Logic and Set Theory: Prof Leader's Example Sheets for 21/22; Sheet 4

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Question 1

Show that $\{Inf, Sep\} \vdash Emp$. Does $Inf \vdash Emp$? Does $Sep \vdash Emp$?

[N.B. You should interpret $\{Inf,Sep\}$ as "the collection of formulæ in the language of sets consisting of the Axiom of Infinity and every instance of the Axiom of Separation" etc.]

The morally correct way to do this is to observe that the axiom of infinity has the form "there is a set with a special property". If there is *even one* set, then—as long as we have separation—there will be an empty set, since the subsets consisting of all those elements of the set that are not equal to themselves will be a set by separation.

Now the axiom of infinity can also come in the form "There is a successor set", or

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$$

In the presence of the axiom scheme of replacement this can be deduced from the bare assertion there is an infinite set (a set not the same size as any proper subset of itself) but the axiom is often taken in this more specialised form because it makes it easy to give immediately an implementation of arithmetic. Fair enough. However, this muddies the waters slightly, in that it enables us to give a different proof of the existence of the empty set. People sometimes say that the axiom of infinity *presupposes* the existence of the empty set, but that's not quite right. Let's get this 100% straight. The axiom in the form "there is a successor set" says

$$(\exists x)((\exists e \in x)(\forall w)(w \notin e) \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x)) \tag{1}$$

I have written out the '$\emptyset \in x$' bit in primitive notation so we can be sure that there are no tricks being played.

The expression (1) is of the form

$$(\exists x)(p \wedge F(x))$$

where p is $(\exists e \in x)(\forall w)(w \notin e)$ and $F(x)$ is $(\forall y)(y \in x \rightarrow y \cup \{y\} \in x)$. Anything of the form $(\exists x)(p \wedge F(x))$ is going to imply $(\exists x)p$, namely

$$(\exists x)(\exists e \in x)(\forall w)(w \notin e)$$

whence

$$(\exists e)(\forall w)(w \notin e)$$

which says that there is an empty set, which is what we wanted.

Actually some of you have made the connection between this question and the point (made in lectures) that we do not admit the empty model. This question of whether or not we admit the empty model is a vexed one that i have striven not to think about for most of my working life—and with moderate success. Aristotle evidently did not accept empty models, since he thought that "All A are B" implied "Some A are B". Recently (i.e., within my lifetime) people (Prof Hyland among them) have started saying that perhaps it would be a good idea to admit empty models. If you want to pursue this, google 'Free Logic'.

My current thinking is that since we evidently cannot deduce the empty set axiom from the separation scheme, and since the only countermodel (= a model that shows that you cannot perform that deduction) is the empty model, then if we want to preserve the completeness theorem, well, we'd better accept empty models. However, as i say, this is something i have tried not to think about!

Question 2

Show that Rep $\vdash$ Sep. Show also that $\{$Emp, Pow, Rep$\}$ $\vdash$ Pair.

First part, deducing the axiom (scheme) of separation from the axiom (scheme) of replacement.

If replacement allows you to use *partial* functions it's easy. If you are only allowed *total* functions then you want Phil Connell's trick (tidied up by me to make it constructive):

Define $f(x)$ to be $\{y : y = x \wedge \phi(y)\}$. This has the effect that f sends to their singletons the things you want to keep, and sends everything else to the empty set. Then $\bigcup f``W$ is $\{x \in W : \phi(x)\}$. Observe that this is constructive.

The other way (preferable in certain circumstances) is to say: *either* there is nothing in W which has ϕ (in which case the set we want is the empty set, and we have an axiom for that) *or* there is an $x \in W$ s.t. $\phi(x)$. For any such x we can define a function f which sends $y \in W$ to y as long as $\phi(y)$, and sends y to x o/w.

What is there to choose between these two proofs? Phil Connell's proof uses the axiom of sumset, but the second method uses excluded middle. (It uses it *twice*; once in the case split, and again in the second of the two cases, testing whether or not $\phi(y)$).

