Model Tripos Questions for Revision

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The first three are model questions. Questions 4 and 5 are old tripos question.

Question 1

A **circular order**[1] is a ternary relation $R(x, y, z)$, whose typical example is the relation that holds between points x, y and z on the unit circle if, starting from x and moving clockwise, one encounters y before z.

(1) *Write down a set of axioms for circular orders.*

A group is **circularly-orderable** if it has a circular ordering that interacts in the obvious way with the multiplication of the group. The typical example is the additive group of integers-mod-p.

(2) *Write down a set of axioms for circularly orderable groups.*

(3) *Prove that a group is circularly orderable iff all its finitely generated subgroups are circularly orderable.*

(4) *Is the multiplicative group of (nonzero) integers mod p (p prime) circularly ordered?*

[1] See Edward V. Huntingdon 'Inter-relations among the four principle types of order' Transactions of the American Mathematical Society **38** (1935) pp 1–9.

Question 2

A **pedigree** is a set P with two unary total functions $f : P \to P$ and $m : P \to P$, with disjoint ranges. ($m(x)$ is x's mother and $f(x)$ is x's father.)

(i) Set up a first-order language $\mathcal{L}$ for pedigrees and provide axioms for a theory T_1 of pedigrees.

A pedigree may be **circle-free**: in a realistic pedigree no one is their own ancestor! Realistic pedigrees are also **locally finite**: no one is the father or mother of infinitely many things.

(ii) One of these two new properties is first-order and the other is not. Give axioms for a theory T_2 of the one that is first-order and an explanation of why the other one is not.

A **fitness function** is a map v from P to the reals satisfying $v(x) = (1/2) \cdot \sum_{f(y)=x} v(y)$ or $v(x) = (1/2) \cdot \sum_{m(y)=x} v(y)$ (depending on whether x is a mother or a father).

(iii) Find a sufficient condition for a pedigree to have a nontrivial fitness function and a sufficient condition for it to have no nontrivial fitness funtion.

(iv) Extend your language $\mathcal{L}$ to include syntax for v. In your new language provide axioms for a new theory T_3 that is to be a conservative extension[2] of T_1 and whose locally finite models are precisely the locally finite pedigrees with a nontrivial fitness function.

There is an obvious concept of *generation* for a pedigree.

(v) Expand $\mathcal{L}$ by adding new predicate(s) and give a first-order theory in this new language for pedigrees that have well-defined generations. Give first-order axioms in $\mathcal{L}$ itself for a theory of pedigrees that have well-defined generations.

(vi) When can one make sense of the idea of the fitness of an entire generation? How can fitness change from one generation to the next?

(vii) Add axioms to your theory of pedigrees admitting-a-concept-of-generation to obtain an $\aleph_0$-categorical theory.

[2]This means that altho' you have theorems expressible in the new vocabulary you have no new theorem expressible in the old vocabulary

Question 3

A *type* in a propositional language $\mathcal{L}$ is a set of formulæ (a countably infinite set unless otherwise specified).

For T an $\mathcal{L}$-theory a T-*valuation* is an $\mathcal{L}$-valuation that satisfies T. A valuation v *realises* a type Σ if $v(\sigma) = \texttt{true}$ for every $\sigma \in \Sigma$. Otherwise v *omits* Σ. We say a theory T *locally omits* a type Σ if, whenever ϕ is a formula such that T proves $\phi \to \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg\phi$.

Now prove the following:

(i) *Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type. If T locally omits Σ then there is a T-valuation omitting Σ.*

(ii) *Let T be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq \mathcal{L}(T)$ be a type. If T locally omits every Σ_i then there is a T-valuation omitting all of the Σ_i.*

[Hint: Show that, if n is such that you can find a family $\langle \phi_i : i \leq n \rangle$, with ϕ_i in Σ_i for every $i < n$ s.t. $T \cup \{\bigwedge_{i \leq n} \neg\phi_i\}$ is consistent, then you can extend this family to one of length $n + 1$.]

For further reading have a look at `yabloomittingtypes.pdf` linked from my home page.

Question 4

(i) State and prove the Tarski-Knaster fixed point theorem for complete lattices.

(ii) Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ be injections. By considering $F : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$F(A) = X \setminus g``(Y \setminus f``A)$$

or otherwise, show that there is a bijection $h : X \to Y$.

Suppose U is a set equipped with a group Σ of permutations. We say that a map $s : X \to Y$ is *piecewise*$-\Sigma$ just when there is a finite partition $X = X_1 \cup \ldots \cup X_n$ and $\sigma_1 \ldots \sigma_n \in \Sigma$, so that $s(x) = \sigma_i(x)$ for $x \in X_i$. Let X and Y be subsets of U, and $f : X \to Y$ and $g : Y \to X$ be piecewise$-\Sigma$ injections. Show that there is a piecewise$-\Sigma$ bijection $h : X \to Y$.

If $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ are two posets with order-preserving injections $f : P \to Q$ and $g : Q \to P$, must there be an isomorphism? Prove or give a counterexample.

Question 5

(2002:B2:11b)

1. State Zorn's lemma.

2. Let U be an arbitrary set and $\mathcal{P}(U)$ be the power set of U. For X a subset of $\mathcal{P}(U)$, the **dual** $X^\vee$ of X is the set $\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}$.

3. Is the function $X \mapsto X^\vee$ monotone? Comment.

4. By considering the poset of those subsets of $\mathcal{P}(X)$ that are subsets of their duals, or otherwise, show that there are sets X such that $X = X^\vee$.

5. What can you say about the fixed points of $X \mapsto X^\vee$ on the assumption that U is finite?