Basis

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November 22, 2012

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Definition 1.
*Let $\langle X, R\rangle$ be a binary structure. An R-**bottomless set** is a subset $Y \subseteq X$ s.t. $(\forall x \in Y)(\exists y \in Y)(yRx)$. We say $B \subseteq X$ is a **basis** if it meets every R-bottomless set.*

Let us say X is dense in A if every bottomless subset of A meets X.

If r is clear from context we shall merely say "bottomless".

This is an important idea for a variety of reasons. The first is the obvious but useful fact that if f is a function defined by $\in$-recursion then it has a unique extension from any basis to the whole of V:

Is this the place to talk about the wellfounded part of a binary structure?

Theorem 2. *Let $\langle X, R\rangle$ be a binary structure and*

- *$g : X \times V \to V$ is an arbitrary (total) function.*
- *f is a (total) function $B \to V$.*[1]
- *B is a basis for the R-bottomless subsets of X;*

Then there is a unique total function $f^* : X \to V$ satisfying

1. *$f^* \restriction B = f$;*
2. *$(\forall x \in (X \setminus B))(f(x) = g(x, f``\{y : R(y, x)\}))$.*

Proof: The idea is very simple. We obtain our best candidate for f^* by closing the graph of f under the operation that adds ordered pairs according to the clause that says that $f(x) =: g(x, f``\{y : R(y, x)\})$.

[1] Here V is the universe, so that when we say "$g : X \times V \to V$" we mean only that we are not putting any constraints on what the values of g (or its second inputs) are to be.

Now suppose (2) fails, so the subset Y of X on which f^* is not uniquely defined is nonempty. This set Y is bottomless because $f(x) =: g(x, f``\{y : R(y,x)\})$. So it meets B, since B is a basis. But then f^* is defined on at least some of Y.

■

This modification of the recursion theorem looks trivial, but there are cases where it is useful. For example, one can add Quine atoms to a model of ZFC in a well controlled way so that every $\in$-bottomless class contains a Quine atom. But then any function defined on the Quine atoms can be extended to the whole universe.

Secondly, as we shall see, it is quite easy to start from a wellfounded model to obtain an illfounded model by RB methods and retain control of the basis. Finally, certain natural conditions on bases enable us to to obtain wellfounded models from illfounded models by RB methods. Another way to motivate the idea of a basis is the factoid that the late lamented Jon Barwise used to call "The solution lemma". (See [**?**] p ???). It is a consequence of the Forti-Honsell antifoundation axiom that every system of equations in the style

$$x_1 = \{\emptyset, x_2, x_3\};\ x_2 = \{\{\emptyset\}, x_3\};\ x_3 = \{\emptyset, x_1\}$$

has a unique solution.

Let's show how to add such bad sets by permutations. Start with three sets x_1, x_2 and x_3—it won't much matter what they are, but let's take them to be von Neumann reals, or something large and remote like that—and consider the product of the three transpositions: $(x_1, \{\emptyset, x_2, x_3\})$, $(x_2, \{\{\emptyset\}, x_3\})$ and $(x_3, \{\emptyset, x_1\})$. In the resulting Rieger-Bernays permutation model x_1, x_2, x_3 form a solution to the system of equations.

What is the feature of interest here? There are these things we're inventing, namely the xs, and we are declaring them in terms of each other and some *wellfounded* sets. In this new model the bad sets x_1, x_2, x_3 form a basis. This basis B has the nice property that $TC(B) \setminus B$ is wellfounded: the bad sets are declared in terms of each other and wellfounded sets only. This is the key to getting rid of them later, as we shall see.

1 Using bases to get rid of illfounded sets by permutations

Now let us think about what has to happen for there to be a permutation σ available in an illfounded model $\mathfrak{M}$ so that $\mathfrak{M}^\sigma$ is wellfounded.

