Number systems of different lengths, and a natural approach to infinitesimal analysis

Richard Pettigrew
Department of Mathematics
University of Bristol

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Outline

The talk in brief

- Describe *EA*, a finitary theory of finite sets.

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- Describe *EA*, a finitary theory of finite sets.
- Define the notion of a natural number system in *EA*.
- Show that there are non-isomorphic natural number systems in *EA*.
- Give a taste of the theory of natural number systems in *EA*.
- Sketch the beginnings of a theory of infinitesimal analysis in an extension on *EA*.

The language of *EA*

Constants, operators, and function and relation symbols

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- Term-forming operator: $\{x \in t : A(x)\}$, whenever A is bounded.
- Relation symbols:
  - $=$ (identity)
  - $\in$ (membership)

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EA in a nutshell

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In other words...

- *EA* is finitely axiomatized by adding the Axioms of Dedekind Finiteness, P, and TC to Jensen's rudimentary functions.
- *EA* is mutually interpretable with $I\Delta_0 + \exp$. (With mutually inverse interpretations? Don't know. Suspect not.)

Definitions

Definition (L generated from 0 by σ)

Suppose σ is a unary global function defined by a term of *EA*, and 0 is a closed term. Then, if L is a linear ordering, we say that *L is generated from* 0 *by* σ if

(1) $\mathrm{First}(L) = 0$, and

(2) $\mathrm{Next}_L(x) = \sigma(x)$, for all x in $\mathrm{Field}(L)$ except $\mathrm{Last}(L)$.

Thus, roughly, if L is generated from 0 by σ, it has the following form:

$$[0, \sigma(0), \sigma(\sigma(0)), \cdots, a]$$

Definitions

Definition (Natural number system)

We say that σ *generates a natural number system from* 0 if

$$\forall L(L \text{ is generated from } 0 \text{ by } \sigma \rightarrow \sigma(\mathrm{Last}(L)) \notin \mathrm{Field}(L))$$

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- If $L = [0_{\mathcal{N}}, \cdots a]$ in $\mathcal{N}$, let $\overline{\sigma_{\mathcal{N}}}(L) = [0_{\mathcal{N}}, \cdots, a, \sigma_{\mathcal{N}}(a)]$.

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- If $L = [0_{\mathcal{N}}, \cdots a]$ in $\mathcal{N}$, let $\overline{\sigma_{\mathcal{N}}}(L) = [0_{\mathcal{N}}, \cdots, a, \sigma_{\mathcal{N}}(a)]$.

Thus, roughly, $\mathcal{N}$ consists of the following linear orderings:

$$[\,], \quad [0_{\mathcal{N}}], \quad [0_{\mathcal{N}}, \sigma_{\mathcal{N}}(0_{\mathcal{N}})], \quad [0_{\mathcal{N}}, \sigma_{\mathcal{N}}(0_{\mathcal{N}}), \sigma_{\mathcal{N}}(\sigma_{\mathcal{N}}(0_{\mathcal{N}}))], \quad \cdots$$

Examples of natural number systems

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- $\mathcal{VN}$ is generated from $\varnothing$ by $\sigma_{\mathcal{VN}} : x \mapsto x \cup \{x\}$

$$[\,],\quad [\varnothing],\quad [\varnothing, \{\varnothing\}],\quad [\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}],\quad \cdots$$

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- $\mathcal{Z}$ is generated from $\varnothing$ by $\sigma_{\mathcal{Z}} : x \mapsto \{x\}$

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$$[\,], \quad [\varnothing], \quad [\varnothing, \{\varnothing\}], \quad [\varnothing, \{\varnothing\}, \{\{\varnothing\}\}], \quad \cdots$$

- $\mathcal{CH}$ is generated from $\varnothing$ by $\sigma_{\mathcal{CH}} : x \mapsto \mathrm{P}(x)$

$$[\,], \quad [\varnothing], \quad [\varnothing, \mathrm{P}(\varnothing)], \quad [\varnothing, \mathrm{P}(\varnothing), \mathrm{P}(\mathrm{P}(\varnothing))], \quad \cdots$$

Induction along natural number systems

Theorem (Bounded induction)

Suppose A is a bounded formula of EA. Then

$$EA \vdash (A(\,[\,]\,)\ \&\ (\forall L \textit{ in } \mathcal{N})[A(L) \to A(\overline{\sigma_{\mathcal{N}}}(L))]) \to (\forall L \textit{ in } \mathcal{N})A(L)$$

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Unbounded induction DOES NOT HOLD

If A is unbounded, then the following does not necessarily hold:

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Both are consequences of the presence only of bounded separation.

