8/27/90

MATH-236 : Metamathematics of Set Theory (Topics course)
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"large cardinals and Determinacy".
Goal: To show:

$$\mathrm{CON}(\mathrm{ZFC} + \exists \text{ a Woodin cardinal}) \Rightarrow \mathrm{CON}(\mathrm{ZFC} + \Delta^1_2\text{-det})$$

Determinacy

For our purposes $\mathbb{R} \approx \omega^\omega = \{f : \omega \to \omega\}$.
ω^ω with product top. So, $\omega^\omega \sim \mathbb{R} \setminus \mathbb{Q}$.
$\underset{\sim}{\Sigma}^1_1 = \{$continuous images of Borel sets$\}$, etc...
Work in the structure $\langle V_{\omega+1}, \in \rangle$

Questions: ① Are the projective sets Lebesgue measurable?
Do the " " have Baire property?
(i.e. how far away are we from the pathological consequence of AC?)

true, assuming $\exists$ inaccessible.

② Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ $(= \omega^\omega \times \omega^\omega \cong \omega^\omega)$
is projective.
Is there a projective choice function for A? i.e.,
is there a function $f : \mathbb{R} \to \mathbb{R}$ s.t.
① For all x, if $\{y : (x,y) \in A\} \neq \emptyset$, then $(x, f(x)) \in A$
② graph(f) is projective. ?
(i.e. how much AC can we have back?)

true if there is a well-ordering of $\mathbb{R}$ which is projective.

Classical results:

1. $\underset{\sim}{\Sigma}^1_1$ sets are Lebesgue measurable and have Baire property.
2. $\underset{\sim}{\Sigma}^1_2$ " of the plane can be uniformized by $\underset{\sim}{\Sigma}^1_2$ functions

This are the best possible in ZFC.

Search for new axioms: Determinacy.

Def: Suppose $A \subseteq \omega^\omega$

Consider the following game:

I	II
n_0	
	m_0
n_1	
	m_1
n_2	
	$\vdots$

Define $x \in \omega^\omega$ by
$$x = (n_0, m_0, n_1, m_1, \ldots)$$

I wins if $x \in A$. OW, II wins.

Def: A winning strategy in this game (G_A) is a function $\tau : \omega^{<\omega} \to \omega$ s.t. for one of the players, if the players plays by τ, then he always wins.

Def: A is determined if $\exists$ winning strategy τ in the game G_A for one of the players.

Def: PD = projective determinacy: Every projective set is determined

$\underset{\sim}{\Sigma}^1_k$-det = every $\underset{\sim}{\Sigma}^1_k$ set is determined.

AD = axiom of determinacy: Every set is determined.

$(ZFC \vdash \neg AD)$

Classical results (ZFC): ① Borel sets are determined (Martin)
[② Open sets are determined (Gale & Stewart)]

(Replacement Ax. is needed in ①)
(Friedman)

Theorem: Assume PD. Then,
① Every projective set is Lebesgue measurable and has the property of Baire.
② Every projective subset of the plane can be uniformized by a projective function.

$$\begin{array}{cc} \vdots & \vdots \\ \boldsymbol{\Sigma}^1_3 & \boldsymbol{\Pi}^1_3 \\ \boldsymbol{\Sigma}^1_2 & \boldsymbol{\Pi}^1_2 \\ \boldsymbol{\Sigma}^1_1 & \boldsymbol{\Pi}^1_1 \end{array}$$

Γ means that every Γ set can be uniformized by a Γ function.

Main open question: Verify or refute the following conjecture:
Conjecture: PD holds iff:
(Woodin) ① Every projective set is Lebesgue meas. & has Baire prop.
② Uniformization.

Kechris: ① & ② $\Rightarrow$ $\boldsymbol{\Pi}^1_1$-det.
① & ② $\Rightarrow$ all "nice" properties: Ramsey, etc..

8/29/90

$AD_{\mathbb{R}}$: Suppose $A \in \mathbb{R}^\omega$. Then, the real game associated to A is determined.

Theorem : $\mathrm{CON}(ZFC + \exists\, \omega$ Woodin cards$) \Rightarrow \mathrm{CON}(ZF + AD)$

Thm : $\mathrm{CON}(ZFC + \exists$ a Woodin card$) \Rightarrow \mathrm{CON}(ZFC + \Sigma^1_2\text{-det})$

? **Remark** : If CH holds in both $V_0 \subseteq V_1$, then $\underset{\sim}{\Sigma}^2_1$-statements are absolute. (??)

$\Sigma^1_2\text{-det} \Rightarrow \mathrm{Th}(L[x])$, $x \in \mathbb{R}$, is freezed in a cone. (Metatheorem)
$\Sigma^1_2\text{-det} + \forall x\ x^\#$exists $\Rightarrow \mathrm{Th}(L[x])$ freezes on a cone (Theorem)

Woodin : If $L[x] \equiv L[x][y]$ ~~and~~ $\forall y$ is ccc over $L[x]$, then $L[x] \models \Sigma^1_2\text{-det}$. Hence, Σ^1_2-det holds in V.

Suppose κ is Woodin. Then, $V^{\mathrm{Coll}(\omega,\kappa)} \models \Sigma^1_2\text{-det}$.

Notation : ① Suppose X is a set. Then, if $k \in \omega$, then $X^k = \{f : k \to X\}$.

~~② A tree on $\omega \times \lambda$, where λ is an ordinal~~

② $(\omega \times \lambda)^{<\omega} = \{(s,t) : s \in \omega^{<\omega},\ t \in \lambda^{<\omega}$ and $\mathrm{dom}\, s = \mathrm{dom}\, t\}$ where $X^{<\omega} = \{f : \mathrm{dom} f \to X : \mathrm{dom} f \in \omega\}$

③ $T \subseteq (\omega\times\lambda)^{<\omega}$ is a tree on $\omega\times\lambda$ if T is closed under "initial segments", in the sense:
$(s,t)\in T \Rightarrow (s\restriction k, t\restriction k)\in T$ for all $k\in \operatorname{dom} s = \operatorname{dom} t$.

④ $T\subseteq X^{<\omega}$ is a tree on X if T is closed under initial segments: $s\in T \Rightarrow s\restriction k\in T$ for all k.

⑤ Suppose $T\subseteq(\omega\times\lambda)^{<\omega}$ is a tree on $\omega\times\lambda$. Suppose $s\in\omega^{<\omega}$. Then, $T_s = \{t\in\lambda^{<\omega} : (s,t)\in T\}$

⑥ If $T\subseteq X^{<\omega}$ is a tree, then $[T]$ denotes the set of all infinite branches of T,

$$[T] = \{f\in X^{\omega} : f\restriction k\in T \text{ for all } k\}.$$

Similarly,

⑦ If $T\subseteq(\omega\times\lambda)^{<\omega}$ is a tree on $\omega\times\lambda$, then $[T]$ is the set $\{(f,g) : f\in\omega^{\omega},\ g\in\lambda^{\omega} \text{ and } (f\restriction k, g\restriction k)\in T \text{ for all } k\}$

⑧ Suppose T is a tree on $\omega\times\lambda$. Then, the projection of T, $p[T] = \{x\in\mathbb{R}=\omega^{\omega} : (x,g)\in[T] \text{ for some } g\in\lambda^{\omega}\}$

⑨ A tree $T\subseteq X^{<\omega}$ is <u>well-founded</u> if $[T]=\emptyset$.
Similarly, a tree T on $\omega\times\lambda$ is <u>well-founded</u> if $[T]=\emptyset$.

<u>Def</u>: Suppose $A\subseteq\mathbb{R}=\omega^{\omega}$. Then, A is <u>λ-Suslin</u> (where λ is an ordinal) if there exists a tree T on $\omega\times\lambda$ such that $A = p[T]$.

Suppose T is a tree on $\omega \times \lambda$ and $x \in \mathbb{R}$. Then,
$T_x = \{ t \in \lambda^{<\omega} : (x \upharpoonright k, t) \in T \}$
where $k = \mathrm{dom}\, t$

So, T_x is a tree on λ.
<u>Notice</u>: $x \in p[T]$ iff the tree $T_x \subseteq \lambda^{<\omega}$ is <u>not</u> well-founded

<u>Notice</u>: (AC) Every set $A \subseteq \mathbb{R}$ is c-Suslin.

<u>Facts</u>: ① $A \subseteq \mathbb{R}$ is ω-Suslin iff A is $\underset{\sim}{\Sigma}^1_1$.
(ZF) ② If $A \subseteq \mathbb{R}$ is $\underset{\sim}{\Sigma}^1_2$, then A is ω_1-Suslin.

<u>Theorem</u> (Jackson): Assume PD. Then, every $\underset{\sim}{\Sigma}^1_n$ set is $\aleph_\omega$-Suslin.
In fact, $\exists f: \omega \to \omega$ s.t. every $\underset{\sim}{\Sigma}^1_n$ set is $\aleph_{f(n)}$-Suslin. (f is primitive recursive).

<u>Note</u>: $\underset{\sim}{\Pi}^1_2$ sets are not ω_1-Suslin, though they are ω_2-Suslin

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8/31/90

<u>Def</u>: $A \subseteq \mathbb{R}$ is <u>λ-Suslin</u> if $A = p[T]$ for some tree T on $\omega \times \lambda$.

