Dear Ed, Allen, Randall and Jean-Paul

I would like to run past you some musings about ultrafinitism. They were prompted by conversations i had with The Gang (mainly Stewart Shapiro) at the Bob Hale Memorial event at UCL on 16/x/18.

I take Finitism to be the belief that everything is finite, and Ultrafinitism to be the belief that not only is everything finite but that there are only finitely many things. Hold on – is Ultrafinitism to be the doctrine that there are only finitely many things? Or perhaps instead the doctrine that there is a finite bound on the size of things? At all events – as we shall now see – the first implies the second.

I take it we are all agreed both that any finite quasiorder has maximal elements, and that comparative size (of sets, in particular) is a quasiorder; so, if every thing is finite and there are only finitely many things, there must be a finite bound on the sizes of things – there must be a thing of maximal size[1].

However that's not what i want to talk about. This text is a result of my turning over in my mind two things that i think people have not paid enuff attention to, namely the existence of a universal set, and the thesis that any finite set can be enumerated, a thesis which needs a nice snappy name; I call it The Counting Principle

- I plan to show in the next section that Ultrafinitism implies the existence of a universal set.
- In the section following that i hope to show that Ultrafinitism is inconsistent with the principle that every finite set is the same size as a set of natural numbers. This further result will use the first bullet.

> One must steel oneself for a certain amount of quibbling. Allen Hazen reminds me of Skolem's paradox, and warns of the possibility of a parallel here. Does the ultrafinite universe know it is finite, in the requisite sense? However, I suspect that there is less scope for this kind of quibbling than one might fear. After all, Ultrafinitism arises from a very hard-nosed mindset that will not have any truck with airy-fairy things like an exponential tower of 2s of height one million that no actual red-blooded hardnose is ever going to have to deal with. Hardnoses don't quibble.[2].

Ultrafinitism implies that there is a Universal Set

Suppose we have what Allen Hazen calls *adjunction*: $x \cup \{y\}$ exists for all y and all sets x. The relation $\subseteq$ is a quasiorder and so its domain (which is finite) has a maximal element, V, say. Let x be arbitrary. Then $V \cup \{x\}$ is a superset of V and – by $\subseteq$-maximality of V – must be equal to V, so $x \in V$. But x was arbitrary, whence $(\forall x)(x \in V)$.

[1]Observe that we are not using trichotomy here (the doctrine that things are comparable in point of size). It is not needed for the argument below that there must be a universal set.

[2]And they probably don't want to mutilate the Logic either. Hardnoses believe in Excluded Middle and Non-Contradiction and sensible stuff like that.

“Every set is the same size as a set of natural numbers”

In the philosophical and psychological literature on counting there is a background assumption that every finite set can be counted (and indeed that that is how one *knows* it's finite) so that every finite set is the same size as an initial segment of $\mathbb{N}$, the natural numbers. See (e.g.) [1]. Everyone seems to make this assumption, and it would be handy to have a snappy name for it: above i call it the *Counting Principle*. It looks obvious, but i think it has nontrivial strength. For one thing, it puts the squeeze on the possible ways of thinking of finiteness. A natural number is a cardinal of a finite set, but if a set is to be finite only as long as it is in 1-1 correspondence with a set of natural numbers then we need a notion of natural number that does not rely on a prior notion of finiteness. Such a notion is to be had, and that's just as well, since the Counting Principle forces us to rely on it. We can set up $\mathbb{N}$ if we have a Dedekind-infinite set. (The existence of a Dedekind-infinite set is one version of the axiom of infinity). If f is a bijection between X and $X \setminus \{x\}$ then $\bigcap\{X' : x \in X' \wedge f``X' \subseteq X' \subseteq X\}$ is the set of natural numbers. And if $g$ is a bijection between $Y$ and $Y \setminus \{y\}$ then $\bigcap\{Y' : y \in Y' \wedge g``Y' \subseteq Y' \subseteq Y\}$ is the set of natural numbers. And in those circumstances (remember we have assumed the axiom of infinity) we can show that the natural numbers we get from X are isomorphic to the natural numbers we get from Y. We obtain an isomorphism between $\bigcap\{X' : x \in X' \wedge f``X' \subseteq X' \subseteq X\}$ and $\bigcap\{Y' : y \in Y' \wedge g``Y' \subseteq Y' \subseteq Y\}$ as

$$\bigcap\{Z \subseteq X \times Y : \langle x, y\rangle \in Z \wedge (\forall u \in X)(\forall v \in Y)(\langle u, v\rangle \in Z \to \langle f(u), g(v)\rangle \in Z)\}$$

If we have a Dedekind-infinite set X we get a set of natural numbers from it, and a notion of finite set as one that is equinumerous with a set (an initial segment, even) of these naturals. These finite sets have cardinals and the set of these cardinals is a Dedekind-infinite set which – by the above – gives us natural numbers that are isomorphic to those we get from X.

So in the presence of the axiom of Infinity there is no circularity problem in defining a finite set as one that is in 1-1 correspondence with a set of cardinals of finite sets (as the Counting Principle requires). I am not claiming that the Axiom of Infinity is the only way of avoiding this circularity (that is for later, if at all). My claim is the slightly weaker one that the Counting Principle refutes Ultrafinitism.

For this refutation we build on the claim that Ultrafinitism implies the existence of a universal set. Suppose further that every set is the same size as a set of naturals; in particular the universe is the same size as a set of naturals. But the only thing the same size as the universe is the universe. So the collection of all natural numbers is the same as the universe. So everything is a natural number.

> I take it we are agreed that the only thing the same size as the universe is the universe? If the universe is the same size as something other than itself then it's a Dedekind-infinite set and the existence of such objects contradicts finitism let alone ultrafinitism. Or it should do. Perhaps an

ultrafinitist might insist that contradicting ultrafinitism requires the *actual existence* of $\mathbb{N}$ as an infinite set. Their hope might be that separation fails so badly that we cannot infer the existence of (an implementation of) $\mathbb{N}$ from the existence of a Dedekind-infinite set X. After all, we would need the intersection of all subsets of X that contain x and are closed under whatever the bijection was between X and $X \setminus \{x\}$. I can see how a good lawyer *just might* get you out of that one (you might be able to claim that there are too many such sets for you to be able to take the intersection of all of them); however the contortions might be instructive.

So everything is a natural number. (So perhaps the number 6 really is Julius Cæsar, just as we always feared). Now consider the function that sends each set to its cardinal. It is clearly not injective (there is more than one singleton after all) and it is onto the universe. So there is a surjection from the collection of all sets onto a (possibly proper) superset of itself (namely the universe). In these circumstances neither the universe nor the collection of all sets can be finite. So the principle that every finite set is the same size as a set of naturals implies that there is no largest finite set.

Here is a shorter argument that might sway some. If everything is a natural number, and the universe is a set, then $\mathbb{N}$, the naturals, is a set, and $\mathbb{N}$-being-a-set is *one* version of the Axiom of Infinity, and the Axiom of Infinity is of course incompatible with Ultrafinitism. However, the existence of $\mathbb{N}$ might be felt to be not strong enough on its own to refute Ultrafinitism.

I think this arguable incompatibility between Ultrafinitism and the Counting Principle is worth flagging because it seems to me highly unlikely that the philosophical / psychological people (e.g. [1]) who are interested in counting suppose that the ability to count finite sets in this way relies on there being no largest set. They probably aren't in-your-face believers in Ultrafinitism but presumably nor do they knowingly subscribe to any beliefs that actually preclude it.

References

[1] Fuson, Karen. C. (1988). Children's Counting and Concepts of Number. Springer Series in Cognitive Development, Springer.