Recursion along a natural number system

Definition (Arithmetical global functions)

Suppose φ is a global function. We say that φ *is arithmetical* if

$$EA \vdash \forall x, y(x \cong y \rightarrow \varphi(x) \cong \varphi(y))$$

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Definition ($\mathcal{N}$ is closed under φ)

Suppose φ is an arithmetical global function. Then we say that $\mathcal{N}$ *is closed under* φ if

$$EA \vdash (\forall x \text{ in } \mathcal{N})(\exists y \text{ in } \mathcal{N})[\mathrm{Field}(y) \cong \varphi(\mathrm{Field}(x))].$$

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Given a natural number system $\mathcal{N}$, the family of arithmetical global functions under which $\mathcal{N}$ is closed is NOT closed under full recursion.

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Given a natural number system $\mathcal{N}$, the family of arithmetical global functions under which $\mathcal{N}$ is closed is NOT closed under full recursion.

Both are consequences of the presence only of bounded induction. Unlimited recursion requires Σ_1 induction.

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  - $\mathcal{VN}$, $\mathcal{Z}$, $\mathcal{CH}$ closed under $\varphi \Leftrightarrow \varphi \in \mathscr{E}^0$.
  - There is a system, $\mathcal{ACK}$, which moves from one set to another in their Ackermann ordering:

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    i.e $\mathcal{ACK}$ closed under exponentiation, but not superexponentiation.

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- But the distinctions are more fine-grained:
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  - There are natural number systems closed under $x\log(\log(x))$ but not under $x\log(x)$.

Made-to-Measure Natural Number Systems

Definition (φ is maximally powerful in $\mathcal{N}$)

φ *is maximally powerful in* $\mathcal{N}$ if, for any arithmetical global function ψ, if $\mathcal{N}$ is closed under ψ, then there is $\mathbf{n}$ such that ψ is eventually majorized by $\varphi^{\mathbf{n}}$.

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Theorem

Suppose there is $\mathbf{C}$ *such that*

(i) $EA \vdash (\forall x)(\mathbf{C} \leq x \rightarrow x < \varphi(x))$

(ii) $EA \vdash (\forall x, y)(\mathbf{C} \leq x \leq y \rightarrow \varphi(x) \leq \varphi(y))$

(iii) $EA \vdash (\forall x, y)(\mathbf{C} \leq x \leq y \rightarrow \varphi(x) - x \leq \mathbf{2}^y - y)$

Then there a natural number system $\mathcal{ACK}_\varphi$ *such that* φ *is maximally powerful in* $\mathcal{ACK}_\varphi$*.*

Relations of length between natural numbers systems

Definition

$\mathcal{M} \preceq \mathcal{N}$ if

$$EA \vdash (\forall x \text{ in } \mathcal{M})(\exists y \text{ in } \mathcal{N})[\mathrm{Field}(y) \cong \mathrm{Field}(x)].$$

i.e. there is an injection from $\mathcal{M}$ into $\mathcal{N}$ that preserves length.

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Theorem

In the presence of Σ_1 induction, and thus unlimited recursion, all natural number systems are of the same length: i.e. $\mathcal{M} \cong \mathcal{N}$, for all $\mathcal{M}$ and $\mathcal{N}$.

The incommensurability of $\mathcal{VN}$ and $\mathcal{Z}$

Theorem

$\mathcal{VN}$ and $\mathcal{Z}$ are incommensurable: that is,

$$\mathcal{VN} \not\preceq \mathcal{Z} \quad \text{and} \quad \mathcal{Z} \not\preceq \mathcal{VN}$$

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I will sketch two proofs:

- One is syntactic.
- The other is model-theoretic.

