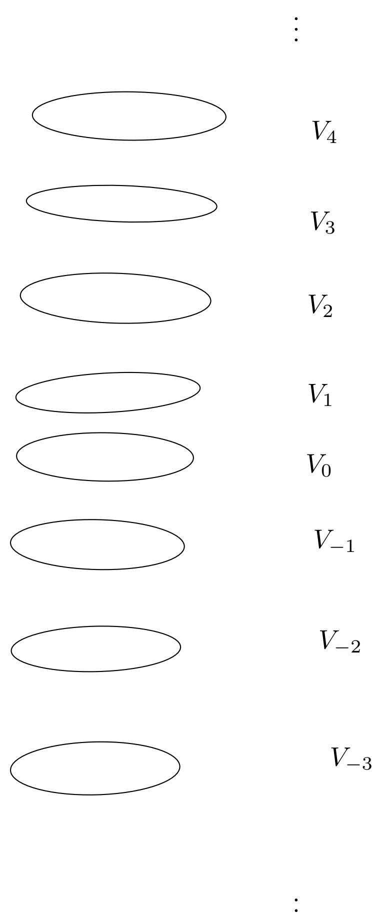
Scaling the Devil's Staircase: Recent Progress in The Theory T$\mathbb{Z}$T of (Positive and) Negative Types

Thomas Forster

November 21, 2021



Contents

1 Introduction

"It's turtles all the way down!!"

Anon (U.S.)

TℤT (first spelled out in [23]) is a many-sorted (strongly typed) set theory. It is a theatre in which certain foundational questions naturally arise and can be naturally treated; it also has strong connections with Quine's set theory of [13].

At NF70 (the 2007 meeting in Cambridge to commemorate the 70th anniversary of the publication of [13], see `https://www.dpmms.cam.ac.uk/~tf/NFUK.html`) Marcel Crabbé suggested that it might be a good idea to collate everything we know about TℤT: the ready availability of a comprehensive survey may perhaps concentrate minds and serve as a useful resource for researchers. Although I embarked on this expository project primarily to get my own thoughts in order, it has recently been borne in on me that there is a community of Russell scholars out there who have ideas about TℤT and who would benefit from having a concise summary. This historically motivated survey—the result of Marcel's suggestion—will attempt to bring the sophisticated beginner up to speed in this interesting backwater. The project to collate what has emerged since [23] has inspired me to do some work on my own account, so some of the results in what follows are mine. I have included all results known to me, and I have striven to attribute them all to their original discoverer.

Before we go any further, a disclaimer! There is a treatment of TℤT as a first-order theory in the language $\mathcal{L}(\in, =)$ of set theory[1]. That investigation is still continuing—under the radar—and more good will come of it, but we will not cover it here.

Although TℤT is manifestly consistent because of compactness, it is nevertheless highly pathological in various interesting and unexpected ways. The somewhat fanciful (diabolical) title is an allusion to the fact that the manifest failure of foundation of the membership relation in TℤT obstructs any and all construction of models for TℤT by recursion on $\in$, in striking contrast to the situation with TST.

2 Definitions

Definitions are the plot-points of expositions. John Cleese once said that the way to write sitcoms properly was to make every plot-point as funny as possible. The correct implementation of this insight in the present context is to accompany every definition by some motivation. It is the pursuit of this policy that has spun out this section to what readers might otherwise consider an absurd length.

Ordered pairs will mostly be Quine ordered pairs. Need to keep an eye on this

[1] The possibility of such an analysis is flagged in [14].

Russellian simple typed set theory TST, as simplified and modified by Carnap [3] and Quine [13], has a family of formally disjoint types of sets, indexed by the naturals, with a type (hereafter "level") of atoms and first-order axioms whose intent is that each subsequent level shall formally be the power set of its predecessor. We can think of it as the ωth-order theory of equality! Carnap [3] seems to be the first reference[2].

A natural enhancement of TST is the type theory ('T$\mathbb{Z}$T') where the levels are indexed by $\mathbb{Z}$, the (positive and negative) integers, and it is *this* theory which is our concern here. It can do no harm to set it out with deafening clarity: this is not because it is difficult to understand—it isn't—but rather because it is easily confused with other things. It cannot be said too often that the levels of sets in the T$\mathbb{Z}$T world view are **not** cumulative; they are **formally disjoint**.

The wellformation rules of $\mathcal{L}$(T$\mathbb{Z}$T) are as follows:

> *The atomic formula* $\ulcorner A \in B \urcorner$ *is wellformed iff the superscript on the variable* A *is one less than the superscript on the variable* B*;*
>
> *The atomic formula* $\ulcorner A = B \urcorner$ *is wellformed iff the superscripts on the variables* A *and* B *are the same.*

(Notice—and this remark is intended to give a flavour of the kind of linguistic pedantry one sometimes needs in the study of the language of T$\mathbb{Z}$T—that the letters 'A' and 'B' in the quoted text above are of course not variables of $\mathcal{L}$(T$\mathbb{Z}$T) but are variables ranging over such variables(!))

We have the usual recursions for quantifiers and connectives. It might seem that this definition of wff is a bit complicated, but it's easy to verify that the set of wffs is decidable.

We write '`XOR`' for the symmetric difference of two sets; this is in order to keep the symbol 'Δ' free as a variable for partitions and types. $B(x)$ is $\{y : x \in y\}$.

2.1 The Axioms of T$\mathbb{Z}$T

The axioms of T$\mathbb{Z}$T are

> *(i) Extensionality. For each* $i \in \mathbb{Z}$*,*
>
> $$(\forall x^{i+1})(\forall y^{i+1})(x^{i+1} = y^{i+1} \longleftrightarrow (\forall x^i)(x^i \in x^{i+1} \longleftrightarrow x^i \in y^{i+1}))$$

and

> *(ii) an Axiom Scheme of Comprehension. For every* i *and* ϕ*, the universal closure of*

$$(\exists x^{i+1})(\forall y^i)(y^i \in x^{i+1} \longleftrightarrow \phi)$$

[2] Possibly Quine's Harvard doctoral dissertation, independently a few years earlier, but i haven't been able to check it.

[It would not be surprising if the preceding discussion had sensitised the borderline obsessionals in the readersip to such an extent that some of them want to complain that the formulæ displayed above are not actually formulæ of $\mathcal{L}$(TℤT), since the variables in formulæ of $\mathcal{L}$(TℤT), while sporting superscripts, never accept numeric variables as superscripts, only concrete numerals.]

Each level is of course a set of the model, because one of our axiom (schemes) is the existence of a universal set at each level.

There are no axioms that *go without saying*, so the above exhausts the list of axioms. Readers may need to be reminded that the axiom of choice is nowhere assumed, no matter how this may fly in the face of their previous custom.

The theory TℤT was first presented to the world in Wang [23] where it is described as a theory of (positive and) negative types, subsequently generally called 'TNT' (for **T**heory of **N**egative **T**ypes—or 'TTN' in French. However, in the climate of closer attention to theories of this kind, we will want to distinguish between the theory of (strictly) negative types (levels indexed by $\mathbb{Z}^-$, the negative integers) and the theory of positive and negative types, hitherto 'TNT' but now TℤT. If every model of the theory of (strictly) negative types could be canonically/naturally extended to a model of TℤT there would be less need to distinguish between these theories, but no-one has so far unearthed any such construction. More on this point in section 4.9 below.

2.2 More Definitions concerning $\mathcal{L}(T\mathbb{Z}T)$

It's worth noting that TℤT has no axioms asserting relations between x^i and x^{i+1}: any alphabetic variant of an axiom of TℤT is an axiom of TℤT.

Definition 1 *If ϕ is a $\mathcal{L}(T\mathbb{Z}T)$ formula then ϕ^+ is the formula obtained from ϕ by raising by 1 all level-superscripts on variables in ϕ.*

2.2.1 Ambiguous Theories

Typical ambguity for a theory $T \supseteq$ TℤT is an ambiguous(!!) locution that could mean $T \vdash \phi \longleftrightarrow \phi^+$ for all **closed** $\phi \in \mathcal{L}$(TℤT) or could mean $T \vdash \phi$ iff $T \vdash \phi^+$ for all $\phi \in \mathcal{L}$(TℤT).

Definition 2
A theory T in $\mathcal{L}(T\mathbb{Z}T)$ is **ambiguous** *if, for all $\phi \in \mathcal{L}(T\mathbb{Z}T)$, $T \vdash \phi$ iff $T \vdash \phi^+$;*
A structure $\mathfrak{M}$ for $\mathcal{L}(T\mathbb{Z}T)$ is **ambiguous** *iff $\mathfrak{M} \models \phi \longleftrightarrow \phi^+$ for all ϕ in $\mathcal{L}(T\mathbb{Z}T)$;*
If $\mathfrak{M}$ is a structure for $\mathcal{L}(T\mathbb{Z}T)$ then $\mathfrak{M}^+$ is the result of relabelling the levels in $\mathfrak{M}$ so that old level $i+1$ becomes new level i.

Thus $\mathfrak{M} \models \phi$ iff $\mathfrak{M}^+ \models \phi^+$.

In some of the TST/TℤT literature an asterisk is sometimes used instead of a plus sign.

Amusing factoid: for any ambiguous theory T and any $\mathfrak{M} \models T$, $\mathrm{Th}(\mathfrak{M})$ and $\mathrm{Th}(\mathfrak{M}^+)$ are synonymous. Beware: they might not be the same theory! **It is not true that** every ambiguous theory has an ambiguous model! The following comes essentially from [21].

We have a propositional language $\mathcal{L}$ with propositional letters p_i, q_i and r_i for all $i \in \mathbb{Z}$. The theory T has two schemes; one to say that, for each $i \in \mathbb{Z}$, precisely one of p_i, q_i and r_i is true. The other scheme says $\neg(p_i \wedge p_{i+1}) \wedge \neg(q_i \wedge q_{i+1}) \wedge \neg(r_i \wedge r_{i+1})$. Note that T is ambiguous.

T, considered as the deductive closure of these axioms (a set of formulæ) has a automorphism group generated by S_3 (permute $\{p, q, r\}$) and $\mathbb{Z}$ (permute the subscripts). This group is the group of automorphisms of T in the sense that if $\sigma(\psi)$ is the result of doing the permutation σ to the formula ψ then $T \vdash \psi$ iff $T \vdash \sigma(\psi)$.

By analogy with T$\mathbb{Z}$T let us write ψ^+ for the result of increasing every subscript in ψ by 1.

> (a) Note that, altho' for all ψ we have: $T \vdash \psi$ iff $T \vdash \psi^+$, nevertheless we do not have $T \vdash \psi \longleftrightarrow \psi^+$ for all ψ. Indeed, if we add the scheme of biconditionals $\psi \longleftrightarrow \psi^+$ to T the resulting theory is inconsistent.
>
> (b) Note that nevertheless each expression of the form $\phi \longleftrightarrow \phi^+$ is individually consistent with T.

