An Introduction to WQO and BQO Theory

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0.1 Lists and streams

The case that will be of most concern to us is where I is an initial segment of $\mathbb{N}$ and where we require f to be **injective**. In this case the relation given by definition **??** is written $\leq_l$. We met ω-sequences from Q on page **??** where they were called Q-**streams**, and finite sequences similarly were Q-**lists**. I shall write the set of Q-streams as Q^ω and the set of Q-lists as $Q^{<\omega}$. Following Mathias (oral tradition) we use the word **stretching** to denote the relation that holds between two Q-lists (or Q-streams) l_1 and l_2 if there is a 1-1 increasing map f from the addresses of l_1 to the addresses of l_2 such that for all addresses a, $a \leq f(a)$. That is to say: think of a (Q-list or a) Q-stream as a map from a (proper) inititial segment of $\mathbb{N}$ to Q. Then f_1 stretches into f_2 iff there is a strictly increasing $f : \mathbb{N} \hookrightarrow \mathbb{N}$ such that .... This relation was probably first identified by Higman ([**?**]) but i'm not sure.

We write this '$l_1 \leq_l l_2$' with a subscript 'l' for '$l$ist', and we say l_1 **stretches** into l_2.

The stretching relation on Q-lists is inductively defined as the $\subseteq$-smallest set of ordered pairs of Q-lists containing $\langle \texttt{nil}, \texttt{nil} \rangle$ and containing $\langle l_1, l_2 \rangle$ if it contains $\langle l_1, \texttt{tl}(l_2) \rangle$, or if $\texttt{hd}(l_1) \leq \texttt{hd}(l_2)$ and it contains $\langle \texttt{tl}(l_1), \texttt{tl}(l_2) \rangle$.

The stretching relation on Q-streams is coinductively defined as the $\subseteq$-largest relation $R \subseteq Q^\omega \times Q^\omega$ such that $R(l_1, l_2) \longleftrightarrow ((\texttt{hd}(l_1) \leq_Q \texttt{hd}(l_2) \wedge R(\texttt{tl}(l_1), \texttt{tl}(l_2))) \vee R(l_1, \texttt{tl}(l_2)))$.

No list stretches into its tail: for all lists l over a quasiorder, $l >_l \texttt{tl}(l)$. $l \geq_l \texttt{tl}(l)$ by the second clause in the recursive definition of $\leq_l$, and we prove by induction that no list can $\leq_l$ something strictly shorter than itself. In contrast to lists, a stream might stretch into its tail.

0.2 Finite and infinite trees

There are also finite and infinite trees over X, and we need to think how to lift quasiorders of X to trees over X.

There is of course an inductive definition for tree-stretching for finite trees.

Definition 1 *$T_a \leq_t T_b$ if*

- *Both are singleton trees $\{a\}$ and $\{b\}$ with $a \leq b$; or*
- *$T_a \leq_t$ some child of T_b; or*
- *The root of $T_a \leq$ root of T_b and the list of children of $T_a \leq_l$ list of children of T_b.*

Proposition 2 *If $\langle Q, \leq \rangle$ is a wellfounded quasiorder, then Q-lists are wellfounded under stretching.*

Proof: Suppose not, and we had an infinite descending sequence of Q-lists under stretching. They can get shorter only finitely often, so without loss of generality we may assume that they are all the same length. But the entries at each coefficient can get smaller only finitely often, so they must eventually be constant. ■

Proposition 3 *If $\langle Q, \leq \rangle$ is a wellfounded quasiorder, then finite Q-trees are wellfounded under tree-embedding.*

Proof: Suppose $\langle Q, \leq \rangle$ is a wellfounded quasiorder and let $\langle t_i : i < \omega \rangle$ be a descending $>_t$-sequence of Q-trees. We will derive a contradiction. The number of children of t_i is a nonincreasing function of i and must be eventually constant: indeed the trees will be of eventually constant shape, and we can delete the initial segment of the sequence where they are settling down. Because the shape is eventually constant there are unique maps at each stage, so for any one address the sequence of elements appearing at that address gets smaller as i gets bigger. ■

PROPOSITION 4 *If $\langle Q, \leq\rangle$ is a wellfounded quasiorder then $\langle \mathcal{P}_{\aleph_0}(Q), \leq^+\rangle$ is wellfounded.*

Proof: Suppose we have an infinitely descending sequence $\langle Q_i : i \in \mathbb{N}\rangle$ of finite subsets of Q under $<^+$. Without loss of generality we can assume that all the Q_i are antichains, by throwing away from each Q_i all elements that are not maximal. This will ensure that any x that appears in both Q_i and in Q_j with $j > i$ must appear in all intermediate levels: if $x \in Q_j$ then it must be $\leq$ something in Q_{j-1} and so on up to Q_i. Since Q_i is an antichain this thing can only be x itself (or something equivalent to it, which will do!) So any x that appears in infinitely many Q_i must appear in cofinitely many of them. But then it can be deleted altogether. So we can assume that each q appears in at most finitely many Q_i.

Nathan queries this: tidy it up

For each $x \in Q_0$ we can build a tree whose paths are sequences s where the ith representative comes from Q_i and for all i, $s(i+1) \leq s(i)$. We need to show that all these paths are finite. If they were not, they would have to be eventually constant, and we have just seen that we can assume that each q can be assumed to appear only finitely often. So the tree whose paths are these sequences is a finite branching tree all of whose paths are finite, so it has only finitely many levels. But there are only finitely many things in Q_0, so eventually the Q_i are empty. ■

Actually we can prove a great deal more than this.

EXERCISE 1 *Define a family of quasiorders of $\mathcal{P}_{\aleph_0}(Q)$ by setting*

$$X \leq_1 Y \text{ if } X \leq^+ Y \text{ and}$$

$$Y \leq_n X \quad \text{iff} \quad (\forall y \in Y)(\exists x \in X)(y \leq x \quad \wedge \quad (Y \setminus \{y\} \leq_{n-1} X \setminus \{x\}))$$

thereafter.

Prove that $\leq_n$ is wellfounded if $\leq$ is.

1 WQOs

It seems that wellfounded quasiorders without infinite antichains are going to be objects of interest, since it seems that—and we will prove this in remark 9—it is the absence of infinite antichains in a wellfounded quasiorder $\langle Q, \leq\rangle$ that enables us to show that $\langle \mathcal{P}(Q), \leq^+\rangle$ is wellfounded.

