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These notes are linked at `www.dpmms.cam.ac.uk/∼tf/NZMAabstract.pdf`

Synonymy Results in the Quine Systems

Two theories are **synonymous** iff
they are mutually interpretable "up to logical equivalence".
Alternatively they "have the same models".

Examples:

Partial Orders and Strict Partial Orders;
Boolean Algebras and Boolean Rings;

Our other (three) examples concern Set Theory:

1. Benedikt Löwe "Set Theory With and Without Urelements, and Categories of Interpretations" Notre Dame Journal of Formal Logic **47** 2006. …shows that the theories ZF and ZFU are synonymous.
2. Peano Arithmetic and ZF with ¬Infinity.
   Richard Kaye and Tin Lok Wong, "On Interpretations of Arithmetic and Set Theory" Notre Dame Journal of Formal Logic **48** Number 4 (2007), 497-510. …show that PA is synonymous with ZF $\setminus$ infinity + ¬Infinity + Transitive Containment.
   This is in virtue of the Ackermann trick:
   n "$\in$" m iff the nth bit in the binary representation of m is set.
3. Oswald's modification: n "$\in$" m iff either
   - m is even and the nth bit of $m/2$ is set; or
   - m is odd and the nth bit of $(m-1)/2$ is clear.

   Church's construction of models of NF_2 (= the 2-stratifiable axioms of NF).
   Fix a bijection $k : V \longleftrightarrow V \times \{0, 1\}$. Then say $x \in' y$ iff either
   - the second component of $k(y)$ is 0 and $x \in$ first component of $k(y)$; or
   - the second component of $k(y)$ is 1 and $x \notin$ first component of $k(y)$.

   This gives a model of NF_2 plus "the wellfounded sets are a model of ZF and every surjective image of a wellfounded set is a set".
   Button (`https://arxiv.org/abs/2103.06715`, submitted to JSL) shows that this theory is synonymous with ZF.
4. Forster-Holmes (submitted to JSL) show that NF is not synonymous with any theory of wellfounded sets.

Tightness

A theory is *tight* if any two extensions of it that are synonymous are identical;
It is *stratified-tight* if any two stratified extensions of it that are synonymous are identical.

PA is tight. ZF is tight. Zermelo + "ranks" is tight.
Tightness is something to do with second-order categoricity:
all these theories are (in a weak sense) second-order categorical.

NF is not tight because
'NF + $\exists!$ Quine atom' and
'NF + there are no Quine atoms'
are synonymous. However NF is stratified-tight.

More suggested Reading for Synonymy

Visser Oxford Slides `www.dpmms.cam.ac.uk/~tf/VisserOxford.pdf`

Hamkins and Freire:
"Bi-interpretation in weak set theories" Journal of Symbolic Logic **86** (2):609-634 (2021)

Enayat:
`https://www.researchgate.net/publication/313910192_Variations_on_a_Visserian_Theme`

Holmes' Proof of Con(NF)

TST is "the ωth order theory of equality".

Jensen's proof of Con(NFU)

Jensen, R.B. "On the consistency of a slight(?) modification of Quine's NF". Synthese **19** (1969) pp. 250-263.

Tangled Type Theory

Tangled Type theory, TTT, is the theory of a structure indexed by of a total order $\langle I, <_I \rangle$ with a set A_i for each $i \in I$, and, for each $i < j$ in I, a binary relation $\in_{i,j}$ such that, whenever $J \subseteq I$ is a subset of order type ω, then the $A_j : j \in J$ equipped with the $\in_{j,j'}$ form a model of TST.

Holmes, Randall "The equivalence of NF-style set theories with "tangled" type theories; the construction of ω-models of predicative NF (and more)" Journal of Symbolic Logic **60** (1995) pp. 178-189.

...shows that NF is equiconsistent with TTT.