

**ZIPF'S LAW
AND L. LEVIN'S PROBABILITY DISTRIBUTIONS**

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ABSTRACT. Zipf's law in its basic incarnation is an empirical probability distribution governing the frequency of usage of words in a language. As Terence Tao recently remarked, it still lacks a convincing and satisfactory mathematical explanation.

In this paper I suggest that at least in certain situations, Zipf's law can be explained as a special case of the *a priori* distribution introduced and studied by L. Levin. The Zipf ranking corresponding to diminishing probability appears then as the ordering determined by the growing Kolmogorov complexity.

One argument justifying this assertion is the appeal to a recent interpretation by Yu. Manin and M. Marcolli of asymptotic bounds for error-correcting codes in terms of phase transition. In the respective partition function, Kolmogorov complexity of a code plays the role of its energy.

0. Introduction and summary

0.1. Zipf's law. Zipf's law was discovered as an empirical observation ([Zi1], [Zi2]): if all words w_k of a language are ranked according to decreasing frequency of their appearance in a representative corpus of texts, then the frequency p_k of w_k is (approximately) inversely proportional to its rank k : see e. g. Fig. 1 in [Ma1] based upon a corpus containing $4 \cdot 10^7$ Russian words.

For various other incarnations of this power exponent -1 law in many different statistical data, cf. [MurSo] and references therein.

Theoretical models of Zipf's distribution also abound. In the founding texts of Zipf himself [Zi2], [Zi1], it was suggested that his distribution “minimizes effort”. Mandelbrot in [Mand] described a concise mathematical framework for producing a model of Zipf's law. Namely, if we postulate and denote by C_k a certain “cost” (of producing, using etc.) of the word of rank k , then the frequency distribution $p_k \sim 2^{-h^{-1}C_k}$ minimizes the ratio $h = C/H$, where $C := \sum_k p_k C_k$ is the average cost per word, and $H := -\sum_k p_k \log_2 p_k$ is the average entropy: see [Ma2].

We get from this a power law, if $C_k \sim \log k$. An additional problem, what is so special about power -1 , must be addressed separately. For one possibility, see [MurSo], sec. III. In this work, we suggest a different mathematical background (see the next subsection).

In all such discussions, it is more or less implicitly assumed that empirically observed distributions concern fragments of a potential countable infinity of objects. I also postulate this, and work in such a “constructive world”: see sec. 1.1 below for a precise definition.

0.2. Zipf’s law from complexity. In this note we suggest that (at least in some situations) Zipf’s law emerges as the combined effect of two factors:

(A) *Rank ordering coincides with the ordering with respect to the growing (exponential) Kolmogorov complexity $K(w)$ up to a factor $\exp(O(1))$.*

More precisely, to define $K(x)$ for a natural number $x \in \mathbf{Z}^+$, we choose a *Kolmogorov optimal encoding*, which is a partial recursive function $u : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$, and put $K(x) = K_u(x) := \min \{y \mid u(y) = x\}$. Another choice of u changes $K_u(\cdot)$ by a factor $\exp(O(1))$.

Furthermore, $K(w)$ for elements w of a constructive world is defined as complexity of its number in a fixed structural numbering (cf. 1.1 below). Changing the numbering, we again change complexity by a $\exp(O(1))$ -factor.

(B) *The probability distribution producing Zipf’s law (with exponent -1) is (an approximation to) the L. Levin maximal computable from below distribution: see [ZvLe], [Le] and [LiVi].*

If we accept (A) and (B), then Zipf’s law follows from two basic properties of Kolmogorov complexity:

(a) *rank of w defined according to (A) is $\exp(O(1)) \cdot K(w)$.*

(b) *Levin’s distribution assigns to an object w probability $\sim KP(w)^{-1}$ where KP is the exponentiated prefix Kolmogorov complexity, and we have, up to $\exp(O(1))$ -factors,*

$$K(w) \preceq KP(w) \preceq K(w) \cdot \log^{1+\varepsilon} K(w)$$

with arbitrary $\varepsilon > 0$.

Slight discrepancy between the growth orders of K and KP is the reason why a probability distribution on infinity of objects cannot be constructed from K : the

series $\sum_m K(m)^{-1}$ diverges. However, on finite sets of data this small discrepancy is additionally masked by the dependence of both K and KP on the choice of an optimal encoding. Therefore, when speaking about Zipf's Law, we will mostly disregard this difference. See also the discussion of the partition function for codes in 0.3 below.

0.3. Complexity as effort. The picture described above agrees with Zipf's motto "minimization of effort", but reinterprets the notion of effort: its role is now played by the logarithm of the Kolmogorov complexity that is by the length of the maximally compressed description of an object. This length is not computable, but it is the infimum of a sequence of computable functions.

