713: Eighteen Lectures for Auckland second Semester 2025

Thomas Forster

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Contents

We have a lot to get through in these 18 lectures; there is not only the material that is specified in the syllabus, there is also the background on which that material relies and which may not have been earlier covered in a way that meets your current needs. Inevitably every lecturer tries to smuggle in extra material on top of what has been stipulated. I am no exception; however I have tried to flag such material for what it is: the final example sheet (which is not examinable) consists entirely of such stuff.

I am going to write out in full quite a lot of what you might otherwise expect to have to copy down off my slides or blackboard. I am going to make these notes available to you so that you can spend the lectures listening and thinking rather than mechanically transcribing.

I am going to assume that you have mastered all the material in **Maths 315**. If you are having difficulties with it, contact me, and i'll see what I can do to make your life easier.

I would like to thank Saul Miller for his close reading of these notes and his willingness to challenge me on omissions, infelicities, typos and the (fortunalely occasional) outright error! These notes have benefitted hugely from his strictures.

18 lectures over six weeks – weeks 1-6.

The rubric says:

> 1. Zorn's Lemma; 2. Propositional logic and the compactness theorem (includes applications of the compactness theorem to mathematics); 3. The Axioms of set theory; 4. Ordinals; 5. Cardinals; 6. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$; 7. First order logic and the compactness theorem; 8. First order theories; 9. Cardinality of models and elementary submodels.

However I shall not follow that order; I shall do them in the order:

• Ordinals (and cardinals);	three lectures
• Propositional Logic: Boolean algebras;	three lectures
• Zorn, Prime ideal theorem;	two lectures
• First order logic and the compactness theorem;	three lectures
• First order theories;	two lectures
• Set Theory;	three lectures
• $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ and their implementation into Set Theory;	one lecture
• Ehrenfeucht-Mostowski; Los's theorem;	one lecture
• Some Basic Model Theory: Countable Categoricity;	one lecture

The treatment of ordinals will use
`www.dpmms.cam.ac.uk/~tef10/ordinalsforwelly.pdf`;
the treatment of propositional Logic will follow
`www.dpmms.cam.ac.uk/~tef10/logiclectures2016.pdf`;
the treatment of set theory will use
`www.dpmms.cam.ac.uk/~tef10/axiomsofsettheory.pdf` and
`www.dpmms.cam.ac.uk/~tef10/ACpedagogy.pdf`.

The Toad never answered a word, or budged from his seat in the road; so they went to see what was the matter with him. They found him in a sort of trance, a happy smile on his face, his eyes still fixed on the dusty wake of their destroyer. At intervals he was still heard to murmer 'Poop-poop!'

Kenneth Graham *The Wind in The Willows*, chapter 2.

The Toad has just discovered Set Theory – i mean motor cars.

Warning! If you are reading this and your name isn't 'Thomas Forster' then you are eavesdropping; these notes are my messages to myself and are made available to you only on the off-chance that such availability might help you in preparing your own notes for this course. This warning doesn't mean that you shouldn't be reading this document, but you should bear it in mind anyway because I do not write out here in detail things I can do off the top of my head. The things that I write out are things that I might, in the heat of the moment, get wrong, or do in the wrong order – or forget altogether. Reading these notes is not a substitute for attending the lectures: it is an adjunct to them.

Chapter 1

Three Lectures on Ordinals and Cardinals

(Some of the arithmetic of ordinals and cardinals cannot really be done properly until we have some set theory under our belt. OTOH it's good to at least *introduce* the students to these ideas early on in the piece. The resulting exposition is inevitably somewhat disjointed.)

1.1 Ordinals

Cantor's discovery of a new kind of number. $1_{\mathbb{R}} \neq 1_{\mathbb{N}}$ etc etc. $1_{\mathbb{R}}$ is a multiplicative unit whereas $1_{\mathbb{N}}$ is the quantum of multiplicity ("how many?"). Brief chat about datatypes.

"How many times do I have to tell you to tidy up your room?" the answer will be an ordinal (possibly finite).

Cantor's discussion of closed sets of reals.

Ordinals measure the length of **discrete deterministic monotone processes**. (synchronous/asynchronous doesn't matter)

Well, we mean something slightly more than discrete . . . the set of stages has a total order, and it's always the case that the set of unreached stages has a first element. (There is always a *next* stage). Monotonicity ensures that it's always clear what the situation is that you are in, and determinism-and-discreteness means that there is always an immediately-next thing to do and that you know what it is.

$\omega,\ \omega + n,\ \omega + \omega,\ \omega \cdot n,\ \omega \cdot \omega,\ \omega^n$.

Ordinals are also the order types (isomorphism classes) of special kinds of total orders. ω is the order-type of $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$. (I write structures as tuples: carrier set followed by operations). Pick 0 off the front and put it on the end, get a bigger ordinal – but the underlying set is the same size – is the same *set*, indeed. The ordinal we get is $\omega + 1$, which illustrates how addition corresponds to concatenation.

But to understand order types we need to put the project into a more general context: *We need to do some logic.*

Congruence relations

You will be familiar with plenty of congruence relations and of their importance. However they might never have been identified for you as an object of study, so you might not know the expression.

A congruence relation is an equivalence relation that has been given a job to do. Every equivalence relation on a set A has a quotient – the set of equivalence classes. An equivalence relation on A is said to be a *congruence relation for* for some operation $f^n : A \to A$ if that f gives rise to an operation on the quotient, often called f too.

The *locus classicus* is the equivalence relation congruence-mod-p on $\mathbb{Z}$, which is a congruence relation for + and × (but not for exponentiation!)

Cardinals are very simple. Read my countability notes [2].

Multiplication, addition and exponentiation.

1.2 Wellfoundedness

Wellfoundedness is a very important property that a binary relation might have. Ordinals are the isomorphism classes of wellorderings, and a wellordering is a wellfounded total order, so there's one reason for paying attention to wellfoundedness. Another reason is that wellfounded relations support a kind of induction, and this kind of induction is quite important generally in Pure Mathematics, and specifically in Set Theory. Let us not delay our encounter with it any longer.

Suppose we have a carrier set with a binary relation R on it, and we want to be able to infer

$$\forall x\, \psi(x)$$

from

$$(\forall x)((\forall y)(R(y,x) \to \psi(y)) \to \psi(x))$$

In words, we want to be able to infer that everything is ψ from the news that you are ψ as long as all your R-predecessors are ψ. y **is an R-predecessor of** x if $R(y,x)$. Notice that there is no "case $n = 0$" clause in this more general form of induction: the premiss we are going to use implies immediately that a thing with no R-predecessors must have ψ. The expression "$(\forall y)(R(y,x) \to \psi(y))$" is called the **induction hypothesis**. The first line says that if the induction hypothesis is satisfied, then x is ψ too. Finally, the inference we are trying to draw is this: **if** x has ψ whenever the induction hypothesis is satisfied, **then** everything has ψ. When can we do this? We must try to identify some condition on R that is equivalent to the assertion that this is a legitimate inference to draw in general (i.e., for any predicate ψ).

Why should anyone want to draw such an inference? The antecedent says "x is ψ as long as all the immediate R-predecessors of x are ψ", and there are plenty of situations where we wish to be able to argue in this way. Take $R(x,y)$ to be "x is a parent of y", and then the inference from "children of blue-eyed parents have blue eyes" to "everyone has blue eyes" is an instance of the rule schematised above. As it happens, this is a case where the relation R in question does *not* satisfy the necessary condition, for it is

in fact the case that children of blue-eyed parents have blue eyes and yet not everyone is blue-eyed.

To find what the magic ingredient is, let us fix the relation R that we are interested in and suppose that the inference

$$\frac{(\forall y)(R(y,x) \to \psi(y)) \to \psi(x)}{(\forall x)(\psi(x))} \qquad R\text{-induction}$$

has failed for some choice ψ of predicate. Then we will see what this tells us about R. To say that R is well-founded all we have to do is stipulate that this failure (whatever it is) cannot happen for any choice of ψ.

Let ψ be some predicate for which the inference fails.

Then the top line is true and the bottom line is false. So $\{x : \neg\psi(x)\}$ is nonempty. Let us call this set A for short. Using the top line, let x be something with no R-predecessors. Then all R-predecessors of x are ψ (vacuously!) and therefore x is ψ too. This tells us that if y is something that is not ψ, *then there must be some y' such that $R(y', y)$ and y' is not ψ either.* If there were not, y would be ψ. This tells us that the collection A of things that are not ψ "has no R-least member" in the sense that everything in that collection has an R-predecessor in that collection. That is to say

$$(\forall x \in A)(\exists y \in A)(R(y,x))$$

To ensure that R-induction can be trusted it will suffice to impose on R the condition that $(\forall x \in A)(\exists y \in A)(R(y,x))$ never hold, for any nonempty $A \subseteq dom(R)$. Accordingly, we will attach great importance to the following condition on R:

DEFINITION 1
R is **well-founded** *iff for every nonempty subset A of $dom(R()$ we have*

$$(\exists x \in A)(\forall y \in A)(\neg R(y,x)).$$

(We say: x is an "R-minimal" element of A.)
A **wellfounded binary structure** *is a pair $\langle X, R\rangle$ where R is a wellfounded relation $\subseteq R^2$.*

This definition comes with a health warning: it is easy to misremember. The only reliable way to remember it correctly is to rerun in your mind the discussion we have gone through: well-foundedness is *precisely* the magic property one needs a relation R to have if one is to be able to do induction over R. No more and no less. The definition is not *memorable*, but it is *reconstructible.*

A fact that is basic but worth recording is:

REMARK 1 *If $\langle X, R\rangle$ is a wellfounded binary structure and is a homomorphic image of $\langle Y, S\rangle$ then $\langle Y, S\rangle$, too, is a wellfounded binary structure.*

Perhaps leave this as an exercise? A homomorphic image of a bad subset (one without a minimal element) also lacks a minimal element.

THEOREM 1 *Wellfounded induction: recursion on wellfounded relations*

Induction over a wellfounded relation is immediate. Justification of recursion requires a little thought.

Let $\langle X, R\rangle$ be a binary structure, with R wellfounded. Then the recursion

$$f(x) = G(x, \{f(x') : R(x', x)\})$$

has a unique solution as long as G is everywhere defined.

A niggle: why does G need to look at x? Why isn't it enough for it to look merely at $\{f(x') : R(x', x)\}$*?*

Answer:

two distinct xs might have the same R-predecessors and we want to keep open the possibility of f sending them to different things.

Fix f. We need the concept of the transitive closure of a relation. The transitive closure of R, written '$R*$' is the $\subseteq$-least transitive relation $\supseteq R$. However the clever idea which is specific to this proof is the concept of an **attempt**. An attempt-at-x is a function f_x which is defined at x and at every y such that $R^*(y, x)$, and obeys the recursion wherever it is defined. That is to say, if f_x is defined for all z s.t. $R(z, y)$, and it is defined at y, then we must have $f_x(y) = G(y, \{f_x(z) : R(z, y)\})$.

The concept of *attempt* is the only clever part of this proof. All that remains to be done is to choose the right thing to prove by induction. We prove by R-induction on 'x' that

(i) every x has an attempt-at-x; and that
(ii) all attempts-at-x agree at x and at all y such that $R^*(y, x)$.

Everything has been set up to make that easy.

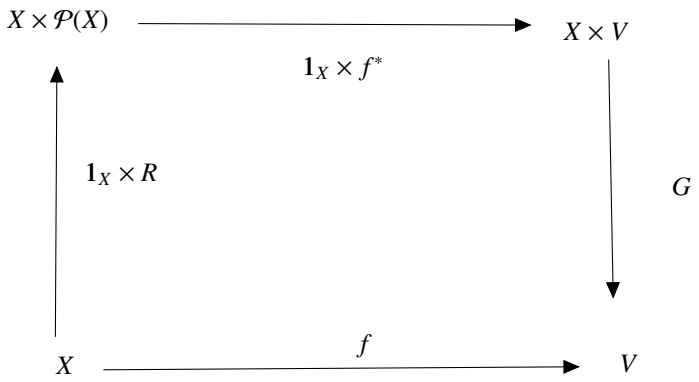
So: suppose the induction hypothesis holds for all y s.t. $R(y, x)$.

That is to say, for every y s.t. $R(y, x)$ there is f_y, an attempt-at-y, and all attempts-at-y agree on all y' s.t. $R^*(y', y)$.

Is there an attempt-at-x? Yes. We take the union of all the f_y for $R(y, x)$ and add the ordered pair that tells us to send x to $G(x, \{f_y(y) : R(y, x)\})$.

Then the function that we are declaring by this recursion is simply the function that, for each $x \in X$, sends it to whatever-it-is that all attempts-at-x want to send x to. This function is defined everywhere and it clearly obeys the recursion.

That is to say, for any set X with a wellfounded relation R on it, and every function $G : X \times V \to V$ there is a unique f making the following diagram commute.



DEFINITION 2 *Wellordering a wellfounded strict total order*

"every terminal segment has a least element" is equivalent. It's the "always an immediate next stage" condition.

COROLLARY 1 *Principle of induction for wellorderings*

COROLLARY 2 *Definition by recursion for wellorderings*

DEFINITION 3 *Ordinals are isomorphism types of wellorderings.*

Make a note of the fact that this is a definition of *ordinal* as an abstract mathematical object; it doesn't tell you what ordinals are *as sets.*

THEOREM 2

1. *Every wellordering is rigid (no nonidentity automorphisms);*
2. *If there is an isomorphism between two wellorderings $\langle A, <_A\rangle$ and $\langle B, <_B\rangle$ then it is unique;*
3. *Given two wellorderings $\langle A, <_A\rangle$ and $\langle B, <_B\rangle$ one is isomorphic to a unique initial segment of the other.*

Proof:

1. The automorphism group of a total order can have no nontrivial finite cycles – every nontrivial cycle looks like $\mathbb{Z}$ and can have no least element.
   If τ is an automorphism of a wellordering consider $\{\tau^n(x) : n \in \mathbb{Z}\}$.
   What is its least element?
2. Suppose σ and τ were two distinct isomorphisms $\langle A, <_A\rangle \to \langle B, <_B\rangle$;
   Then $\sigma \cdot \tau^{-1}$ would be a nontrivial automorphism of $\langle B, <_B\rangle$.

3. We define an isomorphism by recursion in the obvious way. It must exhaust either $\langle A, <_A\rangle$ or $\langle B, <_B\rangle$ and, by the earlier parts, it will be unique.

   To be slightly more formal about it, define $f : A \to B$ by the recursion $f(a) =: \sup\{f(a') : a' <_A a\}$ and $g : B \to A$ *mutatis mutandis*. We prove by wellfounded induction that $f \cdot g$ is the identity wherever it is defined. One of f and g must be total. If not, let a be the first thing not in the domain of f and b the first thing not in the domain of g. Then $\langle a, b\rangle$ should have been in f and $\langle b, a\rangle$ should have been in g.

■

We will give a slightly more detailed proof of part (3) later.

(3) tells us that the obvious order relation on ordinals is a total order. This is a nontrivial fact. The order relation on cardinals is not obviously a total order. We will talk about the order relation on cardinals later.

DEFINITION 4 $\langle X, \leq_X\rangle$ **is an end-extension of** $\langle Y, \leq_Y\rangle$ *iff*
(i) $Y \subseteq X$,
(ii) $\leq_Y \subseteq \leq_X$ *and*
(iii) $(\forall y \in Y)(\forall x \in X)(x \leq y \to x \in Y)$.

Alternatively (equivalently):
"$\langle Y, \leq_Y\rangle$ **is an initial segment of** $\langle X, \leq_X\rangle$*"*

Informally: "New stuff cannot be earlier than old stuff".

For the moment we use this only where $\langle Y, \leq_Y\rangle$ and $\langle X, \leq_X\rangle$ are wellorderings, but the idea is susceptible of generalisations to arbitrary posets and even to binary structures (models of set theory) where the binary relation ($\in$) is not even transitive. But that is for later.

[It might be worth making a song and dance about how a union of a chain of wellorderings under end-extension is a wellordering, and that one needs the family to be ordered by end-extension. We need this if we are to give a rigorous treatment of the limit step in the proof of theorem 4. It alos crops up in the proof of remark 8.]

LEMMA 1 *Every suborder of a wellorder is isomorphic to an initial segment of it.*

The suborder inherits totality and wellfoundedness so is a wellorder. Apply theorem 2.

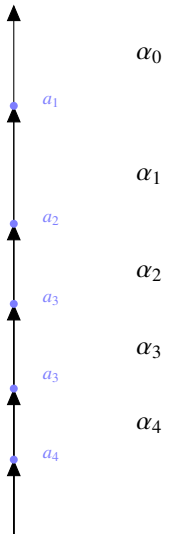
You might like to try Sheet 1 q 4 at this point.

Notice that this is not true of arbitrary total orders: $\mathbb{N}$ is a subordering of $\mathbb{Z}$ but is not iso to an initial segment.

DEFINITION 5 $\alpha \leq_{On} \beta$ *if every wellordering of length* β *(every wellordering whose equivalence class is* β*) has an initial segment of length* α.

(The two ways you might define it are equivalent)

And that initial segment is unique, as we have just seen.



THEOREM 3 *$<_{On}$ is wellfounded.*

Proof:

Let α be an ordinal. We will show that the ordinals below α are wellfounded. The long arrow represents a wellordering $\langle A, <_A \rangle$ of length $\alpha = \alpha_0$. If (*per impossibile*) there is a family $\{\alpha_i : i \in I\}$ of ordinals with no least member (and all of them $< \alpha$) then, for each $i \in I$, $\langle A, <_A \rangle$ has a (unique) proper initial segment of length α_i. For $i \in I$ let a_i be the supremum of that (unique) initial segment of $\langle A, <_A \rangle$ of length α_i. Then $\{a_i : i \in I\}$ is a subset of A with no $<_A$-least member. ■

I have drawn the picture as if the index set I were $\mathbb{N}$ and the sequence is strictly descending. We don't actually need this assumption but it does make the picture easier to draw. The assumption can be justified by appeal to a principle called DC: the principle of *dependent choice* of which more in section 4.1.

This result is nontrivial: it's not always true that the family of isomorphism types of widgets has a widget structure. Recall linear order types without wellfoundedness; not linearly ordered. (Not even antisymmetrical – think of $(0, 1)$ and $[0, 1]$.)

Beware! Some textbooks contain theorems with statements that sound like theorem 3 but are actually much weaker. Later we will see an implementation of ordinals into Set Theory, due to Von Neumann, and the textbooks contain proofs that the order relation on Von Neumann ordinals is wellfounded. This is true for completely banal reasons: the order relation on von Neumann ordinals is actually $\in$ – set membership. One of the axioms of the usual set theory (ZFC) is the axiom of foundation, which says that $\in$ is wellfounded ... so *of course* the order relation on (von Neumann) ordinals is wellfounded. (In fact if we define the class of von Neumann ordinals as the least class containing $\emptyset$ and closed under $\bigcup$ and $x \mapsto x \cup \{x\}$ then we can prove that the order relation on Von Neumann ordinals is wellfounded without using the axiom of foundation.)

However, a proof that the order relation on von Neumann ordinals is wellfounded is not a proof that $\langle On, <_{On}\rangle$ is wellfounded any more than a check that UBUNTU runs properly on my laptop means that it will run safely on yours. The fact that UBUNTU runs safely on my laptop is not a fact about UBUNTU but rather a fact about the binary for my machine, and that says nothing about the binary for your machine. One does not prove facts about abstract mathematical entities by reasoning about their implementations. Proving that von Neumann ordinals are wellordered by $\in$ doesn't prove that ordinals are wellordered by $<_{On}$ – it doesn't prove anything about ordinals at all: what it shows is that – to that extent at least – Von Neumann ordinals behave according to spec. In that sense it's a bit like the famous demonstration of Russell-and-Whitehead that the natural numbers in their system obey $1 + 1 = 2$. It's not a fact about *numbers*, it's a fact about *their system*.

Nevertheless the policy of claiming that the order relation on ordinals is well-founded by appealing to their implementation as von Neumann ordinals looks like a sensible short-cut if you are proposing to do all your mathematics by implementing it in ZF. Since this is the chief context in which people study ordinals people can be forgiven for looking for a quick hacky way of getting one of the basics sorted. *Tout comprendre, c'est tout pardonner.*

THEOREM 4 *Vital, central fact! (Cantor)*

Every ordinal is the order type of the set of ordinals below it in their natural order.

Equivalently, the order type of an initial segment of the ordinals is the least ordinal not in it.

Proof: You prove this by induction.

> I am endebted to Saul Miller for pressing me about the proof for the limit stage.
>
> Let λ be a limit ordinal. (Limit ordinals are always called λ, just like chief engineers are always called Scottie).
>
> Now suppose, for the induction, that for every $\beta < \lambda$, β counts the ordinals below β. What about the set of ordinals below λ? This is the union
>
> $$\bigcup_{\beta<\lambda}\{\gamma : \gamma < \beta\}$$
>
> It's a nested union. Each set $\{\gamma : \gamma < \beta\}$ that we are unioning has a wellordering, and the wellorderings on these sets agree, and each one is an end-extension of the earlier ones so when we take the union of them the order type of the union is the supremum of the order types. (There are exercises about this if you want to think harder about it: 8 and 12 on sheet 1 are related: you are right to be suspicious!) The order-types of these sets are just all the β below λ, so that supremum is λ as desired.
>
> The point is that the fact that these are end-extensions really matters. If i take the union of all the finite wellorderings of the intervals $[-n, 0]$ then their union is $\mathbb{Z}^-$ - the negative integers and that ain't a wellordering!
>
> The union of a chain of wellorderings under end-extension is another wellordering. It will do you no harm to think about why this is true ... what would a bottomless subset of the union look like?

■

COROLLARY 3 *(The Burali-Forti Paradox)*

The collection On of all ordinals cannot be a set.

Proof:

By thm 3 $\langle On, <_{on}\rangle$ is a wellordering. Since it is downward-closed, thm 4 tells us that its order type must be the least ordinal not in it. The least ordinal that is not a ordinal? I don't need this! Beam me up, Scottie. ■

Strictly speaking we cannot correctly state and prove these last two allegations until we know what a set of ordinals is. So this is a promissory note... to be redeemed when we do some set theory. In any case one can argue that corollary 3 goes deeper than set theory. That fact that *On* turns out not to be a set is an artefact of set-theoretic foundationalism. If we'd decided to think of ordinals in a less set-theoretic way we would've ended up with a different theorem. Here be dragons. see [4].

DEFINITION 6
Preorderings are transitive and reflexive;
A preorder is a set equipped with a preordering.
A Partial ordering is an antisymmetric preordering.

We assume the reader is familiar with disjoint unions, products and lexicographic products of posets.

Products not just of posets: remember we do products of groups. or of rings....

DEFINITION 7 *Addition and Multiplication of ordinals defined synthetically.*

[omitted; done live at the board]

Uniqueness of ordinal subtraction. What might we mean by '$\alpha - \beta$'? If $\beta \leq \alpha$ then whenever $\langle B, <_B\rangle$ belongs to β and $\langle A, <_A\rangle$ belongs to α then there is an isomorphism $\pi : \langle B, <_B\rangle$ to a unique initial segment of $\langle A, <_A\rangle$. The truncation

Explain $f``x$ notation

$$\langle A \setminus (\pi``B),\ <_A\restriction(A \setminus (\pi``B))\rangle$$

is our wellordering of length $\alpha - \beta$. This definition ensures that $\beta + (\alpha - \beta) = \alpha$.

Part 3 of theorem 2 reassures us that ordinal subtraction is uniquely defined.

We really do need wellfoundedness here. You'd think that $\omega^* - \omega^*$ would be 0, wouldn't you? But it can be any natural number. The set of negative integers has lots of initial segments of length ω^*.

We remark without proof that it is immediate from the definitions of addition and multiplication in terms of disjoint union and lexicographic product that both operations are associative, and that multiplication distributes over addition.

We need ordinal subtraction for Cantor Normal Forms.

So now we can do induction/recursion on ordinals.

DEFINITION 8 *cofinality; regular ordinal*

('regular' is topological jargon) You have never seen anything of cofinality $> \omega$.

Now might be a good time to attempt sheet 1 question 6 that says that every countable limit ordinal has cofinality ω.

DEFINITION 9
Recursive definition of addition, multiplication and exponentiation of ordinals.

$\alpha + 0 = \alpha$;
$\alpha + (\beta + 1) = (\alpha + \beta) + 1$, *and*
$\alpha + \sup(B) = \sup(\{\alpha + \beta : \beta \in B\})$.

$\alpha \cdot 0 = 0$; $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$, *and*
$\alpha \cdot \sup(B) = \sup(\{\alpha \cdot \beta : \beta \in B\})$.

$\alpha^0 = 1$; $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, *and* $\alpha^{sup(B)} = \sup(\{\alpha^\beta : \beta \in B\})$.

Remember which way round to write multiplication. Not commutative!!!

Wellorderings of length ω^ω: The set of polynomials in one variable with nonnegative integer coefficients ordered by dominance...

...and ϵ_0. A bit harder!

DEFINITION 10 *Countable ordinal*
A countable ordinal is the order type of a wellordering of $\mathbb{N}$.

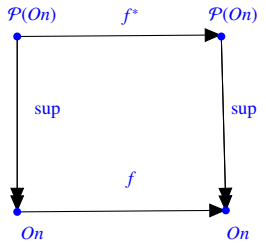
It's an immediate consequence of this definition, in conjunction with theorem 4, that an ordinal is countable iff there are countably many ordinals below it. This fact is too elementary to merit a label, but you need to internalise it. This absolutely must underpin your understanding of countable ordinals. Without it you would be entirely lost.

DEFINITION 11 *Normal functions*
A total function $f : On \to On$ *is* **normal** *if it is total, strictly increasing and continuous.*

The range of a normal function is a **clubset** *"closed unbounded set"*

The function that enumerates the members of a clubset in increasing order is normal.

"Continuous"? It means that the following diagram commutes.



"f^*" is a nonce notation for the function $X \mapsto f``X$. I don't expect to use it again.

Addition, multiplication and exponentiation on the Right are normal. Not on the Left!

LEMMA 2 *Division Algorithm for Normal Functions.*
If $f : On \to On$ *is normal, and* α *is any ordinal, then there is* β *such that*

$$f(\beta) \leq \alpha < f(\beta + 1).$$

Proof:

The β we want is $\sup\{\gamma : f(\gamma) \leq \alpha\}$. What is $f(\beta)$? By normality it must be $\sup\{f(\gamma) : f(\gamma) \leq \alpha\}$, which is clearly $\leq \alpha$. So β is not merely the *supremum* of $\{\gamma : f(\gamma) \leq \alpha\}$, it is actually the *largest element* of $\{\gamma : f(\gamma) \leq \alpha\}$. But then $f(\beta + 1)$ must be strictly greater than α. ■

ω is a *countable ordinal*. Observe that $\omega + 1$, ω^2 and lots of other ordinals are also countable. Are *all* ordinals perhaps countable . . . ? No!

THEOREM 5 *Hartogs' Lemma.*
For every set X there is a wellordered set Y s.t. $Y \not\hookrightarrow X$.

Proof:

Notice that – despite Cantor's theorem – $\mathcal{P}(X)$ will not do, beco's there is no reason to suppose that it can be wellordered. We can wellorder it if we have AC of course, but we want to keep our assumptions to a minimum.

We exhibit a uniform construction of such a Y.

Consider $\mathcal{P}(X \times X)$. This is the set of all binary relations on X. We define a map $f : \mathcal{P}(X \times X) \to On$. If $R \in \mathcal{P}(X \times X)$ is a wellordering we send it to its order type, its length; if it is not a wellordering we send it to 0. The range f"$(\mathcal{P}(X \times X))$ of f is the set Y that we want.

What is the cardinality of Y? Y is naturally wellordered, so what is its order-type in this ordering? Y is *downward-closed* so, by theorem 4 its order-type is the least ordinal not in Y. The ordinals in Y are precisely the ordinals of wellorderings of subsets of X. So the order type of Y is the least ordinal not the length of a wellordering of any subset of X. So Y is not the same size as any subset of X. *It's too big.*

■

This function is sometimes called 'Hartogs' aleph function'. Do not confuse this notation with the notation that gives subscripts to alephs: $\aleph_0$ is not $\aleph(0)$!

It's natural to ask specifically what happens if we do the construction of theorem 5 in the particular case where $X = \mathbb{N}$. The answer is that we get the set of countable ordinals, a set that Cantor called the *second number class*. It's an expression not much used nowadays, but it's helpful and evocative, particularly when you bear in mind that $\mathbb{N}$, the set of finite ordinals (or cardinals, in the finite world they're the same thing) is the *first* number class[1]. We need a name for the cardinal of this set: $\aleph_1$. The supremum of the second number class is the ordinal ω_1, the least uncountable ordinal.

We take up this thread again on page 72.

DEFINITION 12 *Rank functions for wellfounded (binary) structures.*

If $\langle X, R\rangle$ is a wellfounded binary structure we define:

$$\rho(x) = sup\{\rho(y) + 1 : R(y, x)\}.$$

(The intention is that $\rho(x)$ shall be the least ordinal > all the $\rho(y)$ for y Related to x.)

[1] It also sets you up to muse over what the third number class might be.

LEMMA 3 *Rank function is uniquely defined.*

Proof: By coroll 1.2.

■

Now would be a good moment to attack Sheet 1 q 11
Hartogs' tells you ∃ enuff ordinals.

Remember to make a big fuss about this

DEFINITION 13
Let us say that a homomorphism $h : \langle X, R\rangle \to \langle Y, S\rangle$ between wellfounded structures is **parsimonious** *if, for all $x \in X$, $h(x)$ is an S-minimal member of $\{y : (\forall x')(R(x', x) \to S(h(x'), y))\}$.*

The rank function on wellfounded sets is parsimonious.
This terminology is not standard, but it is useful.

If you want to know more about ordinals read
`www.dpmms.cam.ac.uk/~tef10/ordinalsforwelly.pdf`

Chapter 2

Posets and Fixed-Point Theorems

Poset you know; likewise toset; poset subsumes toset.
Pointwise and lexicographic products of posets already done.

DEFINITION 14
A complete poset is one every subset of which has a least upper bound
The expressions 'complete poset' and 'complete lattice' are used interchangeably.
A poset $\langle P, \leq \rangle$ *is directed if* $(\forall x, y)(\exists z)(x \leq z \wedge y \leq z)$.
A subset X *of a poset* $\langle P, \leq \rangle$ *is directed if* $(\forall x, y \in X)(\exists z \in X)(x \leq z \wedge y \leq z)$.
A poset is

- *directed-complete if every directed subset has a least upper bound.*
- *chain-complete if every chain has a least upper bound.*

If X is a countable set then $\langle \mathcal{P}_{\aleph_1}(X), \subseteq \rangle$ is simply $\langle \mathcal{P}(X), \subseteq \rangle$, which is a complete poset. (See definition 45 for this $\mathcal{P}_\kappa$ notation).

Every complete poset is directed. If X is uncountable then $\langle \mathcal{P}_{\aleph_1}(X), \subseteq \rangle$ is directed but not complete.

2.1 Lattices

We can define $\leq$ from $\wedge$, $\vee$ and $=$. In a distributive lattice $\vee$ and $\wedge$ distribute over each other. distributive: examples and non-examples. Subspaces of a vector space not distributive, nor is the lattice of partitions of a fixed set under refinement[1].

[1] Someone in lectures asked about the meaning of distributivity. A very good question!
If we have two binary operations o_1 and o_2 where o_1 distributes over o_2:

$$(\forall xyz)(o_1(x, o_2(y, z)) = o_2(o_1(x, y), o_1(x, z)))$$

what we are saying is that, for any x, the operation $y \mapsto o_1(x, y)$ is an endomorphism of the o_2 structure. If it's injective it'll be a *scaling factor*. Obvious when you think of it. For example, on the integers, multiplication by a fixed integer is an endomorphism of the additive structure of the integers.

Complemented lattice.
Boolean Algebra

Complete lattice. Power sets and topologies.
The regular open sets form a complete poset that is actually a b.a.
Aside here to explain subalgebra.

2.2 Separative poset – not examinable

A separative poset is one that is "as undirected as possible". In a directed poset any two points have an upper bound. Clearly we cannot say that no two (distinct) points have an upper bound, beco's if $x \leq y$ then anything $\geq y$ is an upper bound for both. What we can say is that in all other cases – where x is **not** $\leq y$ – there is $y' \geq y$ s.t. x and y' have no upper bound. Thus we say:

DEFINITION 15 *$\langle X, \leq \rangle$ is separative iff*

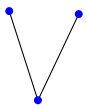
$$(\forall x, y \in X)(x \not\leq y \rightarrow (\exists z \geq y)(\forall w)(w \not\geq z \vee w \not\geq x))$$

We will abbreviate $(\forall w)(w \not\geq z \vee w \not\geq x))$ ("z and x have no upper bound") as $z \perp x$.
I think this notation is standard.

Actually that definition should probably be described in more detail as "upwardly separative". There is an analogous notion of *downwardly separative*. Naturally they are interchangeable in applications.

There are lots of separative posets and they matter: the idea of separative poset underlies the technique of *forcing* which Paul Cohen exploited to prove the independence of the Continuum Hypothesis from ZFC in 1963. All those posets are infinite.

Is the poset in this Hasse diagram separative?



2.3 Fun with Complete Posets

The Big Theorem concerning complete posets is Tarski-Knaster, theorem 6 below. It's more useful than you might at first guess, beco's complete posets are everywhere.
The reals are a complete poset under $<_{\mathbb{R}}$;
every power set is a complete poset under $\subseteq$;
every wellordering is a complete poset;
in any topological space the open sets are a complete poset under $\subseteq$.

In fact there is quite a nice approach to various topological concepts (product topology, quotient topology, that sort of thing ...) through the idea of complete posets, which i will now show you.

2.3.1 Topology

A topology on X is a sub-poset of the complete poset $\mathcal{P}(X)$. It is also a *complete* poset, though it is *not* a sub–complete-poset, co's the $\bigwedge$ operations of $\mathcal{P}(X)$ and of a topology on X are not the same (though the $\bigvee$ operations are the same.)

We think of a topology as the set of its open sets.

The collection of *regular* open subsets (a set is regular open if it is the interior of its closure) form a boolean algebra. These two posets are important in logic.

For each set X the set of topologies on X is a complete poset.

(So each topology is a complete poset and the collection of topologies on a fixed set is itself a complete poset. Do not get confused. Hold on to your hat!)

Huge mileage can be made from the fact that the property of being a topology is intersection-closed

See section 7.5.1 for a definition of intersection-closed.

For example: for any X and any $B \subseteq \mathcal{P}(X)$, the property of being a topology on X that contains all members of B is intersection-closed (obviously). The intersection of all topologies containing every element of B is clearly what we would ordinarily call the topology with basis B.

> The more open sets a topology has, the *finer* it is;
> The fewer open sets a topology has, the *coarser* it is.
>
> Every open set distinguishes its members from its nonmembers. So a topology with *more* open sets makes *finer* distinctions.
>
> Since 'coarser' means 'fewer open sets' so the coars**est** topology such that Φ should be the intersection of all topologies such that Φ (always assuming that Φ is intersection-closed).

The fact that the class of topologies on a fixed set is closed under arbitrary intersections underpins lots of definitions. We will consider the subspace topology, the product topology and the quotient topology. All three definitions rely on the set of topologies on a fixed set being a complete poset under inclusion, and each relies on some relevant property of topologies being intersection-closed. You may know the three definitions already, but a unifying story can gievn for them by exploiting the fact that the family f topologies an a fixed set is a complete poset under $\subseteq$.

DEFINITION 16 *The subspace topology*
The subspace topology on $Y \subseteq X$ is the $\subseteq$-least ("coarsest") topology on Y making the inclusion map continuous.

How do we know there is such a topology on Y?
Any topology on Y that makes the inclusion embedding continuous must contain $(\mathrm{id}_Y)^{-1}$"X' for any open $X' \subseteq X$. And $(\mathrm{id})_Y^{-1}$"X' is just $Y \cap X'$.
So the subset topology must be the topology generated by all the $Y \cap X'$ for X' open.

DEFINITION 17 *The Product Topology.*
Suppose $\{Y_i : i \in I\}$ *is a family of topologies.* $\prod_{i\in I} Y_i$ *is the set of all functions* h *with domain* I *s.t.* $h(i) \in Y_i$ *for each* $i \in I$*, and we want to give it a topology. What do we want this topology to do? For each* $j \in I$ *there is a "'projection map"* $\prod_{i\in I} Y_i \to Y_j$ *that sends each* $h \in \prod_{i\in I} Y_i$ *to* $h(j)$*. Then the product topology on* $\prod_{i\in I} Y_i$ *is defined to be the* $\subseteq$*-least ("coarsest") topology making all these projection maps continuous.*

Let's see what this actually amounts to. Let f_j be the projection map $\prod_{i\in I} Y_i \to Y_j$. We require that f_i is to be continuous. So let U be an open set in Y_i; then f_i^{-1}"U had better be an open subset of $\prod_{i\in I} Y_i$. That is to say

$$\{f \in \prod_{i\in I} Y_i : f(i) \in U\} \qquad (**)$$

must be an open set. And that is our only constraint. So the product topology on $\prod_{i\in I} Y_i$ is the intersection of all topologies on $\prod_{i\in I} Y_i$ that contain all the sets $(**)$ above. But that is simply to say that it is the topology generated by the sets $**$. "Coarsest"? Yes, the topology generated by a basis B is the coarsest topology containing all the open sets in B.

DEFINITION 18 Quotient topology.
The idea is this. Suppose I have a set X *equipped with a topology, and a set* Y*, not yet so equipped. I also have a set* F *of functions* $X \to Y$ *and i'm going to call them all continuous, just beco's I happen to feel like it. So i have to cook up a sensible topology on* Y *that makes them all continuous. When is a subset* $Y' \subseteq Y$ *to be open in this new topology on* Y*? Well,* Y' *had better not be open unless* f^{-1}"Y' *is an open subset of* X *for every* $f \in F$*. We couldsolve this by making no* Y' *(other than* Y *and* $\emptyset$ *of course) open, but that's not sensible. But if* f^{-1}"Y' *is, indeed, an open subset of* X *for every* $f \in F$ *then it is safe to admit* Y' *as an open set. So let's do that: make* Y' *open as long as* f^{-1}"Y' *is an open subset of* X *for every* $f \in F$*. By doing this we have plumped for the finest topology on* Y *that makes all the functions in* F *continuous. It's the union of all those topologies.*

With the definition of quotient topology and subset topology one is the finest and the other is the coarsest, and that's beco's they are on different sides of the arrow!

2.3.2 Knaster-Tarski: Fixed Points, Inductively-defined Sets and Cantor-Bernstein–like theorems

THEOREM 6 *Tarski-Knaster*
Let $\langle X, \leq\rangle$ *be a complete lattice and* f *an order-preserving map* $\langle X, \leq\rangle \to \langle X, \leq\rangle$*. Then* f *has a fixed point.*

Proof:

Set $A = \{x : f(x) \leq x\}$ and $a = \bigwedge A$. (A is nonempty because it must contain $\bigvee X$.)

That's the only part of the proof you need to *remember*, co's you can work the rest of it out from the definition of a.

But, for the sake of completeness, we continue ...

Since f is order-preserving, we certainly have $f(x) \leq x \to f^2(x) \leq f(x)$, and so $f(a)$ is also a lower bound for A as follows. Let $x \in A$ be arbitrary; we have $f(x) \leq x$, whence $f^2(x) \leq f(x)$, so $f(x) \in A$ and $a \leq f(x)$.

$$f(a) \leq^{(1)} f^2(x) \leq^{(2)} f(x) \leq^{(3)} x$$

(1) holds beco's $a \leq f(x)$ (as we've just showed) and f is order-preserving;
(2) holds beco's $f(x) \leq x$ and f is order-preserving;
(3) holds beco's $x \in A$.

...giving $f(a) \leq x$ as desired. But a was the *greatest* lower bound, so $f(a) \leq a$ and $a \in A$. But then $f(a) \in A$ since $f``A \subseteq A$, and $f(a) \geq a$ since a is the greatest lower bound. ■

Observe that this a is not only a fixed point, it is the *least* fixed point. Proving that it is the least fixed point is a useful miniexercise.

There are other fixed-point theorems of this flavour "a slick function from a nice poset into itself has lots of fixed points" (for example "every normal function from On to On has a fixed point") and we will deal with them as they come up, not all together. Notice that in theorem 6 we did not assume that f was continuous in the order topology. With that extra assumption we can get sharper results. However there is no space for them here.

Observe further that if $\langle X, \leq\rangle$ is a complete poset then so too, for any $a \in X$, is $\{x \in X : x \geq a\}$ equipped with the restriction of $\leq$, and it has the same sup and inf operations. It is a genuine sub–complete-poset. This has the immediate consequence that

COROLLARY 4
Let $\langle X, \leq\rangle$ be a complete lattice; let a be a member of X and let f be an order-preserving map $\langle X, \leq\rangle \to \langle X, \leq\rangle$. Then f has a fixed point $\geq a$.

and this in turn has the further corollary

COROLLARY 5
Let $\langle X, \leq\rangle$ be a complete lattice and f an order-preserving map $\langle X, \leq\rangle \to \langle X, \leq\rangle$.
Then f has a complete poset of fixed points.

Proof:

We need to show that every set of fixed points for f has a sup. So let A be a set of fixed points for f. Clearly A has a sup $\bigvee A$ beco's $\langle X, \leq\rangle$ is a complete lattice. Is this the thing we want? The obvious thing to do is to try to prove that it is a fixed point.

