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Oxo tutorials

Thomas Forster

December 11, 2021

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1 Sheet 1

Some exercises involve showing that two sets are identical. What is the fundamental axiom of Set Theory, the one that tells you you are dealing with sets rather then lists or multisets? Yes, it's the axiom of extensionality, the axiom that tells you that two sets are identical iff they have the same members. So if you want to prove $X = Y$ you want to prove $(\forall z)(z \in X \longleftrightarrow z \in Y)$. By this means you can often reduce a question about the identity of sets to an exercise in predicate or perhaps even propositional logic. That's certainly the case for question 1. Just join up the dots and don't panic.

1.1 Question 1

1.2 Question 2

1.3 Question 3

$\bigcup W$ is of course the universe, aka V. If the universe is a set then the axiom (scheme) of separation (subsets) will tell you that the Russell class is a set, and it isn't. Part (iii) looks different but it isn't. If there were such a set its sumset would be V.

1.4 Question 4

The first four parts are really a riff on the axiom of pairing. The fifth part needs the axiom scheme of separation (subsets).

1.5 Question 5

Notice that he's not asking you to prove that there is an infinite set, merely that there are infinitely many sets.

1.6 Question 6

There's a lot of meat in this question. I think i am going to do the lazy thing and copy-and-paste some material i wrote up for my students at The Other Place.

If P is a property of sets, a set x is said to be hereditarily *P if every member of* $\mathrm{TC}(\{x\})$ *has property P. Consider the classes HF, HC and HS of hereditarily finite,*

Prove that the class HF of hereditarily finite sets coincides with V_ω.

Which of the axioms of ZF are satisfied in the structure HF?

This is quite a good question to look at. You have two inclusions to prove, one in each direction.

To prove $V_\omega \subseteq HF$ you do an ordinary mathematical induction: you prove by induction on n that everything in V_n is in HF.

For the other direction you have to use $\in$-induction to show $HF \subseteq V_\omega$. The property $\phi(x)$ you prove by $\in$-induction is "$x \in HF \to x \in V_\omega$". The need for foundation/induction stems from the fact that if foundation fails then potentially a Quine atom[1] is a counterexample to the inclusion. A nice idiomatic illustration of $\in$-induction.

Take care when asking yourself whether or not an axiom is true in a structure. Yer typical set theoretic axiom states that the universe is closed under some operation (as it might be power set, or sumset). **Saying that a structure is a**

[1] A Quine atom is a set $x = \{x\}$.

model for that axiom is not the same as saying that it's closed under the corresponding operation.

When wondering whether or not an axiom is true in a model $\mathfrak{M}$ *the thing to ask yourself is "Suppose* $\mathfrak{M}$ *were the whole world, and i am living inside* $\mathfrak{M}$*: does the axiom appear to be true?"*

(This is all about absolute properties versus non-absolute properties—a topic which Dr Knight is wary of broaching. My student Ha Thu Nguyen gives a very simple illustration ... If $\mathfrak{M} = \langle M, \in \rangle$ *is a model of ZF then it doesn't think that its carrier set* M *is a set, but we can see from outside that it is. If* $\mathfrak{M}$ *is a countable model then the thing that* $\mathfrak{M}$ *believes to be the power set of* $\mathbb{N}$ *is, indeed, countable seen from outside; however* $\mathfrak{M}$ *does not know of any bijection between that set and the set it believes to be* $\mathbb{N}$*. My student Tim Talbot puts it very well: the countable model is a Tardis!)*

While we are about it: Which axioms of ZF hold in $V_{\omega+\omega}$*?*

All of them except replacement. Consider the function class $n \mapsto V_{\omega+n}$*. Replacement would make the image of* $\mathbb{N}$ *in this function class—namely* $\{V_{\omega+n} : n \in \mathbb{N}\}$*—into a set of the model, and it can't be, beco's it is of rank* $\omega + \omega$*.*

Observe the following ramifications of the fact that $\mathbb{R}$ *has a subset that is wellordered to length* $\omega + \omega$ *in the inherited order. If replacement held in* $V_{\omega+\omega}$ *then* $V_{\omega+\omega}$ *would contain, for every wellordering in* $V_{\omega+\omega}$*, the corresponding von Neumann ordinal. It is easy to check that the rank of the von Neumann ordinal* α *is* α *itself, which means that* $V_{\omega+\omega}$ *cannot contain any ordinal from* $\omega + \omega$ *onwards. So replacement fails. In general* V_α *will contain at least some wellorderings that are far too long for their von Neumann ordinals to be in* V_α*. It happens only rarely that* α *"catches up" with the ordinals in* V_α*.*

Another question on this from The Other PLace.