Such a picture makes sense especially if the objects satisfying Zipf's distribution, are *generated* rather than simply *observed*.

Intuitively, whenever an individual mind, or a society, finds a compressed description of something, this something becomes usable, and is used more often than other "something" whose description length is longer. For an expanded version of this metaphor applied to the history of science, see [Man3].

For words in the initial Zipf's observation, this principle refers to ways in which mind/brain generates and uses language.

0.4. Relation to previous works. I am aware of two works where complexity is invoked in relation to Zipf's law: [Ve] and [MurSo].¹

(a) Briefly, viewpoint of [MurSo] is close to ours, but, roughly speaking, the authors focus on the majority of objects consisting of "almost random" ones: those whose size is comparable with Kolmogorov complexity, and which therefore cannot be compressed. This is justified by the authors' assumption that the data corpus satisfying Zipf's Law comes from a sequence of successive observations over a certain system with stochastic behaviour.

To the contrary, our ranking puts in the foreground those objects whose size might be very large in comparison with their complexity, because we imagine systems that are *generated* rather than simply *observed*, in the same sense as texts written in various languages are generated by human brains.

¹By the time this text was essentially written, one more article [Del] appeared that suggests essentially the same relation between Zipf and complexity as this paper. Prof. Jean-Paul Delahaye kindly drew my attention to it after my article was posted in arXiv.

To see the crucial difference between the two approaches on a well understood mathematical example, one can compare them on the background of error-correcting codes, following [ManMar]. Each such code C (over a fixed alphabet) determines a point in the unit square of the plane (*transmission rate, minimal relative Hamming's distance*). The closure of all limit code points is a domain lying below a certain continuous curve which is called *asymptotic bound*.

If one produces codes in the order of growing size, most code points will form a cloud densely approximating the so called Varshamov–Gilbert bound that lies strictly below the asymptotic bound.

To the contrary, if one produces codes in the order of their Kolmogorov complexity, their code points will well approximate the picture of the whole domain under the asymptotic bound: see details in [ManMar]. Moreover, Levin's distribution very naturally leads to a thermodynamic partition function on the set of codes, and to the interpretation of asymptotic bound as a phase transition curve: this partition function has the form $\sum_C K(C)^{-s(C)}$ where $s(C)$ is a certain function defined on codes and including as parameters analogs of temperature and density. Here one may replace K with KP , and freely choose the optimal family defining complexity: this will have *no influence at all* on the form of the phase curve/asymptotic bound.

Moreover, $\log K(w)$ that is the bit-size of a maximally compressed description of w , plays precisely the role of energy in this partition function, thus validating our suggestion to identify it with “effort”.

It is interesting to observe that the mathematical problem of generating good error-correcting codes historically made a great progress in the 1980's with the discovery of algebraic geometric Goppa codes, that is precisely with the discovery of greatly compressed descriptions of large combinatorial objects.

To summarize, the class of a priori probability distributions that we are considering here is *qualitatively distinct* from those that form now a common stock of sociological and sometimes scientific analysis: cf. a beautiful synopsis of the latter by Terence Tao in [Ta] who also stresses that “mathematicians do not have a fully satisfactory and convincing explanation for how the [Zipf] law comes about and why it is universal”.

(b) We turn now to the paper [Ve], in which T. Veldhuizen considers Zipf's law in an unusual context that did not exist in the days when Kolmogorov, Solomonov and Chaitin made their discoveries, but which provides, in a sense, landscape for an industrial incarnation of complexity. Namely, he studies actual software and

software libraries and analyzes possible profits from software reuse. Metaphorically, this is a picture of human culture whose everyday existence depends on a continuous reuse of treasures created by researchers, poets, philosophers.

Mathematically, reuse furnishes new tools of compression: roughly speaking, a function f may have a very large Kolmogorov complexity, but the length of the library address of its program may be short, and only the latter counts if one can simply copy the program from the library.

In order to create a mathematical model of reuse and its Zipf's landscape along the lines of this note, I need to define the mathematical notion of *relative Kolmogorov complexity* $K(f|F)$. This notion goes back to Kolmogorov himself and is well known in the case when f, F are finite combinatorial objects such as strings or integers (cf. [LiVi]).

In the body of the paper, I generalize this definition to the case of a *library* F of programs. The library may even contain uncomputable oracular data, and thus we include into the complexity landscape oracle-assisted computations.

0.5. Some justifications. Consider some experimental data demonstrating the dependance of Zipf's rank from complexity in the most natural environment: when we study the statistics not of all words, but only *numerals*, the names of numbers.