You will fail! However, all is not lost, because you use corollary 4 to show that there is a least fixed point above $\bigvee A$, and that fixed point is the one we want. ■

There is an echo here of the fuss I was making earlier about how the complete poset of open sets in a topology on a set X is not a sub–complete-poset of the power set of X. It's subtle but it matters. In both case the second structure (the open sets, the set of fixed points for the given increasing function) is a subposet of the first – it is a poset with the inherited structure. However (in both cases) the second structure does not have the same infinitary sup operations as the first *even tho' it does have an infinitary sup operation!*

The following is a perhaps more familiar-looking example. Fix an (infinite) vector space. Its power set is a complete poset under $\subseteq$. The collection of its *subspaces* is also a poset – a subposet of that power set indeed. It is also a complete poset. However the sup operation in the subposet of subspaces is not the same as in the power set. The sup operation in the power set is $\bigcup$; the sup of a hatful H of subspaces is not $\bigcup H$, but is the space *spanned by* $\bigcup H$.

2.3.3 Inductively defined Sets

This proof of theorem 6 shows not only that order-preserving functions have fixed points but that they have *least* fixed points. This gives us the existence of inductively defined sets because the operation of taking a set and adding to it the result of applying all the constructors once to all its members is order-preserving (with respect to $\subseteq$). The above definition of the element a echoes precisely the declaration of $\mathbb{N}$ as an intersection of a family of sets. Compare

$$\bigwedge\{x : f(x) \leq x\} \text{ with } \bigcap\{X : (S\text{“}X \cup \{0\}) \subseteq X\}.$$

There are three things you might worry about here:

(i) Is $\{X : (S\text{“}X \cup \{0\}) \subseteq X\}$ a set? Co's, if not, it isn't there for us to take $\bigcap$ of it;

(ii) if we want to use theorem 6 to deduce the existence of $\mathbb{N}$ then we seem to be using T-K on the complete poset of the power set of the set of cardinals, and is *that* a set?;

(iii) what is $S(\alpha)$ when α is a cardinal about which we know nothing?

(i) and (ii) you are not to worry about for the moment. These are set-theoretic issues which we will sort out later.

The answer to (iii) is that actually you know this already: $S(\alpha)$ is just $\alpha + 1$ which is $|x \cup \{y\}|$ whenever $|x| = \alpha$ and $y \notin x$. When we come to the axiom of choice we shall see that typically $S(\alpha) = \alpha$ for infinite α.

2.3.4 Cantor-Bernstein–like theorems

COROLLARY 6 *Cantor-Bernstein*
The preorder $\leq$ on cardinals is antisymmetric.

Proof:

Suppose $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$. Then the function $a \mapsto A \setminus g``(B \setminus f``a)$ is an order-preserving function $\mathcal{P}(A) \to \mathcal{P}(A)$ and will have a fixed point – call it $\mathcal{A}$.

Then the function

$$\texttt{if } a \in \mathcal{A} \texttt{ then } f(a) \texttt{ else } g^{-1}(a)$$

bijects A with B.

■

There are lower-tech proofs of corollary 6 that do not involve assuming that $\mathcal{P}(A)$ is a set – and you will find them in the older textbooks – but they are fiddly.

It may be worth noting that this proof of Cantor-Bernstein relies on excluded middle ("¿$a \in \mathcal{A}$?"). Cantor-Bernstein is not a constructive thesis.

Have a look at Sheet 3 question 1 at this point.

Other applications include Banach-Tarski.

Chapter 3

Three Lectures on Propositional Logic

The language. Propositional letters (aka *literals*): p, q, r ... or (better!) p, p', p'' ..., so that the set of literals forms a regular language. NB: the internal structure of the literals given by the prime symbol is not going to be visible to the semantics for the logic. 'p'''' is a single symbol not a string of four.

The letters point to things that evaluate to `true` and `false`. I am going to reserve the symbols '$\top$' and – even more important – '$\bot$' for propositional constants NOT for truth-values. The symbols '$\top$' and '$\bot$' are symbols in a propositional language. They are *reserved words* and they evaluate to `true` and `false`.

The letters are glued together with **connectives**: $\vee$, $\wedge$, $\rightarrow$, $\neg$, $\longleftrightarrow$, `XOR` ...
Set of wffs is context-free

Truth-functionality. Valuations and truth-tables.
Interdefinability of connectives.
Intension and extension. Now we can talk about $\rightarrow$.

3.1 if-then

A conditional is a binary connective that is an attempt to formalise a relation of implication. The word 'conditional' is also used (in a second sense) to denote a formula whose principal connective is a conditional (in the first sense). Thus we say both that '$\rightarrow$' is a conditional and that '$A \rightarrow B$' is a conditional. The conditional $\neg B \rightarrow \neg A$ is the **contrapositive** of the conditional $A \rightarrow B$, and the **converse** is $B \rightarrow A$. (cf., converse of a relation). A formula like $A \longleftrightarrow B$ is a **biconditional**.

The two components glued together in a conditional are the **antecedent** (from which one infers something) and the **consequent** (which is the something that one infers). In *modus ponens* one *affirms* the antecedent and then *infers* the consequent, thus:

$$\frac{A \to B \quad A}{B}$$

Modus tollens is the rule:

$$\frac{A \to B \quad \neg B}{\neg A}$$

where one denies the consequent and thereby refutes the antecedent.

Affirming the consequent and inferring the antecedent:

$$\frac{A \to B \quad B}{A}$$

is the **fallacy of affirming the consequent**. A **fallacy** is a defective inference. This particular fallacy is an important fallacy, for reasons that will emerge later.

Clearly we are going to have to find a way of talking about implication, or something like it. Given that we are resolved to have a purely truth-functional logic we will need a truth-functional connective that behaves like implies. ('Necessarily' is a lost cause but we will attempt to salvage *implies*). Any candidate must at least obey *modus ponens*:

$$\frac{A \quad A \to B}{B}$$

A useful clue to the answer comes from the combination of two thoughts:

(i) the thought that 'A if and only if B' (you will often see 'if and only if' written as 'iff') comes out true as long as A and B agree on their truth-value; and

(ii) the thought that A iff B must be $(A \to B) \wedge (B \to A)$.

So both $A \to B$ and $B \to A$ must come out true when A and B agree.

If A and B are both true or both false then $A \to B$ comes out true. We also know that $A \to B$ comes out false when A is true and B is false. This leaves to be decided only the case where A is false and B is true. A useful further thought is that $A \to B$ and $B \to A$ have to be different, but also that we can get each from the other by swapping A and B. This tells us that $A \to B$ must come out true when A is false and B is true.

Thus our resolve that 'if A then B' should be extensional has determined that $A \to B$ will be equivalent to '$\neg(A \wedge \neg B)$' which itself is equivalent to '$\neg A \vee B$'. $A \to B$ evaluates to `true` unless A evaluates to `true` and B evaluates to `false`; the connective $\to$ thus defined is the **material conditional**.

This does not solve the problem of identifying the intensional conditional (it doesn't even try) but it is surprisingly useful, and we can go a long way merely with an extensional conditional. Understanding the intensional conditional is a very hard problem, since it involves thinking about the internal structure of intensions and nobody really

has a clue about that. (This is connected to the fact that we do not really have robust criteria of identity for intensions). The literature it has spawned is vast and inconclusive, but we cannot avoid it altogether.

The idea is that we can use this strictly truth-functional stuff to codify arguments that only involve *and*, *or* and *not*, and don't involve *all* or *some*. The following example is from Kalish and Montague. It's a bit contrived but you get the idea.

> If God exists then He is omnipotent.
> If God exists then He is omniscient.
> If God exists then He is benevolent.
> If God can prevent evil then – if He knows that evil exists – then He is not benevolent if He does not prevent it.
> If God is omnipotent, then He can prevent evil.
> If God is omniscient then He knows that evil exists if it does indeed exist.
> Evil does not exist if God prevents it.
> Evil exists.
> ———————————
> *God does not exist.*

Here are the basic propositions and the letters we are going to abbreviate them to.

God exists	E
God is omnipotent	P
God is omniscient	O
God is benevolent	B
God can prevent Evil	D
God knows that Evil exists	K
God prevents Evil	J
Evil exists	V

If God exists then He is omnipotent.	$E \to P$	(1)
If God exists then He is omniscient.	$E \to O$	(2)
If God exists then He is benevolent.	$E \to B$	(3)
If God can prevent Evil then – if He knows that Evil exists – then He is not benevolent if He does not prevent it.	$D \to (K \to (\neg J \to \neg B))$	(4)
If God is omnipotent, He can prevent Evil.	$P \to D$	(5)
If God is omniscient then He knows that Evil exists if it does indeed exist.	$O \to (V \to K)$	(6)

Evil does not exist if God prevents it. $J \rightarrow \neg V$ (7)

Evil exists. V (8)

We want to persuade ourselves that God does not exist[1]. Well, suppose he does. Let's deduce a contradiction

Assume E. Then (1), (2) and (3) give us

P (9),

O (10)

and

B (11)

Now that we know O, (7) tells us that

$V \rightarrow K$ (12)

But we know V (that was (8)) so we know

K (13)

We know P, so (5) tells us that

D (14)

We can feed D into (4) and infer

$K \rightarrow (\neg J \rightarrow \neg B)$ (15)

But we know K (that was line 13) so we get

$\neg J \rightarrow \neg B$ (16)

(8) and (7) together tell us $\neg J$, so we get $\neg B$. But we got B at line 11.

DEFINITION 19 *Recursive definition of satisfaction*

A valuation *is a function from literals to truth-values. We define what it is for a valuation to satisfy a compound formula by recursion on the subformula relation which (you will have noticed) is wellfounded.*

v sat l [l a literal] iff v(l) = `true`*;*
v sat $\phi \wedge \psi$ *iff (v sat* ϕ *and v sat* ψ *)*
v sat $\phi \vee \psi$ *iff (v sat* ϕ *or v sat* ψ *)*
v sat $\phi \rightarrow \psi$ *iff (either not(v sat* ϕ*) or v sat* ψ *)*
v sat $\neg\phi$ *iff not(v sat* ϕ*).*

We say $\phi \models \psi$ *iff every valuation that sat* ϕ *also sat* ψ*.*

[1] Purely for the sake of argument, you understand!

Semantic entailment and validity

"true under all valuations"; "tautology"

Logical equivalence: two formulæ are logically equivalent iff they are satisfied by the same valuations.

DEFINITION 20

A **theory** *is a set of sentences, closed under some notion of deducibility clear from context.*
A **Logic** *is a theory closed under substitution.*

A Logic that contains (to take a pertinant example) $A \to (B \to A)$ must contain all substitution instances of it, such as: $p \to (p \to p)$, or $(p \vee q) \to ((q \vee r) \to (p \vee q))$

Here is an example of a propositional theory. We might call it the theory of adding two eight-bit words (with overflow). It has 24 propositional letters, p_0 to p_7, p_8 to p_{15} and p_{16} to p_{23}, and axioms to say that p_{16} to p_{23} represent the output of an adding the binary number p_0 to p_7 to the binary number p_8 to p_{15}. `true` is 1 and `false` is 0, so it contains things like $((p_0 \wedge p_8) \to \neg p_{16})$ (because an odd plus an odd is an even!).

	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0
+	p_{15}	p_{14}	p_{13}	p_{12}	p_{11}	p_{10}	p_9	p_8
=	p_{23}	p_{22}	p_{21}	p_{20}	p_{19}	p_{18}	p_{17}	p_{16}

Notice that this is a theory not a logic, co's it's not closed under substitution. It contains $(p_0 \wedge p_8) \to \neg p_{16}$ but not (for example) $(p_1 \wedge p_9) \to \neg p_{17}$ which is obtained from it by the substitution: $p_0 \mapsto p_1$, $p_8 \mapsto p_9$ and $p_{16} \mapsto p_{17}$.

DEFINITION 21 *Any set T (a theory or a Logic) of axioms-and-rules-of-inference gives rise to a deducibility relation written '$\vdash$': "$T \vdash \phi$" to mean that ϕ can be deduced using the T-machinery. Sometimes people write "$\psi \vdash_T \phi$" to mean that ϕ can be deduced from ψ using the T-machinery.*

Theories and Logics usually (tho' not always) arise from a set of axioms and a set of rules of inference. Thus, considered as sets of formulæ they are what we call *recursively enumerable*. We say they are *axiomatised*.

The logic consisting of all valid formulæ of propositional logic (all tautologies) does not on the face of it arise in this way. It is a nontrivial fact that by a judicious choice of theory (Logic) we can get $\models$ and $\vdash$ to coincide. Particular set of axioms-and-rules doesn't matter; what matters is that it can be done ... *proof of concept*.

We'll have a brief look at some alternative rules of inference, so that we get some idea of the generality of a propositional theory.

Natural deduction!

Brief chat about completeness theorems. Kuratowski's theorem about planar graphs.

Can you detect semantic validity just by looking at the syntax, without looking at the models? Talk about the biconditional fragment.

REMARK 2 *Completeness for the Biconditional Fragment*
For a formula ϕ in the language with only $\longleftrightarrow$ and $\bot$ the following are equivalent

- *ϕ is valid (= satisfied by all valuations);*
- *Every literal in ϕ appears an even number of times;*
- *ϕ is derivable from the three axiom schemes (all substitution instances of)*
  $A \longleftrightarrow A$;
  $(A \longleftrightarrow B) \longleftrightarrow (B \longleftrightarrow A)$ and
  $(A \longleftrightarrow (B \longleftrightarrow C)) \longleftrightarrow ((A \longleftrightarrow B) \longleftrightarrow C)$.
  And your sole rule of inference is modus ponens.

Whatever your axioms and rules of inference are, it's going to be pretty easy to show that $\Gamma \vdash \phi$ implies $\Gamma \models \phi$; it's the other direction that is hard. In the biconditional logic case it's easy to see that anything deduced from the three axiom schemes by *modus ponens* has an even number of occurrences of every propositional letter. It's the other direction that is hard. [There is an extension of this to the logic with negation as well, but I can't remember what the axiom for $\neg$ is: it may be $\neg(A \longleftrightarrow \neg A)$. It may even be that if we have '$\bot$' as a propositional constant in the language we don't actually need any more axioms. You may like to check.]

Remark 2 gives us a very nice illustration of a completeness theorem. It says that the *semantic* property of being valid (which involves checking a formula against valuations (which are of course external to the formula) is the same as the *syntactic* property of containing an even number of occurrences of every letter (which can be checked merely by looking inside the formula) are the same! There is another respect (which i have already mentioned) in which this completeness result is typical: the direction [syntactic property] implies [semantic property] is an easy induction; it's the other direction that is hard.

If we are going to prove that $\models$ and $\vdash$ coincide, we'd better have precise mathematical definitions of them. We know what $\models$ is. So we need to be clear about $\vdash$.

We also need to be crystal-clear about what a proof is.

Because we are short of time I am going to use an axiomatisation-with-rule-of-inference kit that gives a slick proof of completeness. I have in fact shamelessly lifted it from [6] ch 2.

Brief chat about **Interdefinability of connectives classically** (could've been done earlier)

We don't exploit interdefinability in natural deduction!

We have three axiom schemes, K, S and T plus *modus ponens*. Or three axioms plus a rule of substitution plus *modus ponens*.

DEFINITION 22
K: $A \to (B \to A)$;
S: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$' and
T: $((A \to \bot) \to \bot) \to A$;
[Define Hilbert-style proof]

A singleton list containing an axiom is a proof. What about the empty list?

T is the characteristic axiom for **classical Logic**: double negation and law of excluded middle. Not everybody likes these two axioms, so it's nice to have an axiomatisation which lists them separately so they can be dropped if we want.

K and S enable us to prove the "deduction theorem"...

First, a notational innovation you will have to get used to: people often write 'L, A' for '$L \cup \{A\}$'.

DEFINITION 23 *The* **Deduction Theorem** *for a logic L is the assertion*

$$\textit{if } L, A \vdash B, \textit{ then } L \vdash A \to B.$$

The converse is trivial as long as L has *modus ponens.*

THEOREM 7 *The deduction theorem holds for L iff L contains (all substitution instances of) K and S.*

Proof:

$L \to R$ The left-to-right direction is easy, for we can use the deduction theorem to construct proofs of K and S. This we do as follows:

$$L \vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

(which is what we want) holds iff (by the deduction theorem)

$$L \cup \{(A \to (B \to C))\} \vdash ((A \to B) \to (A \to C))$$

iff (by the deduction theorem)

$$L \cup \{(A \to (B \to C)), (A \to B)\} \vdash (A \to C)$$

iff (by the deduction theorem)

$$L \cup \{(A \to (B \to C)), (A \to B), A\} \vdash C.$$

But this last one we can certainly do, since

$$[(A \to (B \to C));\ (A \to B);\ A;\ (B \to C);\ B;\ C]$$

is a Hilbert-proof of C from $L \cup \{(A \to (B \to C)), (A \to B), A\}$ (and we have already seen how to do this by natural deduction).

We also want $L \vdash A \to (B \to A)$. This holds (by the deduction theorem) iff $L \cup \{A\} \vdash (B \to A)$ iff (by the deduction theorem again) $L \cup \{A, B\} \vdash A$.

$R \to L$ Suppose $L, A \vdash B$. That is to say, there is a (Hilbert) proof of B in which A is allowed as an extra axiom. Let the ith member of this list be B_i. We prove by induction on i that $L \vdash A \to B_i$. $B_i \to (A \to B_i)$ is always a (substitution instance of) an axiom (because of K), so if B_i is an axiom, we have $L \vdash A \to B_i$ by *modus ponens*. If B_i is A, this follows because $L \vdash A \to A$. If B_i is obtained by *modus ponens* from two earlier things in the list, say B_j and $B_j \to B_i$ then, by induction hypothesis, we have $L \vdash A \to B_j$ and $L \vdash A \to (B_j \to B_i)$. But by S this second formula gives us $L \vdash (A \to B_j) \to (A \to B_i)$ and then $L \vdash A \to B_i$ by *modus ponens*.

■

From now on we are going to assume that our only rules of inference are *modus ponens* and substitution. Thus when we write "$\Gamma \vdash \phi$" we mean that if we add to Γ all substitution-instances of K, S and T, and close under *modus ponens* then we can find a Hilbert-style proof of ϕ.

THEOREM 8 *The Adequacy Theorem*
Let Γ be a set of expressions in a propositional language.
If $\Gamma \models \bot$ then $\Gamma \vdash \bot$.

Proof:

"Contrapositive"? (first occurrence)

We prove the contrapositive. Suppose $\Gamma \nvdash \bot$. We propose to infer $\Gamma \not\models \bot$. $\Gamma \models \bot$ says that any valuation that satisfies Γ satisfies $\bot$, but of course no valuation satisfies $\bot$, so $S \models \bot$ says that no valuation satisfies Γ. So the challenge is to find a valuation that satisfies Γ, given that we cannot deduce $\bot$ from Γ.

The idea is to construct a sequence $\Gamma_0, \Gamma_1, \Gamma_2 \ldots$ s.t. $\Gamma_i \nvdash \bot$ for each i, and such that $\Gamma_\omega = \bigcup_{i<\omega} S_i$ "decides" every formula... by which we mean that, for each formula ϕ, either $\Gamma_\omega \vdash \phi$ or $\Gamma_\omega \vdash \neg\phi$. To do this we enumerate the expressions of the language in order type ω as $\langle t_i : i \in \mathbb{N}\rangle$.

(If you have done your first-year revision exercises you will be aware that the set of expressions is countable. This is because the set of finite sequences from a countable set is countable. You can prove this using the prime powers trick.)

Given Γ_i we obtain Γ_{i+1} by asking whether or not $\Gamma_i \cup \{t_{i+1}\} \vdash \bot$. If $\Gamma_i \cup \{t_{i+1}\} \nvdash \bot$ then set $\Gamma_{i+1} = \Gamma_i \cup \{t_{i+1}\}$, `else` $\Gamma_{i+1} = \Gamma_i$.

The valuation we want is now the valuation that sends every literal in the deductive closure of $\bigcup_{i\in\mathbb{N}} \Gamma_i$ to `true` and sends all others to `false`. ■

We can now obtain the completeness theorem as a corollary. We prove only the hard direction.

COROLLARY 7 *The Completeness Theorem for Propositional Logic.*
If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

Proof:

Suppose $\Gamma \models \phi$. Then we must have $\Gamma \cup \{\neg\phi\} \models \bot$. (No valuation can satisfy both ϕ *and* $\neg\phi$ even if it tries with both hands). Then, by theorem 8, the Adequacy Theorem, we have $\Gamma \cup \{\neg\phi\} \vdash \bot$.

Now, by the Deduction Theorem, we have $\Gamma \vdash \neg\phi \rightarrow \bot$. Finally axiom T allows us to infer ϕ. ■

The combination of axioms and rule of inference used here was chosen precisely to expedite this particular proof of completeness: K and S give you the deduction theorem, and axiom T provides the final step. Other combinations will give different proofs. There are presentations of propositional logic that are more natural and easier to *use* but they make the completeness theorem much harder to prove.

We obtain as a corollary the compactness theorem.

Consider a propositional language with a countable infinity of literals. We can topologise the set of all valuations by declaring, for each finite set x of pairs $\langle l, t\rangle$ where l is a literal and t is a truth value, that the set $\{v : x \subset v\}$ (thinking of valuations as sets of ordered pairs) is a basis element[2]

Why is the compactness theorem for propositional logic like the compactness of the space of valuations? The space of valuations is compact. That is beco's it is the product of lots of copies of the two-point space (one copy for each propositional letter) and the two-point space is compact. And a product of compact spaces is compact. (That's Tikhonov – in fact a subtly weaker version of Tikhonov that sez that a product of compact *Hausdorff* spaces is compact *Hausdorff*). For any propositional formula ϕ the set $[[\phi]]$ of valuations making it true is closed (in fact clopen). Suppose now that Γ is an inconsistent set of formulæ. Then $\{[[\phi]] : \phi \in \Gamma\}$ is a family of closed sets with empty intersection. So some finite subset of it has empty intersection. So there is a finite $\Gamma' \subseteq \Gamma$ with $\Gamma' \models \bot$.

Clearly, proofs being finite objects, if there is a proof of ϕ from Γ, then there is a proof that uses only finitely many formulæ in Γ. But, by corollary 7 (which tells us that $\vdash$ and $\models$ are the same relation) it then follows that if $\Gamma \models \phi$ then $\Gamma' \models \phi$ for some finite subset $\Gamma' \subseteq \Gamma$. We'd better give this a name and a number:

COROLLARY 8 *The Compactness Theorem (for Propositional Logic)*
If $\Gamma \models \phi$ *then there is* $\phi' \subseteq \Gamma$, Γ' *finite, with* $\Gamma' \models \phi$.

One consequence of the completeness theorem for propositional logic is that both "ϕ is a tautology" and "ϕ is not a tautology" become what one might call *existential* sentences. "ϕ is a tautology" becomes "there is a p s.t. p is a proof of ϕ" and "ϕ is not a tautology" becomes "there is a valuation that refutes ϕ".

This two-pronged attack looks useful if we are looking for efficient engines that answer whether or not a propositional formula is a tautology. Clearly we have a deterministic algorithm that runs in time exponential in the number of distinct propositional letters in ϕ: simply examine all valuations. Clearly any valuation that refutes ϕ can

[2] You might think that we need these finite sets of pairs to be partial functions but actually we don't: it all comes out in the wash.

be shown to do so in time linear in the length of ϕ. Thus non-tautologousness is what they call *nondeterministic polynomial*. What about tautologousness? ϕ is tautologous iff there is a proof of it. But can a correct proof be verified in time polynomial in the length of ϕ? The question is: "is there a system of rules and axioms with the feature that there is a polynomial P in one variable such that every tautology of length n has a proof in that system of length less than $P(n)$?". Curiously this question seems to be open.

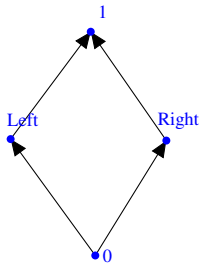
Can detect tautologousness by exhaustive search co's only finitely many cases. Spuriously easy. So consider the truth value algebra $\mathcal{P}(\mathbb{N})$.

3.2 Boolean Algebras detect propositional Tautologies

The most salient boolean algebra is the two-element boolean algebra, sometimes (as here) notated $\mathbb{2}$.

What do we mean by this title? Suppose you try building truth tables using a boolean algebra B instead of $\mathbb{2}$. Each row of such a truth-table corresponds to what one might call a *B-valuation* – a thing like an ordinary valuation except that it takes values in B. You'll end up with $|B|^n$ rows (assuming your formula has n distinct letters in it) instead of 2^n rows, so it's not the kind of thing you would want to do unless you had a compelling reason! Let us say that a boolean algebra B **authenticates** a propositional formula ϕ if every row of this truth table puts `true` under the main connective of ϕ.

Here is a Hasse diagram of the four-element Boolean Algebra.



and a four-valued truth-table for $p \vee (\neg p)$

p	$\vee$	$(\neg$	$p)$
1	1	0	1
Left	1	Right	Left
Right	1	Left	Right
0	1	1	0

We know what $\wedge$, $\vee$ and $\neg$ are in a boolean algebra, but we defined $\rightarrow$ purely in terms of its two-valued truth-table. So let us say that $p \rightarrow q$ is short for $(\neg p) \vee q$.

So s'pose I fill in a truth table for a formula ϕ using a boolean algebra B. I have $|B|^n$ rows. 2^n of those rows are rows in which the letters take only values $\top$ and $\bot$ (or `true` and `false`). So, if B authenticates ϕ, then $\mathbb{2}$ likewise authenticates ϕ; in plain English, ϕ is a tautology.

So "authenticated by B" implies "authenticated by $\mathbb{2}$" (=tautology)

Now (for the other direction) suppose that B does not authenticate ϕ. Then there is $b \in B$, $b \neq \top$, and a row of the truth-table (a valuation, call it v) under which ϕ receives the value b. Consider now any maximal ideal I that contains b and the homomorphism h onto the quotient (which is $\mathbb{2}$) whose kernel is I. This homomorphism h *kills* b – sends it to $\bot$. Then $h \cdot v$ is a $\mathbb{2}$-valuation that makes ϕ false.

So "refuted by B" implies "refuted by $\mathbb{2}$" (= not a tautology)

Conclusion:

We defined a tautology to be anything authenticated by the two-element boolean algebra $\mathbb{2}$, but we could, for any boolean algebra B, have defined a tautology to be anything authenticated by B. Authenticated by one is the same as authenticated by all. So it suffices to check authentication by $\mathbb{2}$.

Thus all boolean algebras detect the same set of tautologies. We'd better minute this fact.

REMARK 3 *We can define a propositional tautology as*
- *"authenticated by all Boolean Algebras" or*
- *"authenticated by even one Boolean Algebra";*

– it makes no difference.

In fact we can characterise boolean algebras as those things with $\wedge$, $\vee$, 0 and 1 etc that authenticate all propositional tautologies, but there is no space to prove this fact here.

This does lead to some interesting mathematics. Think of 'boolean algebra' in the above discussion as occupying a variable slot. There may be other kinds of algebras with $\vee$, $\wedge$ etc. Each such kind gives us the set of things that it authenticates, and these can be used to characterise other logics.

3.3 Applications of Propositional Compactness

Other examples of propositional theories, and applications of propositional compactness.

A group is (right)-orderable if it admits an order $\leq$ such that

$$(\forall a, b, c)(a \leq b \ \rightarrow \ a \cdot c \ \leq \ b \cdot c).$$

(*Think* ... additive group of $\mathbb{Q}$, multiplicative group of $\mathbb{R}^+$ ...) They tend to be abelian so we write the group operation with a '+'.

REMARK 4
A group is right-orderable iff all its finitely generated subgroups are right-orderable.

Proof:

One direction is easy: if G is right-orderable so are all its subgroups, in particular all its finitely generated subgroups.

Obvious tho' this is, it's a useful snotty-logician-opportunity to make the point that this happens beco's the theory of orderable groups is $\forall^*$, so the class of its models is closed under substructure.

For the converse we consider an arbitrary group G and set up a propositional language and exploit propositional compactness. The language has, for each pair of distinct elements $a, b \in G$, a propositional letter $p_{a,b}$. (Secretly the meaning of $p_{a,b}$ is that $a < b$). The propositional theory to which we are going to apply compactness has the three axiom schemes:

For all $a, b, c \in G$

$p_{a,b} \rightarrow p_{ac,bc}$
$p_{a,b} \rightarrow (p_{b,c} \rightarrow p_{a,c})$
$p_{a,b}$ XOR $p_{b,a}$ for $a \neq b$

(thanks to Dean Miller for spotting that the third clause needs the extra condition). The first scheme states that the order respects group multiplication, and the second and third assert that the order is total.

If we assume that every finitely generated subgroup of G is orderable then any finite set T' of these axioms is consistent beco's each finite subset mentions only finitely many elements of G. For each such T' consider the subgroup $G^{(T')}$ of G generated by the elements mentioned in the subscripts of the propositional letters appearing in T'. This is a finitely generated subgroup of G and is accordingly orderable by hypothesis. Any ordering of $G^{(T')}$ gives a valuation which satisfies T'. ■

This is one of various standard applications of propositional compactness. Others are ...

(i) *The order extension principle*: every partial order on a set can be refined to a total order;

(ii) If every finite subgraph of a graph is n-colourable then the graph itself is n-colourable.

We will keep these up our sleeve for example-sheet questions. However there are two that we will write out in detail.

3.3.1 The Interpolation Lemma

More of a remark than a lemma but it's always called a lemma so i'll go with the flow.

But first, a bit of notation. For any set Γ of formulæ, let $\mathcal{L}(\Gamma)$ be the set of propositional formulæ built up from literals in Γ. The '$\mathcal{L}$' connotes $\mathcal{L}$anguage.

LEMMA 4 *Let P, Q, R be three pairwise-disjoint sets of literals.*

Let $\phi \in \mathcal{L}(P \cup Q)$ and $\psi \in \mathcal{L}(Q \cup R)$ be formulæ such that $(\phi \to \psi)$ is a theorem of the propositional calculus.

Then there is a formula $\theta \in \mathcal{L}(Q)$ such that both $(\phi \to \theta)$ and $(\theta \to \psi)$ are theorems.

The formula θ is an **interpolant**.

Proof:

Consider the set $\Gamma = \{\gamma \in \mathcal{L}(Q) : \vdash (\phi \to \gamma)\}$ of Q-consequences of ϕ. The idea is to show that this set entails ψ, and that therefore (by compactness) some finite subset of it entails ψ, and the conjunction of that finite subset will be the θ we seek.

If Γ is to entail ψ we want every valuation that satisfies Γ to satisfy ψ. Now we do know that every valuation that satisfies ϕ also satisfies ψ, so it will be sufficient to show that any Q-valuation that satisfies Γ can be extended to a $(P \cup Q)$-valuation that satisifies ϕ.

We argue by contradiction. Suppose there is a Q-valuation v that satisfies Γ but cannot be extended to one that satisfies ϕ. Consider the set

$$\{p : v(p) = \texttt{true}\} \cup \{\neg p : v(p) = \texttt{false}\} \qquad \text{(A)}$$

This set entails all Q-consequences of ϕ but refutes ϕ itself. So some finite subset Δ of it refutes ϕ. Contraposing we have $\phi \to \neg \bigwedge \Delta$. But $\neg \bigwedge \Delta$ is a Q-consequence of ϕ and is therefore satisfied by v ... which is to say is implied by (A).
So there is no such v.

■

Observe that this proof does not tell us how to find such a θ: it merely tells us there is one. Example sheet 3 question 5 will guide you through a more effective proof that enables you to actually compute θ from ϕ and ψ. I quote:

> "(a)$^+$ Suppose A is a propositional formula and 'p' is a letter appearing in A. Explain how to find formulæ A_1 and A_2 not containing 'p' such that A is logically equivalent to $(A_1 \wedge p) \vee (A_2 \wedge \neg p)$.
>
> (b) Hence or otherwise establish that, for any two propositional formulæ A and B with $A \vdash B$, there is a formula C, containing only those propositional letters common to both A and B, such that $A \vdash C$ and $C \vdash B$. (Hint: for the base case of the induction on the size of the common vocabulary you will need to think about expressions over the empty vocabulary)"

There is the possibility of a nice further exercise at this point. Challenge the student: suppose the propositional letter p' has only positive occurrences in A and in B. Can you be sure of finding an interpolant in which it appears only positively? What results can you prove along these lines..?

Notice the way in which we wellorder the language in the proof of the completeness theorem. That's all right if we have only countably many literals, but it will require nontrivial choice assumptions if the set of literals is uncountable. Can we recover any of the extra strength of those nontrivial choice assumptions that we have to put in if we are to prove compactness for uncountable propositional languages, and use it to prove results of interest? Yes!

REMARK 5
Compactness for arbitrary propositional languages implies that every boolean algebra has an ultrafilter.

Actually we don't yet know what an ultrafilter is. Look at definition 26.

Proof:

Let B be a Boolean Algebra. For each $b \in B$ create a propositional letter $\mathcal{U}_b$ whose meaning is secretly that b belongs to the ultrafilter whose existence we are trying to prove. We set up a propositional theory $\mathcal{U}_B$. It contains $\mathcal{U}_\top$, $\neg\mathcal{U}_\bot$; for each $b \in B$ it contains $\mathcal{U}_b$ XOR $\mathcal{U}_{\neg b}$; whenever $a = b \vee c$ then it contains $\mathcal{U}_a \to (\mathcal{U}_b \vee \mathcal{U}_c)$ and if $a \leq b$ it contains $\mathcal{U}_a \to \mathcal{U}_b$.

The cardinality of this theory is at least the cardinality of B. Any finite subset is consistent, since any finite set of the $\mathcal{U}_b$ can mention only finitely many elements, and every finite boolean algebra has an ultrafilter. [This is because every finite boolean algebra has minimal nonzero elements[3], and any principal filter generated by such an element is maximal]. So, by compactness, $\mathcal{U}_B$ is consistent. Any valuation v satisfying $\mathcal{U}_B$ gives you an ultrafilter. An element b of B belongs to this ultrafilter iff $v(U_b) = 1$. ■

3.4 CNF and DNF

DEFINITION 24 *A formula in a propositional language with only $\wedge$, $\vee$ and $\neg$ is in* **conjunctive normal form** *("CNF") iff it is a conjunction of disjunctions of atomics and negatomics ('$\neg$' is attached only to propositional letters, and there is no '$\wedge$' inside a '$\vee$'); it is in* **disjunctive normal form** *("DNF") iff it is a disjunction of conjunctions of atomics and negatomics ('$\neg$' is attached only to propositional letters, and there is no '$\vee$' inside a '$\wedge$').*

REMARK 6
Every formula is logically equivalent to (is satisfied by the same valuations as) both a formula in CNF and a formula in DNF.

The CNF and DNF representations are unique up to commutativity of $\wedge$ and $\vee$.

I am not proposing to provide a full proof. The manipulations needed to obtain a CNF or a DNF for a formula rely on the distributivity of $\wedge$ over $\vee$ and of $\vee$ over $\wedge$, plus the de Morgan laws (ask Wikipædia) to "import" the '$\neg$'s.

It *may* be worth remarking that it can take exponential time to put a formula into CNF/DNF. This is because the length of the CNF/DNF of ϕ can be exponentially longer than ϕ. Suppose our propositional language has letters $\{p_i : i \in \mathbb{N}\}$. Let Φ_0 be p_0 and let Φ_{2n+1} be $\Phi_{2n} \vee p_{2n+1}$ and let Φ_{2n+2} be $\Phi_{2n+1} \wedge p_{2n+2}$. Then the length of Φ_i is i but its CNF/DNF is of length $\sim 2^i$.

Miniexercise: What is the CNF of a tautology? What is the DNF of the negation of a tautology?

Now could be a good moment to tackle question 5 from Sheet 3:

[3] called **atoms**.

> "Establish that the class of all propositional tautologies is the maximal propositional logic in the sense that any proper superset of it that is a propositional logic (closed under $\models$ and substitution) is trivial (contains all well-formed formulæ)."

This maximal propositional logic is always called "classical", and the salient feature that distinguishes it from most subsystems of interest is axiom T, which gives us the de Morgan laws, Excluded Middle ($A \vee \neg A$) and Double Negation ($\neg\neg A \to A$).

3.4.1 Resolution Theorem Proving

Worth a very brief mention: a proof system for Classical Propositional Logic.

A *clause* is a disjunction of atomics and negatomics. The sole rule of inference is "resolution":

$$\text{From } \Gamma \vee p \text{ and } \Theta \vee \neg p \text{ infer } \Gamma \vee \Theta.$$

The method of proof is: Take your axioms, and turn them all into CNF, and thus turn each into a set of clauses. For example, if one of your axioms was $A \longleftrightarrow B$, this has CNF $(\neg A \vee B) \wedge (\neg B \vee A)$ giving the two clauses $\neg A \vee B$ and $\neg B \vee A$. Thereafter, on being challenged to prove ϕ, you turn $\neg\phi$ into CNF, which gives a set of clauses. You add these clauses to the clauses you already have, and you attempt to obtain the empty clause by using the rule of resolution. The empty clause is the `false` ... so if you obtained it you have deduced the `false` from $\neg\phi$ and thereby proved ϕ as desired.

This is the logical basis of the programming language PROLOG.

Now could be a moment to attempt Sheet 5 question 6.

Chapter 4

One lecture on The Axiom of Choice

AC = axiom of Choice;
ZL is Zorn's Lemma[1]
WO is the Wellordering Principle: every set can be wellordered.

Dean Miller makes the point that in a huge proportion of cases where we use ZL, the chain complete poset is ordered by $\subseteq$.

REMARK 7 *WO implies AC.*

Proof:

Suppose you can wellorder anything that is shown to you, and you want a choice function on a family X of nonempty sets. You wellorder $\bigcup X$ by some wellorder which you call '$<$' and then, for each $x \in X$, declare $f(x)$ to be the $<$-least element of x. ■

REMARK 8 *ZL implies WO.*

Proof:

You are given a set X and you want to wellorder it. Your weapon is ZL, which means that whenever you have a chain-complete poset, it will have a maximal element. How do you use ZL? Well, you seek a chain-complete poset such that a maximal element of it is a wellordering of X. How about taking your chain-complete poset to be the poset of wellorderings of subsets of X (thought of as subsets of $X \times X$) ordered by $\subseteq$? Not *quite*. The problem is that a union of a chain of wellorderings under $\subseteq$ might not be a wellordering. You need to partially order the wellorderings by **end-extension**. (Recall definition 4.) ■

You might like to look at Sheet 1 questions 8 and 12 in this connection.

REMARK 9 *AC implies WO*

[1] What is yellow and equivalent to the axiom of choice?

Again we have a matching challenge. We want to wellorder a set X and we are told we can have a choice function on any family of nonempty sets that we like. The obvious suspect is $\mathcal{P}(X) \setminus \{\emptyset\}$. We now define, by recursion on the ordinals, a sequence s of elements of X indexed by ordinals. By AC, the family $\mathcal{P}(X) \setminus \{\emptyset\}$ of nonempty sets has a choice function f. Then we declare $s(\alpha)$, the αth member of our sequence, to be $f(X \setminus \{s(\beta) : \beta < \alpha\})$.

How can we be sure that we do not run out of ordinals? Hartogs' lemma (theorem 5) tells us that there is a wellordering too big to be embedded in X. So we must have used up all of X by the time we reach the order type of any such wellordering. ■

DEFINITION 25
A function $f : \langle X, \leq_X \rangle \to \langle X, \leq_X \rangle$ *is* **inflationary** *if* $(\forall x \in X)(x \leq_X f(x))$.

Inflationary is NOT the same as *increasing*! Explain the difference to your friends.

For AC → ZL we need

THEOREM 9 *The Bourbaki-Witt theorem*
Every inflationary function from a chain-complete poset into itself has a fixed point.

Proof:
Let $\langle X, \leq_X \rangle$ be a chain complete poset, and let $f : X \to X$ be inflationary. The idea is to build a chain, starting at some (any) $x \in X$, extend it at successor stages by doing f to the latest element obtained, and at limit stages by taking sups – $\langle X, \leq_X \rangle$ is chain complete. If we reach a fixed point at any stage we have our hearts' desire. But Hartogs' lemma (theorem 5) tells us that we cannot run out of ordinals.

■

COROLLARY 9 *AC implies ZL*

Proof:
Let $\langle X, \leq_X \rangle$ be a chain-complete poset. By AC we have a choice function f on $\mathcal{P}(X) \setminus \{\emptyset\}$. Then the function

$$x \mapsto \texttt{ if } x \texttt{ is } \leq_X\text{-maximal } \texttt{then } x \texttt{ else } f(\{x' \in X : x <_X x'\})$$

is inflationary and must have a fixed point by theorem 9. That fixed point will be a maximal element by construction. ■

4.1 Weak versions: countable choice, and a classic application thereof; DC

ctbl U of ctbls is ctbl. Do the same with DC

[not being written up for the notes: I can do this in my sleep]

A Dedekind-Infinite set is one the same size as some proper subset of itself. If countable choice fails there may be infinite sets that are not Dedekind-infinite.

König's Lemma

4.2 Applications of Zorn's Lemma

We look for chain-complete posets.

Comparability of cardinals

Independent sets in a vector space

Filters in a boolean algebra. Filters? Wossat?!

DEFINITION 26 *Filters and ideals in boolean algebras*

A **filter** *in a boolean algebra is a subset closed under $\wedge$ and $\geq$;*
it's a collection of "big" elements.

A filter is **proper** *iff it does not contain $\bot$.*
$\subseteq$ −maximal proper filters are called "ultrafilters",

The dual concept is **Ideal***: closed under $\vee$ and $\leq$; an ideal dual to an ultrafilter is a* **prime** *ideal. (The word 'prime' comes from ring theory)*

For any b.a. the set of its filters partially ordered by $\subseteq$ is a complete poset; set of proper filters similarly ordered is chain-complete. (In fact it's directed-complete). Arbitrary intersection of a nonempty family of proper filters is a proper filter.

Yer typical boolean algebra is a power set algebra, which is to say a product of lots of copies of the two-element boolean algebra, commonly written '$\mathbb{2}$'. yes/no. Hence the connection to logic.

Ideals are called ideals because they are ideals in boolean rings.

A boolean algebra becomes a ring if we take $\times$ to be $\wedge$ and $+$ to be `XOR`: $x + y = (x \wedge \overline{y}) \vee (y \wedge \overline{x})$. The boolean elements 0 and **1** become the 0 and **1** of the ring.