Second part.

Two applications of power set to $\emptyset$ gives you $\{\{\emptyset\}, \emptyset\}$ which we then whack with the function class

$$(u = \{\emptyset\} \wedge v = x) \vee (u = \emptyset \wedge v = y)$$

which will give us the pair $\{x, y\}$.

Question 3

Write down sentences in the language of set theory to express the assertions that, for any two sets x and y, the product $x \times y$ and the set y^x of all functions from x to y exist. You may assume that your pairs are Wiener-Kuratowski.

If you use Wiener-Kuratowski pairs then $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ and is a subset of $\mathcal{P}^2(\{x, y\})$. Similarly $x \times y$ is a subset of the power set a couple of times of $x \cup y = \bigcup\{x, y\}$. Clearly the set of functions from x to y can be obtained in the same way.

What if you want to establish that these things are sets without knowing what your pairing function is? Imagine the following situation: i want $X \times Y$ and i know that there is a set-theoretic construct $\langle x, y\rangle$, tho' i don't know what it is and i'm not allowed to assume anything other than that it is there and is available. We do the following: fix $y \in Y$ and consider the function class that sends x to $\langle x, y\rangle$. The image of X in this function exists by replacement and it is of course $X \times \{y\}$. So $X \times \{y\}$ exists for all y. Now consider the function class that sends y to $X \times \{y\}$. The image of Y in this function exists by replacement and its sumset is $X \times Y$.

So: if we have replacement we can prove that $X \times Y$ exists *whatever implementation of pairing-with-unpairing we use*. You might like to prove the converse: if $X \times Y$ always exists for all implementations of pairing-with-unpairing then replacement follows.

Question 4

Is it true that if x is a transitive set then the relation $\in x$ is a transitive relation? Does the converse hold?

Neither direction works.

The restriction $\in \restriction \{\{\emptyset\}\}$ is the empty relation, and therefore transitive, but $\{\{\emptyset\}\}$ is not a transitive set.

For the other direction, consider

$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is a transitive set, but $\in \restriction \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is not a transitive relation: $\emptyset \in \{\emptyset\}$ and $\{\emptyset\} \in \{\{\emptyset\}\}$ but $\emptyset \not\in \{\{\emptyset\}\}$.

I think the danger in this second part is to reason "Suppose $z \in y \in x$; then, since $\in \restriction x$ is transitive, we infer $z \in x$". This doesn't work. It assumes that $\in \restriction x \cup \{x\}$ is transitive which sounds the same but isn't.

Question 5

Use $\in$-induction to show that the only $\in$-autormorphism definable by a function-class is the identity.

Assume that f is an $\in$-automorphism, which is to say $f(x) = f``x$ for all $x$. But "every member of $x$ is fixed by $f$" is just $x = f``x$, whence $f(x) = x$. So x is fixed as long as all its members are; $\in$-induction does the rest.

Discussion

Notice we could be doing one of two inductions here, and it just might matter which. We could specialise to a given $\in$-automorphism f and then prove by $\in$-induction that everything is fixed by it; alternatively we could prove by $\in$-induction that "x is fixed by all $\in$-automorphisms". This second formulation has a universal quantifier ranging over *classes* and that raises technical difficulties for ZF since it involves quantifying over proper classes. This is something that you don't want to think about, at least not yet.

Question 6

What is the rank of $\{2, 3, 6\}$? What is the rank of $\{\{2, 3\}, \{6\}\}$? Work out the ranks of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, using your favourite constructions of these objects from $\mathbb{N}$.

First part: 7 and 8. He's obviously taking natural numbers to be the corresponding Von Neumann ordinals and the rank of a von Neumann ordinal is itself[1].

The second part of this question makes several points. One of them is the point that there are lots of ways of implementing $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ etc as sets; another is that—mathematically at least—it doesn't much matter which one you use. The other is to get you to do some set-theoretic calculations—computing the ranks of particular sets.

