Properties of $j, (M,E)$: (in ZFC)

① $a \in G$ iff $j''(\cup a)$ "$\in$" $j(a)$
i.e., there exists $z \in M$ s.t. $z \in j(a)$ and
$\{x \in M : x E z\} = \{j(t) : t \in \cup a\}$.

Assuming κ is Woodin, then
② (M,E) is well-founded and if N = transitive collapse of (M,E), then $N^{<\kappa} \subseteq N$ in $V[G]$.
(by Steel (?) if ② holds, then there is a inner model for a Woodin card.)

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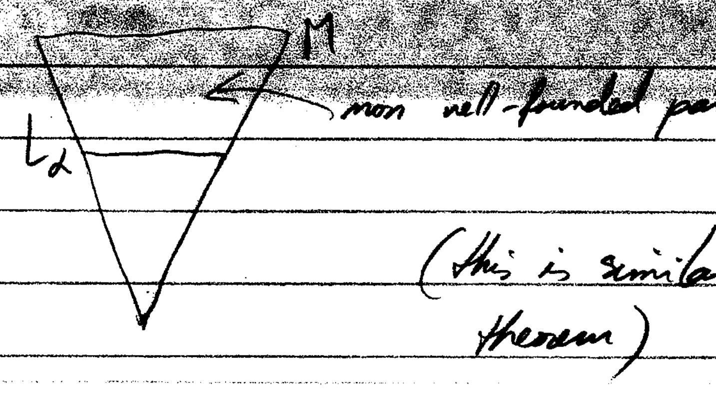
ex. (ZFC)
let $\mathbb{P}^{\infty} = \{a : a$ is stationary$\}$.
let $G \subseteq \mathbb{P}^{\infty}$ be V-generic.
$j : V \to (M,E)$, $j''(\cup a)$ "$\in$" $j(a)$, $V \subseteq (M,E)$.

let $L_\alpha \models ZFC$, α countable , $j : L_\alpha \to (M,E)$
$j''(\cup a)$ "$\in$" M , $L_\alpha \subseteq M$.



(this is similar to Barwise's compactness theorem)

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H. Woodin

λ strongly inacc (any λ would work)
$G \subset \mathbb{P}_{<\lambda}$ is V-generic.
$j: V \longrightarrow (M, E) = \lim_{a \in G} M_a$

$U_a = \{ b \subseteq a : b \in G, \cup b = \cup a \}$
Thus, U_a is an ultraf. on $\mathcal{P}(a)^V$
$j_a : V \longrightarrow (M_a, E_a) = V^a / U_a$

<u>Claim</u>: ① For each $a \in G$, there is $z \in M$ s.t.
$z E j(a)$ and $\{ t : t E z \} = \{ j(x) : x \in \cup a \}$
② $a \in G$ iff $j''\cup a$ "$\in$" $j(a)$ (in the sense of ①)

<u>Key lemma</u>: Suppose a is stationary and $F: a \longrightarrow \cup a$ is a choice function on a. i.e., $F(\sigma) \in \sigma$, if $\sigma \neq \emptyset$
Then, there is $t \in \cup a$ s.t. $\{ \sigma : F(\sigma) = t \}$ is stationary in a.

Pf: If not, for each $t \in \cup a$ there is $H_t : \cup a^{<\omega} \longrightarrow \cup a$ s.t. if $F(\sigma) = t$, then σ is not closed under H_t.
Define $H : (\cup a)^{<\omega} \longrightarrow \cup a$ by
$H(\langle t_0, \ldots t_n \rangle) = H_{t_0}(\langle t_1, \ldots t_n \rangle)$
a is stationary, so there is $\sigma \in a$, $\sigma \neq \emptyset$ s.t. σ is closed under H.
Let $t = F(\sigma)$. Then, σ is closed under H_t. ($\rightarrow\leftarrow$).