For the other direction (turning a boolean ring into a boolean algebra) we define $x \wedge y$ to be $x \cdot y$, and $x \vee y$ is $x + y + xy$. How does that work?! If we are thinking of our boolean algebra as a subalgebra of a power set algebra then $+$ is `XOR` and $\cdot$ is $\cap$. It follows that $x + y + xy$ is $(x$ `XOR` $y)$ `XOR` $(x \cap y)$. Now reflect that every member of $x \cup y$ belongs to precisely one of x `XOR` y and $x \cap y$.

You may be interested in giving a convincing mathematical account of the way in which boolean algebras and boolean rings are the same thing. There is a literature on identities of this kind. If you are interested in this, have look at sheet 4 question 13 which introduces another case of two things that are "the same".

DEFINITION 27 *Principal and nonprincipal ideals and filters.*

Boolean homomorphism h must send $a \wedge b$ to $h(a) \wedge h(b)$ and so on for all the other operations.

Ideals are kernels of boolean algebra homomorphisms .

The kernel of the homomorphism $y \mapsto y \wedge \overline{x}$ is **the ideal generated by** x. The filter dual to such an ideal is a **principal filter** (generated by $\overline{x}$).

A principal ideal is a boolean algebra in its own right.

DEFINITION 28 *Quotient over an ideal, or filter*

$x \sim_I y$ *if* $(x$ `XOR` $y) \in I$;

or

$x \sim_F y$ *if* $(x \wedge y) \vee (\overline{x} \wedge \overline{y}) \in F$.

Consider the ideal of finite sets in $\mathcal{P}(\mathbb{N})$, and the quotient algebra.

General patter about representation theorems.

THEOREM 10 *Stone's Representation Theorem*
Every Boolean algebra is isomorphic to one where

- *the order relation is* $\subseteq$ *– set inclusion;*
- $\wedge$ *is* $\cap$*;*
- $\vee$ *is* $\cup$*, and*
- *complementation* ($x \mapsto \overline{x}$) *is set complementation.*

Proof:

The challenge is to associate to each element of the algebra a set in such a way that elements higher in the algebra get sent to bigger sets (more elements). (Of course we also have to respect the boolean operations $\wedge$, $\vee$ and complement). In principle these sets could be anything, but in fact we don't have to look very far from home. It turns out that we can send each element b to the set of maximal filters F such that $b \in F$. Clearly the higher up in the algebra you are the more filters you belong to: if $a \leq b$ then any maximal filter containing a contains b. It remains to check that the inclusion is strict. That is ...

We need a lemma that says that

LEMMA 5 *If* $b \not\geq a$ *then there is a maximal filter containing* a *but not* b*.*

Proof:

Consider the collection of those filters that are supersets of the principal filter generated by $a \wedge \overline{b}$, and partially order by $\subseteq$. This is a chain-complete poset and must have a maximal element by Zorn.

■

Lemma 5 is one of many equivalent formulations of a a theorem known as the **Prime Ideal Theorem**. Other versions say: *every boolean algebra contains a prime ideal* or *every ideal in a Boolean algebra can be extended to a prime ideal*, or versions of the last two in terms of ultrafilters rather than prime ideals. Of course since Boolean rings and boolean ideals are really the same we also have the formulation *every ideal in a Boolean ring can be extended to a prime ideal*. It is significant that the Prime Ideal theorem – altho' obviously a consequence of ZL – does not return the favour: The Prime ideal theorem does not imply ZL. However the slightly mopre general "every ideal in a ring can be extended to a prime ideal *does* imply ZL.

It's easy to check that the boolean operations are respected by the map $b \mapsto$ the set of maximal filters F such that $b \in F$. Any maximal filter must contain precisely one of b and $\overline{b}$ so that ensures that complementation is respected. [It also explains why we need *maximal* [proper] filters not just any-old-proper-filters... which is why we need the the prime ideal theorem]

If lemma 5 reminds you of the definition of separative poset (definition 15) the rat you think you have smelt is a real one. However we have no time to explain the connection.

Chapter 5

Three lectures on First order Logic and the Compactness Theorem

Stuff that it would be fun to fit in

> Kleene's thm on conservative extensions;
> many-sorted theories;
> relettering vbls not context-free;
> Skolemisation;
> categoricity: both 1st and 2nd order.

Explain the syntax before anything else

DEFINITION 29
Predicate/Relation symbol
'=' is a reserved word
arity
function symbol
constant symbol
atomic formula
quantifier

5.1 The Syntax of First-order Logic

All the apparatus for constructing formulæ in propositional logic works too in this new context: If A and B are formulæ so are $A \vee B$, $A \wedge B$, $\neg A$ and so on. However we now have new ways of creating formulæ, new gadgets which we had better spell out:

There is really an abuse of notation here: we should use quasi-quotes ...

5.1.1 Constants and variables

Constants tend to be lower-case letters at the start of the Roman alphabet ('a', 'b' ...) and variables tend to be lower-case letters at the end of the alphabet ('x', 'y', 'z' ...). Since we tend to run out of letters we often enrich them with subscripts to obtain a larger supply: 'x_1' etc.

5.1.2 Predicate letters

These are upper-case letters from the Roman alphabet, usually from the early part: 'F' 'G' .... They are called *predicate* letters because they arise from a programme of formalising reasoning about predicates and predication. '$F(x, y)$' could have arisen from 'x is fighting y'. Each predicate letter has a particular number of terms that it expects; this is the **arity** of the letter. **Unary** predicates have one argument, **binary** predicates have two; n-**ary** have n. 'loves' has arity 2 (it is binary) 'sits-on' is binary too. If we feed it the correct number of terms – so we have an expression like $F(x, y)$ – we call the result an **atomic formula.**

The equality symbol '=' is a very special predicate letter: you are not allowed to reinterpret it the way you can reinterpret other predicate letters. The Information Technology fraternity say of strings that cannot be assigned meanings by the user that they are **reserved**; elsewhere such strings are said to be **part of the logical vocabulary**. The equality symbol '=' is the only relation symbol that is reserved. In this respect it behaves like '$\wedge$' and '$\forall$' and the connectives, all of which are reserved in this sense.

Similarly arity of functions. [say a bit more about this]

Atomic formulæ can be treated the way we treated literals in propositional logic: we can combine them together by using '$\wedge$' '$\vee$' and the other connectives.

5.1.3 Quantifiers

Finally we can **bind** variables with **quantifiers**. There are two: $\exists$ and $\forall$. We can write things like

$(\forall x)F(x)$:	Everything is a frog;
$(\forall x)(\exists y)L(x, y)$	Everybody loves someone

The syntax for quantifiers is variable-preceded-by quantifier enclosed in brackets, followed by stuff inside brackets:

$$(\exists x)(\ldots) \text{ and } (\forall y)(\ldots)$$

We sometimes omit the pair of brackets to the right of the quantifier when no ambiguity is caused thereby.

The difference between variables and constants is that you can bind variables with quantifiers, but you can't bind constants. The meaning of a constant is fixed. Beware! This does not mean that constants are reserved words! The constant 'a' can denote anything the user wants it to denote, it doesn't wander around like the denotation of a

variable such as 'x'. Confusingly that's not to say that there are no reserved constants; there are plenty in formalised mathematics, the numerals '0', '1' ... for starters.

Perhaps one should spell this out a bit more. A constant (symbol) is a thing of the same syntactic type as a variable but which – unlike the variable – cannot be bound, since it denotes the same single thing throughout an environment. A reserved word is a constant symbol that means the same thing in every environment. (Merely not-being-bindable doesn't make you a constant!)

For example, in a formula like

$$(\forall x)(F(x) \to G(x))$$

the letter 'x' is a variable: you can tell because it is bound by the universal quantifier. The letter 'F' is not a variable, but a predicate letter. It is not bound by a quantifier, and cannot be: the syntax forbids it. In a first-order language you are not allowed to treat predicate letters as variables: you may not bind them with quantifiers. Binding predicate letters with quantifiers (treating them as variables) is the tell-tale sign of **second-order** Logic. We also have

5.1.4 Function letters

These are lower-case Roman letters, typically 'f', 'g', 'h' .... We apply them to variables and constants, and this gives us **terms**: $f(x)$, $g(a, y)$ and suchlike. In fact we can even apply them to terms: $f(g(a, y))$, $g(f(g(a, y), x))$ and so on. So a term is either a variable or a constant or something built up from variables-and-constants by means of function letters.

The other tell-tale sign of second-order logic is having function symbols that act on function symbols, or relation symbols that relate other relation symbols. For obvious reasons such gadgets are called *second order* function symbols/relation symbols. An obvious one is "apply", which takes a function symbol and a term as inputs.

5.1.5 Quantifiers

In Analysis we often use the cofinite quantifier $\forall^\infty$ which reads "for all but finitely many..." – think of the definition of *convergent sequence*. The sequence $f : \mathbb{N} \to \mathbb{Q}$ is convergent iff $(\forall \epsilon > 0)(\forall^\infty m)(\forall^\infty m)(|f(m) - f(n)| < \epsilon)$

You may remember the proof that there are infinitely many primes congruent to -1 mod 4[1]. It uses the quantifier "there is an odd number of ..."

Duality

The usual quantifiers $\exists$ and $\forall$ are *dual* in the sense of question 10 on sheet 2: $\neg\forall\neg = \exists$ and $\neg\exists\neg = \forall$

"dense" and "open" in topology are dual notions.

[1] Suppose there are only finitely many of them. Take 4 times their product and subtract 1. It must have an odd number of factors that are congruent to -1 mod 4, and they must be different from all those in what you tho'rt was the complete set.

5.1.6 Higher-order

Difference between 1st and 2nd order theories.

DEFINITION 30 *In second-order languages you are allowed quantifiers over function symbols and predicate letters.*

Topology is a second-order concept.

Complete ordered fields is a 2nd order theory; Simple groups ...

Possibility of many-sorted theories – not the same! Vector spaces

For people trying to get entirely straight in their minds what a first-order formula is, examples like the following can be quite confusing. (It's the answer to a question on a compsci example sheet that i had to teach.)

$$(\exists x_1 \dots x_n)(\bigwedge_{1 \le i \ne j \le n} x_i \ne x_j) \tag{H}$$

The example sheet question asked for a first-order sentence that is true only in structures with at least n distinct inhabitants.

In a straightforward official sense this sentence (H) is not first-order, in that the recursions that generate first-order formulæ do not output it. For one thing, it exploits the fact that the variables have internal structure; for another the '$\bigwedge$' is a binder that – in some sense – is a universal quantifier over the subscripts on the variables. But for all that it's not second-order either.

My take on this is that (H) is (obviously) not *literally* a sentence in a first-order language, tho' one could perhaps think of it as a program that, when provided with a numeral as input, outputs a genuine first-order sentence. Which first-order sentence you get will of course depend on the numeral you gave it. Alternatively you can think of it as a uniform parametrised description (in a metalanguage) of an infinite family of first-order sentences. That's probably the simplest way to cope with this kind of slangy mathematical shorthand. It is probably safe to think of formulæ like (H) as first-order, if only by courtesy: trying to spice up the definition of first-order formula so that (H) becomes a first-order formula would be a very messy exercise.

Bear in mind that, whatever your kit of relation symbols, function symbols etc etc is, the subformula relation between the formulæ you get is going to be wellfounded and you can perform inductions and recursions on it.

5.1.7 Signatures

DEFINITION 31
Signature: a structure is a set ('carrier set' better than 'domain') with knobs on.
The signature is the thing that tells you what the knobs are. Languages have signatures too. A structure is a structure "for" a language iff they have the same signature.

A substructure of the structure $\mathfrak{M}$ is a subset of the carrier set of $\mathfrak{M}$ equipped with the same knobs and closed under the relevant operations.

Reducts/expansions

We need the concept of signature for basic sense-making reasons. It doesn't make sense to ask whether a formula ϕ is true in a structure $\mathfrak{M}$ unless all the gadgets in ϕ appear also in $\mathfrak{M}$.

Typically signatures tend to be finite, altho' sometimes – for special reasons – one expands a structure to one with infinitely many constant symbols. In fact we do this in theorem 21.

I try to to use upper-case $\mathfrak{FRAKTUR}$ font for variables ranging over structures, but it doesn't come out very well on a blackboard! I will write the carrier set of the structure $\mathfrak{M}$ as M, the corresponding upper-case roman letter. get these in the right order

$\mathcal{L}(T)$, for T a theory.

5.2 Axioms of LPC

This section will be very short; since we are not going to spend much time actually deducing theorems of LPC we are not going to be very concerned about what the axioms are. In any case, the details of the proof of the completeness theorem do not seem to be very sensitive to one's choice of axioms.

I have copied the following from PTJ's book [6], and I provide them only for the sake of completeness.

We need the concept of a **free variable**. Brief chat.

We need the axioms:

• $((\forall x)p) \to p[t/x]$
where p is a formula with 'x' free in it, and t is any term with no free occurrences of 'x'

• $(\forall x)(p \to q) \to (p \to (\forall x)q)$
('x' not free in p)

• $(\forall x)(x = x)$

• $(\forall xy)(x = y \to p \to p[y/x])$
p any formula with x

My proof of the completeness theorem will also use the rules:

• **Universal Generalisation**: if we have proved $\phi(x)$ with 'x' free, then we have proved $\forall x\phi(x)$. ("Let x be arbitrary ...")

• and a rule that says: if we have a proof of $F(a)$ for some 'a' and a proof of $(\exists x)(F(x)) \to p$ then we have a proof of p.

5.3 Semantics

In this section we develop the ideas of truth and validity (which we first saw in the case of propositional logic) in the rather more complex setting of predicate logic.

What we will give is – for each language $\mathcal{L}$ – a definition of what it is for a formula of $\mathcal{L}$ to be true in a structure-for-$\mathcal{L}$.

The first thing we need is the concept of a signature from definition 31: for a formula ϕ to have a prayer of being true in a structure $\mathfrak{M}$, the signature of the language

that ϕ belongs to must be the same as the signature of $\mathfrak{M}$. It simply does not make sense to ask whether or not (for example) the transitivity axiom $(\forall xyz)(x < y \wedge y < z. \rightarrow x < z)$ is true in a structure unless that structure has a binary relation in it.

First we need to decide what our carrier set is to be. Next we need the concept of an **interpretation**. An interpretation is the thing that married up the gadgets in the signature at the structure with the gadgets in the signature in the language. More formally it is a function $\mathcal{I}$ assigning to each predicate letter, function letter and constant in the language of ϕ a subset of M^n, or a function $M^k \rightarrow M$, or element of M *mutatis mutandis*. That is to say, to each syntactic device in the language of ϕ, the interpretation assigns a component of $\mathfrak{M}$ of the appropriate arity.

For example, one can interpret the language of arithmetic by determining that the "domain of discourse" (the carrier set) is to be $\mathbb{N}$, the set of natural numbers, and that the interpretation of the symbol '$\leq$' will be the set of all pairs $\langle x, y \rangle$ of natural numbers where x is less than or equal to y, and so on.

We have now equipped the language with an interpretation so we know what the symbols mean, but not what the values of the variables are. In other words, settling on an interpretation has enabled us to reach the position from which we started when doing propositional logic. It's rather like the position we are in when contemplating a computer program but not yet running it. When we run it we have a concept of instantaneous state of the program: these states (snapshots) are allocations of values to the program variables. Let us formalise a concept of state.

A **finite assignment function** is a finite (partial) function from variables in $\mathcal{L}$ to M, the carrier set of $\mathfrak{M}$. These will play a rôle analogous to the rôle of valuations in propositional calculus. I have (see above) carefully arranged that all our variables are orthographically of the form x_i for some index i, so we can think of our assignment function f as being defined either on *variables* or on *indices*, since they are identical up to 1-1 correspondence. It is probably better practice to think of the assignment functions as assigning elements of M to the *indices* and write "$f(i) = \ldots$", since any notation that involved the actual *variables* would invite confusion with the much more familiar "$f(x_i) = \ldots$", where f would have to be a function defined on the things that the variables range over.

Next we define what it is for a partial assignment function to satisfy a sentence p (written "$sat(f, p)$"). We will do this by recursion on the set of formulæ (which comes equipped with a wellfounded subformula relation that justifies induction) so naturally we define *sat* first of all on atomic sentences.

Notice that in

$$sat(f, x_i = x_j)$$

we have a relation between a function and an expression, not a relation between f and x_i and x_j. That is to say that we wish to **mention** the variables (talk about them) rather than **use** them (to talk about what they point to). This contrast is referred to as the **use-mention distinction**.[2] This is usually made clear by putting quotation marks of some kind round the expressions to make it clear that we are mentioning them but not using them. Now precisely what kind of quotation mark is a good question. Our first

[2]It has been said that the difference between logicians and mathematicians is that logicians understand the use-mention distinction.

clause will be something like

$$sat(f, \text{`}x_i = x_j\text{'}) \text{ iff}_{\text{df}} f(i) = f(j). \tag{5.1}$$

But how much like? Notice that, as it stands, 5.1 contains a name of the expression which follows the next colon: $x_i = x_j$. Once we have put quotation marks round this, the i and j have ceased to behave like variables (they were variables taking indices as values) because quotation is a referentially opaque context.

A context is **referentially opaque** if two names for the same thing cannot be swapped within it while preserving truth. Quotation is referentially opaque because when we substitute one of the two names for Dr. Jekyll/Mr. Hyde for the other in

'Jekyll' has six letters

we obtain the falsehood

'Hyde' has six letters

even though Jekyll and Hyde are the same person. The intuition behind the terminology is that one cannot "see through" the quotation marks to the thing(s) pointed to by the words 'Jekyll' and 'Hyde', so one cannot tell that they are the same. There are other important contexts that are referentially opaque: belief, for example. I might have different beliefs about a single object when it is identified by different names, and these beliefs might conflict.

But we still want the 'i' and 'j' to be variables, because we want the content of clause 5.1 to read, in English, something like: "for any variables i and j, we will say that f satisfies the expression whose first and fourth letters are 'x', whose third and fifth are i and j, respectively and whose middle letter is '=', iff $f(i) = f(j)$". It is absolutely crucial that in the piece of quoted English text 'x' and '=' appear with single quotation marks round them while 'i' and 'j' do not. Formula (5.1) does not capture this feature. To correct this Quine invented a new notational device which he called "corners" and which are nowadays known as "Quine quotes" (or "quasi-quotes"), which operate as follows: the expression after the next colon:

$$\ulcorner x_i = x_j \urcorner$$

being an occurrence of '$x_i = x_j$' enclosed in Quine quotes is an expression that does not, as it stands, name anything. However, i and j are variables taking natural numbers as values, so that whenever we put constants (numerals) in place of i and j it turns into an expression that will name the result of deleting the quasi-quotes. This could also be put by calling it a variable name.

A good way to think of quasi-quotes is not as a funny kind of quotation mark – for quotation is referentially opaque and quasi-quotation is referentially transparent – but rather as a kind of diacritic, not unlike the LaTeX commands I am using to write these notes. Within a body of text enclosed by a pair of quasi-quotes, the symbols '$\wedge$', '$\vee$' and so on, do not have their normal function of composing *expressions* but instead compose *names of expressions.* This also means that Greek letters within the scope of quasi-quotes are not dummies for expressions or abbreviations of expressions but are

variables that range over expressions (not sets, or natural numbers). Otherwise, if we think of them as a kind of funny quotation mark, it is a bit disconcerting to find that – as Quine points out – $\ulcorner\mu\urcorner$ is just μ (if μ is an expression with no internal structure). The interested reader is advised to read pages 33-37 of Quine's [7], where this device is introduced.

It might have been easier to have a new suite of operators that combine names of formulæ to get names of new formulæ so that, as it might be, putting '&' between the names of two formulæ gave us a name of the conjunction of the two formulæ named. However, that uses up a whole font of characters, and it is certainly more economical, if not actually clearer, to use corners instead.

Once we have got that straight we can declare the following recursion, where 'α' and 'β' are variables taking expressions as values.

DEFINITION 32 *First the base cases, for atomic formulæ*

$sat(f, \ulcorner x_i = x_j \urcorner)$ *iff* $f(i) = f(j)$*;*
$sat(f, \ulcorner x_i \in x_j \urcorner)$ *iff* $f(i) \in f(j)$*.*

Then the inductive steps
if $sat(f, \alpha)$ *and* $sat(f, \beta)$*, then* $sat(f, \ulcorner \alpha \wedge \beta \urcorner)$*;*
if $sat(f, \alpha)$ *or* $sat(f, \beta)$*, then* $sat(f, \ulcorner \alpha \vee \beta \urcorner)$*;*
if for no $g \supseteq f$ *does* $sat(g, \alpha)$ *hold, then* $sat(f, \ulcorner \neg \alpha \urcorner)$*;*
if there is some $g \supseteq f$ *such that* $sat(g, \ulcorner F(x_i) \urcorner)$*, then* $sat(f, \ulcorner (\exists x_i)(F(x_i)) \urcorner)$*;*
if for every $g \supseteq f$ *with* $i \in dom(g)$*,* $sat(g, \ulcorner F(x_i) \urcorner)$*, then* $sat(f, \ulcorner (\forall x_i)(F(x_i)) \urcorner)$*.*

Then we say that ϕ *is* **true in** $\mathfrak{M}$*, written* $\mathfrak{M} \models \phi$ *iff* $sat(\perp, \phi)$*, where* $\perp$ *is the empty partial assignment function.*

Finally, a formula is **valid** *iff it is true in every interpretation.*

5.4 Completeness theorem for LPC: the set of valid sentences is semidecidable

The full significance of this material will not become apparant until the second part of this course, where you learn to master computable function theory, since 'semidecidable' is a technical term from that area. The informal notion that 'semidecidable' captures is that a set x is semidecidable iff there is a finite engine which, on being presented with a candidate for membership of x will – sooner or later – say 'yes' if the candidate is a member of x but will dither eternally if it isn't.

The completeness theorem for LPC tells us that the set of valid sentences is semidecidable. The engine in question here is a device/gremlin/whatever that searches all possible proofs until it finds a proof of the candidate.

You might think it is obvious that there should be such an engine. That would be a big mistake: it's true but it's not obvious. If you think it is obvious then you have misunderstood something, and you need to retrace your steps.

5.4.1 $\in$-terms

Suppose $T \vdash (\exists x)(F(x))$. There is nothing to stop us adding to $\mathcal{L}(T)$ a new constant symbol 'a' and adding to T an axiom $F(a)$. Clearly the new theory will be consistent if T was. Why is this? Suppose it weren't, then we would have a deduction of $\perp$ from $F(a)$. But T also proves $(\exists x)(F(x))$, so we can do a $\exists$-elimination to have a proof of $\perp$ in T. But T was consistent.

Notice that nothing about the letter 'a' that we are using as this constant tells us that a is a thing which is F. We could have written the constant 'a_F' or something suggestive like that. Strictly it shouldn't matter: variables and constant symbols do not have any internal structure that is visible to the language, and the 'F' subscript provides a kind of spy-window available to anyone *mentioning* the language, but not to anyone merely *using* it. The possibility of writing out novel constants in suggestive ways like this will be useful later.

Check for yourself that $(\exists x)(\forall y)(F(y) \rightarrow F(x))$ is always true. It tells us that for any F with one free variable we can invent a constant whose job it is to denote an object which has property F as long as anything does. If there is indeed a thing which has F then this constant can denote one of them, and as long as it does we are all right. If there isn't such a thing then it doesn't matter what the constant denotes.

This constant is often written $(\epsilon x)F(x)$. Since it points to something that has F as long as there is something that has F, we can see that

$$(\exists x)(F(x)) \quad \text{and} \quad F((\epsilon x)F(x))$$

are logically equivalent (true in the same structures). So we have two rules

$$\frac{(\exists x)(F(x))}{F((\epsilon x)F(x))} \quad \text{and} \quad \frac{F((\epsilon x)F(x))}{(\exists x)(F(x))}$$

THEOREM 11 *Every consistent theory in a countable language has a model.*

Proof:

Let T_0 be a consistent theory in a countable language $\mathcal{L}(T_1)$.

We do the following things

1. Add axioms to T_0 to obtain a complete extension; Explain Complete Extension
2. Add ϵ terms to the language.

We execute the task in (1) the way we proved theorem 8 – The Adequacy Theorem,

Notice that when we add ϵ-terms to the language we add new formulæ: if '$(\epsilon x)F(x))$' is a new ϵ-term we have just added then '$G((\epsilon x)F(x))$' is a new formula, and T_0 doesn't tell us whether it is to be true or to be false. That is to say $\mathcal{L}(T_0)$ doesn't contain '$(\epsilon x)F(x)$' or '$G((\epsilon x)F(x))$'. Let $\mathcal{L}(T_1)$ be the language obtained by adding to $\mathcal{L}(T_1)$ the expressions like '$(\epsilon x)F(x)$' and '$G((\epsilon x)F(x))$'.

We extend T_0 to a new theory in $\mathcal{L}(T_1)$ that decides all these new formulæ we have added. This gives us a new theory, which we will – of course – call T_1. Repeat and take the union of all the theories T_i we obtain in this way: call it T_∞. (Easy to see that all the T_i are consistent – we prove this by induction).

It's worth thinking about what sort of formulæ we generate. We added terms like $(\epsilon x)(F(x))$ to the language of T_1. Notice that if H is a two-place predicate in $\mathcal{L}(T)$ then we will find ourselves inventing the term $(\epsilon y)H(y,(\epsilon x)F(x)$ which is a term of – one might say – *depth* 2. And there will be terms of depth 3, 4 and so on as we persist with this process. All atomic questions about ϵ terms of depth n are answered in T_{n+1}.

T_∞ is a theory in a language $\mathcal{L}_\infty$, and it will be complete. The model $\mathfrak{M}$ for T_∞ will be the structure whose carrier set is the set of ϵ terms we have generated *en route*[3]. All questions about relations between the terms in the domain are answered by T_∞. The interpretation of an n-ary relation symbol 'R' from $\mathcal{L}(T)$ will be the set of all tuple $\langle t_1 \ldots t_n\rangle$ such that $T_\infty \vdash R(t_1 \ldots t_n)$ and function symbols similarly.

Does this make $\mathfrak{M}$ into a model of T? We will establish the following:

LEMMA 6 $\mathfrak{M} \models \phi(t_1, \ldots t_n)$ *iff* $T_\infty \vdash \phi(t_1, \ldots t_n)$

Proof:

We do this by induction on the complexity of ϕ. When ϕ is atomic this is achieved by stipulation. The induction step for propositional connectives is straightforward. (Tho' for one direction of the '$\vee$' case we need to exploit the fact that T_∞ is complete, so that if it proves $A \vee B$ then it proves A or it proves B.)

The remaining step is the induction step for the quantifiers. They are dual, so we need consider only $\forall$. We consider only the hard direction $(L \rightarrow R)$.

Suppose $\mathfrak{M} \models (\forall x)\phi(x, t_1, \ldots t_n)$. Then $\mathfrak{M} \models \phi(t_0, t_1, \ldots t_n)$ for all terms t_0. In particular it must satisfy it even when $t_0 = (\epsilon x)(\neg\phi(x, t_1, \ldots t_n))$, which is to say

$$\mathfrak{M} \models \phi((\epsilon x)(\neg\phi(x, t_1, \ldots t_n)), t_1, \ldots t_n)$$

So, by induction hypothesis we must have

$$T_\infty \vdash \phi((\epsilon x)(\neg\phi(x, t_1, \ldots t_n)), t_1, \ldots t_n)$$

whence of course $T_\infty \vdash (\forall x)\phi(x, t_1, \ldots t_n)$.

■

This completes the proof of theorem 11. Observe the essential rôle played by the ϵ terms.

This is a result of fundamental importance. Any theory that is not actually self-contradictory is a description of *something*. It's important that this holds only for first-order logic. It does not work for second-order logic, and this fact is often overlooked.

COROLLARY 10 *Compactness for first-order logic.*

If T is a first-order theory all of whose finite fragments have models then T has a model.

Proof:

Such a T is obviously consistent (proves no contradictions) so, by theorem 11 it has a model. ■

[3] And we really do mean the set of epsilon terms, not the denotations of those terms... Our models really are created entirely out of syntax.

This theorem looks cute and it has many, many, consequences, but most of them are unattractive, and say things like "first-order logic cannot capture this concept". The most striking of them is that there is no first-order way of saying what a natural number is.

THEOREM 12 *There are "nonstandard" models of arithmetic.*

Proof:

What does that mean? Let T be a first-order theory of arithmetic of $\mathbb{N}$, with $+$, $\times$, $=$, anything you like, really. Then it has a model which is not the "standard" model. Add a constant symbol – '*' (I don't want to use anything standard and suggestive like 'ω' or '∞'.) Then we add axioms for '*' to say $* \neq 0$, $* \neq 1$, $* \neq 1 + 1 \ldots$. Clearly any finite subset of T with these new axioms is consistent as long as T was, and so has a model. ■

To be specific (and this might help you get your thoughts about interpretations in order), the first n of these new axioms will tell you only that * must be at least $n + 1$. That is to say, for each n, there is an interpretation $\mathcal{I}_n$ of the language of arithmetic which interprets that language *into the standard model* and $\mathcal{I}_n(\text{'*'}) = n + 1$. $\mathcal{I}$ treats '0', '1' etc as usual. The sequence of $\mathcal{I}_n$s "kicks the can down the road".

This technique is used on an industrial scale to show that certain theories are not axiomatisable, by which we mean (as in this case)...

> "There is no first-order theory whose models are precisely the standard model of Arithmetic."

In general we prove things like

> "There is no first order theory the class of whose models is the class K"

where K is something natural like (for example) the singleton of the standard model of arithmetic, or the class of all simple groups, or the class of all fields of finite characteristic.

This train of thought is a rich source of example sheet questions and exam questions.

5.5 Decidability

Propositional logic is *decidable*: there is an algorithm that tells us whether or not a candidate formula is a tautology. First-order Logic is not decidable in this sense. It's semidecidable because it is axiomatisable: every valid sentence is spat out by our axiomatisation, so if a sentence is valid we learn this fact in finite time. What about if it isn't valid? We would learn that – too – in finite time if every falsifiable first-order formula had a finite countermodel (there are only countably many possible such countermodels and we can wellorder them in order type ω and examine them one by one) but that is not true (consider the negation of the theory DLO of dense linear orders, which we encounter on p. 7.2.) It is falsifiable, but the only structures that falsify it are

infinite!) We have no time to prove that in this course, but a special case is tractable. Question 9 on sheet 3 invites the reader to show that the monadic fragment of first-order logic (one-place predicate letters only, no function symbols) is decidable.

Chapter 6

Three Lectures on Set Theory

We give a historically motivated introduction.

Every time Mathematics discovers it needs a new suite of entities it is liable to get into a tizz about whether or not it's safe to assume that the new entities really are around to be used as desired. Famously there is/was a problem about complex numbers. (You can see there was a problem from the use of the word 'imaginary') We usually sort things out in the end, fortunately. Now in quite a lot of cases the new entities can be introduced to the world dressed up as *sets*. Indeed that is why sets seem so useful. Two relevant and helpful examples are *points at infinity* and *ideal divisors*.

Why do we need points at infinity? A point at infinity is a point where two parallel lines meet. In a sense that's all we know about it, the parallel lines that meet there. So how about we just identify the desired point at infinity with – the bundle ("pencil") of parallel lines that are supposed to go through it? Points at infinity are *concretised* as pencils of lines.

There is a similar story to be told about ideal divisors in rings. One way of approaching this topic is through something which by now is familiar to you. What are integers mod 17? How do you think of them? Do you think of them as a special kind of number? Or do you think of them as equivalence classes of integers under congruence-mod-17? Both points of view are legitimate. If you think of them as equivalence classes of integers, then you will be looking for a way of feeling comfortable about the process of *abstraction* that will give you this new kind of number. If – on the other hand – you already think of them as this new kind of number then you might be interested in *concretising* them in terms of older and already familiar things – integers... in fact *sets* of integers. This is where sets come in.

Imaginary divisors get concretised as ideals (= sets). Here is the standard example: $\mathbb{Z}[\sqrt{-5}]$ is sold to us as the substructure of $\mathbb{C}$ generated by $\mathbb{Z}$ and $\sqrt{-5}$. Famously it does not have unique factorisation: in $\mathbb{Z}[\sqrt{-5}]$ we can factorise 6 as $2 \cdot 3$ and also as $(1+\sqrt{-5})\cdot(1-\sqrt{-5})$ – we can compute these products in $\mathbb{C}$ – and all these four factors are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

So we invent "lower" factors – four of them in fact. One to be a common factor of 2 and $1+\sqrt{-5}$, a second to be a common factor of 2 and $1-\sqrt{-5}$, the third to be a common factor of 3 and $1+\sqrt{-5}$, and finally the fourth to be a common factor of 3 and

$1 - \sqrt{-5}$.

How are we to concretise these ficticious factors? The key observation is that, although we (think) we do not know what these new roots are, we do know exactly what their nontrivial multiples are. After all, the multiples of HCF(a, b) are precisely the numbers of the form $ax + by$; this fact gives us a way in.

Different ideal divisors will correspond to different sets of multiples, so we concretise the ideal divisor of 3 and $1 + \sqrt{-5}$ *as that set*: $\{a \cdot 3 + b \cdot (1 + \sqrt{-5}) : a, b \in \mathbb{Z}\}$.

The set of multiples of a fixed prime is closed under addition, and also by multiplication by arbitrary ring members. And that of course is the definition of an ideal in a ring.

Integers and rationals similarly. Also equivalence classes of Cauchy sequences as reals. We need this beco's *prima facie* there seems to be a question about whether or not it is consistent to assume that natural numbers have additive inverses; or that integers have multiplicative inverses; or that *all* holes in the rationals can be simultaneously filled.

'extension"? OK, so the set of multiples of the ideal divisor exists as a comprehended object, some suitably concrete object-in-extension. Ditto the pencil-of-lines. So there is an unproblematic object-in-extension corresponding to the two intensions (ideal divisor, point at infinity). Does this always work? Does every set-in-intension have a corresponding set-in-extension? No! Russell was able to show this, using very old ideas going back at least to the Greeks. Russell's paradox. It's an interesting object proof-theoretically but for us it's just a pain. We are going to have to come up with some subset of the set of axioms of naïve set theory plus a good story.

There are various subsets one can use, but – altho' I am an expert on one particular one, due to Quine – i am not going to tell you about that subset, but talk only about the mainstream version which everyone uses. It's known as **Zermelo-Fraenkel Set theory** or 'ZF' for short.

A guiding principle in trying to suss out the most suitable subset of axioms to use is the recurring thought that set theory started off (as outlined above) as a way of concretising abstract mathematical objects. Thus the axioms of set theory arose largely out of a desire to manipulate sets and prove the existence of such sets as might serve as *simulacra* for mathematical objects. Thus the axioms largely consist of assertions that sets can be manipulated in certain ways to obtain new sets, and that the world of sets is closed under certain operations.

Most set theories do not have axioms giving us sets that are interesting in their own right *as sets* – such as the set of all sets, or the set of all cardinals... largely because the existence of such sets is not compatible with axioms saying that sets can be manipulated freely. In particular they tend not to be compatible with separation ...

But first we deal with the most fundamental axiom: extensionality.

$$(\forall x, y)(x = y \longleftrightarrow (\forall z)(z \in x \longleftrightarrow z \in y)).$$

It's called 'extensionality' because a binary relation R is called 'extensional' as long as $(\forall xy)(x = y \longleftrightarrow (\forall z)(R(z, y) \longleftrightarrow R(z, x)))$.
(Do not confuse this use of 'extensional' with 'extensional' meaning 'truth-functional', contrasted with *intensional*.) The thought behind the axiom of extensionality is that sets

are the datatype with absolutely minimal internal structure: sets *without* knobs on. Sets are what you get if you start with a mathematical structure and erase all information about the relations that hold between its members and anything that might identify them ('anonymisation") so that the only information is presence/absence information. You don't do anything to their members so the only way of telling two sets apart is by seeing if they have different members.

Let's pursue this idea of concretisation and see what axioms it leads us to. We concretise functions as sets of ordered pairs so let's concretise ordered pairs. We want a total [binary] function `pair` and two [unary] partial functions `fst` and `snd` (or π_1 and π_2 if you prefer) s.t.

$(\forall xy)(\texttt{fst}(\texttt{pair}(x,y)) = x)$ and
$(\forall xy)(\texttt{snd}(\texttt{pair}(x,y)) = y)$.

One that works is

DEFINITION 33 *The* **Wiener-Kuratowski** *pair*
$\texttt{pair}(x,y) = \{\{x\},\{x,y\}\}$.
$\texttt{fst}(p) = \bigcap\bigcap p$ *and*
$\texttt{snd}(p)$ = *the unique member of* $\bigcup p$ *belonging to exactly one member of* p.

$$x = \texttt{snd}(p) \longleftrightarrow (\exists! z)(z \in p \wedge x \in z).$$

If ordered pairs are concretised as above, what axioms do we need if we are to construct and deconstruct them?

Pairing: $(\forall xy)(\exists z)(\forall w)(w \in z \longleftrightarrow (w = x \vee w = y))$
Sumset: $(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow (\exists w)(z \in w \wedge w \in x))$
Power set: $(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow (\forall w)(w \in z \rightarrow w \in x))$
Separation: $(\forall \vec{y})(\forall x)(\exists z)(\forall w)(w \in z \longleftrightarrow w \in x \wedge \phi(w,\vec{y}))$

I have written these out in primitive notation as far as possible. Set theory is a first-order theory in the language with just '$\in$' and '$=$'.

Now might be a good time to tackle Sheet 4 Q (11), p. 115

Separation implies that there is no universal set, lest we get Russell's paradox.

What other axioms are we going to need…? Well, it shouldn't matter how we concretise ordered pairs. Let's try to prove the existence of $X \times Y$ (which is a set, after all, even if it's not a set *of sets*) without knowing how we concretised ordered pairs.

For any $x \in X$ we consider the function $f_x : y \mapsto \langle x,y\rangle$. Then f_x"Y is just $\{x\} \times Y$. Consider now the function $F_x : x \mapsto \{x\} \times Y$. Then F_x"X is $\{\{x\} \times Y : x \in X\}$ and $\bigcup$ of this is just $X \times Y$. ■

Notice that we have not made any assumptions about what particular object the ordered pair $\langle x,y\rangle$ might be for $x \in X$ and $y \in Y$. However we have assumed (twice, with f_x and F_x) that *the image of a set in a function is a set.* This assumption is the **Axiom Scheme of Replacement**.

(If you want to prove the existence of $X \times Y$ in the special case where your ordered pairs are Wiener-Kuratowski you don't need replacement, tho' you do need power set.

This is an old example sheet question, and you might like to try it – it'll help you to get a feel for set-theoretic manipulation.)

The formulation of the replacement scheme in the language of set theory is slightly fiddly, because we do not want variables ranging over global functions:

$$(\forall y)(\exists! x)(\phi(y,x)) \to (\forall Y)(\exists X)(\forall x)(x \in X \longleftrightarrow (\exists y \in Y)(\phi(y,x)))$$

Of course this can be done with parameters, but stating that makes it even harder to read.

The image of a set in a class is a set

The upper case 'X' and 'Y' are not second-order variables; i'm using upper case to make it easier to read. [There is actually a converse to this: if $X \times Y$ always exists however you implement pairing and unpairing then the axiom scheme of replacement follows. It's question 15 on sheet 4.]

Now that we have replacement various things become possible. We can give a proper definition of transitive closure and we can construct the cumulative hierarchy.

Let us take these in turn.

6.1 Transitive Closures and Transitive Sets

The justification I gave of R-induction on the assumption that R is wellfounded was an informal one. Now that we are doing set theory formally the time has come to formally deduce R-induction from the assumption that R is wellfounded.

Suppose $(\forall x)((\forall y)(R(y,x) \to F(y)) \to F(x))$. Suppose further (with a view to obtaining a contradiction) that $\neg F(a)$ for some a. Naturally we want a to give rise to a set with no R-minimal element, thereby contradicting wellfoundedness of R. The obvious candidate is the collection $\{z : R^*(z,a) \wedge \neg F(z)\}$ which is a subset (so we use separation) of the collection $\{z : R^*(z,a)\}$ of things related to a by R^* – aka the *transitive closure* of R. How are we going to prove that $\{z : R^*(z,a)\}$ is a set?

If the structure over which we are doing R-induction is a set, so we are looking at the structure $\langle X, R\rangle$ then $\{z : R^*(z,a)\}$ is a subset of X and we can obtain it by separation. However in a lot of cases where we are doing induction the domain over which we are doing the induction is not a set, so this line will not work.

If we are to obtain $\{z : R^*(z,a)\}$ by means of the axioms we have seen so far we are clearly going to have to use replacement, as follows. We write 'R"$\{a\}$' for '$\{x : R(x,a)\}$'. Use the function $n \mapsto R^n$"$\{a\}$ and take the image of $\mathbb{N}$ in it; then do $\bigcup$ to the result. The trouble with this is that the 'n' is not a variable in the language. We need a relation Φ that relates n to R^n"$\{a\}$. We do this by saying $\phi(n,X)$ if every set that contains $\langle n, X\rangle$ and contains $\langle m, Y\rangle$ whenever it contains $\langle m+1, R$"$Y\rangle$ also[1] contains $\langle 0, \{a\}\rangle$.

$$\Phi(n,X) \longleftrightarrow (\forall A)((\langle n,X\rangle \in A \wedge (\forall m, Y)(\langle m+1, R\text{"}Y\rangle \in A \to \langle m, Y\rangle \in A)) \to \langle 0, \{a\}\rangle \in A)$$

Then, writing '$x = \phi(n)$' for '$\Phi(n,X)$', ϕ becomes the function $n \mapsto R^n$"$\{a\}$ tghat we needed. (I do not supply a proof that Φ is functional in this way. Perhaps i should)

It takes a while to get your head round this definition! This is *Quine's trick*.

[1] Thank you Saul MIller!

DEFINITION 34 *Transitive closure*
The collection

$$TC(x) = \bigcup_{n\in\mathbb{N}}(\bigcup^n x)$$

is the **Transitive Closure** *of x.*
We also say: x is **transitive** *if* $x \subseteq \mathcal{P}(x)$.