Consider the model $\mathfrak{M}$ obtained from a wellfounded model of something like Z by swapping $\emptyset$ with $\{\emptyset\}$. This model has a single Quine atom a which is a basis for the bottomless sets. Now we can get get rid of this object by means of the transposition $(a, \{\emptyset\})$—σ for short. Suppose X is a collection of sets which—in the sense of $\mathfrak{M}^\sigma$—has no $\in$-minimal element. That means X must contain things in the support of σ. For suppose it did not. Then $\in_\sigma \upharpoonright X$ would be the same as $\in \upharpoonright X$ and X would have to contain a since $\{a\}$ is a basis. But if $a \in X$, X does indeed have an $\in_\sigma$-minimal element, namely a. And if $\emptyset \in X$ then $a \in X$, since $\emptyset$ is $\{a\}$ in the sense of $\mathfrak{M}^\sigma$. The moral of this is that a single Quine atom added to a wellfounded model by a single transposition can be got rid of by a single transposition and the state of nature restored.

Let's think about how we might in general be able to use the existence of a basis to cook up a permutation model in which the illfounded sets have been killed off.

THEOREM 3.
Suppose $\mathfrak{M}$ is a model with a basis B for the bottomless sets and σ is a permutation such that $(\forall b \in B)(\emptyset \in TC(\sigma(b)))$ and $(\forall b_1, b_2 \in B)(\sigma(b_1) \notin TC(b_2))$. Then $\mathfrak{M}^\sigma$ contains no illfounded sets.

Proof:
Let σ be an involution swapping every basis element with something yet to be determined, but otherwise leaving non-basis elements alone. Suppose now that A is a family of sets in V^σ that has no $\in_\sigma$-least member. It cannot consist entirely of fixed elements beco's if it did it would be a family of sets of V with no $\in$-least member and would meet the basis—and everything in the basis is moved. So it contains either a basis element or σ of a basis element. (Notice that this *reductio* argument doesn't prove that it must contain a basis element, merely that if *per impossibile* everything in it were fixed, it would contain a basis element). To take care of the possibility that it contains a basis element (but not $\sigma(b)$ for b a basis element) we must ensure that $\emptyset \in TC(\sigma(b))$ for every basis element b. But if it contains $\sigma(b)$ for b a basis element we find that it contains everything inside the basis element and we seem to be back where we started. We can argue as before that something must be moved. If the moved thing we have found is a $\sigma(b)$ then we have made no progess. What we want to ensure is that one of the moved things that we find inside it is another basis element—not σ of a basis element. So it will suffice to ensure that whenever b_1 and b_2 are basis elements then $\sigma(b_1) \notin TC(b_2)$. ■

We can actually get by with slightly weaker conditions on σ than this. Let us say a basis element b **has rank 0** if $\sigma(b)$ is wellfounded and everything in its transitive closure is fixed. Then if basis elements of rank 0 are dense in a bottomless$^\sigma$ set A we obtain a contradiction. Indeed we can say that a basis element b has rank β if there is a set X of basis elements of lower rank s.t. $\sigma``X$ is dense in $TC(b)$. All we have to do is ensure that every basis element has a rank.

It might be worth getting a picture of where the atoms go in the wellfounded RB model. A Quine atom q becomes the set it's swapped with, and the set becomes its singleton, and that singleton becomes the singleton^2 and so on as in Hilbert's Hotel.

The conditions sound quite strong, but they are satisfied in the case that really matters to us, namely the case where the illfounded sets all arise from Quine atoms. We'd better check this!

Suppose X is a collection of sets without a σ-minimal element, where σ is a product of transpositions $(x, \{x\})$ over a wellfounded model. X cannot consist entirely of fixed sets, beco's all fixed sets are wellfounded. So it must contain either an x or a $\{x\}$. Clearly it has to be an x, since $\{x\}$ is really x (in the sense of $\mathfrak{M}^\sigma$). So that's OK.

We will need the factoid that if B is a basis for $\mathfrak{M}$, then it is a basis for any transitive submodel of $\mathfrak{M}$.

References

[1] Jon Barwise and Laurence Moss, Vicious Circles Cambridge University Press 1996