Fix $a \in G$.
We must find z.
Let z be given by the identity function on a. i.e.,
$I_a : a \to a$ be the identity.
$[I_a]_{U_a} \in M_a$.
$z = j_{a\infty}([I_a]_{U_a})$.

Suppose $t \in M$ and $t \in z$.
Then, $t = j_{b\infty}([F]_{U_b})$ for some $b \in G$. WMA $b \leq a$.
$t \in z$ iff $[F]_{U_b} \; E_b \; j_{ab}([I_a]_{U_a})$

$j_{ab}([I_a]_{U_a}) = [I]$, where $I(\sigma) = \sigma \cap \cup a$
$= I_a(\sigma \cap \cup a)$.

So, $t \in z$ iff $\{\sigma \in b : F(\sigma) \in I(\sigma)\} \in G$ and is stationary in b.
iff $b^* \in G$, $\cup b^* = \cup b$, where $b^* = \{\sigma \in b : F(\sigma) \in \sigma \cap (\cup a)\}$

$b^* \in G$, $F \upharpoonright b^*$ is a choice function on b^*.
By genericity, it follows $\exists b^{**} \in U_b$, $\exists s \in U_a$ s.t.
$\sigma \in b^{**} \to F(\sigma) = s$
(so $s \in \cup a$).
So, $[F]_{U_b} =_{M_b} [H_s]$ where $H_s : b \to V$ is the constant function $H_s(\sigma) = s$, all σ.
i.e., $[F]_{U_b} = j_b(s)$

But $j_{b\infty}([F]_{U_b}) = t \quad (t \in z)$

So, $j_{b\infty}([F]_{U_b}) = j(s)$

i.e., $t = j(s)$ for some $s \in \cup a$. □ (this proves ①)

as for ② ~~that~~: $j''\cup a$ "$\in$" $j(a)$ iff $a \in G$.

By the argument for ①, $j''(\cup a) = z$, where $z = j_{a^*\infty}([I_{a^*}]_{U_{a^*}})$ where $a^* = \mathcal{P}(\cup a)$. Note, $a^* \in G$. (Why?: by genericity, $\exists b \in G$ with $\cup a \subseteq \cup b$. But then, $\{\sigma \cap \cup a : \sigma \in b\} = \bar{a}$, $\bar{a}$ is in G since $b \le \bar{a} \le a^*$)

(To see that $\exists b \in G$ with $\cup a \subseteq \cup b$, fix $\bar{b} \in G$ and set $\bar{\bar{b}} = \{\sigma \subseteq \cup a \cup \cup\bar{b} : \sigma \cap \cup\bar{b} \in \bar{b}\}$, $\bar{\bar{b}} \le \bar{b}$ and $\bar{\bar{b}}$ gives b.)

Suppose that $z \mathrel{E} j(a)$. But $z \mathrel{E} j(a)$ iff $[I_{a^*}]_{U_{a^*}} \mathrel{E_{a^*}} [H_a]$, $H_a(\sigma) = a$ iff $\{\sigma : \underbrace{I_{a^*}(\sigma)}_{=\sigma} \in a\} \in U_{a^*}$

iff $a \in U_{a^*}$ iff $a \in G$. □

Applications: let $\mathbb{P}^\infty = \{a : a \text{ stationary}\}$.

Suppose G is V-generic for $\mathbb{P}^\infty$.

(So, $G \cap D \neq \emptyset$, where D is definable and dense in $\mathbb{P}^\infty$)

Then, $\exists j : V \to (M, E)$ s.t.

① $\forall a$, $j''\cup a$ "$\in$" M, $j''\cup a$ "$\in$" $j(a)$ iff $a \in G$.

So, $j \restriction V_\lambda$ "$\in$" M, for all λ and $G \cap V_\lambda$ "$\in$" M

i.e., $V[G] \subseteq M$.

To compute $G \cap V_\lambda$ one needs only $j \restriction V_\lambda$.
(Why? : $a \in G \cap V_\lambda$ iff $j'' \cup a \in j(a)$.)

