Light-face games and their strategies

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1: a light-face coanalytic game

Here is a Π^1_1 game. The players write, respectively and bitwise, α and ε in $\mathcal{N}$; then

Rule: II wins if $\alpha \notin WORD$ or if ε codes a subset R of $\omega \times \omega$ such that $M_\varepsilon =_{\mathrm{df}} (\omega, R)$ forms an ill-founded ω-model of `KPL` with $|\alpha| \cong$ some ordinal of M.

Theorem *No winning strategy for this game can lie in any set-generic extension of L.*

For one direction we use a classical argument of Solovay (1967):

1·0 Proposition *There is no winning strategy for the first player in our game.*

Proof: let σ be such. By the boundedness theorem, the ordinals of the form $|\sigma \star [\varepsilon]|$ are bounded by some countable ordinal, θ say; let α code θ and by Gandy find M with sto(M) $= \theta^+$, the first admissible ordinal strictly greater than θ. $\dashv$ (1·0)

For the other direction we shall use a diagonal argument.

Some remarks on ω models

Let M be a model of some reasonable but possibly weak set theory, for example, Kripke–Platek with the axiom of infinity. Some of the "ordinals" of M will be well-founded and therefore isomorphic to von Neumann ordinals. These "ordinals" will be called *standard*. We write $sto(M)$ for the supremum of the von Neumann ordinals isomorphic to "ordinals" of M.

We suppose that M can define the rank of its members, and we write $stp(M))$ for the *standard part* of M, the transitive set isomorphic to the set of members of M whose rank, computed in M, is standard.

An ω-model means one with standard integers; equivalently, an M with $sto(M) > \omega$. In the coding of ω-models, we require each $k \in \omega$ to be represented by $2k$ in the model. We shall use the following notation: if x is in the transitive set isomorphic to the standard part of M we shall write $\llcorner x \lrcorner$ for the integer that represents x in M. Conversely, if k is an integer, we write $\ulcorner k \urcorner$ to indicate the object that it represents in M, or in the collapsed well-founded part of M: this second definition is more a reminder to the reader than a mathematical definition. Thus $2k = \llcorner k \lrcorner$, and $\ulcorner 2k \urcorner = k$.

Here is an easy instance of the diagonal argument to be applied.

1·1 PROPOSITION *ε cannot be in the standard part of the model it codes, provided the set of reals represented in that model is downwards closed under "recursive in".*

Proof : let $X = \{k \mid \varepsilon \text{ says } \ulcorner 2k \urcorner \text{ is not in } \ulcorner k \urcorner\}$. If X were in the model, represented by ℓ say, we would have a classical paradox by considering whether ℓ is in X or not:

$$\begin{aligned} \ell \notin X &\iff \varepsilon \text{ says } \ulcorner 2\ell \urcorner \text{ is not in } \ulcorner \ell \urcorner \\ &\iff \ell \in X. \end{aligned}$$

So X cannot be in the model, but it is recursive in ε, so ε cannot be, either. $\dashv$ (1·1)

1·2 PROPOSITION *There is no winning strategy for the second player in $L[G]$, where G is an L-generic for some notion of forcing which is a set in L.*

Proof : We suppose otherwise, and derive a contradiction. Note that by Shoenfield, once a strategy τ has come into being it remains a winning strategy in each further expansion of the universe. Thus we may simplify the picture by using the universal properties of the collapsing algebras to find a regular uncountable cardinal θ of L and G a generic for $\mathrm{Coll}(\omega, \theta)$ such that there is a winning strategy $\tau \in L_\theta[H]$ for the second player.

We write η for θ^+, the next admissible ordinal after θ.

We work in the model $L_\eta[G]$. Since the Lévy partial ordering $\mathrm{coll}(\omega, \theta)$, which we shall denote by $\mathbb{P}$, is a member of L_η, $L_\eta[G]$ is admissible. Note that there is in $L_\eta[G]$ an α coding θ. We play that α against τ, obtaining $\varepsilon = [\alpha] \star \tau$. Write M for the model coded by ε. Note that $sto(M)$ will be exactly η, by M^lle^ Ville's Lemma that $stp(M)$ is admissible.