Assuming AC, every set is $2^{\aleph_0}$-Suslin.

<u>Def</u>: Suppose X is a set ~~#~~. Then, $m(X^{<\omega})$ is the set of all countably complete ultrafilters on $X^{<\omega}$.

<u>Recall</u>: An ultrafilter $\mathcal{U}$ on I (so $\mathcal{U} \subseteq \mathcal{P}(I)$) is δ-complete if $\bigcap S \in \mathcal{U}$ whenever $S \subseteq \mathcal{U}$, $|S| < \delta$.

The convention is that we say $\mathcal{U}$ is countably complete if $\mathcal{U}$ is ω_1-complete.

<u>Remark</u>: $X^{<\omega} = \bigcup \{X^k : k \in \omega\}$ and $X^{k_1} \cap X^{k_2} = \emptyset$ iff $k_1 \neq k_2$, (or $X = \emptyset$).

Thus, if $\mathcal{U} \in m(X^{<\omega})$, there must exist uniquely k s.t. $X^k \in \mathcal{U}$. (use ω_1-completeness)
(we say $\mathcal{U}$ concentrates on k)

<u>Recall</u>: If $\mathcal{U}$ is a countably complete ultrafilter on an index set $I \neq \emptyset$, there corresponds to $\mathcal{U}$ an elementary embedding $j : V \to M$, where M is a transitive class $M \cong V^I/\mathcal{U}$.
(For well-foundedness, use ω_1-completeness).

<u>Def</u>. Suppose $\langle \mathcal{U}_k : k \in \omega \rangle$ is a sequence from $m(X^{<\omega})$.
The sequence $\langle \mathcal{U}_k : k \in \omega \rangle$ is a <u>tower</u> if

① $\mathcal{U}_k(X^k) = 1$ for all k (cosmetic requirement)
(i.e. $\mathcal{U}_k$ concentrate on k)

② For all $k_1 \leq k_2$ and for all $A \subseteq X^{k_1}$, $A \in \mathcal{U}_{k_1}$ iff $\{ s \in X^{k_2} : s \restriction k_1 \in A \} \in \mathcal{U}_{k_2}$

The tower $\langle \mathcal{U}_k : k<\omega \rangle$ is <u>well-founded</u> if for any sequence $\langle A_k : k<\omega \rangle$ such that $A_k \in \mathcal{U}_k$ for all k, there exists $f \in X^\omega$ s.t. $f \restriction k \in A_k$, for all k.

~~Remark:~~
~~A tower is well-founded~~

<u>Remark</u>: Suppose $\langle \mathcal{U}_k : k<\omega \rangle$ is a tower of measures on $X^{<\omega}$. For each k, let M_k be the transitive collapse of $V^{X^k}/\mathcal{U}_k$.

Let $j_{0k} : V \longrightarrow M_k$ be the associated embedding.
$V = M_0$

For each $k_1 < k_2$, $\mathcal{U}_{k_2}$ projects to $\mathcal{U}_{k_1}$ in the sense: if $A \subseteq X^{k_1}$, $A \in \mathcal{U}_{k_1}$ iff $A^* = \mathcal{U}_{k_2}$, where $A^* = \{ s \in X^{k_2} : s \restriction k_1 \in A \}$.

So, there is a natural elementary embedding $j_{k_1,k_2} : M_{k_1} \longrightarrow M_{k_2}$

~~Thm~~ <u>Fact</u>: The tower is countably complete iff $\lim M_k$ is well-founded.

<u>Another view</u>: Let E_k, $\sim_k$ be the induced relations of "$\in$", "$=$" on V^{X^k} by $\mathcal{U}_k$. Let $k_1 < k_2$

Have a map $V^{k_1} \to V^{k_2}$, take direct limit, etc..

<u>Proof of Fact</u>:

For each k, let s_k be the element of M_k given by the identity function on X^k. ($I_k : X^k \to X^k$ s.t. $I_k(a) = a$)

observe: $s_k \in (j_{0k}(X))^k = j_{0k}(X^k)$

(note that $j(X^Y) = j(X)^{j(Y)}$)

because the measures cohere: if $k_1 < k_2$, $s_{k_2} \restriction k_1 = j_{k_1 k_2}(s_{k_1})$.

$$\begin{array}{ccccccc} V & & & & & & \\ \parallel & & & & & & \\ M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots \\ & & s_1 & & s_2 & & \\ & & \cap & & \cap & & \\ & & j_{01}(X) & & \underbrace{j_{02}(X) \times j_{02}(X)} & & \end{array}$$

observe: $A \in \mathcal{U}_k$ iff $s_k \in j_{0k}(A)$.

This follows from:

General fact: Suppose $\mathcal{U}$ is an ult. on I.

look at $(V^I/\mathcal{U}, E)$.

let s be the representative of the identity function on I.

Then, $A \in \mathcal{U}$ iff $s \in j(A)$.

So, the s_k's fit together to define s^∞ (an "ω-sequence") of elements of $M^\infty = \lim M_k$.

Caution!: s^∞ need not be in M^∞. (in fact, almost never does)

(even if M^∞ is well-founded)

(1) Suppose M^∞ is well-founded.

Let $\langle A_k : k < \omega \rangle$ be a sequence with $A_k \in \mathcal{U}_k$, for all k.

We seek $f \in X^\omega$ s.t. $f \restriction k \in A_k$, for all k.

We have $j_{0\infty} : V \longrightarrow M^\infty$

Consider $j_{0\infty}(\langle A_k : k<\omega\rangle) = \langle j_{0\infty}(A_k) : k\in\omega\rangle$.

But $j_{k\infty}(s_k) \in j_{0\infty}(A_k)$ because $A_k \in \mathcal{U}_k$ and hence $s_k \in j_{0k}(A_k)$

Define (in V) $f^\infty = \bigcup \{ j_{k\infty}(s_k) : k\in\omega\}$

$f^\infty \in j_{0\infty}(X)^\omega$, but in general, $f^\infty \notin M^\infty$.

Note $f^\infty \restriction k \in j_{0\infty}(A_k)$.

So, except for the fact $f^\infty \notin M^\infty$, f^∞ works in M^∞.

Let's assume A_{k_2} projects to A_{k_1}, $k_1 < k_2$.

i.e., $\{ s\restriction k_1 : s\in A_{k_2}\}$.

$A^*_k = A_k \cap$ proj. A_j to X^k, all $j<k$.

Define a tree $T \subseteq X^{<\omega}$ by

$$T = \{ s\in X^{<\omega} : s\in A_k \text{, where } k = \operatorname{dom} s\}$$

We want to show $[T] \neq \emptyset$.

$T^\infty = j_{0\infty}(T)$.

So, $f^\infty \in [T^\infty]$.

But, since M^∞ is well-founded, $M^\infty \models T^\infty$ is not well-founded

i.e. $M^\infty \models [T^\infty] \neq \emptyset$.

By elementarity of the embedding, $M \models T$ has a branch. Done.

9/5/90

Claim: Suppose $\langle \mathcal{U}_k : k<\omega\rangle$ is a tower of measures on $X^{<\omega}$.

Then, the tower is countably complete (or well-founded) iff the direct limit of $\langle M_k : k<\omega\rangle$ is well-founded.

Remark: Recall: each $\mathcal{U}_k$ concentrates on X^k and naturally defines a filter $\mathcal{U}_k^*$ in X^ω (for each $A \in \mathcal{U}_k$, $A^* = \{f \in X^\omega : f \restriction k \in A\} \in \mathcal{U}_k^*$)

The tower condition is in essence the requirement:

$\mathcal{U}_{k_1}^* \supseteq \mathcal{U}_{k_2}^*$ iff $k_1 \le k_2$.

So, $\langle \mathcal{U}_k^* : k < \omega \rangle$ is a decreasing sequence of "filters" (actually, filterbases) on X^ω.

The tower is countably complete if $\bigcap \mathcal{U}_k^*$ is a filter.

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last time we "did": If lim is well-founded, then the tower is countably complete.

Let $M_k^* = (V^{X^k}, \sim_k, E_k)$.

If $k_1 \le k_2$, then $M_{k_1}^*$ "$\subseteq$" $M_{k_2}^*$.

(Suppose $f \in V^{X^{k_1}}$, then define $\bar{f} \in V^{X^{k_2}}$ by $\bar{f}(s) = f(s \restriction k_1)$).

Have map $V^{X^{k_1}} \to V^{X^{k_2}}$, etc ...

For the converse:

Assume the tower is countably complete.

We must show that $\lim M_k$ is well founded.

Suppose OW. Then, we can choose a sequence $\alpha_k \in OR$ such that $j_{k,k+1}(\alpha_k) > \alpha_{k+1}$.