Then in our model we expect that:

(i) Most of the numbers n , those that are Kolmogorov "maximally complex", will appear with probability comparable with $n^{-1}(\log n)^{-1-\varepsilon}$, with a small ε : "most large numbers appear with frequency inverse to their size" (in fact, somewhat smaller one).

(ii) However, frequencies of those numbers that are Kolmogorov very simple, such as 10^3 (thousand), 10^6 (million), 10^9 (billion), must produce sharp local peaks in the graph of (p_n) .

The reader may compare these properties of the discussed class of Levin's distributions, which can be called *a priori distributions*, with the observed frequencies of numerals in printed and oral texts in several languages, summarized in Dehaene, [De], p. 111, Figure 4.4. (Those parts of the Dehaene and Mehler graphs in the book [De] that refer to large numbers, are somewhat misleading: they might create an impression that frequencies of the numerals, say, between 10^6 and 10^9 smoothly interpolate between those of 10^6 and 10^9 themselves, whereas in fact they abruptly

drop down. See, however, a much more detailed discussion in [DeMe].) The article [Del] also quotes ample empirical data obtained by Google search.

To me, the observable *qualitative* agreement between suggested theory and observation looks convincing: brains and their societies do follow predictions of a priori probabilities. Of course, one has to remember that compression degrees that can be achieved by brains and civilisations might produce *quantitatively* different distributions at the initial segments of a Kolmogorov Universe, because here dependence of the complexity of objects on the complexity of generating them mechanisms (“the culture code”) becomes pronounced.

There is no doubt that many instances of empiric Zipf’s laws will not be reducible to our complexity source. Such a reduction of the Zipf law for all words might require for its justification some neurobiological data: cf. [Ma1], appendix A in the arXiv version.

Another interesting possible source of Zipf’s law was considered in a recent paper [FrChSh]. The authors suggested that Zipf’s rank of an object, member of a certain universe, might coincide with its PageRank with respect to an appropriate directed network connecting all objects. This mechanism generally produces a power law, but not necessarily exactly Zipf’s one.

In any case, the appeal to the uncomputable degree of maximal compression in our model of Zipf–Levin distribution is exactly what can make such a model an eye-opener.

0.6. Fractal landscape of the Kolmogorov complexity and universality of Zipf’s law. A graph of logarithmic Kolmogorov complexity of integers k (and its prefix versions) looks as follows: most of the time it follows closely the graph of $\log k$, but infinitely often it drops down, lower than any given computable function: see [LiVi], pp. 103, 105, 178. The visible “continuity” of this graph reflects the fact that complexity of $k + 1$ in any reasonable encoding is almost the same as complexity of k .

However, such a picture cannot convey extremely rich self-similarity properties of complexity. The basic fractal property is this: if one takes any infinite decidable subset of \mathbf{Z}^+ in increasing order and restricts the complexity graph on this subset, one will get the same complexity relief as for the whole \mathbf{Z}^+ : in fact, for any recursive bijection f of \mathbf{Z}^+ with a subset of \mathbf{Z}^+ we have $K(f(x)) = \exp(O(1)) \cdot K(x)$.

Seemingly, this source of “fractalization” might have a much wider influence: see [NaWe] and related works.

If we pass from complexity to a Levin's distribution, that is, basically, invert the values of complexity, these fractal properties survive.

This property might be accountable for "universality" of Zipf's law, because it can be read as its extreme stability with respect to the passage to various sub-universes of objects, computable renumbering of objects etc. In the same way, the picture of random noise in a stable background is held responsible for universality of normal distribution.

0.7. Plan of the paper. In the main body of the paper, I do not argue anymore that complexity may be a source of Zipf's distribution. Instead, I sketch mathematics of complexity in a wider context, in order to make it applicable in the situations described in [Ve].

In sec. 1, I define the notion of Kolmogorov complexity relative to an *admissible family* of (partial) functions. The postulated properties of such admissible families should reflect our intuitive idea about library reuse and/or oracle-assisted computations.

In sec. 2, I suggest a formalization of the notion of computations that (potentially) produce admissible families. It turns out that categorical and operadic notions are useful here, as it was suggested in [Man1], Ch. IX.

Acknowledgements. I first learned about Zipf's Law from (an early version of) the paper [Ma] by D. Yu. Manin, and understood better the scope of my constructions trying to answer his questions. The possibility that Zipf's law reflects Levin's distribution occurred to me after looking at the graphs in the book [De] by S. Dehaene. Professor Dehaene also kindly sent me the original paper [DeMe]. C. Calude read several versions of this article and stimulated a better presentation of my arguments. An operadic description of a set of programs for computation of (primitive) recursive functions was discussed in [Ya], and my old e-mail correspondence with N. Yanofsky helped me to clarify my approach to the formalization of the notions of reuse and oracles. Finally, L. Levin suggested several useful revisions. I am very grateful to all of them.

