Set theory & Logic: exercise sheet

Thomas Forster

June 28, 2013

Contents

Most of this stuff is not actually examinable in the sense that anybody is going to ask you these questions in the tripos: rather it is intended to give you a taste of what sort of thing you can do with what you are learning in this course. Very few of these exercises are genuinely hard, (and those are marked as such) but you may have difficulty locating the (on the whole fairly straightforward) proofs. You are not to feel you have good reason for despair if you find many of them difficult or obscure: plenty of them are both. This sheet is still experimental, and comments and reports of mistakes or infelicities are welcome.

Beware here set-theoretic notation: $f`x$ is the value that the function f assigns to the argument x. $f``x$ is the set of values of f for arguments in x. In set theory we do not have the kind of clear typing information (e.g. you know which variables range over reals and which over integers) needed to disambiguate the notation "$f(x)$". It should be avoided.

1 Boolean Algebras

In what follows a *fuzzy* is a reflexive symmetrical relation. Prove a representation theorem for fuzzies.

How many (free) bases are there for the free boolean algebra with n generators?

2 Back-and-forth constructions

1. Take two countable dense linear orders without endpoints. (For example two copies of the rationals considered as a ordered set.) In each paint each point red or blue at random, but subject to the condition that the red points are to be dense in the blue and *vice versa*. Prove that there is an order-isomorphism between the two copies which respects the colouring. You have now proved that any two countable dense linear orders without endpoints are isomorphic. What does this tell you about the theory of dense linear order? (*Hint*: remember the completeness theorem and the Skolem-Löwenheim theorem).

2. Prove that any two countable atomless boolean algebras are isomorphic.

3. A countable atomic boolean algebra is *saturated* if every element dominating infinitely many atoms is the join of two elements dominating two disjoint infinite sets of atoms. Prove that any two countable saturated atomic boolean algebras are isomorphic. (It's easy with the hint: show that this condition is equivalent to the requirement that the quotient modulo finite elements is atomless.)

4. Some graph theory to try if you are entirely happy about your answer to the first one: (You do not need any results from graph theory to do this).

let A_n be the assertion that if X and Y are disjoint sets of vertices both of cardinality at most n, then there is a vertex x not in $X \cup Y$ joined to every member of X and to no member of Y. Prove that any two countable graphs satisfying A_n for each finite n are isomorphic.

5. What does this tell you about the relationship between the A_n and an arbitrary sentence in the language of graph theory? (see hint for Q1)

6. Observe that for each k the proportion of graphs with n vertices that satisfy A_k tends to 1 as n gets large. Deduce the zero-one law for first-order properties of graphs.

7. What is the smallest (nontrivial, i.e., at least two members) graph satisfying A_1? (easy); satisfying A_2? (I do not know the answer to this!)

8. If you have done Q1 you have a proof that any two countable dense linear orders are isomorphic *which is symmetrical in the two linear orders that you are considering*. Ask yourself next: what condition do you need to impose on dense linear orders of cardinality $\aleph_1$ (for example) to make the same proof work for them? (*Not hard if you have got this far*). And what in general for dense linear orders of cardinality $\aleph_n$? (*harder*) (It isn't true that any two dense linear orders without endpoints of the same cardinality are isomorphic without extra conditions).

9. (The model companion of ZF^-). Let $\gamma(x, y_1 \ldots y_n)$ be a finite conjunction of some of the following atomic formulas and their negations: $x \in x$; $x \in y_i$ $(i \leq n)$; and $y_i \in x$ $(i \leq n)$. We define the theory T as follows. If

$$\bigwedge_{1 \leq i < j \leq n} y_i \neq y_j \wedge x \neq y_i \wedge \gamma(x, y_1 \ldots y_n)$$

is satisfiable then

$$(\forall y_1 \ldots y_n)(\exists x)[\bigwedge_{1 \leq i < j \leq n} y_i \neq y_j \to \bigwedge_{1 \leq i \leq n} x \neq y_i \wedge \gamma(x, y_1 \ldots y_n)]$$

is an axiom of T.

Prove that T has a unique (up to isomorphism) countable model.

3 Set Theory

1. Consider the following property: every y such that $x \in y$ is disjoint from a finite member of itself. Is the collection of sets with this property a set? What is this property anyway?

2. Define E on the natural numbers by: $n\ E\ m$ iff the n^{th} bit in the binary expansion of m is 1 (Remember to start counting at the 0th bit!!) Do you recognise this structure?

3. If you got that easily consider the following more complicated version: $n\ E_{\mathcal{O}}\ m$ iff either m is even and the n^{th} bit in the binary expansion of $m/2$ is 1 or m is odd and the n^{th} bit in the binary expansion of $(m-1)/2$ is 0. You have almost certainly never seen this structure before: what can you say about it?

