The Baltimore model HS of Hereditarily Symmetric sets

Thomas Forster

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0.1 Stuff to fit in

Work in HS. What about the inner model HIO of sets hereditarily for every n the size of a set^n of singletons? If this is a proper inner model then it cannot be a model of ZF o/w we would get a contradiction if we constructed HS inside L: HIO would be a model of ZF that is properly included in L. So perhaps it's the same as HS. . . ?

There seem to be three possibilities

1. $HIO = HS$;
2. $HIO = V_\omega$;
3. HIO has members of arbitrarily high rank;
4. HIO is bounded but contains sets of infinite rank.

Let's look at (3). HIO is a model of strZF + IO, and so interprets ZF. This ought to be impossible if the original model contained no inaccessibles. (I'm assuming that HIO is a model of strZF: perhaps this needs to be checked)

(1) seems unlikely. When i reflect on how hard it was to show that V_ω was the size of a set of singletons it would be very galling were there a simple proof that every set is the size of a set of singletons! But i might have been wading through treacle instead of feasting on it.

0.2 A message from Nathan Bowler

Your (Randall's) modification of my idea can be extended to something that works, I think.

Let $\mathbb{N}$ be the set of natural numbers, implemented as you suggested. For technical reasons we ought to run the risk of not including $0 = \{\emptyset\}$ as a natural number, though this may annoy half the mathematical community. Take the set consisting of even permutations of $\mathbb{N}$, even permutations of the set of singletons of $\mathbb{N}$, even permutations of the set of singletons of singletons of $\mathbb{N}$, etc..

This set is hereditarily symmetric with respect to the group of even permutations of $\mathbb{N}$, but not with respect to the group of permutations of $\mathbb{N}$ moving only finitely much stuff.

I'd be happy to hear any comments, even if they happen to be refutations.

Nathan

Chapter 1

Background and Introduction

I have been thinking about the construction of [2] and the similarities/contrasts with FM methods and Rieger-Bernays methods.

Do not read any further unless you understand the rudiments of FM models (You don't need the subtleties of normal filters). Other background is is [2].

The idea that we all understand is a group G of permutations of a set A of urelemente. For $\sigma \in G$ we extend σ to an $\in$-automorphism of $V(A)$ by setting $\sigma(x) =: \sigma``x$ and appealing to $\in$-recursion. This explains why we started with a set of atoms. If $a \in A$ had started off with some members, then the recursive definition of σ might eventually have caught up with a and set it to something different from what σ originally sent it to. It's not hard to see that if the members of A are not atoms then the only σ for which this will not eventually give us a collision is the identity. It's annoying, because it means that the goodies the FM method brings us all come with a heavy price: we have to let *urelemente* into the tent.

That seems pretty conclusive, and it's so obvious that nobody setting out an exposition[1] of the FM method bothers to spell it out. But it turns out that there just might be some point, after all, in trying to permute sets of sets not sets of atoms. (It seems that the Cohen method for proving the independence of AC does this: i must go and read it!)

[1] I nearly comitted a Gallicism there. . .

1.1 A variant of HS?

Start with a group G of permutations of V_ω in the way we started with the Baltimore model. If we were to try to do a FM construction we would be thinking of the elements of V_ω as atoms, which of course they aren't. They have internal structure and when we extend a permutation σ of them to sets of higher rank we find that the recursion that defines σ higher up will result in contradictions. But it's safe if we do this recursion only to sets of infinite rank. So we extend σ to the rest of V by setting $\sigma(x) =: \sigma``x$ **as long as x is of infinite rank**. That way we have σ being very nearly an $\in$-automorphism, in the sense that

$$(\forall y)(\forall x \in V \setminus V_\omega)(x \in y \longleftrightarrow \sigma(x) \in \sigma(y))$$

How sensible is it now to consider the class of hereditarily stable sets? We'd better get a sensible definition of 'stable'. I think we have to deem everything in V_ω to be stable, and sets of higher rank are stable iff they are fixed by everything in G.

So the question now is: what happens in the inner model HS of hereditarily stable sets?

1.2 Definitions

First, some definitions: (x, y) is the transposition that swaps x and y while fixing everything else. The "nth action of σ (on X)" is σ acting on X by moving elements of $\bigcup^n X$. Let two sets be *n-similar* if one is sent to the other by the nth action of the alternating group on V_ω. Similarly *n-symmetric*. $[x]_n$ is the equivalence class of x under n-similarity (it's "n-equivalence class"). A *stratification graph* is a digraph where all paths between any two vertices are the same length. (You count one unit of length for following an edge in its preferred sense, and -1 for travelling in the opposite sense). We need stratification graphs here beacuse they are the best way to visualise the various actions of $\mathrm{Alt}(V_\omega)$. For any set x we construct a stratification graph by having vertices for all the members of $\bigcup^n x$ a distance $n + 1$ from the vertex for x. Thus members of $TC(x)$ may have multiple occurrences. Naturally we put an edge from a to b if $a \in b$.

1.3 Some Useful Basic Lemmas about HS

If X is a set all of whose members are k-symmetric then X is $k+1$-symmetric. So

Lemma 1. *If X is a subset of HS all of whose members are k-symmetric then X itself is in HS.*

In particular, since $V_\alpha \cap HS$ is 2-symmetric, note that for any set of ordinals X whatever, the set $Y =: \{V_\alpha \cap HS : \alpha \in X\} \in HS$.

Now suppose $V \neq L$ with X a nonconstructible set of ordinals. We will show that $Y \in HS \setminus L$. Suppose Y were in L. Then we would also find in L the set $\{\alpha : V_\alpha \cap HS \in Y\}$, and this set is of course X. This shows that

Remark 2. *If $HS \subseteq L$ then every set of ordinals is constructible.*

This answers a question of Vu Dang's. Lemma 1 will come in useful later when we consider how to embed V into HS.

Another corollary of this observation is this. We can implement `BFEXT`-types in HS by Scott's trick. If IO holds we can use it to show that the Scott's-trick `BFEXT`-types are an isomorphic copy of the model we started with. Let X be an arbitrary set of Scott's-trick `BFEXT`-types. We impose a `BFEXT`-structure on it in the obvious way, use IO to make a copy a couple of types lower and bingo.

This next result got me in a tangle for a while, until i remembered that Γ-collection does not imply Γ-replacement. You need Γ-separation as well. Do not forget that any set theory with a universal set satisfies full collection!

