Logic and Proof Exercises

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Two Definitions

Definition 1 Input Resolution
One of the resolving clauses should always be from the input (i.e. from the knowledge base or the negated query).

Complete for Horn clauses but not in general. (In fact we will see below two examples where Input Resolution doesn't deliver the goods, namely exercises 3 and 1)

Definition 2 Linear Resolution
A generalization of Input Resolution. Allows resolutions of clauses P and Q if P is in the input or is an ancestor of Q in the proof tree.

I think it's complete. I inadvertently fibbed to some of you earlier: the definition i gave of linear resolution was the definition of *input* resolution. I told you (correctly) that Linear Resolution is complete, but i gave you the wrong definition, so that the thing i gave you was not complete.

Some Exercises

Excluded Middle

Excluded middle is the principle $A \vee \neg A$ that says there are only two truth-values. One might expect it to be equivalent to

$$\neg(\neg(A \longleftrightarrow B) \wedge \neg(B \longleftrightarrow C) \wedge \neg(A \longleftrightarrow C)) \qquad \text{(\textit{Tertium non datur} (variant form))}$$

but actually this variant form is weaker, and is constructively correct.

Exercise 1
Provide sequent, natural deduction and resolution proofs of the above variant form of Tertium not datur. Establish that there is no input resolution proof of it.

There are extra points for supplying proofs that respect the constraints of constructive logic, so in particular your sequent proof should only ever have one formula on the right.

(I have a constructive sequent proof of the variant form of *Tertium non datur* but it is in WORD and the turnstile doesn't print. The usual bribes are available for a nice LaTeX-ed proof.)

Wellfounded Relations

Exercise 2
Provide demonstrations, using sequent calculus, natural deduction or resolution that

$$\forall x(\forall y(R(y,x) \to \phi(y)) \to \phi(x)) \to \forall z(\phi(z)) \textit{ and } (\forall xy)(R'(x,y) \to R(x,y))$$

together imply

$$\forall x(\forall y(R'(y,x) \to \phi(y)) \to \phi(x)) \to \forall z(\phi(z)).$$

Maria Gorinova of Clare has kindly provided two model answers: a sequent calculus proof, and a resolution proof. Daniel Spencer (also of Clare) has provided a natural deduction proof.

Graphs

A graph is a set of vertices with undirected edges, and no loops at vertices. It is *connected* if one can get from any vertex to any other vertex by following edges. The *complement* of a graph is what you think it is: it has the same vertices, but a complementary set of edges.

Exercise 3
Use propositional resolution to show that a graph and its complement cannot both be disconnected.

Your first thought might be to try to do it in first-order logic, but that is actually quite hard—and i haven't managed to get it to come out yet. Doing it in propositional logic requires some preliminary thought. I have a proof, but i haven't managed to find a proof using linear resolution—tho' i know there must be one. My proof is not compliant with the restrictions surrounding *Input Resolution*. In fact it's not hard to show that there is no proof thus compliant:

Exercise 4
Show that there is no input resolution proof for exercise 3.

It ought to be possible to do Exercise 3 in first-order logic as well as in propositional logic, but i haven't managed to get a model answer. (This possibility came up beco's Nick Spooner (top of 1a in 2011) misread the then current version of exercise 3 as an instruction to find a resolution proof in first-order logic!) In this setting you have a binary relation $E(x,y)$ to say there is an edge between x and y, a binary relation $P(x,y)$ to say there is a path between x and y in G, and a binary relation $Q(x,y)$ to say there is a path between x and y in $\overline{G}$.

Exercise 5
Using the above vocabulary write down an expression of first-order logic that says that a graph and its complement cannot both be disconnected.

The exercise should continue "Then prove it by any method of your choice: natural deduction, sequent calculus or resolution." However, as of 25/iii/2014 i haven't got a resolution proof (which annoys me, beco's i should) and i haven't attempted the other two—yet. They look a bit hard to do by hand, tho' with a proof assistant it should be fairly easy.

Skolemisation

The two formulæ in this section are related to the fact that Skolemisation is in some sense logically conservative. The reader might like to look long and hard at the formulæ below with a view to seeing what they mean.

Exercise 6
Prove the two following formulæ by resolution or sequent calculus.

1.
$$(\forall x \exists y) R(x, y) \to (\forall x \exists y)(\forall x' \exists y')(R(x, y) \land R(x', y') \land (x = x' \to y = y'))$$

2.
$$(\forall x \exists y) R(x, y) \to (\forall x \exists y)(\forall x' \exists y')(\forall x'' \exists y'') \bigwedge \begin{pmatrix} R(x, y) \\ R(x', y') \\ R(x'', y'') \\ x = x' \to y = y' \\ x = x'' \to y = y'' \\ x' = x'' \to y' = y' \end{pmatrix} \tag{1}$$

In the proof by resolution you will need resolution rules for functions and equality, such as

$$\{\neg(x = y), f(x) = f(y)\}$$

I have sequent proofs but—annoyingly—no resolution proofs at this stage.