To see this, notice that since $\lim M_k$ is not well-founded, then there exists an infinite sequence $k_1 < k_2 < \dots < k_i < \dots$ and ordinals β_i s.t. $j_{k_i,k_{i+1}}(\beta) > \beta_{i+1}$.

now, playing with the ordinals β_i, get the α_i's.
Choose $F_k : X^k \to OR$ to represent α_k.
So, the fact that $j_{k,k+1}(\alpha_k) > \alpha_{k+1}$ corresponds to
$\{t \in X^{k+1} : F_{k+1}(t) < F_k(t \upharpoonright k)\} \in \mathcal{U}_{k+1}$.
For each k, let $A_k = \{t \in X^k : F_0(\emptyset) > F_1(t\upharpoonright 1) > \ldots > F_k(t)\}$
Then, $A_k \in \mathcal{U}_k$, for all k.
Since $\langle \mathcal{U}_k : k \in \omega \rangle$ is countably complete, $\exists f \in X^\omega$ s.t. $f \upharpoonright k \in A_k$ for all k, Hence, $F_k(f \upharpoonright k)$ is a descending sequence of ordinals. $(\to\leftarrow)$.

Def: (A version of λ-Suslin, not trivial in ZFC)
Suppose T is a tree on $\omega \times \lambda$. Then, T is <u>weakly homogeneous</u> if there exists a countable set $\sigma \subseteq m(\lambda^{<\omega})$ s.t. for all $x \in R$, $x \in p[T]$ iff there exists a countably complete tower $\langle \mathcal{U}_k : k < \omega \rangle$ of measures in σ with the property that $\mathcal{U}_k(T_{x\upharpoonright k}) = 1$ for all k.

<u>Notice</u>: Suppose $\langle \mathcal{U}_k : k < \omega \rangle$ is countably complete tower, T is a tree on $\omega \times \lambda$, $x \in \mathbb{R}$ and $\mathcal{U}_k(T_{x\upharpoonright k}) = 1$ for all k. Then, $x \in p[T]$.

<u>Example</u>: Any tree on $\omega \times \omega$ is weakly-homogeneous.
(let σ = measures$(\omega^{<\omega})$ = principal ult. on $\omega^{<\omega}$)

Remark: weakly-homo on $\omega\times\lambda$ $\to$ $\boxed{}$
tree on $\omega\times\omega$ $\to$ Σ^1_1 sets
Σ^1_1 sets have nice properties
$\boxed{}$ " " "

Def: $A \subseteq \mathbb{R}$ is weakly-homogeneously Suslin if $A=p[T]$ where T is a weakly-homogeneous tree.

Def: A tree T on $\omega\times\lambda$ is δ-weakly homogenous if σ can be chosen to contain only δ-complete ultrafilters.

Def: ① $A\subseteq\mathbb{R}$ is δ-weakly homogeneously Suslin if there is a δ-weakly-homo T s.t. $A=p[T]$.

② $A\subseteq\mathbb{R}$ is ∞-weakly homoge. Suslin if A is δ-weakly homo. Suslin, for all δ.

(Note: if e.g. λ is singular, then T is δ-weak-homo for all $\delta<\lambda$, but not λ-weak. homo.
if λ is measurable, then T is δ-w.-h. for all $\delta<\lambda$ iff T is λ-weak.-homo. (Probably true).)

③ A is $<\delta$ weakly-homo. Suslin if A is α-weakly homo for all $\alpha<\delta$.

Properties of weakly-homo. Suslin sets:

① They are Lebesgue measurable and have Baire property.
(" have all usual regularity properties).

② They are countable or contain a perfect set.

(All these follow from)

Theorem: Suppose T is a tree on $\omega\times\lambda$ which is δ-weakly homo. Then, there exists a tree S on $\omega\times\kappa$ for some κ $(=2^{2^\lambda})$ such that:

9/7/90

H. WOODIN

last time:

<u>Theorem</u>: Suppose κ is a Woodin cardinal and T is a tree on $\omega \times \lambda$. Then, $\exists \delta < \kappa$ such that if G is generic for $Coll(\omega, \delta)$, then $V[G] \models T$ is $<\kappa$ weakly-homogeneous.

<u>Corollary</u>: ~~Suppose that κ is inaccessible and $\{\delta < \kappa : \delta$ is Woodin$\}$ is stationary. (i.e., κ is Mahlo to Woodin cardinals.) (This implies that κ is itself Woodin). Suppose $G \subseteq Coll(\omega, <\kappa)$ is generic. Let $\mathbb{R} = \mathbb{R}^{V[G]}$.~~

<u>Corollary</u>: Suppose that κ is a measurable Woodin cardinal. Suppose $G \subseteq Coll(\omega, <\kappa)$ is generic. Let $\mathbb{R} = \mathbb{R}^{V[G]}$. Then, $V(\mathbb{R}) \models$ Every tree is weakly homogeneous.

<u>Def</u>: A tree T on $\omega \times \lambda$ is <u>homogeneous</u> if there exists $\pi : seq \to meas(\lambda^{<\omega})$ such that: $(seq = \omega^{<\omega})$

1. $\pi(s)(T_s) = 1$, all s s.t. $T_s \neq \phi$.
2. if $s \subseteq t$, then $\pi(t)$ projects to $\pi(s)$.
3. for all $x \in \mathbb{R}$, $x \in p[T]$ iff the tower $\langle \pi(x \restriction k) : k < \omega \rangle$ is countably complete.

Similarly, define δ-homogeneous, $<\delta$-homogeneous and define $A \subseteq \mathbb{R}$ is homogeneously Suslin, δ-homogeneously Suslin, etc...

<u>Note</u>: If $C \subseteq \omega^\omega$ is a closed set, then C is ∞-homogeneously Suslin.
Why?: Let $T = \{(s,s) : C \cap [s] \neq \phi\}$
$\pi(s)$ = principal measure generated by s.

<u>Lemma</u>: Suppose $A \subseteq \mathbb{R}$ is δ-homogeneously Suslin and $f: \mathbb{R} \to \mathbb{R}$ is continuous.
Then, $f''A$ is δ-weakly homogeneously Suslin.

(So, the weakly-homogeneously Suslin sets are the continuous images of the homogeneously Suslin sets.)
To see the equivalence,
<u>Alternate definition of weak homogeneity</u>: A tree T on $\omega \times \lambda$ is weakly homogeneous iff there exists $\pi : \text{seq} \times \text{seq} \to \text{meas}(\lambda^{<\omega})$
(where $\text{seq} \times \text{seq} = \{ (s,t) : s,t \in \omega^{<\omega} \wedge |s| = |t| \}$)
such that:

① $\pi(s,t)(T_s) = 1$ if $T_s \neq \emptyset$
② if $s^* \subseteq s$, $t^* \subseteq t$, then $\pi(s,t)$ projects to $\pi(s^*, t^*)$
③ $x \in p[T]$ iff $\exists y \in \mathbb{R}$ s.t. $\langle \pi(x \restriction k, y \restriction k) : k < \omega \rangle$ is ω_1-complete

Clearly, this def. $\Rightarrow$ old def. : let $\sigma = \text{range}(\pi)$.
To see old def $\Rightarrow$ this def : by induction
$\pi(0,0) = \emptyset$
etc....

<u>Notice</u>: Suppose T is weakly homo. and $\pi : \text{Seq} \times \text{Seq} \to \text{meas}(\lambda^{<\omega})$ is a witness.
Let $A = \{ (x,y) \in \mathbb{R} \times \mathbb{R} :$ the tower $\langle \pi(x \restriction k, y \restriction k) : k < \omega \rangle$ is countably complete $\}$
<u>Claim</u>: A is homogeneously Suslin.

<u>Lemma</u>: Suppose $\pi : \text{seq} \to \text{meas}(\lambda^{<\omega})$ s.t. $\pi(s)(\lambda^{\ell(s)}) = 1$ and $s \subseteq t \Rightarrow \pi(t)$ projects to $\pi(s)$.
Define $A = \{x :$ the tower $\langle \pi(x \restriction k) : k < \omega \rangle$ is countably complete.
Then, A is homogeneously Suslin.

<u>Pf</u>: We must find T, homogeneous, with $A = p[T]$.
Choose T so that π witnesses homogeneity.
For each real x such that $\langle \pi(x \restriction k) : k < \omega \rangle$ is not countably complete, choose $\langle A^*_k : k < \omega \rangle$ s.t. $\pi(x \restriction k)(A^*_k) = 1$ $(A^*_k \in \lambda^k)$ and $\langle A^*_k : k < \omega \rangle$ witnesses the tower is not ω_1-complete. Set $A^x_k = \lambda^k$ if the tower is ω_1-complete.
For each $s \in \text{seq}$, let $T_s = \bigcap \{A^x_k : s \subseteq x \text{ and } k = \ell(s)\}$
Each such A^x_k is of measure 1 for $\pi(s)$, $s = x \restriction k$.
$\pi(s)$ is countably complete ult. on $\mathcal{P}(\lambda^{<\omega})$ so $\pi(s)$ is certainly $\mathfrak{c}^+$-complete. Hence, $\pi(s)(T_s) = 1$.
Assume, WLOG, $A^x_{k_1}$ projects to $A^x_{k_2}$ for $k_2 < k_1$.

$T = \{(s,t) : t \in T_s\}$, T is a tree on $\omega \times \lambda$. $\square$

<u>Lemma</u>: Suppose A is homogeneously Suslin. Then, A is determined.

<u>Lemma</u>: Suppose A is homogeneously Suslin (weakly homo. suslin). and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(A)$ is homogeneously Suslin (weakly homo. suslin).

