Automorphisms and Antimorphisms in NF

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It is an old question in NF studies whether or not it is consistent that there should be an internal $\in$-automorphism of the universe, a *set*. It has been known for a long time [1] that the existence of such an object is consistent relative to the axiom of choice for pairs. Although the existence of an internal automorphism may be felt to be of independent interest, it is of interest to us specifically because it bears on the question of which NF-like theories can be synonymous with which theories of wellfounded sets. If NF is to be synonymous with a theory of wellfounded sets it must admit a definable wellfounded extensional relation on the universe, and such a thing is plainly in conflict with the existence of an internal $\in$-automorphism. And one would like to be able to state a result that isn't complicated by any use of AC. (There will be no use of AC anywhere in this note). We are now in a position to offer a proof of the consistency of the existence of $\in$-automorphisms without assuming AC_2. This answer to an old question came as a side-effect of a (partial) answer to another—equally old—question, namely: "Does NF admit an (internal) *anti*morphism, a permutation σ of V such that $x \in y \longleftrightarrow \sigma(x) \notin \sigma(y)$?". We present this partial answer below and explain its significance.

Definition 1 *The dual $\widehat{\phi}$ of a formula ϕ is the formula obtained from ϕ by replacing all occurrences of '$\in$' in ϕ by '$\notin$'.*

It has been known for some time that $\phi \longleftrightarrow \widehat{\phi}$ is a theorem of NF whenever ϕ is a closed stratifiable formula. Permutation models can be found in which $\phi \longleftrightarrow \widehat{\phi}$ *fails* for some unstratifiable ϕ, but it remains an open question whether or not there are models in which $\phi \longleftrightarrow \widehat{\phi}$ *holds* for all ϕ. The natural conjecture is that there should be such models.

We do not prove the full conjecture here but we can prove the relative consistency of the scheme $\phi \longleftrightarrow \widehat{\phi}$ ("the duality scheme") at least for all ϕ that are stratifiable-mod-2. There is a general notion of a formula being stratifiable-mod-n but for the moment we concern ourselves only with the special case of stratifiability-mod-2

A formula ϕ is stratifiable-mod-2 iff its variables can be assigned to two types `yin` and `yang` in such a way that in subformulæ like '$x = y$' the two variables receive the same type and in subformulæ like '$x \in y$' the two variables receive different types. Naturally all occurrences of any one variable receive the same decoration.

Naturally there is also the analogous *stratification-mod-n*—whose definition we leave to the reader. It sounds like an innocent—if complex—relaxation of the concept of stratification (itself under a PR cloud) but there is an important difference. Stratification can be used to justify a restriction of the naïve comprehension scheme[1]; not so stratification-mod-n: $\{x : x \notin^n x\}$ is a paradoxical object. This failure of comprehension takes the gloss off the observation that *transitive set* and *wellfounded set* (and therefore even *von Neumann ordinal*!) can—for every n—be captured by formulæ that are stratifiable-mod-n. We omit the proof.

A **transversal** for a disjoint family is a set that meets every member of the family on a singleton.

Lemma 1 *Any two involutions-without-fixed-points whose corresponding partitions-of-V-into-pairs have transversals are conjugate.*

Proof:

First we establish that if $\mathcal{T}$ is a transversal for a partition $\mathbb{P}$ of V into pairs then its cardinality is $|V|$. Clearly $|\mathbb{P}| = T|\mathcal{T}|$, since we can send each piece of $\mathbb{P}$ to the unique singleton $\subset \mathcal{T}$ that meets it. Observe that there is a bijection between ι"V and $\mathbb{P} \times \{0, 1\}$, as follows. For each x there is a unique $p_x \in \mathbb{P}$ with $x \in p_x$. If $x \in \mathcal{T}$ we send $\{x\}$ to $\langle p_x, 0\rangle$; if $x \notin \mathcal{T}$ we send $\{x\}$ to $\langle p_x, 1\rangle$.