(a) is pretty obvious; for (b) fix an arbitrary S; we will find a valuation satisfying $S \longleftrightarrow S^+$. Suppose (with a view to obtaining a contradiction) that every valuation satisfies precisely one of S and S^+. Think of the valuation that goes $\ldots p\, q\, r\, p\, q\, r \ldots$[3] (with period 3) as you ascend through the levels (call it f) and the two valuations f^+ and f^{++}. Observe that f sat S iff f^+ sat S^+ and so on. Do any of these valuations actually satisfy $S \longleftrightarrow S^+$? If they do, we are happy. If not, then each of them satisfies precisely one of $\{S, S^+\}$. WLOG f satisfies S but not S^+; then f^+ satisfies S^+ but not S, f^{++} satisfies S but not S^+ and f^{+++} satisfies S^+ but not S. But $f^{+++} = f$.

■

We will often practice typical ambiguity in the following sense. If the theorem we want to state can be proved (and has the same proof!) at every level, then we will omit the level superscripts in the statement of the theorem.

T$\mathbb{Z}$TI is T$\mathbb{Z}$T with the axiom of infinity added. Any formulation will do, and we will not here say anything that relies on one formulation being used rather than another.

Wang makes the point that, although every level of every model of T$\mathbb{Z}$T is infinite, nevertheless T$\mathbb{Z}$T does not prove the axiom of infinity. This is because there is a simple compactness proof of the consistency of T$\mathbb{Z}$T+ $\neg$infinity. This has caused unneccessary confusion in some circles, and from time to time people have thought that they have proved the consistency of T$\mathbb{Z}$T+ infinity without straw. How strong does a theory have to be to prove the consistency of T$\mathbb{Z}$T+ infinity relative to the consistency of T$\mathbb{Z}$T? Must such a system be strong enough to prove the consistency of T$\mathbb{Z}$T+ infinity outright? The question has never been properly investigated as far as i know, and some people might be helped if it were.

This might be the point at which to insert a discussion of Landon thingie's mistake

DEFINITION 3
(i) The **height** *of a variable x^i in a formula $\phi \in \mathcal{L}(T\mathbb{Z}T)$ is computed as follows: if j is the lowest level of any variable in ϕ then the height of the variable x^i is $i - j$. Clearly heights cannot be negative, but they can be zero.*

(ii) If ϕ is a formula with one free variable, and that variable is of height n, we say that ϕ is an n-formula.

Height is an important notion for us, for the following reason—part of the folklore of NF studies. Suppose $\phi(x^i)$ is a k-formula with one free variable. Then $(\forall x^i)(\phi(x^i) \longleftrightarrow \phi(j^k(\pi)(x^i)))$ for all permutations π of level $i - k$. Consider: the universal set at each level is fixed by all permutations acting on the next level down; $\{\{\emptyset\}\}$ at each level is fixed by all permutations acting two levels down, and so on. There are generalisations to formulæ with more than one variable, all of them easier to understand than to state.

2.3 Duality

$\widehat{\phi}$ (the *dual* of ϕ) is the result of replacing '$\in$' by '$\notin$' throughout ϕ. The significance of this for us is that T$\mathbb{Z}$T$\vdash \phi$ iff T$\mathbb{Z}$T$\vdash \widehat{\phi}$, since evidently the dual of an axiom of T$\mathbb{Z}$T is a theorem of T$\mathbb{Z}$T. The dual of a structure $\mathfrak{M}$ for $\mathcal{L}$(T$\mathbb{Z}$T) (or the language of set theory for that matter) is the structure with the same carrier set M but the complement of the membership relation[4]. A model of T$\mathbb{Z}$T is **self-dual** (or just plain "dual") if it satisfies $\phi \longleftrightarrow \widehat{\phi}$ for all $\phi \in \mathcal{L}$(T$\mathbb{Z}$T). We say a permutation σ is an **antimorphism** if it satisfies $(\forall xy)(x \in y \longleftrightarrow \sigma(x) \notin \sigma(y))$.

ZF is clearly not a natural setting for ideas about duality and antimorphisms . . . one looks for Set Theories with a universal set. Church's set theory has models with antimorphisms; it is not known if NF has models with antimorphisms, tho' there are partial results in that direction.

2.4 Symmetry

Any permutation of a set X acts in an obvious way on the iterated power sets $\mathcal{P}^n(X)$ for any and all n. Sometimes we want to think of these induced permutations as actions of permutations of X, but sometimes it is helpful to identify an operation j (for *j*ump) that injects the symmetric group Symm(X) on a set X into the symmetric group Symm($\mathcal{P}(X)$) on $\mathcal{P}(X)$, so that we think of the action of Symm(X) on $\mathcal{P}(X)$ as the action on $\mathcal{P}(X)$ of the subgroup

[3] Actually there are three such valuations so just fix one of them.
[4] this might need some clarification

j"Symm(X) on $\mathcal{P}(X)$, and we can think of the action of Symm(X) on $\mathcal{P}^n(X)$ as an action of j^n"Symm(X) similarly. We will make frequent use of this notation in what follows.

We say $x^i \sim_n y^i$ if there is a(n internal!) permutation π of level $i - n$ that sends x^i onto y^i in the action described in the preceding paragraph (that is to say: $(j^n\pi)(x^i) = y^i$)) and we write '$[x^i]_n$' for the equivalence class of x^i under this relation. We say "x and y are n-equivalent". We observe without proof that

Lemma 1 *If x and y are n-equivalent they satisfy the same n-formulæ.*

It is easily verified that $\sim_n$-equivalence classes are sets according to T$\mathbb{Z}$T.

A set x^i is k-symmetric if it is fixed by every [internal] permutation of level $i - k$; alternatively x^i is k-symmetric if it is fixed by $j^k(\pi)$ for every [internal] permutation π of level $i - k$. For each i and k, the property of being a set at level i that is k-symmetric is first-order. (Uniformly in i but not in k.) Thus (for example) the universal set and the empty set at each level are 1-symmetric. A set is plain **symmetric** *tout court* iff it is k-symmetric for some k.

A model is **symmetric** if every element of it is symmetric. We do not know of any symmetric models of T$\mathbb{Z}$T; this is a major open question.

Holmes writes about other applications of these ideas in his contribution to this volume.

2.5 Truncation and Relabelling

For $\mathfrak{M}$ a model of TST the n**th truncation** $\mathfrak{M}^{(n)}$ of $\mathfrak{M}$ is $\mathfrak{M}$ with the bottom n levels deleted and the surviving levels relabelled. We have a similar notion of a model of TST being a truncation of a model of T$\mathbb{Z}$T.

Any model of T$\mathbb{Z}$T can be **relabelled**: take some level other than level 0 to be the new level and renumber all other levels appropriately. The result is a new model, with the same carrier set but different decorations. This might be a good moment to peek ahead at the notion of ambiguous model in section 4.7 and reflect that if $\mathfrak{M}$ is an ambiguous model of T$\mathbb{Z}$T then it is elementarily equivalent to all its relabellings.

2.6 Ultraproducts

The reader is assumed to be familiar with ultraproducts. Ultraproducts of many-sorted structures can be messy, at least if the signature of the structures concerned allows infinitely many sorts, since there is no way of asserting in the many-sorted language that a element in the ultraproduct must belong to a sort—let alone precisely one sort. An element in the product the sorts of whose values jump around all over the place might not be of any proper sort in the ultraproduct. It may be that the correct notion of product of a family of structures for a many-sorted language contains only those functions that do *not* disconcert us by jumping around between sorts. That would certainly be the view of products of vector spaces taken by people who think of these things as two-sorted structures. What we need is the notion of the *standard* part of an ultraproduct, the collection of those (equivalence classes of) functions in the product that almost everywhere are of one level. Perhaps one should prove a version of Łoś's theorem for this modified concept of ultraproduct, but I crave the reader's indulgence. Thus the standard part of an ultrapower contains all the "constant functions" and more besides. In fact the nonstandard part of any model of TST will consist of lots of models of TℤT.

Definition 4 *The* **shifting ultraproduct** *of a model* $\mathfrak{M}$ *of* TST *is the model of TℤT obtained by making* $\mathbb{Z}$*-many copies of* $\mathfrak{M}$*, relabelling in all possible ways, and taking an ultraproduct using an ultrafilter* $\mathcal{U}$ *on* $\mathbb{Z}$*.*

In fact this shifting ultraproduct is a substructure of the ultrapower $\mathfrak{M}^{Z}/\mathcal{U}$.

2.7 Natural Models

A **natural** or **standard** model (of TℤT or TST) is one where V_{i+1} is (genuinely) the same size (seen from outside) as $\mathcal{P}(V_i)$. A natural model of TST is determined up to isomorphism by its bottom level; $\langle\langle x\rangle\rangle$ is the natural model whose bottom level is x. The **canonical** model of TST is $\langle\langle\emptyset\rangle\rangle$. It might be felt to be more—well—*natural* to say that a natural model of TST is one where each level is the actual power set of the level below it. For the moment let us say that such a model is *strongly* natural. Clearly the existence of strongly natural models of TℤT trivially contradicts the axiom of foundation in the metatheory. The point (as we shall see later—section 4.2) is that the lack of natural models of TℤT has nothing to do with the presence of the axiom of foundation in the metatheory. It is in fact the case that there are no models of TℤT that are natural even in the weaker original sense with which this paragraph started, and that is no triviality but a fact worth noting. It is for that reason that we prefer the slightly unnatural definition of natural model given here.

2.8 ω-standard and κ-complete models

An ω-standard model is one whose natural numbers are standard. Since models of TℤTI have a set of natural numbers at each level one might expect that these various $\mathbb{N}$ might turn out to behave differently, with higher $\mathbb{N}$s being "wiser" than lower $\mathbb{N}$s. Fortunately we are spared this complication: we can prove that, at every level, every natural number (= equipollence class of finite sets) contains a set of singletons and thus the naturals at level $i+1$ are (externally) isomorphic to the naturals of level i *via* the (external) singleton function. Any nonstandard natural at level i will have a copy at level $i+1$. Presumably this means that an ambiguity scheme holds for sentences in the arithmetic of TℤT, so that anything that can be proved in the arithmetic of one level can be proved in the arithmetic of any level, but—as far as i know—the matter has never been properly examined.

A model is κ**-complete** iff every (externally visible) subset of V_i of size $\leq \kappa$ is (coded by) an element of V_{i+1}. To put it another way, at each level the universe of that level is a κ-complete boolean algebra.

In particular a model $\mathfrak{M}$ is **countably complete** iff, at every level k, whenever $f : V_k \to V_k$, $f``\mathbb{N}_k^{(\mathfrak{M})}$ (i.e., the image in f of the natural numbers of $\mathfrak{M}$ at level k—the graph of f is of course *not* assumed to be a set of $\mathfrak{M}$) is a set of $\mathfrak{M}$. Countable completeness is a kind of higher-order replacement axiom for countable sets, just as κ-completeness is a kind of higher-order replacement scheme for sets of size κ. In the ZF-iste literature one often hears this characterised as *closed under κ-sequences*.