> *(i)* S_1 *is the class* HF *of hereditarily finite sets;*
>
> *(ii)* S_2 *is the class* $\{x : \mathrm{TC}(\{x\})$ *is finite*$\}$ *of* strongly hereditarily finite *sets;*
>
> *(iii)* S_3 *is the smallest set containing* $\emptyset$ *and closed under* $\mathcal{P}$ *and under formation of arbitrary subsets;*
>
> *(iv)* $S_4 = \bigcup\{V_n : n < \omega\}$ *is the* ω*th stage in the von Neumann hierarchy.*
>
> *(v)* S_5 *is the collection of hereditarily D-finite sets*[2] *Naturally you are not to use AC in this endeavour!*
>
> *(vi) Let* S_6 *be the* $\subseteq$*-least set containing* $\emptyset$ *and closed under* $x, y \mapsto x \cup \{y\}$*.*

Prove that all these sets are the same.

Deduce in particular that the class HF *is a set. Is* HC *a set? If so, does it coincide with* V_α *for any* α*?*

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[2]If AC fails then it might happen that not every infinite set has a countable subset. A set which has no countably infinite subset is *Dedekind-finite* or "D-finite" for short.

The key to these questions is induction, both structural and wellfounded. A good thing to read is Logic, Induction and Sets, ch 2.

For every inductively defined set X there is an in-house induction principle, which we might as well call "X"-induction. S_6-induction is the principle:

If $F(\emptyset)$
and
$F(y) \to (\forall z \in S)(F(y \cup \{z\}))$
then
$(\forall x \in S)(F(x))$.

How do we know that S even exists? Answer: we use replacement. In fact we always use replacement to show that the closure of a set under an operation exists. In this case we consider the function f that sends 0 to the empty set, and thereafter sends n to $\{x \cup \{y\} : x, y \in f(n)\}$.

Then we use replacement followed by sumset to obtain $\bigcup f``\mathbb{N}$ which (as the reader can verify) is what we want. (The reader might want at this point to think ahead to the later question that asks us how we might use a construction like this to obtain HC, the set of hereditarily countable sets.)

S_4 is another inductively defined set. It, too, has an induction principle, So we can prove by induction that all its members are in $S_?$. And we prove by induction on S_4 that all its members are in S'. Hence $S_4 = S_?$ by extensionality.

Prove by $\in$-induction that if you are hereditarily finite then you are strongly hereditarily finite, and conversely.

Let's have a look at a couple of these inductions in detail: specifically 6 (ii) and 6(iii).

RTP: S is closed under binary union. (That is, $(\forall x, y \in S)(x \cup y \in S)$.)

Fix $x \in S$ and prove by induction on y that $x \cup y \in S$.

Certainly true for $y = \emptyset$. For the induction suppose $x \cup y \in S$ and deduce $x \cup (y \cup \{z\}) \in S$. $x \cup (y \cup \{z\}) = (x \cup y) \cup \{z\}$. $x \cup y \in S$ by induction hypothesis, and so $(x \cup y) \cup \{z\} \in S$ by closure properties of S.

Next we prove that $x \in S \to \bigcup x \in S$. Naturally we use S-induction.

Certainly true for $x = \emptyset$. For the induction step assume $\bigcup x \in S$ and infer $\bigcup(x \cup \{y\}) \in S$. Now $\bigcup(x \cup \{y\}) = \bigcup x \cup y$ and we know S is closed under binary union, so $\bigcup(x \cup \{y\}) \in S$.

Several of these sets are not explicitly given as inductively defined. (HF and the set of strongly hereditarily finite sets). For these we can exploit $\in$-induction. (Note that we have not used $\in$-induction so far!) We prove by $\in$-induction that every set that is hereditarily finite is strongly hereditarily finite. It is by $\in$-induction, too, that we prove that everything in HF is in S or S' or

S'' or whatever. This direction really needs $\in$-induction (aka foundation). If foundation fails then there might be a Quine atom and of course there are no Quine atoms in S. We can prove this last fact by ... S-induction!