Finally if π_1 and π_2 are two involutions-without-fixed-points equipped with transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, then not only do we have $|\mathcal{T}_1| = |\mathcal{T}_2| = |V|$ but π_1 and π_2 are conjugate, as follows. $\mathcal{T}_1$ and $\mathcal{T}_2$ are in bijection, by a map π^*, say. Any such π^* can be extended to a permutation π of the universe by adding all the ordered pairs $\langle \pi_1(x), \pi_2(\pi^*(x))\rangle$ for $x \in \mathcal{T}_1$.

■

The key novel idea here is that of a *universal involution*. That in turn relies on a notion of *permutation morphism*.

Definition 2

For permutations σ and τ of sets X and Y, a **map of permutations** *from σ to τ is a function $\pi : X \to Y$ such that $\pi \cdot \sigma = \tau \cdot \pi$.*

If π is injective, we call it an **embedding of permutations***, and we will write "$\sigma \leq_B \tau$" to say that there is an embedding of permutations from σ to τ.*

An involution is **universal** *if every involution embeds into it.*

In all the cases of interest to us we have here $X = Y = V$ but we shouldn't forget that definition 2 is more general.

For the moment we need definition 2 only for involutions, and we will speak of *involution-embeddings* or *embeddings of involutions*. In due course we will prove (lemma 3) that there are universal involutions, and give examples. We do not address the question of whether there are permutations that are universal for other classes of permutations, interesting though that question is.

[1]Because of recent work by Randall Holmes' work it is increasingly suspected that NF is consistent.

Think of a permutation as a digraph wherein every vertex has indegree one and outdegree one, and loops at vertices are allowed. An embedding-of-permutations must send n-cycles onto n-cycles, and when you look at it like that the Cantor-Bernstein theorem that we are about to prove becomes much more obvious.

Lemma 2 *If σ and τ are permutations of V with $\sigma \leq \tau \leq \sigma$ then σ and τ are conjugate.*

Proof:

Suppose $\sigma \leq \tau$ in virtue of $\rho : V \hookrightarrow V$ and $\tau \leq \sigma$ in virtue of $\pi : V \hookrightarrow V$. Consider the map $\mathcal{P}(V) \hookrightarrow \mathcal{P}(V)$ defined by $S \mapsto V \setminus \rho``(V \setminus \pi``S)$. By Tarski-Knaster this map has a least fixed point, which we will call P. Then the map $V \hookrightarrow V$ given by $\pi \restriction P \cup \rho^{-1} \restriction (V \setminus P)$ conjugates σ to τ. ■

Notice that the map that conjugates σ and τ has a stratifiable definition in terms of them, so if they are definable it is too, and so is its least fixed point. The significance of the lfp lies not in its leastness but rather in the fact that is definable in terms of ρ and π.

In fact—for the moment—we will need lemma 2 only for involutions.

Corollary 1 *Any two universal involutions of V are conjugate.*

■

Lemma 2 is telling us that the intersection of the quasiorder $\leq_B$ and its converse $\geq_B$ is the equivalence relation of conjugacy. This justifies us in additionally using '$\leq_B$' to denote the partial ordering induced on the quotient.

The notation $j(\sigma)$ denotes the permutation $x \mapsto \sigma``x$.

The following elementary facts will loom large.

Remark 1
(i) Conjugacy is a congruence relation for j;
(ii) j is $\leq_B$-order-preserving.

Proof:

(i) is obvious;

For (ii) observe that if π is an embedding of permutations from σ to τ then $j(\pi)$ is an embedding of permutations from $j(\sigma)$ to $j(\tau)$. ■

We will need this in the proof of the second part of lemma 3.

We begin by giving some examples of universal involutions of V. c is the complementation permutation: $x \mapsto V \setminus x$.

Lemma 3 *For all i, $j^i(c)$ is a universal involution.*

Proof:

First we prove that $j(c)$ is universal.