Observe that the property of being a transitive set is intersection-closed.
By Quine's trick plus replacement we prove that $TC(x)$ is always a set.
Beware overloading of this terminology! (We have "transitive closures" of relations too! – see p. 12). Evidently $TC(x)$ is the $\subseteq$-least transitive superset of x.

6.2 The Cumulative Hierarchy

DEFINITION 35 *The* **Cumulative Hierarchy**
is defined by recursion on the ordinals:

$$V_\alpha =: \bigcup_{\beta<\alpha}\mathcal{P}(V_\beta).$$

We need, perhaps, to say a little bit about why this definition is legitimate. *Prima facie* there is a worry because we are doing a recursion over all the ordinals (which is not a set – see corollary 3) rather than merely over an initial segment of it (which is). It's OK because of two things: (i) for each initial segment the function defined on that initial segment is unique, and (ii) the functions we define by recursion on those initial segments all agree.

V_α is the collection of things on God's bench at dawn on day α. We want to be sure that V_α exists (*i.e.*, is a set) for all α. Obviously we want to do an induction over the ordinals. No problem at successor ordinals, co's we use Power set. The justification at limit ordinals needs replacement. We need to take the image of the collection of ordinals below α in the function $\gamma \mapsto V_\gamma$.

Can do no harm to take some time out to think about what the various V_αs look like. V_0 is empty; $V_1 = \{\emptyset\}$; $V_2 = \{\emptyset, \{\emptyset\}\}$ .... (How big is V_n?). What does V_ω consist of?

We now define a rank function on members of the cumulative hierarchy:

DEFINITION 36 *$\rho(x)$ is the least ordinal α such that $x \subseteq V_\alpha$.*

Conway used to say of the rank of a set that it was the set's *birthday*. This comes from a lovely image he used ....

Every morning God wakes up, makes himself a nice strong cup of tea, and sets[2] to work creating sets. Each morning he creates – simultaneously – all the sets that can be made from the sets that he finds on his bench at dawn that morning. Thus each set gets created at the first opportunity, on the first day after all its members have been created.

[2] Joke; JOKE!!

Thus the birthday of a set is the first day after the birthdays of all its members. The set of birthdays is wellordered, so each birthday has an ordinal.

The alert and suspicious reader will notice that I am using the same letter 'ρ' here as in definition 12, and will wonder whether or not this is legitimate. It is: the two rank functions are the same, and I think it is safe to bounce this back to the reader.

(i) We prove by induction on the ordinals that $\langle V_\alpha, \in\restriction V_\alpha\rangle$ is a wellfounded binary structure, so it has a rank function.
(ii) Then we prove that all the rank functions we obtain, for all α, agree.
(iii) Finally we prove that they agree with the *other* (novel) rank function that we have just defined in definition 36.

Now is as good a place as any to record the fact that every V_α is transitive. So every set in the cumulative hierarchy has a transitive closure in the cumulative hierarchy.

LEMMA 7 $(\forall\alpha)(V_\alpha \subseteq \mathcal{P}(V_\alpha))$

Proof:

Of course you do this by induction. I think I can safely leave the details to the reader. ■

There is an intimate connection between the cumulative hierarchy and the concretisation project...

We cannot straightforwardly concretise/implement cardinals as equivalence classes if we have separation beco's $\bigcup \alpha = V$ whenever α is an equipollence class, so separation would give us Russell's paradox.

REMARK 10 *If α is an equipollence class (other than $\{\emptyset\}$) then $\bigcup \alpha = V$.*

Proof:

Suppose not. Then there is b s.t. $(\forall A \in \alpha)(b \notin A)$. Let A be a member of α (any will do). For any $a \in A$, the set $(A \cup \{b\}) \setminus \{a\}$ is in bijection with A and is therefore in α. But then $b \in \bigcup \alpha$ after all.

■

COROLLARY 11 *The only equipollence class that is a set is $\{\emptyset\}$.*

In fact something analogous happens for any any equivalence relation $\sim$ with a natural global definition: if $[X]_\sim$ is an equivalence class then $\bigcup^n [X]_\sim = V$ for some small concrete n depending only on $\sim$. For example, if $\sim$ is equipollence, so that $[\{x\}]_\sim$ is the number 1, the equivalence class of singletons, then its sumset is V. However the details of the proof depend very sensitively on the definition of the equivalence relation, so we don't bother with the details, but just draw the moral: equivalence classes are not the way to concretise mathematical objects arising from equivalence relations.

However, now that we have the cumulative hierarchy, we are in a position to solve the problem of implementing objects that arise from equivalence relations.

6.3 Scott's Trick

DEFINITION 37 Scott's trick

When trying to concretise/implement a mathematical entity that arises naturally from equivalence classes [of entities already implemented as sets] for an equivalence relation, then instead of using $[x]_\sim$ *the* $\sim$*-equivalence class of a set x, we use instead the collection of things that are* $\sim$ *to x and are of minimal rank with that property.*

Thus, if $\sim$ is an equivalence relation we instantiate the (as it might be, cardinal) not as the *true* equivalence class – which might not be available – but instead as $[x]_\sim \cap V_\alpha$ where α is the least ordinal α s.t. $[x]_\sim \cap V_\alpha$ is nonempty. Observe that x might not be a member of its (as-it-might-be) cardinal thus construed! Observe that deployment of Scott's trick relies on everything being in the cumulative hierarchy (or at least equivalent to something in the cumulative hierarchy). If x is not in the cumulative hierarchy and nothing $\sim x$ is in the cumulative hierarchy, then Scott's definition turns up the empty set, which is not what we want.

For this to work we need to be sure that, for all x and all equivalence relations $\sim$, there is some y in the cumulative hierarchy with $y \sim x$. There are various axioms that deliver this (one of them is the antifoundation axiom of Forti and Honsell, which you may have heard of: "every set picture is a picture of a unique set"[3]) but the simplest way to ensure it is to brutally assume that every set is in the cumulative hierarchy.

This is one of the various forms of the axiom of foundation.

> In practice Scott's trick is only ever invoked to cut down to a set an equivalence class that is a *proper class*. That's when it is really useful. However it can be used in settings where everything is a set. For example, if we think of natural numbers as finite von Neumann ordinals, and then integers as equivalence classes of ordered pairs of naturals under the relation $\langle a, b\rangle \sim \langle x, y\rangle$ iff $a+y = b+x$ then the equivalence classes are sets and each integer has rank ω. But of course if one uses Scott's trick one can think of an integer as a set of finite rank. This was brought to my attention by my student Saul MIller – the only person know to me to think of it! Thank you Saul!

6.4 The Axiom of Foundation

This axiom takes various forms, and it's worth taking some time to straighten them out.

One form is the assertion that every set is wellfounded. What do we mean by a wellfounded set? We know what a wellfounded *relation* is, but a wellfounded *set*? The most intuitively appealing way to characterise wellfounded sets is to say that x is a wellfounded set iff there is no ω-sequence $\langle x_i : i \in \mathbb{N}\rangle$ with $x = x_0$ and $(\forall n \in \mathbb{N})(x_{i+1} \in x_i)$, but this is equivalent to the correct definition only if we have dependent choice. The correct definition is that x is a wellfounded set iff $\in\restriction TC(\{x\})$ is a wellfounded relation.

So we want an axiom that says that all sets are wellfounded. We can do this by saying that $\in$ is a wellfounded relation, but that's a bit suspect because the universe is not a set if foundation holds, so we are cutting off the branch we are sitting on.

[3] A set picture is a digraph (set of ordered pairs) that looks as if it could be the graph of $\in$ restricted to a transitive set. See definition 46

We can adopt a scheme of $\in$-induction.

We can say that every set is wellfounded, as above.

There is also the axiom of restriction ...

The axiom of restriction says $(\forall x)(\forall y)(x \in y \to (\exists z \in y)(z \cap y = \emptyset))$. "$(\exists z \in y)(z \cap y = \emptyset)$" ... sounds a bit more like foundation if you read it as "y has an $\in$-minimal element". But what about the "$(\forall x)(\forall y)(x \in y \to$" bit? This harks back to the proof by mathematical induction that every nonempty set of natural numbers has a $<_{\mathbb{N}}$-least element. You prove by induction on n that every subset of $\mathbb{N}$ containing n has a $<_{\mathbb{N}}$-minimal element.

The axiom of restriction is an attempt to say that every nonempty set has an $\in$-minimal element:

$$(\forall y)(y \neq \emptyset \to (\exists z \in y)(z \cap y = \emptyset)).$$

$$(\forall y)((\exists x)(x \in y) \to (\exists z \in y)(z \cap y = \emptyset)).$$

By standard manipulation of first-order formulæ this becomes

$$(\forall y)(\forall x)(x \in y \to (\exists z \in y)(z \cap y = \emptyset)).$$

and then you permute the quantifiers

$$(\forall x)(\forall y)(x \in y \to (\exists z \in y)(z \cap y = \emptyset)).$$

and then it appears to say that every x has a certain property, which we call 'regular'.

This is why the axiom of restriction/foundation is important: by unleashing Scott's trick it enables us to always concretise any mathematical entity arising from an equivalence relation. If we do *not* have the axiom of foundation then models can be found in which there are illfounded sets that are not the same size as any wellfounded set. That would mean, at the very least, that we cannot use Scott's trick to implement cardinals.

Thus Scott's trick in conjunction with the axiom of foundation has solved the concretisation problem for objects arising from equivalence relations.

There are still two axioms we haven't mentioned, at least not in this connection. One is the axiom of choice, which we saw earlier. The other is the Axiom of Infinity, and it arises from the need to implement $\mathbb{N}$ and $\mathbb{R}$. It's clear than any set that implements $\mathbb{N}$ must be infinite, and we have not so far had an axiom that tells us there are infinite sets and we can no longer postpone postulating them. The axiom of infinity will tell us that there is an infinite set. It comes in various forms, and if we have the axiom scheme of replacement and foundation and AC then all the forms you might think of turn out to be equivalent. One specially fiddly version that is often seen in the literature is

Axiom of Infinity: $(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))$

Quite why it should take this form has something to do with the implementation of ordinals, to which we now turn.

We can of course use Scott's trick to implement ordinals but with ordinals we have an extra trick up our sleeve. Every equivalence class (= abstract ordinal) contains a wellordering whose order relation is set membership, and this wellordering is unique. We prove this using ...

6.5 Mostowski Collapse

LEMMA 8 *(Mostowski's collapse lemma)*
If $\langle X, R\rangle$ is a well-founded structure, then there is a transitive set Y and a homomorphism $\pi{:}\langle X, R\rangle \twoheadrightarrow \langle Y, \in\rangle$.

Proof:
We use the theorem about wellfounded induction and recursion, theorem 1.
Define $\pi(x) := \{\pi(y) : R(y, x)\}$. The *definiens* (The RHS) is a set by replacement.
The desired Y is simply the range of π. Y is transitive because nothing ever gets put into Y unless all its members have been put in first. ■

The map given by this construction is parsimonious (see p. 20).

Mostowski collapse shows that every wellfounded structure $\langle X, R\rangle$ has a homomorphism π onto a structure $\langle \pi\text{“}X, \in\rangle$ where $\pi\text{“}X$ is a transitive set.

It is worth recording that If we execute the Nostowski collapse construction simultaneously on two isomorphic wellfounded structures we get the same transitive set. Using terminology we introduced on p. 10 we can say that the equivalence relation of isomorphism between wellfounded binary structures is a congruence relation for the function that sends a structure to its Mostowski collapse. It may be worth recording also that this means that the function that sends each wellfounded binary structure to its Mostowski collapse is a *classifier* for the equivalence relation of isomorphism between wellfounded binary structures.

We haven't defined classifiers yet so here goes (should probably move this earlier)

DEFINITION 38 *A classifier for an equivalence relation $\sim$ is a function f s.t. $(\forall x, y)(x \sim y \longleftrightarrow f(x) = f(y))$.*

Plenty of mathematical entities arise from equivalence relations, more specifically from equivalence classes wrt such a relation. For example cardinal numbers correspond to equivalence classes of sets under equinumerosity. If we want to concretise cardinals as sets we will need a function (the "cardinal-of" function) that sends two sets to the same thing iff they are equinmerous. So whenever we have a suite of mathematical objects that arise from an equivalence relation in this way we desire a classifier for that equivalence relation.

In general there is no reason to expect that the homomorphism π is injective. It's simple to give illustrations where it is and also illustrations where it isn't. If $\{y : R(y, x_1)\} = \{y : R(y, x_2)\}$ then clearly $\pi(x_1) = \pi(x_2)$. Clearly if there is no such pair x_1 and x_2 then π will be injective. Recall from page 62 that in these circumstances we say that R is **extensional**. Reflect that the axiom of extensionality says that $\in$ is extensional.

If R is extensional, then no two things in X have the same set of R-predecessors and so no two things ever get sent to the same thing by π. So if R is extensional then π is injective. This gives us the special case:

REMARK 11 *If $\langle X, R\rangle$ is a well-founded extensional structure, then there is a* **unique** *transitive set Y and a unique* **iso***morphism between $\langle X, R\rangle$ and $\langle Y, \in\rangle$.*

Why 'collapse'? A good question (and one not often asked!) It's a piece of mathematical slang, but none the worse for that. I think it's something to do with the thought that the move from the wellfounded structure to its homomorphic image under π *destroys information.* Any two isomorphic wellfounded structures get collapsed to the same thing. This is important, because it means that the things-they-get-collapsed-to can be taken as concretisations of their isomorphism type.

Mostowski collapse is a crucial lemma in the study of wellfounded sets, and it gets used all the time, but we mustn't lose track of the fact that we are encountering it in the context of a story about how to implement ordinals. So we ask: What happens in the cases where $\langle X, R\rangle$ is a wellordering? Wellorders are total orders so distinct things have distinct predecessors so the homomorphism is an isomorphism.

Thus every wellordering is isomorphic to a wellordering whose order relation is $\in$! And this wellordering is of course unique. [Why?] We then take this canonical representative to be our ordinal.

DEFINITION 39
Every wellordering is isomorphic to a unique wellordering $\langle X, \in\rangle$ where X is a transitive set. Such a wellordering is a **von Neumann ordinal**.

(often just called plain 'ordinals' [which is naughty]).

The fact, noted above, that two isomorphic wellorderings get sent by Mostowski collapse to the same transitive set is why it's safe to think of these collapsed wellorderings as ordinals.

REMARK 12

- *The order relation $<_{On}$ on von Neumann ordinals is $\in$;*
- *Each ordinal is identical to the set of its predecessors;*
- $\alpha + 1 = \alpha \cup \{\alpha\}$ *if our ordinals are Von Neumann.*

Notice that in showing that every wellordering is isomorphic to a unique von Neumann ordinal we have used replacement but have not used foundation.

The fact that every von Neumann ordinal coincides with the set of the ordinals below it fits very cutely with theorem 4 that says that every ordinal counts the set of ordinals below it in their natural order.

Notice also that although we have shown that every wellordering is isomorphic to a special one (which we can use as its ordinal) namely the wellordering whose order relation is $\in$, there doesn't seem to be a similar move available for cardinals. Given a set x is there an obvious special set in bijection with x, something that we can use as its cardinal? Not clear. We will return to this later.

Now is the moment to observe that the peculiarly specific form of the axiom of infinity we saw on p. 68 has a purpose. It precisely gives us a set containing 0 and closed under successor, and we can obtain the $\subseteq$-least such set from it by separation, as follows:

Let A be a set given by that fancy version of the axiom of infinity. Then the set we want is

$$\{x : x \in A \wedge (\forall y)(\emptyset \in y \wedge (\forall w)(w \in y \to w \cup \{w\} \in y) \to x \in y)\}$$

which is a set by separation. That set is of course the set of finite von Neumann ordinals, which will do for our implementation of $\mathbb{N}$.

Observe that – on this implementation – the set $\mathbb{N}$ of natural numbers is identical with the set of finite ordinals which – since we are using von Neumann ordinals – turns out to be the same as the (von Neumann) ordinal ω. These three objects are of course distinct as mathematical objects, but for all that they all get implemented in set theory as this one set. You will often see set theorists (and others who drink Kool-aid with them) write 'ω' instead[4] of '$\mathbb{N}$'. This phenomenon is suceptible of sociolinguistic analysis: no number theorist ever uses this notation.

> If the set of finite von Neuann ordinals (aka the von Neumann ω) is to do duty as the set of natural numbers we will have to define arithmetic operations on its members. Successor we have seen: $\mathtt{succ}(n) = n \cup \{n\}$. But what about + and $\times$? Do not fret if you can't see an obvious natural way of doing it: there isn't an obvious nice natural way. However we can define these operations by recursion, and that will have to do

Once we've implemented ordinals we can implement integers, rationals, reals and complexes. In lots of different ways, in fact.

> Naturals can be von Neumann naturals or Zermelo naturals or Scott's trick naturals;
>
> Integers can be signed naturals or equivalence classes of ordered pairs of naturals ("field of fractions");
>
> Rationals can be signed ordered pairs of naturals or equivalence classes of ordered pairs of integers ("field of fractions");
>
> Reals can be Dedekind cuts in rationals or equivalence classes of Cauchy sequences of rationals;
>
> Complex numbers typically are thought of as ordered pairs of reals.
>
> Perhaps a word is in order on the "field of fractions" construction.Start with $\mathbb{N}\times\mathbb{N}$ and define an addition operation on these pairs pointwise. We say that two pairs are equivalent if they "have the same difference". More formally (since we don't want to talk about negative integers yet so we can't yet say that difference is a negative number even if it is) we say $\langle a, b\rangle \sim \langle x, y\rangle$ iff $a + y = b + x$. This equivalence relation is a congruence relation for the pointwise addition so when we take the quotient we get an addition on it for free. The quotient is $\mathbb{Z}$.

And in every case where you are using equivalence classes to implement something there is the possibility of using Scott's trick to cut the class down to something smaller.

It's a very helpful exercise to crunch out the ranks of the sets that implement these various mathematical objects under the assorted possible implementations. Question 11 on sheet 4 invites you to do that.

[4] It gets worse. We will see soon how the cardinal number $\aleph_0$ becomes the fourth mathematical object to be implemented as this set. I'm not making this up

The answers we get do not matter in the slightest – the ordinals obtained are properties of the implementing sets, not of the mathematical entities themselves[5] – but the exercise will give you experience in manipulating some purely set theoretic quantities, and prepare you for doing some more idiomatic set theory in the days to come – something you will not have done before.

Set-theoretic foundationalism offers us both the Cauchy reals and the Dedekind reals, so it had better prove that they are isomorphic. One way of doing this is to prove that the Cauchy reals are complete and then appeal to the categoricity of the theory of complete ordered fields. So a useful exercise is to prove that the Cauchy reals are complete. It is an important fact that this does not need the axiom of choice. I think i shall put this on sheet 5.

6.6 Ordinals again

We now pick up the thread dropped on page 19. .

DEFINITION 40
(i) An aleph is the cardinality of a (usually infinite) wellordered set;
(ii) $\aleph(\alpha)$*, for* α *a cardinal, is the least aleph* $\not\leq \alpha$*.*

(I think we first saw this aleph-without-a-subscript notation on p 19).

Look again at the proof of lemma 5, Hartogs' lemma, which told us that $\aleph(\alpha)$ is always defined. The proof I gave there used replacement (tho' we didn't bring out the use of replacement!) It is possible to give a proof without replacement (as Hartogs in fact originally did, the axiom scheme of replacement not having been formulated at that stage) as follows.

Given X we seek a wellordered set Y with $|Y| \not\leq |X|$.

Consider $\mathcal{P}(X \times X)$ (use Wiener-Kuratowski ordered pairs if you want to be specific); throw away every subset that isn't a wellordering; quotient out what's left under isomorphism. The result is (a concretisation) of the set of ordinals of wellorderings of subsets of X – as it were equivalence-classes-local-to-X – and is the Y we desire.

This argument gives us an upper bound for $\aleph(|X|)$: $\aleph(\alpha) \leq 2^{2^{\alpha^2}}$. By modifying the construction you can obtain better bounds (such as $\aleph(\alpha) \leq^* 2^{\alpha^2}$ – where the asterisk means surjection) but we don't need them.

6.6.1 Initial Ordinals

DEFINITION 41 *If* $\langle X, <_X \rangle$ *is a wellordering of order type* α *then* $card(\alpha)$ *is the cardinality* $|X|$ *of the underlying ("carrier") set* X*.*

You could also say $card(\alpha)$ *is, for example,* $|\{\beta : \beta <_{On} \alpha\}|$.

[5] People sometimes use the phrase "essential rank" of a mathematical entity. The essential rank of a structure $\mathcal{A}$ is the least ordinal α s.t. $|A| \leq \beth_\alpha$. (The funny symbol is *beth*, the second letter of the Hebrew alphabet. Like *aleph*, the first letter, it s used to denote (some) cardinals.) Alternatively it's the least ordinal α s.t. V_α is big enuff to contain a copy of A (the carrier set of $\mathcal{A}$). It's a nice phrase, and it should be standard, but it isn't.

This 'card' notation is in the literature, but it is not in common use, and you will not need to know it once you have digested definition 42.

> The chief reason why it is not in common use is the thoroughly bad, hacky, reason that: if α is a Von Neumann ordinal, then $card(\alpha)$ is in fact $|\alpha|$, the cardinality of the actual set α. So, if our ordinals are Von Neumann, we can always write '$|\alpha|$' instead of '$card(\alpha)$' – with the effect that we don't need the latter notation!

DEFINITION 42
An ordinal α is **initial** *if* $(\forall \beta <_{On} \alpha)(card(\beta) <_{card} card(\alpha))$.
We enumerate the initial ordinals as $\omega_0, \omega_1, \ldots \omega_\alpha \ldots$, *and*
We define $\aleph_\alpha$ *to be* $card(\omega_\alpha)$ *which of course was* $|\{\beta : \beta <_{On} \omega_\alpha\}|$;
An **aleph** *is a cardinal of a wellorderable set.*

Observe that all items in definition 42 are genuinely definitions not an implementation.

The following should be evident:

> $\aleph_\alpha$ is also the αth aleph;
> $\aleph_{\alpha+1}$ is $\aleph(\aleph_\alpha)$;
> The alephs are wellordered by $<_{card}$.

It might do the reader no harm to explain why these things are true. Alephs are cardinals of wellorderable sets; why are they wellordered by $<_{card}$?

This notation is legitimate because, if X is wellorderable, the Y that we obtain from the construction in the proof of theorem 5 is of minimal size $\not\leq |X|$. So, if $|X|$ is the αth aleph, $|Y|$ is the $(\alpha + 1)$-th aleph. Is this OK? Yes: each aleph corresponds to a unique initial ordinal, so – by theorem 3 – the alephs are wellordered by $<_{card}$, so we can enumerate them using ordinals.

We can use initial ordinals to implement alephs as sets. Every aleph corresponds to a unique initial ordinal, so we can implement an aleph as the corresponding (von Neumann) initial ordinal. If we are willing to adopt AC then every cardinal is an aleph, and we have in fact implemented all cardinals. Could we not have implemented cardinals by Scott's trick? Yes, if we have foundation, or even if we have the (weaker) assertion that every set is the same size as a wellfounded set. This route *via* von Neumann initial ordinals doesn't need either of these assumptions, but it does use AC.

However it is blindingly cute, and has become the industry standard.

REMARK 13 *Every regular ordinal is initial.*

Proof:

It's not a particularly deep or important fact but it's basic and will help you orient yourself. And the proof is idiomatic. Actually we prove the contrapositive.

We need a factoid. Suppose $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ are (strict) total orders, with $<_A$ a wellorder, and there is a bijection $f : A \twoheadrightarrow B$. (We probably want B to not have a last element; must check what else we might need). We are *not* assuming that f is order-preserving! Nevertheless f does have a maximal order-preserving restriction, a

rather special one: there is $A' \subseteq A$ s.t $f \restriction A'$ is order-preserving, and f"A' is cofinal (unbounded) in $\langle B, <_B \rangle$.

We obtain A' by recursion on $\langle A, <_A \rangle$. The first member of A' is the bottom element of $\langle A, <_A \rangle$. Thereafter the next member is always the $<_A$-least element a of A. s.t. $f(a) >_B f(a')$ for all $a' <_A a$ that we have already put into A'. Suppose f"A' were bounded in $\langle B, <_B \rangle$. Consider the subset $B' \subseteq B$ consisting of things not dominated by any $f(a)$ for $a \in A'$, and consider the $b \in B'$ s.t. $f^{-1}(b)$ is $<_A$-minimal. $f^{-1}(b)$ should have been put into A'.

End of factoid: ■

Now suppose β is not an initial ordinal. (As I said, we are proving the contrapositive). Then there is $\alpha < \beta$ s.t. α has as many predecessors as β. Let $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ (as in the factoid) be the ordinals below α and the ordinals below β respectively. The factoid gives us a set of ordinals cofinal in β whose order type $\leq \alpha < \beta$. So β is not regular. ■

Here is an illustration of a particular case.

The picture shows why every countable limit ordinal has cofinality ω. The long right-pointing arrow represents a countable ordinal manifested as a wellordering of naturals ($\mathbb{N}$ in a funny order). The (unbounded!) increasing sequence of natural numbers reading from the left are the numbers chosen as in the recursion ... 1001 is the least natural number > 257 that is above 257 in both orders. The semicircle represesents where this increasing sequence of naturals comes to a halt, closes off. Are there any natural numbers in the region flagged by the question marks? Suppose there were – 347, say. OK, so what were doing declaring 1001 to be the 6th member of the sequence? We should have used 347!

Thus every countable limit ordinal λ is the sup of an ω-sequence $\langle \lambda_i : i < \omega \rangle$ of smaller ordinals.

> An apparently minor point that i want to emphasise here ... To show that a countable limit ordinal λ has cofinality ω we need to have a counting of the ordinals below λ or – which is in effect the same thing – a wellordering of $\mathbb{N}$ of order type λ. *You can't compute an ω-sequence $\langle \lambda_n : n < \omega \rangle$ whose sup is λ simply from λ itself.*

DEFINITION 43
Such a sequence of smaller ordinals is a **fundamental sequence** *for λ.*

Fundamental sequences give you a way of using ordinals to measure how rapidly growing a function $f : \mathbb{N} \to \mathbb{N}$ is. One can define a sequence f_α over countable ordinals α by something like $f_0(n) = n + 1$; $f_{\alpha+1}(n) = (f_\alpha)^n(n)$ and (and this is the clever bit) if λ is the sup of $\langle \lambda_n : n < \omega \rangle$ set $f_\lambda(n) = f_{\lambda_n}(n)$.

[Something to think about ... every regular ordinal is initial ... is every initial ordinal regular... ? ω is initial and is regular; you saw in an example sheet question that ω_1 (which is obviously initial) is regular ...]

6.7 $\aleph^2 = \aleph$

(Using the letter '$\aleph$' as a variable to range over alephs...)

We start by noting that $\aleph = \aleph + \aleph$. (Well, all we will *actually* need is $\aleph + \aleph + \aleph = \aleph$, but never mind). Beginners might like to have this spelled out, and it holds because $2 \cdot \omega_\alpha = \omega_\alpha$. How so? Any order of limit order-type consists of lots of concatenated copies of $\mathbb{N}$, each of length ω. You can interleave two (or indeed three) worders of length ω to get a worder of length ω so you can do this for all the copies simultaneously.

We start by defining a function $\mathfrak{S} : On \to On$. Given an ordinal α, take a wellordering $\langle A, <_A \rangle$ of order type α, make disjoint copies of all its proper initial segments, and then concatenate the copies ... with longer things appended after shorter things.

The result is a wellordering and its order type is defined to be $\mathfrak{S}(\alpha)$. [This notation is not standard, and I am not going to use it outside this proof so i'm not numbering its definition]. Thus – for example – $\mathfrak{S}(\omega) = 1 + 2 + 3 + 4 + \ldots = \omega$

LEMMA 9

(i) $\mathfrak{S} : On \to On$ *is a normal function;*
(ii) Every initial ordinal is a value of $\mathfrak{S}$.

Proof:

(i) $\mathfrak{S} : On \to On$ evidently also has a recursive definition:

$\mathfrak{S}(\alpha + 1) = \mathfrak{S}(\alpha) + \alpha$ and
$\mathfrak{S}(\lambda) = \text{Sup}\{\mathfrak{S}(\alpha) : \alpha < \lambda\}$ for λ limit.

... from which it is clear that $\mathfrak{S}$ is a normal function.

(ii) Use the division algorithm for normal functions to show that there is a β s.t. $\mathfrak{S}(\beta) \leq \omega_\alpha < \mathfrak{S}(\beta + 1)$. If $\mathfrak{S}(\beta) < \omega_\alpha$ then we have $\omega_\alpha \leq \mathfrak{S}(\beta + 1) = \mathfrak{S}(\beta) + \beta$ which is impossible, since $\mathfrak{S}(\beta)$ and β both have cardinality below $\aleph_\alpha$. ■

We want to show that $(\aleph_\alpha)^2 = \aleph_\alpha$. $\aleph_\alpha$ is defined as $card(\omega_\alpha)$, so it is the cardinal $|\{\beta : \beta < \omega_\alpha\}|$, which means that the canonical set of size $(\aleph_\alpha)^2$ is the cartesian product

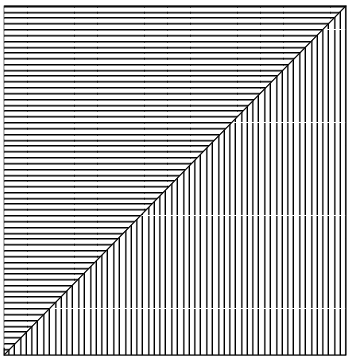
$$\{\beta : \beta < \omega_\alpha\} \times \{\beta : \beta < \omega_\alpha\}.$$

We partition this last set into three pieces:

(i) the [graph of] the identity relation restricted to $\{\beta : \beta < \omega_\alpha\}$, and

(ii), (iii)

the two triangles above-and-to-the-left, and below-and-to-the-right of the diagonal.



To be slightly more formal about it, we partition the cartesian product

$$\{\beta : \beta < \omega_\alpha\} \times \{\beta : \beta < \omega_\alpha\}$$

into the three pieces

$$\{\langle\beta,\gamma\rangle : \beta < \gamma < \omega_\alpha\}, \quad \{\langle\beta,\gamma\rangle : \beta = \gamma < \omega_\alpha\} \quad \text{and} \quad \{\langle\beta,\gamma\rangle : \gamma < \beta < \omega_\alpha\}.$$

It is clear that the third piece is of order type $\Im(\omega_\alpha)$ in the lexicographic order.

The idea is to show that these three pieces all have cardinality $\aleph_\alpha$. That's obvious for the second piece, the identity relation. Also there is an obvious bijection between the first and third piece ("flip your ordered pairs") so it will suffice to prove that the third piece ("the bottom-right triangle") has cardinality $\aleph_\alpha$.

Now we can prove

THEOREM 13 $(\forall\alpha)(\aleph_\alpha = (\aleph_\alpha)^2)$.

Proof:

By induction on α. The fact that it holds for $\alpha = 0$ you learnt in first year.

Assume true for all alephs $< \aleph_\alpha$. By lemma 9, ω_α is a value of $\Im$; we want to show that it is actually a fixed point. Now ω_α is an initial ordinal, which is to say that for any $\beta < \omega_\alpha$, the cardinal $|\{\gamma : \gamma < \beta\}|$ is less than $\aleph_\alpha$, and (by induction hypthesis) is equal to its own square. Suppose ω_α were $\Im(\beta)$ for some $\beta < \omega_\alpha$. This would entail that the size of the cartesian product $\{\gamma : \gamma < \beta\} \times \{\gamma : \gamma < \beta\}$ is at least $\aleph_\alpha$, contradicting the induction. So ω_α is a fixed point of $\Im$. This means that the lower-right triangle of the cartesian product $\{\gamma : \gamma < \omega_\alpha\} \times \{\gamma : \gamma < \omega_\alpha\}$ – which can be wellordered to length $\Im(\omega_\alpha) = \omega_\alpha$ – is of cardinality $\aleph_\alpha$. It's clearly naturally isomorphic to the upper-left triangle (as remarked earlier) so the cartesian product is now a union of three sets each of size $\aleph_\alpha$, giving $(\aleph_\alpha)^2 = \aleph_\alpha + \aleph_\alpha + \aleph_\alpha = \aleph_\alpha$ as desired.

■

Thus if the axiom of choice holds (so that every infinite cardinal is an aleph) then $\alpha = \alpha^2$ for *all* infinite[6] cardinals α. There is a converse!

[6] In fact we can prove that $\alpha = \alpha^2$ for *all* infinite cardinals directly using Zorn's lemma. The reader might like to prove this for their own satisfaction. In fact i might set it as an exercise. But our trajectory givs more information, since we get a choice-free proof that $\alpha = \alpha^2$ for infinite alephs.

COROLLARY 12 *If $\alpha = \alpha^2$ for all infinite cardinals, then AC follows.*

Proof:

Let α be an arbitrary infinite cardinal, and suppose $\beta^2 = \beta$ for all infinite cardinals β. Then we have

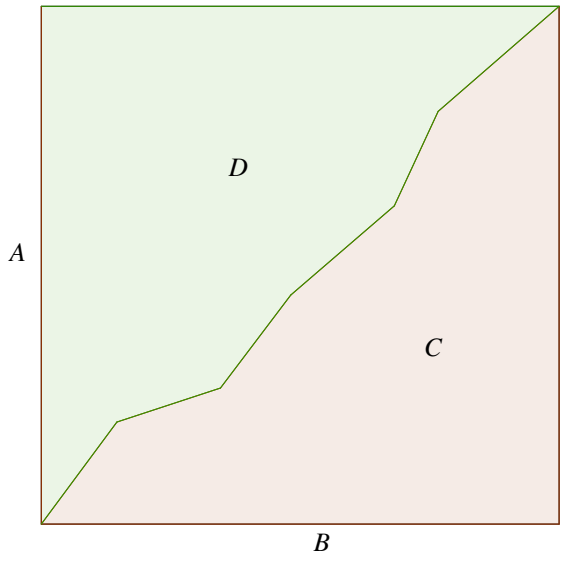
$$\begin{aligned}
\alpha + \aleph(\alpha) &= (\alpha + \aleph(\alpha))^2 \\
&= \alpha^2 + 2 \cdot \alpha \cdot \aleph(\alpha) + (\aleph(\alpha))^2 \\
&= \alpha + 2 \cdot \alpha \cdot \aleph(\alpha) + \aleph(\alpha) \\
&= \alpha + \alpha \cdot \aleph(\alpha) + \aleph(\alpha) \\
&= \alpha(1 + \aleph(\alpha)) + \aleph(\alpha) \\
&= (\alpha \cdot \aleph(\alpha)) + \aleph(\alpha) \\
&= (\alpha + 1) \cdot \aleph(\alpha) \\
&= \alpha \cdot \aleph(\alpha)
\end{aligned}$$

Now we use Bernstein's lemma. . . .

REMARK 14 *Bernstein's lemma*

$$\gamma + \delta = \alpha \cdot \beta \;\rightarrow\; \alpha \leq^* \gamma \;\vee\; \beta \leq \delta$$

Here '$\alpha \leq^* \gamma$' means that there is a surjection from a set of size γ to a set of size[7] α.
Proof:



Proof:

Suppose A and B are two sets (of size α and β). Suppose further that we have split $A \times B$ (represented by the square figure above) into two pieces, C and D (of size γ and δ), so that $C \cap D = \emptyset$ and $C \cup D = A \times B$. Now project the C region onto the A axis.

Does it cover the whole of the A-axis? (I've tried to draw the picture so that it's not clear whether it does or not!) If it does, then $|A| \leq^* |C|$. If it doesn't, then there is a line through D parallel to the B axis, whence $|B| \leq |D|$.

■

We return to the proof of corollary 12. On the face of it there are two ways in which we could apply Bernstein's lemma here.

(i) We can infer $\aleph(\alpha) \leq^* \alpha \vee \alpha \leq \aleph(\alpha)$. The second disjunct is the one we want so we would like to exclude the first disjunct: $\aleph(\alpha) \leq^* \alpha$. For all we know this could happen if α is not an aleph, so we have to use Bernstein the other way round:

(ii) $\aleph(\alpha) \leq \alpha \vee \alpha \leq^* \aleph(\alpha)$.

The first disjunct is of course impossible – by definition of $\aleph(\alpha)$ – so we infer the second, which tells us that any set of size α is a surjective image of a wellordered set. But any such surjective image can be wellordered, and this gives us our result. ■

Bernstein's Lemma looks like an annoying factoid of no interest. However it is an essential prop in the proof that the reals are the same size as the power set of the naturals.

We can also use theorem 13 to show that a lot of initial ordinals are regular.

THEOREM 14 *(uses AC) Every ordinal $\omega_{\alpha+1}$ is regular.*

Proof:

If $\omega_{\alpha+1}$ were the sup of fewer than $\aleph_{\alpha+1}$ (which is to say the sup of no more than $\aleph_\alpha$ smaller) ordinals – then the set of ordinals below it (which is of size $\aleph_{\alpha+1}$) would be a union of at most $\aleph_\alpha$ things each of size $\aleph_\alpha$ at most. We saw in Sheet 2 question 4 how to use AC to show that such a union is of size $(\aleph_\alpha)^2$ at most, and theorem 13 now tells us it is of size $\aleph_\alpha$ at most, which is impossible. ■

The obvious follow-up question is: if λ is limit can ω_λ be regular? It is if $\lambda = 0 \ldots$. The context in which to consider this question is the context of independence proofs, to which we now turn.

6.8 Independence of the Axioms from each other

We've spent quite a lot of time and energy rolling out set theory as a platform on which to do mathematics; it can do no harm to do something a bit more idiomatic; set theory does, after all, have a life of its own. The schedules require me to cover problems of consistency and independence of the axioms, so let's do that.

Do they?

We prove independence results by exhibiting models. We emphasise that for philosophical reasons we are interested only in transitive models. The idea is that, if I give you a set x, I must also give you all its members. So any sensible model with an element x must contain everything in the transitive closure of x as well. Hence our restriction to transitive models only.

That is one reason why Mostowski collapse is so important: it gives us transitive models. Any every transitive set is a model of the axiom of extensionality.

6.8.1 Δ_0 formulae and the Lévy Hierarchy

First we define Δ_0 formulæ and a quantifier hierarchy associated with them

DEFINITION 44

A **restricted quantifier** *in the language of set theory is '$(\forall x)(x \in y \rightarrow \ldots)$' or '$(\exists x)(x \in y \wedge \ldots)$'.*

A Δ_0-formula in the language of set theory is a formula built up from atomics by means of boolean connectives and restricted quantifiers.

Thereafter a Σ_{n+1} (respectively Π_{n+1}) formula is the result of binding variables in a Π_n (respectively Σ_n) formula with existential (respectively universal) quantifiers.

We immediately extend the Σ_n and Π_n classes by closing them under interdeducibility-in-a-theory-T, and signal this by having 'T' as a superscript, so our classes are Σ_n^T and Π_n^T. As usual, we omit the superscripts when they are clear from context.

We find that Δ_0 formulæ behave in many ways as if they contained no quantifiers at all. An unrestricted quantifier is an injunction to scour the whole universe in a search for a witness or a counterexample; a restricted quantifier invites us only to scour that part of the universe that lies in some sense "inside" something already given. The search is therefore "local" and should behave quite differently: that is to say, restricted universal quantification ought to behave like a finite conjunction and ought to distribute over disjunction in the approved de Morgan way. (And restricted existential quantification similarly of course).

Set the definition of 'absolute' up in lights

One effect of this is that Δ_0 predicates are **absolute** between transitive models. This merits a short discussion. If $\phi(x)$ is a formula with one free variable and no quantifiers, and $\mathfrak{M}$ believes there is an x such that $\phi(x)$, then any $\mathfrak{M}' \supseteq \mathfrak{M}$ will believe the same. This much is obvious. The dual of this is similarly obvious: If $\phi(x)$ is a formula with one free variable and no quantifiers, and $\mathfrak{M}$ believes that $\phi(x)$ holds for every x, then any $\mathfrak{M}' \subseteq \mathfrak{M}$ will believe the same. We say that existential formulæ **generalise upwards** and universal formulæ **generalise downwards**. Something analogous holds for Σ_1 formulæ and Π_1 formulæ. They generalise upwards and downwards in the same way *as long as $\mathfrak{M}$ and $\mathfrak{M}'$ are both transitive models.* Δ_0 formulæ of course generalise both upward and downward and are therefore **absolute**.

We need this gadgetry if we are to cope with what is usually the first problem students have with finding models for fragments of ZF. The first thing to note is that if $\mathfrak{M} = \langle M, \in \rangle$ is a model of set theory then '$\mathfrak{M} \models \phi$' is actually a formula of set theory. Which formula? The formula we obtain from 'ϕ' by replacing every quantifier '$(\forall x)(\ldots)$' by '$(\forall x)(x \in M \rightarrow \ldots)$' and replacing every quantifier '$(\exists x)(\ldots)$' by '$(\exists x)(x \in M \wedge \ldots)$'.

The problem I have just spoken of is this: most of the axioms of ZF take the form of an assertion that the universe is closed under some operation or other. If we are to get straight which sets (or classes) are models of which axioms we will need to be absolutely clear about the difference between being closed under an operation and being a model for the axiom that says you are closed under that operation. You might think that for a set to be a model of the axiom that says the world of sets is closed under

operation blah it is necessary and sufficient for that set to be closed under operation blah. But you'd be wrong! We consider two contrasting cases, pairing and power set.

$\mathfrak{M} \models$ the axiom of pairing iff

$$(\forall x \in M)(\forall y \in M)(\exists z \in M)(\forall w \in M)(w \in z \longleftrightarrow w = x \vee w = y)$$

$\mathfrak{M}$ is closed under the pair set operation iff $(\forall x, y \in M)(\{x, y\} \in M)$.