To prove that HF $= V_\omega$ prove by $\in$-induction on HF that all its members are in V_ω. For the opposite direction prove by induction on n that $V_n \subseteq$ HF.

Consider the classes HF, HC and HS of hereditarily finite, hereditarily countable and hereditarily small sets (where we call a set small *if it can be injected into one of the sets $V_\omega, V_{\omega+1}, V_{\omega+2}, \ldots$): in each case determine which axioms of ZF hold, and which fail, in the structure obtained from the class as in the previous question.*

Observe that $\{V_{\omega+n} : n \in \mathbb{N}\}$ is hereditarily small. It's a countable set of hereditarily small sets. (Prove by induction on n that all the $V_{\omega+n}$ are hereditarily small). But $\bigcup\{V_{\omega+n} : n \in \mathbb{N}\}$ is $V_{\omega+\omega}$ which is not hereditarily small. So HS is not a model of sumset.

One worry one can banish at the outset: one is assumed to be working in ZFC with foundation, so that foundation is true in all the classes concerned. If $\langle V, \in\rangle$ is wellfounded, so is any substructure of it.

When checking to see whether an axiom ϕ holds in a class H, one key thing to be sure you get right is: restrict all the variables in ϕ to H!!

Is HC a set?

There is a proof using replacement, where HC_0 is $\{\emptyset\}$ and thereafter HC_α is the set of all countable subsets of $\bigcup\limits_{\beta<\alpha} HC_\beta$. (When does this process close off and why?)

However there is a cuter proof. With the help of countable choice we can find a bijection $\sigma : \{x \subseteq \mathbb{R} : |x| \leq \aleph_0\} \to \mathbb{R}$.

We can now define a function i by $\in$-recursion.

$$i(x) =: \sigma(i``x)$$

We prove by $\in$-induction that i is defined on all hereditarily countable sets and is injective. This establishes that HC is a set since it is $i^{-1}``\mathbb{R}$, and also establishes that there are at most $2^{\aleph_0}$ hereditarily countable sets. In fact there are precisely $2^{\aleph_0}$ hereditarily countable sets: for the other direction observe that $V_{\omega+1} \subseteq HC$ and is of size $2^{\aleph_0}$.

Jech proved (JSL 1980) that HC is a set and is of rank ω_2 at most, and he did it without choice. Randall Holmes has shown that—similarly without AC—for any X, the class of sets hereditarily smaller than X is always a set. Have a look at `www.dpmms.cam.ac.uk/~tf/cam_only/randallhereditary.pdf`. *It's a nice, idiomatic, delicate piece of set-theoretic combinatorics.*

Some detailed thoughts about hereditarily small sets

There is a cluster of independence results concerning the set of hereditarily small sets. There is a certain amount of equivocation going on.

By "hereditarily small" one might mean that

(i) $TC(x)$ is small;

or one might mean that

(ii) Everything in $TC(x)$ is small;

or even

(iii) x belongs to every set that contains all its small subsets.

(iii) is an inductive definition and supports an induction principle.

By "small" one might mean that $|x| < \beth_\omega$, or one might mean that $(\exists n)(|x| < \beth_n)$. If AC fails, then $\beth_\omega$ might not be an aleph and these two assertions about $|x|$ might not the same. This gives us six sets to consider:

1. $\{x : (\forall y \in TC(\{x\}))(|y| < \beth_\omega)\}$
2. $\{x : (|TC(\{x\})| < \beth_\omega)\}$
3. $\{x : (\forall y \in TC(\{x\}))(\exists n \in \mathbb{N})(|y| < \beth_n)\}$
4. $\{x : (\exists n \in \mathbb{N})(|TC(\{x\})| < \beth_n)\}$
5. $\{x : (\forall y)((\forall z)((z \subseteq y \land |z| < \beth_\omega) \to z \in y) \to x \in y)\}$
6. $\{x : (\forall y)((\forall z)(\forall n \in \mathbb{N})((z \subseteq y \land |z| < \beth_n) \to z \in y) \to x \in y)\}$

$V_{\omega+\omega}$ is included in all these sets, but is a member of none of them.

I am omitting proofs of the following inclusions, in the belief that they will be obvious to the reader:

$4 \subseteq 2 \subseteq 1$;
$4 \subseteq 3 \subseteq 1$;
$6 \subseteq 5$.