There are bijections $V \longleftrightarrow \{x : \emptyset \notin x\}$; in what follows fix θ to be one of them—it won't matter which.

There is a standard construction of a type-level ordered pair due to Quine in which we set $\langle x, y\rangle := j(\theta)(x) \cup j(c \cdot \theta)(x)$. Note that $j(c)$ simply reverses the order of elements of such ordered pairs: $j(c)(\langle x, y\rangle) = \langle y, x\rangle$.

For any involution σ we can define an embedding of involutions π from σ to $j(c)$ by

$$x \mapsto \langle x, \sigma(x)\rangle$$

The function π is injective, since $\langle -, -\rangle$ is an implementation of ordered pairs.

To see that π is a map of involutions from σ to $j(c)$ we calculate as follows:

$$\begin{aligned}
(j(c) \cdot \pi)(x) &= j(c)(\pi(x)) && \text{Expand } \pi(x) \text{ to get}\\
&= j(c)[\langle x, \sigma(x)\rangle] \\
&= \langle \sigma(x), x\rangle \\
&= \langle \sigma(x), \sigma(\sigma(x))\rangle && \text{because } \sigma \text{ is an involution}\\
&= \pi(\sigma(x))
\end{aligned}$$

For the main result we argue as follows.

Clearly any involution into which a universal involution can be embedded is also universal, and any involution conjugate to a universal involution is again universal.

$j(c)$ is universal, so there is an embedding of c into $j(c)$. By part (ii) of Remark 1 this embedding lifts to embeddings of $j^i(c)$ into $j^{i+1}(c)$, and composing these embeddings we get embeddings of $j(c)$ into $j^i(c)$ for any $i \geq 1$. Thus $j^i(c)$ is universal for any $i \geq 1$.

■

Of some interest will be the sequence of permutations: $\mathbb{1}$, c, $jc \cdot c$, $j^2c \cdot jc \cdot c$ …, where c is the complementation permutation. We will write them (as is customary) as 'c_i', thus: $c_1 := c$; $c_{i+1} := j(c_i) \cdot c$.

Lemma 4

(i) All the c_i are involutions;

(ii) All the c_i commute with each other.

Proof:

(i) We prove this by induction on i. Suppose c_i is an involution. $c_{i+1} = jc_i \cdot c$. So $(c_{i+1})^2 = (jc_i \cdot c)^2 = jc_i \cdot c \cdot jc_i \cdot c$. Now by the key triviality we can rearrange to $jc_i \cdot jc_i \cdot c \cdot c = \mathbb{1}$.

In fact this even shows that all products of the c_i are involutions.

(ii) We prove by induction on i that, for all j, c_i commutes with c_j.

Case $i = 0$. $c_0 = c$ and c commutes with $j(\pi)$ for all π. But every c_j is $j(\pi) \cdot c$ for some π, and (compose with c on the right) $j(\pi) \cdot c \cdot c = j(\pi)$ and if we compose with c on the left we get $c \cdot j(\pi) \cdot c$ which, too, is $j(\pi)$ because c commutes with $j(\pi)$.

Now for the induction.

$$c_{i+1} \cdot c_j = j(c_i) \cdot c \ \cdot \ j(c_{j-1}) \cdot c$$

and the RHS simplifies to

$$j(c_i) \cdot j(c_{j-1})$$

which is

$$j(c_i \cdot c_{j-1})$$

which by induction hypothesis is

$$j(c_{j-1} \cdot c_i)$$

which is

$$j(c_{j-1}) \cdot j(c_i).$$

We now sprinkle a couple of cs judiciously—by the triviality we know we can insert them anywhere—obtaining

$$j(c_{j-1}) \cdot c \ \cdot \ j(c_i) \cdot c$$

which is of course

$$c_j \cdot c_{i+1}.$$

■

In what follows we write $B(x)$ for $\{y : x \in y\}$.

Remark 2
Let σ and τ be involutions of V.