Suppose $\omega \le \alpha < \beta$.
Let $\mathcal{P}^\alpha(\beta) = \{S \subseteq \beta : ot(S) = \alpha\}$.
Then, $j(\alpha) = \beta$ iff $\mathcal{P}^\alpha(\beta) \in G$
(so, in particular, $\mathcal{P}^\alpha(\beta)$ is stationary)
So, $\delta = cp(j)$, $j(\delta) \in OR$, iff $\exists \kappa > \delta$ s.t. $\mathcal{P}^\delta(\kappa)$ is stationary and $\mathcal{P}^\delta(\kappa) \in G$.
(To see this, suppose $\mathcal{P}^\alpha(\beta) \in G$. Then, $j''(\cup \mathcal{P}^\alpha(\beta)) \in j(\mathcal{P}^\alpha(\beta))$.
So, $j''(\beta) \in j(\mathcal{P}^\alpha(\beta)) = (\mathcal{P}^{j(\alpha)}(j(\beta)))^M$. i.e., $j(\alpha) = \beta$.)

Claim: $c.p(j) = \delta$ iff $\exists S \subseteq \delta$ s.t. $S \ne \delta$ and $S \in G$.
Pf: $\Rightarrow$ Suppose $cp(j) = \delta$.
Then, $j''\delta \in j(\mathcal{P}(\delta))$ (since $\mathcal{P}(\delta) \in G$, $\mathcal{P}(\delta)$ is stationary)
$\delta = cp(j)$ iff $j''\delta = \delta \wedge j(\delta) \ne \delta$.
$cp(j) = \delta \Rightarrow \delta \in j(\mathcal{P}(\delta)) \wedge j(\delta) \ne \delta$.
Split $\delta = S_0 \cup S_1$, $S_0 \cap S_1 = \phi$. S_0, S_1 both stationary.
Then, since $j''\delta = \delta \wedge \delta < j(\delta)$, it follows $\delta \in j(S_0)$ or $\delta \in j(S_1)$. So, $S_0 \in G$ or $S_1 \in G$.
$\Leftarrow$
$S \subseteq \delta$, $\delta \setminus S$, S stationary, $S \in G$.

Note that $\cup S = \delta$ since S is stat. So, $j''\cup S = j''\delta$

$j''\delta \in j(S)$, $S \subseteq \delta$, $j(S) \subseteq j(\delta)$.
So, $j''\delta$ must be an initial segment of δ. i.e. $j''\delta = \delta$
Need : $j(\delta) \ne \delta$.

If $j(\delta)=\delta$, then consider $U_{\mathcal{P}(\delta)}$, $\mathcal{P}(\delta)\in G$.

Since $j''(\delta)=\delta$, $a\in U_{\mathcal{P}(\delta)}$ iff $j''\delta\in j(a)$

iff $\delta\in j(a)$

iff $\delta\in a$

Note: $U_{\mathcal{P}(\delta)}=\{a\subseteq\mathcal{P}(\delta):\delta\in a\}$

So $\uparrow$ is principal.

($\bigcup a\in a\Rightarrow a$ is stationary

these are the degenerate stationary sets) ∎

So, the potential critical points are regular cardinals.

Thus, if $V=L$, then $j(\alpha)\notin OR$ if $\alpha\geq cp(j)\geq\omega_1^L$

Why?: $\mathcal{P}^\alpha(\beta)$ is never stationary if $\alpha<\beta$

OW, $j:L_\alpha\to L_\beta$, $\alpha\geq\omega_1$ and $j\restriction\alpha\neq id$.

Def: κ is Jonsson if $\mathcal{P}^\kappa(\kappa)\setminus\{\kappa\}$ is stationary.

Recall: Measurable cardinals are Jonsson.