The first class—$\{x : (\forall y \in TC(\{x\}))(|y| < \beth_\omega)\}$—can be used to prove the independence of sumset from the other axioms of ZF. This is because one of its members is $\{V_{\omega+n} : n \in \mathbb{N}\}$. The same goes for the third set: $\{x : (\forall y \in TC(\{x\}))(\exists n \in \mathbb{N})(|y| < \beth_n)\}$. The fifth and sixth sets, too, contain $\{V_{\omega+n} : n \in \mathbb{N}\}$ but not $V_{\omega+\omega}$ and so prove the independence of sumset.

The second class—$\{x : (|TC(\{x\})| < \beth_\omega)\}$—on the other hand is clearly a model of sumset: clearly $TC(\bigcup x) \subseteq TC(x)$ so if $|TC(x)| < \beth_\omega$ then $|TC(\bigcup x)| < \beth_\omega$ as well. For the same reason the fourth set $\{x : (\exists n \in \mathbb{N})(|TC(\{x\})| < \beth_n)\}$ will also be a model of sumset.

6 is a model of replacement. If f is a map 6 $\to$ 6, then $f``x$ is always a set of hereditarily small sets in the appropriate sense. Is it also small itself? $|f``x| \leq_* |x| \leq \beth_n$ so $|f``x| < 2^{|f``x|} \leq 2^{|x|} \leq \beth_{n+1}$ so $f``x$ is small too. This doesn't seem to work for 5.

The axiom of power set is a problem. The classes in 3, 4 and 6 will satisfy power set, but for the others we have to assume a certain amount of AC if we want power set to hold. We need $\alpha < \beth_\omega \to 2^\alpha < \beth_\omega$.

My Doktorvater Adrian Mathias claims that 2 is a model of stratified replacement, or even that it is an extension of $V_{\omega+\omega}$ that is elementary for stratified formulæ. How might one prove this, or results like it? In $V_{\omega+\omega}$ one can encode APGs that are pictures of sets in 1-6. If x is a set whose APG is a set in $V_{\omega+\omega}$ then $|TC(\{x\})| \leq \beth_n$ for some $n \in \mathbb{N}$. That is to say, 4 contains precisely the sets whose pictures appear in $V_{\omega+\omega}$. The construction one then performs uses the ideas of Rieger-Bernays permutations from question 5 of this sheet. Concretise the isomorphism classes of set pictures using Scott's trick. Then reflect that there is a natural "membership" relation on the [isomorphism classes of] set pictures. Swap every [Scott's-trick] isomorphism class with the collection of isomorphism classes that are "'members" of it, and extend this to a permutation of the universe by fixing everything else. Look at the Rieger-Bernays permutation model given by this permutation.

Consider the binary relation E on $\mathbb{N}$ defined by: $n\, E\, m$ iff the $(n+1)$st bit (counting from the right) in the binary expansion of m is 1. What can you say about the structure $\langle \mathbb{N}, E\rangle$?

Once you notice that any finite set of naturals can be coded by another natural in this way you quickly realise that $\langle \mathbb{N}, E\rangle \simeq \langle V_\omega, \in\rangle$.

This bijection was first noticed by Wilhelm Ackermann, and is a useful gadget for interpreting arithmetic in set theory and vice versa.

The collection H_κ of sets hereditarily of size less than κ is practically guaranteed to be a model of replacement, as follows. Suppose $X \in H_\kappa$, and $f : H_\kappa \to H_\kappa$. Then $f``X$ is a subset of H_κ. How big is it? It's a surjective image of thing of size $< \kappa$. We want it to be of size $< \kappa$ itself. So all we need is a surjective image of something of size less than κ is itself of size $< \kappa$. This is certainly true if κ is an aleph, and even in many cases when it isn't. So certainly if κ is an aleph then H_κ is a model of replacement.

1.7 Question 7

This question is an old chestnut, but it's been given a twist i hadn't seen before. No dog is too old to learn new tricks. Even if there is an infinite sequence of $\mathfrak{M}$s (and i'm not convinced that they form a descending sequence under $\in$ even if there is such a sequence) if we earnestly desired a contradiction we would have to establish that this sequence is a set of the model $\mathfrak{M}_0$ with which we started.

Can we describe a construction of this infinite sequence while remaining inside the model? Presumably not. Infinite descending $\in$ *sequences do not contradict the axiom of foundation unless they are actually sets of the model. IF they are visible only from outside then there is no contradiction with foundation.*

2 Sheet 2

3 Sheet 3

4 Sheet 4