1. Admissible sets of partial functions and relative complexity

1.1. Notations and conventions. We recall here some basic conventions of [Man1], Ch. V and IX. Let X, Y be two sets. A *partial function* from X to Y is a pair $(D(f), f)$ where $D(f) \subset X$, $f : D(f) \rightarrow Y$. We call $D(f)$ the *domain* of f ,

and often write simply $f : X \rightarrow Y$. If $D(f) = \emptyset$, f is called an empty function. If $D(f) = X$, we sometimes call f a *total* function. If X is one-element set, then non-empty functions $X \rightarrow Y$ are canonically identified with elements of Y . Partial functions can be composed in an obvious way: $D(g \circ f) := f^{-1}(D(g) \cap \text{Im}(f))$. Thus we may consider a category consisting of (some) sets and partial maps between them.

Put $\mathbf{Z}^+ :=$ the set of positive integers. Then $(\mathbf{Z}^+)^m$ for $m \geq 1$ can be identified with the set of vectors (x_1, \dots, x_m) , $x_i \in \mathbf{Z}^+$. By definition, $(\mathbf{Z}^+)^0$ is an one-element set, say, $\{*\}$. Any partial function $f : (\mathbf{Z}^+)^m \rightarrow (\mathbf{Z}^+)^n$ will be called an (m, n) -function. For $m = 0$, such a non-empty function may and will be identified with a vector from $(\mathbf{Z}^+)^n$.

Let X be an infinite set. A structure of *constructive world* on X is given by a set of bijections \mathcal{N}_X called *structure numberings*, $X \rightarrow \mathbf{Z}^+$, such that any two bijections in it are related by a (total) recursive permutation of \mathbf{Z}^+ , and conversely, any composition of a structural numbering with a recursive permutation is again a structure numbering. Explicitly given finite sets are also considered as constructive worlds. (A logically minded reader may imagine all our basic constructions taking place at the ground floor of the von Neumann Universe).

Intuitively, X can be imagined as consisting of certain finite Bourbaki structures that can be unambiguously described and encoded by finite strings in a finite alphabet that form a decidable set of strings, and therefore also admit a natural numbering. Any two such natural numberings, of course, must be connected by a computable bijection.

Morphisms between two constructive worlds, by definition, consist of those set-theoretical maps which, after a choice of structural numberings, become partially recursive functions. Thus, constructive worlds are objects of a category, Constructive Universe.

In order to introduce a formalization of oracle-assisted computations, we will have to extend the sets of morphisms allowing partial maps that might be non-computable.

1.2. Admissible sets of functions. Consider a set Φ of partial functions $f : (\mathbf{Z}^+)^m \rightarrow (\mathbf{Z}^+)^n$, $m, n \geq 0$. We will call Φ an *admissible set*, if it is countable and satisfies the following conditions.

(i) Φ is closed under composition and contains all projections (forget some coordinates), and embeddings (permute and/or add some constant coordinates).

Any $(m+1, n)$ -function can be considered as a family of (m, n) -functions (u_k) : $u_k(x_1, \dots, x_m) := u(x_1, \dots, x_m, k)$. From (i) it follows that for any $u \in \Phi$ and $k \in \mathbf{Z}^+$, also $u_k \in \Phi$. Similarly, if $u(x_1, \dots, x_m)$ is in Φ , then

$$U(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n}) \equiv u(x_1, \dots, x_m)$$

is in Φ .

(ii) For any (m, n) , there exists such an $(m+1, n)$ -function $u \in \Phi$ that the family of functions $u_k : (\mathbf{Z}^+)^m \rightarrow (\mathbf{Z}^+)^n$, contains all (m, n) -functions belonging to Φ .

We will say that such a function u (or family (u_k)) is ample.

(iii) Let f be a total recursive function f whose image is decidable, and f defines a bijection between $D(f)$ and image of f . Then Φ contains both f and f^{-1} .

From now on, Φ will always denote an admissible family.

1.3. Complexity relative to a family. Choose an $(m+1, n)$ -function $u \in \Phi$ and consider it as a family of (m, n) -functions u_k as above. For any (m, n) -function $f \in \Phi$, put $K_u^\Phi(f) = \min \{k \mid f = u_k\}$. The r.h.s. is interpreted as ∞ if there is no such k . We will call such a family u *Kolmogorov optimal in Φ* , if for any other $(m+1, n)$ -family v there is a constant $c_{u,v}$ such that for all (m, n) -functions $f \in \Phi$ we have $K_u^\Phi(f) \leq c_{u,v} K_v^\Phi(f)$.