Def: κ is completely Jonsson if for all stationary a with $\bigcup a\in\kappa$ ($\bigcup a$ is a proper initial segment of κ) the set $\{S\subset\kappa: S\cap\bigcup a\in a\wedge S\in\mathcal{P}^\kappa(\kappa)\}$ is stationary.

<u>Lemma</u>: Suppose $\kappa \to (\kappa)^{<\omega}_{<\kappa}$. Then κ is completely Jonsson.

<u>Corollary</u>: Measurable cardinals are completely Jonsson, and if κ is measurable, then $\{\delta<\kappa : \delta$ is completely Jonsson$\}$ is non-empty.

<u>Recall</u>: $\kappa \to (\kappa)^{<\omega}_{<\kappa}$ if for all $\delta<\kappa$ and all functions $f:\kappa \to \kappa^{<\omega}$, there exists $S \subseteq \kappa$ with $|S|=\kappa$ s.t. for each $i<\omega$, $f \restriction [S]^i$ is constant.

$\kappa \to (\kappa)^{<\omega}_{<\kappa} \implies \kappa$ is strongly inaccessible.

<u>Application of $\mathbb{P}^\infty$</u>:

Suppose there is a proper class of completely Jonsson cardinals.

Let $G \subseteq \mathbb{P}^\infty$ be V-generic. Then, we get $j: V \to V[G]$

So, $M \equiv V[G]$. i.e. M is well-founded.

We get $j: V \to (M, E)$, $V[G] \subseteq M$.

Let's show $j(\delta) \in OR$ for all δ.

Suppose b, δ are given. Let's find $b \leq a, \kappa$, $b \Vdash j(\delta)<\kappa$.

Let $\alpha = |\cup a|$.

Let κ be a Completely Jonsson card $>\alpha, \delta$.

WLOG, $\cup a = \alpha$. (permute $V \xrightarrow{\pi} V$ to $\cup a \to X$ etc.)

$b = \{ s \subseteq \kappa : s \cap \alpha \in a ,\ |s| = \kappa \}$

Since κ is completely Jonsson, b is stationary in $\mathcal{P}(\kappa)$.

$b \leq a$.

$b \Vdash j(\kappa) = \kappa$. $\Big(b \Vdash j'' \cup b \in j(b)$

$\qquad\qquad j''\kappa \in \left(\mathcal{P}^{j(\kappa)}(j(\kappa)) \right)^M \Big)$

$\delta < \kappa \longrightarrow j(\delta) < j(\kappa) = \kappa$.

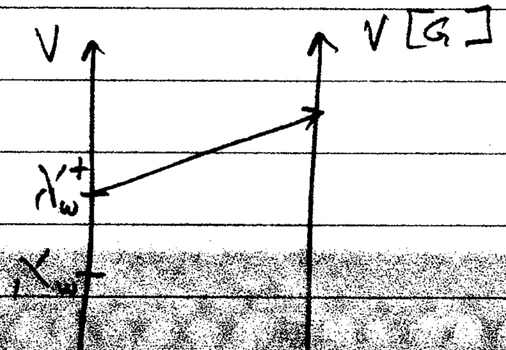
ex: Assume GCH.

let $\delta = \aleph_\omega^+$

let $S = \{ \alpha : \alpha < \aleph_\omega^+ \ \ cf(\alpha) = \omega_3 \}$

Pick G with $S \in G$.

So, $S \Vdash cp(j) = \aleph_\omega^+$



So, $cf((\aleph_\omega^+)^V)^{V[G]} = \omega_3$.

and nothing below $\aleph_\omega^+$ has been collapsed.

(49)

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H. Woodin (P-version. Q-version is similar)

Theorem: Suppose κ is Woodin. Suppose $G \subseteq \mathbb{P}_{<\kappa}$ is V-generic. Let $j: V \to (N, E)$ be the induced elementary embedding. Then, N is well-founded, and if M = transitive collapse of N, then $M^{<\kappa} \subseteq M$ in $V[G]$.
So, j yields $j: V \to M$, $M^{<\kappa} \subseteq M$ and for all $a \in \mathbb{P}_{<\kappa}$, $a \in G$ iff $j''(\cup a) \in j(a)$.