In contrast $\mathfrak{M} \models$ the axiom of power set iff

$$(\forall x \in M)(\exists y \in M)(\forall z \in M)(z \in y \longleftrightarrow (\forall w \in M)(w \in z \rightarrow w \in x))$$

Now, since $\mathfrak{M}$ is transitive, the last bit – $(\forall w \in M)(w \in z \rightarrow w \in x)$ – is equivalent to $z \subseteq x$, so the displayed formula simplifies slightly to

$$(\forall x \in M)(\exists y \in M)(\forall z \in M)(z \in y \longleftrightarrow z \subseteq x)$$

$\mathfrak{M}$ is closed under the power set operation iff

$$(\forall x \in M)(\mathcal{P}(x) \in M)$$

Are these two equivalent? Clearly not. Reflect that, by Downward Skolem-Löwenheim (theorem 17) and Mostowski collapse (lemma 8) ZF has a countable transitive model $\mathfrak{M}$. In a countable transitive model every set must be countable. So the thing in $\mathfrak{M}$ that $\mathfrak{M}$ believes to be the power set of $\mathbb{N}$ has only countably many members (they're all in $\mathfrak{M}$ beco's $\mathfrak{M}$ is transitive) and it therefore cannot possibly be the true power set of the naturals of $\mathfrak{M}$.

The point is that "$x = \{y, z\}$" is just "$y \in x \ \wedge \ z \in x \ \wedge \ (\forall w \in x)(w = y \vee w = z)$" which is Δ_0 and is absolute;

In contrast $x = \mathcal{P}(y)$ is

$(\forall w \in x)(\forall z \in w)(z \in y) \wedge (\forall w)((\forall u)(u \in w \rightarrow u \in y) \rightarrow y \in x)$ which is not Δ_0!

We are now in a position to look at some actual independence results.

6.8.2 Some actual Independence Results

Let's start with the simplest possible example. It exploits V_ω, a set I talked about earlier, and whose existence I proved in lectures. For which axioms ϕ can we establish that $\langle V_\omega, \in\rangle \models \phi$?

Well, it's transitive so it's a model for extensionality. It's a model for pairing and power set, and is actually closed under pairing and under power set. It's a model of separation because any subset of a member of V_ω is also a member of V_ω. What about replacement? Is the image of a set in V_ω in some function also a set in V_ω? Well, obviously not, beco's such a function could send its arguments from V_ω into the wide blue yonder. However we have overstated what is needed for being a model of replacement. For V_ω to be a model of replacement all that is necessary is that if we have a function from V_ω to V_ω *which is definable with all its parameters in V_ω and all its bound variables constrained to range over things in V_ω* then the image of an element

of V_ω in such a function is also in V_ω. And *that* is clearly true – we don't even need the italicised condition.

Reflect that $\mathfrak{M} \not\models \bot$, for all $\mathfrak{M}$, so no inconsistent theory can have a model. Therefore the fact that V_ω is a set means that we have proved the consistency of *something*, that something being whatever the set of things is that are all true in $\langle V_\omega, \in\rangle$.

To cut a long story short it's pretty clear that it is a model of all the axioms except infinity: V_ω not only does not contain any infinite set, it doesn't even contain any set that it mistakenly believes to be infinite. However it satisfies all the other axioms. In fact it's even a model of the Axiom of Choice, and it's a model of the Axiom of Choice even if the theory in which we are conducting this discussion does not assume AC.

This shows that the Axiom of Infinity does not follow from the other axioms of ZFC.

Another structure to consider is $V_{\omega+\omega}$. This is transitive, so it's a model of extensionality. It's obviously a model of pairing, sumset and power set. Also separation This time it's clearly a model of infinity. Not only does it contain an infinite set, it contains an infinite set which is infinite in the sense of the model. ("x is infinite" is not Δ_0 so we have to be careful.)[8]

Put them all on the board and tick them off one by one

It's going to be a model of sumset because something gets into $V_{\omega+\omega}$ as long as its rank is less than $\omega + \omega$ ... and $\bigcup$ decreases rank. (and "$y = \bigcup x$" is Δ_0).

It will be a model of AC as long as the theory in which we are conducting the analysis has AC as an axiom. As long as our ordered pairs are Wiener-Kuratowski any wellordering of a member of $V_{\omega+\omega}$ will also be a member of $V_{\omega+\omega}$, a couple of layers higher up. (W-K pairs increase rank by 2).

So: which axiom or axiom scheme is left? Replacement!

You want to say ... "it can't be a model of replacement, beco's – if it were – it would then be a model of the whole of ZF, and so we would have proved the consistency of ZF inside ZF" and you read somewhere about the Incompleteness theorem of Gödel that says that can't happen. And you'd be right of course. However it would be nice to have an actual instance of replacement that fails. Ideally i'd let you think about it but time is short. Consider the function f that sends n to $V_{\omega+n}$. You will need Quine's trick (page 64) to define it properly.

Observe that the image f"$\mathbb{N}$ of the set of naturals in this function is $\{V_{\omega+n} : n \in \mathbb{N}\}$ and the rank of this set is clearly $\omega + \omega$ so it cannot be a member of $V_{\omega+\omega}$.

Both the models we have considered so far are V_αs. However there are other structures we can use.

DEFINITION 45
$\mathcal{P}_\phi(x) = \{y \subseteq x : \phi(y)\}$;
H_ϕ *is the least fixed point for* $x \mapsto \mathcal{P}_\phi(x)$;
Alternatively $H_\phi = \{x : (\forall y \in TC(\{x\}))\phi(y)\}$.

[8] There is a subtlety here, because the specially sexed-up version of the axiom of infinity $(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))$ that we saw on page 68 asserts that there is an x with a special property, and that special property is Δ_0. The point is that you have to do a bit of work to show that if $\emptyset \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x)$ then x really is Dedekind-infinite.

We also write '$\mathcal{P}_\kappa(x)$' (where κ is a cardinal) for $\{y \subseteq x : |y| < \kappa\}$, and H_κ for the least fixed point of this function, so that $H_\kappa = \{x : (\forall y \in TC(\{x\}))(|y| < \kappa)\}$

The 'H' means 'hereditarily'.

Observe that $V_\omega = H_{\aleph_0}$. $V_{\omega+\omega}$ is not H_α for any cardinal α.

The next H after $H_{\aleph_0}$ is $H_{\aleph_1}$, the set of hereditarily countable sets, commonly notated 'HC'.

It's perhaps not blindingly obvious that HC is a set. However if you have countable choice (so that ω_1 is regular) then every hereditarily countable set is in V_{ω_1} and then HC is a set by separation.

> Saul MIller has asked me to prove this in more detail. We prove that no hereditarily countable set can have uncountable rank. Suppose *per impossibile* that there is a hereditarily countable set x of rank $\alpha > \omega_1$. By the result in Sheet 1 exercise 11 there must be a set in the transitive closure of x that is of rank precisely ω_1. Everything in the transitive closure of x is hereditarily countable. So there is a hereditarily countably set y of rank ω_1. Think about the set of ranks of members of y. This is a countable set (beco's y is countable) of countable ordinals (all members of y are of countable rank) and its supremum is $\rho(y)$ – which is ω_1. This contradicts the regularity of ω_1.

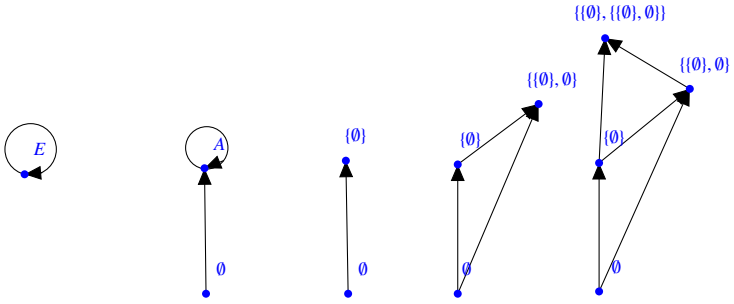
However we can prove the existence by the natural device of set pictures.

Set Pictures

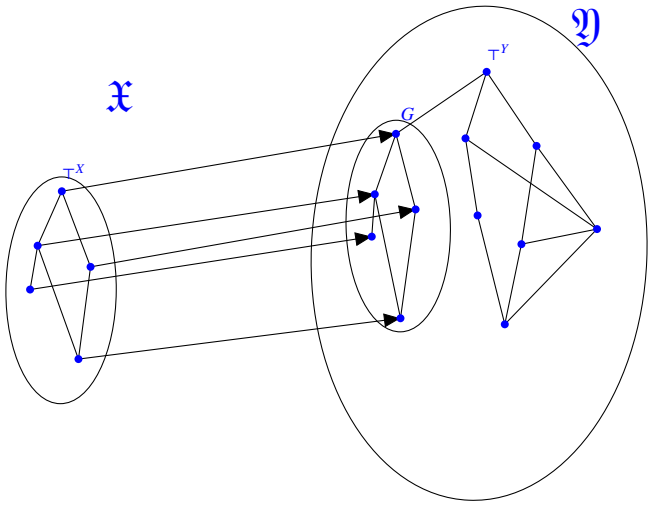
DEFINITION 46 *A* **set picture** *(also known as an* **accessible pointed (di)graph** *or* **APG***) is a special kind of decorated digraph, an extensional binary structure $\langle X, R, \top^R\rangle$ where $\top^R$ is a designated ("top") element satisfying $X = (R^*)^{-1}\text{``}\{\top^R\}$, a condition which says that $\top^R$ is* **accessible** *from any vertex in X.*

The idea is that it's a binary structure that looks as if it might be the graph of $\in^*$ restricted to the transitive closure of a singleton. The binary relation in an APG may be wellfounded, but much use is made of the fact that there are APGs that aren't wellfounded. Like digraphs out in the Big Wide World these digraphs may be decorated, and the decorations tend to be sets, pasted on in such a way that the membership relation between them echoes R. (Graphs can have either their edges or the vertices decorated; here we decorate only the vertices.) See pictures below.

Here are some graphics for APGs:



This second graphic shows the embedding relation which we define below.



This diagram[9] depicts the obvious embedding relation between set pictures: $\mathfrak{X} = \langle X, R, \top^R\rangle$ embeds into $\mathfrak{Y} = \langle Y, S, \top^S\rangle$ if there is $y \in Y$ with $S(y, \top^S)$ and $\langle X, R, \top^R\rangle \simeq \langle S^{-1}\text{“}\{y\}, S, \top^S\rangle$.

Clearly isomorphism is a congruence relation for this embedding relation and we write '$\mathcal{E}$' for the relation between the isomorphism classes.

If you forget this definition you can reconstruct it if you remember that it's trying to say that the set that $\langle X, R, \top^R\rangle$ is a picture of is a member of the set that $\langle Y, S, \top^S\rangle$ is a picture of.

Now to prove the existence of HC we consider $V_{\omega+1}$ and wellfounded set pictures whose carrier sets are in $V_{\omega+1}$, and Scott's-trick isomorphism classes thereof. Evidently the family of (Scott's-trick) equivalence classes is a wellfounded binary structure, so we can take the Mostowski collapse. The Mostowski collapse is HC.

Which axioms are true in HC?

I'm hoping that by now you can be trusted, Dear Reader, to rise to this challenge. We need countable choice to show that a union of countably many countable sets is countable (and we'll need that if we are to verify sumset). We can verify all of them except power set. Why is power set not true in HC? Well, everything in HC is countable, and the power set of a countably infinite set is uncountable. But life is not that simple. Remember that by downward Skolem-Löwenheim ZF must have a countable model and indeed (by Mostowski collapse) a countable *transitive* model. In any such model every set is countable! However not all the inhabitants of such a model are countable *in the sense of the model*: the model contains some (externally) countable sets for which it does not supply a bijection to the naturals of the model. In HC, in contrast, every set is internally countable, so the axiom of power set really does fail.

[9] I have to confess that the binary relation in the picture isn't extensional. With any luck the reader won't notice. What matters is the isomorphism between the two smaller ellipses.

We still have to prove the independence of extensionality, AC, pairing, sumset and foundation. Let's press on.

6.8.3 Independence of Sumset

We define $\beth_\alpha$ to be $|V_{\omega+\alpha}|$. To prove the independence of sumset we consider $H_{\beth_\omega}$. This is a set for the same reason that HC is. This time we consider the set of all wellfounded set pictures in $V_{\omega+\omega}$ and consider the set of Scott's-trick equivalence classes of them. The embedding just described is inherited by the quotient, and we write the inherited embedding as '$\mathcal{E}$'. Evidently the family of (Scott's-trick) equivalence classes is a wellfounded binary structure, so we can take the Mostowski collapse. The Mostowski collapse is $H_{\beth_\omega}$.
$H_{\beth_\omega} \not\models$ sumset because $\{V_{\omega+n} : n < \omega\}$ is in $H_{\beth_\omega}$ but $\bigcup\{V_{\omega+n} : n < \omega\} = V_{\omega+\omega}$ is not.

It is a model of all the other axioms for familiar reasons. It satisfies AC as long as the universe within which we constructed it satisfies AC.

6.8.4 Independence of the Axiom of Foundation

Let σ be the transposition $(\emptyset, \{\emptyset\})$. Equip the universe with a *new* membership relation $x \in_\sigma y$ defined as $x \in \sigma(y)$. Observe that $\emptyset \in_\sigma \emptyset$, so foundation does not hold in $\langle V, \in_\sigma\rangle$. What about the other axioms? The first thing to note is that all the axioms of ZFC (except foundation) are preserved whatever permutation you use.

DEFINITION 47 *ϕ^σ is the result of replacing '$\in$' in ϕ throughout by '$\in_\sigma$'.*

Then $(\langle V, \in_\sigma\rangle \models \phi) \longleftrightarrow \phi^\sigma$.
When ϕ is an axiom other than foundation we want to prove $\vdash (\forall\sigma)(\phi \longleftrightarrow \phi^\sigma)$.

OK, so you look at ϕ^σ, and you notice that – *prima facie* – distinct occurrences of a given variable can have different prefixes. Variables that never appear to the right of an '$\in$' you say are of level 0, and you don't have a problem with them. Variables that appear to the right of an '$\in$' only when the variables to the left of the $\in$ are of level 0 are of level 1 and you don't have a problem with them – they always have a 'σ' applied to them – unless they also appear to the left of an $\in$. Let 'y' be such a variable. Then we have subformulæ like $x \in \sigma(y)$ and $y \in \sigma(z)$.

We make the elementary observation that '$x \in \sigma(y)$' is equivalent to '$\sigma(x) \in \sigma$"$(\sigma(y))$' and so can be replaced by it in ϕ where appropriate. σ"z is $\{\sigma(w) : w \in z\}$ and the function $z \mapsto \sigma$"z is of course just yet another permutation. We might find that we have to "lift" σ in this way more than once .... So the notation '$j(\sigma)$' for this new permutation might come in handy.

The key is to manipulate the formulæ you are dealing with so as to ensure that, for every variable, all occurrences of that variable have the same prefix ... the point being that – for any permutation τ whatever – $(\forall x)(\ldots\tau(x)\ldots)$ is equivalent to $(\forall x)(\ldots x\ldots)$ beco's τ is a permutation.

This is a description of the recursive step in an algorithm for rewriting atomic formulæ in such a way that, for each variable, all its occurrences end up with the same prefix, so we can reletter. We now say that a formula is *stratifiable* iff this algorithm succeeds.

It's now simple to verify that ϕ^σ is equivalent to ϕ as long as ϕ is stratifiable. Not all instances of replacement are stratifiable but it turns out not to matter.

$$(\forall x \exists! y)\phi(x,y) \;\rightarrow\; (\forall X)(\exists y)(\forall z)(z \in Y \longleftrightarrow (\exists w)(w \in X) \wedge \phi(w,z))$$

becomes

$$(\forall x \exists! y)\phi^\sigma(x,y) \;\rightarrow\; (\forall X)(\exists y)(\forall z)(z \in \sigma(Y) \longleftrightarrow (\exists w)(w \in \sigma(X)) \wedge \phi^\sigma(w,z))$$

We can drop the σs preceding 'X' and 'Y' to obtain

$$(\forall x \exists! y)\phi^\sigma(x,y) \;\rightarrow\; (\forall X)(\exists y)(\forall z)(z \in Y \longleftrightarrow (\exists w)(w \in X) \wedge \phi^\sigma(w,z))$$

which is merely another instance of replacement (as long as σ is a function class). Thus the map (on the syntax) sending each ϕ to ϕ^σ sends every stratifiable formula ϕ to (something logically equivalent to) ϕ, and sends every instance of replacement to something logically equivalent to another instance.

We now check that every axiom other than foundation is either stratifiable or interdeducible with a stratifiable formula, and accordingly remains true in the new model. Observe that in the new model $\emptyset$ has become an object equal to its own singleton. Such objects are called **Quine atoms**. We added only one Quine atom, but if (say) we had swapped every natural number with its singleton we would have added countably many. We will need this when we come to prove the independence of AC.

6.8.5 Independence of the Axiom of Choice

Proving the independence of the axiom of choice from ZF is hard work, and was finally cracked by Cohen in 1963 with the advent of *forcing*. Forcing is too demanding for a course like this (tho' there are two exercises that are sleepers for it, Sheet 5 questions 7 and 8) but there are other ideas that go into the independence proof, and some of them can be profitably covered here.

One useful thought is that the axiom of choice says that the universe contains some highly asymmetrical objects. After all, as we saw in theorem 2 on page 13, any wellordering is rigid. If we can arrange matters so that everything in the universe has some symmetries then we will break AC. I've made it sound easier than it is, but that's the idea.

We start with a model of ZF + foundation, and use the permutation methods seen above to obtain a permutation model with a countable set A of Quine atoms. The permutation we use to achieve this is the product of all transpositions $(n, \{n\})$ for $n \in \mathbb{N}^+$.

A will be a **basis** for the illfounded sets in the sense that any class X lacking an $\in$-minimal element contains a member of A. Since the elements of A are Quine atoms every permutation of A is an $\in$-automorphism of A, and since they form a basis we can extend any permutation σ of A to a unique $\in$-automorphism of V in the obvious way: declare $\sigma(x) := \sigma``x$. Notice that the collection of sets that this definition does not reach has no $\in$-minimal member if nonempty, and so it must contain a Quine atom. But σ by hypothesis is defined on Quine atoms.

Any permutation σ of the atoms can be extended to an $\in$-automorphism of the universe (also written σ, by slight abuse of notation) by declaring $\sigma(x) = \sigma``x$. Now (a, b) is of course the transposition swapping a and b, and we will write '(a, b)' also for the unique automorphism to which the transposition (a, b) extends. Every set x gives rise to an equivalence relation on atoms. Say $a \sim_x b$ if (a, b) fixes x. We say x is of (or has) **finite support** if $\sim_x$ has a cofinite equivalence class. (At most one equivalence class can be cofinite).

The union of the (finitely many) remaining (finite) equivalence classes is the **support** of x. Does that mean that x is of finite support iff the transitive closure $TC(x)$ contains finitely many atoms? Well, if $TC(x)$ contains only finitely many atoms then x is of finite support (x clearly can't tell apart the cofinitely many atoms not in $TC(x)$) but the converse is not true: x can be of finite support if $TC(x)$ contains cofinitely many atoms. (Though that isn't a sufficient condition for x to be of finite support!!)[10]

It would be nice if the class of sets of finite support gave us a model of something sensible, but extensionality fails: if X is of finite support then $\mathcal{P}(X)$ and the set $\{Y \subseteq X : Y \text{ is of finite support}\}$ are distinct sets both of finite support, but they have the same members with finite support. We have to consider the class of elements *hereditarily* of finite support. Let's call it HS. This time we *do* get a model of ZF.

LEMMA 10 *The class of sets of finite support is closed under all the definable operations that the universe is closed under.*

Proof:

When x is of finite support let us write '$A(x)$' for the cofinite equivalence class of atoms under $\sim_x$. For any two atoms a and b the transposition (a, b) induces an $\in$-automorphism which for the moment we will write (a, b), too.

Now suppose that $x_1 \ldots x_n$ are all of finite support, and that f is a definable function of n arguments. $x_1 \ldots x_n$ are of finite support, and any intersection of finitely many cofinite sets is cofinite, so the intersection $A(x_1) \cap \ldots A(x_n)$ is cofinite. For any a, b we have

$$(a, b)(f(x_1 \ldots x_n)) = f((a, b)(x_1) \ldots (a, b)(x_n))$$

since (a, b) is an automorphism. In particular, if $a, b \in A(x_1) \cap \ldots A(x_n)$ we know in addition that (a, b) fixes all the $x_1 \ldots x_n$ so

$$(a, b)(f(x_1 \ldots x_n)) = f(x_1 \ldots x_n).$$

So the equivalence relation $\sim_{f(x_1 \ldots x_n)}$ induced on atoms by $f(x_1 \ldots x_n)$ has an equivalence class which is a superset of the intersection $A(x_1) \cap \ldots A(x_n)$, which is cofinite, so $f(x_1 \ldots x_n)$ is of finite support. ∎

This takes care of the axioms of empty set, pairing, sumset and power set. To verify the axiom scheme of replacement we have to check that the image of a set hereditarily of finite support in a definable function (with parameters among the sets hereditarily of finite support and all its internal variables restricted to sets hereditarily of finite

[10] A counterexample: wellorder cofinitely many atoms. The graph of the wellorder has cofinitely many atoms in its transitive closure, but they are all inequivalent.

support) is hereditarily of finite support too. The operation of translating a set under a definable function (with parameters among the sets hereditarily of finite support and all its internal variables restricted to sets hereditarily of finite support) is definable and will (by lemma 10) take sets of finite support to sets of finite support.

So if X is in HS and f is a definable operation as above, f"X is of finite support. And since we are interpreting this in HS, all members of f"X are in HS, so f"X is in HS too, as desired.

To verify the axiom of infinity we reason as follows. Every wellfounded set x is fixed under all automorphisms, and is therefore of finite support. Since all members of x are wellfounded they will all be of finite support as well, so x is hereditarily of finite support. So H will contain all wellfounded sets that were present in the model we started with. In particular it will contain the von Neumann ω.

It remains only to show that AC fails in HS. Consider the set of (unordered) pairs of atoms. This set is in HS. However no selection function for it can be. Suppose f is a selection function. It picks a (say) from $\{a, b\}$. Then f is not fixed by (a, b). Since f picks one element from every pair $\{a, b\}$ of atoms, it must be able to tell all atoms apart; so the equivalence classes of $\sim_f$ are going to be singletons, $\sim_f$ is going to be of infinite index, and f is not of finite support.

So the axiom of choice for countable sets of pairs fails. Since this axiom is about the weakest version of AC known to man, this is pretty good. The slight drawback is that we have had to drop foundation to achieve it. On the other hand the failure of foundation is not terribly grave: the only illfounded sets are those with a Quine atom in their transitive closures, so there are no sets that are gratuitously illfounded: there is a basis of countably many Quine atoms. On the other hand it is only the illfounded sets that violate choice! To show the independence of AC from ZF with foundation we would have to do more work, and it's a bit much for here.

6.9 The Modern Theory of Wellfounded Sets

Modern Set Theory, since its flowering in the 1960s, has been concerned with the fine structure of the cumulative hierarchy. That is to say, the modern tradition assumes the axiom of foundation without question. On the whole the tradition has also embraced the axiom of choice. The only area where the axiom of choice is occasionally eschewed is in the study of **The axiom of determinacy**.

To a significant extent developments in modern set theory have been driven by a set theoretic foundationalism that believes that all the problematic chickens flying around mathematics will eventually come home to roost in set theory, and that Set Theorists need to be ready with answers. To this end certain decisions have been made about how Mathematics is to be understood-as-Set-Theory: ordinals are von Neumann ordinals, reals are taken to be sets of natural numbers (aka finite (von Neumann) ordinals). By means of such identifications problems in Analysis become problems in Set Theory. In principle one might worry that the translation being used might not be faithful to the interpreted mathematics (Frege worried about this) but generally Set Theorists do not.

In 1940 Gödel exhibited a construction of a very impoverished model of ZFC which demonstrably satisfies AC and GCH. Modifications of his construction loom large in

subsequent work.

The roots that Set Theory had in Analysis have had profound and lasting effects on the direction it has taken. Analysis has been a rich source of axioms, or conjectures. The axiom of determinacy is one. This is an exciting and powerful axiom that is very attractive to Set Theorists ... so attractive in fact that they are willing (part-time) to overlook the fact that it contradicts AC and pursue some of the weirdnesses that it opens up. Another is the conjecture that there is a set with a countably additive two-valued measure vanishing on singletons. Questions in (the) Descriptive Set Theory (of the reals) loom large too.

Another rich source of axioms and ideas has been generalisations of the axiom of infinity. How "long" is the cumulative hierarchy?

Roughly there are two ways of generalising the axiom of infinity.

One is to dream up ever more improbable things and postulate that the universe contains examples of them. The earliest example of this kind of move is probably the axiom of Mahlo that every normal function from ordinals to ordinals has a regular fixed point. A more striking example is the postulate that there should be a set admitting a countably additive two-valued measure vanishing on singletons. The cardinal of such a set is said to be **measurable**. It's not obvious that such a set must be *large*, but it becomes clear that such sets are very *rare*, so the smallest one is probably quite large.

The other is to spice up a theory T by adding an axiom that says, in a natural way, that T has a model. Thus we discover an axiom that says thare is an ordinal κ s.t. V_κ is a model of ZF. We can repeat the trick, and with a little ingenuity we can devise axioms that will perform lots of such extensions with one stroke. This can require a great deal of ingenuity, and one set of ideas for how to do it grew out of the proposition that there is a a measurable cardinal. This turns out to imply lots of things that look like strong axioms of infinity. The idea has been incredibly fertile.

6.10 Games!!

We have defined a game to be a payoff function $G : A^\omega \to \{\mathtt{I}, \mathtt{II}\}$. However, in order to win a play of a game we need a *strategy*, namely a function defined on *positions* rather than a function—like G—defined on *plays*. Fortunately in certain circumstances we can process a payoff function G into a function defined on at least some positions, and we can do this when there are positions such that every play through them is won by the same player. This certainly happens if the payoff set G^{-1}"$[\mathtt{I}]$ is closed or open, but it will also happen even if it merely has nonempty closed or open subsets. Let us use the word **valuation** for (possibly partial) functions sending positions to $\{\mathtt{I}, \mathtt{II}\}$.

If $G(\pi) = \mathtt{I}$ for every play π that is an end-extension of p then we can sensibly process G into a valuation sending p to $\mathtt{I}$. For a fixed arena and payoff set let us call the valuation v obtained from G in this way the **base valuation**.

Now there is an obvious way of extending valuations. I call it E (for "extend label"), and it is defined as follows. If v : positions $\to \{\mathtt{I}, \mathtt{II}\}$ then

DEFINITION 48

E v p =:

`if` *p is even* `then (if` *there is a child p′ of p with v*(*p′*) = `I then I`*;*

`else if` *v*(*p′*) = `II` *for all children p′ of p* `then II`*;* `else if`

p is odd `then if` *there is a child p′ of p with v*(*p′*) = `II then II`*;*

`else if` *v*(*p′*) = `I` *for all children p′ of p* `then I else fail`.

Clearly for any fixed arena and payoff set, the collection of valuations forms a chain-complete poset under inclusion (valuations tho'rt of as sets of ordered pairs) and the base valuation is the bottom element of this poset. E is clearly a monotone function from this poset into itself, so there will certainly be fixed points for E. The fixed points form a chain-complete poset, so there will even be fixed points by Bourbaki-Witt.

Any fixed point v for E will give rise to a pair of canonical nondeterministic strategies. I call them **soot** strategies. It is the stay out of trouble strategy, which, for player i, is to play nodes labelled i wherever possible and please yourself otherwise.

Now suppose G is an open (if player `I` is to win this has become apparent by some finite position) or closed (if player `II` is to win this has become apparent by some finite position) game these maximal fixed points become interesting, and for two reasons.

(i) In an open or closed game a soot strategy defined at the empty position is winning;

(ii) In an open or closed game a maximal valuation must be total.

Proof of (ii) Suppose not, and let $v = E(v)$ be a maximal fixed point that is not a total function. If i is the player that wins every play whose fate is not determined by a finite initial segment, then add to v all ordered pairs $\langle p, i\rangle$ for all positions p at which v is undefined. The result is a total function, and is still a fixed point for E.

In fact we get

THEOREM 15 *(Gale-stewart) Every open or closed game is determined.*

It suffices to show that every open game is determined. An open game is one where player `I` wins – if at all – after only finitely many moves, and II wins all the infinite plays. (If player `II` wins a finite position p we can just pretend that she hasn't won, and go on playing forever and she wins all infinite paths that start with p). So the only label we ever give a position is '`I`'. We propagate the labelling up the tree by iterating E, and if the empty position gets labelled '`I`' then `I` has a winning strategy. If not, then `II` has a winning strategy which is "always move to an unlabelled position".

■

I should perhaps say a bit about nondeterministic versus deterministic strategies in these games. (I was hoping to avoid the subject altogether but clarification of it involves a nice recursion which is paedogogically useful). In the proof of Gale-Stewart we start off by labelling '`I`' all positions p s.t. every infinite play extending p is a win for player `I`. Then we propagate the labelling up the tree $A^{<\omega}$ of finite strings from A (aka positions) by iterating the function i called E. The idea is that if the empty position is labelled '`I`' then player `I` has a winning strategy which is simply to move always to a position labelled '`I`'. On the face of it this is a nondeterministic strategy. In principle there is the possibility that he could be always in a position *from which he can force*

a win but keeps putting it off and putting it off and eventually II wins instead. This can actually happen if the payoff set is not open, so we have to be very careful with the concept of *nondeterministic winning strategy*. However in the case where the payoff set is open we have the following argument to calm our nerves. We can define a rank function on the positions labelled 'I'. The positions p s.t. every infinite play extending p is a win for player I are of rank 0. Things that get labelled later on get higher ranks. (The rank of a labelled position is the number of times we have to iterate E to get it labelled). The point of this ranking is that every time I moves from a labelled position to a new labelled position the move takes him to a position of lower rank, and every descending sequence of ordinals is finite. So any play following this nondeterministic strategy is a win for I.

6.10.1 Axiom of Determinacy and AC

The **Axiom of Determinacy** is the assertion that all games played over the arena $\mathbb{N}$ are determinate. Striking fact: AD contradicts AC.

> I think the following proof is due to Scott Johnson. https://en.wikipedia.org/wiki/Murder_of_Scott_Johnson from his time in Cambridge. He studied briefly under my *DoktorVater* Adrian Mathias.

THEOREM 16 $\neg(AC \wedge AD)$

Proof:

Consider the equivalence relation on $\mathbb{N}^\omega$ defined by:

$$f \sim g \text{ if } (\exists n_0 m_0 \in \mathbb{N})(\forall x \in \mathbb{N})(f(n_0 + x) = g(m_0 + x)).$$

If $f \sim g$ then there are minimal naturals we can take as witnesses to the '$\exists n_0, m_0 \in \mathbb{N}$' and the difference between the two things in this minimal pair will be either even or odd. Thus if $f \sim g$ they are either "an odd distance apart" or "an even distance apart" *but not both*. We now use AC to pick a representative from each equivalence class. We say a sequence is *odd* iff it is an odd distance from the chosen representative of its equivalence class.

Now let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be the set of odd sequences, and consider G_A. We will show that G_A has no winning strategy for either player.

Suppose II has a winning strategy τ. The game starts with player I making the 0th move. Consider a play where II uses τ (so we know she is going to win) and player I uses τ as well. He cannot do this straightforwardly, since τ eats sequences of *odd* length and I is confronted with sequences of *even* length. To use τ he turns the even sequence he is confronted with into an odd one by hanging a 0 on the front and *then* using τ. In particular his first move will be to play whatever τ ordains as the response to the sequence $\langle 0 \rangle$.

The result of this play must be a win for II, since τ is winning for her. Let us call it X. Now consider what happens if I starts, not by playing $\tau\langle 0 \rangle$, but 0 itself. II is still going to use τ, so II's response will then be $\tau\langle 0 \rangle$ (which was I's first move in the first play we considered), and I's response to that will be whatever II's second move

in the original play would have been. Clearly this infinite sequence generated by this play will be just like the infinite sequence X generated by the original play but with a 0 hung on the front. But if X was odd this sequence must be even and *vice versa* so they can't possibly *both* be wins for `II` as they should. Therefore τ was not Winning.

Now suppose player `I` has a winning strategy σ. Consider the play where he uses σ and player `II` uses σ too, rather the way in which `I` used τ in the previous thought-experiment. σ only works on sequences of *even* length whereas `II` wants answers to problems of *odd* length, and she gets round this by inserting $x_0 + 1$ after x_0 in each sequence she is confronted with. (x_0 is `I`'s first move, ordained by σ). This play – call it Y – is a win for `I`. Now consider the play where `I` uses σ as before (and therefore starts with x_0) and `II` instead of playing $\sigma\langle x_0, x_0 + 1\rangle$ plays $x_0 + 1$. `I`'s next move will be $\sigma\langle x_0, x_0 + 1\rangle$. Since `I` is continuing to use σ the remaining moves will be the same as in the first play with σ we considered, except that they are now being made by the "other" player. Clearly this play cannot be a win for `I` if the last one was, since each is odd iff the other is even.

■

Chapter 7

Two Lectures on Model Theory

Not to be lectured

Can we fit in quantifier-elimination?

All this model theory is about first-order theories No second-order stuff thank you!

7.1 The Skolem-Löwenheim Theorems

Notice that the proof of theorem 11 gives us something slightly more than I have claimed. If the consistent theory T we started with was a theory in a countable language then the model we obtain by the above method is also countable. It's worth recording this fact:

COROLLARY 13
Every consistent theory in a countable (first-order) language has a **countable** *model.*

We can actually prove something more general. Think about what happens to the construction in the proof of theorem 11 if our language has uncountably many constant symbols or function symbols or predicate letters. The proof will procede as before by wellordering the language, and we will build uncountably many ϵ-terms. Clearly the set of terms we generate will be no bigger than the size of the language. This is the

THEOREM 17 *Downward Skolem-Löwenheim*
A consistent theory in a language $\mathcal{L}$ has a model of size $|\mathcal{L}|$ at most.

THEOREM 18 *Upward Skolem-Löwenheim*
Any theory with infinite models has arbitrarily large models.

Proof: Add lots of constants and appeal to compactness. ■

Actually we can do significantly better. The point is that theorem 17 tells us that if we add enough constants to ensure that the model is of size at least κ, then the model will be no bigger than κ.

7.2 Categoricity

DEFINITION 49
A theory is categorical iff it has only one model up to isomorphism;
*A theory is categorical-in-κ (or κ-***categorical***) if it has precisely one model of size κ up to isomorphism.*
*A structure is said to be κ-***categorical** *if its theory is κ-categorical*[1].

Duplication

No interesting examples of categorical first-order theories: they all have only finite models (indeed only *one* finite model!)

Plenty of examples of first-order theories categorical-in-$\aleph_0$. Such theories are always called *countably categorical.* Here's the standard example, the theory of dense linear order without endpoints.

7.2.1 Back and forth

The theory of dense linear order has one primitive nonlogical symbol $\leq$ and the following axioms:
$\forall xyz(x \leq y \rightarrow (y \leq z \rightarrow (x \leq z)))$;
$\forall xz(x \leq y \rightarrow y \leq x \rightarrow x = y)$;
$\forall xy\exists z(x < y \rightarrow (x < z \wedge z < y))$;
$\forall x\exists y(y > x)$;
$\forall x\exists y(x > y)$;
$\forall xy(x \leq y \vee y \leq x)$.

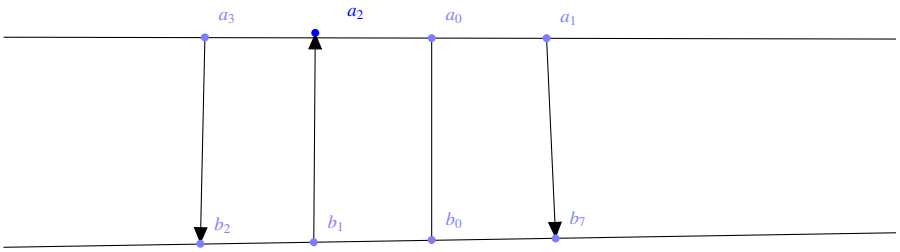
> I have to confess i'm being a bit naughty here .... I defined only the symbol '$\leq$' but then proceeded to use '$<$' and '$>$' as well, knowing that you would know what I meant. It's naughty in the sense that one shouldn't exploit linguistic conventions without spelling them out first. Admittedly mathematicians do it all the time, and it takes a logician to make a fuss about this ... but then this *is* a logic course.

THEOREM 19 *All countable dense linear orders without endpoints are isomorphic.*

I shall provide a proof because it is possible to prove the theorem the wrong way.
Proof: Suppose we have two countable dense linear orders without endpoints, $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ and $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$. They are both countable, so the elements of $\mathcal{A}$ can be enumerated as $\langle a_i : i \in \mathbb{N} \rangle$ and the elements of $\mathcal{B}$ can be enumerated as $\langle b_i : i \in \mathbb{N} \rangle$.

We start by pairing off a_0 with b_0. Thereafter we procede by recursion. At each stage we have paired off some things in $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ with some things in $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$. Let us now consider the first thing in $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ not already paired off. (We mean: first in the sense of $\langle a_i : i \in \mathbb{N} \rangle$.) This lies between two things we have already paired, and we must find a mate for it in $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$ that lies in the interval between their mates. Since the ordering is dense, this interval is nonempty, and we pick for its mate the first (in the sense of the $\langle b_i : i \in \mathbb{N} \rangle$) in it.

[1] Readers should be warned that many people confusingly write 'ω-categorical' when talking about $\aleph_0$-categorical structures like the countable dense linear order without endpoints.



In the illustration we paired a_0 with b_0. Then we sought the first b that is to the right of b_0 so it can match a_1 and (in the picture) that first b is b_7, since $b_1 \ldots b_6$ are to the left of b_0. Next we had to find a match for b_1, and that turned out to be a_2. Then the a of smallest subscript that hasn't been mated so far is a_3 – and its mate is going to be b_2.

That is the recursive process we use to build the bijection. It goes *back and forth*: $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ to $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$ and then $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$ to $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$. That way we can be sure that by the time we have gone back and forth n times we have used up the first n things in the canonical enumeration of $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ *and* the first n things in the canonical enumeration of $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$. We will have used n other things as well on each side, but we have no control over how late or early they are in the canonical orderings.

The union of all the finite partial bijections we thus construct is an isomorphism between $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ and $\langle \mathcal{B}, \leq_{\mathcal{B}} \rangle$. ■

Note that this construction shows that the group of order-automorphisms of the rationals acts transitively on unordered n-tuples.

Theorem 19 tells us that the theory of dense linear orders without endpoints is complete. Suppose it were not. Then there would be a formula ϕ that is undecided by it, and by the completeness theorem there would be dense linear orders without endpoints that were ϕ and dense linear orders without endpoints that were not ϕ. But then these dense linear orders would not be elementarily equivalent, and *a fortiori* not isomorphic either.

The study of countable structures that are unique up to isomorphism is a pastime widespread among logicians and has interesting ramifications.

There is a remarkable and deep theorem of Morley that says that a theory that is κ-categorical for even one uncountable κ is κ-categorical for *all* uncountable κ. It is beyond the scope of these notes. However, there are a number of natural and important countably categorical theories.

Now might be a good moment to look at question 8 on sheet 3.

7.3 Results related to completeness, exploiting completeness

7.3.1 Prenex Normal Form and Quantifier-Counting

DEFINITION 50 *A formula is in* **prenex normal form** *if it is of the form string-of-quantifers followed by stuff containing no quantifiers. All quantifiers hae been "pulled to the front" or "exported".*

THEOREM 20 *Every formula is equivalent to one in PNF.*

Sketch of proof:
Quantifiers can be "pulled to the front". $(\forall x)(A(x)) \wedge (\forall y)(B(y))$ is clearly equivalent to $(\forall x)(\forall y)(A(x) \wedge B(y))$, and there is an analogous equation for '$\exists$'.
Less obvious that
$(\exists x)(A(x)) \rightarrow p$ is equivalent to $(\forall x)(A(x) \rightarrow p)$. ■

The significance of this is that it gives us a nice measure of the logical complexity of a formula: count the length of the quantifier prefix once it's in PNF. Better, count the number of **quantifier blocks** in the prefix. There are theorems connecting the quantifier prefixes that you find in the axioms of a theory T with the operations that the class of models of T is closed under. We shall prove the simplest of them to give a flavour.

DEFINITION 51
A sentence is **universal** *iff it is in PNF and its quantifier prefix consists entirely of universal quantifiers.*
By a natural extension we say a theory is "universal" iff, once you put its axioms into PNF, their quantifier prefixes consist entirely of universal quantifiers.
We define similarly **universal-existential** *sentences, and theories*[2] *as theories all of whose axioms, when in PNF, have a block of universal quantifiers followed by a block of existential quantifiers, and so on.*

DEFINITION 52 *Equational Theories and Equational Languages*
A language is equational if it has no nonlogical relation symbols;
A equational theory is a theory in an equational language.

Equational theories can have *function* symbols of course, and constant symbols. Group theory is equational; ring theory is equational ...

DEFINITION 53 *The* **diagram** $D_{\mathfrak{M}}$ *of a structure* $\mathfrak{M}$ *is the theory obtained by expanding* $\mathfrak{M}$ *by giving names to every* $m \in M$*, and collecting all true atomic assertions about them.*

For a theory T*,* $T_\forall$ *is the set of universal consequences of* T*.*

> Dear Reader, in case you were thinking that $T_\forall$ is the kind of thing that only a sad logician would dream up, it might be worth pointing out that the theory of integral domains is precisely the universal fragment of the theory of fields: if T is the theory of fields then $T_\forall$ is the theory of integral domains. A structure is an integral domain iff it is a substructure of a field. See theorem 21 below.

[2] Such theories are sometimes called "inductive"; Dunno why.