A β-model $\mathfrak{M}$ (of a set theory) is one that is honest about wellfoundedness: if $\langle X, R\rangle$ is a binary structure in $\mathfrak{M}$ that $\mathfrak{M}$ believes to be wellfounded (that is: $\mathfrak{M}$ hosts no subsets of X that lack R-minimal elements) then $\langle X, R\rangle$ really is wellfounded seen from outside. Thus: if $\mathfrak{M}$ is a β-model and $\langle X, R\rangle \in M$ is a structure that is illfounded seen from outside then $\mathfrak{M}$ hosts a subset of X witnessing this fact. Notice that this does not oblige $\mathfrak{M}$ to contain *all* subsets-of-X-that-lack-R-minimal-elements, merely that it should contain *some*—if there are any.

2.9 Cardinals

We write '$|x|$' for the cardinal number of x.

We will use generalised beth numbers: $\beth_0(\alpha) := \alpha$; $\beth_{n+1}(\alpha) := 2^{\beth(\alpha)}$. (No transfinite subscripts!). This is in addition to the usual notation where $\beth_\alpha := |V_{\omega+\alpha}|$. An *aleph* is the cardinal of a wellordered set. $\aleph(\kappa)$ is the least aleph $\not\leq \kappa$; $\aleph^*(\kappa)$ is the least aleph $\not\leq^* \kappa$. The fact that we use these notations should alert the reader to the fact that we do not silently assume the axiom of choice; if we ever do make that assumption we will flag it.

In T$\mathbb{Z}$T we naturally take $|x^i|$ to be $\{y^i : y^i \text{ is in bijection with } x^i\}$. Although one normally (when working in ZF, say) thinks of 2^α as $|\mathcal{P}(A)|$ where $|A| = \alpha$, in T$\mathbb{Z}$T one has an alternative definition, namely 2^α as $|\mathcal{P}(A)|$ where $|\iota``A| = \alpha$ ($\iota$ is the singleton function). In T$\mathbb{Z}$T there is of course no sense to be made of the externally obvious fact that there is a bijection between $A$ and $\iota``A$. Nevertheless we need a way of alluding to the relation between the cardinals of these sets, and for this we use the letter 'T', so that $T|x| = |\iota``x|$. This definition has the disconcerting (but actually harmless) effect that 2^α is not always defined. If A is not the same size as any set of singletons then there is no cardinal $2^{|A|}$.

The **tree** $\mathcal{T}(\alpha)$ is a downward-branching rooted tree whose top element (root) is α. For each $\kappa \in \mathcal{T}(\alpha)$ the set $\{\beta : 2^\beta = \kappa\}$ is a **litter** and its members are κ's **children**, and they are all in $\mathcal{T}(\alpha)$.

We can think of trees as binary structures in more than one way, and we will equivocate between these ways. Although it's usually most convenient to think of these trees as digraphs, they can also be thought of as posets, and when we think of them as posets we write the order relation with the symbol '$\ll$'.

Observe the crucial rôle played here by the cunning definition of exponentiation that we gave above: If α is a cardinal (equipollence class) at level i, the cardinal 2^α (at least if it exists) is also at level i! This means that all the elements of $\mathcal{T}(\alpha)$ live at the same level, so that $\mathcal{T}(\alpha)$ is always a set of the model: all the routine reasoning about trees can be conducted inside T$\mathbb{Z}$T.

Holmes calls them 'Specker trees' because, although Specker never published anything about them, he used them in his (unpublished[5]) proof of the axiom of infinity from the axioms of Quine's NF, and it is hard to imagine a proof that doesn't use them (though the proof in the second edition of [15] is a heroic attempt); accordingly one can surmise that he did at any rate invent them. However their first *citable* appearance is in [8]. I'd always rather hoped that Holmes would relent and take to calling them 'Forster trees' but he is Not To Be Moved.

We will appeal often to various related facts which we will collectively call "Sierpinski-Hartogs":

LEMMA 2 *(Sierpinski-Hartogs)*
$\aleph(\alpha) < \aleph(2^{2^{2^\alpha}})$.
$\aleph^{(}\alpha) \leq^* 2^{\alpha^2}$.

If α is an aleph we have

$$\aleph^*(\alpha) \leq^* 2^\alpha \text{ and } \aleph^*(\alpha) < \aleph^*(2^\alpha).$$

These results are standard and we supply no proof. We quote without proof the result from [8] that $\mathcal{T}(\alpha)$ is always wellfounded and that if α is an aleph then the rank of $\mathcal{T}(\alpha)$ is finite. It is open in T$\mathbb{Z}$T—as indeed it is in ZF—whether there can be cardinals α such that $\mathcal{T}(\alpha)$ has infinite rank. The techiques that Holmes used in his unpublished consistency proof for NF will presumably provide models of ZF and T$\mathbb{Z}$T in which there are cardinals whose trees are of infinite rank.

2.10 Initial Extensions

A model of TST has a bottom level, level 0. Can we add a level -1? If we can, and the result is (after relabelling, of course) a model of TST, then we call the result an *initial extension*. There is a certain amount of bureaucracy concerning the numbering of levels, and i, for one, tend to think of the bottom level of an initial extension as being labelled with a negative integer, so that one has to go back and redefine the language of TST so that the bottom level can be labelled with any integer at all. Naturally one tries not to think about such things.

The operation of initial extension is right-inverse to truncation, but it is not everywhere defined. For example if the cardinality of the bottom level of a

[5]Specker proved the axiom of Infinity in NF using cardinal trees, and soon afterwards he found a proof of the stronger result that AC fails in NF; naturally he published the stronger result not the weaker one. . . even tho' the two proofs are quite different and the weaker one is of independent interest

model $\mathfrak{M} \models$ TST is a natural number other than a power of 2 then $\mathfrak{M}$ cannot be extended downwards. Clearly a model of TST with 2^n atoms at level 0 can be extended downwards by an external observer simply by scraping together n objects and bijecting the old atoms with sets of the scrapings. Interestingly we can do this internally as well, as the following definition reveals.

DEFINITION 5 *An initial extension of a model $\mathfrak{M}$ (of length k) is a sequence $v_0, v_{-1}, \ldots v_{-(k-1)}$ of subsets of V_0 where $v_0 = V_0$ and thereafter the v_i are chosen so that $|\mathcal{P}(v_{i-1})| = |\iota``v_i|$, and the $v_i$ are representatives chosen from the cardinals in a branch of $\mathcal{T}(|V_0|)$. The $v_i$ are the levels of the initial extension, and we define a membership relation between them in the obvious way. We are told there is a bijection $f_i : \mathcal{P}(v_i) = \iota``v_{i-1}$ so we say that x (in v_{i-1}) is a member (in the new sense) of y (in v_i) if $x \in f_i^{-1}(\{y\})$.*

Clearly, for this definition, we have to have chosen suitable f_i but there are only finitely many needed so there is no problem with AC.

Notice (and this will matter) that this definition is internal to the theory: we can quantify over all the initial extensions from that level of any length. "X is an initial extension of length k" is a wff of $\mathcal{L}$(T$\mathbb{Z}$T) *with 'k' free*. Notice, further, that + of this formula is also a wff of $\mathcal{L}$(T$\mathbb{Z}$T): there is nothing in the definition of initial extension that relies on everything being of level 0. One can do downward extensions from higher levels as well ... tho' one tends to call them *sideways* extensions.

In what sense does this structure merit the description "initial extension"? It has new levels below V_0—the v_i—and a membership relation between them. Variables of level $-i$ of course are variables of level 0 restricted to v_{-i}.

It is important to note that, for any formula ϕ of $\mathcal{L}$(T$\mathbb{Z}$T), "there is an initial extension in which ϕ holds" is likewise a formula of $\mathcal{L}$(T$\mathbb{Z}$T), so the process can be iterated.

It remains to be shown that this new structure is actually a model of TST.

Extensionality is enforced by *fiat* since the f_i were all bijections. To verify the comprehension axioms, let $\phi(x)$ be an expression with one free variable (of level $-i$ for some $i < k$, otherwise there is nothing to prove). We want there to be a set which does the job of being $\{x : \phi\}$ in the new model.. The key fact is that the relativisation of ϕ to the new setting is a wff of $\mathcal{L}$(T$\mathbb{Z}$T).

one could say a bit more about this

It's not hard to see that

REMARK 1 *A model $\mathfrak{M} \models TST$ has a downward extension of length k (k concrete) precisely if the rank of $|V_0|$ is at least k.*

Proof:

One direction is easy: if there is an initial extension of length k then the cardinals $|v_i|$ form a branch of length at least k in $\mathcal{T}|V_0|$.

For the other direction suppose $\rho(|V_0|)$ is at least k. Then there is a branch in $\mathcal{T}|V_0|$ of length k, and we pick representatives from the nodes on the branch. We also have to pick bijections. Only finitely many choices needed, so no problem with AC. ■

The situation becomes much more complicated if $\rho(|V_0|)$ is infinite and we want to talk about all possible initial extensions.

2.10.1 Another way of coding Initial Extensions

Suppose we succeded in imagining an initial extension—a lot of things of level -1. If we apply the B (principal ultrafilter) function to any of these new atoms we get a thing at level 1. The members of B of one of these fictional atoms will be precisely those atoms to which the fictional atom belonged. (Recall that $x \in y \longleftrightarrow y \in B(x)$). At any level, the set of Bs of things two levels down is a set of free generators for the complete boolean algebra of things at that level. This means that, if we can find a set X of elements of level 1 that freely generates the complete boolean algebra that is level 1 then we can think of the things in X as Bs of the fictional level -1 objects. Then we can interpret the new atomic formula "$x^{-1} \in y^0$" as the old formula "$y^0 \in x^1$" where the variable 'x^1' is restricted to range over (members of) X.

The whole process is faintly reminiscent of the manufacture of ideal divisors.

2.11 Games

We will encounter two kinds of Games in what follows below.

1. The **∈-game** as in [9] and [19]. `I` and `II` play $G(x)$ as follows: each player (`I` starting) picks a member of the other player's last choice (`I` starts by picking a member of x) until the game is ended by one player or the other trying to pick a member of $\emptyset$ and thereby losing. ∈-determinacy says that one player or the other must have a winning strategy. Any symmetric model of T$\mathbb{Z}$T must satisfy ∈-determinacy. (It's an old result of mine [7] that if x is n-symmetric then $G(x)$ can be wrapped up in at most $n+2$ moves).

   In section 6.2 we are interested in something a little stronger: the possibility that, for some k, one player or the other can force a win within k moves.

2. The **Strong extensionality** game, $G(x = y)$ played by players $=$ and $\neq$ to determine whether or not $x = y$. The rules are as follows: $\neq$ picks $x' \in x$ (or $y' \in y$, it's up to him to decide which) and $=$ replies with $y' \in y$ (or $x' \in x$ *mutatis mutandis*). The first player unable to move loses.

   Evidently if $x = y$ then $=$ has a strategy to avoid losing. A axiom of Strong extensionality would assert the converse. Hence the name of the game.

2.11 Grundy Rank

In this connection we bring up the notion (from combinatorial game theory) of Grundy rank. In any model $\mathfrak{M}$ a function G (not a function internal to $\mathfrak{M}$, obviously!) obeying the recursion "$G(x) =:$ least ordinal not G of any member of x" is a **Grundy rank** function. See, for example, [2] or—a good place to start—the Wikipædia article.