Pf: It suffices to show (because of what we have already done) that N is well-founded and that its transitive collapse is closed under $<\kappa$ sequences.
This is equivalent to showing that N is closed under $<\kappa$ sequences in the sense that if $Y \subseteq N$, $Y \in V[G]$ and $|Y| < \kappa$ in $V[G]$, then there exists $Z \in N$ s.t. $Y = \{t \in N : t E Z\}$. (So this implies well-foundedness and closure under $<\kappa$ sequences).
Work in V. Let $\delta < \kappa$ and $\langle \tau_\alpha : \alpha < \delta \rangle$ of terms of elements of N.
Continue to work in V. For each α, choose a maximal antichain A_α and a function $F_\alpha : A_\alpha \to V$ s.t. for each $a \in A_\alpha$, $a \Vdash \tau_\alpha = [F_\alpha(a)]$.
We should work below a given condition a_0.
Since κ is Woodin, there is $\lambda < \kappa$, λ strongly inacc. $a_0 \in V_\lambda$ and $A_\alpha \cap V_\lambda$ is semiproper in $\mathbb{P}_{<\lambda}$ for all $\alpha < \delta$.

Hence, $A_\alpha \cap V_\lambda$ is a maximal antichain in $\mathbb{P}_{<\lambda}$ for all α.

Define: $a = \{ X \prec V_{\lambda+1} : |X| < \lambda, \delta, a_0 \in X$; for each $\alpha \in X \cap \delta$ there exists $b \in X \cap A_\alpha \cap V_\lambda$ s.t. $X \cap \cup b \in b$. And $X \cap \cup a_0 = a_0 \}$

Key Claim: a is stationary in $\mathcal{P}_\lambda(V_{\lambda+1})$.

Pf: let $H : V_{\lambda+1}^{<\omega} \to V_{\lambda+1}$ be given.

Fix $\lambda^* > \lambda$, $cf(\lambda^*) > \lambda$. (let $\lambda^* = \kappa$ if you want)

We shall find $Z \prec V_{\lambda^*}$ with $\{\delta, \lambda, a_0, H\} \subseteq Z$ and $Z \cap V_{\lambda+1} \in a$. This will prove a is stationary.

Build Z by an elementary chain. $\langle A_\alpha : \alpha < \delta \rangle$

Fix $Z_0 \prec V_{\lambda^*}$. $|Z_0| < \lambda$, $\{\delta, \lambda, H, a_0\} \subseteq Z_0$.

(Since a_0 is stationary, $a_0 \in V_\lambda$. So, Z_0 exists)

The elementary chain is indexed by $Z_0 \cap \delta$, $\langle Z_\alpha : \alpha \in Z_0 \cap \delta \rangle$ and has the properties.

(0) $Z_\alpha \prec V_{\lambda^*}$, $|Z_\alpha| < \lambda$

(1) $\alpha < \beta \Rightarrow Z_\beta$ end-extends $Z_\alpha \cap V_\lambda$

(2) if β is limit, then $Z_\beta = \bigcup_{\alpha<\beta} Z_\alpha = \bigcup \{Z_\alpha : \alpha < \beta, \alpha, \beta \in Z_0 \cap \delta\}$

(3) for all $\alpha \in Z_0 \cap \delta$, there is $b \in Z_{\alpha+1} \cap A_\alpha \cap V_\lambda$ s.t. $Z_{\alpha+1} \cap (\cup b) \in b$.

So, suppose $\beta \in Z_0 \cap \delta$ and $\langle Z_\alpha : \alpha \leq \beta \rangle$ has been defined. So, $Z_\beta \prec V_{\lambda^*}$, $|Z_\beta| < \lambda$ and $A_\beta \cap V_\lambda \in Z_\beta$.