One should start with a warning : the (set-theoretic) rank of a set equipped with an ordering cannot be computed from the order-type of the ordering: it's a property of the set, not of any ordering of it. And again, it's nothing to do with cardinality either, or very little. There are small sets (*singletons* indeed) of arbitrarily high rank.

So the rank of a mathematical object implemented as a set is not a attribute of that object; it's an attribute of the *implementation* of the object. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ etc do not have ranks, it's their implementations that do. This comes as a surprise to many students, so it is worth making a big song-and-dance about it. Admittedly it is true that the *minimum possible* rank of an implementation of that object *is* a mathematical invariant of that object [I think Prof Leader sometimes calls it "essential rank"] but it's a curiously uninteresting one, being controlled entirely by cardinality. You cannot implement $\mathbb{R}$ as an object of rank ω beco's there are too few things of lower rank for all the reals to be implemented by those things of lower rank. There are

[1] Actually, here i should show my true self; the rank of a set is an ordinal, and ordinals are not sets, they are numbers. So what I should really say is the rank of the von Neumann ordinal α is α—by which i mean the ordinal itself, not the von Neumann *simulacrum* of it.

uncountably many reals but only countably many things of finite rank. This cardinality consideration is the only constraint.

Actually in parts of the set theoretic literature reals are taken to be functions from $\mathbb{N}$ to $\mathbb{N}$, things sometimes called *set theorists' reals*. They have rank ω—which is best possible, for the reason given above. (They're something to do with continued fractions.)

Discussion

This question makes several points. One of them is the point that there are lots of ways of implementing $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ etc as sets; another is that—mathematically at least—it doesn't much matter which one you use. The other is to get you to do some set-theoretic calculations—computing the ranks of particular sets.

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So: pick an implementation, and compute the ranks of the sets you end up with. For bonus points, pick more than one implementation, and compute all of them! If you know what p-adic numbers are, compute their rank too. (The p-adics are the completion of $\mathbb{Q}$ w.r.t the p-adic metric. How are you to think of the completion set-theoretically?)

A number of you have asked me if there is a suite of implementations of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$ etc etc that is in any sense *canonical*. The answer i think has to be *no*: all implementations are morally of equal value. After all, you don't want the Riemann Hypothesis to be provable in ZF if you take your reals to be equivalence classes of Cauchy sequences but false if you take them to be Dedekind cuts. You definitely want all the implementations to be isomorphic. This is one of the reasons why one needs replacement. But no time to think about that here! "*das ist ein weites Feld*" as my mother used to say when confronted with a difficult question.

Question 7

A set x is called hereditarily finite *if each member of $TC(\{x\})$ is finite. Prove that the class* HF *of hereditarily finite sets coincides with V_ω.*

Which of the axioms of ZF are satisfied in the structure HF?

The key to this question is induction, both structural and wellfounded.

One direction is easy: you prove by induction on n that everything in V_n is in HF. For the other direction you have to use $\in$-induction to show $HF \subseteq V_\omega$. The property $\phi(x)$ you prove by $\in$-induction is "$x \in HF \to x \in V_\omega$".

The need for foundation/induction stems from the fact that if foundation fails then potentially a Quine atom[2] is a counterexample to the inclusion.

This is quite a good question to look at. You have two inclusions to prove, one in each direction. To prove $V_\omega \subseteq HF$ you do an ordinary mathematical induction—on rank; to prove $HF \subseteq V_\omega$ you do an $\in$-induction. A nice idiomatic illustration. Beco's of the possibility of finding Quine atoms in HF (which violate foundation) you *have* to use foundation to prove that $HF \subseteq V_\omega$.

HF is more usually known as V_ω. It is a model of all the axioms of ZFC except infinity.

Codicil

Take care when asking yourself whether or not an axiom is true in a structure. Yer typical set theoretic axiom states that the universe is closed under some operation (as it might be power set, or sumset). **Saying that a structure is a model for that axiom is not the same as saying that it's closed under the corresponding operation**.