Clearly no model of T$\mathbb{Z}$T can admit a global set-theoretic rank function, because of the infinite descending $\in$-chains. Grundy rank is more permissive, in the sense that although $\mathcal{L}(\in)$-structures that have self-membered sets do not admit Grundy rank functions, the presence of infinite descending $\in$-chains does not preclude Grundy rank, so it may be that Grundy rank is of interest in the context of the model theory of T$\mathbb{Z}$T.

3 Natural questions

We close this introductory section with a list of the questions that will naturally occur to the thoughtful reader on being exposed to T$\mathbb{Z}$T. We will answer those we can, and will report on known progress on those we so far can't.

(i) Can every model of TST (with V_0 infinite) be extended downwards?
(ii) Can every model of TNT be extended upwards?
(iii) Are there natural models of T$\mathbb{Z}$T?
(iv) Are there countably complete models of T$\mathbb{Z}$T?
(v) Are there ω-standard models of T$\mathbb{Z}$T?
(vi) Are there symmetric models of T$\mathbb{Z}$T?
(vii) Are there models of T$\mathbb{Z}$T that satisfy strong extensionality?
(viii) Are there ambiguous models of T$\mathbb{Z}$T?
(ix) Are there models of T$\mathbb{Z}$T with a Grundy rank function?
(x) Are there models of T$\mathbb{Z}$T with antimorphisms?
(xi) Does T$\mathbb{Z}$T decide all $\forall^*\exists^*$ sentences?
(xii) Are there "term" models of T$\mathbb{Z}$T: models wherein every element is the denotation of a closed set abstract?
(xiii) Can we safely add any infinitary comprehension axioms?

There now follow some brief comments about the above, offered mostly without proof.

The answer to (i) is "no". This is partly because—somewhat surprisingly—"there is an initial extension" is first-order. For $n \in \mathbb{N}$ let $\mathfrak{M}_n$ be the natural model of TST with precisely n atoms at level 0. If n is not a power of 2 then $\mathfrak{M}_n$ does not have an initial extension, so neither will any ultraproduct of $\{\mathfrak{M}_n : (\forall m \in \mathbb{N})(n \neq 2^m)\}$.

We will show below that the answers to (iii) and (iv) are "no", and that the answer to (vii) is "yes".

"Yes to (viii)" implies Con(NF). This result of Specker's [21] is fundamental in the literature on NF, and the reader is assumed familiar with it, so we will not treat it here. “Yes to (xii)” implies “Yes to (vi)”. You might think that “Yes to (vi)” implies “Yes to (v)”, but there can be definable nonstandard naturals: if there are only finitely many alephs the cardinality of the set of alephs is a nonstandard natural.

We shall see that there are no *very*-nice models (standard or countably complete) and all the *moderately*-nice models (ω-standard, symmetric, and ambiguous) would contradict choice (if there are any!) but the various failures of choice seem to be unrelated!

It is a major challenge to students of T$\mathbb{Z}$T to explain what is going on with AC. We need to explain both (i) why these three nicenesses should all contradict choice, and (ii) why the three refutations of choice are so utterly different. This problem has been around for a long time. See [6].

At this stage we seem to know nothing about (ii) at all.

We outline a partial solution to (xi) below, in theorem 2.

In the next section we provide detailed analyses of the remaining questions under separate headings... except that proofs that use Omitting Types will be treated in the following—final—section.

3.0.1 Question (xiii)

In [5], there is a discussion of the extent to which one can safely relax the syntactic constraints on the comprehension scheme in NF. Some of that discussion is applicable here, but of course in this context there is no possibility of relaxing the stratification constraint—the language will not allow it. Branching quantifiers lead to contradiction (and in a way that suggests that the same would happen in T$\mathbb{Z}$T). However one can consider accepting some formulæ in $L_{\omega_1,\omega}$, particularly those where only finitely many levels are allowed. It is proved that the models of NF that are also models of stratifiable $L_{\omega_1,\omega}$ comprehension using only finitely many levels and mentioning only finitely many parameters are precisely the models of NF where the $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ of the model are the same as the naturals of the metatheory and the power set thereof. One would expect that the same holds for T$\mathbb{Z}$T, tho’ the matter has not been checked.

It’s easy to see that no model of T$\mathbb{Z}$T can contain at any level n the set WF_n of wellfounded sets of that level[6]. $\mathrm{WF}_{i+1} = \mathcal{P}(\mathrm{WF}_i)$ and $\mathrm{WF}_{i-1} = \bigcup \mathrm{WF}_i$, so if WF exists at even one level it exists at all levels. The point then is that, for all levels n, $\mathrm{WF}_n \in \mathrm{WF}_{n+1}$, so the set $\{\rho(\mathrm{WF}_n) : n \in \mathbb{Z}\}$ of set-theoretic ranks of the various WF has no least member. Similarly `I` and `II` (the two collections of sets for which `I` (resp. `II`) have winning strategies in the $\in$-game introduced in section 2.11) cannot be sets—at any level. (This is in [9]).

It seems that one can safely allow infinitary comprehension for formulæ that mention only finitely many levels.

[6]Dear Reader, here we can only mean *wellfounded-seen-from-outside*; the model has no internal concept of wellfounded set.

4 Various kinds of models

4.1 Dual models

In this section we show that every model of T$\mathbb{Z}$T is dual, and that there are models with antimorphisms. T$\mathbb{Z}$T is dual because every instance of $\phi \longleftrightarrow \widehat{\phi}$ is a theorem of TST. ■

Theorem 1 *T$\mathbb{Z}$T has models with (internal) antimorphisms.*

Proof:

By compactness we can extend T$\mathbb{Z}$T by adding a constant symbol σ and axioms to say that $(\exists\tau)(\sigma = j\tau \cdot c)$, $(\exists\tau)(\sigma = j^2\tau \cdot jc \cdot c)$ and so on down. (c is the set-theoretic complementation function, at each level.) Equivalently we can extend T$\mathbb{Z}$T by adding constant symbols σ^i and axioms to say that $\sigma^{i+1} = j(\sigma^i) \cdot c$. We can even allow the comprehension axioms to contain the new constant(s).

■

Observe however that we cannot use this technique of adding constants to show that T$\mathbb{Z}$T+ Ambiguity + duality is consistent if T$\mathbb{Z}$T+ Ambiguity is consistent: by enlarging the language we obtain new ambiguity axioms, and there is no reason to suppose that the new ambiguity axioms are satisfied. If anything, the opposite: suppose that at each level i we have a constant symbol 'σ^i' to denote a permutation, and we want to add axioms to say that $\sigma^{i+1} = j(\sigma^i) \cdot c$—and this can be done. However (for simplicity's sake) consider the case where all the σ are involutions. Observe that if σ^i has a fixed point then σ^{i+1} has no fixed point. Assuming AC_2 we can show that if σ^i has a fixed point then σ^i has none. Ambiguity for formulæ containing sigmas will enforce that either all σ^i have fixed points or none of them do: after all "(σ^i has a fixed point)$^+$" is "(σ^{i+1} has a fixed point)" The inescapable conclusion is that none of the σ^i have fixed points ... so AC_2 must fail!

Thus, if we want ambiguity to hold for formulæ containing sigmas then we will have to assume that $\neg AC_2$, and it is not clear how to exploit that assumption in constructing a model.

4.2 Natural Models

If we think of T$\mathbb{Z}$T as a higher-order theory (and it would be superhuman not to feel tempted!) then we expect level $n+1$ to be genuinely the size of the power set of level n. (We do not want to get ourselves embroiled in any pointless fights about the axiom of foundation in our metatheory by asking that level $n+1$ should genuinely *be* the power set of level n!) That is why when, in section 2.7, we gave a definition of *natural model of T$\mathbb{Z}$T*, we did it in the way we did. That has the effect that a model is natural iff, for every level V_n of $\mathfrak{M}$ every (externally visible) subset X of V_n is encoded by an element x of V_{n+1} in the sense that X is the $\mathfrak{M}$-set $\{y : y \in x\}$.

Another way of expressing this is to say that $\mathfrak{M}$ is complete iff every level of $\mathfrak{M}$ is a complete boolean algebra [seen from outside].

We recapitulate from [10] the proof that T$\mathbb{Z}$T has no natural models. We reason in the theory that is contemplating a structure $\mathfrak{M}$ for $\mathcal{L}$(T$\mathbb{Z}$T), and we show that it cannot be a natural model of T$\mathbb{Z}$T. Suppose $\mathfrak{M}$ is a natural model of T$\mathbb{Z}$T and, for each $i \in \mathbb{Z}$, let α_i be the cardinality of level i of $\mathfrak{M}$; then $\aleph(\alpha_{i-1}) < \aleph(\alpha_i)$ by lemma 2 the Sierpinski-Hartogs' lemma, and $\aleph``\{\alpha_i : i \in \mathbb{Z}\}$ is a set according to the theory we are reasoning in, and—by Sierpinski-Hartogs again—can have no least element, contradicting the fact that it is a set of alephs.

4.3 Countably Complete Models

Recall the definition of κ-complete model from section 2.8. Clearly any natural model is κ-complete for all κ, and there are no natural models of T$\mathbb{Z}$T. It turns out that there aren't even any countably complete models!

Remark 2 *T$\mathbb{Z}$T does not have any countably complete models.*

Proof:

The corresponding assertion about NF—that NF has no countably complete models—is in [16]. It's worth noting here because—as Holmes puts it—this is a T$\mathbb{Z}$T-fact not an NF-fact. The point is that in any countably complete model $\mathfrak{M}$ of T$\mathbb{Z}$T all structures believed by $\mathfrak{M}$ to be wellorderings genuinely are wellorderings. (Any binary structure in $\mathfrak{M}$ that is not a wellordering [seen from outside] but is mistakenly believed by $\mathfrak{M}$ to be a wellordering would have a countable subordering with no bottom element, and by countable completeness this subordering would be in $\mathfrak{M}$). Now, for each $i \in \mathbb{Z}$, let α_i be the supremum of the lengths of the (internal) wellorderings in V_i. A Burali-Forti argument shows that $\alpha_i < \alpha_{i+1}$ always, and this gives us a contradiction since the α_i are genuine ordinals of the metatheory—since, by assumption of countable completeness—wellorderings in $\mathfrak{M}$ are genuine wellorderings. ■

In some sense this is the same as the proof that T$\mathbb{Z}$T has no natural models.

4.4 β-Models

The same considerations establish that T$\mathbb{Z}$T has no β-models. The ordinals at any level are illfounded seen from outside, but it's a theorem of T$\mathbb{Z}$T that the ordinals are wellfounded. Could say more about this. . . ?

4.5 ω-standard Models

The difference between—on the one hand—*countably complete model* and β-model and—on the other—*ω-standard* model is subtle but important. We are considering structures for the language of set theory (typed or untyped, it makes no difference). If $\mathfrak{M}$ is ω-standard, it means that every subclass of $\mathbb{N}^{(\mathfrak{M})}$ (the natural-numbers-from-$\mathfrak{M}$) is a set of $\mathfrak{M}$.