(1) Let τ be an involution without fixpoints. Then $\mathcal{T}$ is a transversal for τ iff $\mathcal{T}$ is a fixpoint for $j\tau \cdot c$;

(2) $\mathcal{T}$ is a fixpoint for σ iff $B(\mathcal{T})$ is a transversal for $j\sigma \cdot c$.

Proof:
(1) Think of τ as a partition of V into pairs. Then, if $\mathcal{T}$ is a transversal, $V \setminus \mathcal{T}$ (which is also a transversal) is precisely $\tau``\mathcal{T}$.

(2) A piece of [the partition] $j\sigma \cdot c$ is a pair $\{x, V \setminus \sigma``x\}$—which of course might be a singleton. If $\sigma(T) = T$ then, for all $x$, precisely one of $x$ and $V \setminus \sigma``x$ will contain T. That is to say, $\{x, V \setminus \sigma``x\} \cap B(\mathcal{T})$ is a singleton, so $B(\mathcal{T})$ is a transversal.

For the other direction ... if $B(\mathcal{T})$ is a transversal for $j\sigma \cdot c$ then, for all x, precisely one of x and $V \setminus \sigma``x$ contains T, which is to say that $\mathcal{T} \in x \longleftrightarrow \sigma(\mathcal{T}) \in x$. In particular let x be $\{\mathcal{T}\}$; then $\mathcal{T} \in \{\mathcal{T}\} \longleftrightarrow \sigma(\mathcal{T}) \in \{\mathcal{T}\}$, so $\sigma(\mathcal{T}) = \mathcal{T}$.

■

Corollary 2

(i) For any ultrafilter $\mathcal{U}$ on V, $B^n(\mathcal{U})$ is a transversal for c_{2n+1};
(ii) All the c_{2n+1} are conjugate;
(iii) For all $n \geq 1$, c_n is conjugate to c_{n+2}.

Proof:

(i) We do an induction on n.

For the case $n = 0$ any ultrafilter is a transversal for c.
Suppose for the induction that $B^{n-1}(\mathcal{U})$ is a transversal for c_{2n-1}.
Consider

$$c_{2n+1}(A) \in B^n(\mathcal{U}).$$

By definition of B this is the same as

$$B^{n-1}(\mathcal{U}) \in c_{2n+1}(A)$$

Now $c_{2n+1}(A) = V \setminus (c_{2n}``A)$, so we can rewrite the displayed formula as

$$c_{2n}(B^{n-1}(\mathcal{U})) \notin A.$$

By induction hypothesis $B^{n-1}(\mathcal{U})$ is a transversal for c_{2n-1}, which is to say that $B^{n-1}(\mathcal{U})$ is a fixed point for c_{2n}. So rewrite '$c_{2n}(B^{n-1}(\mathcal{U}))$' as '$B^{n-1}(\mathcal{U})$'; this turns our formula-in-hand into

$$B^{n-1}(\mathcal{U}) \notin A$$

which (by definition of B) becomes

$$A \notin B^n(\mathcal{U}).$$

So we have proved

$$c_{2n+1}(A) \in B^n(\mathcal{U}) \longleftrightarrow A \notin B^n(\mathcal{U})$$

... which is to say that $B^n(\mathcal{U})$ is a transversal for c_{2n+1}.

(ii) now follows by lemma 1.

(iii) By induction on n.

The case $n = 1$ we know from (ii).
For the induction step suppose π conjugates c_n to c_{n+2}, which is to say

$$\pi \cdot c_n \cdot \pi^{-1} = c_{n+2}$$

Lift by j:

$$j\pi \cdot j(c_n) \cdot (j\pi)^{-1} = j(c_{n+2})$$

compose both sides with c on the right:

$$j\pi \cdot j(c_n) \cdot (j\pi)^{-1} \cdot c = j(c_{n+2}) \cdot c$$

But c commutes with $(j\pi)^{-1}$ so we can rearrange the LHS, and $j(c_{n+2}) \cdot c = c_{n+3}$ on the RHS giving

$$j\pi \cdot \underline{j(c_n) \cdot c} \cdot (j\pi)^{-1} = c_{n+3}$$

Now $j(c_n) \cdot c$ (underlined) $= c_{n+1}$ giving

$$j\pi \cdot c_{n+1} \cdot (j\pi)^{-1} = c_{n+3}$$

as desired.

