Extracted models and the Independence of Extensionality

Thomas Forster

February 25, 2007

First, some slang. If T is a name for a system of axiomatic set theory (with extensionality of course), then TU is the name for the result of weakening extensionality to the assertion that *nonempty* sets with the same elements are identical. 'U' is for 'Urelemente'—German for 'atoms'.

We start with a model $\langle V, \in \rangle$ of ZF. The traditional method is to define a new membership relation by taking everything that wasn't a singleton to be empty, and then set $y\ IN\ z$ iff $z = \{x\}$ for some x such that $y \in x$: it turns out that the structure $\langle V, IN \rangle$ is a model of ZFU. However there is nothing special about the singleton function here. Any injection from the universe into itself will do. So let's explore this. We start with a model $\langle V, \in \rangle$ of ZF, and an injection $f : V \to V$ which is not a surjection (such as ι).

We then say $x \in_f y$ is false unless y is a value of f and $x \in f^{-1}(y)$. (So that everything that is not an (as it might be) singleton has become an empty set (an *urelement*) in the sense of $\in_f$).

This gives us a new structure: its domain is the same universe as before, but the membership relation is the new $\in_f$ that we have just defined.

Now we must prove that the structure $\langle V, \in_f \rangle$ is a model of ZF with extensionality weakened to the assertion that *nonempty* sets with the same elements are identical.

What is true in $\langle V, \in_f \rangle$? Try pairing, for example: what is the pair of x and y in the sense of $\in_f$? A moment's reflection shows that it must be $f\{x, y\}$: if you are a member of $f\{x, y\}$ in the sense of $\in_f$ then you are a member of $f^{-1} \cdot f\{x, y\}$, so you are obviously x or y. Think about this until you are happy about it. Then try power set and sumset. (The power set of x in the new sense must be f of the set of those things that are subsets-of-x-in-the-new-sense ...). Only later should you start worrying about proving a theorem about what statements are preserved.

For use later

We define an embedding $e : V \to V$ by recursion on $\in$ by

$$e(x) =: f(e``x).$$

It's easy to show that $x \in y$ iff $e(x) \in_f e(x)$ so e is rather nice. You are probably comfortable with the idea of an **end-extension** in connection with, say, linear orders. "All the new stuff is put *on the end*". There is a corresponding notion of an end-embedding: the thing being embedded-into is an end-extension of the range of the embedding. There is a corresponding notion of end-extension in models of set theory. You have a model of set theory. Add some new sets to it. As long as none of the original sets acquire new members when you do this you say we have an *end-extension*. "No new members of old sets!". This is an important notion.

The injection we have just defined—e—is an end-embedding. It's also what we call a **$\mathcal{P}$-embedding**, namely an embedding that not only adds no new members of old sets but doesn't even add any new subsets of old sets. A $\mathcal{P}$-embedding preserves $\Delta_0^{\mathcal{P}}$ formulæ, where the class of $\Delta_0^{\mathcal{P}}$ formulæ is the smallest class containing atomics and closed under boolean operations and restricted quantification AND $(\forall x \subseteq Y)(\ldots$ and $(\exists x \subseteq Y)(\ldots$. A $\mathcal{P}$-embedding not only adds no new members of old sets (so it's an end-extension) it also adds no new subsets.

There are $\mathcal{P}$-embeddings all over the place: the embedding from a well-founded model of ZF into any Rieger-Bernays permutation model of it is always a $\mathcal{P}$-embedding.

I claim e is a $\mathcal{P}$-embedding. It is a simple matter to check that the range of e is the whole of M. It is easy too to check that anything which $\mathcal{M}$ believes to be a subset of a thing in the range of e is also in the range of e, and this makes e into a $\mathcal{P}$-embedding.

For T a theory in the language of set theory let T^* be T with an extra unary function letter: *, and two new axioms, (i) $(\forall x)(x^*$ is an *urelement*); (ii) $(\forall x, y)(x^* = y^* \longleftrightarrow x = y)$. We interpret T^* in T by means of a map σ defined on formulæ in the extended language as follows:

- σ of $\in$ is $\in_f$.
- σ of $y = x^*$ is to be $y = g(x)$, where g is any old injective function whose range is disjoint from the range of f. .

and we extend σ to all other formulæ by recursion.
We then check that σ of an axiom of ZF(C)U is a theorem of ZF(C)*.