We have seen (remark 2) that T$\mathbb{Z}$T does not have any countably complete models, but we haven't shown that it has no ω-standard models. Any such model would be interesting because—as I shall now outline—AC would fail in it.

Nor any β-models: rearrange this material; supply more proofs

REMARK 3

(1) Every tree $\langle \mathcal{T}(\alpha), \ll \rangle$ is wellfounded, and therefore has a rank;
(2) If $\rho(\alpha)$ is infinite then α is not an aleph;
(3) $\rho(|V|)$ is not a standard natural number;
(4) If $\alpha \in \mathbb{N}$ then $\langle \mathcal{T}(\alpha), \ll \rangle$ is a linear order;

Proof:

1. This is a consequence of Sierpinski-Hartogs; the proof in [8] goes through in T$\mathbb{Z}$T.

2. Suppose this were not the case, and α were an aleph that was a counterexample, so that $\rho(\alpha)$ was infinite. Let β be the least member of $\mathcal{T}(\alpha)$; $\alpha = \beth_n(\beta)$ for some n. However, if $\mathcal{T}(\alpha)$ has arbitrarily long descending chains, we will have $\alpha = \beth_{n+2}(\gamma)$ for some γ. This gives us $\gamma \geq \beta$ whence $\beth_n(\gamma) \geq \beth_n(\beta) = \alpha$ contradicting $\alpha = \beth_{n+2}(\gamma)$.

3. If $\rho(|V_i|)$ is a standard natural then, arguing inside the metatheory, we can say $\rho(|V_i|) > \rho(|V_{i-1}|)$ and so on down through all the negative integers, thereby giving an infinite descending sequence of naturals. The metalanguage will not countenance this.

4. This is merely a rephrasing of the uncontroversial observation that $(\forall n, m \in \mathbb{N})(2^n = 2^m \rightarrow n = m)$.

$\blacksquare$

Conventionally we use the letter 'ρ' to denote the rank function for wellfounded structures, so we shall here write '$\rho(\alpha)$' for the rank of $\mathcal{T}(\alpha)$.

COROLLARY 1 *AC fails in every ω-standard model of T$\mathbb{Z}$T.*

Proof:

If every natural number is standard, then $\rho(|V_i|)$ cannot be a natural number. It is an ordinal, so it must be an infinite ordinal. But no cardinal of infinite rank can be an aleph. So $|V_i|$ is not an aleph, so AC fails. $\blacksquare$

REMARK 4 *(Holmes)*
Any model of T$\mathbb{Z}$T which contains a genuinely countably infinite set must be ω-standard and refute choice.

Proof: [7]

We work in some uncontroversial metatheory in which we can survey a model $\mathfrak{M}$ of T$\mathbb{Z}$T. Suppose we can prove that $\mathfrak{M}$ contains—somewhere, it doesn't matter at which level—a set x which we can see to be countably infinite. (If there is such a set at even one level then every level will have one). Now think about the $\mathfrak{M}$-cardinals of the finite subsets of x (that are in $\mathfrak{M}$). These cardinals will be the natural numbers of $\mathfrak{M}$. And it is clear that they look standard to us.

Then, by corollary 1, we infer ¬AC.

■

Holmes' remark was the same observation about specifically V_ω, which makes the fact much more striking.

What this does mean is that there is no routine method of—as it were—*capping off* a model of Zermelo set theory (or rather capping off $\mathbb{Z}$ copies) to obtain a model of T$\mathbb{Z}$T. It is very striking that any model of T$\mathbb{Z}$T that contains the actual V_ω violates AC. This is really extraordinary.

Should connect this with the thoughts on p **??**.

At this stage we do not know if T$\mathbb{Z}$T has any models containing genuinely wellfounded sets of infinite rank, and *a fortiori* we don't know if T$\mathbb{Z}$T has ambiguous models containing wellfounded sets of infinite rank. This has consequences for the study of Quine's NF. It is known that, if NF remains consistent if we add to it a kind of weak standardness assumption for $\mathbb{N}$, then it has models containing infinite sets that are *internally* wellfounded... but of course such sets might not be wellfounded seen from outside.

Could supply a ref for this but it would make the bib top-heavy with references to my work!

4.6 Models with Grundy rank?

Recall that if $\langle X, R\rangle$ is a wellfounded binary structure we define the Grundy rank of $x \in X$ to be the least ordinal not the Grundy rank of any x' s.t. $x' \, R \, x$. Wellfoundedness of R is sufficient for the existence of a Grundy rank function for $\langle X, R\rangle$ (and of course it will be unique) but it is not necessary. However it is obviously neccessary that $(\forall x \in X)\neg(x \, R \, x)$.

The fact that Grundy rank precludes self-membership means that no model of NF can admit a Grundy rank function. However there is nothing obviously preventing us from decorating a model of T$\mathbb{Z}$T with a Grundy rank function, even a model that is ambiguous because it arises from $\mathbb{Z}$-many copies of a model of NF.

Grundy rank functions arise from the study of discrete finite two-person games; see [2] for more details. Given this connection, one might expect that any model of T$\mathbb{Z}$T that obeys $\in$-determinacy (and there are such models, as we see in section 6.2) would admit a Grundy rank function. However things are not that simple.

There now follows an insert from Nathan Bowler.

[7] Thanks to Tim Button for prodding me to be clearer about this

4.6.1 Every model of TZT has a Grundy rank function

We assign Grundy ranks greedily. That is, we recursively construct a sequence of predicates Γ_α for α an ordinal number, where $\Gamma_\alpha(x)$ holds precisely when x has elements y_β with $\Gamma_\beta(y_\beta)$ for all $\beta < \alpha$ but has no element y with $\Gamma_\alpha(y)$. To do this, we need the following lemma:

Lemma 3 *Let V be a model of T$\mathbb{Z}$T and let Δ be a predicate on V. Then there is a predicate Γ on V such that for any element x of V we have $\Gamma(x)$ precisely when [$\Delta(x)$ and there is no $y \in x$ with $\Gamma(y)$].*

Proof:

We consider the game $G(\Delta)$, played between two players O and E. The positions are elements x^i of V with $\Delta(x^i)$. If i is even then it is E's turn to play and if i is odd then it is O's turn. Whoever's turn it is must play some $y^{i-1} \in x^i$ with $\Delta(y^{i-1})$. If they cannot do so, then the other player wins.

We define $\Gamma(x^i)$ to mean '$\Delta(x^i)$ and one of the following two options holds: either i is even and O has a winning strategy starting from x^i in $G(\Delta)$ or else i is odd and O has no winning strategy starting from x^i in $G(\Delta)$'. It is clear that this Γ is as required. ■

Now we can define the predicates Γ_α recursively using this lemma applied to the predicates Δ_α, where $\Delta_\alpha(x)$ means 'x has elements y_β with $\Gamma_\beta(y_\beta)$ for all $\beta < \alpha$'. For any x let α be the least ordinal such that x has no element y with $\Gamma_\alpha(y)$. By the construction of Γ_α, it follows that $\Gamma_\alpha(x)$. So for any x there is a unique α such that $\Gamma_\alpha(x)$, which we can take to be the Grundy rank of x.

4.6.2 Some model of T$\mathbb{Z}$T has a Grundy rank function in which all ranks are at most $\omega + 1$

In any model of T$\mathbb{Z}$T with a Grundy rank function G it is easy to show that every layer contains elements of all finite Grundy ranks. Thus all elements of the form V_i have infinite Grundy rank, and in particular there is some element x^i with $G(x^i) = \omega$. Then $G(V_i) \geq \omega + 1$ and so there is always some element of Grundy rank $\omega + 1$. Our aim now is to construct a model of T$\mathbb{Z}$T and a Grundy rank function in which no element has Grundy rank greater than $\omega + 1$.

We begin by applying the methods of the last Section inside a non-ω-standard model W of a sufficiently large fragment of ZFC. We interpret things W believes to be sets, predicates, relations etc. as actual sets, predicates, relations etc. in the usual way. We obtain some V which W believes to be a model of T$\mathbb{Z}$T and some G which W believes to be a Grundy rank function on V. We now build a new model of T$\mathbb{Z}$T, which we will also call V. This shouldn't lead to any confusion, since for any standard integer n we take V_n to be the n^{th} layer of V. We ignore the nonstandard layers. The containment relation will be the same as before.

We define a Grundy rank function $\overline{G}$ on V by

$$\overline{G}(x^i) = \begin{cases} G(x^i) & \text{if } G(x^i) \text{ is a standard natural number.} \\ \omega & \text{if } G(x^i) \text{ isn't a standard natural number and } i \text{ is odd.} \\ \omega+1 & \text{if } G(x^i) \text{ isn't a standard natural number and } i \text{ is even.} \end{cases}$$

To prove that this is a Grundy rank function, we must show that for any x^i the ordinal $\overline{G}(x^i)$ is the least ordinal not in $\overline{G}``x^i$. If $\overline{G}(x^i)$ is finite then this is clear from the fact that (W believes that) G is a Grundy rank function. If $\overline{G}(x^i)$ is ω then x^i must have a sequence $(y_n^{i-1}|n \in \mathbb{N})$ of elements with $\overline{G}(y_n^{i-1}) = n$, otherwise $G(x^i)$ would be the least n for which there is no such y_n. On the other hand, x^i has no element y^{i-1} with rank ω, since $i-1$ is even. Thus ω is the least ordinal not in $\overline{G}``x^i$.

The hardest case is $\overline{G}(x^i) = \omega + 1$. In this case we obtain as before a sequence $\langle y_n^{i-1} \in x^i : n \in \mathbb{N}\rangle$ with $\overline{G}(y_n^{i-1}) = n$. Since $G(x^i)$ isn't a standard natural number, there is some nonstandard natural number ν with $\nu < G(x^i)$. Thus x^i has an element y_ν^{i-1} with $G(y_\nu^{i-1}) = \nu$ and so $\overline{G}(y_\nu^{i-1}) = \omega$ since $i-1$ is odd. Since x^i has no element y^{i-1} with $\overline{G}(y^{i-1}) = \omega+1$ (once more since $i-1$ is odd) it follows that $\omega+1$ is the least ordinal not in $\overline{G}``x^i$. This completes the proof. ■

Models of T$\mathbb{Z}$T with Grundy rank functions are virgin territory. Lots of questions that have not been investigated: Can an ω-model of T$\mathbb{Z}$T have a Grundy rank function?

4.7 Ambiguous Models

Clearly the axioms of T$\mathbb{Z}$T are (what one would call) *polymorphic*. It is a celebrated result of Specker [20] that if there is a model of T$\mathbb{Z}$T that is ambiguous then Quine's NF is consistent and conversely.