LEMMA 11 *For any consistent theory T and any model $\mathfrak{M}$ of $T_\forall$, the theory $T \cup D_{\mathfrak{M}}$ is consistent.*

Proof:

Let $\mathfrak{M}$ be a model of $T_\forall$, with carrier set M. Add to $\mathcal{L}(T)$ names for every member of M. Add to T all the (quantifier-free) assertions about the new constants that $\mathfrak{M}$ believes to be true. This theory is $T \cup D_{\mathfrak{M}}$. We want this theory to be consistent. How might it not be? Well, if it isn't, there must be an inconsistency to be deduced from a conjunction ψ of finitely many of the new axioms. This rogue ψ mentions finitely many of the new constants. We have a proof of $\neg\psi$ from T. T knows nothing about these new constants, so clearly we must have a UG proof of $(\forall \vec{x})\neg\psi$. But this would contradict the fact that $\mathfrak{M}$ satisfies every universal consequence of T. ■

THEOREM 21 *T is universal iff every substructure of a model of T is a model of T.*

Proof:

L → R is easy. We prove only the hard direction.

Suppose that T is a theory such that every substructure of a model of T is also a model of T. Let $\mathfrak{M}$ be an arbitrary model of $T_\forall$. We will show that it must be a model of T. We know already from the foregoing that the theory $T \cup D_{\mathfrak{M}}$ is consistent, and so it must have a model – $\mathfrak{M}^*$, say. $\mathfrak{M}^*$ is a model of T, and $\mathfrak{M}$ is a submodel of $\mathfrak{M}^*$ and therefore (by assumption on T) a model of T – as desired.

But all we knew about $\mathfrak{M}$ was that it was a model of the universal consequences of T. So any old $\mathfrak{M}$ that was a model of the universal consequences of T is a model of T. So T is axiomatised by its universal consequences. ■

There are lots of theorems with this flavour: "The class of models of T is closed under operation burble iff T has an axiomatisation satisfying syntactic condition blah"

But wait! If we have the axiom of choice then, whenever we have an axiom that says $(\forall y)(\exists x)(F(x, y))$ then we can invent a function symbol and an axiom that says $(\forall x)(F(x, f(x)))$. In fact we don't even need the axiom of choice. [you might like to think about why, and have a look at question (xi) on Sheet 3]. If we do this often enough we can invent enough function symbols to turn any theory we like into a universal theory, and then all its substructures are also models of it!

Yes you can, but when you add new function symbols you restrict your notion of substructure!

The next theorem after theorem 21 will say that a theory is universal-existential iff the class of its models, partially ordered by isomorphic embeddability, is directed complete. One direction is easy – you might even like to prove it – but the converse (The Chang-Łoś-Suszko lemma) is hard. Indeed I don't know how to pronounce it, let alone prove it!

7.4 Omitting Types

You were set up for this topic by question 15 on sheet 3. That question concerned types for propositional logic. There is an analogous result for predicate logic but it is much

harder, and we are not going to prove it (most textbooks will have a proof). However we do need to engage with the ideas, so we will at least *state* it.

DEFINITION 54
A **type** *in a first-order language $\mathcal{L}$ is a (usually) infinite set of formulæ.*
*A type Σ is an n-*__type__ *if the formulæ in it all have n free variables.*
A model $\mathfrak{M}$ **realises** *an n-type Σ if there is a tuple $m_1 \dots m_n$) in M s.t. $\mathfrak{M} \models \sigma(m_1 \dots m_n$ for every $\sigma \in \Sigma$. Otherwise $\mathfrak{M}$* **omits** *Σ.*
We say a theory T **locally omits** *an n-type Σ if, whenever ϕ is a formula with n free variables such that T proves $(\forall \vec{x})(\phi(\vec{x}) \rightarrow \sigma(\vec{x}))$ for every $\sigma \in \Sigma$, then $T \vdash (\forall \vec{x})(\neg\phi(\vec{x}))$.*

THEOREM 22 *Omitting Types Theorem*
If T locally omits a type Σ, then it has a model omitting Σ.

OTT sounds a bit recherché but it does touch on some mainstream concerns: the standard model of arithmetic omits the 1-type

$$\{x \neq 0;\ x \neq 1;\ x \neq 1 + 1;\ x \neq 1 + 1 + 1 \dots\}$$

7.5 Direct Products and Reduced Products

If $\{\mathcal{A}_i : i \in I\}$ is a family of structures, we define the product

$$\prod_{i \in I} \mathcal{A}_i$$

to be the structure whose carrier set is the set of all functions f defined on the index set I such that $(\forall i \in I)(f(i) \in A_i)$ and the relations of the language are interpreted *"pointwise":* the product believes $f\ R\ g$ iff $(\forall i \in I)(f(i)\ R\ g(i))$.

The $\{\mathcal{A}_i : i \in I\}$ are said to be the *factors* of the product $\prod_{i \in I} \mathcal{A}_i$.

For this operation to make sense it is of course necessary that all the $\mathcal{A}_i$ should have the same signature!

Products are nice in various ways.

DEFINITION 55 *Let Γ be a class of formulæ.*

Products **preserve** *Γ if whenever $\prod_{i \in I} \mathcal{A}_i$ is a product of a family $\{\mathcal{A}_i : i \in I\}$ and $\phi \in \Gamma$ and $(\forall i \in I)(\mathcal{A}_i \models \phi)$ then*

$$\prod_{i \in I} \mathcal{A}_i \models \phi.$$

By definition of product, products preserve atomic formulæ. Clearly they also preserve conjunctions of anything they preserve, and similarly universal quantifications over things they preserve.

What about more complex formulæ? You know that products preserve equational theories (a product of rings is a ring, after all). They also preserve Horn formulæ.

DEFINITION 56 .

A Horn clause	*is a disjunction of atomics and negatomics of which at most one is atomic.*
A Horn property	*is a property captured by a [closure of a] Horn expression;*
A Horn theory	*is a theory all of whose axioms are universal closures of (conjunctions of) Horn clauses.*

Yes, but what is a horn formula?

7.5.1 Intersection-closed properties

A property F of sets is *intersection-closed* if an arbitrary intersection of sets with property F also has property F. Examples: the property of being a transitive relation (tho'rt of as a set of ordered pairs) is intersection-closed; ditto equivalence relation, symmetric relation.... An arbitrary intersection of convex figures in the plane is another such figure. An arbitrary intersection of subgroups of a group is a subgroup of that group. And many more.

The properties *symmetric reflexive*, and *transitive* are all Horn. The idea is an important one because for Horn properties one has an idea of **closure**. Take transitivity for example. R is transitive iff

$$(\forall xyz)((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$$

That is to say, the graph of R (R thought of as a set of ordered pairs) is closed under the operation that accepts the pair $\langle x, y\rangle$ and the pair $\langle y, z\rangle$ as inputs and outputs the pair $\langle x, z\rangle$.

As usual, when one has a set and an operation that can be applied to its members, one has a notion of canonical unique closure of that set under that operation. The point about Horn properties is that in the Horn clause there are lots of premisses (all positive, each saying – as it were – that the relation contains a certain tuple) and one conclusion, saying that in that case the relation contains this other tuple.

Observe that "is a total order" is not a Horn property.

REMARK 15 *Products preserve Universal Horn formulæ*

Proof:

Suppose every factor $\mathcal{A}_n$ believes $(\forall \vec{x})((\bigwedge_{i<j} \phi_i(\vec{x})) \rightarrow \phi_j(\vec{x}))$, where all the ϕ are atomic. We want to show that the product believes it too. So let $\vec{f} = f_1 \dots f_k$ be a tuple of things in the product satisfying the antecedent. That is to say, for each factor $\mathcal{A}_n$, we have $\mathcal{A}_n \models \phi_i(f_1(n), f_2(n) \dots f_k(n))$ for each $i < j$. But then every $\mathcal{A}_n$ believes $\phi_j(f_1(n), f_2(n) \dots f_k(n))$ so the product believes $\phi_j(f_1, f_2 \dots f_k)$ as desired. ■

In particular an arbitrary product of transitive relations is a transitive relation. [This is a good point of departure]. An arbitrary product of posets is a poset (being a poset is horn) but an arbitrary product of tosets is not reliably a toset because the totality condition (trichotomy, connexity) is not Horn.

This illustrates how products do not always preserve formulæ containing $\vee$ or $\neg$. This suggests that remark 15 is best possible. (We won't prove it) How so? If ϕ is preserved, then the product will fail to satisfy it if even *one* of the factors does not satisfy it even tho' all the rest do. (cf Genesis [19:23-33] where not even one righteous man is enough to save the city. The product is not righteous unless *all* its factors are). In these circumstances the product $\models \neg\phi$ but it is not the case that all the factors $\models \neg\phi$. As for $\vee$, if ϕ and ψ are preserved, it can happen that $\phi \vee \psi$ is not, as follows. If half the factors satisfy ϕ and half satisfy ψ, then they all satisfy $\psi \vee \phi$. Now the product will satisfy $\psi \vee \psi$ iff it satisfies one of them. But in order to satisfy one of them, that one must be true at *all* the factors, and by hypothesis it is not. Something similar happens with the existential quantifier.

7.5.2 Reduced products

We will need filters and ultrafilters from definition 26.

Given a filter F over the index set, we can define $f \sim_F g$ on elements of the product if $\{i \in I : f(i) = g(i)\} \in F$. Then we *either* take this $\sim_F$ to be the interpretation of '=' in the new product we are defining, keeping the elements of the carrier set of the new product the same as the elements of the old *or* we take the elements of the new structure to be equivalence classes of functions under $\sim$. These we will write $[g]_{\sim_F}$ or $[g]_F$ or even $[g]$ if there is no ambiguity.

This new object is denoted by the following expression:

$$(\prod_{i\in I} \mathcal{A}_i)/F$$

Similarly we have to revise our interpretation of atomic formulæ so that

$$(\prod_{i\in I} \mathcal{A}_i)/F \models \phi(f_1, \ldots f_n) \text{ iff } \{i : \phi(f_1(i), \ldots f_n(i)\} \in F.$$

REMARK 16 *$\sim_F$ is a congruence relation for all the operations that the product inherits from the factors.*

Can't do any harm to write out a proof. [Not lectured but supplied for the notes]

Let H be an operation, of arity h, and let $\vec{f}$ and $\vec{g}$ be two h-tuples in the product, with $f_i \sim_F g_i$ for each $i \leq h$. That is to say: for each $i \leq h$, $\{n : f_i(n) = g_i(n)\} \in F$. Since h is finite, we can conclude that $\{n : \bigwedge_{i\leq h} f_i(n) = g_i(n)\} \in F$.

We want $H(\vec{f}) \sim_F H(\vec{g})$. That is to say we desire that

$$\{n : H(f_1(n) \cdots f_h(n)) = H(g_1(n) \cdots g_h(n))\} \in F.$$

But we know (by our assumption that $f_i \sim_F g_i$ for each $i \leq h$) that $\bigwedge_{i\leq k}(f_i(n) = g_i(n))$ holds for an F-large set of n, so if H is given the same tuple of arguments it can hardly help but give back the same value.

■

It may be worth bearing in mind that to a certain extent the choice between thinking of elements of the carrier set of the reduced product as the $\sim_F$-equivalence classes of functions and thinking of them as those functions is a real one and might matter. I have proceeded here on the basis that the carrier set is the set of $\sim_F$-equivalence classes because that seems more natural. However, in principle there are set-existence issues involved in thinking of a product this way – how do we know that the $\sim_F$-equivalence classes are sets? – so we want to keep alive in our minds the possibility of doing things the second way. This will matter when we come to consider reduced products where the factor structures are proper classes (= have carrier sets that are proper classes). This happens in the extensions of ZF(C) with large cardinal axioms (specifically measurable cardinals). In practice these issues are usually swept under the carpet; this is a safe strategy only because it is in fact possible to sort things out properly! There is of course also the possibility of picking representatives from the equivalence classes, possibly by means of AC.

(For those of a philosophical turn of mind, there is an interesting contrast here with the case of quotient structures like, say, integers mod p. I have the impression that, on the whole, mathematicians do not think of integers-mod-p as sets of integers, nor as integers equipped with a nonstandard equality relation, but rather think of them as objects of a new kind. These reflections may have significance despite not really belonging to the study of *mathematics*: the study of *how we think about mathematics* is important too. And of course this also takes us back to the roots of set theory as discussed at the beginning of chapter 6.)

The reason for proceeding from products to reduced products was to complicate the construction and hope to get more things preserved. In fact nothing exciting happens (we still have the same trouble with $\vee$ and $\neg$ – think: *tosets*) unless the filter we use is ultra. Then everything comes right.

7.6 Ultraproducts and Łoś's theorem

At this point an excellent book to look at is [1]. It's comprehensive, clear, and probably at the right level.

THEOREM 23 *(Łoś's theorem)*
Let $\mathcal{U}$ be an ultrafilter on I. For all expressions $\phi(f, g, h \ldots)$,

$$(\prod_{i\in I} \mathcal{A}_i)/\mathcal{U} \models \phi(f, g, h \ldots) \text{ iff } \{i : \mathcal{A}_i \models \phi(f(i), g(i), h(i) \ldots)\} \in \mathcal{U}.$$

Proof:

We do this by structural induction on the rectype of formulæ. For atomic formulæ it is immediate from the definitions.

[wouldn't hurt to write out the details for the fainthearted!]

The proofs for $\wedge$ and $\forall$ are just as with products.

As we would expect, the only hard work comes with $\neg$ and $\vee$, though $\exists$ merits comment as well.

Disjunction

Suppose we know that
$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \phi$ iff $\{i : \mathcal{A}_i \models \phi\} \in \mathcal{U}$ and
$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \psi$ iff $\{i : \mathcal{A}_i \models \psi\} \in \mathcal{U}$.
We want to show

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models (\phi \vee \psi) \text{ iff } \{i : \mathcal{A}_i \models \phi \vee \psi\} \in \mathcal{U}.$$

The steps in the following manipulation will be reversible. Suppose

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \phi \vee \psi.$$

Then

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \phi \text{ or } (\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \psi.$$

By induction hypothesis, this is equivalent to

$$\{i : \mathcal{A}_i \models \phi\} \in \mathcal{U} \text{ or } \{i : \mathcal{A}_i \models \psi\} \in \mathcal{U},$$

both of which imply

$$\{i : \mathcal{A}_i \models \phi \vee \psi\} \in \mathcal{U}.$$

$\{i : \mathcal{A}_i \models \phi \vee \psi\}$ is $\{i : \mathcal{A}_i \models \phi\} \cup \{i : \mathcal{A}_i \models \psi\}$. Now we exploit the fact that $\mathcal{U}$ is ultra: for all A and B it contains $A \cup B$ iff it contains at least one of A and B, which enables us to reverse the last implication.

Negation

We assume

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \phi \text{ iff } \{i : \mathcal{A}_i \models \phi\} \in \mathcal{U}$$

and wish to infer

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \neg\phi \text{ iff } \{i : \mathcal{A}_i \models \neg\phi\} \in \mathcal{U}.$$

Suppose $(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \neg\phi$. That is to say,

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \not\models \phi.$$

By induction hypothesis this is equivalent to

$$\{i : \mathcal{A}_i \models \phi\} \notin \mathcal{U}.$$

But, since $\mathcal{U}$ is ultra, it must contain I' or $I \setminus I'$ for any $I' \subseteq I$, so this last line is equivalent to

$$\{i : \mathcal{A}_i \models \neg\phi\} \in \mathcal{U},$$

as desired.

Existential quantifier

The step for $\exists$ is also nontrivial:

$$(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \exists x\phi$$

$$(\exists f)(\prod_{i\in I}\mathcal{A}_i)/\mathcal{U} \models \phi(f)$$

$$(\exists f)\{i \in I : \mathcal{A}_i \models \phi(f(i))\} \in \mathcal{U},$$

and here we use the axiom of choice to pick a witness at each factor

$$\{i \in I : \mathcal{A}_i \models \exists x\phi(x)\} \in \mathcal{U}.$$

■

You will notice that in the induction step for the existential quantifier you use the axiom of choice to pick a witness from each factor, and this use of AC seems unavoidable. This might lead you to suppose that Łoś's theorem is actually equivalent to AC (after all, it implies that an ultraproduct of nonempty structures is nonempty and that sounds very like AC) but this seems not to be the case. Try it! I am endebted to Phil Freeman for drawing my attention to Paul Howard, Proc Am Math Soc Vol. 49, No. 2, Jun., 1975.

This has the incredibly useful corollary (which we shall not prove) that

COROLLARY 14 *A formula is equivalent to a first-order formula iff the class of its models is closed under elementary equivalence and taking ultraproducts.*

Theorem 23 enables us to show that a lot of things are not expressible in any first order language. Since, for example, an ultraproduct of finite p-groups (which are all simple) is not simple, it follows that the property of being a simple group is not capturable by a language in which you are allowed to quantify only over elements of the object in question.

Miniexercise: If the ultrafilter is principal ($\{J \subseteq I : i \in J\}$), then the ultraproduct is isomorphic to the ith factor. So principal ultrafilters are no use.

In contrast if the ultrafilter is nonprincipal you can make good use of the construction even if all the models you feed into it are the *same*.

DEFINITION 57 *If all the factors are the same, the ultraproduct is called an* **ultrapower***, and we write '$A^\kappa/\mathcal{U}$' for the ultraproduct where there are κ copies of A, indexed by a set K of size κ and $\mathcal{U}$ is an ultrafilter on K.*

Not only are $\mathfrak{M}$ and the ultrapower $\mathfrak{M}^\kappa/\mathcal{U}$ elementarily equivalent by Łoś's theorem , we also have the following, of which we will make frequent use.

DEFINITION 58 *Elementary Embedding*
An embedding $i : \mathfrak{M} \hookrightarrow \mathfrak{N}$ is **elementary** *iff*

$$\mathfrak{M} \models \phi(x_1, \ldots x_k) \text{ iff } \mathfrak{N} \models \phi(i(x_1) \ldots i(x_k))$$

for all tuples $x_1 \ldots x_k$ in $\mathfrak{M}$ and all $\phi \in \mathcal{L}(\mathfrak{M})$.

For example: the identity map is an elementary embedding from $\mathbb{Q}$ into $\mathbb{R}$, where both these structures are thought of as ordered abelian groups. Not as fields! (Why not?)

LEMMA 12 *The function that sends an m in $\mathfrak{M}$ to the equivalence class (in $\mathfrak{M}^\kappa/\mathcal{U}$) of the function in $\mathfrak{M}^\kappa$ that takes the value m everywhere is elementary.*

This embedding i is just a typed version of the K combinator!

Proof:

It will be sufficient to show that, for any $m \in \mathfrak{M}$, if there is an $x \in \mathfrak{M}^\kappa/\mathcal{U}$ such that $\mathfrak{M}^\kappa/\mathcal{U} \models \phi(x, i(m))$ then there is $x \in \mathfrak{M}$ s.t. $\mathfrak{M} \models \phi(x, m)$. Consider such an $x \in \mathfrak{M}^\kappa/\mathcal{U}$. It is the equivalence class of a family of functions which almost everywhere (in the sense of $\mathcal{U}$) are related to m by ϕ so – by Łoś's theorem – there must be something x in $\mathfrak{M}$ such that $\mathfrak{M} \models \phi(x, m)$. Then $i \mapsto x$ will do.

■

If you are doing Set Theory you will see the utility of this later in connection with measurable cardinals.

The following observation of Dana Scott's in the 1960's was the insight that unlocked all of modern large cardinal theory.

PROPOSITION 1
If V is wellfounded and $\mathcal{U}$ is countably complete over κ then $V^\kappa/\mathcal{U}$ is wellfounded.

Proof. Suppose $\langle [f_i] : i \in \mathbb{N} \rangle$ satisfies $(V^\kappa/\mathcal{U}) \models [f_{i+1}] \in [f_i]$ for each $i \in \mathbb{N}$. Use AC_ω to pick f_i for each i. Then, for each $i \in \mathbb{N}$, let A_i be $\{\alpha < \kappa : f_{i+1}(\alpha) \in f_i(\alpha)\}$. All A_i are in $\mathcal{U}$ by hypothesis. But then, by countable completeness of $\mathcal{U}$, the intersection $\bigcap_{i \in \mathbb{N}} A_i$ is nonempty (it is actually in $\mathcal{U}$), and for any address β in it, it is the case that $(\forall i)(f_{i+1}(\beta) \in f_i(\beta))$, contradicting the assumption that there are no ω-descending $\in$-chains in V.

■

This is *Huge*! If $V^\kappa/\mathcal{U}$ is wellfounded we can take a Mostowski collapse and we get something that lives comfortably inside the world of sets. We have an elementary embedding of the universe into a substructure of itself – and altho' people knew about elementary embeddings – the possibility of an elementary embedding from the set theoretic universe into something was a possibility that hitherto nobody had ever considered. It opened up a feast of possibilities; Set Theory went beserk at this point.

Ultraproducts enable us to give a particularly slick proof of the compactness theorem for predicate calculus.

THEOREM 24 *(Compactness theorem for predicate logic)*
Every finitely satisfiable set of sentences of predicate calculus has a model.

Proof: Let Δ be a set of wffs that is finitely satisfiable. Let $\mathcal{S}$ be the set of finite subsets of Δ (elsewhere in these notes notated $\mathcal{P}_{\aleph_0}(\Delta)$), and let $X_s = \{t \in \mathcal{S} : s \subseteq t\}$. Pick $\mathfrak{M}_s \models s$ for each $s \in \mathcal{S}$. Notice that $\{X_s : s \in \mathcal{S}\}$ generates a proper filter. Extend this to an ultrafilter $\mathcal{U}$ on $\mathcal{S}$. Then

$$(\prod_{s\in\mathcal{S}} \mathfrak{M}_s)/\mathcal{U} \models \Delta.$$

This is because, for any $\phi \in \Delta$, $X_{\{\phi\}}$ is one of the sets that generated the filter that was extended to $\mathcal{U}$. For any $s \in X_{\{\phi\}}$, $\mathfrak{M}_s \models \phi$, so $\{s : \mathfrak{M}_s \models \phi\} \in \mathcal{U}$.

■

Notice we are not making any assumption that the language is countable.

Notice the relation between Arrow's paradox and the nonexistence of nonprincipal ultrafilters on finite sets. Consider an ultraproduct of finitely many linear orders: it must be isomorphic to one of the factors. This is Arrow's "dictatorship" condition.

Chapter 8

Example Sheets

Questions marked with a '+' are brief reality-checks;
Questions marked with a '*' are for enthusiasts/masochists only;
 means what you think it means, and
 signals a particularly tasty question.

8.1 Sheet 0: Countability

Explain briefly why the diagonal argument that shows that $\mathcal{P}(\mathbb{N})$ is uncountable doesn't show that there are uncountably many finite sets of naturals.

8.2 Sheet 1: Mainly Ordinals

1. Write down subsets of $\mathbb{R}$ of order types $\omega + \omega$, ω^2 and ω^3 in the inherited order.

2. Which of the following are true?

   (a) $\alpha + \beta$ is a limit ordinal iff β is a limit ordinal;
   (b) $\alpha \cdot \beta$ is a limit ordinal iff α or β is a limit ordinal;
   (c) Every limit ordinal is of the form $\alpha \cdot \omega$;
   (d) Every limit ordinal is of the form $\omega \cdot \alpha$.

   For these purposes 0 is a limit ordinal.

3. Consider the two functions $On \to On$: $\alpha \mapsto 2^\alpha$ and $\alpha \mapsto \alpha^2$. Are they normal?

4. Prove the converse to lemma 1.2: if $\langle X, <_X \rangle$ is a total order satisfying "every subordering is isomorphic to an initial segment" then it is a wellordering.

5. What is the smallest ordinal you can not embed in the reals in the style of question (1) on this sheet?

6. Prove that every [nonzero] countable limit ordinal has cofinality ω. What about ω_1?

7. Recall the recursive definition of ordinal exponentiation:

$$\alpha^0 = 1;\ \alpha^{\beta+1} = \alpha^\beta \cdot \alpha,\ \text{and}\ \ \alpha^{sup(B)} = \sup(\{\alpha^\beta : \beta \in B\}).$$

   Ordinal addition corresponds to disjoint union [of wellorderings], ordinal multiplication corresponds to lexicographic product, and ordinal exponentiation corresponds to ...? Give a definition of a suitable operation on wellorderings and show that your definition conforms to the spec: $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

8. Let $\{X_i : i \in I\}$ be a family of sets, and Y a set. For each $i \in I$ there is an injection $X_i \hookrightarrow Y$.

   Give an example to show that there need not be an injection $(\bigcup_{i\in I} X_i) \hookrightarrow Y$.

   But what if the X_i are nested? [That is, $(\forall i, j \in I)(X_i \subseteq X_j \vee X_j \subseteq X_i)$.]

9. Prove that every ordinal of the form ω^α is **indecomposible**:

   $\gamma + \beta = \omega^\alpha \ \rightarrow\ \gamma = \omega^\alpha \ \vee\ \beta = \omega^\alpha$.

10. Show that an arbitrary intersection of transitive relations is transitive.

11. Let $\langle X, R\rangle$ be a wellfounded binary structure, with rank function ρ.
    Prove that $(\forall x \in X)(\forall \alpha < \rho(x))(\exists y \in X)(\rho(y) = \alpha)$.

12. Let $\{X_i : i \in \mathbb{N}\}$ be a nested family of sets of ordinals.

    (a) Give an example to show that the order type of $\bigcup_{i\in\mathbb{N}} X_i$ need not be the sup of the order types of the X_i.
    (b) What condition do you need to put on the inclusion relation between the X_i to ensure that the order type of $\bigcup_{i\in\mathbb{N}} X_i$ is the sup of the order types of the X_i?
    (c) Show that the ordered set of the rationals can be obtained as the union of a suitably chosen ω-chain of some of its finite subsets.

    (The point is that any structure whatever can be obtained as a direct limit ("colimit") of its finitely generated substructures.)

13. Using the uniqueness of subtraction for ordinals, and the division algorithm for normal functions, show that every ordinal can be expressed uniquely as a sum

$$\omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots \omega^{\alpha_n} \cdot a_n$$

    where all the a_i are finite, and where the α_i are strictly decreasing.

14. Let f be a function from countable [nonzero] limit ordinals to countable ordinals satisfying $f(\alpha) < \alpha$ for all (countable limit) α. (f is "*pressing-down*".) Can f be injective?

8.3 Sheet 2: Posets

1. For $n \in \mathbb{N}$,

   (a) How many antisymmetrical binary relations are there on a set of cardinality n?
   How many binary relations satisfying *trichotomy*: $(\forall xy)(R(x,y) \vee R(y,x) \vee x = y)$?
   How are your two answers related?

   (b) How many *symmetric* relations are there on a set of cardinality n?
   How many *antisymmetric trichotomous* relations are there on a set of cardinality n?
   How are your two answers related?

   (c) Contrast (a) and (b).

2. Draw Hasse diagrams of all the partial orders of sets with four elements. Indicate which of them are complete posets.

3. Consider the set of equivalence relations on a fixed set, partially ordered by $\subseteq$. Show that it is a lattice. Must it be distributive? Is it complete?

4. Recall that $\alpha \cdot \beta$ is $|A \times B|$ where $|A| = \alpha$ and $|B| = \beta$.
   Show that a union of α disjoint sets each of size β has size $\alpha \cdot \beta$.
   Explain your use of AC.

5. Use Zorn's Lemma to prove that

   (i) Every partial ordering on a set X can be extended to a total ordering of X;

   (ii) For any two sets A and B, there exists either an injection $A \hookrightarrow B$ or an injection $B \hookrightarrow A$.

6. Let $\langle P, \leq_P \rangle$ be a chain-complete poset with a least element, and $f : P \to P$ an order-preserving map. Show that the set of fixed points of f has a least element and is chain-complete in the ordering it inherits from P. Deduce that if $f_1, f_2, \ldots, f_n$ are order-preserving maps $P \to P$ which commute with each other (i.e. $f_i \circ f_j = f_j \circ f_i$ for all i, j), then they have a common fixed point. Show by an example that two order-preserving maps $P \to P$ which do not commute with each other need not have a common fixed point.

7. $\mathbb{N} \rightharpoonup \mathbb{N}$ is the set of partial functions from $\mathbb{N}$ to $\mathbb{N}$, thought of as sets of ordered pairs and partially ordered by $\subseteq$.

   Is it complete? Directed-complete? Separative? Which fixed point theorems are applicable?

8. For each of the following functions $\Phi : (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N})$

   (i) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, undefined otherwise.

   (ii) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, $\Phi(f)(n) = 0$ otherwise.

   (iii) $\Phi(f)(n) = f(n-1) + 1$ if $f(n-1)$ is defined, $\Phi(f)(n) = 0$ otherwise

   determine

(a) whether Φ is order-preserving, and
(b) whether it has a fixed point.

(i) is order-preserving, but its sole fixed point is the empty function.

Values of the last two operations are always total functions so the operations can't be order-preserving!

Part (iii) requires care. Suppose $f = \Phi(f)$; what is $\Phi(f)(0)$? To ascertain the value of $\Phi(f)(0)$ we have to look at $f(0-1)\ldots$ and that will crash, so $\Phi(f)(0)$ traps the failure and returns 0. Thereafter the recursion procedes smoothly to give us the identity function.

9. Let $\langle A, \leq\rangle$ and $\langle B, \leq\rangle$ be total orderings with $\langle A, \leq\rangle$ isomorphic to an initial segment of $\langle B, \leq\rangle$ and $\langle B, \leq\rangle$ isomorphic to a terminal segment of $\langle A, \leq\rangle$. Show that $\langle A, \leq\rangle$ and $\langle B, \leq\rangle$ are isomorphic.

10. Let U be an arbitrary set and let $\mathcal{P}(U)$ be the power set of U.
For X a subset of $\mathcal{P}(U)$, the **dual** $X^\vee$ of X is the set

$$\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}.$$

   (a) Is the function $X \mapsto X^\vee$ monotone? Comment.

   (b) By considering the poset P of those subsets of $\mathcal{P}(U)$ that are subsets of their duals, or otherwise, show that there are sets $X \subseteq \mathcal{P}(U)$ with $X = X^\vee$.

   (c) $X^{\vee\vee}$ is clearly a superset of X, in that it contains every superset of every member of X. Does it consist solely of supersets of members of X? (That is, do we have $Y \in X^{\vee\vee} \to (\exists Z \in X)(Z \subseteq Y)$?)

   (d) Is $A^{\vee\vee\vee}$ always equal to $A^\vee$?

11. Players `I` and `II` alternately pick elements (`I` plays first) from a set A (repetitions allowed: A does not get used up) thereby jointly constructing an element s of A^ω, the set of ω-sequences from A. Every subset $X \subseteq A^\omega$ defines a game $G(X)$ which is won by player `I` if $s \in X$ and by `II` otherwise. Give A the discrete topology and A^ω the product topology.

    $A^{<\omega}$ is the set of finite sequences from A. By considering the poset of partial functions $A^{<\omega} \to \{\mathtt{I}\}$ or otherwise prove that if X is open then one of the two players must have a winning strategy.

12. $\mathbb{R} = \langle 0, 1, +, \times, \leq\rangle$ is a field. Consider the product $\mathbb{R}^{\mathbb{N}}$ of countably many copies thereof, with operations defined pointwise. Let $\mathcal{U}$ be an ultrafilter $\subseteq \mathcal{P}(\mathbb{N})$ and consider $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$. Prove that it is a field. Is it archimedean?

13. (i)$^+$ How many order-preserving injections $\mathbb{R} \to \mathbb{R}$ are there?
    (ii)☠ Let $\langle X, \leq_X\rangle$ be a total order with no nontrivial order-preserving injection $X \to X$. Must X be finite?

8.4 Sheet 3: Propositional and Predicate Logic

1. (This question is a sleeper for Banach-Tarski. Look at [8].)

   (i) State and prove the Tarski-Knaster fixed point theorem for complete lattices.

   (ii) Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ be injections. By considering $F : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

   $$F(A) = X \setminus g``(Y \setminus f``X)$$

   or otherwise, show that there is a bijection $h : X \to Y$.

   Suppose U is a set equipped with a group Σ of permutations. We say that a map $s : X \to Y$ is *piecewise-Σ* just when there is a finite partition $X = X_1 \cup \ldots \cup X_n$ and $\sigma_1 \ldots \sigma_n \in \Sigma$, so that $s(x) = \sigma_i(x)$ for $x \in X_i$. Let X and Y be subsets of U, and $f : X \to Y$ and $g : Y \to X$ be piecewise-Σ injections. Show that there is a piecewise-Σ bijection $h : X \to Y$.

   If $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ are two posets with order-preserving injections $f : P \to Q$ and $g : Q \to P$, must there be an isomorphism? Prove or give a counterexample.

2. Show how $\wedge$, $\vee$ and $\neg$ can each be defined in terms of $\to$ and $\bot$.
   Why can you not define $\wedge$ in terms of $\vee$?
   Can you define $\vee$ in terms of $\to$?
   Can you define $\wedge$ in terms of $\to$ and $\vee$?

3. (a) Show that for every countable set A of propositions there is an independent set B of propositions with the same deductive consequences.

   (b) If A is finite show that we can find such a B with $B \subseteq A$.

   (c) Give an example to show that we should not expect $B \subseteq A$ if A is infinite.

   (d) Show that if A is an infinite independent set of propositions then there is no finite set with the same deductive consequences.

4. Explain very briefly the relation between truth-tables and Disjunctive Normal Form.

5. Explain briefly why every propositional formula is equivalent both to a formula in CNF and to a formula in DNF.

   Establish that the class of all propositional tautologies is the maximal propositional logic in the sense that any proper superset of it that is a propositional logic (closed under $\models$ and substitution) is trivial (contains all well-formed formulæ).

6. A formula (of first-order Logic) is in **Prenex Normal Form** if the quantifiers have been "pulled to the front" – every propositional connective and every atomic subformula is within the scope of every quantifier.

   Explain briefly why every first-order formula is equivalent to one in PNF.

Axiomatise the theory of groups in a signature with '=' and a single three-place relation "x times y is z". Put your axioms into PNF. What are the quantifier prefixes?

Find a signature for Group Theory which ensures that every substructure of a group is a semigroup-with-**1**.

7. Show that the theory of equality plus one wellfounded relation is not axiomatisable.

8. Write down axioms for a first-order theory T with equality plus a single one-place function symbol f that says that f is bijective and that for no n and no x do we have $f^n(x) = x$.

   (a) Is T finitely axiomatisable?
   (b) How many countable models does T have (up to isomorphism)?
   (c) How many models of cardinality of the continuum does it have (up to isomorphism)? (You may assume that the continuum is not the union of fewer than $2^{\aleph_0}$ countable sets, a fact whose proof – were you to attempt it – would need AC.)
   (d) Let κ be an uncountable aleph. How many models does T have of size κ?
   (e) Is T complete?

9. Show that monadic predicate logic (one-place predicate letters only, without equality and no function symbols) is decidable.

10. 

    (a)$^+$ Suppose A is a propositional formula and 'p' is a letter appearing in A. Explain how to find formulæ A_1 and A_2 not containing 'p' such that A is logically equivalent to $(A_1 \wedge p) \vee (A_2 \wedge \neg p)$.
    (b) Hence or otherwise establish that, for any two propositional formulæ A and B with $A \models B$, there is a formula C, containing only those propositional letters common to both A and B, such that $A \models C$ and $C \models B$. (Hint: for the base case of the induction on the size of the common vocabulary you will need to think about expressions over the empty vocabulary).

11. Why does T not follow from K and S?

    Show that Peirce's Law: $((A \to B) \to A) \to A$ cannot be deduced from K and S.

12. Look up *monophyletic*. Using only the auxiliary relation "is descended from" give a definition in first-order logic of what it is for a set of lifeforms to be monophyletic.

13. Is

    $$(\forall x)(\exists y)(F(x,y)) \to (\forall x)(\exists y)(\forall x')(\exists y')[F(x,y) \wedge F(x',y') \wedge (x = x' \to y = y')]$$

    valid?

14. (a) Show that the theory of fields of characteristic zero is (first-order) axiomatisable but not finitely axiomatisable.

    Show that the theory of fields of finite characteristic is not first-order axiomatisable.

    (b) Recall that a simple group is one with no nontrivial normal subgroup. Is the theory of simple groups first order?

    (c) A local ring is a ring with a unique maximal ideal. Is the theory of local rings first-order? [Hint: what might the unique maximal ideal be?]

    (d) Is the theory of posets in which every element belongs to a unique maximal antichain first-order?

    (e) A theory T is **algebraic** iff every axiom of T is of the form $(\forall \vec{x})\Phi$ where ϕ is a conjunction of equations between T-terms. Prove that, if T is algebraic, then a pointwise product of models of T is another model of T, and substructures and homomorphic images of models of T are models of T.

    Which of the theories in (a)–(d) are algebraic?

15. 

    A *type* in a propositional language $\mathcal{L}$ is a countably infinite set of formulæ.

    For T an $\mathcal{L}$-theory a *T-valuation* is an $\mathcal{L}$-valuation that satisfies T.
    A valuation v *realises* a type Σ if v satisfies every $\sigma \in \Sigma$. Otherwise v *omits* Σ.

    We say a theory T *locally omits* a type Σ if, whenever ϕ is a formula such that T proves $\phi \to \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg\phi$.

    (a) Prove the following:

    Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type.
    If T locally omits Σ then there is a T-valuation omitting Σ.

    (b) Prove the following:

    Let T be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq \mathcal{L}(T)$ be a type.

    If T locally omits every Σ_i then there is a T-valuation omitting all of the Σ_i.

16. This question is a sleeper for **NP-completeness**.

    Prove that, for every formula ϕ in CNF, there is a formula ϕ' which

    (i) is satisfiable iff ϕ is;

    (ii) is in CNF where every conjunct contains at most three disjuncts.

    (Hint: there is no presumption that $\mathcal{L}(\phi') = \mathcal{L}(\phi)$.)

8.5 Sheet 4: Set Theory

1. Let us say $A@B$ is $\{\{a, b\} : a \in A \wedge b \in B\}$.

   If A and B are not disjoint then $A@B$ might contain singletons.

   If you are given a set X of pairs-and-singletons that happens to be of the form $A@B$ can you recover A and B?

2. Show that if x is a transitive set, then so are $\bigcup x$ and $\mathcal{P}(x)$.

   Are the converses true?

3. Show that the Pair-set axiom is deducible from the axioms of empty set, power set, and replacement.

4. Show that $\{z : \neg(\exists u_1, \ldots, u_n)((z \in u_1) \wedge (u_1 \in u_2) \wedge \cdots \wedge (u_n \in z))\}$ is not a set for any n. What assumptions have you made?

5. Write down sentences in the language of set theory to express the assertions that, for any two sets x and y, the product $x \times y$ and the set y^x of all functions from x to y exist. You may assume that your pairs are Wiener-Kuratowski.

   Which axioms of set theory are you going to have to assume if these assertions are to be provable?

6. (a) Prove that every normal function $On \to On$ has a fixed point.
   (b) Prove that the function enumerating the fixed points of a normal function $On \to On$ is itself normal.
   (c) If α is an ordinal and f is a normal function show that f has a fixed point of cofinality $cf(\alpha)$.
   (d) Are any of your fixed points regular?
   (e) If α is a regular ordinal and f is a normal function show that f has a fixed point of cofinality α.

7. Show that the axiom of choice follows from the assumption that cardinals are totally ordered by $\leq_{card}$. (This is the other direction of sheet 2 question 5.)

8. Explain briefly the equivalence of the four versions of the axiom of foundation given in lectures:

   (i) The axiom scheme of $\in$-induction;
   (ii) Every set is wellfounded;
   (iii) Axiom of Regularity;
   (iv) Every set belongs to the cumulative hierarchy.

9. f is an $\in$-automorphism if f is a permutation of V that preserves $\in$:

   $$(\forall xy)(x \in y \longleftrightarrow f(x) \in f(y)).$$

   Show that a model of ZF (with foundation of course) can have no nontrivial $\in$-automorphisms.

   Give an example to show that the surjectivity condition on f is necessary; that is to say, there are non-trivial injective $\in$-homomorphisms.

10. Recall that ρ is set-theoretic rank. If $\langle x, y\rangle$ is the Wiener-Kuratowski ordered pair show that $\rho(\langle x, y\rangle) = \max(\rho(x), \rho(y)) + 2$.

    (a) Can you define a ordered pair such that $\rho(\langle x, y\rangle) = \max(\rho(x), \rho(y)) - 1$?

    (b) Can you define a ordered pair such that $\rho(\langle x, y\rangle) = \max(\rho(x), \rho(y)) + 1$?

    (c)* Can you define a ordered pair such that $\rho(\langle x, y\rangle) = \max(\rho(x), \rho(y))$ for all but finitely many x and y?

11. There are various ways of constructing implementations (as sets) of $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ from an implementation (as sets) of the naturals. For one of these constructions compute the ranks of the sets that have the rôles of $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$.

    Different implementations will almost certainly give you different answers. Are there any lower or upper bounds on the answers you might get?

12. Let G be a graph where, for each vertex v, the collection $N(v)$ of neighbours of v is a set. (v' is a neighbour of v iff there is an edge between v and v').

    Give an example to show that G might be a proper class.

    Now suppose G is connected; prove that it is a set.

    What axioms have you used?

13. Consider the binary relation E on $\mathbb{N}$ defined by: $n\,E\,m$ iff the nth bit (counting from the right, starting at 0) in the binary expansion of m is 1. What can you say about the structure $\langle \mathbb{N}, E\rangle$?

14. Prove that, for each $n \in \mathbb{N}$, there is a set of size $\aleph_n$. Is there a set of size $\aleph_\omega$?

15. Assume that the cartesian product $x \times y$ always exists however you implement ordered pairs. Infer the axiom scheme of replacement.

16. Assume that every normal function $On \to On$ has a regular fixed point. Consider the function that enumerates the initial ordinals and deduce that there is a "weak inaccessible" κ. Which axioms of ZF hold in V_κ?

17. Suppose $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ are families of sets such that for no $i \in I$ is there is a surjection $A_i \twoheadrightarrow B_i$. Show that there is no surjection $\bigcup_{i\in I} A_i \twoheadrightarrow \prod_{i\in I} B_i$.