This motivates the following definition:

DEFINITION 5 *$\langle Q, \leq\rangle$ is a* **wellquasiorder** *(hereafter "WQO") iff whenever $f : \mathbb{N} \to Q$ is an infinite sequence of elements from Q then there are $i < j \in \mathbb{N}$ s.t. $f(i) \leq f(j)$.*

A natural example of a WQO is the set of (unordered) pairs of natural numbers with $\{x, y\}$ related to $\{n, m\}$ if $\max(x, y) \leq \min(n, m)$. It is the wellfoundedness of this quasiorder that ensures termination of Euclid's algorithm. (For the moment i shall continue to use both notations for sequences)

Definition 6 *A* **bad sequence** *(over $\langle Q, \leq \rangle$) is a sequence $\langle x_i : i \in \mathbb{N} \rangle$ such that for no $i < j$ is it the case that $x_i \leq x_j$. A sequence that is not bad is* **good***. A sequence $\langle x_n : n \in \mathbb{N} \rangle$ is* **perfect** *if $i \leq j \to x_i \leq x_j$.*

Finite sequences $\langle x_i : i < k \in \mathbb{N} \rangle$ too will sometimes be said to be **bad** *as long as they satisfy the remaining condition: $i < j < k \to x_i \not\leq x_j$.*

Thus a wellquasiorder is a quasiorder with no bad sequences. With the help of Ramsey's theorem we can prove that in a WQO not only is every sequence good but that it must have a perfect subsequence. (Notice that this is not the same as saying that in any quasiorder every good sequence has a perfect subsequence!)

Lemma 7 *In a WQO every sequence has a perfect subsequence.*

Proof:

Although this theorem is very easy to prove, the usual (indeed only) proof using Ramsey's theorem is so natural and idiomatic, and so important *qua* prototype for so many other applications of Ramsey's theorem, that it is worth doing in full.

Let $\langle Q, \leq_Q \rangle$ be a WQO, and $f : \mathbb{N} \to Q$ a sequence. Partition $[\mathbb{N}]^2$ into the two pieces $\{\{i < j\} : f(i) \leq_Q f(j)\}$ and $\{\{i < j\} : f(i) \not\leq_Q f(j)\}$. An infinite subset monochromatic for the first piece would give us a bad sequence, contradicting the assumption that $\langle Q, \leq_Q \rangle$ was a WQO, and the set monochromatic for the second piece is a perfect subsequence. ■

A quibble: a set monochromatic for the first piece would be an infinite subset $X \subseteq \mathbb{N}$ such that whenever $i < j$, both in X, we have $f(i) \not\leq_Q f(j)$. Now this is not *literally* a bad sequence, since a bad sequence is *inter alia* a function defined on $\mathbb{N}$ not on an infinite subset of it. What we have just seen is a situation where we have to do a bit of renumbering of elements of an index set in order to make a claim literally true. This particular case is so trivial that one hardly notices one is doing it, and there would appear to be nothing to be gained (at the time) by flagging it, but we will find later examples where we really have to be explicit about it.

A quasiorder is a WQO iff the strict version of the corresponding partial order is wellfounded and has no infinite antichains. (miniexercise) Notice this does *not* mean that for each x in a WQO there are only finitely many things incomparable with x, nor even that there are only finitely many equivalence classes of things incomparable with x. What it does say is that if there are infinitely many things incomparable with x, some of them will be comparable with some others.

[*HOLE Is this the right place to make the point that the rank of $\langle \mathcal{P}(X), \leq^+ \rangle$ cannot be much bigger than the rank of $\langle X, \leq \rangle$. After all, every subset of X*

has a finite set of minimal elements, and to compare two subsets X' and X'' of X wrt $\leq^+$ it is sufficient to compare their finite bases. Of course you need DC to choose a basis!]

Now some basic facts about WQO's, some with an algebraic flavour.

Proposition 8

(i) Substructures of WQOs are WQO;

(ii) Homomorphic images of WQOs are WQO;

(iii) The pointwise product of finitely many WQOs is WQO;

(iv) The intersection of finitely many WQOs is WQO;

(v) Disjoint unions of finitely many WQO are WQO;

(vi) If $\leq_1$ and $\leq_2$ are both quasiorders of a set Q, and the graph of $\leq_1$ is a subset of the graph of $\leq_2$, and $\leq_1$ is a WQO, then so is $\leq_2$.

Proof:

(i). Any bad sequence in a substructure is a bad sequence in the whole structure.

(ii). Suppose $f : \langle Q, \leq \rangle \to \langle X, \leq \rangle$ is a quasiorder homomorphism and $S : \mathbb{N} \to X$ a bad sequence of members of X. Consider $Q^\dagger = \{q \in Q : (\exists n \in \mathbb{N})(f(q) = S(n)\}$. (We are of course assuming that f is a surjective homomorphism). Let R be the binary relation $R(q, q')$ iff $(\exists n \in \mathbb{N})(S(n) = f(q) \wedge S(n+1) = f(q'))$. $R \subseteq (Q^\dagger \times Q^\dagger)$ satisfies the conditions for the application of *DC* (in that $(\forall q \in Q^\dagger)(\exists q' \in Q^\dagger)(R(q, q'))$, and the output sequence will be a bad sequence of members of Q.

For (iii) (iv) and (v) it is clearly sufficient to deal with the case of two WQOs. The proofs of all three use Ramsey's theorem with exponent 2, or the perfect subsequence lemma (lemma 7). For (iii) consider the product of two WQOs $\langle Q, \leq_Q \rangle$ and $\langle X, \leq_X \rangle$, and suppose we have a bad sequence $\langle \langle x_i, q_i \rangle : i \in \mathbb{N} \rangle$. By the perfect subsequence lemma there must be an infinite $I \subseteq \mathbb{N}$ such that for $i < j$ both in I we have $x_i \leq_X x_j$. Now consider the sequence of q_i for $i \in I$. This must be a good sequence, since $\langle Q, \leq_Q \rangle$ is WQO, so there are $i < j$ both in I with $q_i \leq_Q q_j$. So $\langle \langle x_i, q_i \rangle : i \in \mathbb{N} \rangle$ was not bad.

The proofs of (iv) and (v) are almost exactly the same. Notice that (iv) tells us in particular the if $\leq_1$ and $\leq_2$ are two WQOs of one carrier set, then the intersection of their graphs is the graph of a WQO of the same set.

Finally (vi) is obvious, but—since it will be generalised later—a bit of detail may be helpful. A quasiorder is a WQO if the complement of its graph does not contain a copy of $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$. This property of (the graph of) a relation is clearly preserved under superset.

■

We have already checked that if $\langle Q, \leq \rangle$ is a quasiorder, so is $\langle \mathcal{P}(Q), \leq^+ \rangle$.