It is natural to ask about the status of restrictions of the scheme of ambiguity biconditionals, and quite a lot of work was done on this question in the middle of the last century. It is known that all models of T$\mathbb{Z}$T are ambiguous for formulæ mentioning only three levels. The most salient open problem in this area is whether or not T$\mathbb{Z}$T $\vdash \phi \longleftrightarrow \phi^+$ for all $\forall^*\exists^*$ sentences. We will show:

Theorem 2 *If Φ is a $\forall^*\exists^*$ sentence then $\Phi \to \Phi^+$ holds in every model of TST with at least n atoms, where n is finite and depends only on Φ.*

That will suffice to establish that $\Phi \to \Phi^+$ is a theorem of T$\mathbb{Z}$T.

Proof:

Throughout this discussion we will try to keep to the cute mnemonic habit—due to Quine—of writing a typical universal-existential sentence with the initial—universally quantified—variables as $\vec{y}$ ('y' for $y$ouniversal) and the existentially quantified variables as $\vec{x}$—for E$x$istential). That was so we can talk about y variables and x variables.

We know of old that when dealing with universal-existential sentences we need concern ourselves only with those Φ that are of the form $(\forall \vec{y})(\psi(\vec{y}) \to (\exists \vec{x})\theta(\vec{x}, \vec{y}))$ where ψ is a conjunction of atomics and negatomics, and θ is quantifier-free; all universal-existential sentences considered below will be assumed to be of this form.

We want to prove that $\mathfrak{M}$ (a model of TST) satisfies $\Phi \to \Phi^+$. We assume that $\mathfrak{M} \models \Phi$ and that the variables in Φ of lowest level are of level 0. We want to infer that $\mathfrak{M} \models \Phi^+$. The new idea is that it is not neccessary to find a particularly clever type-raising injection that deals with all Φ; it isn't even necessary to find an h for each Φ. Our h will depend on the instantiations of the y variables in Φ.

We require of our injection h that it lift levels and that it respect $\in$: $(\forall u, v)(u \in v \longleftrightarrow h(u) \in h(v))$. For this it is necessary and sufficient that $h(v)$ always be a (not neccessarily proper) superset of $h``v$, with the property that $h(v) \setminus h``v$ be disjoint from the range of h.

So, let $\vec{y}$—elements of $\mathfrak{M}$—be some tuple of instances of the '$\forall \vec{y}$' in Φ^+. Clearly if we can find a level-raising injection $h : \mathfrak{M} \hookrightarrow \mathfrak{M}^+$ with the feature that every y in our tuple is a value of h then we are home and hosed. To start with, things are comparatively straightforward. For reasons which will become clear (they may be clear already) h is going to have to be setlike, and the best way of doing that is to ensure that it is definable. So, for each $y_1 \ldots y_n$, (where $y_1 \ldots y_n$ are the y objects of level 1 in Φ^+ (the lowest level)) we designate a thing of level 0 to serve as $h^{-1}(y_i)$ *and we give it a name*—'a_i', say. This will ensure that h is definable with parameters (the various ys and the a_i) and is therefore setlike. We are now in a position to announce what h does to things in level 0: it sends a_i to y_i and sends everything else of level 0 to its singleton (or anything definable—it really doesn't matter as long as it's an injection).

That was painless. Suspiciously easy, you might think! Thereafter how do we define h on (things of) level $n+1$—on the assumption that we have defined it on (things of) level n? Well, there are various y objects of level $n+2$ that have to be values of this h. So y is h of something... but of what? Here the clue is that h is an $\in$-homomorphism. This tells us that $h(v)$ is always a superset of $h``v$. What do we know about $h(v) \setminus h``v$? We have already remarked that it mustn't contain any values of h. So, if y is to be h of anything it must be h of $h^{-1}``(y \cap h``V)$. If y is at level $n+2$ then $h^{-1}``(y \cap h``V)$ is of level $n+1$, so that reveals to us h of at least *some* things of level $n+1$. (Notice that for this to work we absolutely need to ensure that h remains setlike at each stage, and this is why we want it to be definable.) The other elements u of level $n+1$ can be sent to $h``u$, but of course any superset of $h``u$ obtained by adjoining *non*values of h will do—as long as the h that results thereby is setlike.

I hope it is now clear how to show that $\forall^*\exists^*$ sentences generalise upward in all sufficiently large models of TST. Let Φ be any universal-existential sentence as above, and fix a sufficiently large model $\mathfrak{M} \models$ TST. For any tuple of ys instantiating $\psi(\vec{y})$ we devise an injection h as in the above construction. Now invoke Φ in $\mathfrak{M}$, obtaining witnesses to the x variables, and apply h to all those witnesses. These will be witnesses to the x variables in Φ^+.

■

This extra flexibility in constructing h seems to be of no use to us, and with our fairly limited aims it isn't, admittedly. However, we might be trying to (upwardly) preserve formulæ in $\forall^*\exists^*\Gamma$ for some class more demanding than just the quantifier-free formulæ, and in such an endeavour the extra flexibility might turn out to be very useful indeed.

Notice that this does not (or at least does not obviously) resolve the question of whether or not T$\mathbb{Z}$T decides all $\forall^*\exists^*$ sentences. It does mean that every $\forall^*\exists^*$ is either true in cofinitely many finitely generated models of TST or is false in cofinitely many finitely generated models of TST—and that is new, if unsurprising. We know that every model of TST is elementarily equivalent to a countable model and that every countable model is a direct limit (colimit) of all finitely generated models, but there does seem to be the possibility that there could be a $\forall^*\exists^*$ sentence that is false in all finitely generated models of TST but nevertheless true in some (but not all) models with an infinite bottom level. Other partial results pointing in the same direction are to be found in [17], [18] and [4].

While we drum our fingers waiting for the last details of the universal existential conjecture to be sorted out, there remain some observations we can make. The possibility we have to exclude is that of an $\exists^*\forall^*$ sentence which is true in some but not all infinitely generated models.

We collect the following known facts.

(i) Every countable model is a direct limit of finitely generated models;

(ii) every $\forall^*\exists^*$ sentence true in arbitrarily large finitely generated models is true in cofinitely many finitely generated models;

(ii) every $\forall^*\exists^*$ sentence true in an infinite natural model is true in all larger infinite models.

The technique used to prove theorem 2 can be used to show that if $x \subseteq y$ where x and y are both infinite ($x \hookrightarrow y$ will do) then every $\forall^*\exists^*$ sentence true in $\langle\langle x\rangle\rangle$ is true also in $\langle\langle y\rangle\rangle$. (Theorem 2 concerns the case where $y = \mathcal{P}(x)$.) This means that any $\forall^*\exists^*$ sentence true in even one natural model $\langle\langle x\rangle\rangle$ with x infinite is true in all sufficiently large natural models.

So any $\forall^*\exists^*$ sentence true in arbitrarily large finite models is true in all sufficiently large natural models. And any "bad" $\exists^*\forall^*$ sentence true some but not all infinite models is false in all sufficiently large natural models. This shows that the theory of all sufficiently large natural models decides all $\forall^*\exists^*$ sentences.

4.8 Initial (downward) extensions of models of TST

This section needs to be tidied up

We can now see how to say "there is an initial extension of length n in which ϕ holds". And we can do it with 'n' free.

Another point to notice—a real bonus—is that there is nothing in this proof that assumes that ϕ has no parameters. We can write down a formula to say that there is an initial extension in which x has any given property.

The (internal to $\mathfrak{M}$) cardinal tree $\mathcal{T}|V^i|$ of the cardinal of the universe at level i encodes all possible *standard* downward extensions of the ith truncation of $\mathfrak{M}$. Thus a model can discuss internally the possibility of initial extensions of arbitrary length.

It's not hard to cook up models of TST with initial extensions of arbitrary length. Any terminal segment of any model of T$\mathbb{Z}$T is such a model. If $\mathfrak{M}$ is any model of TST then an ultraproduct of all truncations of $\mathfrak{M}$ is a model that has initial extensions of arbitrary length.

The *huge* problem with T$\mathbb{Z}$T is that any terminal segment of $\mathfrak{M}$ a model of T$\mathbb{Z}$T can see all possible downward extensions—or thinks it can. However, because of Sierpinski-Hartogs, it cannot see the downward extension of order-type ω^*.

perhaps this remark does not belong precisely here...

It might be an idea to minute the following facts:

Remark 5

Every initial extension of a countably complete model is countably complete, and Every initial extension of a model with all integers standard is a model with all integers standard.

4.9 TNT—the Theory of Negative Types

Recall that TNT is the theory of types indexed by the negative integers.

It has been known for ever—I think it is in [22]—that the theory of a terminal segment of T$\mathbb{Z}$T is axiomatisable: it is TST with the extra rule "from ϕ^+ infer ϕ". (One might wonder how to axiomatise it with extra axioms instead of the extra rule of inference.) Presumably the theory of an initial segment of a model of T$\mathbb{Z}$T is precisely TNT with the extra rule "from ϕ^- infer ϕ". (Again, one might wonder how to axiomatise it with extra axioms instead of an extra rule of inference.) It is a reasonable bet that the theory of an initial segment of a model of T$\mathbb{Z}$T can be axiomatised as TNT plus axioms to say that the rank of the tree of $|V|$ at level -2 is not 0, is not $S(0)$ ...

Is it the case that every model of TNT can be extended upwards by adding a level 0? The answer is presumably 'no'. A new object at level $n+1$ has to have a preëxisting sumset at level n, so—in extending a model upwards—we have to be careful not to stick on top anything that would force the model we started with to be more complete than it is.

It may be that the "upward extension problem" is easier for the type theories corresponding to NFP and NFI, but I don't think the matter has ever been investigated.

Beyond this there is not much to be said about TNT. On the whole the pathologies that mark T$\mathbb{Z}$T off as different from TST are manifested also in

TNT. (remark 2 and so on). TNT has no natural models, no countably complete models, and so on. And for the same reasons, so the proofs are the same.

5 FM models for T$\mathbb{Z}$T?

At this stage there is no good notion of FM model for T$\mathbb{Z}$T. This is a thoroughly unsatisfactory *lacuna*, since without FM models we really have no good way of getting models of T$\mathbb{Z}$T+ $\neg$AC. In the first instance one wants what might call *finite support models*.

What might one mean by that expression? One thing one might mean is that: for every x there is n such that x is finitely supported n levels down. Another thing one might mean is that every x is finitely supported at cofinitely many lower levels.

Perhaps we should introduce some terminology at this juncture. Let us say that $\mathfrak{M}$ is a **finite support model** if every x in $\mathfrak{M}$ has finite support at some lower level, and that $\mathfrak{M}$ is a **strong finite support model** if every x in $\mathfrak{M}$ has finite support at cofinitely many lower levels. Notice that we do not need intermediate notation for the case where every x in $\mathfrak{M}$ has finite support at *infinitely* many lower levels, since every finite support model has that feature.

There is an odd fact about this that struck me. It might not mean anything of course, but here goes. The odd fact is that, for any model $\mathfrak{M} \models$ T$\mathbb{Z}$T, if it is a finite support model then, if it has an initial segment that is a strong finite support model of TNT, then $\mathfrak{M}$ is a strong finite support model (of T$\mathbb{Z}$T). Suppose every set at level $-n$ (with $-n < 0$) is finitely supported at cofinitely many levels. Let x be an object of level 0 that is finitely supported at level $-n$. Then, for all sufficiently large m, everything in the support of x at level $-n$ is supported at level $-m$ and this of course means that x, too, is finitely supported at any such level $-m$. This powers an induction that sweeps up through all of $\mathfrak{M}$.