    You will need the axiom of choice. Is there a converse?

    Using these ideas you can show that $\aleph_\omega \neq 2^{\aleph_0}$ *without* using AC.

18. Let $\{A_i : i \in \mathbb{N}\}$ be a family of finite structures, and $\mathcal{U}$ a nonprincipal ultrafilter on $\mathbb{N}$. Show that the ultraproduct is finite if there is a finite bound on the size of A_i and is of size $2^{\aleph_0}$ if every infinite subset of $\{A_i : i \in \mathbb{N}\}$ contains arbitrarily large elements.

8.6 Sheet 5

This sheet is for enthusiasts who want to take this stuff further; it's a mixture of revision, consolidation-with-backfill and looking-ahead.

1. Explain to members of your tutorial group (or to anyone listening who might be confused) the difference between

   (i) Nonstandard naturals
   (ii) Countable ordinals
   (iii) Infinite Dedekind-finite cardinals

   Why is there no such thing as an infinite Dedekind-finite *ordinal*?

2. For P a poset, let P^* be the poset of chains-in-P partially ordered by end-extension. (Chains are allowed to be empty). Show that there is no injective homomorphism $P^* \hookrightarrow P$.

3. Any two countable dense linear orders without endpoints are isomorphic.
   Give an illustration to show how your back-and-forth construction might not work for dense linear orders of size $\aleph_1$.
   How do you have to spice up the denseness condition to prove an analogous result for linear orders of size $\aleph_1$?

4. A wellordering of $\mathbb{N}$ is *recursive* iff its graph (subset of $\mathbb{N} \times \mathbb{N}$) is decidable ("recursive");
   An ordinal is *recursive* iff it is the order type of a decidable ("recursive") wellordering of $\mathbb{N}$.

   Which of the countable ordinals you have learnt to know and love are recursive? Come to think of it, are *all* countable ordinals recursive?

5. Prove Trakhtenbrot's theorem that if S is a signature with equality and at least one binary relation symbol then the set of S-sentences true in all finite structures is not semidecidable ("r.e.").

6. Using propositional logic only, show that a(n undirected) graph and its complement cannot both be disconnected. *[Hint: propositional letters will correspond to edges].*

7. (This question and the next are tasters for *forcing*)

   A poset $\langle P, \leq_P \rangle$ is [upwards] separative if

   $$(\forall x, y \in P)(x \not\leq y \rightarrow (\exists z \geq y)(\forall w)(w \not\geq z \vee w \not\geq x)).$$

   For each of the following posets say whether or not it is (i) separative (ii) directed (iii) chain-complete.

- The set of finite sequences of countable ordinals (thought of as sets of ordered pairs) partially ordered by $\subseteq$.
- The set $\{f : f \text{ is an injection from some set of countable ordinals} \hookrightarrow \mathbb{R}\}$ ordered by $\subseteq$. Think of f as a set of ordered pairs.

8. A poset $\langle P, \leq \rangle$ is called *downwards separative* if for all $x \not\leq y$ there is $z \leq x$ with z incompatible with y. ("incompatible" means "have no lower bound").
A poset is *downwards splitting* if for every x there are y and z such that $y, z \leq x$, and y and z are incompatible.

   (a) Show that not every downwards separative poset is downwards splitting.

   (b) Show that if a poset has no minimal elements and is downwards separative, then it is downwards splitting.

   A set $D \subseteq P$ is called *downwards dense* if for every p in P there is a d in D such that $d \leq p$.

   Suppose $\mathcal{X}$ is a collection of subsets of P. We say that $G \subseteq P$ is $\mathcal{X}$-*generic* if G has nonempty intersection with every downwards dense element of $\mathcal{X}$.

   We say that G is a *filter* if

   1. for any x, y in G there is z in G such that $z \leq x$ and $z \leq y$, and
   2. for any x in G and $x \leq y$, we have y in G.

   (c) If $\mathcal{X}$ is countable, show that there is an $\mathcal{X}$-generic filter.

   (d) Let $\langle P, \leq \rangle$ be a downwards separative poset with no minimal elements and let $\mathcal{X}$ be a collection of subsets of P closed under complementation (i.e., if $X \in \mathcal{X}$, then also $(P \setminus X) \in \mathcal{X}$). Show that if G is an $\mathcal{X}$-generic filter, then $G \notin \mathcal{X}$.

   (e) Let $\langle P, \leq \rangle$ be $\{0, 1\}^{<\omega}$, the set of finite sequences of zeros and ones, ordered by reverse inclusion. Show that this is a downwards separative poset without minimal elements.

   (f) Let $\mathcal{X}$ be the collection of recursive sets of finite sequences of zeros and ones. Show, using (c), (d), and (e), that there is a non-recursive such set.

9. (Concrete constructions of limits in ZF)

   Let $\langle I, \leq_I \rangle$ be a directed poset and, for each $i \in I$, let A_i be a set and, for all $i \leq_I j$, let $\sigma_{i,j} : A_i \hookrightarrow A_j$ be an injection, and let the injections commute.

   Show that there is a set A_I with, for each $i \in I$, an injection $\sigma_i : A_i \hookrightarrow A_I$ and the $\sigma_{i,j}$ commute with the σ_i.

   Show also that A_I is minimal in the sense that if B is any set such that for each $i \in I$ there is an injection $\tau_i : A_i \hookrightarrow B$ and the τ_i commute with the $\sigma_{i,j}$, then there is a map $A_I \hookrightarrow B$.

   Let $\langle I, \leq_I \rangle$ be a directed poset and, for each $i \in I$, let A_i be a set and, for all $i \leq_I j$, let $\sigma_{j,i} : A_j \twoheadrightarrow A_i$ be a surjection, and let the surjections commute.

Show that there is a set A_I with, for each $i \in I$, a surjection $\pi_i : A_I \twoheadrightarrow A_i$.

Show also that A_I is minimal in the sense that, if B is any set such that for each $i \in I$ there is a surjection $\tau_i : B \twoheadrightarrow A_i$ and the τ_i commute with the $\sigma_{i,j}$, then there is a map $B \twoheadrightarrow A_I$.

10.

Let G be the alternating group of permutations of V_ω. Any $\pi \in G$ can be extended to a permutation acting on the whole universe by fixing everything not in V_ω, and we still write it π.
For each $n \in \mathbb{N}$ and any x whatever such a π can move x by acting "n levels down", on[1] $\bigcup^n x$.
A set that is fixed by everything in G under the nth action of G is said to be n-**symmetric**; if it is n-symmetric for all sufficiently large n it is just plain **symmetric**.

Consider the collection of sets that are hereditarily symmetric. Which axioms of ZFC are true in this structure?

11. (A taster for Large Cardinals)

Prove Łoś's theorem (You may assume AC)

Suppose there is a set K with a nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(K)$ that is closed under countable intersections. By using Scott's trick concretise the elements of the ultrapower $V^K/\mathcal{U}$. Prove that it is wellfounded. What can you say about the Mostowski collapse?

12. 

(a) Let T be a consistent theory in a language containing a (possibly complex) expression $\phi(-,-)$ which T proves to be an infinite total order. Let $\mathcal{I} = \langle I, <_I \rangle$ be a total ordering. Show that T has a model in which $\mathcal{I}$ is embedded as part of the graph of ϕ.

(b) $\mathcal{I} = \langle I, \leq_\mathcal{I} \rangle$ is a **set of indiscernibles** for a model $\mathfrak{M}$ for a language $\mathcal{L}$ iff $\leq_\mathcal{I}$ is a total order, and for all $\phi \in \mathcal{L}$, if ϕ is a formula with n free variables in it then for all distinct n-tuples $\vec{x}$ and $\vec{y}$ from $\mathcal{I}$ **taken in $\leq_\mathcal{I}$-increasing order** we have $\mathfrak{M} \models \phi(\vec{x}) \longleftrightarrow \phi(\vec{y})$.

Now let $\mathcal{I}$ be a total order, T a theory with infinite models and a formula $P()$ with one free variable s.t. T thinks that the extension of P is an infinite total order. Then T has a model $\mathfrak{M}$ in which $\mathcal{I}$ is embedded in (the interpretation of) P as a set of indiscernibles.

(Notice that there is no suggestion that the copy of $\mathcal{I}$ in $\mathfrak{M}$ is a set of $\mathfrak{M}$, or is in any way definable.)

It is comparatively straightforward, given $\mathcal{I}$ and T and $P()$, to find $\mathfrak{M}$ as in the theorem if we do not ask that I should be embedded as a set of indiscernibles:

[1] Any group of permutations of a set X acts on $\mathcal{P}_{\aleph_0}(X)$ (the set of finite subsets of X) in an obvious way.

compactness does the trick. To get the set of indiscernibles you need to use Ramsey's theorem from Graph theory.

This might make a good open-book exam question!

13. Wikipædia says:

> Commutative Rings ⊇ Integral Domains ⊇ Integrally Closed Domains ⊇ GCD domains ⊇ Unique Factorization Domains ⊇ Principal Ideal Domains ⊇ Euclidean Domains ⊇ Fields

All these families-of-structures can be thought of as belonging to the one signature: 0, **1**, +, · and −. Which of them are first-order axiomatisable? In each case provide axiomatisations or explain why there are none. Identify the quantifier complexity of the axiomatisations you find. Comment on whether or not each theory is finitely axiomatisable.

14.  How many countable [linear] order types are there whose automorphism group is transitive on singletons?

15. (i) How many transitive subsets of V_ω are there?

(ii) How many transitive sets are there all of whose members are countable?

16. Prove that a directed limit (colimit) of wellfounded structures under end-extension is wellfounded.

17. Prove that the Cauchy reals (the family of equivalence classes of Cauchy sequences in the rationals) form a complete ordered field. And do it without using AC.

18. A **circular order** (see [5]) is a ternary relation $R(x, y, z)$, whose typical example is the relation that holds between points x, y and z on the unit circle if, starting from x and moving clockwise, one encounters y before z.

(1) *Write down a set of axioms for circular orders.*

A group is **circularly-orderable** if it has a circular ordering that interacts in the obvious way with the multiplication of the group. The typical example is the additive group of integers-mod-p.

(2) *Write down a set of axioms for circularly orderable groups.*

(3) *Prove that a group is circularly orderable iff all its finitely generated subgroups are circularly orderable.*

(4) *Is the multiplicative group of (nonzero) integers mod p (p prime) circularly ordered?*

19. Prove that the axiom of choice is equivalent to the assertion that every set can be given a group structure.

    Hint: If i want to welllorder a set X, what set do i decorate with a group structure? It's not X itself, in case you were wondering.

Chapter 9

Answers to Selected Exercises

9.1 Sheet 0

Question 0.(i)

"Explain briefly why the diagonal argument that shows that $\mathcal{P}(\mathbb{N})$ is uncountable doesn't show that there are uncountably many finite sets of naturals."

Answer

The diagonal set might not be finite. Suppose $f : \mathbb{N} \to \mathcal{P}_{\aleph_0}(\mathbb{N})$. The diagonal set is $\{n \in \mathbb{N} : n \notin f(n)\}$. The usual argument will show that it is not a value of f. So we have proved:

Suppose $f : \mathbb{N} \to \mathcal{P}_{\aleph_0}(\mathbb{N})$. Then $\{n \in \mathbb{N} : n \notin f(n)\}$ is infinite.

9.2 Sheet 1

Question 1.11

Let $\langle X, R\rangle$ be a wellfounded binary structure, with rank function ρ.

Prove that

$$(\forall x \in X)(\forall \alpha < \rho(x))(\exists y \in X)(\rho(y) = \alpha).$$

Answer

The obvious weapon is R-induction. We want to show that

$$(\forall \alpha < \rho(x))(\exists y \in X)(\rho(y) = \alpha)$$

when being told that

$$(\forall y)(R(y, x) \to (\forall \alpha < \rho(y))(\exists z \in X)(\rho(z) = \alpha)).$$

There are two cases to consider, depending on whether or not $\rho(x)$ is successor. If it is, then there is y with $R(y, x)$ with $\rho(x) = \rho(y) + 1$.

The other case is where $\rho(x)$ is limit. Suppose $\alpha < \rho(x)$. Then, since $\rho(x) = sup\{\rho(y) : R(y, x)\}$, there is y s.t. $R(y, x)$ with $\alpha < \rho(y)$.

But now the induction hypothesis on 'y' tells us that α is the rank of something in X. ■

This result is telling you that, for any wellfounded binary structure $\langle X, R\rangle$, its rank function is a surjection to an initial segment of the ordinals ("nothing left out") which is sort-of neat and what you'd expect. I use it as an exercise because the proof is a natural example of wellfounded induction. Even if you managed it by other means, go over this proof - to gain familiarity with wellfounded induction.

Question 1.9

Prove that every ordinal of the form ω^α is **indecomposible**:
$\gamma + \beta = \omega^\alpha \ \rightarrow \ \gamma = \omega^\alpha \ \vee \ \beta = \omega^\alpha$.

Answer

Do not attempt to do this by induction!

If γ and β are both less than ω^λ then they are both less than ω^δ for some $\delta < \lambda$. Then $\gamma + \beta$ is less than $\omega^{\delta+1} < \omega^\lambda$.

If γ and β are both less than $\omega^\lambda \cdot \omega$ then they are both less than $\omega^\lambda \cdot n$ for some n, and then $\gamma + \beta$ is less than $\omega^\lambda \cdot (2n) < \omega^\lambda \cdot \omega$.

Question 1.12

"Let $\{X_i : i \in \mathbb{N}\}$ be a nested family of sets of ordinals.

(a) Give an example to show that the order type of $\bigcup_{i\in\mathbb{N}} X_i$ need not be the sup of the order types of the X_i.
(b) What condition do you need to put on the inclusion relation between the X_i to ensure that the order type of $\bigcup_{i\in\mathbb{N}} X_i$ is the sup of the order types of the X_i?
(c) Show that the ordered set of the rationals can be obtained as the union of a suitably chosen ω-chain of some of its finite subsets.

(The point is that any structure whatever can be obtained as a direct limit ("colimit") of its finitely generated substructures.)"

Answer

(a) Let $X_i = \{\omega\} \cup \{n < \omega : 0 < n < i\}$. Each X_i has order type i but the union has order type $\omega + 1$.
(b) You need X_{i+1} to be an end-extension of X_i.
(c) Let f be a bijection between $\mathbb{N}$ and $\mathbb{Q}$, and let the ith finite subset be $\{f(n) : n < i\}$.

9.3 Sheet 2

Question 2.1

"For $n \in \mathbb{N}$,

(a) How many antisymmetrical binary relations are there on a set of cardinality n?

How many binary relations satisfying *trichotomy*: $(\forall xy)(R(x,y) \vee R(y,x) \vee x = y)$? How are your two answers related?

(b) How many *symmetric* relations are there on a set of cardinality n? How many *antisymmetric trichotomous* relations are there on a set of cardinality n? How are your two answers related?

(c) Contrast (a) and (b)."

Answer

(a) $2^n \cdot 3^{\binom{n}{2}}$ antisymmetric binary relations on a set of size n.

$2^n \cdot 3^{\binom{n}{2}}$ trichotomous binary relations on a set of size n.

These two answers are the same.

(b) $2^{\binom{n+1}{2}}$ antisymmetric and trichotomous binary relations on a set of size n.

$2^{\binom{n+1}{2}}$ symmetrical binary relations on a set of size n.

These two answers are the same.

(c)

To get a bijection for (a) you need a choice function on the set of pairs of elements of the underlying set: take the matrix for a symmetrical relation, and flip all the bits in the upper right triangle. This is just `XOR`-ing with the graph of a total order. Thus every total order naturally gives rise to a bijection between the set of symmetrical relations and the set of antisymmetrical trichotomous relations.

To get a bijection for (b) reason as follows. For both symmetrical relations and antisymmetrical trichotomous relations you have n independent choices of whether or not to put in the "diagonal" ordered pairs $\langle x, x\rangle$. Now consider the pair $\langle x, y\rangle$ with $x \neq y$ and its mate $\langle y, x\rangle$. A symmetrical relation either has both these pairs or neither (two possibilities); an antisymmetrical trichotomous relation has precisely one of them (again, two possibilities). So, either way you have $\binom{n}{2}$ choices of two outcomes. So the two answers are the same. How are we to find a bijection between the set of symmetrical relations and the set of antisymmetrical trichotomous relations? It's pretty clear that there is no natural way of doing it. However, if we arm ourselves with a total order $<$ of our set we can do something. If R is antisymmetrical and trichotomous we want to obtain a symmetrical relation R' from it. Put into R' all (and only) the "diagonal" pairs in R. Then if R contains $\langle x, y\rangle$ where $x < y$, then put into R' *both* the pairs $\langle x, y\rangle$ and $\langle y, x\rangle$. If instead it contains $\langle y, x\rangle$ then put neither pair into R'. It should now be clear how to go in the opposite direction.

There doesn't seem to be any natural bijection between the set of all permutations of a set A (also known as the *symmetric group on* a set and notated $\Sigma(A)$) and the set of total orders of A. However, if we fix a total order of a set A, any other total order of A can be thought of as the application of a permutation of A to that fixed total order. So

there is a natural map from the set of total orders of A to the set of bijections between $\Sigma(A)$ and the set of total orderings of A.

Question 2.2

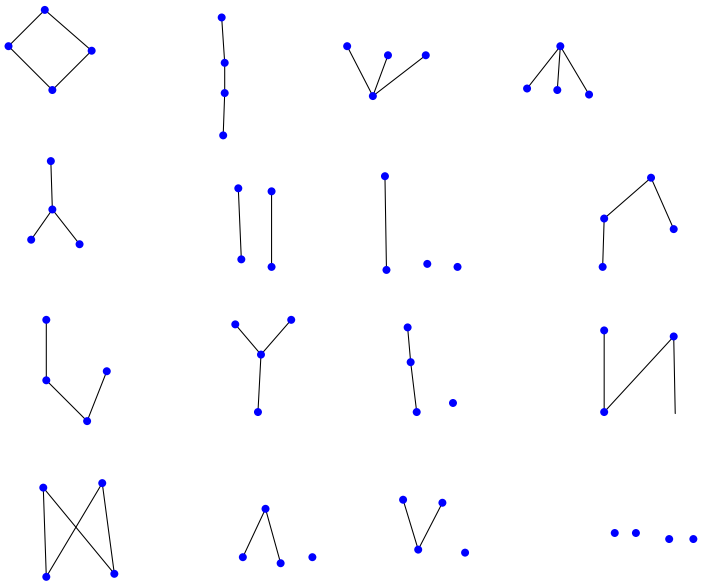
"Draw Hasse diagrams of all the partial orders of sets with four elements. Indicate which of them are complete posets."

Answer

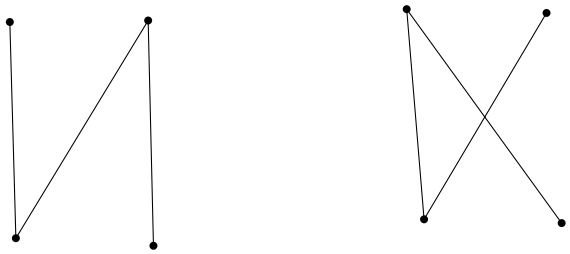
Is there a systematic way of generating them? No easy way, that's for sure. (The set of Hasse diagrams for posets with four elements is a quotient of the set of posets with four elements and in general the cardinalities of quotients are hard to compute, so you really don't know how many you are looking for, and you can't easily know when you have found them all.) I find myself wondering how many isomorphism types of partial orders there are on a set of n elements. Not *exactly* of course, for I see no prospect for an exact formula, but it would be nice to know whether or not there is an exponential lower bound, or a polynomial upper bound. There are $2^{\binom{n}{2}}$ reflexive relations on n elements. How many of them are transitive? My guess is: exponentially many, but I have no exact figure. Let me see... How many transitive relations on $n+1$ things does a given transitive relation on n things extend to? There are $2n$ places where one might put in an edge. The only constraints arise as follows.

> (i) Suppose there is no edge from x to y, where x and y are of the original party of n. Then we cannot have an new edge from x to a (a is our new chap) as well as a new edge from a to y.
>
> (ii) If there is an edge from x to y and we add a new edge from y to a then we have to add a new edge from x to a.

Anyway, there are 16 isomorphism classes of posets on 4 elements, 2 of which are complete (the two with both a top and a bottom element, co's the empty subset has to have a sup!)



One of my students distinguished



. . . which are two embeddings of the same poset into the plane. This makes the same interesting point that my Pittsburgh colleague Ken Manders likes to make. When you formalise (= represent something concretely, or *concretise*) you add extra structure and this structure may be spurious.

There is a general question here: *How do I know when i've got them all?* This particular instance (before us) of this *general* question isn't *so* hard that we are prompted to think much about the general question, but a bit of thought won't go amiss. The answer of course is that you have to find a fairly robust way of thinking of these things as mathematical objects and then find a way of classifying them. In this case the obvious thing to do is to identify them with their Hasse diagrams and then classify them – perhaps – in terms of the number of edges they have. Or the number of maximal elements. But the question still lurks in the shadows: "How can I give a *mathematical*

proof that I have got all of them?".

Back in the 1930s there was an American crime writer called *John Dickson Carr*, who specialised in locked-room murders. In *The Hollow Man* (said by many to be the best locked room murder of all time) his detective delivers himself of a long disquisition in the form of a classification of all locked-room murders[1]. There is a small finite set of them apparently. I don't know how he could be sure, and I keep hoping to find a new one. It's the same with tragedies. Some Russian structuralist in the 1920's has a classification of them – again a small finite number.

A live version of this problem was the problem solved – within my lifetime – of the classification of all finite simple groups. How did the Monster crew know they'd got all of them? There is an answer to this, but I don't know it.

Question 2.4

"Cardinals: Recall that $\alpha \cdot \beta$ is $|A \times B|$ where $|A| = \alpha$ and $|B| = \beta$. Show that a union of α disjoint sets each of size β has size $\alpha \cdot \beta$. Explain your use of AC."

Answer

The way in is to ask yourself "How do i make a union of α pairwise disjoint things each of size β look like $A \times B$ where $|A| = \alpha$ and $|B| = \beta$?"

We are given a family $\{B_a : a \in A\}$ where $|A| = \alpha$ and $|B_a| = \beta$ for all $a \in A$. We distinguish one of the B_a and call it Bertie. For each other B_b we pick a bijection f_b between B_b and Bertie. (This is where AC comes in). Then any element x in $\bigcup_{a\in A} B_a$ can be thought of as the ordered pair $\langle B_x, f_x(x)\rangle$ where B_x is the unique B to which x belongs (and it belongs to only one, co's the B_a are all disjoint). That way $\bigcup_{a\in A} B_a$ can be thought of as the cartesian product of a set of size α (namely A) with a set of size β (namely Bertie).

This may be a way of motivating indiscrete categories: in the above we use choice to expand $\{B_a : a \in A\}$ into an indiscrete category.

Question 2.6

"Let $\langle P, \leq_P\rangle$ be a chain-complete poset with a least element, and $f : P \to P$ an order-preserving map. Show that the set of fixed points of f has a least element and is chain-complete in the ordering it inherits from P. Deduce that if $f_1, f_2, \ldots, f_n$ are order-preserving maps $P \to P$ which commute with each other (i.e. $f_i \circ f_j = f_j \circ f_i$ for all i, j), then they have a common fixed point. Show by an example that two order-preserving maps $P \to P$ which do not commute with each other need not have a common fixed point."

[1] Of course the actual murder in the story is not accommodated in his scheme.

Answer

PTJ[2] says: "For the second part, first show that each of $f_2, \ldots, f_n$ maps the set of fixed points of f_1 into itself, and then use induction on n. (It's worth pointing out the similarity with the technique – which they should have seen before – for finding a common eigenvector of a family of commuting endomorphisms.) The simplest counterexample for the last part is to take two distinct constant maps."

> If $f_1 \ldots f_n$ are order-preserving maps $P \to P$ which commute with each other then their set of common fixed points is chain-complete and has a least element.

Proof:

By induction on n. Base case easy! If $n > 1$ by induction hypothesis the set F of common fixed points of $f_1 \ldots f_{n-1}$ is chain complete and has a least element. Since f_n commutes with all f_k with $k < n$, f_n sends F into itself and then we apply the result above.

Au fond what's going on here is the following. We have a theorem that says that if f_1 is a slick function from a nice poset P into itself then P has a nice subposet (or sub–nice-poset) P_1 of fixed points for f_1. Now let f_2 be a second slick function $P \to P$. Whack P_1 with f_2 to obtain a nice subposet P_2 (fixed points for f_2) of P_1. Keep on trucking. To make it work it turns out that you need the fs to commute ....

(Can you make it work if the fs form an ω-sequence, so there are countably many of them...?)

Question 2.8

"For each of the following functions $\Phi : (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N})$

(i) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, undefined otherwise.

(ii) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, $\Phi(f)(n) = 0$ otherwise.

(iii) $\Phi(f)(n) = f(n-1) + 1$ if $f(n-1)$ is defined, $\Phi(f)(n) = 0$ otherwise

determine

(a) whether Φ is order-preserving, and

(b) whether it has a fixed point."

Answer

(i) is order-preserving, but its sole fixed point is the empty function. (Be sure to write out your answer so that it's clear that you mean the empty function not the identically zero function!)

Values of the last two operations are always total functions so the operations can't be order-preserving!

(ii) clearly has no fixed point!

Part (iii) requires care. Suppose $f = \Phi(f)$; what is $\Phi(f)(0)$? To ascertain the value of $\Phi(f)(0)$ we have to look at $f(0-1)$... and that will crash, so $\Phi(f)(0)$ traps the

[2] My Cambridge colleague Peter Johnstone, who put this question on an example sheet.

failure and returns 0. Thereafter the recursion procedes smoothly to give us the identity function.

Question 2.9

Let $\langle A, \leq\rangle$ and $\langle B, \leq\rangle$ be total orderings with $\langle A, \leq\rangle$ isomorphic to an initial segment of $\langle B, \leq\rangle$ and $\langle B, \leq\rangle$ isomorphic to a terminal segment of $\langle A, \leq\rangle$. Show that $\langle A, \leq\rangle$ and $\langle B, \leq\rangle$ are isomorphic.

Answer

The proof is an almost perfect analogue of the Tarski-Knaster–style proof of Cantor-Bernstein. Conway used to call this the *co-co-co theorem*: *Co*initial plus *co*final gives *co*extensive. Secretly its a theorem about ternary ("circular") orders, but I won't tell you about that unless you ask.

Question 2.10

"Let U be an arbitrary set and let $\mathcal{P}(U)$ be the power set of U.
For X a subset of $\mathcal{P}(U)$, the **dual** $X^\vee$ of X is the set

$$\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}.$$

1. Is the function $X \mapsto X^\vee$ monotone? Comment.
2. By considering the poset P of those subsets of $\mathcal{P}(U)$ that are subsets of their duals, or otherwise, show that there are sets $X \subseteq \mathcal{P}(U)$ with $X = X^{\vee}$.
3. $X^{\vee\vee}$ is clearly a superset of X, in that it contains every superset of every member of X. Does it consist solely of supersets of members of X? (That is, do we have $Y \in X^{\vee\vee} \to (\exists Z \in X)(Z \subseteq Y)$?)
4. Is $A^{\vee\vee\vee}$ always equal to $A^\vee$?"

Answer

A word on motivation...People often speak of $\forall$ and $\exists$ as 'dual". If you think of $\forall$ and $\exists$ as sets – so that $(\exists x)F(x)$ iff $\{x : F(x)\} \in \exists$ and $(\forall x)F(x)$ is $\{x : F(x)\} \in \forall$ – then $\forall$ is the singleton of the domain, and $\exists$ is the set of nonempty subsets of the domain, then *literally* $\forall^\vee = \exists$ and $\exists^\vee = \forall$.

1. $X \mapsto X^\vee$ is obviously *anti*monotone. (Sometimes called *antitone*). It is also continuous in the order topology.
2. This question has got 'Zorn' written all over it, hasn't it? We should be looking for the poset P to be chain-complete. Fortunately it is. Let $\{a_i : i \in I\}$ be a chain, with I an ordered set, so that $i \leq_I j \to a_i \subseteq a_j$. Consider the chain dual to this, namely $\{(a_i)^\vee : i \in I\}$. We want the dual to $\bigcup\{a_i : i \in I\}$ to be $\bigcap\{(a_i)^\vee : i \in I\}$.

First we prove that $\bigcup\{a_i : i \in I\} \subseteq \bigcap\{(a_i)^\vee : i \in I\}$. To establish this we need to show that, for all $i, j \in I$, $a_i \subseteq (a_j)^\vee$.

If $i <_I j$ then $a_i \subseteq a_j \subseteq (a_j)^\vee$;

if $j <_I i$ then $a_j \subseteq a_i$ whence $(a_i)^\vee \subseteq (a_j)^\vee$, but we have $a_i \subseteq (a_i)^\vee$ giving $a_i \subseteq (a_j)^\vee$ as desired.

By part (1) we can now infer that the sup $(\bigcup\{a_i : i \in I\})^\vee$ of the chain is a subset of its dual $\bigcap\{(a_i)^\vee : i \in I\}$.

So, by Zorn's lemma, this poset has maximal elements. Such an element is its own dual.

3. Yes. Suppose not, and that $Y \in A^{\vee\vee}$ but Y is not a superset of anything in A. Then consider $\bigcup\{x \setminus Y : x \in A\}$. It's clearly in $A^\vee$ since it meets every $x \in A$ but it is disjoint from Y, so Y cannot be in $A^{\vee\vee}$. Thus $A^{\vee\vee}$ is the closure of A under superset.

4. For the last part ("Is $A^{\vee\vee\vee}$ always equal to $A^\vee$?") reflect that we have shown that $B \subseteq B^{\vee\vee}$ so we must have $A^\vee \subseteq A^{\vee\vee\vee}$. For the reverse inclusion suppose $Y \in A^{\vee\vee\vee}$. That is to say, Y meets everything in $A^{\vee\vee}$. How does this differ from meeting everything in A? Not one whit, because the only difference between things in A and things in $A^{\vee\vee}$ is that any new things that appear in $A^{\vee\vee}$ are supersets of things in A, and if you meet every superset of y (y a member of A) then you meet y. This proves the inclusion.

If there is a moral to this question it is that in certain circumstances one can find fixed points for antimonotonic (antitone) functions as long as certain nice things happen, such as the function being sufficiently continuous. Plenty of things are fixed points for continuous antimonotone (antitone) functions: $\sqrt{2}$ is a fixed point for the antimonotone (antitone) function $x \mapsto 2/x$. In Biology, a *species* is a fixed point for ... well, i'll let you work that one out by yourself. Something to do with "can mate to produce fertile offspring".

Question 2.11

First thing to clear up is the correct definition of open set in the product topology.

The next thing to do is to think of $A^{<\omega}$ as the set of *positions* in the game... *odd* positions when it is `II`'s turn to move, and *even* positions when it is `I`'s turn to play. Observe that, if `I` wins at all, he wins after finitely many moves. Accordingly, if s is a position such that all elements of A^ω that kick off with s are wins for `I`, then we are not interested in any proper end-extensions of it. We just label it with a '`I`' (to mean that `I` has won) and have done with it

At this point there are two ways of joining up the dots.

(i) We can define a labelling function by recursion on the end-extension relation between positions. Yes, this means that the empty position is at the top!

The recursion is:

> If s is an even position and it has an end-extension by 1 that is labelled '`I`' then label it '`I`';
>
> If s is an odd position and every end-extension of it by 1 is labelled '`I`' then label it '`I`'.

Then it becomes an exercise in recursion.

(ii) You can take the hint: "...by considering the poset of partial functions $A^{<\omega} \to \{\mathtt{I}, \mathtt{II}\}$...". How so? Well, any labelling can be processed by one step of the recursion outlined in (i), so we have a function from labellings to labellings which is monotone, or inflationary – or something. One way or another we reach a fixed point, which will be the labelling we are trying to define by recursion in (i).

Let's think about this fixed point, this *maximal labelling*. The key question is: "Does it label the empty position?". If it does, then `I` has a winning strategy, which is to always pick a labelled node when it is his turn. If not, then `II` has a strategy which is always to play *unlabelled* nodes when it is her turn. That way she stays alive forever and thereby wins.

This is the **Gale-Stewart theorem**, aka "*Open Determinacy*". It is far from best possible. Best possible is Borel Determinacy, a theorem of D.A. Martin from the 1970's. A big fat juicy transfinite induction.

9.4 Sheet 3

Question 3.1

"(This question is a sleeper for Banach-Tarski. Look at [8].)

(i) State and prove the Tarski-Knaster fixed point theorem for complete lattices.

(ii) Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ be injections. By considering $F : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$F(A) = X \setminus g``(Y \setminus f``X)$$

or otherwise, show that there is a bijection $h : X \to Y$.

Suppose U is a set equipped with a group Σ of permutations. We say that a map $s : X \to Y$ is *piecewise-Σ* just when there is a finite partition $X = X_1 \cup \ldots \cup X_n$ and $\sigma_1 \ldots \sigma_n \in \Sigma$, so that $s(x) = \sigma_i(x)$ for $x \in X_i$. Let X and Y be subsets of U, and $f : X \to Y$ and $g : Y \to X$ be piecewise-Σ injections. Show that there is a piecewise-Σ bijection $h : X \to Y$.

If $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ are two posets with order-preserving injections $f : P \to Q$ and $g : Q \to P$, must there be an isomorphism? Prove or give a counterexample."

Answer

The proof is essentially the same as the Tarski-Knaster–style proof of Cantor-Bernstein, corollary 6. Consider the function $A \mapsto U \setminus g``(U \setminus f``A)$. This is an order-preserving map $\mathcal{P}(U) \to \mathcal{P}(U)$ and therefore has a fixed point, A, say. Then

$A = U \setminus g``(U \setminus f``A)$.

We now consider the map

$\texttt{if } x \in A \texttt{ then } f(x) \texttt{ else if } x \in g\text{“}(U \setminus f\text{“}A) \texttt{ then } g^{-1}(x) \texttt{ else fail}$

We want to show that this map (or its restriction) maps X onto Y and that it is piecewise-Σ.

It's pretty clear that it is piecewise-Σ. To be continued

For the final part the answer is clearly no: consider $[0, 1)$ and $(0, 1]$.

Question 3.2

"Show how $\wedge$, $\vee$ and $\neg$ can each be defined in terms of $\rightarrow$ and $\bot$.

Why can you not define $\wedge$ in terms of $\vee$?

Can you define $\vee$ in terms of $\rightarrow$?

Can you define $\wedge$ in terms of $\rightarrow$ and $\vee$?"

Answer

You can define $\vee$ in terms of $\rightarrow$, perhaps surprisingly. $p \vee q$ is truth-functionally the same as $(p \rightarrow q) \rightarrow q$.

You can't define $\wedge$ in terms of $\vee$ because any formula built up solely using $\vee$ is true in more than half of its rows and $p \wedge q$ is true in only one quarter of its rows.

The following proof that you can't define '$\wedge$ in terms of '$\rightarrow$' and '$\vee$' is due to one of my students.

Think of the four element boolean algebra (picture on p. 38) with its two extra elements `left` and `right`. Reflect that $\texttt{left} \rightarrow \texttt{right}$ is `right` and that $\texttt{right} \rightarrow \texttt{left}$ is `left`. And $\texttt{left} \vee \texttt{right}$ is of course $\top$. Consider any complex expression $\texttt{fake-and}(p, q)$ with the two letters 'p' and 'q' in it – built up solely with $\rightarrow$ and $\vee$ – that comically aspires to be conjunction. Consider the valuation that sends p to `left` and sends q to `right`. There is no way it can send $\texttt{fake-and}(p, q)$ to $\bot$, but that's what it would have to do if $\texttt{fake-and}(p, q)$ really were $p \wedge q$.

Question 3.5

"Explain briefly why every propositional formula is equivalent both to a formula in CNF and to a formula in DNF.

Establish that the class of all propositional tautologies is the maximal propositional logic in the sense that any proper superset of it that is a propositional logic (closed under $\models$ and substitution) is trivial (contains all well-formed formulæ)."

Answer

Without loss of generality we can suppose that our language contains '$\top$' and '$\bot$'. This involves no loss of generality co's we can always introduce them by definition if they aren't already there.

Suppose our Logic contains a formula Φ which is not a tautology. Since Φ is not a tautology, its CNF is not the empty conjunction, so Φ is a conjunction of finitely many ϕ_i, each of which is a disjunction of propositional letters and negations of propositional letters. So each of these ϕ_i is a theorem of our logic. Now we use the rule of substitution. Let ϕ be any of the ϕ_i. Replace all the letters with a positive occurrence in ϕ by $\bot$, and all those with negative occurrences by $\top$. ϕ now simplifies to $\bot$. So $\bot$ is a valid expression of our logic. But then anything follows.

Notice that this proof relies on every formula having a CNF, and therefore doesn't work for constructive logic... which is just as well!

Question 3.7

Show that the theory of equality plus one wellfounded relation is not axiomatisable.

Answer

Add countably many constants and axiom to say that they constitute an infinite descending chain.

Question 3.8

"Let L be the language consisting of a single function symbol f, of arity 1. Write down a theory T that asserts that f is a bijection with no finite cycles, and describe the countable models of T. Prove that T is a complete theory."

Answer

The theory T has the following axioms:

$$(\forall x)(\forall y)(f(x) = f(y) \to x = y)$$
$$(\forall y)(\exists x)(f(x) = y)$$
$$(\forall x)(\overbrace{f(f(f\cdots}^{n \text{ times}}(x)\cdots)) \neq x) \text{ (for each } n \in \mathbb{N})$$

Any countable model $\mathfrak{M}$ of T is a disjoint union of at most countably many f-cycles, all of which are of the form $\{\ldots f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}$ for some x.

Imagine you are living in a world where there is nothing going on other than lots of points joined together by f edges, and all you can ever do is move along f edges (in either direction) from one point to another. What do you discover? By the end of time you have discovered that you are living on a copy[3] of $\mathbb{Z}$. And that's *all* you have discovered: if the model contains another copy of the $\mathbb{Z}$-gon that you could have been on you never learn this fact. There is no way, in this language, of saying that two vertices lie on distinct $\mathbb{Z}$-gons.

[3] Actually it's not really $\mathbb{Z}$ beco's $\mathbb{Z}$ has additive and multiplicative structure, which this thing hasn't. It's really just a digraph. One might call it the $\mathbb{Z}$-**gon**.

This is an informal picture and is definitely not a proof, but it might lead us to one.

But there is a proof using only techniques available to you. You observe that, altho' T can have nonisomorphic *countable* models (one, two or many copies of $\mathbb{Z}$), all its models of size $2^{\aleph_0}$ are isomorphic. This may not be immediately obvious. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two models both of size $2^{\aleph_0}$ then they both consist of $2^{\aleph_0}$ $\mathbb{Z}$-gons. (A detailed proof of this fact needs a little bit of AC but i'll spare you the details). So there is a bijection between the (set of) $\mathbb{Z}$-gons-in-$\mathfrak{M}_1$ and the (set-of) $\mathbb{Z}$-gons-in-$\mathfrak{M}_2$. This isn't *quite* a bijection between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, but we are nearly there. All we have to do is pick, for each pair of a-$\mathbb{Z}$-gon-in-$\mathfrak{M}_1$-with-a-$\mathbb{Z}$-gon-in-$\mathfrak{M}_2$, a digraph isomorphism between the two $\mathbb{Z}$-gons, and take the union of all those isomorphisms. This union will be an isomorphism between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. If T were not complete we would be able to find ϕ such that $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ were both consistent. Add $2^{\aleph_0}$ constants and deduce (by compactness) that $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ both have models of size at least $2^{\aleph_0}$. Indeed (by downward Skolem-Löwenheim) they must both have models of size *precisely* $2^{\aleph_0}$. These models would have to be nonisomorphic beco's one of them believes ϕ and the other believes $\neg\phi$. But they are both models of T so they are isomorphic.

Instead of $2^{\aleph_0}$ one can use $\aleph_1$. Students would be unlikely to try doing it that way beco's $\aleph_1$ is a mysterious phobic object for them. But it works better. In particular one does not need AC, at least not after the use of AC to prove upward Skolemheim to show that there is a model of size $\aleph_1$. What does a model of T of size $\aleph_1$ look like? Lots of copies of $\mathbb{Z}$ of course, but precisely how many? The set of copies of $\mathbb{Z}$ is a surjective image of a set of size $\aleph_1$ and so is of cardinality $\leq \aleph_1$. The copies of $\mathbb{Z}$ have a global wellordering, so the size of their union (which is $\aleph_1$) is $\aleph_0$ times something; it can only be $\aleph_1$. We can show that any two models A and B of size $\aleph_1$ are isomorphic by a transfinite back-and-forth construction. Wellorder each model in order-type ω_1 .... At each stage you look at the first thing in A not already mated and seek a mate for it in B. If this thing you have picked up belongs to a $\mathbb{Z}$-gon that you have already sampled then there is precisely one match for it in B. If it comes from a virgin $\mathbb{Z}$-gon then there will be $\aleph_1$ virgin $\mathbb{Z}$-gons containing suitable partners so pick the first one. And the same coming back in the other direction. It's not actually terribly difficult, but it looks a bit scary to a student.

Sometimes students can be *soooo* annoying. The point of this question (as you have probably guessed by now) is to direct your attention to theories that are categorical in some *un*countable cardinal. However there is a way of answering this question that doesn't exploit this possibility, and some of you found it. That was not in the script at all. Grrr! Suppose $T \nvdash \phi$ and $T \nvdash \neg\phi$. Add countably many constants to the language of T, and add axioms to $T \cup \{\phi\}$ and to $T \cup \{\neg\phi\}$ to say that the denotations of these constants all belong to different $\mathbb{Z}$-gons. These two theories (call them T_1 and T_2) both have countable models by downward Skolemheim. What can countable models of these theories look like? They must consist of precisely $\aleph_0$ $\mathbb{Z}$-gons infinitely many of which have a distinguished element in each $\mathbb{Z}$-gon. There's no way of compelling every $\mathbb{Z}$-gon to have a distinguished element, so this doesn't completely wrap things up for us. However, you weren't supposed to do it that way anyway!