4. An *antimorphism* is a permutation π of V so that $\forall x\ y\ x \in y \iff \pi`x \notin \pi`y$. Prove (*without* using the axiom of foundation) that no model of *ZF* has an antimorphism.

   (i) Find an antimorphism of the second structure in this section.

   (ii) Is it unique? (hint: Consider the dual of the preceding structure, i.e., the natural numbers with the relation $n\ E_{\mathcal{O}^*}\ m$ iff either m is even and the n^{th} bit in the binary expansion of $m/2$ is 0 or m is odd and the n^{th} bit in the binary expansion of $(m-1)/2$ is 1. Prove that this is isomorphic to the naturals with $E_{\mathcal{O}}$)

5. Define a ring-with-1 structure on V. (*hint: dolly up the construction on the second question in this section*) Then define a field structure and an algebraicly closed field structure. (*hint: the previous hint no longer helps*).

6. How many transitive subsets does V_ω have?

7. Find a partition $\Pi = A_1 \cup A_2 \cup \ldots$ of V_ω into countably many pieces which are $\in$-isomorphic. (*i.e.*, there is a bijection between A_i and A_j which respects $\in$).

8. Von Neumann proposed an axiom to the effect that a class is a set iff it is strictly smaller than the universe. Prove that adding this to a class

version of Zermelo set theory is the same as adding replacement and Global Choice.

4 Recursion

1. What is a wellfounded relation? The following rule

$$\frac{\forall x[(\forall y)(yRx \to \psi(y)) \to \psi(x)]}{\forall x\psi(s)}$$

   is R**-induction**. Justify R-induction from the assumption that R is wellfounded. Use R-induction to prove that R is wellfounded.

2. Prove by $\in$-induction that there are no non-trivial $\in$-automorphisms. (easy)

3. Prove by $\in$-induction that no set is a member^n of itself.

4. Prove by $\in$-induction that there is no universal set.

5. Show how to define a rank function from the domain of a wellfounded relation which takes values in the ordinals. (easy)

6. Prove that every discrete (i.e. the players make alternate moves at discrete intervals) two-player game (without draws) in which every play is of finite length has a winning strategy for one player or another. (hint: this is an exercise in a section entitled "recursion". Make sure you understand your answer to the previous question.)

7. Does the same hold for discrete two-player games (without draws) in which a play that goes on forever is won by White, and Black wins, if at all, after finitely many moves?

8. In question 5 you have shown that every member of the set $\mathcal{G}$ of discrete two-player games (without draws) in which every play is of finite length has a winning strategy for one player or another. *Hypergame* is the game in which the first player chooses a member of $\mathcal{G}$ which the two players then play, player two starting. Is *Hypergame* in $\mathcal{G}$? Which player, if any, has a winning strategy?

9. Define a relation $\leq$ on V by recursion on $\in$ as follows: The empty set $\leq$ everything else, and when x and y are nonempty $x \leq y$ iff $(\exists z \in y - x)(\forall w \in x - y)(z \leq w)$. Lose some sleep over it for a while. Is this relation a partial order? What can you say about the restriction of $\leq$ to V_ω?

10. The game G_x is played as follows: `I` picks a member x' of x, and loses if he can't; if he can, they then play $G_{x'}$, with `II` starting.

    (a) Show that G_x is determined if x is wellfounded.

    (b) If $x \leq y$ (in the sense of the previous question) and you know who has a winning strategy for G_x (resp. G_y), what does this tell you about who has a winning strategy for the other?

5 The axiom of choice

AC^α_β is that form of the axiom of choice which says that every set (of cardinality α) all of whose members are of cardinality at most β has a selection (choice) function. If either β or α is omitted it is to be understood that there is no restriction in that parameter. Thus AC^ω is AC for countable sets (of anything), AC_{fin} is AC for sets of finite sets, and AC is AC!

1. What are α and β such that you need AC^α_β to prove that a union of countably many countable sets is countable?

2. Let S_X be the full symmetric group on a set X. Prove that in S_X any two permutations of the same cycle type are conjugate in S_X. For which α and β do you need AC^β_α to prove this; how do they depend on X?

3. Prove that any permutation of any set X is the product of two involutions. For which α and β do you need AC^β_α to prove this? How do they depend on X?

4. Prove $\forall n \in \mathbb{N}\ AC^n$. Think about how you might try (and fail!) to use this to prove AC^ω.

5. The *cofinality* of an ordinal α is the least ordinal β such that any wellordering of length α has an unbounded subset of length β. For

example the cofinality of ω is obviously ω. Ordinals equal to their own cofinality are *regular*, otherwise *singular*. What is the cofinality of ω_1, the least uncountable ordinal?

