A Languages-and-Automata Examination for Rebecca Ye

November 28, 2021

Question 1

$$Fib(n+1) =: Fib(n) + Fib(n-1)$$

Explain why this is not a primitive recursive definition. Show that nevertheless there is a primitive recursive declaration of this function.

Question 2

We think of 0 as false and 1 as true, so we can define if-then-else by

$\texttt{IF-THEN-ELSE}(0, x, y) = y;$

$\texttt{IF-THEN-ELSE}(1, x, y) = y.$

Show that if-then-else is primitive recursive.

A primitive recursive predicate is a property of natural numbers, or—if you prefer—a formula $F(x_1 ... x_n)$, such that the function $f(x_1 .... x_n)$ which = 1 if $F(x_1 ... x_n)$ is true and = 0 if $F(x_1 ... x_n)$ is false is primitive recursive.

Deduce that if f, g and h are primitive recursive functions then the function

$$F(x) = \text{IF } f(x) = 0 \text{ THEN } g(x) \text{ ELSE } h(x)$$

is also primitive recusive

Question 3

Let ODD be the function sending odd numbers to 1 and even numbers to 0.

Show that ODD is primitive recursive.

Let DIVTWO be the function sending n to the integer part of n/2 (so that DIVTWO(0) = DIVTWO(1) = 0; DIVTWO(2) = DIVTWO(3) = 1, DIVTWO(4) = DIVTWO(5) = 2 etc..)

Show that DIVTWO is primitive recursive.

Question 4

Are all primitive recursive functions total? Are all total functions primitive recursive? Explain your answer.

Question 5

Here are four definitions of what it is for X to be a "semidecidable set":

(i) X is the range of a μ-recursive function;

(ii) X is the range of a total μ-recursive function;

(iii) For some μ-recursive function f, X is the set of arguments on which f halts;

(iv) There is a computable partial function f such that f of a member of X is 1, and f of a nonmember is 0 or is undefined.

Explain why they are all equivalent.

Question 6

Recall that a set X is decidable iff X and $\mathbb{N} \setminus X$ are both semidecidable. Show that X is decidable if and only if there is a computable total function that sends members of X to 1 and members of $\mathbb{N} \setminus X$ to 0.

Question 7

What does "$A \leq_m B$" mean?

There are two definition of the halting set. One is the set of all $i \in \mathbb{N}$ s.t. the ith function halts on input i; The other is the set of pairs $\langle p, i \rangle \in \mathbb{N}^2$ s.t. the pth program halts on input i. Show that each of these two sets many-one reduces to the other.

What is the halting problem? Why is it unsolvable?

Show that the set of (codes of) machines that compute total functions is not semidecidable.

Question 8

Give context-free grammars generating the following languages:

1. $\{a^p b^p : p \geq 0\}$
2. $\{a^p b^q : p < q\}$
3. $\{a^p b^q : p \neq q\}$
4. $\{a^p b^* c^p : p \geq 0\}$
5. $\{a^p b^p c^* : p \geq 0\}$

6. $\{a^p b^q c^r : p \neq q \text{ or } q \neq r\}$
7. $\{a^p b^q c^r : p \neq q \text{ or } q \neq r\}$
8. $\{w \in \{a, b\}^* :\ w$ contains exactly twice as many as as bs$\}$

Question 9

Which of these statements are correct and which incorrect?

1. If a machine has no more than n states then there is a $k > n$ such that it cannot distinguish between strings of length k: it either accepts all of them or none;
2. If a machine has no more than n states, and it accepts at least one string, then it must accept a string with no more than $n - 1$ characters;
3. If a machine has no more than n states, and it accepts at least one string with more than n characters, then it must accept at least one string with no more than $n - 1$ characters;
4. If a machine has no more than n states, and it accepts at least one string with more than n characters, then it must accept at least one string with more than $2n$ characters;
5. Every subset of a regular language is regular.
6. Every subset of a regular language is context-free.
7. The complement of a regular language is regular.
8. The intersection of two regular languages is regular.
9. The union of two regular languages is regular.
10. $L((r + s)^*) = L((r^* s^*)^*)$.
11. $L((rs^*)^*) \subseteq L((r^* s^*)^*)$.
12. $L((r^* s^*)^*) \subseteq L((rs^*)^*)$.
13. $\{ww : w \in \Sigma^*\}$ is regular.
14. $\varnothing$, (the empty language) is regular.
15. Σ^* (the universal language over the alphabet Σ) is regular.
16. $\{\epsilon\}$, the language containing only the empty string, is regular.
17. $\{w \in \Sigma^* : w$ contains an even number of 0's and an even number of 1's$\}$ is regular.
18. $\{w \in \Sigma^* : w$ contains an odd number of 0's and an odd number of 1's$\}$ is regular.

19. $\{w \in \Sigma^* : w$ contains the same number of 0's as 1's$\}$ is regular.

20. $\{w \in \Sigma^* : w$ contains an odd number of 0's and an even number of 1's $\}$ is context-free.

21. $\{w \in \Sigma^* : w$ contains more 0's than 1's$\}$ is context-free.

22. $\{w \in \Sigma^* :$ every initial segment of w has at least as many 0's as 1's$\}$ is context-free.

23. The set of strings without three consecutive zeroes is a regular language.

24. The set of all those strings whose 10th character is a '0' is a regular language.

Question 10

There is an alphabet Σ with six letters a, b, c, d, e and f that represent the six rotations through $\pi/2$ radians of each face of the Rubik cube. Everything you can do to the Rubik cube can be represented as a word in this language. Let L be the set of words in Σ^* that take the cube from its initial state back to its initial state. Is L regular?

Question 11

Let q be a number between 0 and 1. Let L be the set of sequences $s \in \{0, 1\}^*$ such that the binary number between 0 and 1 represented by s is less than or equal to q. Show that L is a regular language iff q is rational. What difference would it have made if we had defined L to be be the set of sequences $s \in \{0, 1\}^*$ such that the binary number between 0 and 1 represented by s is less than q?

Question 12

The **interleaving** of two languages L and M is the set of all words that can be obtained from a word in L and a word in M by interleaving the two words in the way that people shuffle together two halves of a pack of cards.

Prove that the interleaving of two regular languages is regular

Give an example to show that the interleaving of two context free languages is not always context-free.