Lemma 9 *Let $\langle Q, \leq \rangle$ be a quasiorder. Then the following are equivalent*

1. *$\langle Q, \leq \rangle$ is WQO;*

2. $\langle \mathcal{P}(Q), \leq^{+} \rangle$ *is wellfounded;*

3. $\langle \mathcal{P}_{\aleph_1}(Q), \leq^{+} \rangle$ *is wellfounded.*

Proof:

(ii) obviously implies (iii).

(iii) $\rightarrow$ (i)

If $\langle q_i : i \in \mathbb{N} \rangle$ is a bad sequence then define $Q_i =: \{q_j : j > i\}$ for each $i \in \mathbb{N}$. Then $\langle Q_i : i \in \mathbb{N} \rangle$ is a $<^{+}$-descending sequence of countable subsets of Q.

(i) $\rightarrow$ (ii)

We will actually prove something slightly more refined, namely that if $\langle Q, \leq \rangle$ is wellfounded, and that $Q_0 >^{+} Q_1 >^{+} \ldots$ is a $>^{+}$ descending chain of subsets of Q, there is an infinite antichain $\subseteq Q$. This will be sufficient to establish (i) $\rightarrow$ (ii).

For each i pick $q_i \in Q_i \not\leq$ anything in Q_{i+1}. So in particular, we immediately have $q_i \not\leq q_{i+1}$. But since $Q_j <^{+} Q_i$ for $j > i$ it follows that if $j > i$ we cannot have $q_i \leq q_j$ since q_j must be less than something in Q_{i+1}, and q_i has been chosen not to be $\leq$ anything in q_{i+1}. An application of Ramsey's theorem to the set $\{q_i : i \in \mathbb{N}\}$ gives either a set of representatives which form an infinite descending sequence under $<$, which is impossible by wellfoundedness, or an antichain, which was what we wanted.

■

The set-of-countable-subsets constructor behaves like the (full) power-set constructor. This is an important fact which will be very useful to us later, as it will enable us to substitute this constructor (which *does* have fixed points) for the power set constructor (which famously does *not* have fixed points). But this is actually a special case of:

1.1 The Minimal Bad Sequence construction

There is a well-defined notion of the *wellfounded part* of a quasiorder $\langle X, \leq_X \rangle$ that we saw in definition **??**. Is there analogously a notion of the WQO part of a (wellfounded) quasiorder? If $\langle X, \leq_X \rangle$ is wellfounded but not WQO then $\langle \mathcal{P}(X), \leq^{+} \rangle$ is not wellfounded. However, it will have a wellfounded part. Does this give rise to a concept of the WQO part of $\langle X, \leq_X \rangle$? It might seem that a sensible thing to say might be that the WQO part of $\langle X, \leq_X \rangle$ is the set of those x such that $X \setminus \uparrow \{x\}$ is in the wellfounded part of $\langle \mathcal{P}(X), \leq^{+}_X \rangle$. Or again that the WQO part of $\langle X, \leq_X \rangle$ is the set of those x such that $\downarrow \{x\}$ is in the wellfounded part of $\langle \mathcal{P}(X), \leq^{+} \rangle$. However, neither of these will work, as the example of the quasiorder $\langle \mathbb{N}, = \rangle$ shows. For this quasiorder the set of those x such that $X \setminus \uparrow \{x\}$ is in the wellfounded part of $\langle \mathcal{P}(X), \leq^{+}_X \rangle$ is $\emptyset$. This is unduly restrictive, since any finite subset of $\mathbb{N}$ is WQO by $=$; on the other hand the set of those x such that $\downarrow \{x\}$ is in the wellfounded part of $\langle \mathcal{P}(X), \leq^{+} \rangle$ is $\mathbb{N}$, which is not WQO by $=$.

Although there is no good notion of *the* WQO part of a relation, there is an ingenious construction which will do some of the work to which we would have put such a notion had there been one. Any quasiorder that is wellfounded but is not WQO has bad sequences, and—as we shall see—has some that are in some sense minimal. This "minimal bad sequence" is a key idea, and its significance for us here is that the set of things below such a minimal bad sequence in a wellfounded quasiorder $\langle X, \leq \rangle$ behaves in some ways as if it were the WQO part of $\langle X, \leq \rangle$. A precise definition will be given later[1]: for the moment our approach is a two pronged one: (i) How do we make one? (ii) What can it do for us once we have got it?[2]

Let $\langle X, \leq_X \rangle$ be a wellfounded quasi order that is not WQO.

Let A be the set of proper initial segments of bad X-sequences, and let $R(s, t)$ hold if t is an end-extension of s by one element x, with x minimal so that s with x on the end is in A. Notice that $(\forall s \in A)(\exists t \in A)(R(s, t))$ so the conditions of DC apply.

Let x_0 be a minimal member of $\{x :$ there is a bad sequence whose first member is $x\}$. Let x_{n+1} thereafter be a minimal member of $\{x :$ there is a bad sequence whose first n members are $\langle x_0 \ldots x_{n-1} \rangle$ and whose $n+1$th member is is $x\}$. Let us say that a sequence constructed by this algorithm is an **MBS**.

The following remark is not needed just yet but crops up naturally here. If we topologise Q^ω in the usual way by giving Q the discrete topology and Q^ω the product topology, we find that if Q is a quasiorder that is not WQO then the set of bad sequences is a closed subset of Q^ω in the product topology. That is why the MBS algorithm—which is a greedy algorithm—works. The set of its outputs is a closed set.

Remark 10 *If Q is a QO that is wellfounded but not WQO then the set of MBSs is a closed subset of Q^ω.*

Observe that although every subsequence of an MBS is bad (obviously!) it is not the case that every subsequence of an MBS is an MBS. I am endebted to Stijn Vermeeren for this example.

Quasiorder $\{0, 1\} \times \mathbb{N}$ by

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } \langle a, b \rangle = \langle c, d \rangle \text{ or } a = c \text{ and } b = 0.$$

This is well-founded, but not WQO, and the following is an MBS:

$$\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 1, 4 \rangle, \ldots \tag{1}$$

But the subsequence

$$\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 1, 4 \rangle, \ldots$$

of (1) is not an MBS, since $\langle 1, 1 \rangle$ is not minimal among the pairs that kick off a bad sequence, since

[1] By the reader!

[2] "Where does she live;
what's her number
and how can I meet her?? - The Everleys

$$\langle 1,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 0,4\rangle, \ldots$$

is a bad sequence that begins with $\langle 1,0\rangle$—and $\langle 1,0\rangle \leq \langle 1,1\rangle$.

Fortunately we do not need every subsequence of an MBS to be an MBS. We need every such subsequence to be bad, and we also need every sequence which is in some suitable sense below an MBS to be good.