The syntactic proof

Lemma (Parikh-style Bounding Lemma)

Suppose A is a bounded formula of EA. Then, if

$$EA \vdash \forall x \exists ! y A(x, y)$$

Then there is a classical natural number, **n***, such that*

$$EA \vdash \forall x \exists ! y (y \in \mathrm{P}^{\mathbf{n}}(\mathrm{TC}(x)) \ \ \& \ \ A(x, y))$$

The syntactic proof

Proof. Suppose $\mathcal{VN} \preceq \mathcal{Z}$. That is,

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\exists! z \text{ in } \mathcal{Z})(\mathrm{Field}(v) \cong \mathrm{Field}(z))$$

Thus, by Parikh-style Bounding Lemma, there is **n** such that

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\exists z! \text{ in } \mathcal{Z})(z \in \mathrm{P}^{\mathbf{n}}(\mathrm{TC}(v))\ \&\ \mathrm{Field}(v) \cong \mathrm{Field}(z))$$

The syntactic proof

But, by (meta-theoretical) induction on **n**,

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\forall z \text{ in } \mathcal{Z})(z \in \mathrm{P}^{\mathbf{n}}(\mathrm{TC}(v)) \to z \in V_{\mathbf{n}+\mathbf{4}})$$

Thus,

$$EA \vdash (\forall v \text{ in } \mathcal{VN})(\exists z! \text{ in } \mathcal{Z})(z \in V_{\mathbf{n}+\mathbf{4}} \ \& \ \mathrm{Field}(v) \cong \mathrm{Field}(z))$$

which is false. $\square$

The model-theoretic proof

Proof. Let M be a model of EA that contains a non-standard member of $\mathcal{VN}$, b. Then define the following submodel of M:

$$C(M,b) = \bigcup_{\mathbf{n}=1}^{\infty}\{x \in M : M \models x \in \mathrm{P}^{\mathbf{n}}(b)\}$$

We call $C(M,b)$ the *cumulation model of EA of b*. Then

$$C(M,b) \models EA$$

But $C(M,b)$ contains only standard members of $\mathcal{Z}$, while it contains non-standard members of $\mathcal{VN}$. Thus, it is not the case that $\mathcal{VN} \preceq \mathcal{Z}$. □

Measuring the universe

Definition ($\mathcal{N}$ measures the universe)

$\mathcal{N}$ measures the universe if

$$EA \vdash (\forall x)(\exists y \text{ in } \mathcal{N})[x \cong \mathrm{Field}(y)]$$

Theorem

In the presence of Σ_1 induction, and thus unlimited recursion, every natural number system measures the universe.

Measuring the universe

Theorem

In EA, no natural number system measures the universe.

Proof. Suppose $\mathcal{N}$ measures the universe. If $\mathbf{k}$ is a classical natural number, let

- $v_{\mathbf{k}}$ be the $\mathbf{k}^{\text{th}}$ member of $\mathcal{VN}$,
- $z_{\mathbf{k}}$ be the $\mathbf{k}^{\text{th}}$ member of $\mathcal{Z}$, and
- $n_{\mathbf{k}}$ be the $\mathbf{k}^{\text{th}}$ member of $\mathcal{N}$.

Since $\mathcal{N}$ measures the universe,

$$EA \vdash (\forall x)(\exists y! \text{ in } \mathcal{N})[x \cong y]$$

Thus, by the Parikh-style Bounding Lemma, there is $\mathbf{n}$ such that

$$EA \vdash (\forall x)(\exists y! \text{ in } \mathcal{N})[y \in \mathrm{P}^{\mathbf{n}}(x) \ \& \ x \cong y]$$

Measuring the universe

Thus, for all classical natural numbers, **k**,

$$n_{\mathbf{k}} \in \mathrm{P}^{\mathbf{n}}(v_{\mathbf{k}}) \quad \text{and} \quad n_{\mathbf{k}} \in \mathrm{P}^{\mathbf{n}}(z_{\mathbf{k}})$$

Thus,

$$n_{\mathbf{k}} \in \mathrm{P}^{\mathbf{n}}(v_{\mathbf{k}}) \cap \mathrm{P}^{\mathbf{n}}(z_{\mathbf{k}})$$

Thus,

$$n_{\mathbf{k}} \in V_{\mathbf{n}+\mathbf{4}}$$

But this gives a contradiction, since $V_{\mathbf{n}+\mathbf{4}}$ cannot contain sufficiently many members of $\mathcal{N}$ to measure all standard members of $\mathcal{VN}$ and $\mathcal{Z}$. $\square$