1.4. Theorem. a) If Φ contains an ample family of $(m+1, n)$ -functions, then it contains also a Kolmogorov optimal family of (m, n) -functions.

b) If u and v are two Kolmogorov optimal families of (m, n) -functions, then

$$c_{v,u}^{-1} \leq K_u^\Phi(f)/K_v^\Phi(f) \leq c_{u,v}.$$

Proof. Let $\theta : \mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ be a total recursive bijection between $\mathbf{Z}^+ \times \mathbf{Z}^+$ and a decidable subset of \mathbf{Z}^+ . Assume moreover that $\theta(k, j) \leq k \cdot \phi(j)$ for some $\phi : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$. Choose any ample family $U \in \Phi$ of $(m+1, n)$ -functions and put

$$u(x_1, \dots, x_m, k) := U(x_1, \dots, x_m, \theta^{-1}(k)).$$

Then u is ample and optimal, with the following bound for the constant $c_{u,v}$:

$$c_{u,v} \leq \phi(K_U^\Phi(v)). \quad (1.1)$$

In fact, it suffices to consider such v that f occurs in (v_k) . Then

$$\begin{aligned} f(x_1, \dots, x_m) &= v(x_1, \dots, x_m, K_v^\Phi(f)) \\ &= U(x_1, \dots, x_m, K_v^\Phi(f), K_U^\Phi(v)) \end{aligned}$$

so that

$$K_u^\Phi(f) \leq \theta(K_v^\Phi(f), K_U^\Phi(v)) \leq K_v^\Phi(f) \phi(K_U^\Phi(v)).$$

1.5. Constants related to Kolmogorov complexity estimates. In the inequality (1.1) estimating the dependance of Kolmogorov complexity on the choice of encoding, two factors play the central roles.

One is $K_U^\Phi(v)$. Its effective calculation depends on the possibility of translating a program for v into a program given by U . In the situation where Φ consists of all partial recursive functions, such a compilation can be performed if U satisfies a property that is stronger than ampleness: cf. [Ro] and [Sch] where such families are discussed and constructed.

Another factor is the growth rate of ϕ . Below we will show how the task of optimization of ϕ can be seen in the context of Levin's distributions, reproducing an argument from [Man2].

1.5.1. Slowly growing numberings of $(\mathbf{Z}^+)^2$. Let $R = (R_k \mid k \in \mathbf{Z}^+)$ be a sequence of positive numbers tending to infinity with k . For $M \in \mathbf{Z}^+$, put

$$V_R(M) := \{(k, l) \in (\mathbf{Z}^+)^2 \mid kR_l \leq M\}.$$

Clearly,

$$\text{card } V_R(M) \leq \sum_{l=1}^{\infty} \left\lceil \frac{M}{R_l} \right\rceil < \infty,$$

where $[a]$ denotes the integral part of a .

We have

$$V_R(M) \subset V_R(M+1), \quad (\mathbf{Z}^+)^2 = \bigcup_{M=1}^{\infty} V_R(M).$$

Therefore we can define a bijection $N_R : (\mathbf{Z}^+)^2 \rightarrow \mathbf{Z}^+$ in the following way: $N_R(k, l)$ will be the rank of (k, l) in the total ordering $<_R$ of $(\mathbf{Z}^+)^2$ determined inductively by the following rule: $(i, j) <_R (k, l)$ iff one of the following alternatives holds:

- (a) $iR_j < kR_l$;
- (b) $iR_j = kR_l$ and $j < l$;

1.5.2. Proposition. *The numbering N_R is well defined and has the following property: all elements of $V_R(M+1) \setminus V_R(M)$ have strictly larger ranks than those of $V_R(M)$. Moreover:*

(i) *If the set $\{(q, l) \in \mathbf{Q} \times \mathbf{Z}^+ \mid q \geq R_l\}$ is enumerable (image of a partial recursive function), then N_R is computable (total recursive).*