But $A_\beta \cap V_\lambda$ is semiproper in $\mathbb{P}_{<\lambda}$.

given after the Def. of semiproper.

$(V_{\lambda^*})^{V_\lambda} \subseteq V_{\lambda^*}$ and $V_{\lambda+2} \in V_{\lambda^*}$ so we can apply the Lemma to find $Z \prec V_{\lambda^*}$ s.t. $Z_\beta \subseteq Z$, $|Z| < \lambda$, Z end-extends $Z_\beta \cap V_\lambda$ and for some $b \in A_\beta \cap V_\lambda$, $Z \cap (\cup b) \in b$. Set $Z_{\beta+1} = Z$.

We must check:

① $\alpha \leq \beta \Rightarrow Z_{\beta+1}$ end-extends $Z_\alpha \cap V_\lambda$.

This is immediate since Z_β end-extends $Z_\alpha \cap V_\lambda$ and $Z_{\beta+1}$ endextends $Z_\beta \cap V_\lambda$.

So, $\langle Z_\alpha : \alpha \in \delta \cap Z_0 \rangle$ exists as desired.

Set $Z = \bigcup \{ Z_\alpha : \alpha \in \delta \cap Z_0 \}$

<u>Claim</u>: ⓪ $|Z| < \lambda$, $Z \prec V_{\lambda^*}$, $H \in Z$.

① Z end-extends $Z_\alpha \cap V_\lambda$ for all $\alpha \in \delta \cap Z_0$. So, $Z \cap \delta = Z_0 \cap \delta$.

② for each $\alpha \in Z \cap \delta$, there is $b \in Z \cap A_\alpha \cap V_\lambda$ with $Z \cap (\cup b) \in b$.

<u>Pf. of Claim</u>: ⓪ immediate.

① immediate since $Z = \bigcup \{ Z_\alpha \}$, and $\alpha < \beta \Rightarrow Z_\beta$ endextends $Z_\alpha \cap V_\lambda$.

② follows from ①: $Z \cap \delta = Z_0 \cap \delta$. So, we may suppose $\alpha \in Z_0 \cap \delta$. Hence, there is $b \in Z_{\alpha+1} \cap A_\alpha \cap V_\lambda$ with $Z_{\alpha+1} \cap (\cup b) \in b$. But Z end-extends $Z_{\alpha+1} \cap V_\lambda$ so $b \in Z$ and $Z \cap (\cup b) = Z_{\alpha+1}$. $\square$

Thus, a is stationary.

Note: $a \le a_0$ in P_κ and if $x \in a$, $\alpha \in X \cap \delta$, then there is a unique b with $X \cap (\cup b) \in b$, $b \in A_\alpha \cap V_\lambda \cap X$.

Recall: we have $\langle \tau_\alpha : \alpha < \delta \rangle$, $\langle A_\alpha : \alpha < \delta \rangle$, $\langle F_\alpha : \alpha < \delta \rangle$,

$F_\alpha : A_\alpha \to V$, $b \in A_\alpha \Rightarrow b \Vdash \tau_\alpha = [F_\alpha(b)]$

$(F_\alpha(b) \in V^b)$

(note $N = \{[g] : g \in V^b, b \in G$. So, given a condition c, find $\bar{c} < c$ to decide g, b for τ_α. i.e., $\bar{c} \Vdash b \in G \wedge \tau_\alpha = [g]$.

since $\bar{c} \Vdash b \in G$, WMA $\bar{c} \le b$. Thus we can replace g by $\bar{g}$ where $\bar{g}(t) = g(t \cap \cup b)$

We now define $f \in V^a$ so that $a \Vdash t \in [f]$ iff $t = \tau_\alpha$, some α.

For $x \in a$, $f(x) = \{F_\alpha(b)(x \cap \cup b) : a \in X \cap \delta$, $b \in X \cap A_\alpha \cap V_\lambda$ and $X \cap (\cup b) \in b\}$

(Since b is unique (because G is generic), this def. is not ambiguous.)