6. Find a sufficient condition on α for $\forall\beta < \alpha\ AC^\beta$ to imply AC^α.

7. Assume the axiom of foundation. When κ is a cardinal H_κ is the collection of all sets hereditarily of power less than κ: that is to say a set is hereditarily of size less than κ iff it is a set of fewer than κ sets all hereditarily of size less than κ. Thus $H_{\aleph_0}$ is precisely V_ω. Is $H_{\aleph_1}$ a set? (This is fairly routine). How much AC have you used?

8. The sequence of beth numbers is defined by transfinite induction: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ taking limits at limits. Prove (and you may use AC to do it) that for all $\alpha \in On$ we have

$$|H_{\beth_\alpha}| = \beth_\alpha$$

6 Algebra

1. An LD-**algebra** has one binary operation $*$ which is left-distributive: $a * (b * c) = (a * b) * (a * c)$. Prove that

   (a) For that each $n \in \mathbb{N}$ there is a unique LD-algebra whose domain is the naturals 0, 1, $\ldots 2^n - 1$ such that $x * 1 = x + 1 \pmod{2^n}$. Call this algebra A_n.

   (b) Show that the map $\lambda x.x \pmod{2^n}$ from A_{n+1} to A_n is an LD-homomorphism.

   (c) Show that the free LD-algebra on one generator has a solvable word problem.

   (d) Let A_∞ be the projective limit of the A_n for $n \in \mathbb{N}$. Is it free?[1]

7 Miscellaneous

1. How many countable total order types are there whose automorphism group is transitive (on singletons)

[1]This is open

2. If $\langle X, \leq \rangle$ is a total order, then a *suborder* of it is a subset $X' \subseteq X$ ordered by the obvious restriction of $\leq$. Prove that $\langle X, \leq \rangle$ is a wellordering iff every suborder of it is isomorphic to an initial segment of it. (easy).

3. Suppose $\langle X, \leq \rangle$ is a total order such that every initial segment is isomorphic to a terminal segment. What can you infer?

4. Show that any partial order in which every subset has a top element is finite. Have you used any choice?

5. Find a dense subset of the plane such that no three points of it are collinear. Is your set countable? Do not attempt to use AC unless you are undertaking to make the set uncountable—it doesn't help)

6. X and Y are two sets with injections $f : X \to Y$ and $g : Y \to X$. In what circumstances can one construct a bijection $\pi : X \to Y$ so that $\pi \circ g \circ \pi = f$? How much AC do you need?

7. $\{n - \frac{1}{m} : n, m \in \mathbb{N}\}$ is a subset of $\mathbb{R}$ which is wellordered (by the usual ordering of $\mathbb{R}$) to length ω^2. The set of ordinals which can be represented in this way by sets of reals is clearly an initial segment of On. Is it a proper initial segment, and if so, what is its sup?

8. Is there a bijection f between the rationals and open intervals in the rationals such that $(\forall$ rationals $p)(p \in f(p))$? Do not use AC.

9. Is it possible to define a partition $\sim$ of $\mathbb{R}$ (the reals) such that every choice set for $\mathbb{R}/\sim$ has the order type of the rationals?

10. Functions $\mathbb{N} \to \mathbb{N}$ can be preordered by *dominance* thus: $f < g$ if for all suff large n, $f`n < g`n$. Consider polynomials in one variable with coefficients in $\mathbb{N}$, totally ordered by dominance. What can you say about this total order? How long is the total order? How big is the underlying set? Now consider the slightly fatter set, containing functions of one variable $\mathbb{N} \to \mathbb{N}$ which is inductively generated from the set of polynomials in one variable with coefficients in $\mathbb{N}$ by allowing x^f whenever we have f. Thus we have things like

$$x^{x^{x^9+3}+x^{x^2}+x^2+5} + x^{x^5+10} + x^{1000}$$

How many functions are there like this? What do you get if you order them by dominance?

11. A Cretan says “Everything I say is false”. (Do not confuse this with the better known version “What I am now saying is false”!) Has she spoken truly or falsely, and what follows in each case?

12. Let X be an infinite set and let $\mathcal{G}$ be a group of permutations of X. For Y, Z subsets of X say $Y \leq_{\mathcal{G}} Z$ iff $\exists n \in N\ \exists g_1 \ldots g_n \in \mathcal{G}\ \exists Y_1 \ldots Y_n$ a partition of Y so that $\bigcup_{i<n} g_i``Y_i \subseteq Z$.

    (i) Consider the case where $\mathcal{G}$ is the full symmetric group on X. Prove that $Y \leq_{\mathcal{G}} Z$ & $Z \leq_{\mathcal{G}} Y$ iff Y and Z are the same size. You will need your answers to Q2 in this section and to Q1 of the next section.