Lemma 11 *Let $\langle X, \leq\rangle$ be a wellfounded quasiorder that is not a WQO and $B = \langle b_0, b_1 \ldots\rangle$ be an MBS. Let $X' = \{x \in X : (\exists n)(x < b_n)\}$. Then $\langle X', \leq\rangle$ is WQO.*

Proof: Suppose $S = \langle s_0, s_1 \ldots\rangle$ is a bad sequence from X'. We will prove by induction on $\mathbb{N}$ that nothing in S is below b_n. This is clearly true for $n = 0$, as follows. If $s_i < b_0$ then the tail of S starting at s_i is a bad sequence beginning with something less than b_0 contradicting minimality of b_0.

For the induction suppose that nothing in S is below any of $b_1 \ldots b_n$. Suppose now that, *per impossibile*, there were $s_i < b_{n+1}$. Consider the sequence that begins $b_0 \ldots b_n$ and continues $s_i, s_{i+1} \ldots$. It can't be bad, because b_{n+1} was minimal among the set of elements that are the are $n+1$th members of bad sequences beginning $b_0 \ldots b_n$, and $s_i < b_{n+1}$ rules s_i out as a candidate. So this sequence contains a good pair. Both S and B are bad, so the good pair must be a $b_j \leq s_k$ with $j \leq n$ and $k \geq i$. Now consider s_k. It is in X' so there is a b_m in B with $s_k < b_m$. This m cannot be $\leq n$, by induction hypothesis, so we must have $m > n$. But then we have $j \leq n < m$ with $b_j \leq b_m$ (in fact $b_j < b_m$) contradicting badness of B. ■

2 Kruskal's theorem

Next we show that (finite) lists over a WQO are WQO.

Lemma 12 *If $\langle X, \leq\rangle$ is a WQO, so is $\langle X^{<\omega}, \leq_l\rangle$.*

Proof: We use *reductio ad absurdum*. Suppose that $\langle X, \leq\rangle$ is a WQO but that $\langle X^{<\omega}, \leq_l\rangle$ is not. We know by now from proposition 2 that $\langle X^{<\omega}, \leq_l\rangle$ is wellfounded, so let us construct a minimal bad sequence $\langle a_i : i \in \mathbb{N}\rangle$ of lists. Look at the heads of the lists in the minimal bad sequence. These are WQO by hypothesis so (by lemma 7) there must be an infinite subsequence $\langle b_i : i \in \mathbb{N}\rangle$ of $\langle a_i : i \in \mathbb{N}\rangle$ such that for $i < j$, $\mathtt{hd}(b_i) \leq \mathtt{hd}(b_j)$. Throw away all the other lists in this bad sequence. We now have a bad sequence of lists whose heads, at least, form an increasing sequence. Now consider the tails. We want to show that the tails are WQO as well, for that will complete the proof for us by using the third clause of definition 0.1. We know from page **??** that $\mathtt{tl}(l) <_l l$ always, so these tails belong to a collection of things *below* this minimal bad sequence, $\langle a_i : i \in \mathbb{N}\rangle$, in the sense of lemma 11. Therefore the sequence of tails of elements of $\langle b_i : i \in \mathbb{N}\rangle$ is not a bad sequence. So there are $i < j$ such that $\mathtt{tl}(b_i) \leq_l \mathtt{tl}(b_j)$. Therefore (by the third clause in the inductive definition of $\leq_l$) $b_i \leq b_j$, so $\langle b_i : i \in \mathbb{N} \rangle$ is not a bad sequence, and $\langle a_i : i \in \mathbb{N} \rangle$ is not bad either. ■

Now we can prove

THEOREM 13 *(Kruskal) Finite trees over a WQO are WQO.*

Proof: By wellfoundedness of $<_t$, if there is a bad sequence there is a minimal bad sequence, and let $\langle a_i : i \in \mathbb{N} \rangle$ be one. Look at the roots of the trees. Since the roots are from a WQO there must be an increasing ω-subsequence $\langle b_i : i \in \mathbb{N} \rangle$ from $\langle a_i : i \in \mathbb{N} \rangle$ such that if $i < j$ then (root of b_i) $\leq$ (root of b_j) (this was lemma 7). Let l_i be the list of children of a_i.

We know that the roots of the a_i form a strictly increasing sequence. What we now have to look at is a countable sequence of lists of children of the trees we started with. These trees form a collection of trees below (in the sense of lemma 11) the minimal bad sequence we started with. So, by lemma 11 they are WQO, so lists over them are WQO as well. Therefore there are $i < j$ with $l_i \leq_l l_j$, so (by the third clause in the definition of $\leq_t$) it follows that $a_i \leq_t a_j$. Thus $\langle a_i : i \in \mathbb{N} \rangle$ is not bad.

■

this needs to be revised in the light of the treatment of MBSs

2.1 Ranks again

We should never forget that ordinals first came to the attention of mathematicians as that-kind-of-number-that-measures-the-length-of-transfinite-processes.

Nobody ever lives long enough to execute a transfinite process but, if even the immortal creatures that do might have the same interest in economy and despatch that finite beings do. If we are trying to define—by recursion on a wellfounded relation R—a function f defined on the domain of R, then the rank of R is the ordinal that is an absolute lower bound on the number of stages in the computation of all the values of f. We can compute $f(x)$ at stage $\rho(x)$ but not before. And we achieve this lower bound by making maximal possible use of parallelism—the ability to compute f simultaneously for all arguments of the same rank. However we might also be interested in spinning out the computation as long as possible, by *not* processing all arguments of the same rank simultaneously, but one after the other.

How long might we take if we make no use of parallelism at all? If we have an infinite antichain then we have a countably infinite set of arguments that can be wellordered to the length of any countable ordinal α, and if we take its members in that order we can clearly take at least α steps. This means that if there is an infinite antichain there is no countable bound on the time we can take. The converse is true too.

> *HOLE Fit this in in the correct place: The less parsimonious the map from a qoset to the ordinals the more distinctions one makes between the things in the domain, so the more informative the map is*

Remark 14 *Let $\langle X, \leq_X\rangle$ be a countable QO. Then the following are equivalent:*

1. *$\langle X, \leq_X\rangle$ is WQO;*
2. *The set $\{\alpha : \exists$ homomorphism $\pi : \langle X, \leq_X\rangle \twoheadrightarrow \langle\{\beta : \beta < \alpha\}, <\rangle\}$ is bounded below ω_1.*

Proof:

$\neg$(i) $\to \neg$ (ii) is proved by the observation above about infinite antichains.