There's probably nothing wrong with this (of course there might not be any such models anyway) but it does look rather odd.

It might be telling us something about upward extensions of models of TNT.

Amusing reflections, but why should there be any finite support models even of the milder flavour? Where would one find them?

6 Obtaining Models of T$\mathbb{Z}$T using Omitting Types

Any fool can realize a type: it takes a model theorist to omit one.

Gerald Sacks.

We start by reminding ourselves of some standard model theory that we are going to use.

A **type** is a set of formulæ all with the same number of free variables. An n-**type** is a type all of whose elements have n free variables.

A model $\mathfrak{M}$ **realises** a type Σ iff there in $\vec{x} \in M$ s.t. $\mathfrak{M} \models \sigma(\vec{x})$ for every $\sigma \in \Sigma$. We say a model **omits** a type if it does not realise it. A formula $\phi(\vec{x})$ **supports** a type $\Sigma(\vec{x})$ if $\vdash (\forall \vec{x})(\phi(\vec{x}) \to \sigma(\vec{x}))$ for every $\sigma \in \Sigma$. We say a theory T in a language $\mathcal{L}$ **locally omits** a 1-type Σ if, whenever $\phi(x) \in \mathcal{L}$ is a formula such that T proves that ϕ supports Σ, then $T \vdash (\forall x)(\neg\phi(x))$.

Mainly we will be concerned with 1-types. We will be working in the language of T$\mathbb{Z}$T (which is countable) so all our types will be countable.

The Omitting Types theorem now says:

Theorem 3 *Let T be a consistent theory in a first-order language $\mathcal{L}$. If T locally omits a type Σ then T has a model that omits Σ.*

There is also the stronger:

Theorem 4 *(Extended Omitting Types Theorem)*

Let T be a consistent theory in a language $\mathcal{L}$. If T locally omits each type Σ in a countable class $\mathfrak{S}$ of types then T has a model that, for each $\Sigma \in \mathfrak{S}$, falsifies at least one σ in Σ.

These are standard model-theoretic results and accordingly we supply no proofs here.

Since T$\mathbb{Z}$T has infinitely many levels it naturally happens that the kinds of things we wish to bring about by Omitting Types tend to be things that need to happen at all levels simultaneously, so we find ourselves wishing to omit a type at each level. Thus it happens that typically an application of omitting types in this context will be an invocation of theorem 4.

It turns out that in T$\mathbb{Z}$T we can can make much more extensive use of Omitting Types than we can in NF. This is because it's very difficult for a a k-formula $\phi(x)$ to control what happens inside $\bigcup^k x$. When using Omitting types in NF we have to consider the possibility of *unstratifiable* ϕ that might support the type we are trying to omit. In the T$\mathbb{Z}$T context we can attempt the following strategy. We are trying to find a model that omits a type Σ (a 1-type so the sake of simplicity). We consider ϕ s.t T$\mathbb{Z}$T $\vdash \phi(x) \to \sigma(x)$ for all $\sigma \in \Sigma$. Such a ϕ is a "k-formula" for some k, which is to say that free variable in ϕ is of height (see definition 3 for definition of height) k. The strategy is to show that if there is a ϕ s.t T$\mathbb{Z}$T $\vdash \phi(x) \to \sigma(x)$ for all $\sigma \in \Sigma$ where ϕ is a k-formula, then there is a ϕ such that T$\mathbb{Z}$T $\vdash \phi(x) \to \sigma(x)$ for all $\sigma \in \Sigma$ where ϕ is a 0-formula. This last is unlikely in the extreme and so enables us to prove that T$\mathbb{Z}$T locally omits Σ. This ruse will not work in NF because ϕ might not be stratifiable.

It may be worth making the point that, in T$\mathbb{Z}$T, countable types come in two very different flavours, depending on whether or not they mention infinitely many levels. Types that mention infinitely many levels ought to be very easy to omit, because no n-formula can be expected to control the behaviour of its arguments more than n levels down.

In what follows we consider the prospect of obtaining models of TℤT with the following nice features, all of which would be obtained by omitting types of the second flavour: types that mention infinitely many levels.

(i) For all x, $TC(x) = V$ or x is hereditarily finite;
(ii) Every set is symmetric;
(iii) There is an Internally Definable Total Order;
(iv) Strong Extensionality;
(v) $\in$-determinacy;
(vi) Omit the type $\{|x| \leq T^n|V| : n \text{ standard}\} \cup \{|x| \not\leq \beth_n : n \text{ standard}\}$
(vii) Omit the type
$\{t \in x : t \text{ a concrete hereditarily finite set}\} \cup \{|x| \leq T^n|V| : n \text{ standard}\}$

(vii) is of course stronger than (vi). I know how to do (i) and (iv) and (v), and we do them below. I'm not so confident that we should be able to do (ii).

(vii) is of course the endeavour to obtain a model with no infinite wellfounded sets by OTT.

6.1 For all x, $TC(x) = V$ or x is hereditarily finite

We have to be careful how we state this, since $TC(x^i)$ is not a TℤT construct. "$TC(x^i) = V$" is a shorthand for an infinitary disjunction

$$\bigvee_{i \in \mathbb{N}} (\bigcup^k x^i = V_{i-k})$$

We will exhibit a model of TℤT wherein, for all x, either $TC(x) = V$ (in this sense) or x is hereditarily finite.

Let Σ be the the 1-type that says of x that, for each (concrete) k, $\bigcup^k x$ is neither the universe nor the empty set. (We omit level superscripts since the argument is the same at each level).

REMARK 6 *Every extension of TℤT in the same language locally omits Σ.*

Proof:

Suppose ϕ supports Σ. That is to say: ϕ is a formula of $\mathcal{L}$(TℤT) with one free variable 'x' s.t., for each k, $\phi(x)$ implies that $\bigcup^k x$ is neither V nor $\emptyset$. Suppose the free variable 'x' has level n in ϕ (so that ϕ is what we call an "n-formula"). Consider an x which has ϕ. Think about $\bigcup^n x$. It is not equal to V, nor to $\emptyset$. In particular $V \notin \bigcup^n x$ since otherwise $\bigcup^{n+1} x$ would be V, contradicting the fact that x has ϕ. Let y be anything in $\bigcup^n x$, and consider the transposition $\tau =: (y, V)$. Then $j^n \tau(x)$ certainly has ϕ too, but its $(n+1)$-times sumset is V.

So no formula ϕ supports Σ. This means that TℤT vacuously satisfies the condition for omitting Σ, namely: TℤT $\vdash (\forall x)(\neg\phi(x))$ for all ϕ realising Σ—since there are no such ϕ! ■

This "every extension of T$\mathbb{Z}$T in the same language" does sound incredibly strong, but it is correct. How are we to insert into a model an object x s.t. $\bigcup^n x$ is not V or $\emptyset$ for any n? Of course we can introduce a constant to denote a counterexample, but there of course in doing that one is expanding the language, and there doesn't seem to be any other way of doing it. a constant, and

COROLLARY 2 *Let T be a theory in $\mathcal{L}(T\mathbb{Z}T)$. Then T has models that omit Σ. In any such model, for every element x, there is a concrete natural number n such that $\bigcup^n x$ is either V or $\emptyset$.*

(Strictly speaking we are not using omitting types to omit Σ but rather using *extended* omitting types to omit every Σ_i—where Σ_i is of course the 1-type that says of x^i that, for each (concrete) k, $\bigcup^k x^i$ is neither the universe nor the empty set at level $i - k$)

This echoes an observation of Boffa's that in NF every definable transitive set is either V or is hereditarily finite, and prompts the thought that one theory T to which corollary 2 applies is T$\mathbb{Z}$T+ the ambiguity scheme, which is equiconsistent with NF. However one must resist the temptation to infer from this that NF (if it is consistent) has models in which every set whose transitive closure is not V is hereditarily finite, since the proof would involve an excursion into a language extended by adding names for a level-shifting automorphism. Of course one can always try to run the proof of remark 6 in NF but then one runs up against the problem that in NF there may be *unstratified* formulæ that support Σ—and the argument to show that formulæ that support Σ must be empty works only on stratified formulæ.

6.2 The Consistency of $\in$-determinacy

Perhaps a little clarification is in order when one considers $\in$-determinacy for T$\mathbb{Z}$T. This is because if $G(x)$ can go on for arbitrarily many steps then no strategy for $G(x)$ can possibly be an element of the model we are working in. The best we can hope for is that we should find an $\mathfrak{M} \models$ T$\mathbb{Z}$T such that, for every x in $\mathfrak{M}$, the metatheory knows of a winning strategy for $G(x)$. Clearly not *every* $\mathfrak{M} \models$ T$\mathbb{Z}$T can obey this, since a simple compactness argument allows us to add to T$\mathbb{Z}$T constants a_i at each level and axioms to say $a_{i+1} = \{a_i\}$. Clearly $G(a_i)$ cannot have a winning strategy—even seen from outside—every play is infinite. However, omitting types will enable us to find models $\mathfrak{M} \models$ T$\mathbb{Z}$T where every $G(x)$ has a winning strategy that is actually a set of the model.

Let Σ be the 1-type $\{\psi_k(x) : k \in \mathbb{N}\}$, where $\psi_k(x)$ says that in $G(x)$ neither player can force a win in k moves. (Both players have a strategy to stay alive for k moves.) Then

THEOREM 5 *T$\mathbb{Z}$TI locally omits Σ.*

Proof: Suppose $\phi(x)$ is a formula with only 'x' free that implies $\psi_k(x)$ for each k. Let n be even such that ϕ is a $(\leq n)$-formula. (I.e., ϕ is an n'-formula for some $n' \leq n$.)

Now assume $\phi(x)$ and consider $\bigcup^n x$. If σ is any permutation at all then, if we let σ act on $\bigcup^n x$ to obtain $(j^n \sigma)(x)$, we must have $\phi((j^n\sigma)(x))$. We will design σ with care so as to obtain a contradiction.

Let X be a set at the same level as $\bigcup^n x$. Consider the game $G'(X)$ which is like $G(x)$ except that player `I` is trying to ensure that `II`'s last move (this is why we want n to be even, so that `II` moves last) is a member of X. This $G'(X)$ is a finite game of perfect information so one of the two players must have a winning strategy. Observe that $\{y : \emptyset \in y\}$ is a moiety,[8] and consists entirely of things that are wins for player `I`. This means that if we can find a set $X \subseteq \bigcup^n x$ such that (i) player `I` has a winning strategy in $G'(X)$ and (ii) X has a superset X' that is a moiety then we take σ to be any permutation mapping X' onto $\{y : \emptyset \in y\}$. Then we have $\phi((j^n\sigma)(x))$, but nevertheless `I` Wins $G((j^n\sigma)(x))$.

Do we want to move the definition of moiety?