(ii) *If the series $\sum_{l=1}^{\infty} R_l^{-1}$ converges and its sum is bounded by a constant c , then*

$$N_R(k, l) \leq c(kR_l + 1). \quad (1.2)$$

(iii) *If the series $\sum_{l=1}^{\infty} R_l^{-1}$ diverges, and*

$$\sum_{l=1}^M R_l^{-1} \leq F(M)$$

for a certain increasing function $F = F_R$, then

$$N_R(k, l) \leq (kR_l + 1)F(kR_l + 1).$$

Proof. The first statements are an easy exercise. For the rest, notice that if M is the minimal value for which $(k, l) \in V_R(M)$, we have $M - 1 < kR_l \leq M$ and

$$N_R(k, l) \leq \text{card } V_R(M),$$

and in the case (ii) we have

$$\text{card } V_R(M) \leq \sum_{m=1}^{\infty} MR_m^{-1} \leq c(kR_l + 1).$$

Similarly, in the case (iii) we have

$$\text{card } V_R(M) \leq M \sum_{m=1}^M R_m^{-1} \leq (kR_l + 1)F(kR_l + 1)$$

1.6. L. Levin's probability distributions. From (1.2) one sees that any sequence $\{R_l\}$ with converging $\sum_l R_l^{-1}$ can be used in order to construct the bijection $\mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$, $(k, l) \mapsto N_R(k, l)$ linearly growing wrt k . Assume that it is computable and therefore can play the role of θ in (b).

In this case, for any l , the set of rational numbers $k/M \leq r_l := R_l^{-1}$ must be decidable.

Even if we weaken the last condition, requiring only enumerability of the set $k/M \leq r_l$ (i. e. asking each r_l to be computable from below), the convergence of $\sum_l r_l$ implies that there is a universal upper bound (up to a constant) for such r_l . Namely, let KP be the exponentiated *prefix Kolmogorov complexity* on \mathbf{Z} defined with the help of a certain optimal prefix enumeration (see [LiVi], [CaSt] for details).

1.6.1. Proposition. ([Le]). *For any sequence of computable from below numbers r_l with convergent $\sum_l r_l$, there exists a constant c such that for all l , $r_l \leq c \cdot KP(l)^{-1}$*

More generally, L. Levin constructs in this way a hierarchy of complexity measures associated with a class of abstract norms, functionals on sequences computable from below.

As I explained in the Introduction, this paper suggests that these *mathematical* distribution laws might lead to a new explanation of statistic properties of some observable data.

2. The computability (pro)perads and admissible families

2.1. Libraries, oracles, and operators. In this section, we will define admissible sets of partial (m, n) functions Φ formalizing intuitive notions of “software libraries and their reuse” and “oracle-assisted computation”. A Kolmogorov complexity relative to such a set will include a formalization of the intuitive notion of relative complexity $K(f|g)$ in the cases when f, g are recursive (“reuse of g ”) and when g might be uncomputable (“oracle-assisted computation”).

In this section our main objects are *not functions* but objects of higher types:

(i) *Programs for computing functions*, eventually even *names* of oracles telling us values of uncomputable functions, and programs including these names.

(ii) *Operators*, that is programs *computing certain functions whose arguments and values are themselves programs*.

The main reason for such shift of attention is this. Already the set of partial recursive (m, n) -functions for $m \geq 1$ *is not a constructive world*, as well as its extensions with which we deal here. To the contrary, the set of programs calculating recursive functions in a chosen programming language, such as Turing machines, or texts in a lambda-calculus, is constructive, but endowed with *uncomputable equivalence relation*: “two programs compute one and the same function”. We will call such worlds “programming methods”.

More precisely, let X, Y be two constructive worlds. A *programming method* is a constructive world $P(X, Y)$ given together with a map $P(X, Y) \rightarrow \text{ParSet}(X, Y)$, $p \mapsto \bar{p}$, where ParSet is the category of sets with partial maps as morphisms. We will systematically put bar over the name of a program to denote the function $\bar{p}: X \rightarrow Y$ which p computes.

A good programming method must have additional coherence properties. For brevity, consider only infinite constructive worlds, and assume that P computes all recursive isomorphisms. We can then extend P to any two infinite constructive worlds, and it is natural to require the existence of at least two additional programs/operators

$$Ev \in P(X \times P(X, Y), Y), \quad \overline{Ev}(x, p) := \bar{p}(x) \text{ for } x \in X, p \in P(X, Y), \quad (2.1)$$

$$Comp: P(P(X, Y) \times P(Y, Z)) \rightarrow P(X, Z), \quad \overline{\overline{Comp}(f, g)} = \bar{g} \circ \bar{f}. \quad (2.2)$$

We will call such objects as Ev and $Comp$ operators and say that they *lift* the respective operations on functions: evaluation at a point and composition.

For more detailed mathematical background, cf. [Man1], Ch. IX, sec. 3–5.

In the following we start with a certain programming method P computing at least all partial recursive (m, n) -functions, and describe ways of extending it necessary to formalize the notions of library reuse and oracles.