■

Also worth minuting is the fact that

Remark 3 *Conjugacy is a congruence relation for the operation $\pi \mapsto j\pi \cdot c$.*

Proof:
Suppose σ and τ are conjugate; so, for some π,

$\pi \cdot \sigma \cdot \pi^{-1} = \tau$;	do j to both sides:
$j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) = j(\tau)$;	compose with c:
$j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) \cdot c = j(\tau) \cdot c$;	but c commutes with j of anything, giving:
$j(\pi) \cdot j(\sigma) \cdot c \cdot j(\pi^{-1}) = j(\tau) \cdot c$	

which says that $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ are conjugate.

■

Remark 4 *c_2 is conjugate to $j(c)$ and so is also universal.*

Proof:
Given a set of the form $x \triangle B(\emptyset)$ we can recover x since it is $(x \triangle B(\emptyset)) \triangle B(\emptyset)$. So $x \mapsto x \triangle B(\emptyset)$ is injective. But the same thought reassures us that it is surjective too, so it is genuinely a permutation of V and, actually, an involution. In fact we can write it $\prod_{x \in V}(x, x \triangle B(\emptyset))$ as a product of disjoint transpositions ... or π for short. To see that π conjugates c_2 to $j(c)$, we calculate as follows:

we cld delete this remark, since it doesn't get used, but it is nice

$$\begin{aligned}
(j(c) \cdot \pi)(x) &= j(c)(x \triangle B(\emptyset)) \\
&= j(c)(x) \triangle j(c)(B(\emptyset)) \\
&= j(c)(x) \triangle (V \setminus B(\emptyset)) \\
&= j(c)(x) \triangle (V \triangle B(\emptyset)) \\
&= (j(c)(x) \triangle V) \triangle B(\emptyset) \\
&= (V \setminus j(c)(x)) \triangle B(\emptyset) \\
&= (c \cdot j(c))(x) \triangle B(\emptyset) \\
&= c_2(x) \triangle B(\emptyset) \\
&= (\pi \cdot c_2)(x)
\end{aligned}$$

■

The medium-term hope is that an analysis of the congruence classes of involutions will lead to a proof that the operation $\sigma \mapsto j\sigma \cdot c$ will have a fixed point (up to conjugacy) which will be enough to provide a permutation containing an antimorphism. But we are not there yet. Meanwhile we report a side-effect of this investigation.

Corollary 3
Every model of NF has a permutation model with an internal $\in$-automorphism.

Proof: It follows from corollary 1 that $j(c)$ and $j^2(c)$ are conjugate, making $j(c)$ an example of a permutation which is conjugate to j of itself. It was shown in [1] that any model containing such a permutation π has a permutation model wherein π has become an (internal) $\in$-automorphism. ■

In [1] it is shown that there must be such a π, but that was on the assumption of AC_2, and of course we have here scrupulously eschewed AC_2.

Very well, we have shown that every model of NF has a permutation model that houses a nontrivial $\in$-automorphism; can we be sure (always assuming that NF has models at all) that it has models *lacking* such automorphisms? Or is it perhaps the case that every model of NF has a permutation model that does *not* have an internal $\in$-automorphism? One would expect so, but any existence proof of a permutation would have to be fairly nonconstructive, for the following reason.