But suppose there is no such X ... what is our strategy then? In this situation, whenever $X \subseteq \bigcup^n x$ is a set big enough for `I` to be able to force `II` to land in it when they play $G'(X)$ then X is too big to be mapped into $\{y : \emptyset \in y\}$ by any permutation. The obvious thing to do is to try to find $X \subseteq \bigcup^n x$ s.t. `II` can play to land in X and there is a permutation mapping X onto a set consisting entirely of wins for player `II`. The difficulty with this is that wins for `II` are pretty thin on the ground, as this table of games in the canonical model shows:

	$\|V\|$	$\|$`I`$\|$	$\|$`II`$\|$	$\|$`I`$\|/\|$`II`$\|$
1	1	0	1	
2	2	1	1	1
3	4	2	2	1
4	16	12	4	3
5	65536	61440	4096	15
6	2^{65536}	$2^{65536} - 2^{61140}$	2^{61140}	$\sim 2^{4096}$

So we have to assume the axiom of infinity.

Observe that, whenever we partition $\bigcup^n x$ into $X_1 \sqcup X_2$, either `I` has a strategy to force `II` to land in X_1, or `II` has a strategy for her to land in X_2. In this second (bad) case we are considering, whenever `I` has a strategy to force `II` to land in X_1, we find that X_1 is too large to have a superset that is a moiety. But in these circumstances X_2 is reassuringly small. All we need is that there should be a moiety M that consists entirely of wins for player `II`, for then we take our σ to be the permutation that sends everything in X_2 into M, and sends $X_1 \cup (V \setminus \bigcup^n x)$ onto $(V \setminus M) \cup (M \setminus \sigma``X_2)$.

But this is easy. We noted above that $\{y : \emptyset \in y\}$ is a moiety, and consists entirely of things that are wins for player `I`. Its power set is also a moiety, consisting entirely of things that are wins for player `II` and it is the M that we want. ■

So TℤTI locally omits the type that says there is an x s.t. for every n,

[8]A moiety is a set x such that $|x| = |V \setminus x| = |V|$.

neither I nor II has a strategy to win in n moves. We now apply Omitting Types to conclude

Corollary 3
TℤTI has models in which, for every inhabitant x, there is a concrete n such that one of the two players I *and* II *has a strategy to win $G(x)$ within n moves.*

The winning strategies given by corollary 3 can actually be encoded inside the model, in the sense that "I or II has a strategy to Win $G(x)$ within k steps" (k concrete) is a formula of $\mathcal{L}(\mathrm{T}\mathbb{Z}\mathrm{T})$.

Corollary 3 is actually stronger than saying that there is a model that satisfies $\in$-determinacy.

Nathan Bowler has pointed out to me that the assumption that $\bigcup^n x = V$ is not enough to imply that one of I and II must have a strategy to win in $n+1$ moves; it isn't even enough to imply that $G(x)$ is determinate. If x is a nondeterminate set with $\bigcup^n x \neq V$ then $y = x \cup \{V\}$ is also nondeterminate, but $\bigcup^n y = V$. Why is y nondeterminate? Because giving I the extra possibility of picking V is clearly no use to him (II will pounce on $\emptyset$); and, if II can't win $G(x)$, then she's not going to be able to win $G(x \cup \{V\})$ either because I is not going to help her out by picking V!

So the fact that TℤTI has models which satisfy $\in$-determinacy is stronger than the fact (corollary 2) that it has a model in which, for every element x, there is a concrete natural number n such that $\bigcup^n x$ is either V or $\emptyset$.

It may be that a refinement of this proof will obviate the need to assume the axiom of infinity.

6.3 Quine Atoms (sort-of)

It's pretty clear that TℤT locally omits the 1-type that says that, for every n, x is $\iota^n(y)$ for some y. But, by the same token, it locally omits the 1-type that says that, for every n, x is $B^n(y)$ for some y. What is this telling us? It says that if, for each n, f_n is an operation that lifts levels by n, then TℤT locally omits the 1-type that says that, for every n, x is $f_n(y)$ for some y. Expand and explain

Now... there are $\aleph_0$ such types so presumably we can omit them all simultaneously.

6.4 Strong Extensionality

We can frame a strong extensionality axiom by means of games but that doesn't work well here, since player $=$ can always win the game $G(V_i = V_i \setminus \{V_{i-1}\})$. Player $\neq$ has to play V_{i-1} from V_i (it's his only chance) and player $=$ can reply with $V_{i-1} \setminus \{V_{i-2}\}$ and they are in the same situation as before. Since we are in TℤT we can descend through the levels for ever.

This is clearly not the way to go; a much more idiomatic version of strong extensionality for TℤT is:

DEFINITION 6
A model $\mathfrak{M}$ of $T\mathbb{Z}T$ satisfies strong extensionality iff it omits the 2-type Σ:

$$\{x \neq y\} \cup \{x \sim_n y : n \in \mathbb{N}\}. \tag{Σ}$$

Equivalently:
In $\mathfrak{M}$, for every x, the intersection of all the $[x]_n$ is $\{x\}$.

(The connection with $G(x = y)$ is that any pair x, y that realises this type will give a game $G(x = y)$ where, even though $x \neq y$, player $=$ can stave off defeat for as long as she wishes—though perhaps not for ω steps.)

Strong Extensionality is a nontrivial condition. Clearly one can use compactness to obtain models containing two distinct objects a and b such that $\bigcup^n a$ and $\bigcup^n b$ are singletons for all n, and this will violate strong extensionality.

On the other hand any symmetric model satisfies strong extensionality. As things stand, we do not know of any symmetric models of TℤT, and if there are any they refute choice, whereas—as we will now show—there are strongly extensional models of TℤT+ AC. In fact, *every* model of TℤT is elementarily equivalent to a strongly extensional model.

THEOREM 6 *Every model of $T\mathbb{Z}T$ is elementarily equivalent to a strongly extensional model.*

Proof:
We work in TℤT.

Suppose ϕ is a formula with two free variables, both at level 0, and it supports Σ. We may assume that ϕ is an equivalence relation. We will show that it must be the identity relation, which is as much as to say that no predicate supports Σ. The choice of level 0 is for ease of exposition: the argument has to be run at all levels simultaneously. The idea of the proof is to obtain conflicting information about the size of the equivalence classes.

The equivalence classes $[x]_\phi$ are sumsets of unions of $(n+1)$-orbits (i.e., orbits under the action of $\mathrm{Symm}(V_{-n})$) of pairs, where $-n$ is the lowest level mentioned in ϕ): $[x]_\phi = \bigcup\{\{x, y\} : \phi(x, y)\}$, and $\{\{x, y\} : \phi(x, y)\}$ is a union of orbits on pairs. By Bowler-Forster [1], if these orbits are not singletons they must be of size something like $T^n|\mathrm{Symm}(V_{-n})|$, and the same goes for their sumsets. This means that the equivalence classes are large.

On the other hand ϕ supports Σ, so each equivalence class $[x]_\phi$ is, for each k, included in a $\mathrm{Symm}(V_{-k})$-orbit. How big are these orbits? Each $\mathrm{Symm}(V_{-k})$-orbit is a surjective image of $\mathrm{Symm}(V_{-k})$ and its size is therefore bounded above by something like $T^k|\mathrm{Symm}(V_{-k})|$, and this gets small as k gets bigger.

Thus every extension of TℤT locally omits Σ. So, if $\mathfrak{M} \models$ TℤT, $\mathrm{Th}(\mathfrak{M})$ locally omits Σ and has a model that omits Σ. Such a model is a strongly extensional model of TℤT that is elementarily equivalent to $\mathfrak{M}$.

■

Have we proved, *en passant*, that every consistent extension of TℤT locally omits the type that denies strong extensionality..?

References

[1] Nathan Bowler and Thomas Forster, "Normal Subgroups of infinite Symmetric Groups, with an Application to stratified Set Theory". Journal of Symbolic Logic **74** (march 2009) pp 17–26.

[2] John Conway: "On Numbers and Games". Academic Press

[3] Rudolf Carnap "Abriss der Logistik" Schriften zur Wissenschaftlichen Weltauffassung **2**, Wien, Julius Springer 1929, 114pp.

[4] Anuj Dawar, Zachiri McKenzie and Thomas Forster. "Decidable Fragments of the Simple Theory of Types with Infinity and NF" Notre Dame J. Formal Logic **58** Number 3 (2017), 433–451.

[5] Olivier Esser and Thomas Forster "Relaxing Stratification". *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), pp. 247–258.

[6] Thomas Forster, "Term Models for a weak Set Theory with a Universal Set" Journal of Symbolic Logic **52** (1987) pp 374–387. (Reprinted in Føllesdal, ed: Philosophy of Quine, V. Logic, Modality and Philosophy of Mathematics).

[7] Thomas Forster "Axiomatising Set Theory with a Universal Set", *in* "La Theorie des ensembles de Quine" Cahiers du Centre de Logique **4**, CABAY, Louvain-La-Neuve 1983.

[8] Thomas Forster "N.F." Ph.D. thesis, Cambridge 1976 `http://www.dspace.cam.ac.uk/handle/1810/223940`

[9] Thomas Forster "Games Played on an Illfounded Membership Relation" in *A Tribute to Maurice Boffa* ed Crabbé, Point, and Michaux. Supplement to the December 2001 number of the Bulletin of the Belgian Mathematical Society.

[10] Thomas Forster "A Consistent second-order Theory with no (second-order) Models". Zeitschrift für Mathematische Logic und Grundlagen der Mathematik **35** (1989) pp 385-6.

[11] Thomas Forster "Set Theory with a Universal Set". OUP second edition.

[12] Henson, C.W. "Permutation Methods applied to NF". Journal of Symbolic Logic **38** pp. 69−76.

[13] W.V. Quine "New Foundations for Mathematical Logic." American Mathematical Monthly 1937 reprinted in FRALPOV

[14] W.V. Quine, "Set Theory and Its Logic" Harvard, the Belknap Press 1963.

[15] J Barkley Rosser "Logic for Mathematicians" second edition Chelsea.

[16] J Barkley Rosser and Hao Wang. "Non-Standard Models for Formal Logics" Journal of Symbolic Logic **15**, Issue 2 (1950), 113–129.

[17] Panagiotis Rouvelas: "Increasing Sentences in Simple Type Theory". Ann. Pure Appl. Logic **168**(10): 1902–1926 (2017)

[18] Panagiotis Rouvelas: "Decreasing Sentences in Simple Type Theory". Math. Log. Q. **63**(5): 342–363 (2017)

[19] D. I. Savelieff "A Game played on the Universe of Sets". Izvestiya Mathematics **72** (part 3) (2008) pp 581–625.

[20] Ernst Specker "The Axiom of Choice in Quine's New Foundations for Mathematical Logic". Proceedings of the National Academy of Sciences of the USA **39** pp. 972−5.

[21] Ernst Specker "Dualität" Dialectica **12** pp. 451−465. See the English translation on: `https://www.dpmms.cam.ac.uk/~tf/dualityquinevolume.pdf`

[22] Ernst Specker "Typical Ambiguity". In *Logic, Methodology and Philosophy of Science*. ed E. Nagel, Stanford.

[23] Hao Wang, "Negative Types" MIND **61** pp. 366−8.