We must show that $a \Vdash f$ works:

Suppose $c \le a$, $h \in V^c$ and $c \Vdash [h] \in [f]$.

i.e., WMA $\forall y \in c$, $h(y) \in f(y \cap \cup a)$.

Set $x = y \cap (\cup a)$, $x \in a$.

Define $J : c \to \cup c \times \cup c$ by $J(y) = \langle \alpha, b \rangle$, where $\alpha \in y \cap \delta$, α least, $b \in y \cap (\cup a) \cap A_\alpha \cap V_\lambda$ and $h(y) = F_\alpha(b)$.

(given α, b is unique)

J is a "choice" function.

choose $d \leq c$ s.t. $J \restriction d$ is constant with value α_0, b_0.

Claim: $d \Vdash \tau_{\alpha_0} = [h]$

Pf of Claim: let $y \in d$

then, $b_0 \in y$ and $y \cap \underbrace{(\cup a) \cap (\cup b_0)}_{= \cup b_0} \in b_0$

So, $d \leq b_0$

$b_0 \in A_{\alpha_0}$. So, $b_0 \Vdash \tau_{\alpha_0} = [F_{\alpha_0}(b_0)]$.

So, $d \Vdash \tau_{\alpha_0} = [F_{\alpha_0}(b_0)]$

Finally, if $y \in d$, then $y \cap (\cup c) \in c$ and

$h(y \cap (\cup c)) = F_{\alpha_0}(b_0)(y \cap (\cup c) \cap (\cup b_0))$

$h \in V^c$

$F_{\alpha_0}(b_0) \in V^{b_0}$. So,

$= F_{\alpha_0}(b_0)(y \cap \cup b_0)$

So, $d \Vdash [h] = [F_{\alpha_0}(b_0)]$

$d \Vdash [h] = \tau_{\alpha_0}$ $\square$.

Conclusion, $a \Vdash$ "$[h] \in [f] \Rightarrow \exists \alpha$ with $[h] = \tau_\alpha$"

All that remains: For all $\alpha < \delta$, $a \Vdash \tau_\alpha \in [f]$.

Fix α.

let $\bar{a} = \{y \in a : \alpha \in y\}$. $\bar{a} \subseteq a$ and $a \Vdash \bar{a}$.

($\bar{a}$ is club in a)

Suppose $c \leq \bar{a}$. i.e., $y \cap (\cup \bar{a}) \in \bar{a}$, $y \in c$.

Define $J : \mathcal{C} \to V$ by : $J(y) = b$ where $b \in A_\alpha \cap V_\lambda \cap y$ and $y \cap (\cup b) \in b$. b exists since $y \cap \cup a \in a$ and $\alpha \in y \cap \cup a \cap \delta$.

Choose $d \in c$ with $J \restriction d$ constant with value b_0.

So, if $y \in d$, $b_0 \in y \cap A_\alpha \cap V_\lambda$ and $y \cap (\cup b_0) \in b_0$.

But then, $d \leq b_0$. $b_0 \in A_\alpha$ hence $b_0 \Vdash \tau_\alpha = [F_\alpha(b_0)]$

But $d \Vdash [F_\alpha(b_0)] E [f]$ since for all $y \in d$, $(F_\alpha(b_0))(y \cap \cup b_0) \in f(y \cap \cup a)$.

The point is : $\alpha \in y$, $b_0 \in y$ for all $y \in d$. So, if $x = y \cap (\cup a)$, then $F_\alpha(b_0)(x \cap \cup b_0) \in f(x)$.

