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QUANTIFICATION AND THE EMPTY DOMAIN

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Quantification theory, or the first-order predicate calculus, is ordinarily so formulated as to provide as theorems all and only those formulas which come out true under all interpretations in all *non-empty* domains. There are two strong reasons for thus leaving aside the empty domain.

(i) Where D is any non-empty domain, any quantificational formula which comes out true under all interpretations in all domains larger than D will come out true also under all interpretations in D. (Cf. [4], p. 92.) Thus, though any small domain has a certain triviality, all but one of them, the empty domain, can be included without cost. To include the empty one, on the other hand, would mean surrendering some formulas which are valid everywhere else and thus generally useful.

(ii) An easy supplementary test enables us anyway, when we please, to decide whether a formula holds for the empty domain. We have only to mark the universal quantifications as true and the existential ones as false, and apply truth-table considerations.

Incidentally, the existence of that supplementary test shows that there is no difficulty in framing an *inclusive* quantification theory (i.e., inclusive of the empty domain) if we so desire. A proof in this theory can be made to consist simply of a proof in the exclusive (or usual) theory followed by a check by the method of (ii). We may, however, be curious to see a more direct or autonomous formulation: one which does not consist, like the above, of the exclusive theory plus a rule of expurgation. And, in fact, such formulations have of late been forthcoming: Mostowski [5], Hailperin [3], and, as part of a broader context, Church [2]. I shall not presuppose acquaintance with these papers, except in my final paragraph (and then only with [3]).

Quantification theory may or may not be so fashioned as to accord significance to *vacuous* quantification, i.e., to the attachment of a quantifier to a formula lacking free occurrences of the variable of the quantifier. The three versions cited above all do recognize vacuous quantification. Now admission of the empty domain raises a question regarding the vacuous universal quantification '$(x)f$' of a falsehood 'f': should we regard this quantification as false for the empty domain (on the ground that a vacuous quantifier is always simply redundant and omissible), or as true for the empty domain (on the ground that all universal quantifications are true for the empty domain)? Hailperin points out that Mostowski implicitly elected

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the first alternative, but that a more elegant system can be obtained by electing rather the second.

Hailperin is right, except that he has understated his case. If a general semantical characterization of the truth conditions of quantification over a domain is phrased along natural lines, without special mention of vacuousness of quantification or emptiness of domain, the verdict of truth of '$(x)f$' for the empty domain is pretty sure to follow. Moreover, this verdict is mandatory if, in general, the vacuous quantification '$(x)p$' is equated to '$(x)(p \,.\, Fx \supset Fx)$'; and we *must* so equate it if we are to preserve extensionality, since '$p \equiv . \, p \,.\, Fx \supset Fx$' is tautologous.

Hailperin recognizes a close kinship of his system of inclusive quantification theory to Church's. But in developing his system and establishing its properties he refers in detail to a system of exclusive quantification theory in my [6]. Now this exclusive system runs, in the revised edition of [6], as follows:

(1) If ϕ is tautologous (by truth tables), $\vdash\phi$.
(2) $\vdash\ulcorner(\alpha)(\phi \supset \psi) \supset . \, (\alpha)\phi \supset (\alpha)\psi\urcorner$.
(3) If α is not free in ϕ, $\vdash\ulcorner\phi \supset (\alpha)\phi\urcorner$.
(4) $\vdash\ulcorner(\alpha)\phi \supset \phi\frac{\beta}{\alpha}\urcorner$.
(5) If $\ulcorner\phi \supset \psi\urcorner$ and ϕ are theorems, so is ψ.

Here $\phi\frac{\beta}{\alpha}$ is understood as like ϕ except for containing free β in place of all free α; and '$\vdash\phi$' means that the *closure of* ϕ is a theorem. Free variables are not allowed in theorems.

In the first edition of [6] a further principle was included, viz.:

(6) $\vdash\ulcorner(\alpha)(\beta)\phi \supset (\beta)(\alpha)\phi\urcorner$.

However, Berry [1] showed that, by modifying an arbitrary detail in the underlying definition of closure, we can dispense with (6). This simplification, which I adopted in the revised edition, is a particularly striking one when we reflect that (6) was the only polyadic axiom-form in the lot.

Taking the system of my first edition as point of departure, hence (1)–(6), Hailperin builds his system of inclusive quantification theory from it by changing (4) to:

(7) $\vdash\ulcorner(\alpha)\phi \supset (\beta)\phi\frac{\beta}{\alpha}\urcorner$

and adding:

(8) If α is not free in ϕ, $\vdash\ulcorner\phi \supset (\alpha)\psi \,.\!\supset (\alpha)(\phi \supset \psi)\urcorner$.

Now the main point which I want to make is that Hailperin's inclusive theory, simpler though it is than Mostowski's, can be much simplified in turn. We can drop (6) by Berry's expedient. Also we can drop (8) if, instead of changing (4) to (7), we merely weaken (4) by prefixing the hypothesis that α is free in ϕ.

The resulting system of inclusive quantification theory is simply the exclusive system (1)–(5) with the words "If α is free in ϕ" inserted in (4). This insertion is wanted so as to except '$(x)f \supset f$', which fails for the empty domain.

The completeness of this system of inclusive quantification theory can be seen by substantially Hailperin's proof, which is easily adaptable to this system. Another fact which Hailperin establishes about his system, viz., that it can be strengthened to exclusive quantification theory by adding '$\vdash \ulcorner \sim(\alpha)f \urcorner$' (or say '$\vdash \ulcorner \sim(\alpha)(\phi \,.\, \sim\phi) \urcorner$'), is also demonstrable for our simpler system, again by substantially Hailperin's proof.

The required adaptation of Hailperin's proofs is two-fold, involving not only a switch from his system to our new one, but a switch also from the first edition of [6], which he cites for certain proof patterns, to the revised edition. In detail the changes are as follows. In his proofs of his Lemma 1 and Theorem 1, change "*104" to "*103". (This change merely reflects a change of numbering in the revised edition of [6].) In his proof of Theorem 1, drop the proof of Case 1 in favor of a mere citation of our weakened (4) above. In his proof of Lemma 3, change "*112" to "*115" and cite our weakened (4) instead of QE1. (These changes in the proof of Lemma 3 reflect both a change of numbering and a radical change of proof in the revised edition of [6].) In his proof of Theorem 2, read "QE0–6" as excluding QE1 and QE4–5 and as including the weakened (4).

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