Lemma 3. *$HS \models$ collection.*

Proof:

Suppose

$$HS \models (\forall x \in X)(\exists y)(\phi(x, y).$$

That is to say

$$(\forall x \in X)(\exists y)(\phi(x, y)^{HS})$$

Now reflection holds in the original model so we infer

$$(\exists \alpha)(\forall x \in X)(\exists y \in V_\alpha)(\phi(x, y)^{HS})$$

But all these witnesses are in HS so we infer

$$(\exists\alpha)(\forall x \in X)(\exists y \in HS_\alpha)(\phi(x,y)^{HS}).$$

But HS_α is in HS so we have proved:

$$(\exists Y)(\forall x \in X)(\exists y \in Y)(\phi(x,y)^{HS}).$$

■

LEMMA 4.

1. *Let X in HS have a k-symmetric total order. Then all members of X are $\leq k$-symmetric.*
2. *Let X in HS be strongly cantorian and k-symmetric. Then all members of X are $\leq k$-symmetric.*

Proof:

(1) If X contains a set x that is j-symmetric with $j > k$ then it must contain the whole j-orbit $[x]_j$. But then its ordering—being k-symmetric—cannot discriminate between members of $[x]_j$.

(2) For such an X the set $\iota \restriction X$ exists. This set, $\iota \restriction X$, is k-symmetric for some k. (Let us assume for definiteness that we are using Wiener-Kuratowski ordered pairs). Then evidently every $y \in x$ is $k-4$-symmetric at worst. (Consider what a permutation must do to the WK pair $\langle x, \{x\}\rangle$: it must fix both x and $\{x\}$, unless for some $y \neq x$ it moves them to y and $\{y\}$ which it won't). ■

[Dang queries this. He also has a nice notation: he says a symmetric set all of whose members are $\leq k$-symmetric for some k is **supersymmetric**.]

In honour of Dang, let's get a better proof...

Suppose X is strongly cantorian. Then $\iota \restriction X$ exists, and it is k-symmetrical for some k. Now consider an ordered pair $\langle x, \{x\}\rangle \in \iota \restriction X$, and an arbitrary permutation σ from the alternating group on V_ω. $j^k(\sigma)$ must move this pair to another pair in $\iota \restriction X$. So this tells us that, for all such σ and all $x \in X$, $\langle j^m(\sigma)(x), j^m(\sigma(\{x\})\rangle$ is also in $\iota \restriction X$. Now $\langle j^m(\sigma)(x), j^m(\sigma(\{x\})\rangle = \langle j^m\sigma(x), \{j^{m+1}\sigma(x)\}\rangle$. So this tells us that for some $m \in \mathbb{N}$ it is the case that for all $\sigma \in A(V_\omega)$ and for all $x \in X$, $j^m\sigma(x) = j^{m+1}\sigma(x)$. I think we can be confident that this can only happen if x is m-symmetric!

This even convinces Randall, but he does point out that it relies on foundation.

Lemma 4 is a drastic constraint on the kind of sets that can be totally ordered or strongly cantorian. The only supersymmetric sets I can think of are sets of relational types: sets of cardinals, sets of ordinals, sets of `BFEXT`s.

Lemma 5. *In HS we can prove the following about V_ω:*
(1) V_ω has no infinite totally ordered subset;
(2) V_ω is not Dedekind infinite;
(3) V_ω is not cantorian;
(4) V_ω has no infinite strongly cantorian subset.

Proof:

(1) Suppose there were an infinite ordered subset $X \subseteq V_\omega$. For each m, there are only finitely many hereditarily finite sets that are m-symmetric. Therefore (since X is infinite) it must contain—for infinitely many $k > m$—at least one element that is k-symmetric. But this contradicts lemma 4 part (1). ■

(2) This is an immediate corollary of (1). Since $\iota``V_\omega \subseteq V_\omega$ we have (3) as a further corollary.

(4) V_ω contains only finitely many members of each symmetry degree. Any infinite subset of V_ω will have members of arbitrarily large symmetry degree, and therefore—by lemma 4 (2)— it cannot be strongly cantorian. ■

1.4 Prologue on the significance of IO

It is standard in the folklore of NF that the existence locally of the singleton function subverts stratification restrictions. For our purposes the significance of this is that in any model of strZF the hereditarily strongly cantorian sets will satisfy unstratified replacement etc. So we have to consider the following sequence of moves. Start with a model $\mathcal{M}$ of ZF + $V = L$. Take the inner model HS of hereditarily symmetric sets. This is a proper substructure of $\mathcal{M}$. Now, inside HS consider the substructure of hereditarily strongly cantorian sets. This should be a model of ZF. But it is a proper substructure of $\mathcal{M}$, which is impossible, since $\mathcal{M}$ is minimal. This leads one to expect that within HS the inner model of hereditarily strongly cantorian sets is trivial in some way. This turns out to be the case.

Remark 6. *The collection of hereditarily strongly cantorian sets is precisely V_ω.*

Proof:

By lemma 5 part (4), V_ω is a set containing all its strongly cantorian subsets and is therefore a superset of every hereditarily strongly cantorian set. But it is also clear that every set in V_ω is strongly cantorian, indeed hereditarily strongly cantorian. ■

However, despite (4), HS does have some infinite strongly cantorian sets. Every hereditarily finite set in the original model is in HS and is strongly cantorian in HS—if x is in V_ω so is the graph of $\iota \restriction x$. So the Frege $\mathbb{N}$ local to V_ω is stcan (but it's not hereditarily stcan, since the local Frege naturals are not stcan.) Lemma 4 part (2) doesn't prevent sets of local Russell-Whitehead ordinals from being strongly cantorian. If all sets of local Russell-Whitehead ordinals are strongly cantorian then every wellordered set is strongly cantorian and there will be no wellordered counterexamples to IO. I can see that happening!

Remark 6 is a straw in the wind for an old query of Holmes and Forster: is it consistent w.r.t. NF or NFU that the collection of hereditarily strongly cantorian sets should be a set? It does rather suggest that the answer might be 'yes'.

IO is the principle:

Every set is the same size as a set of singletons. (IO)

You will all be familiar with the idea that one can interpret one set theory in another by interpreting $x \in y$ as $\mathcal{X}\ E\ \mathcal{Y}$ where the calligraphic letters range over isomorphism types of suitable binary structures such as APGs or wellfounded extensional relations. In this connection there are two apparently fairly banal observations one can make which on close inspection turn out not be banal at all.