(i) $\to$ (ii)

Consider the (downward-branching) tree $\langle B, \not\geq\rangle$ of bad sequences from X ordered by reverse end-extension (each bad sequence is above all its bad end-extensions). $\langle X, \leq_X\rangle$ is a WQO (no infinite bad sequences) so $\langle B, \not\geq\rangle$ is well-founded and therefore has a rank. The carrier set X is countable so the set B is countable, and the rank of $\langle B, \not\geq\rangle$ is therefore countable too. We will show that this rank is an upper bound for $\{\alpha : \exists$ homomorphism $\pi : \langle X, R\rangle \twoheadrightarrow \{\beta : \beta < \alpha\}\}$.

Let π be such a homomorphism. We quasi-order X by the relation $x \leq_\pi y$ iff $\pi(x) \leq \pi(y)$. Evidently (the graph of) $\leq_\pi$ is a superset of (the graph of) $\leq_X$. Accordingly there are fewer bad sequences in $\langle X, \leq_\pi\rangle$ than there are in in $\langle X, \leq_X\rangle$. Consider the tree of descending chains in $\langle X, <_\pi\rangle$ ordered by reverse end-extension in the style of $\langle B, \not\geq\rangle$.

It is easy to see that the rank—in this tree—of any descending sequence is simply π of its last member, so the tree must have rank α. It is also straightforward that any descending chain in $\langle X, <_\pi\rangle$ is a bad sequence in $\langle X, \leq_X\rangle$, so this tree is a subtree of $\langle B, \not\geq\rangle$, so $\langle B, \not\geq\rangle$ has rank at least α too.

This is a characterisation of well relations. ■

Now where in this proof do we use transitivity of $\leq_X$?

H I A T U S

It would be nice if in addition it were to preserve the condition on ω-sequences so that $\langle\mathcal{P}(A), \leq^+\rangle$ is WQO as long as $\langle A, \leq\rangle$ is. We shall see a counterexample due to Rado which will show that the Hoare ordering of the power set of a WQO is not always a WQO. It is natural to ask what extra conditions one has to add to those comprising WQOness to get a property that is preserved under this construction.

First let us suppose that $\langle\mathcal{P}(Q), \leq_Q{}^+\rangle$ is not WQO, and see what implications this has for $\langle Q, \leq_Q\rangle$. We know immediately that there is a bad sequence $\langle Q_i : i \in \mathbb{N}\rangle$ where for $i < j$, $Q_i \not\leq^+_Q Q_j$.

It would be nice if for each i we could pick a member q_i in Q_i to get a bad sequence on Q, but there is no reason to suppose we can. After all, each q_i would have to "do infinitely many things". Later we will see examples where we definitely cannot pick a single q_i in this way.

However, we can at least do the following. $Q_i \not\leq^+ Q_j$ for $i < j$ which is to say that $\neg(\forall q \in Q_i)(\exists q' \in Q_j)(q \leq q')$ so for each pair i, j with $i < j$ we can pick an element $q_{i,j} \in Q_i$ s.t. $(\forall q \in Q_j)(q_{i,j} \not\leq_Q q)$. So, using countable choice we can pick a family of elements of Q indexed by pairs of distinct natural numbers, such that $(\forall i < j < k)(q_{i,j} \not\leq_Q q_{j,k})$. This isn't exactly a bad sequence: it's a thing which we will call a bad *array*, and the definition of 'array' will emerge later. With hindsight, the (bad) sequences we have just encountered will come to be seen to have been merely a special kind of (bad) array. Just as a sequence (of widgets) is a map from $\mathbb{N}$ to widgets, so an array (of widgets) will be a map from a **block** to widgets. We will see the exact definition of block later. For the moment an operational understanding will have to do, and we take as our current interesting example of a block the structure whose carrier set is $\{\{i,j\} : i,j \in \mathbb{N}\}$ equipped with a binary relation $\lhd$ which for all $i < j < k$ relates $\{i,j\}$ to $\{j,k\}$ and to nothing else. The reader should try and think of the bad array that we constructed at the start of this paragraph not as a family of elements of Q with bizarre subscripts but as a map f from the block $\{\{i < j\} : i,j \in \mathbb{N}\}$ to Q such that $(\forall b, b' \in B)(b \lhd b' \to f(b) \not\leq f(b'))$.

Sequences are special kinds of arrays, and the structure $\langle \mathbb{N}, < \rangle$ is a special kind of block. The block we saw in the previous paragraph is the first nontrivial example of a block, and it's a block of a kind that one might call *quadratic*: it is a set of ordered *pairs*, and in the lexicographic order it is of length ω^2. (One can think of *arrays* as quadratic as well, when they are functions defined on quadratic blocks). One could think of the block $\langle \mathbb{N}, < \rangle$ as a *linear* block and take note that it is of length ω in the lexicographic order, but these italicised *aides memoires* are not used formally and I mention them only to help the reader see that $\langle \mathbb{N}, <, \rangle$ and $\langle \{\{i < j\} : i,j \in \mathbb{N}\}, \lhd \rangle$ are creatures of the same kind, but of different lengths. When we go up one stage, as we will soon, we shall see *cubic* blocks. However there is another point that needs to be made at this early stage, before we do that.

Given a bad (quadratic) array on Q we can construct a bad sequence on $\mathcal{P}(Q)$ all of whose elements are countable sets: simply set $Q_i =: \{q_{i,j} : j > i\}$. (The idea here "If there is a bad sequence of subsets there is a bad sequence of *countable* subsets" is the first reappearance of the idea first flagged on page 6.)

Now consider the case where $\langle \mathcal{P}(\mathcal{P}(Q)), (\leq^+)^+ \rangle$ is not WQO. We can do exactly what we did in the case where $\langle \mathcal{P}(Q), {\leq_Q}^+ \rangle$ was not WQO to get a bad array $\{X_{i,j} : i < j \in \mathbb{N}\}$, but this time of course the $X_{i,j}$ are subsets of Q, not elements of Q. So we repeat the process. $X_{i,j} \not\leq^+ X_{j,k}$, so there must be something in $X_{i,j}$ which is $\not\leq$ anything in $X_{j,k}$. We will pick one such and call it $X_{i,j,k}$. Thus we get an analogous condition on increasing triples from Q, namely: $(\forall i < j < k < l)(q_{i,j,k} \not\leq_Q q_{j,k,l})$. This is the condition which fails if $\langle \mathcal{P}^2(Q), (\leq^+)^+ \rangle$ is *not* WQO. This gives us our third example of a block: $\{\{i < j < k\} : i,j,k \in \mathbb{N}\}$. Similarly, given a bad array $\{q_{i,j,k} : i < j < k \in \mathbb{N}\}$ of triples we can get a bad sequence $X_0, X_1 \ldots X_n \ldots$ on $\mathcal{P}^2(Q)$ of countable sets of countable subsets of Q. X_i will be $\{X_{i,j} \subseteq Q : j > i\}$ where $X_{i,j} = \{q_{i,j,k} : k > j > i\}$.