Suppose π is an $\in$-automorphism, so

$$\pi^{-1} \cdot j\pi = \mathbb{1}$$

and let σ be any permutation which is definable in the sense of commuting with every $\in$-automorphism. Then

$$\pi^{-1} \cdot \underline{j\pi \cdot \sigma} = \sigma.$$

Now π is an $\in$-automorphism, so σ commutes with it and we can permute the underlined terms to obtain successively

$$\pi^{-1} \cdot \sigma \cdot j\pi = \sigma.$$

$$\pi^{-1} \cdot \sigma \cdot j\pi \cdot \sigma^{-1} = \mathbb{1}.$$

$$\sigma \cdot j\pi \cdot \sigma^{-1} = \pi.$$

…which says that π is an $\in$-automorphism in V^{σ}.

Or do we mean $V^{\sigma^{-1}}$ …?

This is a situation that invites further investigation. At present we know how to find models with $\in$-automorphisms starting from models without them, but not how to show that there are models without them to start from. Suppose there were a model without an $\in$-automorphism; we can proced to a permutation model that does have one, by means of the permutation we have seen—which is definable. However, the fact that definable permutations cannot get rid of automorphisms means that the "return" permutation in the new permutation model cannot be definable in that model. Unless NF actually proves the existence of $\in$-automorphisms (which we do not believe for a moment) this must mean that there are definable permutations whose "return" permutations are not definable.

The duality scheme is consistent at least for formulae that are stratifiable-mod-2

The duality scheme $\phi \longleftrightarrow \widehat{\phi}$ is of course implied by the existence of an antimorphism, but antimorphisms contradict AC_2 and we want to derive the consistency of this scheme without having to assume $\neg AC_2$, so trying to obtain duality by getting an antimorphism is not the way to go.

Lemma 5 *Suppose we have two permutations σ and τ such that*

$$(\forall x, y)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y)) \qquad \textit{and} \qquad (\forall x, y)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y))$$

The existence of such a pair entails duality for expressions that are stratifiable-mod-2.

Proof:

Recall that if a formula ϕ is stratifiable-mod-2 then its variables can be assigned to two types `yin` and `yang` in such a way that in subformulæ like '$x = y$' the two variables receive the same type and in subformulæ like '$x \in y$' the two variables receive different types.

Let us associate σ to variables given type `yin` in the assignment and associate τ to variables given type `yang` in the assignment. '$x \in y$' is equivalent to '$\sigma(x) \notin \tau(y)$' and if x is of type `yin` we make this substitution. ' $x \in y$' is also equivalent to '$\tau(x) \notin \sigma(y)$' and if x is of type `yang` we make this substitution. We deal with equations analogously. ϕ is clearly logically equivalent to this rewritten version of ϕ, and in this rewritten version we find that every variable 'x' of type `yin` now appears only as '$\sigma(x)$' and that every variable 'y' of type `yang` now appears only as '$\tau(y)$'. So we can reletter '$\sigma(x)$' as 'x', and '$\tau(y)$' as 'y' and the result is $\widehat{\phi}$. ■

Accordingly we desire a permutation model containing two permutations σ and τ such that

$$(\forall x, y)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y)) \qquad \text{and} \qquad (\forall x, y)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y))$$

It follows easily that we must have

$$\sigma = c \cdot j\tau \qquad \text{and} \qquad \tau = c \cdot j\sigma$$

If this is to happen in a permutation model V^π, what does this tell us about π? We need