<u>Cultural remark</u>:
Example of additional properties of weakly homogen. Suslin sets:

<u>Lemma</u>: Suppose $S \subseteq \omega_1$ and $A = \{x : x \text{ codes } S \cap \alpha \text{ for some } \alpha < \omega_1\}$
(Kechris)

Suppose A is weakly homogeneously Suslin.
Then, $S \in L[t]$ for some $t \in \mathbb{R}$.

(Kechris): Also assume $\exists$ measurable card. Then, the lemma is iff.

H. Woodin

<u>Thm</u>: Suppose $A \subseteq \mathbb{R}$ is homogeneously Suslin. Then, A is determined.

<u>Lemma</u>: Suppose $B \subseteq X^\omega$ is open (X discrete top, product top. on X^ω). Then, the game on X corresponding to B is determined.

<u>Pf</u>:

I	II	
a_0		$a_i \in X$
	a_1	I wins if $\langle a_0, a_1, \ldots \rangle \in X$.
a_2		OW, II wins.
	a_3	

Either I has a winning strategy, in which case we are done, or II's winning strategy is simply to play to preserve that I has no winning strategy ~~for I~~ from that position.

<u>Notice</u>: this argument needs choice. $\square$

<u>Pf. of Theorem</u>: We associate to A a set A^* as follows: Choose a homogeneous tree T on $\omega \times \lambda$ with $A = p[T]$. Let $\pi : \mathrm{Seq} \to \mathrm{meas}(\lambda^{<\omega})$ witness the homogeneity of T. The auxiliary game is the following:

$G^*_{A^*}$

I	II
$\alpha_0\ n_0$	
	m_0
$\alpha_1\ n_1$	
	m_1
$\vdots$	

<u>Rules</u>: $n_i \in \omega$, $m_i \in \omega$ and I wins if $\langle (n_0, m_0, \ldots), (\alpha_0, \alpha_1, \ldots) \rangle \in [T]$; otherwise, II wins.

The game G_A^* is open for II.
(II wins an infinite run iff II wins at some finite stage)

The game G_A^* is determined (since it is open).
If I has a winning strategy, then clearly I has a winning strategy in the original game G_A, and we are done.

So, assume that II has a winning strategy.
Let τ be a winning strategy for II.
We can think of τ as a function from ~~$(\omega \times \lambda)^{<\omega}$~~
~~with~~ the set $\{(s,t,u) : s \in \lambda^{<\omega}, t \in \omega^{<\omega}, u \in \omega^{<\omega}$ and
$\ell(s) = \ell(t) = \ell(u)+1 \}$ into ω.
(history of II)

We can now find a strategy $\bar{\tau}$ for II in the original game by induction, as follows:
$\bar{\tau}(\langle k \rangle) = \tau(\langle \alpha \rangle, \langle k \rangle, \phi)$ for $\pi(\langle k \rangle)$ almost all α.
etc...

We must check that $\bar{\tau}$ is winning for II in G_A.
Suppose not.
So, let $\langle n_0, m_0, n_1, m_1, \ldots \rangle$ be a play that defeats $\bar{\tau}$.
i.e., $\langle n_0, m_0, n_1, \ldots \rangle \in A$.

<u>Notice</u>: for each k, $C_k = \{ t \in \lambda^k : \tau$ produces $\langle m_i : i < k-1 \rangle$
from $\langle t(i), n_i ; i < k \rangle \}$
is of measure 1 for $\pi(x \restriction k)$, $x = \langle n_0, m_0, n_1, \ldots \rangle$

Thus, $\langle C_k : k < \omega \rangle$ is a seq. of sets of measure 1.
Since the tower is countably complete (for $x \in A$)
$(\langle \pi(x \restriction k) : k < \omega \rangle)$
So, $\exists f \in \lambda^\omega$ with $f \restriction k \in C_k$, for all k.
Then,

$$\begin{array}{ll} f(0)\ n_0 & \\ & m_0 \\ f(1)\ n_1 & \\ & m_1 \\ \vdots & \end{array}$$

defeats τ. ($\Rightarrow\Leftarrow$)

Weak version of Martin-Steel theorem:

Thm: Suppose κ is a Woodin cardinal and $A \subseteq \mathbb{R}$ is κ^+-weakly homogeneously Suslin. Then, $\mathbb{R} \setminus A$ is $<\kappa$-homogeneously Suslin.

(Martin): If κ is measurable, then the standard

9/12/90

H. WOODIN

Thm: Suppose T is a tree on $\omega \times \lambda$ which is δ-weakly homogeneous. Then, there is a tree S on $\omega \times \kappa$, some κ $(=2^{2^\lambda})$, such that if G is V-generic for a partial order $\mathbb{P}$ such that $|\mathbb{P}|^V < \delta$, then

$$V[G] \models p[T] = \mathbb{R} \setminus p[S]$$

Pf: let σ witness the δ-weak homogeneity of T.

So, $\sigma \subseteq \mathrm{meas}(\lambda^{<\omega})$ and each measure in σ is δ-complete.

WMA that δ is measurable. Why?: Clear.

Note: $x \in p[T]$ iff $\exists \langle \mathcal{U}_k : k < \omega \rangle \subseteq \sigma$, a ~~tower which is~~ countably complete.

Claim: Suppose $x \notin p[T]$.

Then, let $\sigma_x = \{\mathcal{U} \in \sigma : T_x \restriction k \in \mathcal{U}$, for some $k\}$

let $<_x$ on σ_x be the projection ($\nu \underset{\nu \neq \mu}{\leq} \mu$ if μ projects to ν,

Thus, $(\sigma_x, <_x)$ is a tree of height at most ω.

We can assume σ is closed under projection.

(So, branches in this tree correspond to towers)

Then, there exists a function $\rho : \sigma_x \to \mathrm{OR}$ s.t. if $\mu, \nu \in \sigma_x$ and $\nu < \mu$, then

$j_{\nu\mu}(\rho(\nu)) > \rho(\mu)$

where $j_{\nu\mu} : M_\nu \to M_\mu$ is the induced embedding.

and $M_\nu = V^{\lambda^{<\omega}}/\nu$ $\quad M_\mu = V^{\lambda^{<\omega}}/\mu$

Pf. of Claim: $x \notin p[T]$. So, T_x is well-founded.

Suppose $\mu \in \sigma_x$ and let $j : V \to M_\mu$ be the associated embedding $M_\mu \cong V^{\lambda^{<\omega}}/\mu$.

Let s_μ be the elt. of M_μ determined by the identity function on $\lambda^{<\omega}$.

So, s_μ is a finite sequence from $j_\mu(\lambda)$ and length of $s_\mu = k$, where $\mu(\lambda^k) = 1$.

Also, if $\nu < \mu$ in σ_x, then $j_{\nu\mu}(s_\nu)$ is an initial segment of s_μ.

Finally, $s_\mu \in j_\mu(T_x)$, for all $\mu \in \sigma_x$.

Let $F : T_x \to OR$ be the rank function.

Define $\rho(\mu) = j_\mu(F)(s_\mu)$.

Suppose $\nu < \mu$. Then we must show $j_{\nu\mu}(\rho(\nu)) > \rho(\mu)$.

$$\begin{tikzcd} V \arrow[r, "j_\mu"] \arrow[dr, "j_\nu"'] & M_\mu \\ & M_\nu \arrow[u, "j_{\nu\mu}"'] \end{tikzcd}$$

$\rho(\mu) = j_\mu(F)(s_\mu)$

$j_{\nu\mu}(\rho(\nu)) = j_{\nu\mu}(j_\nu(F)(s_\nu)) = j_\mu(F)(j_{\nu\mu}(s_\nu))$

Hence, $j_{\nu\mu}(\rho(\nu)) > \rho(\mu)$. (The rank function decreases)

Note: We only used DC.

This proves the Claim. $\square$

Corollary to Claim: $x \notin p[T]$ iff there exists $\rho : \sigma_x \to OR$ as in the Claim.

Pf: (The Claim proves $\Rightarrow$),
$\Leftarrow$ follows since σ witnesses weak-homogeneity. $\square$

Remarks: ① If T_x is well-founded, then $\operatorname{rank}(T_x) < \lambda^+$.

② $|j_\mu(\lambda^+)| \leq |(\lambda^+)^\lambda|$

So, $j_\mu(\lambda^+) < ((\lambda^+)^\lambda)^+$

So, let $\kappa = (2^\lambda)^+$

To prove the theorem:

Fix an enumeration $\mu_0, \mu_1, \ldots, \mu_k, \ldots$ of σ such that no measure occurs before its proper projections.

Define S by specifying $[S]$ (S will be a tree on $\omega \times \kappa$).

$[S] = \{(x, f) : \rho \text{ "="} f \restriction \sigma_x$ satisfies the condition above $j_{\nu\mu}(\rho(\nu)) > \rho(\mu)$ for $\nu, \mu \in \sigma_x\}$

Clearly, in V, $p[S] = \omega^\omega \setminus p[T]$.

Suppose G is V-generic for $\mathbb{P}$ and $|\mathbb{P}|^V < \delta$.