2.2. Constructing admissible sets. Each such set Φ will be defined in the following way.

(i) Choose a constructive world S consisting of programs for computing (m, n) -functions. It will be a union of two parts: (programs of) *elementary functions* and *library functions*. All elementary functions will be (Turing) computable, i. e. (partial) recursive. Library functions must form a constructive world (possibly, finite), with some fixed numbering. The number of a library program is called its *address*. Both library functions and elementary functions will be decidable subsets of S .

(ii) Define a set of operators that can be performed on finite strings of partial (m_i, n_i) functions. They will be obtained by iterating several basic operators, such as *Comp* and *Ev* above. The world OP of programs/names of such operators will be easy to encode by a certain constructive world of labelled graphs.

(iii) Take an element $P \in OP$ and specify a finite string $s := (f_1, \dots, f_r)$ of (addresses) of functions from S that can serve as input to P . The pair (P, s) is then a program for calculation of a concrete string of (m, n) -functions.

(iv) Finally, Φ will be defined as the minimal set of functions computable by the programs in S and closed with respect to the applications of all operators in OP .

We will now give our main examples of objects informally described in (i)–(iv).

2.3. The language of directed graphs. In our world OP , each operator ρ can take as input a finite sequence of partial functions $f_i : (\mathbf{Z}^+)^{m_i} \rightarrow (\mathbf{Z}^+)^{n_i}$, $m_i, n_i \geq 0$, $i = 1, \dots, k$ and produce from it another finite sequence of partial functions $g_i : (\mathbf{Z}^+)^{p_i} \rightarrow (\mathbf{Z}^+)^{q_i}$, $m_i, n_i \geq 0$, $i = 1, \dots, l$. We will call *the signature* of ρ the family

$$\text{sign}(\rho) := [(m_1, n_1), \dots, (m_k, n_k); (p_1, q_1), \dots, (p_l, q_l)]. \quad (2.3)$$

Two operations ρ, σ can be composed if their signatures match: $\rho \circ \sigma$ takes as input the output of σ .

Below we will describe explicitly a set of *basic* operations OP_0 . After that the whole set OP will be defined as the minimal set of operations containing OP_0 and closed under composition.

A visually convenient representation of elements of OP is given by (isomorphism classes of) directed labeled graphs: see formal definitions in [BoMan], Section 1.

More precisely, each basic operator ρ of signature (2.3) is represented by a *corolla*: graph with one vertex labelled by ρ ; k flags oriented towards the vertex (inputs) and l flags oriented from the vertex (outputs). Moreover, inputs and outputs must be totally ordered, and labelled by respective pairs (m_i, n_i) .

Labelled directed graphs with more vertices are obtained from a disjoint finite set of corollas by grafting some outputs to some inputs. Grafted pair (output, input) must have equal labels (m, n) . Flags that remain ungrafted form the inputs/outputs of the total graph.

Notice that directed graphs ([BoMan], 1.3.2) do not admit oriented loops.

2.4. Basic operators. In this subsection, we describe operations on strings of functions that must be represented by the respective operators. For brevity, we will denote by single Greek letters these operators.

(a) *Composition of functions.* This basic operator, say γ , has signature of the form $[(m, n), (n, q); (m, q)]$. It produces from two partial functions $f : (\mathbf{Z}^+)^m \rightarrow (\mathbf{Z}^+)^n$ and $g : (\mathbf{Z}^+)^n \rightarrow (\mathbf{Z}^+)^q$ their composition $g \circ f : (\mathbf{Z}^+)^m \rightarrow (\mathbf{Z}^+)^q$. Recall that $D(g \circ f) := f^{-1}(D(g))$. It is a special case of operator *Comp*.

(b) *Juxtaposition.* This basic operator, say σ , has signature of the form

$$[(m, n_1), \dots, (m, n_k); (m, n_1 + \dots + n_k)].$$

It produces from k partial functions $f_i : (\mathbf{Z}^+)^m \rightarrow (\mathbf{Z}^+)^{n_i}$ the function (f_1, \dots, f_k) :

$$D((f_1, \dots, f_k)) = D(f_1) \cap \dots \cap D(f_k),$$

$$(f_1, \dots, f_k)(x_1, \dots, x_m) = (f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m)).$$

(c) *Recursion.* This basic operator, say ρ , has signature of the form

$$[(m, 1), (m + 2, 1); (m + 1, 1)]$$

It produces from partial functions $f : (\mathbf{Z}^+)^m \rightarrow \mathbf{Z}^+$ and $g : (\mathbf{Z}^+)^{m+2} \rightarrow \mathbf{Z}^+$ the function $h : (\mathbf{Z}^+)^{m+1} \rightarrow \mathbf{Z}^+$ such that

$$h(x_1, \dots, x_m, 1) = f(x_1, \dots, x_m),$$

$$h(x_1, \dots, x_m, k + 1) = g(x_1, \dots, x_m, k, h(x_1, \dots, x_m, k))$$

for $k \geq 1$.