I think that the model consisting of a single $\mathbb{Z}$-gon injects elementarily into all models of T (this should be easy to check); this would make it what they call a **prime**

model.

I've been thinking about other possible pædogogical uses to which this question can be put. Group-theoretical ones for example. Think of an arbitrary model $\mathfrak{M}$ of this theory, consisting of lots of $\mathbb{Z}$-gons, and think of its automorphism group. In the course of picking Wikipædia's brains on this I discovered the **Lamplighter group** and I think there is a connection.

I think that this automorphism group of a model $\mathfrak{M}$ of T is a wreath product of $\mathbb{Z}$ (the additive group on the integers) and the full symmetric group on the set of $\mathbb{Z}$-gons included in $\mathfrak{M}$. I also think that this automorphism group acts imprimitively on the carrier set of $\mathfrak{M}$.

So what this theory gives us is both a nice natural example of a wreath product and a nice natural example of an imprimitive group action.

Question 3.9

"Show that monadic predicate logic (one place predicate letters only, without equality and no function symbols) is decidable."

Answer

The key observation is that if you haven't got any predicate letters of degree higher than 2 in your language then you have no way of telling objects apart by their relations to other things. So how do you tell them apart? Well, if you have k monadic predicate letters you can distinguish at most 2^k things. That means that any model of a monadic theory with k predicate letters can be partitioned into at most 2^k equivalence classes. You can then pick one representative from each equivalence class to obtain an *elementary*[4] submodel of size 2^k at most.

Question 3.11

"Why does T not follow from K and S?

Show that Peirce's Law: $((A \to B) \to A) \to A$ cannot be deduced from K and S."

Answer

If we can deduce an expression ϕ from the first two axioms, where ϕ has occurrences of '$\bot$', then we can also deduce the result of replacing in ϕ every occurrence of '$\bot$' by some random propositional letter not appearing anywhere in the proof. So if we could deduce $((p \to \bot) \to \bot) \to p$ we would be able to deduce $((p \to q) \to q) \to p$. At the risk of making a mountain out of a molehill i will, at this point, say that the set of things deducible from axioms 1 and 2 is an inductively defined set and supports an induction principle, and we can use this induction principle to show that everything in this set is a tautology: the two axioms are tautologies, and tautologousness is preserved by *modus*

[4] Remember what this word means! See definition 58 p. 103.

ponens. $((p \to q) \to q) \to p$ is not a tautology and therefore cannot be deduced from the first two axioms. So we can't deduce Peirce's law either.

In earlier editions of this sheet there was a further question along these lines . . . "if A is a tautology not containing '$\bot$' must it be deducible from the first two axioms?". This is a hard question. You might wish to pursue it. If you do, here is a slightly cuddlier version of it. "Find a tautology not containing '$\bot$' which is not derivable from the first two axioms, and use structural induction on the inductively defined set of deductive consequences of the first two axioms to prove that underivability." I have handouts on this with pretty pictures that it cost me blood to draw, so i'm hoping some of you will ask me about it.

But i'm going to insert here my discussion of that earlier impossible question . . .

The answer is 'no' and the proof(s) is (are) very cute, but there is no obvious way in; you just have to know. If you wanted to guess that the answer is 'no' you could reflect that the collection of deductive consequences of the first two axioms using *modus ponens* is an inductively defined set and so supports a kind of induction, so you might try to find some property possessed by the first two axioms that is preserved by *modus ponens* that is not possessed by some special tautology. And this is in fact exactly what we will do.

The counterexample is $((A \to B) \to A) \to A$, commonly known as *Peirce's law*. (*One* of the reasons why this question is ridiculously hard is that – even if you guess that the answer to this question is 'no' there is no way for you to know that Peirce's law is a counterexample. . . let alone guess that it is, in fact, the *simplest* counterexample.) Easy to check that it is a tautology. . . less easy to see that it does not follow from K and S.

Axiom K: $A \to (B \to A)$.

Axiom S: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

One of my students asks me what Peirce's Law means. Good question. I find myself replying that one reason why it's hard to grasp its meaning is that it isn't really a fact about *implication* at all; it's a fact about negation and disjunction. Classical Logic has this odd feature that all the connectives are definable in terms of each other, so $\to$ is definable in terms of $\vee$ and $\neg$, giving us the rewrite rule:

$$(\neg p) \vee q \implies p \to q$$

It turns out that there are some classical truths about $\neg$ and $\vee$ that can be rewritten by repeated applications of this rule into expressions purely in the language of $\to$. Such expressions can masquerade as facts about $\to$ when in fact they are nothing of the sort. So Peirce's Law starts off as

$$\neg(\neg(\neg A \vee B) \vee A) \vee A \qquad (P')$$

which you can easily check to be a tautology. (Mind you, even P' is not exactly a model of lucidity either). It just so happens that it is in the domain of the interpretation that sends $\neg p \vee q$ to $p \to q$.

The idea that is key to cracking this question is the thought that there might be more than one notion of validity, *i.e*, there might be some other property that is possessed by K and S and which is preserved by *modus ponens* but is not possessed by Peirce's Law. There is a ready supply of these notions in the form of *many-valued truth-tables*. We will use the following three-valued truth-table for the connective '$\rightarrow$'.

$\rightarrow$	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

(The figures in the column below the '$\rightarrow$' are the truth-values of the antecedent, and the figures in the row to the right of the '$\rightarrow$' are the truth-values of the consequent, and the figure in the matrix array is the truth-value of the conditional with that antecedent and that consequent.)

For our purposes, think of truth-value 1 as `true` and the other two truth-values as two flavours of `false`.

Notice that, in this truth table, if A and $A \rightarrow B$ both take truth-value 1, so does B. Notice also that K and S take truth-value 1 under all assignments of truth-values to the letters within them. So if ϕ is deducible from K and S, it must take value 1 under any assignment of truth-values to the literals within it (by structural induction).

Then check that, if A is given truth-value 2 and B is given truth-value 3, $((A \rightarrow B) \rightarrow A) \rightarrow A$ then gets truth-value 2, rather than 1.

So Peirce's law is not deducible from K and S.

(Notice that if we ignore the truth-value 2 (so that we discard the second row and the second column) what remains is a copy of the ordinary two-valued table, with 3 as `false` and 1 as `true`. Also, if we similarly ignore the truth-value 3 what remains is a copy of the ordinary two-valued table with 1 as `true` and 2 as `false`.)

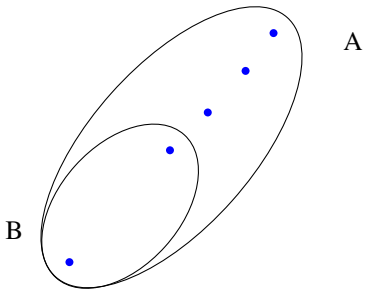
This three-valued logic caper looks entirely *ad hoc* – and indeed it is. Or was. Originally. It turned out later that the funny truth-values have genuine mathematical meaning. (Something to do with possible world semantics). But that wasn't clear to the people who dreamt them up. There's a moral there ... (If you want to know about possible world semantics look at the chapter in `www.dpmms.cam.ac.uk/~tef10/chchlectures.pdf`)

The other moral of this example is that some kinds of Mathematics really need formalisation. Unless we had a concept of *proof* – and of proof by induction on the structures of proofs, indeed – we would have no way of demonstrating that $((A \rightarrow B) \rightarrow A) \rightarrow A$ cannot be derived from K and S.

There is a more subtle, more beautiful and more enlightening – but much harder – proof using Curry-Howard. I have written it up from a brief paragraph in an article of Dana Scott's [9] partly for my own good, and it may well benefit from critical eyes such as yours, Dear Reader.

Dana Scott's clever proof

Suppose *per impossibile* that there were a uniformly definable (and, accordingly, invariant) function P for Peirce's law. Let B be a two-membered set, and let A be obtained from B by adding three new elements.



A has five members and B has two, so any function $A \to B$ identifies a distinguished member of B, namely the one with larger preimage. This defines a function from $A \to B$ to B, which is to say (since $B \subseteq A$) a function from $A \to B$ to A. So what we have, in this rather special case, is a distinguished function $(A \to B) \to A$. Let us call this function F. F exists only because of the special circumstances we have here contrived, and it's not the sort of thing that P would normally expect to have to deal with, so we should expect P to experience difficulty with it ... which of course is exactly what we want! But, if we have a term P, we can apply it to F to obtain a distinguished member of A. But clearly there is no way of picking a member of A in this way. The alleged existence of a uniformly definable P is trying to tell us that whenever we have a set of five things divided into two parts, one with two things in it and the other with three, then one of the five things is distinguished. And that's clearly not true.

On what features of A and B does this counterexample rely? A function $A \to B$ has to give us (*via* the pigeonhole principle) a distinguished element of B, so we need B to have two elements, and A (and therefore $A \setminus B$) to have an odd number. $|A \setminus B| = 1$ is no good, beco's then A has a distinguished element, which we don't want. $|A \setminus B| = 3$ is the smallest number that will do, and that is what Dana Scott gives us.

I append an interesting observation from Mr Matthews of Pembroke, hacked about by me. It is an observation that is intended to prepare one for the thought that there should be expressions in the implicational fragment which do not follow from the first two axioms.

Suppose that any tautology not containing $\bot$ could be deduced from K and S. Let s be an arbitrary proposition, possibly containing $\bot$. Let p be a primitive proposition not in s. Let s' be the proposition obtained from s by replacing all occurences of $\bot$ with p. Let s_2 be $(s' \to p) \to p$. That is to say, replace all the $\bot$s in s by 'p' and put '$\to p) \to p$' on the end. Call it s_2.

Claim: s_2 is a tautology iff s is a tautology.

Proof: :

Suppose s is a tautology and let v be an arbitrary valuation. It will make s true. Will it make $_2$ true? It either believes p, and either way it makes s_2. If it makes p false then the s' in $(S' \to p) \to p$ becomes s and is true according to v, making s_2 true. OTOH if $v(p) = \texttt{true}$ then s_2 has a true consequent and is therefore true.

For the other direction, suppose s_2 is a tautology; *i.e.* any valuation of $\mathcal{L}(s_2)$ will make s_2 true. Think of any valuation of $\mathcal{L}(s)$ and extend it to a valuation of $\mathcal{L}(s_2)$ by making p false. This valuation must make s_2 true by assumption of tautologousness of s_2. But if $(s' \to p) \to p$ is to come out true while p is false, s' has to come out true. But this was an arbitrary valuation. So s' was a tautology, and therefore s – being a substitution-instance of it – must be a tautology too.

Question 3.15 

"A *type* in a propositional language $\mathcal{L}$ is a countably infinite set of formulæ.

For T an $\mathcal{L}$-theory a *T-valuation* is an $\mathcal{L}$-valuation that satisfies T. A valuation v *realises* a type Σ if v satisfies every $\sigma \in \Sigma$. Otherwise v *omits* Σ. We say a theory T *locally omits* a type Σ if, whenever ϕ is a formula such that T proves $\phi \to \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg\phi$.

(a) Prove the following:

Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type. If T locally omits Σ then there is a T-valuation omitting Σ.

(b) Prove the following:

Let T be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq \mathcal{L}(T)$ be a type. If T locally omits every Σ_i then there is a T-valuation omitting all of the Σ_i."

Answer

(a)

THEOREM 25 *The Omitting Types Theorem for Propositional Logic*
Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type.
If T locally omits Σ then there is a T-valuation omitting Σ

Proof:

By contraposition. Suppose there is no T-valuation omitting Σ. Then every formula in Σ is a theorem of T so there is an expression ϕ (namely '$\top$') such that $T \vdash \phi \to \sigma$ for every $\sigma \in \Sigma$ but $T \nvdash \neg\phi$. Contraposing, we infer that if $T \vdash \neg\phi$ for every ϕ such that $T \vdash \phi \to \sigma$ for every $\sigma \in \Sigma$ then there is a T-valuation omitting Σ. ■

However, we can prove something stronger.

(b)

THEOREM 26 *The Extended Omitting Types Theorem for Propositional Logic*
Let T be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq \mathcal{L}(T)$ be a type.
If T locally omits every Σ_i then there is a T-valuation omitting all of the Σ_i.

Proof:

We will show that whenever $T \cup \{\neg\phi_1, \ldots \neg\phi_i\}$ is consistent, where $\phi_n \in \Sigma_n$ for each $n \leq i$, then we can find $\phi_{i+1} \in \Sigma_{i+1}$ such that $T \cup \{\neg\phi_1, \ldots \neg\phi_i, \neg\phi_{i+1}\}$ is consistent.

Suppose not, then $T \vdash (\bigwedge_{1 \leq j \leq i} \neg\phi_j) \rightarrow \phi_{i+1}$ for every $\phi_{i+1} \in \Sigma_{i+1}$. But, by assumption, T locally omits Σ_{i+1}, so we would have $T \vdash \neg \bigwedge_{1 \leq j \leq i} \neg\phi_j$ contradicting the assumption that $T \cup \{\neg\phi_1, \ldots \neg\phi_i\}$ is consistent.

Now, as long as there is an enumeration of the formulæ in $\mathcal{L}(T)$, we can run an iterative process where at each stage we pick for ϕ_{i+1} the first formula in Σ_{i+1} such that $T \cup \{\neg\phi_1, \ldots \neg\phi_i, \neg\phi_{i+1}\}$ is consistent. This gives us a theory $T \cup \{\neg\phi_i : i \in \mathbb{N}\}$ which is consistent by compactness. Any model of $T \cup \{\neg\phi_i : i \in \mathbb{N}\}$ is a model of T that omits each Σ_i.

■

Propositional omitting types is helpful when considering Yablo's Paradox.
See `https://en.wikipedia.org/wiki/Yablo's_paradox` and perhaps
`http://www.dpmms.cam.ac.uk/~tef10/yabloomittingtypes.pdf`

9.5 Sheet 4

Question 4.1

"Let us say $A@B$ is $\{\{a, b\}, a \in A \wedge b \in B\}$. If A and B are not disjoint then $A@B$ might contain singletons.

If I give you a set X of pairs-and-singletons that happens to be of the form $A@B$ can you recover A and B?"

Answer

Yes. Look at all the singletons in X. They are the singletons of the things in $A \cap B$. Call this set of singletons S. Now consider the set P of those pairs that do not meet S. If $\{x, y\}$ is in P then one of x and y belongs to $A \setminus B$ and the other belongs to $B \setminus A$. Now consider the relation that holds between x and z iff $(\exists y)(\{x, y\} \in P \wedge \{y, z\} \in P)$. Since $\{x, y\} \in P$ x and y belong to different components; y and z similarly; so x and z belong to the same component. So this relation is an equivalence relation. It is of index 2 (since there are two components, A and B) and the two equivalence classes are $A \setminus B$ and $B \setminus A$.

Notice however that not every set-of-pairs-and-singletons is of the form $A@B$.

Further, notice that even if one can recover A and B there is no way of telling which is which: after all @ is symmetrical!

Question 4.4

Show that $\{z : \neg(\exists u_1, \ldots, u_n)((z \in u_1) \wedge (u_1 \in u_2) \wedge \cdots \wedge (u_n \in z))\}$ *is not a set for any* n. *What assumptions have you made?*

Answer

Originally i didn't write out an answer to this beco's i tho'rt it was too easy. It's a generalisation of the Russell paradox, but i s'pose that is less obvious to you than it is to me. Lots of people have made heavy weather of it and used too much machinery. You don't need any set theory at all – it's a theorem of first-order logic! Suppose there were such a set, call it b (for bad). b is not a member of itself beco's if it were it would belong to an $\in$-loop of circumference n, a loop consisting of itself many times over, and this contradicts the membership condition for b. OK, so b is not a member of itself. Now b contains precisely those things that do not belong to an $\in$-loop of circumference n, so this must mean that b *does* belong to an $\in$-loop of circumference n. Think about one such loop. It features an object $x \in$ b. So this x belongs to an $\in$-loop of circumference n while being at one and the same time a member of b – which is the set of things that do *not* belong to such a loop. Contradiction. So there is no such b.

Question 4.5

"Write down sentences in the language of set theory to express the assertions that, for any two sets x and y, the product $x \times y$ and the set y^x of all functions from x to y exist. You may assume that your pairs are Wiener-Kuratowski.

Which axioms of set theory are you going to have to assume if these assertions are to be provable?"

Answer

If you use Wiener-Kuratowski pairs then $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ and is a subset of $\mathcal{P}^2(\{x, y\})$. Similarly $x \times y$ is a subset of the power set a couple of times of $x \cup y = \bigcup\{x, y\}$. Clearly the set of functions from x to y can be obtained in the same way.

What if you want to establish that these things are sets without knowing what your pairing function is? Imagine the following situation: I want $X \times Y$ and I know that there is a set-theoretic construct $\langle x, y\rangle$, tho' I don't know what it is and i'm not allowed to assume anything other than that it is there and is available. We do the following: fix $y \in Y$ and consider the function class that sends x to $\langle x, y\rangle$. The image of X in this function exists by replacement and it is of course $X \times \{y\}$. So $X \times \{y\}$ exists for all y. Now consider the function class that sends y to $X \times \{y\}$. The image of Y in this function exists by replacement and its sumset is $X \times Y$.

So: if we have replacement we can prove that $X \times Y$ exists *whatever implementation of pairing-with-unpairing we use*. You might like to prove the converse: if $X \times Y$ always exists for all implementations of pairing-with-unpairing then replacement follows.

Question 4.6

"(a) Prove that every normal function $On \to On$ has a fixed point.
(b) Prove that the function enumerating the fixed points of a normal function $On \to On$ is itself normal.
(c) If α is an ordinal and f is a normal function show that f has a

fixed point of cofinality $cf(\alpha)$.
(d) Are any of your fixed points regular?
(e) If α is a regular ordinal and f is a normal function show that f has a fixed point of cofinality α."

Answer

For (a) iterate ω times and take the sup. The reason why this was not on sheet 1 is that one needs replacement if the collection of iterates is to be a set.

For (b), let f be your normal function; the new function you need is declared by $g(\alpha + 1) = \sup\{f^n(g(\alpha) + 1) : n < \omega\}$, taking sups at limits.

This trick of "add one and keep on trucking" enables you to manufacture strings of fixed points of f that are as long as you please; pause at any limit and take a sup. This deals with (c).

The answer to (d) is that this procedure will never give you a *regular* fixed point. If you iterate α times and take a sup you get a fixed point, but its cofinality is $cf(\alpha)$ which may not be the same as α, but is certainly going to be less than the ordinal you have reached. Not at all clear how we might prove that every normal function has a *regular* fixed point, and it turns out that this is a very strong assumption – stronger by far than the consistency of ZF.

Question 4.9

"f is an $\in$-automorphism if f is a permutation of V that preserves $\in$, so that:

$$(\forall x)(\forall y)(x \in y \longleftrightarrow f(x) \in f(y)).$$

Show that a model of ZF (with foundation of course) can have no nontrivial $\in$-automorphisms.

Give an example to show that the surjectivity condition on f is necessary; that is to say, there are non-trivial injective $\in$-homomorphisms."

Answer

The first bit yields to extensionality plus $\in$-induction. For the second part consider the function f which we define by $\in$-recursion that sends $\emptyset$ to $\{\emptyset\}$ and thereafter sends x to $f``x$. Incidentally this setting is a nice illustration of why it is good practice to write $f``x$ for $\{f(y) : y \in x\}$ rather then to (ambiguously) write $f(y)$ and rely on context to disambiguate. The f we want here is defined by $\in$ recursion: $f(x) =: f``x$. If you write the definiens as $f(x)$ then the declaration of the recursion becomes unintelligible.

Possibly worth pointing out to interested students that this establishes the independence of the axiom of extensionality.

Question 4.11

"There are various ways of constructing implementations (as sets) of $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ from an implementation (as sets) of the naturals. For one of these constructions compute the ranks of the sets that have the rôles of $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$.

Different implementations will almost certainly give you different answers. Are there any lower or upper bounds on the answers you might get?"

Answer

This question makes several points. One of them is the point that there are lots of ways of implementing $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ etc as sets; another is that – mathematically at least – it doesn't much matter which one you use. The other is to get you to do some set-theoretic calculations – computing the ranks of particular sets.

One should start with a warning : the (set-theoretic) rank of a set equipped with an ordering cannot be computed from the order-type of the ordering: it's a property of the *set*, not of any ordering of it. And again, it's nothing to do with cardinality either, or very little. There are small sets (*singletons* indeed) of arbitrarily high rank.

So the rank of a mathematical object implemented as a set is not a attribute of that object; it's an attribute of the *implementation* of the object. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ etc do not have ranks, it's their implementations that do. This comes as a surprise to many students, so it is worth making a big song-and-dance about it. Admittedly it is true that the *minimum possible* rank of an implementation of that object *is* a mathematical invariant of that object [It's the "essential rank" mentioned in the footnote on p. 72] but it's a curiously uninteresting one, being controlled entirely by cardinality. You cannot implement $\mathbb{R}$ as an object of rank ω beco's there are too few things of lower rank for all the reals to be implemented by those things of lower rank. There are uncountably many reals but only countably many things of finite rank. This cardinality consideration is the only constraint.

Actually in parts of the set theoretic literature reals are taken to be functions from $\mathbb{N}$ to $\mathbb{N}$, things sometimes called *set theorists' reals*. They have rank ω – which is best possible, for the reason given above. (They're something to do with continued fractions.)

So: pick an implementation, and compute the ranks of the sets you end up with. For bonus points, pick more than one implementation, and compute all of them! If you know what p-adic numbers are, compute their rank too[5].

Have to edit this thoroughly!

The obvious way to implement natural numbers is as finite von Neumann ordinals. Thus the (set that is to be) the natural number n has rank n. This second 'n' is of course different from the first! It's an ordinal (a number) not a set! The rank of every von Neumann ordinal is the (abstract) ordinal of which it is the implementation.

That way the set $\mathbb{N}$ of (implemented) natural numbers has rank ω, beco's ω is the smallest ordinal bigger than all finite ordinals. And that the best you can do: $\mathbb{N}$ is countable and the least ordinal α s.t. there are infinite sets of rank α is ω.

What about $\mathbb{Z}$? There are several ways to do $\mathbb{Z}$. You could implement an integer as a signed natural number – as an ordered pair of a natural number and a sign bit. Let's assume we are using Wiener-Kuratowski ordered pairs; they increase rank by 2. Not sure what the sign bits + and − are but they are presumably of finite rank. Let's take them to be $\emptyset$ and $\{\emptyset\}$ for the sake of definiteness. Then every integer has finite

[5]The p-adics are the completion of $\mathbb{Q}$ w.r.t the p-adic metric. How are you to think of the completion set-theoretically?

rank so $\mathbb{Z}$ has rank ω, like $\mathbb{N}$. And, again, that is best possible. OTOH you could implement integers by thinking about the field-of-fractions construction. That is to say, you consider the equivalence relation on $\mathbb{N}\times\mathbb{N}$ given by $\langle a,b\rangle \sim \langle x,y\rangle$ iff $a+y=b+x$. The integers are now equivalence classes. So what is the rank of an integer? The rank of an ordered pair of two naturals is finite, but each equivalence class must have rank ω since there is no finite bound on the ranks of the pairs in any equivalence class. So each integer has rank ω, and our set implementing $\mathbb{Z}$ has rank $\omega+1$.

What about $\mathbb{Q}$? You could think of a rational as an ordered pair of integers at least one of which is positive, as long as they have no common factor. If you combine that with thinking of integers as signed naturals then every rational has finite rank and the rank of $\mathbb{Q}$ is ω which, again, is best possible. Or you could think of them in the field-of-fractions way.

Now for $\mathbb{R}$. There are two standard moves here: Dedekind cuts and Cauchy reals. A Dedekind real is a pair of sets of rationals, a *cut*. A Cauchy real is an equivalence class of Cauchy sequences (functions $\mathbb{N}\to\mathbb{Q}$) under the equivalence relation of converging to the same real. This equivalence relation requires thought, since one has to say what it is for two sequences to converge to the same real without saying what the limit is! (You haven't concretised limits yet!) But it's not too hard ... Let f and g be two sequences. They are equivalent iff, for all $\epsilon>0$ there is $n\in\mathbb{N}$ such that, for all $m>n$, $|f(n)-g(n)|<\epsilon$.

How about $\mathbb{C}$? This is pretty straightforward: the obvious way to think of complexes is as ordered pairs of reals.

What about p-adics? The p-adics are the completion of $\mathbb{Q}$ using the p-adic metric, so this ought to be like obtaining $\mathbb{R}$ from $\mathbb{Q}$, which we did above. In that setting we had two options, Dedekind cuts and equivalence classes of Cauchy sequences. However Dedekind cuts rely crucially on the order structure of the rationals, and the p-adic topology on $\mathbb{Q}$ is not an order topology. In this setting our sole weapon is Cauchy sequences.

Question 4.12

"Let G be a graph where, for each vertex v, the collection $N(v)$ of neighbours of v is a set. (v' is a neighbour of v iff there is an edge between v and v').

Give an example to show that G might be a proper class.

Now suppose G is connected; prove that it is a set.

What axioms have you used?"

Answer

The example we want is the graph whose vertex set V is the whole universe and whose edge set E is empty. I like the two puns: 'V' for '$v$ertex' and 'V' for universe, and 'E' for 'Empty' as well as 'edge'.

For the second part suppose G is connected. By assumption $N(v)$ is a set for all vertices v. For any vertex v let $N_n(v)$ be the collection of n-neighbours of v, the vertices whose shortest path to v is of length at most n. We prove, by induction on n, that: for all v, $N_n(v)$ is a set. Evidently true for $n=1$, by assumption. For the induction, reflect

that – for any vertex v – $N_{n+1}(v)$ is $N_n(v) \cup \bigcup\{N_1(u) : u \in N_n(v)\}$. $\{N_1(u) : u \in N_n(v)\}$ is a set by replacement, so its sumset is a set and then we use binary union. So, for any v, the function $n \mapsto N_n(v)$ is well defined, and its range is a set, and the sumset of the range is also a set. This sumset is the entire vertex set.

Without replacement the second part no longer holds. Work in $V_{\omega+\omega}$. Let $G = V$ as before. Join every set of finite rank to V_ω. For sets x, y of infinite rank, if $x \in y$ and $\text{rank}(y) = \text{rank}(x) + 1$ then put an edge between x and y. Then every neighbourhood $N_1(v)$ is a set, and the graph is connected – but it is not a set. (Neither the vertex set nor the edge set is a set.)

I suspect the axiom scheme of replacement is equivalent to the claim that if all neighbourhoods of a graph G are sets and there is only a set of neighbourhoods then the vertices of G form a set.

I also suspect that this is something to do with Isbell's criterion for concretisability of categories, tho' i can't justify this feeling!

Question 4.13

(i)$^+$ How many order-preserving injections $\mathbb{R} \to \mathbb{R}$ are there?

(ii) ☠ Let $\langle X, \leq_X \rangle$ be a total order with no nontrivial order-preserving injection $X \to X$. Must X be finite?

Answer

The point of the hint in (i) is that the number of such injections is precisely $2^{\aleph_0}$ ... which is the same as the number of reals. This fact enables one to construct a counterexample to part (ii) in the form of a set X of reals obtained by a diagonal construction. One wellorders the set of order-preserving injections $\mathbb{R} \hookrightarrow \mathbb{R}$ in order-type ω_α where $\aleph_\alpha$ is the cardinality of $\mathbb{R}$ [so of course you have to wellorder the continuum] and at each stage β one puts something into X (or into $\mathbb{R} \setminus X$ – i'll leave that to you to sort out; you use the βth real to bugger up the βth map) to ensure that [the restriction of] the βth order-preserving map $\mathbb{R} \to \mathbb{R}$ does not inject X into X. You may need to say something about why every order-preserving map $X \to X$ is a restriction of an order-preserving map $\mathbb{R} \to \mathbb{R}$. Of course it's easy if X is dense, but I forget the details. If you want details have a look at: Sierpinski, W. "Sur les types d'ordres des ensembles linéaires". Fundamenta Mathematica **37** (1950) pp 253–264.)

Question 4.17

"Suppose $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ are families of sets such that for no $i \in I$ is there is a surjection $A_i \twoheadrightarrow B_i$. Show that there is no surjection $\bigcup_{i \in I} A_i \twoheadrightarrow \prod_{i \in I} B_i$.

You will need the axiom of choice. Is there a converse?

Using these ideas you can show that $\aleph_\omega \neq 2^{\aleph_0}$ *without* using AC."

Answer

This is bookwork. Answer in lots of textbooks.

THEOREM 27 *The Jordan-König theorem (*AC*):*

If $\langle A_i : i \in I\rangle$ *and* $\langle B_i : i \in I\rangle$ *are families of sets such that* $(\forall i \in I)(|A_i| \not\geq^* < |B_i|)$ *then*

$$|\bigcup_{i\in I} A_i| \not\geq^* |\prod_{i\in I} B_i|$$

Proof:

Suppose not, and that $f : \bigcup_{i\in I} A_i \to \prod_{i\in I} B_i$. We show that f is not onto. For each $i \in I$ let $f_i : A_i \to B_i$ be $\lambda x_{A_i}.(f(x))(i)$. f_i cannot be onto by hypothesis, so pick n_i to be a member of $B_i \setminus f_i$"A_i. (This is where we use AC). Now we find that the function $\lambda i.n_i$ is not in the range of f, for otherwise if $f(a) = \lambda i.n_i$ where $a \in A_i$ say, then $f_i(a) = (\lambda x.(f(x))(i))(a) = (f(a))(i) = (\lambda i.n_i)(i) = n_i$ contradicting choice of n_i.

To show $2^{\aleph_0} \neq \aleph_\omega$ we assume $2^{\aleph_0} = \aleph_\omega$ and derive a contradiction. We take I to be $\mathbb{N}$ and for each $i \in \mathbb{N}$ we take A_i to be a standard set of size $\aleph_i$ (we won't need choice for that) and each B_i to be $\mathbb{R}$. By assuming $2^{\aleph_0} = \aleph_\omega$ we have armed ourself with a wellordering of $\mathbb{R}$, *and that means that we don't need AC to pick* n_i. Jordan-König now tells us that

$$\Sigma_{i\in\mathbb{N}}\aleph_i \not\geq^* (2^{\aleph_0})^{\aleph_0}$$

but $\Sigma_{i\in\mathbb{N}}\aleph_i = \aleph_\omega$ and $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ whence

$$\aleph_\omega \not\geq^* 2^{\aleph_0}$$

contradicting our assumption that $2^{\aleph_0} = \aleph_\omega$.

■

9.6 Sheet 5

Question 5.1

"Explain to members of your tutorial group (or to anyone listening who might be confused) the difference between

(i) Nonstandard naturals
(ii) Countable ordinals
(iii) Infinite Dedekind-finite cardinals"

Why is there no such thing as an infinite Dedekind-finite *ordinal*?

Answer

These three wild-and-woolly things that live in the desolate marches beyond $\mathbb{N}$ often sound similar to beginners, but they are all completely different things!

Infinite Dedekind finite cardinals (aka *Dedekind cardinals*) are cardinals not ordinals – they measure bulk, not order. Whether or not there are such things depend on whether or not countable choice holds. At all events there are no definable Dedekind cardinals, none you can name. So there is no way of reidentifying Dedekind cardinals across models, and there is no system of notation for them. And there might not be any in your neck of the woods.

Nonstandard naturals are a pox brought to us by compactness. They're ordinals, beco's they are natural numbers and natural numbers are finite ordinals... but of course they are *nonstandard* ordinals. Like Dedekind-cardinals they are a product of a malfunction, and none of them can be definable. Again, there is no way of reidentifying them across models, and there is no system of notation for them. And – again – there might not be any in your neck of the woods.

In contrast, countable ordinals are not creatures of the night. Their relationship to natural numbers is that natural numbers are the finite things of this flavour. Unlike the Dedekind cardinals and nonstandard naturals, countable ordinals can be reidentified across models, and there are systems of notation for them.

For the last part, reflect that an infinite Dedekind-finite set cannot be wellordered. (It's infinite but doesn't have a countably infinite subset) so there is no way of connecting it to an ordinal.

Question 5.2

"For P a poset, let P^* be the poset of chains-in-P partially ordered by end-extension. (Chains are allowed to be empty). Show that there is no injective homomorphism $P^* \hookrightarrow P$."

Answer

Fix P and $f : P \hookrightarrow P^*$. We recursively define a map $F : On \hookrightarrow P$ by $F(0)$ =empty chain. Thereafter we declare $F(\alpha) = f(\{F(\beta) : \beta < \alpha\})$. $F(\alpha)$ is defined since $\{F(\beta) : \beta < \alpha\}$ is a chain in P and by assumption f sends it into P. We have now injected the ordinals into P!

This is really a version of Burali-Forti or Hartogs.

Question 5.3

"Any two countable dense linear orders without endpoints are isomorphic.
Give an illustration to show how your back-and-forth construction might not work for dense linear orders of size $\aleph_1$.
How do you have to spice up the denseness condition to prove an analogous result for linear orders of size $\aleph_1$?"

Answer

John Howe's answer to the first part: ω_1-copies of $\mathbb{Q}$ compared with $\omega_1 + \omega$ copies of $\mathbb{Q}$. These two structures are both dense linear orders of cardinality $\aleph_1$ but they aren't isomorphic. No isomorphism can pair a point that has $\aleph_1$ things below it with a point that has $< \aleph_1$ (= countably many) things below it.

For the second part you want the denseness condition tightened to: If X is a countable subset of the ordering, and x is an element strictly above everything in X, then there is $x' < x$ also strictly above everything in X. (Ordinary denseness is like this only with 'finite' instead of 'countable').

Question 5.6

"Using propositional logic only, show that a(n undirected) graph and its complement cannot both be disconnected. (Hint: propositional letters will correspond to edges)"

Answer

Supplied by my student Rob Thatcher (no relation).

Suppose G is a graph such that G and $\overline{G}$ are both disconnected. We will derive a contradiction by resolution.

If G is disconnected then there are vertices a and b which are not connected in G. If $\overline{G}$ is disconnected then there are vertices c and d which are disconnected in $\overline{G}$.

Let us have six propositional letters: ab, ac, ad, bc, bd, cd. The intended interpretation is that ab (for example) means that the edge ab belongs to the edge set of G.

First consider G. The first thing we know is that the edge ab is **not** in the edge set of G, hence we have the clause $\neg ab$. Since we know that a and b are disconnected in G, we cannot allow any indirect paths from a to b. There are two possible lengths of indirect path involving 1 or 2 indirect vertices. (It will turn out that we can get our desired contradiction without considering any indirect paths that are longer, but we don't know that yet, and are just hoping for the best!) This tells us that $(\neg ac \vee \neg bc)$, or, in resolution jargon, $\{\neg ac, \neg bc\}$. Similarly we may add $\{\neg ad, \neg bd\}$. The paths involving 2 vertices are $acdb$ and $adcb$. We already know that the path cd must be present (since it cannot be in $\overline{G}$). Therefore we may add the clauses $\{\neg ac, \neg bd\}$ and $\{\neg ad, \neg cb\}$.

Now consider $\overline{G}$. This cannot have cd, so we can add $\{cd\}$ (this is not negated, since we are now considering edges that are not in $\overline{G}$, and hence must be in G). Similarly, we cannot have any indirect connections from c to d, so we cannot have the paths cad, cbd, $cabd$ or $cbad$. Since we know that ab cannot be in the graph, we can write these as the clauses: $\{ac, ad\}$, $\{bc, bd\}$, $\{ac, bd\}$ and $\{bc, ad\}$. Note that the last two do not contain ab since we know that we **must** have $\neg ab$ by choice of a and b.

So now we have a set of clauses representing the conditions that need to be satisfied if both G and $\overline{G}$ are to be disconnected. To recapitulate, these are:

$$\{\neg ab\};\ \{\neg ac, \neg bc\};\ \{\neg ad, \neg bd\};\ \{\neg ad, \neg bc\};\ \{\neg ac, \neg bd\};$$
$$\{cd\};\ \{ac, ad\};\ \{bc, bd\};\ \{ac, bd\} \text{ and } \{ad, bc\}.$$

Now we may combine these clauses (carefully) using resolution – the choice of clauses to resolve is crucial, since it is very easy to end up with many useless clauses of the form $\{A, \neg A\}$.

$$\dfrac{\dfrac{\{\neg ac, \neg bc\} \qquad \{bc, bd\}}{\{\neg ac, bd\}} \qquad \{\neg ac, \neg bd\}}{\{\neg ac\}} \tag{9.1}$$

$$\dfrac{\dfrac{\{\neg ad, \neg bd\} \qquad \{bd, bc\}}{\{\neg ad, bc\}} \qquad \{\neg ad, \neg bc\}}{\{\neg ad\}} \tag{9.2}$$

Now we have two literal clauses, we can use them to derive a contradiction:

$$\dfrac{\dfrac{\{ac, ad\} \qquad \{\neg ac\}}{\{ad\}} \qquad \{\neg ad\}}{\bot} \tag{9.3}$$

We have derived the empty clause.

Question 5.8

[This question is a sleeper for forcing]

"A poset $\langle P, \leq \rangle$ is called *downwards separative* if for all $x \not\leq y$ there is $z \leq x$ with z incompatible with y. ("incompatible" means "have no common lower bound"). We say that a poset is *downwards splitting* if for every x there are y and z such that $y, z \leq x$, and y and z are incompatible.

(a) Show that not every downwards separative poset is downwards splitting.

(b) Show that if a poset has no minimal elements and is downwards separative, then it is downwards splitting.

A set $D \subseteq P$ is called *downwards dense* if for every p in P there is a d in D such that $d \leq p$.

Suppose $\mathcal{X}$ is a collection of subsets of P. We say that $G \subseteq P$ is $\mathcal{X}$*-generic* if G has nonempty intersection with every downwards dense element of $\mathcal{X}$.

We say that G is a *filter* if

1. for any x, y in G there is z in G such that $z \leq x$ and $z \leq y$, and
2. for any x in G and $x \leq y$, we have y in G.

(c) If $\mathcal{X}$ is countable, show that there is an $\mathcal{X}$-generic filter.

(d) Let $\langle P, \leq \rangle$ be a downwards separative poset with no minimal elements and let $\mathcal{X}$ be a collection of subsets of P closed under complementation (i.e., if $X \in \mathcal{X}$, then also $P \setminus X \in \mathcal{X}$). Show that if G is an $\mathcal{X}$-generic filter, then $G \notin \mathcal{X}$."

Answer

We claim that $P \setminus G$ is downwards dense. Let $x \in P$. By (b), $\langle P, \leq \rangle$ is downwards splitting, so there are incompatible $y, z \leq x$. But G was a filter, so at most one of them can be in G, so the other one must be in $P \setminus G$. If $G \in \mathcal{X}$ then, by closure of $\mathcal{X}$, $P \setminus G \in \mathcal{X}$. So, by definition, G and $P \setminus G$ have nonempty intersection. Contradiction!

(e) Let $\langle P, \leq \rangle$ be the set of finite sequences of zeros and ones, ordered by reverse inclusion. Show that this is a downwards separative poset without minimal elements.

(f) Let $\mathcal{X}$ be the collection of recursive sets of finite sequences of zeros and ones. Show, using (c), (d), and (e), that there is a non-recursive such set.

Comment about Mendel

EXERCISE 1 *Look at this table* **??** *very hard. Of the various properties (reflexivity etc. etc.) of binary relations that you know of, which does this one exhibit? Is there any significance to the fact that there are eight blood groups? There is actually quite a lot of information you can extract from this simple table. Think about it and see what you can get out of it.*

Question ??.??

Solve

$$x^{x^{x^{x^{x^{\cdots}}}}} = 2$$

and comment on the notation. Then think about

$$x^{x^{x^{x^{x^{\cdots}}}}} = 4.$$

The exponent on the LHS is $x^{x^{x^{x^{x^{\cdots}}}}}$ which is 2, so $x^2 = 2$ and $x = \sqrt{2}$. That was easy. The problem with this is that the second equation gives $x^4 = 4$ and thence $x = \sqrt{2}$ again. They can't both be right!

Of course the answer is that the reasoning that led us to conclude that $x = \sqrt{2}$ in the first place doesn't prove that that is the answer. All we have done is show that **if** there is a solution it must be $\sqrt{2}$. We haven't shown that there **is** a solution. In fact it is a simple matter to show by induction that the approximants to the LHS, which we generate as follows

$$a_0 := \sqrt{2}; \quad a_{n+1} := \sqrt{2}^{a_n}$$

... are all less that 2. So the sequence has a limit which is ≤ 2.

Let's see what we can do that is more general.

Let $F(x) =_{df} x^{x^{x^{x^{x^{\cdots}}}}}$

We have $x^{F(x)} = F(x)$. The inverse to this function is the function $\lambda x.x^{1/x}$. This is much easier to understand. For example we can differentiate it. It is the same as $e^{(log\ x)/x}$ whose differential is of course $e^{(log\ x)/x} \cdot (1/x^2 - (log\ x)/x^2)$. This is zero when $x = e$, and this is clearly a maximum. The fact that the differential is zero there of

course means that F reaches a maximum at $e^{1/e}$ and that $F'(e^{1/e})$ is infinite. This gives us the amusing but (as far as I know) useless fact that

$$(e^{1/e})^{(e^{1/e})^{(e^{1/e})^{(e^{1/e})^{(e^{1/e})^{\cdots}}}}} = e$$

(Check this: if the LHS is to evaluate to x we must have $(e^{(1/e)})^x = x$ and e is certainly a solution to this equation.)

We can get a power series expansion of F for values of x not much bigger than 1. Let Σ be the power series for $F(1 + x)$. Then we have

$$(1 + x)^{\Sigma} = \Sigma$$

and we can use the binomial theorem to expand the left hand side. This gives us a sequence of equations expressing later coefficients of Σ in terms of earlier coefficients in a wellfounded way. I haven't worked out the general formula for a_n the coefficient of x^n in $F(1 + x)$ tho' in principle it could be done. ($a_0 = 1$ for a start!)

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