Berkeley's Master Argument for Idealism

The following text is a celebrated argument by Bishop Berkeley which purports to show that nothing exists unconceived. It's a fairly delicate exercise in formalisation.

> HYLAS : What more easy than to conceive of a tree or house existing by itself, independent of, and unperceived by any mind whatsoever. I do at present time conceive them existing after this manner.
>
> PHILONOUS : How say you, Hylas, can you see a thing that is at the same time unseen?
>
> HYLAS : No, that were a contradiction.
>
> PHILONOUS : Is it not as great a contradiction to talk of *conceiving* a thing which is *unconceived*?
>
> HYLAS : It is
>
> PHILONOUS : This tree or house therefore, which you think of, is conceived by you?
>
> HYLAS : How should it be otherwise?
>
> PHILONOUS : And what is conceived is surely in the mind?
>
> HYLAS : Without question, that which is conceived exists in the mind.
>
> PHILONOUS How then came you to say, you conceived a house or a tree existing independent and out of all mind whatever?
>
> HYLAS That was I own an oversight ...

The exercise here is to formalise the above conversation and construct a natural deduction proof that everything is conceived (as Berkeley wants) and perhaps even a sequent calculus proof. This has been discussed in print by my friend Graham Priest, and this treatment draws heavily on his.

You may, if you wish to think through this exercise very hard, try to work out what new syntactic gadgets one needs to formalise this argument, but i don't recommend it. The best thing is to use the gadgetry Priest introduced.

Priest starts off by distinguishing, very properly, between **conceiving objects** and **conceiving propositions**. Accordingly in his formalisation he will have *two* devices. One is a sentence operator T which is syntactically a modal operator and a predicate τ whose intended interpretation is that $\tau(x)$ iff x is conceived. $T\phi$ means that the proposition

ϕ is being entertained. (*By whom* is good question: is the point of the argument that for every object there is someone who conceives it? or that everybody thinks about every object?)

At this point you could, if you like, work out your own natural deduction rules. Here are the rules Priest came up with.

$$\frac{\phi \to \psi}{T(\phi) \to T(\psi)}$$

which says something to the effect that T distributes over conditionals. Priest calls this "affixing". The other rule is one that tells us that if we conceive an object to have some property ϕ then we conceive it.

$$\frac{T(\phi(x))}{\tau(x)}$$

Let us call it the **mixed rule**.

Now it's your turn to do some work.

EXERCISE 7

1. *Devise a natural deduction proof of* $(\forall x)(\tau(x))$*, or of* $(\forall x)((\tau(x) \to \bot) \to \bot)$*. You are allowed to use the undischarged premiss* Tp *where* p *is an arbitrary propositional letter. You may wish to use a natural deduction version of the law of excluded middle. My model answer doesn't, and accordingly i prove only that* $(\forall x)((\tau(x) \to \bot) \to \bot)$*. You might try to prove* $(\exists x)(\tau(x) \to \bot) \to \bot$ *as well.*

   At this point you could, if you like, work out your own sequent calculus rules. Here are the rules i came up with.

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma, TA \vdash \Delta, TB}$$

   and

$$\frac{\Gamma \vdash \Delta, T(\phi(x))}{\Gamma \vdash \Delta, \tau(x)}$$

2. *Prove the sequent* $Tp \vdash (\forall x)(\tau(x))$
3. *Prove that a premiss of the form* Tp *really is needed.*

A Special Kind of Poset

EXERCISE 8

(i) What is an antichain in a poset?
(ii) Show that the theory of posets in which every element belongs to a unique maximal antichain can be axiomatised by adding to the theory of posets either of:
A: $(\forall xyz)(z > x \not\leq y \not\leq x \to z < y)$*; or*
B: $(\forall xyz)(z > x \not\leq y \not\leq x \to z > y)$*.*
(iii) Show that A and B are equivalent, by natural deduction, sequent calculus or resolution.

I have no sample answers at this stage. The usual bribes are available.

This is of some interest, since "posets in which every element belongs to a unique maximal antichain" is *prima facie* second order (it appears to be Σ^2_2, with two second order quantifiers) but it is in fact first order. The question of whether or not a given *prima facie* second-order theory is in fact first order is presumably unsolvable.

Another (random) exercise

Exercise 9 *Attempt 1993:3:3*

Must write this out properly at some point. Try to prove the nonexistence of the Russell class by resolution. You need what Larry calls *factoring*. . . which seems to be a kind of contraction

Derive a contradiction from

$$(\exists y)(\forall x)(x \in y \longleftrightarrow x \notin x)$$

Skolemising gives $x \in a \longleftrightarrow x \notin x$ which is a conjunction of $x \in a \to x \notin x$ and $x \notin x \to x \in a$ whence two clauses

$$\{x \notin a \vee x \notin x\} \text{ and } \{x \in x \vee x \in a\}$$

Clearly you have to bind 'x' $\mapsto$ 'a', whereupon the first clause becomes $\{a \notin a \vee a \notin a\}$ and the second becomes $\{a \in a \vee a \in a\}$.

At this point you have to contract the two occurrences in each clause to obtain singleton clauses $\{a \notin a\}$ and $\{a \in a\}$.