Thus, for all $\alpha < \delta$, $a \Vdash \tau_\alpha E [f]$. $\square$

As before, we can choose $G \subseteq P_{<\kappa}$ s.t. $cp(j)$ is any given regular cardinal $< \kappa$.

ex: let $a = \{ \alpha < \aleph_\omega^+ : cf(\alpha) = \omega_7 \}$

Then, $a \Vdash cp(j) = \aleph_\omega^+$ and

$$a \Vdash (cf((\aleph_\omega^+)^V) = \omega_7)^M.$$

But $j(\kappa) = \kappa$, and $M_\kappa = V[G]_\kappa$.

So, $a \Vdash cf((\aleph_\omega^+)^V) = \omega_7$ in $V[G]$.

$$a \Vdash (\aleph_i)^V = (\aleph_i)^{V[G]}, \text{ all } i \in \omega.$$

(the point is that this is a set forcing, not a class forcing as usual)

10/12/90

H. WOODIN

Suppose κ is Woodin, and suppose that $\delta < \kappa$ is a measurable card. Fix $\delta < \lambda < \kappa$.

Let $a = \{ X \prec V_\lambda : |X \cap V_\delta| = \delta$ and the transitive collapse of X is constructible from a real$\}$

(i.e. if $M = Coll(X)$, then $M \in L[y]$, some y)

<u>Claim</u>: a is stationary

<u>Pf. of Claim</u>: Fix $H : V_\lambda^{<\omega} \to V_\lambda$.

It suffices to find $X \prec V_\lambda$, $X \in a$ with X closed under H.

Fix $\lambda^* > \lambda$, $cf(\lambda^*) > \delta$. Choose $X_0 \prec V_{\lambda^*}$, X_0 countable with $\{\delta, \lambda, \mu, H\} \subseteq X_0$; where μ is a normal measure on δ.

Let $M_0 = Coll(X_0)$. Clearly, M_0 is constructible from a real.

Let $Z = \bigcap \{ A \subseteq \delta : A \in \mu \cap X_0 \}$

Let $Y = \{ F(s) : F \in X_0,\ s \subseteq Z,\ s \text{ finite} \}$

Since $cf(\lambda^*) > \delta$, it follows that $Y \prec V_{\lambda^*}$.

Also, $Z \subseteq Y$. So, $|Y \cap V_\delta| = \delta$.

Let $M = coll(Y)$.

We must check M is construct. from a real.

<u>Claim</u>: $M \in L(M_0)$.

Pf: $M = j_{0\delta}(M_0)$ where $j_{01} : M_0 \to Ult(M_0, \mu_0)$

μ_0 is the collapse of μ.

Also, elements of Z are indiscernibles in V_{λ^*} relative to elements of X_0.

So, this proves a is stationary. $\square$

Note: $V_\lambda \notin a$ since V_λ cannot be constructed from a real ($t^\#$ exists for all $t \in \mathbb{R}$).

Suppose $G \subseteq \mathbb{P}_{<\kappa}$ is V-generic and $a \in G$.
Let $j : V \to M \subset V[G]$ be the associated embedding.
It is easy to verify that $\mathrm{cp}(j) = \omega_1$.

$a \in G$. So, $j''(\cup a) \in j(a)$.
$x \in a \Rightarrow \mathrm{coll}(x)$ is constructible from a real.
$M \models x \in a \Rightarrow \mathrm{coll}(x)$ is const. from a real.

So, in $V[G]$, $\mathrm{coll}(j''\cup a)$ is construct. from a real.
$\cup a = V_\lambda$. So, $\mathrm{coll}(j''\cup a) = V_\lambda$.
Thus, in $V[G]$, $V_\lambda \in L[t]$ for some real t.
Further, $x \in a \Rightarrow |X \cap \delta| = \delta$. So, $a \Vdash j(\delta) = \delta$.
Hence, $M \models$ "δ is a measurable card."
But $\mathcal{P}(\delta)^M = \mathcal{P}(\delta)^{V[G]}$. So, in $V[G]$, δ is measurable.
So, by set forcing, we collapsed everything below λ to δ and preserved "δ is measurable".