We must show that $V[G] \models p[T] = \omega^\omega \setminus p[S]$

Key points:

① each ~~[illegible]~~ $\mu \in \sigma$ is δ-complete and hence it determines uniquely a δ-complete ultrafilter on $\lambda^{<\omega}$ in $V[G]$.

Why?: because any new subset is in ....

② let $\bar{\sigma} = \{\bar{\mu} : \mu \in \sigma\}$. $\bar{\sigma} \in \operatorname{meas}(\lambda^{<\omega})$ in $V[G]$.

$\bar{\sigma}$ witnesses the δ-weak-homogeneity of T <u>in</u> $V[G]$.

③ In $V[G]$, let $j_{\bar{\mu}} : V[G] \to N_{\bar{\mu}}$ be the associated embedding. $N_{\bar{\mu}}$ = trans. coll of $V[G]^{\lambda^{<\omega}} / \bar{\mu}$.

Then, $j_{\bar{\mu}} \restriction V = j_{\mu}$

$j_{\bar{\mu}}(V) = M_{\mu}$

$$\begin{array}{ccc} V & \xrightarrow{j_\mu} & M_\mu \\ \Downarrow & & \\ V[G] & \xrightarrow[j_{\bar{\mu}}]{} & N_{\bar{\mu}} \end{array}$$

For ③ : Suppose $g : \lambda^{<\omega} \to V$ is in $V[G]$.

Then, there exists $f : \lambda^{<\omega} \to V$, $\underline{f \in V}$ s.t. $f = g \pmod{\bar{\mu}}$;

So, ① & ③ gives:

The enumeration $\mu_0, \dots, \mu_k, \dots$ gives $\bar{\mu}_0, \dots, \bar{\mu}_k, \dots$ in $V[G]$.
Using this enumeration to construct $\bar{S}$ in $V[G]$ as we constructed S in V (using same k), we get the same tree.

[We are also using:

④ $j_{\bar{\nu}\bar{\mu}} : N_{\bar{\nu}} \to M_{\bar{\mu}}$ restricts to $j_{\nu\mu} : M_\nu \to M_\mu$

($M_\nu \subseteq N_{\bar{\nu}}$, $j_{\bar{\nu}\bar{\mu}} \restriction M_\nu = j_{\nu\mu}$)

(really, what we are using is: $j_{\bar{\nu}\bar{\mu}} \restriction OR = j_{\nu\mu}$)

(24)

<u>Thm</u>: Suppose T is wkly homogeneous [T is a tree on $(\omega\times\lambda)$]. Then there is a tree S on $(\omega\times\kappa)$ (some κ) s.t.

$$p[S] = \omega^\omega \setminus p[T]$$

in all "small" forcing extensions. So if add a real in this forcing, say then $\alpha\in p[S]$ or $\alpha\in p[T]$

<u>Martin-Steel Thm</u>: Suppose κ is a wc, T is a tree on $(\omega\times\lambda)$ and $\sigma\subset meas(\lambda^{<\omega})$ witnesses that T is κ^+-wkly hom. Let S be the tree for $\omega^\omega\setminus p[T]$ constructed from σ. Then S is $<\kappa$ hom.

<u>Remark</u>: $p[T]^{V[G]} \neq p[T]^V$ iff $\exists$ a tree $\overline{T}\subset\omega^{<\omega}$ and a fn $p:\overline{T}\to\lambda^{<\omega}$ $(T\subset(\omega\times\lambda)^{<\omega})$ such that if $s\subset t \to p(s)\subset p(t)$ $\ell(p(s))=\ell(s)$ where $\exists G?$

1) $(s,p(s))\subset T\ \forall s\in\overline{T}$

2) $\overline{T}\cong 2^{<\omega}$

for, if $x\in[\overline{T}]$, then $(x,f)\in[T]$, so $f\restriction k=p(x\restriction k)$ all k

So if create a new real, that real traces out a new path (thru $\overline{T}$ and thus T?)

And conversely, if create a new branch, i.e., a new real in $p[$ then can get ~~new~~ perfect set.

<u>Remark</u>. Martin Steel gives PD ~~from~~ ω-many WC.

<u>Def</u>: Suppose κ is SI Then κ is a wc iff$_{df}$ for all functions $f:\kappa\to\kappa$ $\exists\delta<\kappa$ and $j:V\to M$ s.t.

1) δ closed under f: $f''\delta\subset\delta$

2) the critical pt of j is δ

3) $V_{j(f)(\delta)}\subset M$

wc is highly Mahlo (each such δ is measurable) but ~~it~~ the least wc is not weakly compact, so it is not measurable

Facts: 1) If there is a WC, then $\exists$ an inner model ($\supset$ ord) with a WC and a Δ^1_3-w.o. of $\mathbb{R}$ (Martin Steel)

Last semester showed $L[x] \models \Delta^1_2$ det

$\rightarrow w_2^{L[x]}$ is a WC in $HOD^{L[x]}$

so $HOD^{L[x]} \models \exists\ \Delta^1_3$ w.o. of $\mathbb{R}$

"Things deteriorate rapidly for reals once add WC"

2) Converse to Martin Steel: $\exists^N$ WC $\wedge$ meas. above $\rightarrow \Pi^1_{N+1}$-det.

Thm: Δ^1_2 det holds iff $\forall x\ \exists x$ inner model $\supset$ ord $\cup\ x$ ~~ord(V[x])~~) of ZFC and α is a WC where $\alpha < \omega_1$.

So a WC is the largest ordinal ~~model~~ consistent with a Δ^1_3-wo of $\mathbb{R}$

If get up to a WC which is meas:
almost all forms of det "live"
between 1st WC and a meas. WC.

meas WC $\longrightarrow$ — super compact
WC $\longrightarrow$ — meas

Thm: Assume $\exists$ meas WC and assume CH. Then Σ^2_1 def. sets of reals are determined!

And Shelah has shown: Assume CH. There is a poset $\mathbb{P}$ s.t.
$V^{\mathbb{P}} \models CH + \exists\ \Delta^2_2$ wo of $\mathbb{R}$

Question: Is "κ a — cardinal" consistent with a $\Sigma^{1\,2}_1(\kappa$-wh$)$,
$\Sigma^{1\,2}_1(<\kappa$ wh$)$ or $\Sigma^{1\,2}_1(\infty$-wh$)$ w.o. of $\mathbb{R}$.

$A \in \Sigma^{1\,2}_1(\Gamma)$ if A can be defined by a $\Sigma^{1\,2}_1$ fla where $\exists$ is restricted to Γ; i.e., instead of $\exists X \subset \mathbb{R}$, we have $\exists X \in \Gamma \wedge X \subset \mathbb{R} \wedge \Psi(X, a)$

This is a more restricted notion of wo than just $\Sigma^{1\,2}_1$:

Assume $V = L(\mathcal{P}(\mathbb{R})) + AD_{\mathbb{R}} + DC + ZF$

Consider $HOD \models AC$;

Let $\Theta = \sup\{\alpha : \exists \pi : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}$

<u>Thm</u>: HOD $\models \exists\, \Sigma^2_1$ ($< \theta^V$ wk) wo of $\mathbb{R}$

Where will we be heading:
Non-stationary tower forcing - pathologies of WC:
Assume κ is WC; then $\exists$ a generic $j: V \to M \subset V[G]$ where
$cp(j) = \omega_1$ or any other Regular cardinal

9/24/90

H. WOODIN

<u>Lemma</u>: Suppose κ is inaccessible. Then, TFAE

(1) κ is a Woodin cardinal.

(2) For every subset $A \subseteq V_\kappa$, there exists $\delta < \kappa$ such that δ is $<\kappa$ A-strong.

i.e., $\exists \delta < \kappa$ s.t. $\forall \lambda < \kappa$ $\exists j : V \longrightarrow M$ with $cp(j) = \delta$, $j(\delta) > \lambda$ and $j(A) \cap M_\lambda = A \cap V_\lambda$

<u>Note</u>: 0-strong or 1-strong say simply that κ is measurable. If $A = V_\kappa$, (2) reduces to $V_\kappa \models \exists$ strong card.

<u>Remark</u>: A cardinal δ is <u>strong</u> if for all λ, there is $j : V \longrightarrow M$ s.t. $cp(j) = \delta$, $j(\delta) > \lambda$ and $V_\lambda \subseteq M$.

δ is λ-strong if $\exists j : V \to M$ as above with $V_\lambda \subseteq M$. Similarly for δ is $<\lambda$-strong.

So, if κ is measurable, then κ is $\kappa+1$-strong. The least measurable κ is not $\kappa+2$-strong.

(Why? Otherwise, $\exists j : V \to M$, $cp(j) = \kappa$ and $V_{\kappa+2} \subseteq M$. So, $M \models \kappa$ is measurable.)

Strong cardinals were introduced by Mitchell as intermediate between measurable cardinals and supercompacts. Woodin cardinals are between strong & supercompact.

To prove the lemma we need extenders.

