116 first coursework answers

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Pseudo-rugby

Rather like the tennis question this is best approached by thinking about it slowly, and not trying to rush things by scrabbling around for a formula to smack it with. If you get your hands dirty first, then you will be more receptive to the formula when you do encounter it. Formula? This question is a sleeper for RSA.

Some of you spotted that as soon as you have seven consecutive scores that you can get, you can get all bigger scores, just by adding sevens. So you can get all but finitely many scores.

The next thing to think about is what happens if you start with different numbers. Suppose you'd started with 12 and 20? Well it's pretty obvious that all you are ever going to get is multiples of 4, since both 12 and 20 are multiples of 4. So what is the analogue—here—of the fact that with 7 and 9 we eventually get every suffiiently large number? Do we eventually get every sufficiently large multiple of four?

Big Intersection

The challenge is to show that if $X \subseteq Y$ then $\bigcap Y \subseteq \bigcap X$.

This question raises the difficulty many students have in coming to terms with the fact that sets can be sets of sets: $\bigcap X$ makes sense only if X is a set of sets, and people are liable to get wrong the difference between $x \cap y$ and $\bigcap X$.

This takes some getting used to, and i thought it would be an idea to get it out of the way early.

You are a member of $\bigcap X$ iff you belong to every member of X. Think of each member of X as a test that a candidate has to pass if it is to get into $\bigcap X$. The more members X has, the more tests a candidate has to pass, and the harder it is to get into $\bigcap X$. So obviously if $X \subseteq Y$ then $\bigcap Y \subseteq \bigcap X$. If you are happy with this, you may stop here: the next paragraph doesn't tell you anything new, but introduces a bit of formal language, easily digestible by those of you who have done 122. Really this is just the same proof, formalised a bit.

Suppose $X \subseteq Y$ and $a \in \bigcap Y$. We will deduce $a \in \bigcap X$. We have assumed $a \in \bigcap Y$, which is to say

$$(\forall y)(y \in Y \to a \in y) \tag{1}$$

Now consider an arbitrary x in Y. $X \subseteq Y$ so $x \in Y$. Now—by formula (1)—since $x \in Y$ we also have $a \in x$. But x was an arbitrary member of X, so a belongs to all members of X, which is to say, $a \in \bigcap X$.

Partitions

Rather like the tennis question this is best approached by thinking about it slowly, and not trying to rush things by scrabbling around for a formula to smack it with. If you get your hands dirty first, then you will be more receptive to the formula when you do encounter it. It's also a check by me to see whether you are omitting partitions with a singleton piece. I will spare you the sermon i wrote about people who forget about singletons: there is a whole section in the notes on errors of overinterpretation, and i don't suppose you like hearing me repeating myself any more than i enjoy it myself. (It's the last section before "Sets and relations")

There are five two-piece partitions with singletons, because a two-piece partition of a five membered set can have at most one singleton piece:
$\{\{1\}, \{2, 3, 4, 5\}\}$,
$\{\{2\}, \{1, 3, 4, 5\}\}$
$\{\{3\}, \{1, 2, 4, 5\}\}$,
$\{\{4\}, \{1, 3, 2, 5\}\}$,
$\{\{5\}, \{1, 3, 2, 4\}\}$.

How many other two-piece partitions? It's not hard to see that there are as many two-piece partitions remaining as there are ways of picking two things from 5, beco's each of these partitions has a piece of size 2, and one of size 3. $\binom{5}{2}$ is 15.

One of you missed some of these partitions; i think that was beco's you thought that the partitions should in some sense respect the order relation $1 < 2 < 3 < 4 < 5$. Overinterpretation again.

Tennis

This question is not so much a mathematical test as a psychological one. As a mathematical test it's easy; the interest and importance of it is that many people approach it in the wrong way and thereby make it very difficult for themselves.

What are these wrong ways? Well, let's start with the very first part. Four people to be divided into two. If you are hasty and think you know what you are doing you jump immediately to $\binom{4}{2}$. If (like my stepdaughter who was doing no science subjects for A-level) you *don't* think you know the answer then you work it out from first princples and you get it right.

What about five people? Well, one person has to sit out, and each time you send one person out the four remaining can play three matches as before. So there are $5 \times 3 = 15$. Most of you got that.

Now suppose you have six. Two people have to sit out. So the answer will be 3 times the number of ways of picking two people from six to sit out. This time it really is $\binom{6}{2} = 15$. So i get 45.

The danger lurking in this question is that you can be tempted to reach for formulæ from the textbook instead of working out from first principles using only your brain.

Balls and families

I'm not going to say anything about balls for the moment.

As for the last question, i think you should all discuss this with Peter. He has got a very good answer. One mistake that was made by more than one person was confusing composition with union. Aunt-of is not the union of parent-of and sister-of: someone doesn't become your aunt by being your sister or your parent, but by being ths sister OF your parent. You need $\circ$ not $\cup$.