1. The expressions '$x \in y$' and '$\mathcal{X}\ E\ \mathcal{Y}$' are both stratified. However the second is homogeneous and the first isn't. This means that it is possible to contruct boolean combinations of formulæ like '$x \in y$' which are not stratified: '$x \in x$' is an example. However, when we write out '$\mathcal{X}E\mathcal{Y}$' in primitive notation we find that not only is it stratified but the two free variables in it receive the same type in any stratification. This has the effect that in *any* molecular combination of expressions like this (and it is to such combinations that expressions in the language of set theory get translated) every variable—and *a fortiori* every *occurrence* of every variable—receive the same type in any stratification, so all such expressions are stratified—*even if they are the translations of unstratified formulæ of set theory.*! This in turn means that we have a method for interpreting unstratified theories in stratified theories.

2. Often in the course of manipulating models one wishes to have numerous disjoint copies of a given structure. This will happen for example, if one wishes to verify the interpretation of the power set axiom, or an instance of the scheme of replacement or collection. There is an obvious—indeed standard—way of creating $|X|$-many disjoint copies of a structure with carrier set A: for each $x \in X$ we can make a copy with carrier set $A \times \{x\}$. How large is this family of disjoint structures? Assuming we have a type-level ordered pair (the best we can do!) the formula that relates $A \times \{x\}$ to $\{x\}$ is homogeneous, and its graph will be a set. So the family is the same size as $\{\{x\} : x \in X\}$, and this is not necessarily the same size as X. To obtain a family the same size as X we have to find a set Y such that X is the same size as $\{\{y\} : y \in Y\}$, and then our family will have carrier sets in $\{A \times \{y\} : y \in Y\}$. We can form a family of $|X|$-many disjoint copies of a structure only if X is the size of a set of singletons. This leads us to IO.

   What if we don't assume we have a type-level ordered pair? Then in the circumstances above we would want X to be the same size as a set^n of singletons for some $n > 1$. Interestingly the assertion that every set is the same size as a set^n of singletons is of course equivalent to IO. Thus we do not have to worry about implementations of ordered pairs after all.

There are two ways of using isomorphism classes of graphs to obtain models of set theories, and the important point of distinction is the treatment of equality. Do we interpret '=' as isomorphisms between the relevant binary structures (so that our variables range over binary structures) or do we interpret it genuinely as '=' between the (implemented) isomorphism types? Both course of action are legitimate. Neither is obviously better than the other, since each has advantages that the other lacks.

If we take the first horn, and do not implement our isomorphism classes, then we have to assume collection in the verification of the translation of replacement (or choice).

The second horn uses what I call the "quotient as witness" construction. We need a representative of a particular ismorphism class α. The obvious candidate is the family of isomorphism classes "below" α. This method works only if you have an implementation to hand and it cannot be employed in the implementation-free approach.

The difficulty with the second horn comes when we try to verify replacement: we wish to have for each $x \in X$ the unique y such that .... So we collect samples of the correct types. Then we form the sum set of the collection and take a quotient.

1.4.1 IO and $\exists NO$

$\exists NO$ is the assertion that there is a set that contains wellorderings of all lengths. This is a theorem of NF but is refuted by ZF. It would be nice to find a stratified fragment of ZF with which $\exists NO$ was consistent. We can do this if there is a wellordered counterexample to IO

Theorem 7. *If there is a wellordered counterexample to IO then $\exists NO$.*

Proof: Suppose IO fails, and let $\langle X, <_X\rangle$ be a wellordering of minimal length not iso to a worder of a set of singletons. Consider the set of its proper initial segments. So by collection followed by stratified separation there is a set Y of wellorderings-of-sets-of-singletons such that every initial segment of $\langle X, <_X\rangle$ is iso to a member of Y and *vice versa*. Then every wellordering is iso to a member of RUSC^{-1}"Y. This is beco's every IO-compliant wellorder is iso to an initial segment of $\langle X, <_X\rangle$. (If you are IO compliant you cannot be the same length as $\langle X, <_X\rangle$ beco's o/w $\langle X, <_X\rangle$ would be IO-compliant too, so you must be shorter. So if Z is any worder, $RUSC(Z)$ is iso to an initial segment of $\langle X, <_X\rangle$ and therefore iso to a member of Y. But then Z is iso to a member of RUSC^{-1}"Y.

Chapter 2

IO and strZF

The interpretation of ZF in strZF + IO which we will see below also gives an interpretation of ZF into strZFU + IO.

So can we interpret strZF in strZFU? (Not obvious: "hereditarily atom-free" isn't stratified.) Randall emphasises that my proof of remark 11 (that AC $\rightarrow$ IO) relies heavily on the fact that NF + AC is inconsistent. Therefore strZFU + AC might not prove IO. So if we are to interpret strZF in strZFU it must be an interpretation of strZF + not-IO.

Don't forget that ZF minus extensionality can be interpreteted in Zermelo. (Scott)

StrZ + IO is not the same theory as strZF + IO since the first is a subsystem of Z and the second is as strong as ZF.

Draw a diagram displaying relations between the following twelve system in the form of a tower of three squares

ZF, ZFU, Z, ZU

strZF+IO, strZFU+IO, strZ+IO, strZU+IO

strZF, strZFU, strZ, strZU

NZF is the theory axiomatised by all the known theorems of ZF and NF. It contains (at the very least!) pairing, power set, sumset, extensionality, existence of a Dedekind-Infinite set, transitive containment, stratified separation, and full collection. The following fact might help us orient ourselves, and may yet even be useful

Remark 8. *HS* $\models$ *NZF*

Proof: By lemma **??** $HS \models$ full collection. It clearly satisfies stratified separation, and the other stratified axioms. The existence of the various S_α ensures transitive containment. ■

But AC and IO are not axioms of NZF!

It is highly desirable that stratified collection should be weaker than full collection. See section 9 of [3] which has some remarks showing that for V_ζ to be a model of strat $\Delta_0^{\mathcal{P}}$ collection, ζ must equal $\beth_\zeta$. (prop 9.4; but there are other relevant remarks scattered about).

Assuming there is a way of interpreting arithmetic in both NF and ZF we can conclude that not both ZF and NF prove Con(NZF). One would expect ZF $\vdash Con(NZF)$ but not NF $\vdash Con(NZF)$.

ABSTRACT

The set theory strZF is a subsystem of both ZF and NF. IO is the axiom that says that every set is the same size as a set of singletons. It is probable (but not yet established) that ZF $\vdash$ Con(strZF). This is in contrast to the two facts (established here) that ZF can be interpreted in both strZF + AC and in strZF + IO.

Health Warning: the strZF of this paper is not the same as the stratified fragment of ZF in [2]: *this* system has stratified collection and *that* one has only stratified replacement.