Extending *EA*

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Definition

EA^{+} is obtained from EA by adding the following axiom:

$$(\exists x)[x \text{ is } \mathcal{ACK}\text{-large}]$$

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Definition

EA^+ is obtained from EA by adding the following axiom:

$$(\exists x)[x \text{ is } \mathcal{ACK}\text{-large}]$$

Theorem

If EA is consistent, then EA^+ *is consistent.*

Infinitesimal analysis in EA^+

Definition (Integers in EA^+)

An integer is an ordered pair (a, b) where a and b are sets. (Intuitively, (a, b) is $a - b$.)

$$(a, b) =_Z (c, d) \leftrightarrow a + d \cong b + c$$

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Definition (Rationals in EA^+)

A rational is an ordered pair (a, b) where a and b are integers, and $b \neq_Z 0$. (Intuitively, (a, b) is $\frac{a}{b}$.)

$$(a, b) =_Q (c, d) \leftrightarrow a \times_Z d \cong b \times_Z c$$

Infinitesimal analysis in EA^+

Definition (Reals in EA^+)

$$r \text{ in } R \leftrightarrow (\exists x)[x \text{ is } \mathcal{ACK}\text{-small } \& \ |r| < x]$$

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Definition (Infinitesimal in EA^+)

$$r \text{ in } I \leftrightarrow (\forall x)\left[x \text{ is } \mathcal{ACK}\text{-small} \rightarrow |r| < \frac{1}{x}\right]$$

Since there is an $\mathcal{ACK}$-large set in EA^+, there are infinitesimals.

R is 'almost' real closed

Definition ($x \simeq y$)

If x and y are in R, then $x \simeq y \leftrightarrow x - y$ in I

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If x and y are in R, then $x \simeq y \leftrightarrow x - y$ in I

Theorem

- *If $0 < a$ is in R, then there is b in R such that $b^2 \simeq a$.*
- *If n is small and odd and $\{a_i\}_{i=0}^{n}$ is a sequence of reals, then there is b in R such that*

$$\sum_{i=0}^{n} a_i b^i \simeq 0$$

Continuous functions in EA^+

Definition (f is continuous)

If $f : J \to R$, then f *is continuous* if

$$(\forall x, y \text{ in } J)[x \simeq y \to f(x) \simeq f(y)]$$

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The following theorems hold:

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The following theorems hold:

- The Intermediate Value Theorem

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Definition (f is continuous)

If $f : J \to R$, then f *is continuous* if

$$(\forall x, y \text{ in } J)[x \simeq y \to f(x) \simeq f(y)]$$

The following theorems hold:

- The Intermediate Value Theorem
- Every continuous function on a closed interval is bounded and attains its bounds.

Differential and integral calculus in EA^+

Definition (f is differentiable)

Suppose $f : J \to R$, x is in J, and α is in R. Then *f is differentiable at x with derivative α* if

$$(\forall \delta \text{ in } I)\left[\frac{f(x+\delta) - f(x)}{\delta} \simeq \alpha\right]$$

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Definition (f is integrable)

Suppose $f : [a, b] \to R$, $a \leq x \leq b$, and α is in R. Then f *is integrable at x with definite integral α* if, for any $\mathcal{ACK}$-large N,

$$\sum_{i=0}^{N} \frac{b-a}{N} \cdot f\left(a + i\frac{b-a}{N}\right) \simeq \alpha$$

Differential and integral calculus in EA^+

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- Rolle's Theorem and the Mean Value Theorem.

Differential and integral calculus in EA^+

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- Rolle's Theorem and the Mean Value Theorem.
- The Fundamental Theorems of the Calculus

Polynomials of large degree

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- *For any large M and N and any x in R, $e_M^x \simeq e_N^x$.*

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Theorem

For any large N, the function $x \mapsto e_N^x$ is differentiable at all points x in R with derivative e_N^x.

Weierstrass' Approximation Theorem

Theorem (Weierstrass)

Suppose $f : [a, b] \to R$ is continuous function. Then there is a polynomial,

$$P(x) = \sum_{i=0}^{N} a_i x^i$$

possibly of large degree, such that

$$(\forall a \leq x \leq b)[P(x) \simeq f(x)]$$

References

All the results here and many more can be found in:

Pettigrew, R. (doctoral thesis)
Natural, Rational, and Real Arithmetic in the Finitary Theory of Finite Sets
http://www.maths.bris.ac.uk/~rp3959/thesis1.pdf/