Suppose $j: V \longrightarrow M$, $cp(j) = \delta$, $\lambda \leq j(\delta)$.
For each $s \in [\lambda]^{<\omega}$, define $\mathcal{U}_s(X) = 1$ iff $s \in j(X)$.
So, $\mathcal{U}_s([\kappa]^{|s|}) = 1$.
$E = \langle \mathcal{U}_s : s \in [\lambda]^{<\omega} \rangle$.
Notation: Denote $\mathcal{U}_s$ by E_s.
E is a (κ, λ)-extender.
Properties of E:

① If $s \subseteq t$, then E_t projects to E_s.
(the way s is a subset of t defines the projection map:
$A \subseteq [\lambda]^{|s|}$, $A^* = \{ b \in [\lambda]^{|t|} :$ for some $a \in A$,
$(a, b, \in) \cong (s, t, \in) \}$

$E_s(A) = 1$ iff $E_t(A^*) = 1$

② (normality): For all $s \in [\lambda]^{<\omega}$ and for all $A \in E_s$,
if $F: A \to \lambda$ is a function s.t. $F(a) \leq \max(a)$ for
all $a \in A$, then there exists $\alpha \leq \max(s)$ such that
$B = \{ b \in [\lambda]^{|s|} : F(b) \in b \wedge \langle b, F(b), \in \rangle \cong \langle a \cup \alpha, \alpha, \in \rangle \in E$
where $t = a \cup \{\alpha\} \}$

this gives $s \in j(A)$. So, $j(F)(s) \leq \max(s)$, $\alpha = j(F)(s)$.

③ (well-foundedness): $Ult(V, E)$ is well-founded.

$E: [\lambda]^{<\omega} \longrightarrow meas([\kappa]^{<\omega})$, E a (κ, λ)-extender.
$$Ult(V, E) = \lim_{s \in [\lambda]^{<\omega}} Ult(V, E(s)) = V^{[\kappa]^{|s|}} / E_s$$

① + ② $\not\Rightarrow$ ③ just as a tower need not be countably complete.

Pf of the lemma: ② $\Rightarrow$ ①:

Suppose $f : \kappa \to \kappa$. Need to find $j : V \to M$ $cp(j) < \kappa$ (say $cp(j) = \delta$), $f''\delta \subseteq \delta$ and $V_{j(f)(\delta)} \subseteq M$.

Need: We can assume $\alpha \leq \beta \Rightarrow f(\alpha) \leq f(\beta)$.

Set $A = f \times V_\kappa$.

So, there is, by ①, a $\delta < \kappa$ which is $< \kappa$ A-strong. Let $\lambda = f(\delta) + 1$. So, δ is λ-A-strong. So, $\exists j : V \to M$ with $cp(j) = \delta$ and $j(A) \cap M_\lambda = A \cap V_\lambda$.

Now, $j(A) = j(f) \times j(V_\lambda)$. So, $j(A) \cap V_\lambda = (j(f) \cap (\lambda \times \lambda)) \times M_\lambda$

$$= A \cap V_\lambda$$
$$= (f \cap (\lambda \times \lambda)) \times V_\lambda$$

So, $M_\lambda = V_\lambda$ and since $\lambda = f(\delta) + 1$, we have $j(f)(\delta) = f(\delta)$. $\square$

① $\Rightarrow$ ②: Let $A \subseteq V_\kappa$ be given. Suppose ② fails. Define $f : \kappa \to \kappa$ by: $f(\delta) = \lambda + \omega$ where λ is least such that δ is not λ-A-strong, $\lambda > \delta$.

Thus, by ①, $\exists j : V \to M$ with $V_{j(f)(\delta)} \subseteq M$, where $\delta = cp(j)$.

Hence, $j(f)(\delta) = \lambda + \omega$, where λ is least with $M \models$ "δ is not λ-$j(A)$-strong".

But $V_{\lambda+\omega} \subseteq M$. Hence, the (κ, λ)-extender E induced by j belongs to M.

$E : [\lambda]^{<\omega} \longrightarrow meas([\delta]^{<\omega})$

$E \subseteq \mathcal{P}(\lambda) \times meas([\delta]^{<\omega}) \subseteq V_{\lambda+\omega}$.

Claim: E witnesses in M that κ is λ-$j(A)$-strong.

Given the Claim we have a contradiction:

$Ult((V_\delta, A \cap V_\delta), E)$ is the same computed in V or in M

$j(A) \cap V_\delta = A \cap V_\delta$ since $\delta = c.p.(j)$

$Ult((V_\delta, A \cap V_\delta), E) = Ult((M_\delta, j(A) \cap M_\delta), E)$

9/26/90

H. Woodin

From now on, we meet W, F 11:30 – 1

Finish last time's proof:

We had: $j : V \longrightarrow M$, $cp(j) = \delta$, $\lambda > \delta$, $V_{\lambda+\omega} \subseteq M$

$E = (\delta, \lambda)$ extender generated by j.

$E \subseteq \lambda^{<\omega} \times meas([\delta]^{<\omega})$

$E \in M$.

$V_{\delta+1} = M_{\delta+1}$

So, E is an extender in M.

let's compute $Ult(M, E)^M$.

$j_E^M : M \longrightarrow N = Ult(M, E)$

$c.p(j_E^M) = \delta$

$j_E^M(V_{\delta+1} = M_{\delta+1}) = j_E(V_{\delta+1})$ where $j_E : V \longrightarrow M = Ult(V, E)$

How does j_E compare with j?

$V \xrightarrow{j} M$

$X = \{j(F)(a) : F \in V, a \in [\lambda]^{<\omega}\}$

$X \prec M$ and $\lambda \subseteq X$, $j''V \subseteq X$

let M_0 be the transitive collapse of X.

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "j_0"'] & & M \\ & M_0 \arrow[ur, "k"'] & \end{tikzcd}$$

k is the inverse of the transitive collapse.

$c.p(k) = \min(OR \setminus X)$

since $\lambda \in X$, $cp(k) \geq \lambda$

<u>Claim</u>: $M_0 \cong Ult(V, E)$

<u>Recall</u>: $E(a)(S) = 1$ iff $a \in j(S)$, $a \in [\lambda]^{<\omega}$
(by def. of E)

<u>Notice</u>: The (δ, λ) extender induced by j_0 is E.
Also, $M_0 = \{ j_0(F)(a) : a \in [\lambda]^{<\omega} \}$

<u>Conclusion</u>: ~~If $A \subseteq V_\delta$, then $j(A) \cap V_\lambda = j_0(A)$~~
If $A \subseteq \delta$, then $j(A) \cap \lambda = j_0(A) \cap \lambda = j_E(A) \cap \lambda =$
$= j^M_E(A) \cap \lambda$.
we pick λ so that this is true
In our situation, $\lambda = |V_\lambda|$. So,
if $A \subseteq V\delta$, then $j(A) \cap V_\lambda = j_0(A) \cap V_\lambda = j_E(A) \cap V_\lambda =$
$= j^M_E(A) \cap V_\lambda$.
So, E witnesses that, in M, δ is λ-$j(A)$-strong. $\square$

Suppose κ is a Woodin cardinal. Then, there exists a poset $\mathbb{P} \subseteq V_\kappa$ s.t. if $G \subseteq \mathbb{P}$ is V-generic, then ($\delta < \kappa$ regular & uncount. in V)
$V[G] \models \exists j : V \to M \subseteq V[G]$ s.t. $M^{<\kappa} \subseteq M$ and
$c.p(j) = \delta$

Def: Suppose $\bigcup a \neq \emptyset$. Then,

except $\langle\rangle$

① a is closed if there exists $F : (\bigcup a)^{<\omega} \to \bigcup a$ s.t. $a = \{\sigma \subseteq \bigcup a : F''\sigma^{<\omega} \subseteq \sigma\}$

② a is stationary if $a \cap b \neq \emptyset$ for all closed sets b with $\bigcup b = \bigcup a$.

i.e., a is stationary if whenever $\bigcup a$ is the underlying set of a model M then a contains the underlying set of an elementary submodel of M.

Example: $S \subseteq \omega_1$ is stationary in the sense of ② iff it is stationary in the usual sense.

(true for regular cards. but not for singular (?))

Suppose κ is a cardinal. Let

$$\mathbb{P}_{<\kappa} = \{a \in V_\kappa : a \text{ is stationary}\}$$

$$\mathbb{Q}_{<\kappa} = \{a \in V_\kappa : a \text{ is stationary and } a \subseteq \mathcal{P}_{\omega_1}(\bigcup a)\}$$

where $\mathcal{P}_{\omega_1}(X) = \{\sigma \subseteq X : |\sigma| < \omega_1\}$.

Lemma: Suppose a is stationary and $X \subseteq \bigcup a$, $X \neq \emptyset$. Then, $b = \{\sigma \cap X : \sigma \in a\}$ is stationary. (Note that $\bigcup b = X$.)

Pf: Clear. □

Define an order on $\mathbb{P}_{<\kappa}$ ($\mathbb{Q}_{<\kappa}$) as follows:

a stronger than b
($a \Vdash b$)

$a \leq b$ if $\bigcup b \subset \bigcup a$ and $\{\sigma \cap \bigcup b : \sigma \in a\} \subseteq b$.