When wondering whether or not an axiom is true in a model $\mathfrak{M}$ the thing to ask yourself is "Suppose $\mathfrak{M}$ were the whole world, and i am living inside $\mathfrak{M}$: does the axiom appear to be true?"

[2] A Quine atom is a set $x = \{x\}$.

Question 8

Which axioms of ZF hold in $V_{\omega+\omega}$?

All of them except replacement. Consider the function class $n \mapsto V_{\omega+n}$. Replacement would make the image of $\mathbb{N}$ in this function class—namely $\{V_{\omega+n} : n \in \mathbb{N}\}$—into a set of the model, and it can't be, beco's it is of rank $\omega + \omega$.

Observe that the fact that $\mathbb{R}$ has a subset that is wellordered to length $\omega + \omega$ in the inherited order has the following ramifications. If replacement held in $V_{\omega+\omega}$ then $V_{\omega+\omega}$ would contain, for every wellordering in $V_{\omega+\omega}$ the corresponding von Neumann ordinal. It is easy to check that the rank of the von Neumann ordinal α is α itself, which means that $V_{\omega+\omega}$ cannot contain any ordinal from $\omega + \omega$ onwards. So replacement fails. In general V_α will contain at least some wellorderings that are far too long for their von Neumann ordinals to be in V_α. It happens only rarely that α "catches up" with the ordinals in V_α.

In contrast, the collection H_κ of sets hereditarily of size less than κ is practically guaranteed to be a model of replacement, as follows. Suppose $X \in H_\kappa$, and $f : H_\kappa \to H_\kappa$. Then $f``X$ is a subset of H_κ. How big is it? It's a surjective image of thing of size $< \kappa$. We want it to be of size $< \kappa$ itself. So all we need is a surjective image of something of size less than κ is itself of size $< \kappa$. This is certainly true if κ is an aleph, and even in many cases when it isn't. So certainly if κ is an aleph then H_κ is a model of replacement.

Question 9

You all think you know that $|\mathbb{R}| = 2^{\aleph_0}$ and you're right of course but finding a bijection between $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ is not an *absolute* doddle. It can do you no harm to track one down.

Jonathan Holmes has the cutest proof of this fact known to me (I have doctored this from his answer to this sheet). Every real number has a unique representation as an ω-string of 0's and 1's *containing arbitrarily late 0s*. The italicised condition removes duplicate representations of dyadic rationals. Each such string corresponds to a set (of addresses where the string has 1s) whose complement is infinite. How many subsets of $\mathbb{N}$ are there whose complement in $\mathbb{N}$ is infinite? Well, there are $\aleph_0$ subsets of $\mathbb{N}$ that do not satisfy this condition, so we are looking at $\mathcal{P}(\mathbb{N})$ minus a countable set. You then use Bernstein's lemma to show that any such set has cardinality precisely $2^{\aleph_0}$

To return to Q9, the clever way to prove this is to observe that the function $F : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{Q})$ that takes a continuous function $f : \mathbb{R} \to \mathbb{R}$ and returns $f \restriction \mathbb{Q}$ (its restriction to $\mathbb{Q}$) is injective. Thus F injects the set of continuous functions $\mathbb{R} \to \mathbb{R}$ into the set of functions $\mathbb{Q} \to \mathbb{R}$. This latter set is of size $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$.

This illustrates a general phenomenon. If the set you are trying to compute the size of is naturally a subset of the n-times power set of $\mathbb{N}$ or $\mathbb{R}$ etc then its size will be an iterated exponent of $\aleph_0$. Sizes of quotients can be very complicated. How many wellorders are there of $\mathbb{N}$, *up to isomorphism?* The answer is $\aleph_1$, which may or may not be equal to $2^{\aleph_0}$. Contrast the two questions:

How many total orders of $\mathbb{N}$ are there whose automorphism group is transitive on singletons? (A)

How many countable order types are there whose automorphism group is transitive on singletons? (B)

The answer to (A) is obviously $2^{\aleph_0}$. The set in (B) is a quotient of the set in (A) and calculating its size is a hard task.