Once the reader is entirely happy with the idea of sifting[3] information about

[3] "I am soft sift in an hourglass, at the wall fast but mined with a motion, a drift ..." *The*

bad sequences in $\mathcal{P}^2(Q)$ or $\mathcal{P}^3(Q)$ to information about bad arrays on Q, they should take on board the idea that this can be done for any finite n.

So the development so far can be summarised as follows.

If $\mathcal{P}^n(Q)$ quasiordered by the result of applying the '+' operation n times to a given quasiorder $\leq_Q$ of Q is not a WQO, then there is a bad $(n+1)$-ary array on Q, which is to say a map f from the set of unordered $n+1$-tuples of natural numbers such that

$$(\forall i_0 < \ldots i_n \in \mathbb{N})(f(\{i_0 \ldots i_{n-1}\}) \not\leq_Q f(\{i_1 \ldots i_n\}))$$

which we discover by sifting.[4]

We showed that from the bad array on Q one can recover a bad sequence on $\mathcal{P}^n(Q)$ whose elements are countable sets of countable sets of^n elements of Q. This is worth minuting.

Proposition 15 *If there is a bad sequence in $\mathcal{P}^n(Q)$ then there is one consisting entirely of (countable sets of)^n elements of Q.*

These sets are **hereditarily countable**.

Proof: We first sift a bad sequence in $\mathcal{P}^n(Q)$ to a bad array of elements of Q, indexed by increasing $n+1$-tuples from $\mathbb{N}$. Then we obtain successively bad arrays on $\mathcal{P}(Q)$, $\mathcal{P}^2(Q)$, and so on by setting Q_s to be $\{Q_t : t = \texttt{butlast}(s)\}$, first for tuples s of length $n-1$, then for tuples s of length $n-2$, and so on up to tuples of length 1, at which point we have a bad sequence of hereditarily countable elements of $\mathcal{P}^n(Q)$. ∎

Our first example of a block was the quadratic block $\{\{i < j\} : i, j \in \mathbb{N}\}$ with the binary relation $\lhd$ which holds between $\{i < j\}$ and $\{j < k\}$. We saw the cubic block too, and its rather more complex definition. Although I am still not planning to give a precise definition of blocks, the reader can see how the process of pulling down a bad sequence in $\mathcal{P}^{n-1}(Q)$ to a bad array on Q gives rise to a block consisting of unordered n-tuples. This block is the **canonical n-block**. A quasiorder that has no bad arrays whose domain (remember an array is a map from a block ...) is the canonical n-block is said to be **ω^n-good**. The ordinal alludes to the length of the canonical n-block in the lexicographic order. (When we give a formal definition of block later it will be an exercise to show that all blocks are wellordered in the lexicographic order.) Thus, in particular, a WQO is a quasiorder that is ω-good.

So we have proved

Proposition 16 *The following are equivalent*

1. *$\langle Q, \leq_Q\rangle$ is ω^n-good;*
2. *$\langle \mathcal{P}^n(Q), (\leq_Q)^{+^n}\rangle$ is ω-good (i.e., WQO);*

Wreck of the Deutschland, Gerald Manley Hopkins.

[4] We are assuming $i_k < i_m$ when $k < m$. That is to say, we are thinking of these objects sometimes as unordered tuples, and sometimes as increasing ordered tuples.

3. $\langle \mathcal{P}^n_{\aleph_1}(Q), (\leq_Q)^{+^n} \rangle$ *is* ω*-good (i.e., WQO).*

(The reader is probably becoming impatient for a proper definition of a block: we will postpone this until we want to make sense of the idea of ω^α-good for $\alpha \geq \omega$. Enthusiasts should for the moment master their impatience and redirect their energies into chewing over the 2-block and attempting exercise **??**.)

The reader should establish, by way of preparation for the harder analogues of infinite exponent that await us, the analogues for ω^n-good quasiorders of the various parts of proposition 8.

Exercise 2 *1. Show that substructures and homomorphic images of* ω^n*-good quasiorders are* ω^n*-good.*

2. Prove the analogue of proposition 8 part (vi) for ω^n*-good quasiorders.*

Notice that the counterexamples of proposition **??** establish also that the class of good quasiorders of finite exponent isn't closed under direct limit or inverse limit either. However it is closed under power set.

Exercise 3 *Prove analogues of the perfect subsequence lemma (lemma 7) for* ω^n*-good quasiorders, and use it to establish the analogues of the later parts of proposition 8, namely (iii) that the product of finitely many* ω^n*-good quasisorders is* ω^n*-good; (iv) that the intersection of (the graphs of) two* ω^n*-good quasiorders on the same carrier set is* ω^n*-good, and (v) that a disjoint union of finitely many* ω^n*-good quasiorders is* ω^n*-good.*

3 BQOs

> *If a man can build a better quasiorder, the world will beat a path to his door.*[5]

Ralph Waldo Emerson *Voluntaries*

That man was the late Crispin Nash-Williams.

At this point we could direct our attention to the class of quasiorders $\langle Q, \leq \rangle$ such that, for all $n \in \mathbb{N}$, the result of doing $+$ n times to it is a WQO, the class we have been calling "good quasiorders of finite exponent", and notice that this class is closed under $+$, unlike the class of WQOs. This would give us a definition of a distinguished class of WQOs: namely the largest class of WQOs closed under $+$, and one could hope that this would turn out to be the resting place for this intuition for tidying up the definition of WQO. However we have to wring this idea out a little further, since there remains much to be gained by considering *transfinite* iterations under $+$. This is because one will then be able

[5]Apologies to Emerson are in order: the correct quotation is "If a man can write a better book, preach a better sermon, or make a better mousetrap than his neighbour, though he build his house in the woods, the world will make a beaten path to his door." In fact further apologies are probably in order: since Emerson was American 'neighbour' should be 'neighbor'.

to generalise the array condition to something that has no finite bound on the length of the sequences. The class of WQOs thus obtained will have even nicer closure properties than the class of good quasiorders of finite exponent.