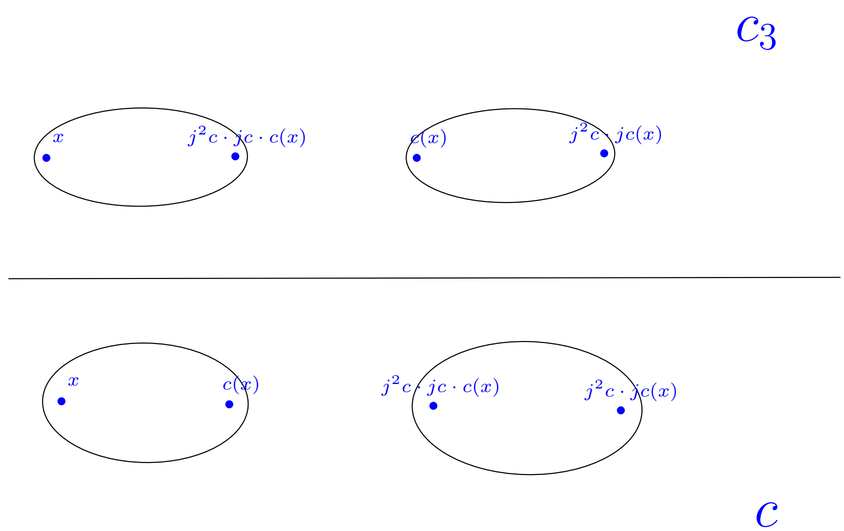
$$\pi\sigma\pi^{-1} = c \cdot j\tau \qquad \text{and} \qquad \pi\tau\pi^{-1} = c \cdot j\sigma$$

…so what are π, σ and τ to be? Material above contains the two permutations c and $c \cdot jc \cdot j^2c$, and it would seem to be worth trying them as candidates for σ and τ. They both lack fixed points—which is a good start. If π conjugates a to b then π^{-1} will conjugate b to a, and we seek a *single* permutation that conjugates c to $c \cdot jc \cdot j^2c$, and *vice versa*, so the optimist's first try is to look for π to be an involution. And we are in luck.

Lemma 6 *There is an involution that conjugates c with c_3.*

Proof:

The universe partitions naturally into bundles closed under both c_1 and c_3. Each such bundle contains precisely four sets. We will define a permutation π in such a way that it fixes each bundle setwise. It will turn out that the π we define is the permutation we seek.



In the accompanying picture we have written a typical bundle twice: once below the line where it is divided into two c-cycles and once above the line where it is divided into two c_3-cycles. We have to biject the set of points below the line with the set of points above the line in a way that respects the two partitions into cycles. Evidently this can be done so we pick one such way for each bundle. By corollary 2 (i) we have transversals for c_3 and c. The transversal for c_3 highlights precisely one element in each pair upstairs, namely that element that contains $B(\emptyset)$. These two highlighted elements cannot—downstairs—belong to different pairs because the downstairs pairs are complements and two complementary sets cannot both contain $B(\emptyset)$.

To illustrate, suppose in the picture that upstairs we highlight x and (therefore) $j^2 c \cdot jc(x)$. We tell π to fix these two sets, and that compels it to swap $c(x)$ and $c_3(x)$.

The other possibility is that we highlight $c_3(x)$ and $c(x)$, and then we tell π to fix those two sets and to swap x and $j^2 c \cdot jc(x)$.

Either way the net result is that π is

$$\texttt{if } B(\emptyset) \in x \texttt{ then } x \texttt{ else } j^2 c \cdot jc(x).$$

Reflect that $B(\emptyset) \in x$ iff $B(\emptyset) \in j^2 c \cdot jc(x)$, and $j^2 c \cdot c$ is an involution. So, if $B(\emptyset) \in x$, it follows that $\pi(x) = x$ and then $\pi^2(x) = x$; if $B(\emptyset) \notin x$ then $\pi(x) = j^2 c \cdot jc(x)$ which does not contain $B(x)$ either. So $\pi^2(x) = \pi(j^2 c \cdot jc(x)) = j^2 c \cdot jc \cdot j^2 c \cdot (x) = x$ and $\pi^2(x) = x$. So π is an involution.

■

COROLLARY 4

Every model of NF has a permutation model that contains two (internal) permutations σ and τ satisfying

$$(\forall xy)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y)) \text{ and } (\forall xy)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y)).$$

Furthermore any such model satisfies duality for formulæ that are stratifiable-mod-2.

References

[1] Thomas Forster "Set Theory with a Universal Set" Oxford Logic guides **20**. Oxford University Press 1992.

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