Question 10

Consider the sequence $S = \omega,\ \omega_\omega,\ \omega_{\omega_\omega}\ \ldots$ of von Neumann ordinals. It's supremum (union) is obviously going to be a fixed point. However, this question is on a *Set Theory* sheet not an *Ordinals* sheet, so you should be thinking quite hard about how we use the resources of set theory to prove that there really is a wellordering of this length. So we should be asking: how do we know the ordinals stretch that far? The proof is a long road...

For a start, how do we even know that the sequence is even there at all for us to take its sup? Clearly we are going to need an instance of the axiom scheme of replacement. Whack $\mathbb{N}$ with the function class that sends n to $\omega_{\cdot_{\cdot_{\omega}}}$ with n dots. How do we know that this function is defined for all natural numbers? Probably by induction on naturals. Start with ω_ω. How do we know that there is a wellordering of this length? Well, ω_ω is the sup of ω, ω_1, ω_2 ..., and we know that each of these exists by Hartogs' lemma. Then we obtain ω_ω by replacement again. (And it is known that you need replacement to prove the existence of wellorderings that long). And how are you going to get from ω_ω to ω_{ω_ω}?

So i don't think he's expecting you to prove that $\alpha \mapsto \omega_\alpha$ is total.

Question 11

The first sentence contains an ellipsis. He doesn't really mean you to show that there is no surjection from $\aleph_n$ onto $\aleph_{n+1}$ beco's you don't know what sets these cardinals are. (In fact in both the von Neumann implementation and the Scott's trick implementation this allegation is, as it happens, true). What he means is that you should show that there is no surjection from **a set of size** $\aleph_n$ onto **a set of size** $\aleph_{n+1}$. This is pretty easy, even if you aren't allowed choice. It follows from the simple observation that if f is a surjection $X \twoheadrightarrow Y$, and X can be wellordered, then f has an obvious right-inverse—which of course will be an injection $Y \hookrightarrow X$.

The rest of the question is the **Jordan-König theorem**, which says that, if you have two families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ of sets, where, for each $i \in I$, there is no surjection of $A_i \twoheadrightarrow B_i$ then the union $\bigcup\limits_{i \in I} A_i$ of the As does not map onto the product $\prod\limits_{i \in I} B_i$ of the Bs.

The idea for the proof is as follows. Suppose

$$f : \bigcup_{i \in I} A_i \to \prod_{i \in I} B_i$$

We will show that f is not surjective. We have to exploit somehow the circumstance that—for each $i \in I$—no function $A_i \to B_i$ can be surjective, so the obvious thing to do is extract from f a family of functions $f_i : A_i \to B_i$, and trade on the fact that f_i is not surjective. To this end we declare that $f_i(a)$ is to be $f(a)$ applied to i. By assumption $B_i \setminus f_i``A_i$ is nonempty, so let $g \in \prod\limits_{i \in I} B_i$ be a function defined so that $g(i) \in B_i \setminus f_i``A_i$ at each i. I'll leave it to you to show that g is not in the range of f.

Actually you don't need the *whole* of the Jordan-König theorem, and i think what Dr Russell wants you to do is use that line of thinking to show that there is no surjection of A onto $\mathbb{N} \to A$ if $|A| = \aleph_\omega$ ... using the fact that $\aleph_\omega$ is the sup of the $\aleph_n$, with $n \in \mathbb{N}$.

There is a proof of Jordan-König in *Logic, Induction and Sets*. The full general version needs choice[3] but if you merely want to show that $2^{\aleph_0} \neq \aleph_\omega$ then you don't. The point is that if a cardinal comes to you with a name like '$\aleph_\alpha$' you may not have the faintest idea what set is the cardinal *of* but you do know that any such set can be wellordered. The cardinal of a(n infinite) wellorderable set is always something of the form $\aleph_\alpha$ for α an ordinal. So $2^{\aleph_0} = \aleph_\omega$ implies that you can wellorder the real line.

