha$ is in $R$. Then $f$* is differentiable at $x$ with derivative $\alpha$ *if*

$$(\forall \delta \text{ in } I)\left[\frac{f(x+\delta)-f(x)}{\delta} \simeq \alpha\right]$$

**Definition 25 ($f$ is integrable)** *Suppose $f : [a,b] \to R$, $a \leq x \leq b$, and $\alpha$ is in $R$. Then $f$* is integrable at $x$ with definite integral $\alpha$ *if, for any $\mathcal{ACK}$-large $N$,*

$$\sum_{i=0}^{N} \frac{b-a}{N} \cdot f\left(a + i\frac{b-a}{N}\right) \simeq \alpha$$

### 3.5 Polynomials of large degree

**Definition 26** *By definition,*

$$e_N^x = \sum_{i=0}^{N} \frac{x^i}{i!}$$

Then $e_N^x$ is in $R$, if $x$ is in $R$ and $N$ is large. Also, $e_M^x \sim e_N^x$, if $x$ is in $R$ and $M$ and $N$ are large. Finally, $\lambda x\, e_N^x$ is differentiable at all $x$ in $R$ with derivative $e_N^x$.

**Theorem 27 (Weierstrass)** *Suppose $f : [a,b] \to R$ is continuous function. Then there is a polynomial,*

$$P(x) = \sum_{i=0}^{N} a_i x^i$$

*possibly of large degree, such that*

$$(\forall a \leq x \leq b)[P(x) \simeq f(x)]$$

### 3.6 References

All the results here and many more can be found in:


Pettigrew, R. *Natural, Rational, and Real Arithmetic in a Finitary Theory of Finite Sets* PhD Doctoral Thesis, University of Bristol. http://www.maths.bris.ac.uk/˜rp3959/thesis1.pdf/