Def: Suppose a is stationary. Then,

① b is <u>closed and unbounded in a</u> if there exists a closed set c, $\bigcup c = \bigcup a$ and $c \cap a = b$.

b is <u>stationary in a</u> if $\bigcup b = \bigcup a$ and b is stationary

(ex : $a = \omega_1$. b is club in ω_1 iff $b = c \subset \omega_1$ some club c)

9/28/90

H. Woodin

<u>Def</u>: $\mathbb{P}^\infty = \{a : a \text{ is stationary}\}$

$\mathbb{Q}^\infty = \{a : a \text{ is stationary and } a \in \mathcal{P}_{\omega_1}(\cup a)\}$

Say $\mathbb{P}^\infty$, $\mathbb{Q}^\infty$ are classes.

<u>Def</u>: A set $D \subseteq \mathbb{P}_{<\kappa}$ is <u>semiproper</u> if

$sp(D) = \{a : a \in \mathcal{P}_\kappa(V_{\kappa+1}),\ a \prec V_{\kappa+1}$, and there is $b \prec V_{\kappa+1}$

such that: ① $D \in a$

② $b \cap V_\kappa$ end-extends $a \cap V_\kappa$

③ for some $x \in b \cap D$, $b \cap \cup x \in x\}$

contains a subset which is relatively closed in $\mathcal{P}_\kappa(V_{\kappa+1})$.

A set X end-extends Y if for all $z \in Y$, $z \cap Y = z \cap X$

(relatively closed means: for $C \subseteq \mathcal{P}(V_{\kappa+1})$, $C \cap \mathcal{P}_\kappa(V_{\kappa+1}) \subseteq sp(D)$

In the case for $\mathbb{Q}_{<\kappa}$, $sp(D) = \{a : a \in \mathcal{P}_{\omega_1}(V_{\kappa+1}) \dots\}$

$sp(D)$ is relatively closed in $\mathcal{P}_{\omega_1}(V_{\kappa+1})$.

<u>Remark</u>: If $D \subseteq \mathbb{P}_{<\kappa}$ is <u>semiproper</u>, then D is predense. i.e, $\{b : b \leq a \text{ for some } a \in D\}$ is dense in $\mathbb{P}_{<\kappa}$. Similarly for $\mathbb{Q}_{<\kappa}$.

To see this, let D be semiproper and let $a_0 \in \mathbb{P}_{<\kappa}$

We must find an element of D compatible with a_0.

Choose $x \in sp(D)$, $a_0 \in x$, and $x \cap (\cup a_0) \in a_0$.

x exists since $sp(D)$ contains a relatively closed subset in $\mathcal{P}_\kappa(V_{\kappa+1})$. So, there is $y \prec V_{\kappa+1}$, $y \in \mathcal{P}_\kappa(V_{\kappa+1})$, $x \subseteq y$

and for some $b \in y \cap D$, $y \cap (\cup b) \in b$ and $y \cap V_\kappa$ end-extends $x \cap V_\kappa$.

So, $a_0 \in x$, $x \cap (\cup a_0) \in a_0$, and thus, $y \cap (\cup a_0) = x \cap (\cup a_0)$ ($x \prec V_{\kappa+1}$, so $\cup a_0 \in x$).

We also have $y \cap (\cup b) \in b$, $b \in D$.

We claim that a_0, b are compatible.

Why?: Let $(\cup a_0) \cup (\cup b) \underset{df}{=} z$.

a_0, b are compatible iff $d \underset{df}{=} \{ \sigma \subseteq z : \sigma \cap (\cup a_0) \in a_0 \wedge \sigma \cap (\cup b) \in b \}$ is stationary in $\mathcal{P}(z)$ (i.e., d is stationary and $\cup d = z$)

But $y \prec V_{\kappa+1}$, $a_0, b \in y$, so $z \in y$.

If d is not stationary in $\mathcal{P}(z)$, then $\exists F : z^{<\omega} \to z$ such that $c_F \cap d = \phi$, where $c_F = \{ \tau \subseteq z : F'' \tau^{<\omega} \subseteq \tau \}$

Since $y \prec V_{\kappa+1}$, κ a limit, there must exist such an F in y. But $y \cap z \in d$ (since $y \cap (\cup a_0) \in a_0$, $y \cap (\cup b) \in b$) and $F''(y \cap z^{<\omega}) \subseteq y$. ($\rightarrow \leftarrow$).

Remark: If κ is supercompact, then every predense set is semiproper. (κ Woodin is not enough)

Lemma: Suppose M is a transitive set and $V_{\kappa+1} \in M$, every $F : V_{\kappa+1}^{<\omega} \to V_{\kappa+1}$ is in M, $M^{V_\kappa} \subseteq M$.

Suppose $D \subseteq \mathbb{P}_{<\kappa}$ (or $\mathbb{Q}_{<\kappa}$) is semiproper.

Then, if $x \underset{\kappa \in x}{\prec} M$, $|x| < \kappa$, $D \in x$, there exists

$y \prec M$, $|y| < \kappa$, $x \subseteq y$, $y \cap V_\kappa$ end-extends $x \cap V_\kappa$ and $y \cap (\cup b) \in b$ for some $b \in y \cap D$.

(So, this is an equivalence, i.e.,

$\{x \cap V_{\kappa+1} : x \prec M,\ \kappa \in x,\ |x| < \kappa\} \subseteq sp(D)$ is relatively closed in $\mathcal{P}_\kappa(V_{\kappa+1})$.)

Pf. of lemma: Suppose D is semiproper, $D \subseteq \mathbb{P}_{<\kappa}$, $x \prec M$, $\{\kappa, D\} \subseteq x$.

D is semiproper. Hence, $sp(D)$ contains a set C relatively closed in $\mathcal{P}_\kappa(V_{\kappa+1})$, i.e., there exists $F : V_{\kappa+1}^{<\omega} \to V_{\kappa+1}$ such that if $\sigma \subseteq V_{\kappa+1}$, $|\sigma| < \kappa$, $F''\sigma^{<\omega} \subseteq \sigma \Rightarrow \sigma \in sp(D)$.

So, such an F belongs to x. (since $x \prec M$, etc.).

Clearly, $F'' x \cap V_{\kappa+1}^{<\omega} \subseteq x$. $|x \cap V_{\kappa+1}| < \kappa$, so $x \cap V_{\kappa+1} \in sp(D)$.

So, there exists $z \prec V_{\kappa+1}$, $|z| < \kappa$, $z \cap V_\kappa$ end-extends $x \cap V_\kappa$, $x \cap V_{\kappa+1} \subseteq z$ and for some $b \in z \cap D$, $z \cap (\cup b) \in b$.

Let $y \cap V_\kappa = z \cap V_\kappa$, $y = \{f(t) : f \in x \wedge t \in z \cap V_\kappa\}$

Claim: ① $x \subseteq y$

② $y \prec M$ and $y \cap V_\kappa = z \cap V_\kappa$.

Pf: ① Clear.

② We have $z \cap (\cup b) \in b$, $b \in z \cap D$, $y \cap (\cup b) = z \cap (\cup b)$.

To see $y \prec M$:

General Claim: if $x \prec M$, $\kappa \in x$ and $s \subseteq V_\kappa$, $y = \{f(t) : f \in x,\ t \in s\}$, then $y \prec M$ and $y \cap V_\kappa = \{f(t) : t \in s$ and $f \in x \cap V_{\kappa+1}\}$.

10/3/90

H. WOODIN — Work with $\mathbb{P}_{<\kappa}$. Everything works also for $\mathbb{Q}_{<\kappa}$.

For $D \subseteq \mathbb{P}_{<\kappa}$, $sp(D) = \{x \in \mathcal{P}_\kappa(V_{\kappa+1}) : D \in X,\ X \prec V_{\kappa+1},$ $\exists y \prec V_{\kappa+1},\ x \subseteq y,\ y$ end-extends $x \cap V_\kappa,\ y \cap (\cup b) \in b$ for some $b \in y \cap D.\}$

we call this last condition " y captures D ".

<u>Remark</u>: Suppose $y \prec V_{\kappa+1}$ and y captures each $D \subseteq \mathbb{P}_{<\kappa}$ such that $D \in y$.

Let $G_y = \{b \in y \cap \mathbb{P}_{<\kappa} : y \cap \cup b \in b\}$.

G_y is a filter in $y \cap \mathbb{P}_{<\kappa}$.

<u>Lemma</u>: y captures D for all $D \in y$ (D dense in $\mathbb{P}_{<\kappa}$) iff G_y is y-generic for $y \cap \mathbb{P}_{<\kappa}$. $\square$

<u>Key Theorem 1</u>: Suppose κ is a Woodin cardinal. Suppose $D \subseteq \mathbb{P}_{<\kappa}$ is dense. Then, $\exists \delta < \kappa$, δ strongly inaccessible such that $D \cap V_\delta$ is dense in $\mathbb{P}_{<\delta}$ and is semiproper in $\mathbb{P}_{<\delta}$.

<u>Key Theorem 2</u>: Suppose κ is a Woodin card. Suppose $\langle D_\alpha : \alpha < \kappa \rangle$ is a sequence of dense sets in $\mathbb{P}_{<\kappa}$. Then, $\exists \delta < \kappa$ strongly inacc. s.t. for all $\alpha < \delta$, $D_\alpha \cap V_\delta$ is semiproper in $\mathbb{P}_{<\delta}$.