The definition domain $D(h)$ is also defined by recursion:

$$(x_1, \dots, x_m, 1) \in D(h) \Leftrightarrow (x_1, \dots, x_m) \in D(f),$$

$$(x_1, \dots, x_m, k + 1) \in D(h)$$

$$\Leftrightarrow (x_1, \dots, x_m, k) \in D(h) \text{ and } (x_1, \dots, x_m, k, h(x_1, \dots, x_m, k)) \in D(g)$$

for $k \geq 1$.

(d) *Operator μ* . Its signature is $[(m+1, 1); (m, 1)]$. Given an $(m+1, 1)$ -function f , it produces the $(n, 1)$ -function h with the definition domain

$$D(h) = \{(x_1, \dots, x_n) \mid \exists x_{n+1} \geq 1 \text{ such that}$$

$$f(x_1, \dots, x_n, x_{n+1}) = 1 \text{ and } (x_1, \dots, k) \in D(f) \text{ for all } k \leq x_{n+1}\}.$$

At the definition domain

$$h(x_1, \dots, x_n) = \min \{x_{n+1} \mid f(x_1, \dots, x_{n+1}) = 1\}.$$

(e) *Identity operation ι* .

2.5. The constructive world of operations OP . Returning now to sec. 2.2, we define OP as the set of finite directed labelled graphs with totally ordered inputs/outputs at each vertex satisfying the following condition: each vertex is labelled by one of the letters $\gamma, \sigma, \rho, \mu, \iota$ and its inputs/outputs are labelled by the respective components of the relevant signature..

We can choose any one of the standard ways to encode such graph by a string over a fixed finite alphabet, and then define a structure numbering of OP by ranking these words in alphabetic order. The subset of well-formed strings that encode graphs ought to form a decidable subset of all strings, and all natural functions such as

graph \mapsto sequence of all inputs of the graph with their (m, n) – labellings

ought to be total recursive.

In the final count, each element of OP determines an operation on finite sets of partial functions, producing another finite set of partial functions. Moreover, OP can be enriched to a free (pro)perad acting on finite strings of partial functions. If the signature of this string does not match the signature of the inputs of the operation, we may and will agree that the operation produces an empty function. The enrichment however requires some care and higher categorical constructions.

2.6. Basic partial functions. Let S be a constructive world of programs/oracles calculating partial (m, n) -functions. Denote by $OP(S)$ the minimal set of such programs containing S and closed with respect to application to them of operators from

OP . In the remainder of this paper, we will always include in S a set of basic recursive functions S_{rec} , such that $OP(S_{rec})$ consists of all (partial) recursive function. In [Man1], Ch. V, Sec. 2, the following set is chosen:

$$\begin{aligned} \text{suc} : \mathbf{Z}^+ &\rightarrow \mathbf{Z}^+, & x &\mapsto x + 1, \\ 1^{(n)} : (\mathbf{Z}^+)^m &\rightarrow \mathbf{Z}^+, & (x_1, \dots, x_m) &\mapsto 1, \quad n \geq 0. \\ \text{pr}_i^m : (\mathbf{Z}^+)^m &\rightarrow \mathbf{Z}^+, & (x_1, \dots, x_m) &\mapsto x_i, \quad n \geq 1. \end{aligned}$$

2.7. Admissible sets and library reuse. The standard Kolmogorov complexity of partial recursive functions is defined relative to the admissible set of functions Φ computable by programs from $OP(S_{rec})$. If we include into the set S only some programs of recursive functions, then the total set of computable functions Φ will not grow, but some functions will be computable by much shorter programs because the price of writing a program for a library program can be disregarded.

2.8. Admissible sets including oracles for uncomputable functions. Here one more complication arises: the requirement (ii) in our definition of the admissible sets of functions is not satisfied automatically in the world $OP(S)$ as it was in the case when the respective set of functions consisted only on recursive functions.

In order to remedy this, we have to add to the list of basic operators the operator Ev from (2.1). Its participation in the iteration of our former basic operations cannot be in an obvious way described by labelled graphs, so more systematic treatment is required: seemingly, we are in the realm of “expanding constructive universe”, some propaganda for which was made in [Man1], Ch. IX, sec. 3.

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