[*HOLE It would be nice to have an example of of some constructor of infinite character under which the class of good quasiorders of finite exponent is not closed. I bet it's infinite trees ...*]

However, to do this, we need to consider expressions like $\mathcal{P}^\alpha(Q)$ where α is a transfinite ordinal. One can hardly imagine a better reason for stopping with the ideas of the preceding paragraph than the obvious fact that such a notation, *prima facie* at least, simply makes no sense. How can it, when the $\mathcal{P}^n(Q)$ were all be taken to be formally disjoint? If we wish to iterate $+$ and $\mathcal{P}$ transfinitely we need the $\mathcal{P}^n(Q)$ to be somehow *cumulative* not disjoint. Given X and Y, both subsets of $\bigcup\{\mathcal{P}^n(Q) : n \in \mathbb{N}\}$, how are we to compare them with respect to the quasiorder we will eventually call $\leq_\infty$? We will have to be able to compare everything in X with everything in Y, and this means comparing things from $\mathcal{P}^n(Q)$ and $\mathcal{P}^m(Q)$ for $m \neq n$. To make sense of this it will be sufficient to identify, once and for all, every element of Q somehow with a subset of Q, for then we can propagate this identification up the cumulative hierarchy of sets built up from Q. Now although there are many ways in which this identification can be done, there is one way that is obviously the simplest, namely to identify each $q \in Q$ with its singleton. Objects identical to their own singletons are called **Quine atoms**. Making this identification has the great advantage that when asking whether or not $q \leq q'$ under the new dispensation (in which every $q \in Q$ is simultaneously a subset of Q and a member of Q) it doesn't make any difference whether we think of q and q' as subsets of Q or elements of Q, since for all q and q' we always have $q \leq q'$ iff $\{q\} \leq^+ \{q'\}$. Notice also the important triviality that if f is an injection from $\langle Q_1, \leq_{Q_1}\rangle$ into $\langle Q_2, \leq_{Q_2}\rangle$ then $\lambda x.f``x$ is an injection from $\langle \mathcal{P}(Q_1), (\leq_{Q_1})^+\rangle$ into $\langle \mathcal{P}(Q_2), (\leq_{Q_2})^+\rangle$. Putting these two together enables us to think of $\langle \mathcal{P}^m(Q), (\leq_Q)^{m+}\rangle$ as an extension of $\langle \mathcal{P}^n(Q), (\leq_Q)^{n+}\rangle$ whenever $n \leq m$.

It is not customary to write '$\mathcal{P}^\alpha(Q)$' to be the result of applying the power set operation α times to a set Q, taking unions at limits to keep things cumulative. In these circumstances it is customary to use the letter 'V' instead, thus:

DEFINITION 17

$V_0(Q) = Q$; $V_{\alpha+1}(Q) =: \mathcal{P}(V_\alpha(Q))$, *taking unions at limit ordinals.*

Then $V_\Omega(Q)$ is the union of all the V_α. $V_\Omega(Q)$ sometimes called the **Zermelo Cone** over Q.)

$\lambda X.(\mathcal{P}(X) \cup Q)$ is thus a monotone function from the complete poset $\langle V, \subseteq\rangle$ into itself. Theorem **??** now tells us that this operation has a greatest and a least fixed point. The least fixed point is of course $V_\Omega(Q)$. The greatest fixed point we notate '$V(Q)$'. $V_\Omega(Q)$ is the wellfounded part of $V(Q)$.

The $+$ operation on quasiorders now becomes a monotone function from the complete poset of quasiorders of $V(Q)$, partially ordered by inclusion, into itself.

This too will have greatest and least fixed points, both of which we will notate '$\leq_\infty$'.

Now why might this be a natural thing to do? And how far should we go, now that we can iterate transfinitely? Over all ordinals, as my invocation of $V_\Omega(Q)$ apparently portends? The legions of the squeamish will complain that $V(Q)$ might not be a set.

It was with just this pending problem in mind that I prepared the ground earlier (p. 6) by making the point that if there is a bad sequence of subsets of X under $\leq^+$, then there is a bad sequence of *countable* subsets of X under $\leq^+$, and indeed, for each n if there is a bad sequence of elements of $\mathcal{P}^n(X)$ there is a bad sequence of (countable sets)^n of members of X. That means that whatever new mathematics we discover by iterating the power set operation, we can discover by iterating instead the set-of-all-countable-subsets operation. This is much less problematic. If we repeatedly apply the function $\lambda X.(\mathcal{P}_{\aleph_1}(X) \cup Q)$, starting at $\emptyset$, and take unions at limits, we will reach a fixed point after ω_1 steps. This (least) fixed point is notated $H_{\aleph_1}(Q)$, and is the **hereditarily countable sets over** Q. Now, by thinking of Q as a set of Quine atoms as before, we can lift $\leq_Q$ transfinitely often by $+$ to be defined on the whole of $H_{\aleph_1}(Q)$. To be precise, we consider the complete poset of quasiorders of $H_{\aleph_1}(Q)$ that extend $\leq_Q$, ordered by inclusion, and note that $+$ is a monotone function from this poset into itself, and must have a fixed point. It is this fixed point that interests us. The more adventurous can relax and accept the application of this process to the poset of quasiorders of $V(Q)$ ordered by inclusion. We will use the same notation—$\leq_\infty$—for both these quasiorders. (The first is simply the restriction of the second to $H_{\aleph_1}(Q)$.)

(It is at this point—where we claim that $\leq_\infty$ can be defined on the whole of $V(Q)$—that we use the fact that '+' is being applied to quasiorders not to partial orders: the collection of partial orders of $V(Q)$ is not a complete poset under inclusion but only a chain-complete poset, and we cannot appeal to Tarski-Knaster (theorem **??**).

Armed with the concepts of $H_{\aleph_1}(Q)$, $V(Q)$ and $\leq_\infty$, we can now define a more robust concept than wellquasiordering. A quasi-ordering $\langle Q, \leq_Q\rangle$ was ω^n-good if the result of lifting $\leq_Q$ n times by $+$ to $\mathcal{P}^n(Q)$ was WQO. Or—which is equivalent by lemma **??**—if the result of lifting $\leq_Q$ $n+1$ times by $+$ to $\mathcal{P}^{n+1}(Q)$ is wellfounded.

We now say

Definition 18 *A quasi-ordering $\langle Q, \leq_Q\rangle$ is BQO if $\langle H_{\aleph_1}(Q), \leq_\infty\rangle$ is wellfounded*

Of course this is equivalent to $\langle V(Q), \leq_\infty\rangle$ or $\langle V_\Omega(Q), \leq_\infty\rangle$ being wellfounded (or indeed BQO-ed!!)

But we *still* haven't given a proper definition of block!

4 Blocks and Games

It is nowadays widely understood that there is a connection between greatest fixed points and open games, and we can indeed characterise $\leq_\infty$ by means of a game, and the game will give us the correct definition of block and a combinatorial definition of BQO that is in the same format as the definition of ω^n-good quasiorder.

Things in $V(Q)$ can be thought of as downward-branching trees (possibly with infinite branches) all of whose leaves are labelled with members of Q. (They satisfy various extensionality conditions which it is not illuminating to dwell on here.)

The game $G_{X \leq_\infty Y}$ is played as follows.

