Part (b)

$$\frac{\Gamma \vdash \Delta, A, B \qquad \Gamma, A, B \vdash \Delta}{\Gamma \vdash \Delta, A \oplus B}$$

If we consider the case where Γ and Δ are both empty the sequent rule becomes

$$\frac{\vdash A, B \qquad A, B \vdash}{\vdash A \oplus B}$$

Now one thing one can infer from the top line is A `XOR` B, since the left upper sequent says at least one of A and B is true, while the right upper sequent says at least one of A and B is false. Could it be anything stronger? $A \wedge B$ for example? $\neg A \wedge \neg B$? These are the only candidates—$A \oplus B$ has to be one of the 16 boolean binary connectives, and it's easy to check that nothingelse will do.

So $A \oplus B$ is A `XOR` B.

To discover what the rule on the left is for A `XOR` B one could try looking for a proof of the sequent $\Gamma, \neg A \rightarrow B, \neg B \rightarrow A \vdash \Delta$, and seeingnwhat sequents one finds at the topof one's tree once on runs out of connectives. Another approach is to observe that A `XOR` B is symmetric in 'A' and 'B' so the upper sequents in a `XOR`$-L$ must be preserved by swapping 'A' and 'B'. One tries various things like

$$\frac{\Gamma \vdash \Delta, A, B \qquad \Gamma, A, B \vdash \Delta}{\Gamma, A \texttt{ XOR } B \vdash \Delta}$$

but i think what one wants is

$$\frac{\Gamma, A \vdash \Delta, B \qquad \Gamma, B \vdash \Delta, A}{\Gamma, A \texttt{ XOR } B \vdash \Delta}$$

and it's a simple matter to check that this is truth-preserving. To do this, assume the upper sequents, assume Γ and assume that precisely one of A, B is true. If it's A that is true we invoke the left upper sequent and infer that either B is true or something in Δ is true. Well, *ex hypothesi* it ain't B wot is true, so it must be something in Δ (as desired); OTOH if it's B that is true we invoke the right upper sequent and infer that either A is true or something in Δ is true. Well, *ex hypothesi* it ain't A wot is true, so it must be something in Δ (as desired).

I think that requires quite a lot of work for 6 marks, but perhaps i'm getting soft in my old age.

References