Thus M has standard part including (but not necessarily equalling) L_η, and so $\mathbb{P}$ is represented in M, by $\llcorner \mathbb{P} \lrcorner$, and we may therefore contemplate forcing over M with these conditions.

We take Shoenfield's approach to forcing in which to each member x of the ground model is associated a constant, $\underline{x}$, of the forcing language, to be used as a name for some object in the generic extension. The inclusion of the ground model, here M, in the extension is achieved by defining for each member m of M a forcing-name $\hat{m}$, ("m-hat"), also in M; for each $(M, \llcorner \mathbb{P} \lrcorner)$-generic H, $\mathrm{val}_H(\hat{m})$ "equals" m, where val_H is the evaluation of forcing names defined by the generic H.

We shall form a subset X of ω, which, we shall see, both is and cannot be a member of $L_\eta[G]$.

First, given $k \in \omega$ we form the sentence φ_k of the forcing language (for $\llcorner \mathbb{P} \lrcorner$) inside M which says that the hat of the object $2k$ (representing k in the coding ε) is a member of the object in the forcing extension named in M by the object represented in M by k (in the coding ε). Thus φ_k is the sentence which says

$$\text{“}\neg\big(\underline{\widehat{\ulcorner 2k\urcorner}} \mathbin{\epsilon} \underline{\ulcorner k\urcorner}\big)\text{”}$$

The map $k \mapsto \varphi_k$ is $\Delta^1_1(\varepsilon)$ and therefore is in $L_\eta[G]$.

We ask whether there is some p in G such that in M the corresponding object $\llcorner p\lrcorner$ forces φ_k: if so, we place k in X; otherwise we exclude it. Thus

$$X \;=_{\mathrm{df}}\; \big\{k \in \omega \;\big|\; \exists p{:}{\in} G\; M \models \llcorner p\lrcorner \mathrel{\dot{\Vdash}}^{\llcorner\mathbb{P}\lrcorner} \varphi_k\big\}$$

The map $\pi : \mathbb{P} \longrightarrow \omega$ given by $\pi(p) = \llcorner p\lrcorner$ is in $L_\eta[G]$. Note further that we run the definition of forcing in M and do not attempt to restrict it to the standard part of M. Therefore X is a Δ_1 subset of ω, in the parameters G, π, ε and the integer $\llcorner\mathbb{P}\lrcorner$, and is therefore in $L_\eta[G]$. Hence there is a $t \in L_\eta$, such that $X = \mathrm{val}_G(t)$. Let the integer ℓ represent t in the coding of M by ε.

We ask if ℓ is in X: and we shall reach a contradiction, for

$$\begin{aligned}
\ell \text{ is not in } X &\iff \exists p{:}{\in} G\; p \Vdash^{\mathbb{P}} \neg(\underline{\hat{\ell}} \mathbin{\epsilon} \underline{t})\\
&\iff \exists p{:}{\in} G\; M \models \llcorner p\lrcorner \mathrel{\dot{\Vdash}}^{\llcorner\mathbb{P}\lrcorner} \neg\big(\underline{\widehat{\ulcorner 2\ell\urcorner}} \mathbin{\epsilon} \underline{\ulcorner \ell\urcorner}\big)\\
&\iff \exists p{:}{\in} G\; M \models \llcorner p\lrcorner \mathrel{\dot{\Vdash}}^{\llcorner\mathbb{P}\lrcorner} \varphi_\ell\\
&\iff \ell \text{ is in } X \text{ after all.}
\end{aligned}$$

The transition from forcing with $\mathbb{P}$ in L_η to forcing with its representative in M relies on the fact that forcing a simple sentence such as φ_k is local to a bounded transitive subset of its standard part. $\dashv$ (1·2)

1·3 Problem Does the existence of a winning strategy for this game imply the existence of $0^\sharp$? or even that ω_1^L is countable ?