Finally you might like to check your understanding of this situation by proving that $2^{\aleph_0}$ cannot be equal to $\aleph_\alpha$ if the cofinality of α is ω.

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I now think i may have misread this question, and that it's not as hard or unreasonable as i first tho'rt. He wants to do the following.

[3]Is *equivalent* to choice indeed: think ... what happens if the product of nonempty sets is not reliably nonempty?

Think of a set A of size $\aleph_\omega$ as $\bigcup_{i \in \mathbb{N}} A_i$ where $|A_i| = \aleph_i$.

Then use a diagonal argument to show that no function $A \to (\mathbb{N} \to A)$ can be surjective.

OK, suppose $f : A \to (\mathbb{N} \to A)$. We must find $g : \mathbb{N} \to A$ which is not in the range of f. We will think of A as $\bigcup_{i \in \mathbb{N}} A_i$ and our function g will send i to a member of A_i.

Given a you desire $g \neq f(a)$, so you seek i s.t. $g(i) \neq f(a)(i)$. What might this i be? Well, by assumption $f(a)$ is not onto A so you pick something not in the range of $f(a)$.

Question 12

If ZF is consistent then, by Downward Löwenheim-Skolem, it has a countable model. Doesn't this contradict the fact that, for example, the power-set of $\mathbb{N}$ is uncountable?

This is all about absolute properties *versus* non-absolute properties. Ha Thu Nguyen gives a very simple illustration ... If $\mathfrak{M} = \langle M, \in \rangle$ is a model of ZF then it doesn't think that its carrier set M is a set, but we can see from outside that it is. The thing that $\mathfrak{M}$ believes to be the power set of $\mathbb{N}$ is, indeed, countable seen from outside; however $\mathfrak{M}$ does not know of any bijection between that set and the set it believes to be $\mathbb{N}$. Tim Talbot puts it very well: the countable model is a *Tardis*!

Question 13

A union of countably many countable sets cannot have size $\aleph_2$.

I think of this as a rather fun question, tho' discerning people such as Prof Leader and Dr Loewe think it's hard. I think it's not so much *hard* as *scary*, since $\aleph_2$ is a radically unfamiliar object, and you are not confident in confronting it. I would bet good money that none of you have ever seen an object of this size in your life. I would even bet good money that you cannot give me an example of a thing of this size, even now. Another feature that makes it difficult for Those With Unfortunate Conditioning Histories is that you are told to *not* use the axiom of choice, and such people haven't learnt to tell when they are using AC and when they aren't. So they reach for the obvious proof—which uses countable choice. And they can't tell they are using countable choice beco's nobody ever told them. So they can't tell that they aren't answering the question.

Nevertheless this question is an idiomatic piece of bare-hands set theoretic manipulation, and you will find it satisfying.

Actually (as i suggested in my `Advance Warning` email) you could rehearse by first proving that a union of countably many finite sets cannot be of size $\aleph_1$. And don't do that by using countable choice to show that a union of countably many countable sets is countable!

You have to give a proof "by contradiction". Suppose your countable family of countable sets has a sumset of size $\aleph_2$. Then you deduce that it must be of size $\aleph_1$ at most. The only fact about $\aleph_2$ in this context that matters is that $\aleph_2$ is an aleph, it's the cardinal of a wellordered set: any set of size $\aleph_2$ can be wellordered.

The point being that if $\displaystyle\bigcup_{\alpha < \omega_\beta} A_\alpha$ is wellordered then you can use the restrictions of that wellordering to the various A_i to exploit the fact that $\aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma$.

If you want a concrete actual set of size $\aleph_2$ you can think of the Von Neumann ordinal ω_2 (or the set I_{ω_2} of ordinals below ω_2 which is actually the same thing if your ordinals are von Neumann). Then you consider what happens if you attempt to express this set as a union of countably many countable sets.

There are analogues of this that you can prove by the same method: a union of countably many finite sets cannot have size $\aleph_1$; a union of $\aleph_1$ countable sets cannot be of size $\aleph_2$.