Introduction

Let strZF be the theory obtained from ZF by discarding the unstratified instances of the axiom schemes of replacement, collection and separation. It is the theory axiomatised by the stratified axioms of ZF-axiomatised-with-collection+replacement+separation. Notice that we do *not* mean the theory axiomatised by the stratified theorems of ZF! We can define strZ and strZFU analogously.

StrZF is a subsystem of both ZF and NF, and NF is strZF + the existence of a universal set.

Despite the (putative) weakness of NZF we can prove

Theorem 9. *Con(NZF + IO) $\rightarrow$ Con(ZF)*

We show this by interpreting ZF in NZF + IO. (In fact in a sratfied fragment of NZF)

What we actually need is not IO but an axiom that says that for all x and y there is a set of singletons the same size as x that is disjoint from y. However this assertion follows from IO as follows.

Remark 10. $(\forall xy)(\exists z)(z \cap y = \emptyset \wedge |x| = |\iota``z|)$

Proof: Suppose x is the same size as a set of singletons, but that every set of singletons the same size as x meets y. Notice that strZF has a type-level ordered pair. Suppose $\iota``z$ is a set of ordered pairs the same size as $x$. Then so is $\iota``(z \times \{w\})$ for any z. So y meets $\iota``(z \times \{w\})$ for any w; so every w is the second component of an ordered pair in $\bigcup y$. But then the set of second components of ordered pairs in $\bigcup y$ is V. But the universal set isn't the same size as any set of singletons, contradicting our assumption that IO. ■

2.1 The main proof

A `BFEXT` is a wellfounded extensional relational structure with a top element—as usual. There is an obvious membership relation between these objects, which we will write E.

We will interpret ZF into strZF + IO. We restrict all quantifiers $(\forall x)\ldots$ to $(\forall x)(\mathtt{BFEXT}(x) \to \ldots$ and all quantifiers $(\exists x)\ldots$ to $(\exists x)(\mathtt{BFEXT}(x) \wedge \ldots$. We read $\in$ as E, and $=$ as $\sim$: isomorphism between `BFEXT`s.

Is every axiom of ZF interpreted by a theorem of the new system?

Clearly we expect no difficulty with Extensionality, Sumset, Infinity and Separation. The axioms that give trouble are those axioms whose instantiating `BFEXT`s can be constructed only by making copies of things. Pairing illustrates in a small way the difficulty we will have with power set and replacement. It isn't enough to take the disjoint union of the two `BFEXT`s in question and add a new top element because there are now two empty sets not one—one in each component—so the result is not extensional. We have to define an equivalence relation by recursion (each equivalence class will have at most two elements) on the union of the two carrier sets. We define a `BFEXT` on the quotient in the obvious way. However the quotient is the wrong type, so we appeal to the fact that it is the same size as a set of singletons, copy it over to this new set, throw away the outer layer of curly brackets and now we have what we want.

We can also prove pairing *without* using IO. Just make the two `BFEXT`s $\langle X, x^*, R\rangle$ and $\langle Y, y^* S\rangle$ disjoint then put together a new one. There is a canonical partial bijection between X and Y induced by the two wellfounded relations. The carrier set of the new structure consists of $X\cup$ those things in Y than are not connected by the canonical bijection to anything in X.

Power set and replacement are a bit harder.

2.1.1 Power set

Suppose we have a BFEXT $\langle X, x^*, R\rangle$. For each subset Y of $R^{-1}``\{x^*\}$ we have to add a new element $Y^R$ to the carrier set $X$, and we have to remove all directed edges going to $x^*$. Then, for each $x$ in $R^{-1}``\{x^*\}$, and each $Y \subseteq R^{-1}``\{x^*\}$ with $x \in Y$, we add a directed edge from x to Y^R. Then we add directed edges from all the Y^Rs to x^*. Exactly how are we to do this? We have to ensure that nothing we choose to be a Y^R is already in X. Simply taking Y^R to be Y is no good: not only might Y already be a member of X, even if—by a miracle—none of the Y were already in X, the definition would be unstratified. However, it is our point of departure: we do obtain the various Y^R from elements $\mathcal{P}(R^{-1}``\{x^*\})$. Here we need remark 10 to tell us that there is a set $W$ disjoint from $X$ and a bijection $f : \mathcal{P}(R^{-1}``\{x^*\}) \to \iota``W$. For $Y \subseteq R^{-1}``\{x^*\}$ we set Y^R to be $\iota^{-1}(f(Y))$. That way, for x in X and $Y \subseteq R^{-1}``\{x^*\}$, we have a directed edge going from x to Y^R iff $x \in Y$ and the collection of these new edges is defined by a stratified condition and is indeed a set.

(If we can use IO, then just take Y wlog to be a set of singletons)

2.1.2 Replacement

We have a BFEXT and a map (definable in the language of $\in$) taking its E-members to other BFEXTs. Since E is homogeneous, the description of this map—translated into the language of E—beomes stratified, so we can use stratified collection to obtain a set X of BFEXTs containing at least one representative of each of the things we want. By assumption, X is the same
! size as a set U of singletons, so we can replace each BFEXT $\langle W, w^*, R\rangle$ in X by a "U-copy" $\langle W \times \{u\}, \langle w^*, u\rangle, R^u\rangle$ where $\{u\}$ is the singleton associated with w. (The third component R^u of this triple is $\{\langle\langle x, u\rangle, \langle y, u\rangle\rangle : \langle x, y\rangle \in R\}$ of course.)

Let us assume this done, so that X is now the set of U-copies of the set we started with. The BFEXTs in X are now all disjoint. What happens if we attempt to construct a new BFEXT whose carrier set is the union of the carrier sets of all the U-copies in X, equip it with a new top point and take the binary relation to be the union of the binary relations associated with the U-copies? Nearly there, but not quite. The result is not extensional, since there is an empty set in each U-copy. We define an equivalence relation by recursion as we did with the axiom of pairing. The quotient is now the
! same size as a set of singletons so we copy the relation over to the sumset of this set of singletons and then we are done.

The need for this last manœuvre stems from the need for there to be, for each member of X, an injection from its carrier set into the carrier set of the new `BFEXT`, and the best way of ensuring that there is such an injection is for it to have a stratified description.

■

The two exclamation marks in the margin indicate the two places where we had to use IO. In the second application the set which we need to be the size of a set of singletons is a family of disjoint sets, and the assertion that every disjoint family is the size of a set of singletons is presumably much weaker. For all I know it could be consistent with NF.