Clearly, 2 implies 1.
(Both theorems are true for $\mathbb{Q}_{<\kappa}$)

Pf. of Theorem 1: Suppose $D \subseteq \mathbb{P}_{<\kappa}$ is dense. Suppose no such δ exists, δ strongly inacc.

Let $S = \{\delta < \kappa : \delta$ is strongly inacc and $D \cap V_\delta$ is dense in $\mathbb{P}_{<\delta}\}$

Let $I = \{\delta < \kappa : \delta$ is strongly inacc.$\}$. Note that $S = I \cap C$ for some C closed and unbounded in κ.

Thus, for each $\delta \in S$, $D \cap V_\delta$ is not semiproper in $\mathbb{P}_{<\delta}$.
i.e., $sp(D \cap V_\delta)$ $(\subseteq \mathcal{P}_\delta(V_{\delta+1}))$ does not contain a subset closed in $\mathcal{P}_\delta(V_{\delta+1})$. i.e., $a_\delta = \mathcal{P}_\delta(V_{\delta+1}) \setminus sp(D \cap V_\delta)$ is stationary for all δ.
So, $a_\delta \in \mathbb{P}_{<\kappa}$ for all δ.
$D \subseteq \mathbb{P}_{<\kappa}$ is dense. So, for each $\delta \in S$, there exists $b_\delta \in D$ with $b_\delta \leq a_\delta$.
For $\delta \in S$, define $f(\delta)$ so that $b_\delta \in V_{f(\delta)}$, $f(\delta)$ strongly inacc, $f(\delta) = rk(b_\delta)+$
For $\delta \notin S$, let $f(\delta)$ = least element of S greater than δ.
Since κ is Woodin, there is $j : V \to M$ s.t. $f \restriction \delta : \delta \to \delta$,
$V_{j(f)(\delta)+1} \subseteq M$, where $\delta = cp(j)$.
We need M to be closed under δ-sequences:
Set $X = \{ j(F)(a) : a \in V_{j(f)(\delta)},\ F \in V \}$. Let $N = \text{collapse}(X)$.

$$\begin{array}{ccc} V & \xrightarrow{j} & M \\ & {\scriptstyle \bar{j}}\searrow & \nearrow{\scriptstyle k} \\ & N & \end{array}$$

$\bar{j} : V \to N$

$cp(\bar{j}) = cp(j)$

$\bar{j}(f)(\delta) = j(f)(\delta)$

$V_{j(f)(\delta)} \subseteq N$

$N^\delta \subseteq N$

(actually, we will probably not need that M is closed under δ-sequences)

Now,

0) $\delta \in S$ $\quad (S = C \cap I)$

i.e., $D \cap V_\delta$ is dense in $\mathbb{P}_{<\delta}$.

1) $sp(D \cap V_\delta)^V = sp(D \cap V_\delta)^M$ $\quad$ (clear, since $\mathcal{P}_\delta(V_{\delta+1})^V = \mathcal{P}_\delta(V_{\delta+1})$)

$M \models$ "$\delta \in j(S)$"

So, there exists $b \in M_{j(f)(\delta)} \cap j(\mathbb{P}_{<\kappa}) \cap j(D)$ such that $b \leq a_\delta$.

$M \models$ "b is stationary" and $b \in M_{j(f)(\delta)} = V_{j(f)(\delta)}$

Thus, $\mathcal{P}(\cup b) \subseteq M$

Therefore, b is stationary.

a_δ is stationary in $\mathcal{P}_\delta(V_{\delta+1})$, so, $\cup a_\delta = V_{\delta+1}$

Thus, if $X \in b$, then $X \cap V_{\delta+1} \in a_\delta = \mathcal{P}_\delta(V_{\delta+1}) \setminus sp(D \cap V_\delta)$

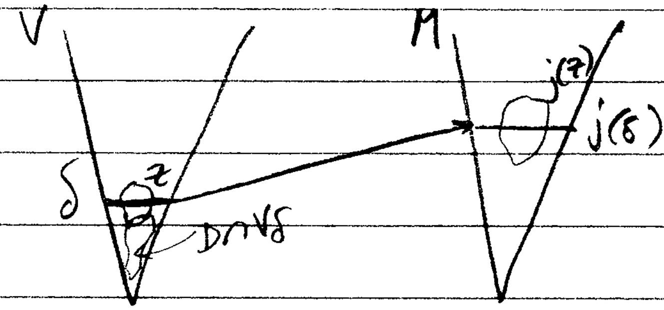
Fix $\lambda >> \kappa$.

b is stationary. So, choose $X \prec V_\lambda$ with $\{b, \kappa, \delta, j \restriction V_{\delta+1}\} \subseteq X$ and $X \cap \cup b \in b$.

Note: since we want $j \restriction V_{\delta+1} \in X$, b may not be stationary any more.

$X \cap \bigcup b \in b$. So, $X \cap \bigcup a_\delta \in a_\delta$. i.e.,
$\underbrace{X \cap V_{\delta+1}}_{\| \atop Z} \in \mathcal{P}_\delta(V_{\delta+1}) \setminus sp(D \cap V_\delta)$

Thus, $j(Z) \in j(\mathcal{P}_\delta(V_{\delta+1}) \setminus sp(D \cap V_\delta))$
$|Z| < \delta$. So, $j(Z) = j''Z$.



Let Y = closure of $j(Z) \cup \{b\} \cup X \cap \bigcup b$ under all functions $H : j(V_\delta) \to j(V_\delta)$.
Notice that $H \in j(Z)$. Why? $Y \subseteq X$, $Z \subseteq X$, $X \cap (\bigcup b) \subseteq X$, $Z = X \cap V_{\delta+1}$, so $j \restriction V_{\delta+1} \in X$, $H \in X$.
For this, $j(Z) = j''Z$, so each such $H = j(h)$, where $h : V_\delta \to V_\delta$, $h \in Z$. Since $j \restriction V_{\delta+1} \in X$, $j''Z \subseteq X$.

$j(Z) \prec j(V_{\delta+1})$. Hence, $Y \prec j(V_{\delta+1})$.
Clearly, $Y \in M$.

(To see that $Y \prec j(V_{\delta+1})$:
We have $Z \prec V_{\delta+1}$. Suppose $t \in V_\delta$. Y = closure of $Z \cup t$ under all functions in Z.)

<u>Claim</u>: Y witnesses that $j(Z) \in sp(j(D) \cap j(V_\delta))^M$
(contradicts $Z \notin sp(D \cap V_\delta)$)

We have: $j(Z) \subseteq Y \prec j(V_{\delta+1})$
Y end-extends $j(Z) \cap j(V_\delta)$
$j(Z) \cap j(V_\delta) = j(Z \cap V_\delta) = Z \cap V_\delta$ (since $|Z| < \delta$ & $\delta = cp(j)$)
So, we need only check that Y end-extends $Z \cap V_\delta$.
$Y \subseteq X$ and $Z = X \cap V_{\delta+1}$. So, $Z = Y \cap V_{\delta+1}$. Hence Y end-extends $Z \cap V_\delta$.
Finally, we have to verify that Y captures $j(D \cap V_\delta)$.
But $b \in Y$, $b \in j(D \cap V_\delta)$ and $Y \cap \cup b = X \cap \cup b \in b$.
So, done. $\square$

<u>Pf. of Theorem 2</u>: Similar.
Let $S = \{\delta : D_\alpha \cap V_\delta$ is dense in $\mathbb{P}_{<\delta}$, $\alpha < \delta$, δ strongly inacc.$\}$
Suppose $S = C \cap T$.
To define $f(\delta)$ use D_α, α least s.t. $D_\alpha \cap V_\delta$ is not semiproper in $\mathbb{P}_{<\delta}$.
etc...

Suppose κ is Woodin.
Let $G \subseteq \mathbb{P}_{<\kappa}$ be V-generic.
Define $j: V \to (M, E) \subseteq V[G]$ as follows:
Suppose $a \in G$.
Define $j_a: V \to (M_a, E_a)$:
$M_a = V^a / \mathcal{U}_a$, where $\mathcal{U}_a = \{b \subseteq a : b \in G\}$

$V^a = \{F : a \to V : F \in V\}$

$\cup b = \cup a$
i.e., b stationary in a.

$\mathcal{U}_a$ is an ultrafilter on $\mathcal{P}(a)^V / NS^V$

Suppose $a \leq b$ in $\mathbb{P}_{<\kappa}$, $a, b \in G$. Then, there is a natural embedding $j_{ba} : (M_b, E_b) \to (M_a, E_a)$

Note that $a \leq b \Rightarrow \{\sigma \in \cup b : \sigma \in a\} \subseteq b$.
Suppose $h : b \to V$, then let $\bar{h} : a \to V$ be given by $\bar{h}(\sigma) = h(\sigma \cap \cup b)$.
So, if $a \in G$ and $d \subseteq a$, $d \in \mathcal{U}_a$, then $\{\sigma \cap \cup b\} \in \mathcal{U}_b$
i.e., $b \in G$.

The class $\{M_a, j_{ba} : a, b \in G, a \leq b\}$ forms a directed system.
Let $M = \lim_{a \in G} M_a$ and $E = \lim_{a \in G} E_a$.

So, $M = \{F : a \to V : a \in G, F \in V\}$