For all of these you need that $(\aleph_\alpha)^2 = \aleph_\alpha$. Some of you got your wires crossed on being told that AC is equivalent to $(\forall \alpha)(\alpha = \alpha^2)$ ("cardinals are idempotent") and came away with the impression that proving that $(\aleph_\alpha)^2 = \aleph_\alpha$ needs the axiom of choice. It doesn't. There are two things going on:

(i) AC is equivalent to "every cardinal is an aleph";
(ii) "alephs are idempotent".

(ii) you prove directly, and without AC. I think it was proved it in lectures. In case it wasn't there is a proof in my lecture notes for this course from 2016, linked from my home page on `https://www.dpmms.cam.ac.uk/~tf/cam_only/partiilectures2016.pdf`. It's not obvious, and it's a suitable size for a tripos question.

Once you've dealt with this question you will have no difficulty with the rather more general:

if every A_α is of cardinality $\aleph_\gamma$ at most, **and** $\bigcup_{\alpha<\omega_\beta} A_\alpha$ can be wellordered

then $|\bigcup_{\alpha<\omega_\beta} A_\alpha| \leq \max\{\aleph_\beta, \aleph_\gamma\}$.

Question 14

Be careful how you read this question. You certainly can't add $\mathbb{1}$ to the ring of V if $\cdot$ is $\cap$ and $+$ is Δ, beco's the $\mathbb{1}$ would have to be V and then one application of separation will give you the Russell class as a set and you'd get Russell's paradox. The question is: are there other ring structures you can impose on V which give you a $\mathbb{1}$?

Actually there are several. Prof Leader's favoured solution is to consider, say, the ring $\mathbb{Z}[V]$ of polynomials over $\mathbb{Z}$ with one variable for every set. A bit of Cantor-Bernstein makes this the same size as V so you can copy the ring structure over to V. That looks to me like an unmathematical trick, but there may be attractive features that i have missed. In fact, knowing Professor Leader, there probably are.

Is there another way? If you know about Conway numbers (and if you don't you should[4]) you will recall that the ordinals can be given the structure of a field of characteristic 2. You might think you can copy this over to V. However the project of copying-the-field-structure needs a bijection between V and On and the axiom that says that there is such a bijection is the axiom of *global* choice, and that axiom is strictly stronger than the axiom you have been given (which is the axiom of *local* choice).

Both these solutions are nice, but there is one that is nicer still. Nicer because it provides a $\mathbb{1}$ while retaining $\cdot$ as $\cap$ and $+$ as Δ. To do that of course you need a different membership relation. This is the device of *Church-Oswald models*. See stuff about them on my home page: `http://www.dpmms.cam.ac.uk/~tf/church2001.pdf`). In brief you define a new membership relation as follows. There is a bijection (call it k) between V and $V \times \{0, 1\}$. We then define

$$x \in_{new} y \text{ iff } \begin{cases} \texttt{snd}(k(y)) = 0 \text{ and } x \in \texttt{fst}(k(y)) & \text{or} \\ \texttt{snd}(k(y)) = 1 \text{ and } x \notin \texttt{fst}(k(y)) \end{cases}$$

This makes the universe into a boolean ring—in fact a boolean algebra—where $\vee$ is $\cup$, $\wedge$ is $\cap$ and complementation is set complementation. (Check: $k^{-1}\langle\emptyset, 1\rangle$ becomes the universal set)

CO models are my favoured solution[5], being natural and interesting structures—models for natural and interesting set theories, indeed.

Question 15$^+$

At last, a starred question! This is hard. I used to lecture it at Part III. If you want to know about this—and you well might—have a look at pp 59 *ff* of `www.dpmms.cam.ac.uk/~tf/cam_only/partiiicomputability2020.pdf`. It shows how to encode lists without using exponentiation.

[4]You may know them under the name 'Surreal numbers'—which is a silly name for them, co's there's nothing surreal about them.

[5]Even tho' i'm sure they weren't what Prof Leader had in mind.