Notice that to interpret *replacement* in strZF + IO we exploited the fact that strZF has stratified *collection*. Life would be a lot simpler if we could get by with using only stratified *replacement*!

We hope that strZF is strictly weaker than ZF, but establishing this might not be entirely straightforward. This is for the following reason:

REMARK 11. *strZF + AC* $\vdash$ *IO*

Proof: .

Suppose there is a wellordered counterexample to IO: a wellordering whose carrier set is not the same size as any set of singletons. Then there is one of minimal length, $\langle X, <_X \rangle$. Then by stratified collection there is a set containing, for anything $\langle X', <_{X'} \rangle$ shorter than $\langle X, <_X \rangle$, a wellordering of a set of singletons isomorphic to $\langle X', <_{X'} \rangle$. We use wellordered choice to pick precisely one representative of each isomorphism class, and we call this collection of representative wellorderings "NO". Every wellordering the same size as a set of singletons is now isomorphic to a wellordering of a set of singletons in NO. We consider the union of the carrier sets of these chosen worders in NO. It is a set of singletons, so we can think of it as $\iota``Y$ for some $Y$. We wellorder $\iota``Y$ as follows:

In the first instance order elements of $\iota``Y$ according to the first element of NO in which they appear. If the first appearances of x and y are in the same element of NO, order them according to the order in which they appear in that member. The result is a worder, and every wellordered set that is the same size as a set of singletons embeds in it, albeit not necessarily in an order-preserving way. Its cardinality is now the maximal cardinality of a wellordered set of singletons, since every wellordered set of singletons can be injected into it. Now consider Y. Every wellordered set can be embedded in it: after all, if Z can be wordered, then $\iota``Z$ can be embedded in $\iota``Y$. And it is itself wellordered. Now we use full AC to infer that Y is the largest cardinal, and we can argue as follows:

$$|\bigcup Y| \leq |Y| \tag{2.1}$$

by maximality of $|Y|$, whence

$$|\mathcal{P}(\bigcup Y)| \leq |\mathcal{P}(Y)|. \tag{2.2}$$

However we also have

$$|Y| \leq |\mathcal{P}(\bigcup Y)| \tag{2.3}$$

which gives us

$$|Y| \leq |\mathcal{P}(Y)|. \tag{2.4}$$

We also have

$$|\mathcal{P}(Y)| \leq |Y| \tag{2.5}$$

by maximality of $|Y|$.

Therefore

$$|Y| = |\mathcal{P}(Y)| \tag{2.6}$$

and a bijection between Y and $\mathcal{P}(Y)$ now gives us a model of NF + AC.

■

2.2 Why might this matter?

As i remarked earlier, one expects strZF to be a lot weaker than ZF. The result above shows that strZF + IO, in contrast, is strong. This means that IO itself is strong. We know that IO is strong in the NF branch of theories (it's refutable in NF, after all) but this is the first sign that it is strong in the other branch as well. If it's strong, then its negation is consistent, and its negation is a first step along the branch that leads to NF.

So the tasks before us are:

1. Establish that strZF really is not significantly stronger than Zermelo.

   What about H_κ? Proposition 9.4 of [3] says that if $\zeta > 0$ and V_κ models stratifiable $\Delta_0^{\mathcal{P}}$ collection THEN $\zeta = \beth_\zeta$. (We cannot drop the stratified collection and make do with stratified replacement instead).

   This isn't very hopeful: it's fairly routine to show that H_κ tends to be a model for replacement as long as κ is regular. OTOH V_κ tends not to be a model for replacement. One would hope that V_κ is a model for strZF for quite a lot of κ. κ would have to be a limit ordinal at least. Might $V_\kappa \models strZF$ turn out to be equivalent to a known sensible property of ordinals? No: it's probably just strong inaccessibility. Consider the following simple modification of the Mathias formula. Work in V_κ. Let $A \in V_\kappa$ be the ordinals below α in some suitably chosen implementation. Suppose for every $\beta < \alpha$ there are sets of alephs of length β. Then stratified collection followed by sumset will give us a set of alephs of length α. But this doesn't take care of cases like the least beth fixed point. If κ is the least beth fixed point is V_κ not a model of stratified collection ...? The conclusion seems to be that the V_κs at least do not give us natural models of stratified collection unless they are also models of full ZF.

2. Anyway, once we have established that ZF $\vdash$ Con(strZF) then we will have established the consistency of strZF + $\neg$IO. This is because if, *per impossibile*, strZF proved IO, this would mean we could interpret ZF in strZF, which we will by then have established is impossible.

3. This would be significant progress. I believe I am right in saying that there are at present no known models of strZF + $\neg$IO (or even KF + $\neg$IO!?). I don't know if the model of the Baltimore paper [2] satisfies IO. I have always supposed that it does, but I have not found a proof one way or the other.

4. Anyway, once we have models of strZF + $\neg$IO we will be pointing in the right direction: in the direction of models of fragments of ZF with properties inconsistent with ZF. It may be too ambitious to dream of a model of NF just yet, but we might get a model of strZF + "$\exists NO$", the assertion that there is a set containing wellorderings of all lengths.

2.3 endpiece

A closing thought: Why is IO strong? The clue lies in the use we make of it. There are certain constructions (for example of the witnesses in the verification of replacement) which require their inputs to be disjoint. IO allows us to make disjoint copies of things and thereby makes these constructions possible. One could say it is a kind of axiom of infinity. Axioms of infinity usually tell us that there are enough ordinals to execute some long recursion, that we will not run out of *time*. IO tells us instead that we will not run out of *space*.

Appendix: the sequel

One interesting question (well, *I* think it's interesting!) is whether or not one can have $\exists NO$ ("there is a set containing wellorderings of all lengths") holding in a wellfounded model of strZF or KF. I think it is interesting because it is a measure of how closely coupled are the two paradoxes of Burali-Forti and Russell. My feeling is that they are not at all closely coupled, and can be resolved simultaneously in radically different ways: you can resolve the Russell paradox by ruling that the pathological collection is not a set, but you can nevertheless at the same time resolve the Burali-Forti paradox in a way that allows the troublesome collection to be a set.