    (ii) Let X be the 3-sphere and let $\mathcal{G}$ be the group of rigid motions. Prove a Schröder-Bernstein theorem for this version of $\leq_{\mathcal{G}}$, (where $Y \sim_{\mathcal{G}} Z$ iff $\exists n \in N\ \exists g_1 \ldots g_n \in \mathcal{G}\ \exists Y_1 \ldots Y_n$ a partition of Y so that $\bigcup_{i<n} g_i``Y_i = Z$.

    Deduce Tarski's theorem on paradoxical decompositions of the sphere. (Recommended for those with a background in analysis and measure theory)

8 Cardinal arithmetic

You are *not* to use the axiom of choice for any of these questions! It would render them trivial.

1. Prove that there can be no cardinal α such that $2^{2^{\alpha}} = 2^{\aleph_0}$

2. How large is $\Pi_{\mathbb{N}}$, the set of partitions of $\mathbb{N}$? How many partitions of $\mathbb{N}$ into finite pieces, into finitely many pieces?

3. Prove the following elementary theorem on cardinal arithmetic due to Tarski:

    If $m + p = m + q$ then there are n, p_1, q_1 such that

$$p = n + p_1$$

$$q = n + q_1$$

$$m = m + p_1 = m + q_1$$

4. Prove that every infinite set of reals has a countable partition. (easy)

5. Suppose $\mathcal{X} \subseteq \mathcal{P}(X)$ is a downward ($\subseteq$-closed) set which is the same size as X. Prove that X has a wellordered subset not in $\mathcal{X}$. (Tarski)

6. Suppose there is an injection $\mathcal{P}(x) \hookrightarrow (x \times x)$ and that x has at least 5 members. Derive a contradiction by showing that x has subsets of all wellordered sizes. (Specker). Attempt the same proof by starting with the assumption that there is a surjection $(x \times x) \to \mathcal{P}(x)$. What becomes of your proof?

7. Let X be a set such that there is a finite-to-one map from $P(X)$ onto X. Prove that X is actually finite, i.e., has cardinal in $\mathbf{N}$ (hint: use Hartogs' theorem to prove by induction on the alephs that if such an X is infinite then X can be mapped surjectively onto every aleph.)

8. Prove that (the set of ordinals below) ω_2 is not a union of countably many countable sets.

   Prove that a union of countably many countable sets cannot have a partition of length ω_2. (quite hard)

9. Suppose ω_1 is a (disjoint) union of countably many countable sets, $\bigsqcup_{i \in \omega} A_i$. Each A_i is a countable set of countable ordinals and has a length in a natural way. What is the sup of these lengths?

10. Prove Hartogs' theorem. For α a cardinal, let T_α be the (downward-branching) tree of cardinals obtained by placing below each node β every cardinal γ such that $2^\gamma = \beta$, starting at α. Prove that T_α is wellfounded.

    If the rank of the tree is infinite, prove that α is not an aleph.

    (This question is not difficult, but you should not attempt it unless you are happy about Hartogs' theorem and rank functions on wellfounded relations, (*vide* section 4)

11. Let α be some infinite cardinal, satisfying conditions you may specify yourself to make the search for an answer easier. Consider the free boolean algebra with α generators. How big is it? How many sets of (free) generators has it got?

9 Some nasty questions for grown-ups

1. Consider the following properties on a poset $\langle P, < \rangle$.

   (a) $P = A \cup B$, the disjoint union of two countably infinite antichains

   (b) whenever x, y, z, x', y', z' are in P and $x < y > z, x' < y' > z'$ then there is an automorphism of $\langle P, < \rangle$ sending x, y, z to x', y', z'

   (c) the same as (b) but this time with $x > y < z$ and $x' > y' < z'$

   The question is: how many possible posets with these properties are there (up to isomorphism) ?

2. (a) Let A be a collection of circular disks in $\mathbb{R}^2$, no two of which intersect. Show A is countable. (Easy!)

   (b) Let B be a collection of circles in $\mathbb{R}^2$, no two of which intersect Need B be countable? (Easy!)

   (c) Let C be a collection of figures-of-eight in $\mathbb{R}^2$, no two of which intersect. Need C be countable?

   A figure of eight is a homeomorphic image of two circles with one point in common. You may assume the Jordan Curve theorem if it helps.

3. How many countable linear orders $\langle A, < \rangle$ are there (up to isomorphism of course) such that $\mathrm{Aut}(\langle A, < \rangle)$ acts transitively on $\langle A, < \rangle$ (i.e. for all a, b in A there is an automorphism g with $g`a = b$)