By interpreting ZF in strZF + IO we established (always assuming that ZF $\vdash$ con(strZF)!) that $\neg$IO is consistent. But to get $\exists NO$ we need not the failure of IO but that there should be wellordered counterexamples to IO. This is because once we have a wellordered counterexample to IO we have a chance of proving $\exists NO$. So what we want now is a new system—let us call it strZF*—whose consistency can be proved in ZF, but into which we can interpret ZF once we add the new axiom:

> Every wellordered set is the same size as a set of singletons. (WIO)

So let strZF* be:

> Sumset;
>
> Pairing;
>
> Infinity;
>
> Wellordered sets have power sets and those power sets are wellordered;
>
> Wellordered stratified collection: If X is a wellordered set, and $(\forall x \in X)(\exists y)(F(x, y))$ with F stratified, then there is a wellordered Y s.t. $(\forall x \in X)(\exists y \in Y)(F(x, y))$;
>
> Wellordered families of wellordered sets have wellordered sumsets.

What we then need is a consistency proof in ZF of strZF*.

If we can do that, then we know that strZF* + $\neg$IO is consistent.

2.4 Joint work with Truss

Let us write '$W(\alpha)$)' for the cardinal of the set of all wellorderable subsets of a set of size α. Tarski proved that $\alpha < W(\alpha)$ by an elegant and complicated diagonal argument, and if one thinks of this result as a refinement of Cantor's theorem it is natural to think about structures analogous to Specker trees. Naturally i wondered if there is a Sierpinski-Hartogs theorem about this construction, saying something like $\aleph(\alpha) < W^{17}(\alpha)$, or something like that. It turns out that there is no such proof, so the obvious way of proving that the analogues of Specker trees are wellfounded doesn't work. Indeed even as i speak John Truss is in the process of putting the final touches to a construction of an FM model of ZF in which there is an infinite descending chain under W.

But it's much easier to show that there can be infinite ascending chains under W where $\aleph(\alpha)$ is the same for all α in the chain. Suppose there are dedekind-finite sets which do not have countably many distinct (inductively) finite subsets. The class of such cardinals is closed under W. Sse x is such a set, and suppose that $\mathcal{P}_{fin}(x)$ has countably many distinct (inductively) finite subsets $\{X_i : i \in \mathbb{N}\}$. Notice that of the elements of $\{\bigcup X_i : i \in \mathbb{N}\}$—all of them (inductively) finite subsets of x—infinitely many must be distinct.

This gives us a construction of a rather nice model. Let D sets be those without a countable infinity of distinct finite subsets, and D-cardinals similarly. (Just an aside: the class of D-cardinals can be coinductively defined as the largest class of finite cardinals closed under W.) Let us consider the class of sets that are hereditarily D. What is true in this model?

Pairing, replacement and separation are easy: they always are in models in sets-that-are-hereditarily something or other. We also get sumset, tho' that is a bit harder. Let X be a hereditarily-D set, and suppose *per impossibile* that $\bigcup X$ has a countable infinity of finite subsets: $\langle X_i : i \in \mathbb{N}\rangle$. Let x be a member of X. How many X_i can it meet? Well, it can meet infinitely many of them, but the meets are all finite, so there can be only finitely many distinct values. So, for each $x \in X$, there are finitely many i such that i is the first index at which $X_i \cap x$ is seen. This allocation of finite sets of natural numbers of members of X enables us to prewellorder X. Each equivalence class under this pwo is finite, so the length of the pwo cannot be finite, so X had a countable partition into finite sets, and so was not a D-set after all. We don't get power set of course, but we do get an axiom saying that for each x, the set of wellordered subsets of x is a set. We also get an axiom that is a bit like $\exists NO$: namely there is a set O of wellorderings such that every wellordering is isomorphic to something in O. There are lots of such O: every set that is not inductively finite is a witness.

To summarise, we drop AC, we weaken power set to something that asserts that for each x, the set of its wellordered subsets is a set, and we get an axiom saying that there is a set O of wellorderings such that every wellordering is isomorphic to something in O. What about foundation? Therein lies the only problem with this development. If we take 'hereditarily-D' to mean 'belongs to the least fixed point for the function sending x to the set of its D-subsets then we get foundation all right but notice that V_ω contains all its D-subsets and so every hereditarily-D set is in V_ω and we don't get $\exists NO$ after all! This looks like a complete disaster. However it does not exclude the possibility of there being *illfounded* nontrivial hereditarily-D sets. There are probably lots of ways of achieving it. Clearly one is to start in a model of ZF + AFA. Another would be to add lots of Quine atoms by a Rieger-Bernays permutation and do an FM construction on top of that to add lots of D-sets, and only then reach for the model of hereditarily D-sets.

Let's try this. Suppose a union of fewer than κ things each of size $< \kappa$ is itself of size less than κ. I think we also need κ to be strongly inacessible.

Let $\phi(x)$ say that $\kappa \not\leq |W(x)|$, and consider H_ϕ. What is this a model for?

Well, it's closed under W. The point is that if $|x| < \kappa$ then $|W(x)| = 2^{|x|}$

Interpreting ZFC in strZF*

What are the differences from the case where we are interpreting ZF in strZF?

Consider power set. The proof is essentially the same, the only difference being that this time the set of Y^R is wellordered so WIO suffices.

Replacement. We are given a `BFEXT` with a wellordered carrier set, and are told that each of its E-members is P-related to another `BFEXT`. Since the set of its E-members is wellordered we can use collection to obtain a set collecting relatives of these E-members, and by wellordered choice this set can be cut down to a wellordered set collecting relatives of these E-members. Since this set of `BFEXT`s is wellordered, we can invoke WIO to find a set of singletons the same size, which we can use to disjoin them.

Chapter 3

Does IO hold in HS?

Consider HS as a submodel of a given model $\mathcal{M}$ of ZF(C) with foundation. Does it satisfy IO? If not, then $\mathcal{M}$ believes that there is a least ordinal α such that HS_α is not the size of a set of singletons. This α cannot be successor, since the power set of a set the same size as a set of singletons is likewise the size of a set of singletons (and this fact is known to HS too). It must be limit. But then, for each $\beta < \alpha$, HS_β is the size of a set of singletons and we can use collection (see lemma 3) and sumset to obtain a set of singletons at least as big as HS_β for $\beta < \alpha$. You'd think this would be enough to show that this set of singletons is at least as big as HS_α, but that needs a bit of AC. What if we know additionally that every HS_β with $\beta < \alpha$ is the same size as a set of singletons $\subseteq HS_\alpha$? Even that doesn't seem to do very much. It shows only that $|V_\alpha|$ and $T|V_\alpha|$ are both minimal upper bounds for $\{|V_\beta| : \beta < \alpha\}$ but that doesn't seem to imply that they are the same cardinal!

Of course HS might satisfy IO but at a terrible cost. Consider the function that sends an ordinal α to the least β such that every set in HS_α is the size of a set of singletons in HS_β. This function—let's call it IO—might increase so fast that there is an α with an inaccessible between α and IO(α). If we cut the model off at this inaccessible then we get something nice.

If we are to have any hope of showing that HS satisfies IO we ought to have no difficulty establishing that V_ω is the size of a set of singletons.

Lemma 5 part (3) told us that V_ω was not cantorian, so if V_ω is the size of a set of singletons it certainly isn't the size of a set of singletons-all-of-finite-rank. However there seems nothing to stop it being the size of a set of singletons some of which are of infinite rank. So the first question before us is:

Can there be an n-symmetric injection from V_ω into a set of singletons?

The important fact about n-symmetric functions is that they "look only n levels down" That doesn't mean to say that no information about the input contained in levels below level n can appear in the value: it might. If f is n-symmetric then altho' it cannot *compute* with information from levels deeper than n, it can nevertheless *copy* information from those levels. The identity function copies *everything*!

(I want someone to think hard about this distinction and say something intelligent to me about it)

THEOREM 12. *In HS V_ω is the same size as a set of singletons.*

Proof: Suppose we have an n-symmetric (injective) function from V_ω to singletons. What does it do? Well, the only thing it can look at is the top n levels of its input, x. It knows that $\bigcup^n x$ contains certain elements of V_ω. It can copy them, but it cannot look inside them. What it thinks it is looking at is a finite n-level stratification graph, with one top vertex.

Each set x defines an equivalence relation $\sim_x$ on V_ω, where $y \ \sim_x \ z$ if the nth action of the transposition (y, z) fixes x. I've omitted the 'n' from the definition of $\sim_x$ beco's we will consider only the case $n = 4$, for reasons which will emerge later. When we build our function f from V_ω to singletons we must ensure that, for every argument x, the equivalence relation $\sim_x$ is the same as the equivalence relation $\sim_{f(x)}$.

So, for the time being, $\sim_x$ is the equivalence relation where $u \sim_x v$ iff $j^4(u, v)$ fixes x. Notice that $\sim_x = \sim_y$ does not imply that x and y are 4-equivalent:

$$\text{Try } x = \{\{\{a, b\}\}\} \text{ and } y = \{\{\{a\}, \{b\}\}\}, \\ \text{for any two elements } a \text{ and } b \text{ of } V_\omega. \tag{A}$$

If f is a 4-symmetric function then $\sim_x = \sim_{f(x)}$. This is not hard to see. Think of f as a set of ordered pairs—WK ordered pairs for the sake of argument. If it contained a pair $\langle x, y\rangle$ where x was altered (to x') by a transposition that fixed y and f then f would also have to contain the pair $\langle x', y\rangle$ and would not be a bijection. But i'm assuming the transposition fixes f ... oh yes, here we use the fact that f is symmetric.

Since the thing we are trying to build is a 4-symmetric function (with domain V_ω and range a set of singletons) what we need—at the very least—is a means of assigning to each x a singleton with the same equivalence relation. Let us see how to do this.

Suppose—for the purposes of illustration—that our equivalence relation for x has—in addition to the one cofinite equivalence class that they all have—two singleton equivalence classes and two pairs. Consider the stratification graph of x and let the six endpoints be labelled with s_1, s_2, p_1, p_2, p_3 and p_4, all of them elements of V_ω. s_1 and s_2 are the two endpoints that have singleton equivalence classes, of course. We now want a 4-shape whose top two levels are single nodes (it's the stratification graph of a singleton, after all) and it is to have the same equivalence relation on V_ω.

How about

$$\{\{\{s_1\}, \{s_2\}, \{p_1, p_2\}, \{p_3, p_4\}\}\}?$$

(I haven't written it out as a tree, but you get the idea). No, because it doesn't separate s_1 from s_2, and the two pairs look the same too. What we need instead is:

$$\{\{\{s_1, a\}, \{s_2\}, \{p_1, p_2, b\}, \{p_3, p_4\}\}\},$$

where a and b are sets of infinite rank (and therefore not moved). So if x is a set s.t. $\bigcup^2 x = \{s_1, s_2, p_1, p_2, p_3, p_4\}$ with $p_1 \sim_x p_2$ and $p_3 \sim_x p_4$ then we set

$$f(x) =: \{\{\{s_1, a\}, \{s_2\}, \{p_1, p_2, b\}, \{p_3, p_4\}\}\}.$$

This illustrates how, on being given a set $x \in V_\omega$, one can return a singleton $\{y\}$ s.t. $V_\omega \cap \bigcup^4 \{y\} = \bigcup^4 x$ so that $\sim_x = \sim_{\{y\}}$. There are of course lots of ways of doing it. This is just as well beco's—as we noted at (A) above—it can happen that $\sim_x$ and $\sim_y$ are the same relation even though x and y are not n-similar for any n. The sets-of-infinite rank that we use in this way i shall call **flags**.

The idea now is that we pick a representative from each 4-equivalence class, and assign to it a singleton in the manner illustrated above. Once we have assigned a singleton to x, we know how to assign one to every other member of $[x]_4$: f of $j^4(\sigma)(x)$ is of course just $j^4(\sigma)(f(x))$. There is the danger that we might assign the same singleton to representatives of different classes. However, as we noted at (A), in every case the allocation of singletons can be done in infinitely many ways. So it is certainly true that, for any finite collection X of 4-equivalence classes, and any way of picking representatives from the elements of X, we can allocate singletons to those representatives in such a way that every representative has the same equivalence relation as the singleton awarded to it, and distinct representatives are awarded singletons that are 4-inequivalent. We then use Zorn's lemma to stitch these partial functions together. "But wait", i hear you cry "AC fails in HS!". Yes, it does, but it is true *in the enveloping model*, and we use AC in the enveloping model to stitch together some hereditarily symmetric functions and get another hereditarily symmetric function. After all, a symmetric union of hereditarily symmetric sets is hereditarily symmetric.

Now we have to check that the function f that we obtain by this method really is symmetric, which is to say k-symmetric for all sufficiently large k. It's clear that it is 4-symmetric, but we need it to be n-symmetric for sufficiently large n. For this we need the extra sets of infinite rank—that we exploit in the construction of the singletons—to be symmetric.

Suppose we apply $j^{k+7}(u, v)$ to f. Let's call the result of this "$f^{(7)}$" (to stop ourselves going crazy!) We want to be sure that the graph of f is the same as the graph of $f^{(7)}$. If f sends x to $\{y\}$, $f^{(7)}$ will send x' to $\{y'\}$, where these two objects are obtained from x and $\{y\}$ by replacing objects $k+4$ layers down in accordance with the transposition (u, v). Let's think about how this acts on the sets 4 levels down, which is where the flags appear. It might move the members of V_ω around. However, as long as $k >$ the degree of symmetry of the labels, it won't move the flags. This means that when the transposition acts on f it can only move ordered pairs in f to ordered pairs that are already there. Now it is easy to find infinitely many labels whose degree of symmetry is, say, 2. (Complements of singletons of members of V_ω will do, for example).

■

So at least some infinite sets (V_ω for example) are the same size as a set of singletons. Are all of them? It is easy to see that if X and Y are both the size of a set of singletons then so is $X \to Y$, the set of functions from X into Y. That's a "successor" step. The limit stage needs choice.

If not all sets are the size of a set of singletons then there will be a first α such that HS does not believe that $V_\alpha \cap HS$ is the size of a set of singletons.

3.1 A little problem to sort out

Ackermann permutations are not just the usual one that creates V_ω; I also think of the involution that swaps $T\alpha$ with $\{\beta : \beta E\alpha\}$ as an Ackermann permutation.

Let's think a bit about how Ackermann permutations add illfounded sets. One implementation of $\mathbb{N}$ (as a quotient of V_ω) is a set and is stcan beco's every member of V_ω is stcan. In these circumstances the (usual) Ackermann permutation is a set by stratified separation, and so in the permutation model V_ω is stcan, and we must have added a wellfounded set. This is

OK beco's the proof that all permutation models have the same WF sets is unstratified so there is no reason to suppose that we cannot add new wellfounded sets. And we have made V_ω into a wellordered set which it wasn't before but that's OK beco's that's unstratified. However OP still fails beco's that's stratified. So what is there that we know isn't ordered? I don't like the look of this ... But i think it's OK: the old V_ω has become an illfounded set and it still isn't orderable.

What happens when you execute the Ackermann permutation depends very sensitively on your choice of implementation of $\mathbb{N}$. For a start you have to ensure that it is well defined: i.e., that the transpositions are disjoint. This is not the case if you are using Von Neumann naturals for example, because every natural number is also a set of natural numbers. In the case of the Baltimore model we note that—at least if we take our natural numbers to be equipollence classes of hereditarily finite sets—then 0 is $\{\emptyset\}$ so that when we swap 0 with the set of its "members" (namely $\emptyset$) we find that the old $\emptyset$ has become a Quine atom. One might expect that this could be avoided but it turns out that something similar will happen whatever is the implementation of $\mathbb{N}$ at work. $\emptyset$ will always be swapped with the numeral 0 and will become an illfounded set of some kind. Unless of course the number 0 is the empty set! 1 then gets swapped with $\{\emptyset\}$.... One thing to bear in mind is that a permutation that preserves rank cannot add any illfounded sets. Is there an implementation that makes the Ackermann permutation such a permutation? In any such implementation 0 will have to be the empty set. 1 then gets swapped with $\{0\}$ and it's easy to see that all the naturals will have to be of finite rank. We know that HS cannot contain an implementation of $\mathbb{N}$ where every natural is of finite rank, beco's HS contains no wellordering of any infinite subset of V_ω. So it looks as if any implementation of $\mathbb{N}$ (or presumably of relational types of `BFEXT`s) will force us to add new illfounded sets with the Ackermann permutation. To be sure of this we will want to know the converse to the observation above, namely we want to know that if π adds no illfounded sets then it preserves rank. Is this true...?

So i'm getting the following picture. Start with a wellfounded model $\mathcal{M}$ (of ZF); create HS, and form the Mostowski collapse of the family of relational types of `BFEXT`s. How does this depend on $\mathcal{M}$?

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3.2

Presumably we can prove the condensation lemma for Sr, and we can use it to prove something like this. Suppose α is an initial ordinal, and $x \in Sr_\alpha$ then $\mathcal{P}(x) \subseteq Sr_{\alpha^+}$, and this might tell us something about the rank of Specker trees.

While we are about it, TFAE: (i) $x \sim_n y$ and (ii) Player `Equal` has a strategy to postpone defeat for n moves in $G_{x=y}$. This is probably worth proving. Worth thinking also about the notion of n-similarity involved in the two-wands version of $G_{x=y}$. That is probably a good idea: it might make it clear why a winning strategy for `Equal` has to be a (special) permutation of V rather than merely a (special) bijection between $\bigcup^n x$ and $\bigcup^n y$. If all the sets that appear in $G_{x=y}$ up to n moves are low, then it doesn't matter whether `Equal` has a bijection or a permutation. As soon as we consider the two-wand construction it becomes clear that it matters—and why.

Time to think about permutation methods applied to the Baltimore model. I'm glad i discovered that nice elementary observation of Coret's(!) It enables us to prove Henson's lemma, so we know that permutations preserve stratified formulæ. We execute the Ackermann permutation on the Baltimore model to obtain B_1 and then throw away the illfounded sets to obtain B_2. Can we be sure that B_2 is a $\mathcal{P}$-extension of B_1 which is elementary for stratified formulæ? I think we need the new model to satisfy Coret's axiom B. I think this will be true as long as the inclusion embedding is setlike. Is it? Now we have—in the form of B_2—a model in which V_ω is well-behaved. Can we now do a higher-up Ackermann-like permutation to add well-behaved wellfounded sets of higher rank?

1. If V exists every sentence is Δ_2. Is the converse true? Is this an omitting types question?

2. Define by recursion: $f(\emptyset) =: \{\emptyset\}$; $f(x) =: \{\{\bigcup f(y)\} : y \in x\}$.

   If this recursion succeeds—so there really is such an f—then we can prove by a weakly stratified $\in$-induction (with a proper class parameter f) that f is injective and $|x| = |f(x)|$ for all x.

   We can even prove—by an *un*stratified $\in$-induction—that $f(x) = \iota``x$. So every set is cantorian.

   Does "Everything is cantorian" imply "Everything is strongly cantorian"?

3. I think that both stratified collection and wellordered choice are enough on their own to prove that the least aleph not the size of a set of singletons is successor. The use of wellordered choice in this connection is standard. To deduce it from stratified collection let $|A|$ be the least aleph not the size of a set of singletons. Stratified collection shows that $|A|$ cannot be limit, as follows. Consider $\mathcal{P}_{|A|}(A)$. Every member of this is the same size as a set of singletons, so let B collect suitable sets of singletons. Then $\bigcup B$ is a set of singletons at least as big as A.