`false` picks a member X' of X, `true` picks a member Y' of Y. If their two choices are both in Q, `true` wins if $X' \leq_Q Y'$ and `false` wins if not. If neither of them are in Q they continue, playing $G_{X' \leq_\infty Y'}$. (If one is in Q and the other isn't then we procede as if neither were: since we have identified each $q \in Q$ with $\{q\}$ we can take elements of Q to be subsets of Q when this is necessary—as now.) If the game goes on forever player `true` wins. The game is open so, by theorem **??**, one or the other player has a winning strategy.

If `false` has a winning strategy in $G_{X \leq_\infty Y}$—and plays according to it!—the play will end with player `true` picking a member of Q.

Let us say $X \leq_\infty Y$ iff player `true` has a winning strategy.

We are now going to turn our attention to identifying those WQO's $\langle Q, \leq_Q \rangle$ such that $\langle V(Q), \leq_\infty \rangle$ is also a WQO. It will turn out that they have a nice combinatorial characterisation.

We start out by noticing that if x is an illfounded member of $V(Q)$ then $(\forall y \in V(Q))(y \leq_\infty x)$. This means that $\langle V(Q), \leq_\infty \rangle$ has (up to equivalence) only one more element than $\langle V_\Omega(Q), \leq_\infty \rangle$. This is not going to make one a WQO when the other is not. Accordingly we can restrict our attention to $\langle V_\Omega(Q), \leq_\infty \rangle$.

Now suppose that $\leq$ wellquasiorders Q but $\leq_\infty$ does not wellquasiorder $V(Q)$. Let us see if we can simplify this to something sensible.

We start with a bad sequence $\langle X_i : i \in \mathbb{N} \rangle$ of members of $V(Q)$. Some of these elements might be members of Q. They cannot all be, because Q is WQO by $\leq$, by hypothesis. We are going to leave alone all X_i that are in Q, and elaborate the others until they, too, turn into members of Q. (The complication in this transfinite case is that we do not know in advance how often we are going to have to unwrap each set).

Start off with $\{X_i : i \in \mathbb{N}\}$, and a digraph which initially is simply the usual wellordering on $\mathbb{N}$, so there is an arrow from X_i to X_j iff $i < j$. We will make ω passes.

When we consider x_s we first check to see if it is a member of Q. If it is, it is then **ratified** which means it will never be replaced. If it is not a member of Q life is a bit more complicated. For each X_t such that there is an arrow from X_s to X_t we choose a member of X_s that is not $\leq_\infty$ anything in X_t, and we call it $X_{(s;t)}$ (for the moment at least). We discard X_s and redirect all arrows ending at X_s to $X_{(s;t)}$ (so we replace each old arrow by a host of new ones) and we replace the arrow from X_s to X_t with a new arrow from $X_{(s;t)}$ to X_t.

After ω passes everything has been ratified or discarded. The wellfoundedness of $\langle V_\Omega(Q), \in \rangle$ ensures that there can be no infinite sequence of Xs with later subscripts always end-extensions of earlier subscripts.

The subscripts are a bit of a mess at the moment: every subscript is an ordered pair of earlier subscripts. Notice that at stage one the only new subscripts we construct are pairs of natural numbers where the first component is smaller than the second, and the only new arrows we generate are things like $X_{(1;3)} \not\leq_\infty X_{(3;5)}$. So there must be a member of $X_{(1;3)}$ that $\not\leq_\infty X_{(3;5)}$ and we call it $X_{((1,3);(3,5))}$. Since this is the only way we can invent new things at this level, no ambiguity will arise in rewriting it as '$X_{1,3,5}$' to remove the duplication of the '3'. The second component of the first pair and the first component of the second pair are always the same!

Now for what subscripts s do we know that $X_{1,3,5} \not\leq_\infty X_s$? (All arrows going *into* $X_{1,3,5}$ arose from arrows going into $X_{1,3}$.) The only arrows going *from* $X_{1,3,5}$ go to $X_{3,5}$ in the first instance, and thereafter to things with subscripts that are end-extensions of $\{3,5\}$ should $X_{3,5}$ not be a member of Q and have to be replaced.

The upshot is that we can take subscripts to be increasing finite sequences of natural numbers, and we only ever arrange for an arrow from X_s to X_t when t is an end-extension of the tail of s.

This will lead us to the correct definition of block.

Now consider a set S of finite sequences from $\mathbb{N}$ that arises from a bad Q-sequence in this way. We will show that every increasing ω-sequence from $\mathbb{N}$ has a unique initial segment in S. Let $f : \mathbb{N} \to \mathbb{N}$ be increasing. Is $\langle f(0) \rangle$ in S? It will be if $X_{f(0)} \in Q$, and if that happens we are done. If $\langle f(0) \rangle$ is not in S this must be because $X_{f(0)} \notin Q$ and in these circumstances we have discarded $X_{f(0)}$ and replaced it by the infinitely many $X_{f(0),j}$ for $j > f(0)$. In particular we will have done this for $j = f(1)$. So is $\langle f(0), f(1) \rangle \in S$? It will be unless $X_{f(0),f(1)} \notin Q$. In those circumstances we discarded $X_{f(0),f(1)}$ and replaced it by each of $X_{f(0),f(1),j}$ for $j > f(1)$. And so on. Eventually we hit a member of Q and at that point we have an element of s that is an initial segment of f. Notice that we only ever put into S a sequence s if we have already discarded all initial segments of s, so that the initial segment in S of our infinite sequence is unique.

This motivates the definitions which follow.

Definition 19

1. *A* **block** *is a set B of strictly increasing finite sequences of naturals with the property that every strictly increasing ω-sequence of natural numbers has a unique initial segment in B.*

2. *if f is a strictly increasing ω-sequence from* $\mathbb{N}$ *and B is a block then f@B is the unique initial segment of f lying in B.*

3. *We write $s \lhd t$ if t is the tail of an end-extension of s.*

Several missable points to note here:

1. $\lhd$ is not transitive except in the single case of a block all of whose elements are singletons;
2. We really do mean "tail of an end-extension" not 'end-extension of the tail': which would allow $\langle 3\rangle \lhd \langle 2\rangle$.
3. There is nothing to prevent blocks containing two tuples like $\langle 2, 3\rangle$ and $\langle 3\rangle$, but this never seems to happen in naturally occurring blocks.

Notice that this agrees with our picture of the canonical n-blocks for finite n.

With an eye to possible future generalisation let us note that we could take this definition as a definition of an $\mathbb{N}-$block: the finite sequences that are the elements of the block are increasing finite sequences from $\mathbb{N}$, but increasing finite sequences from X will do do as long as X is an infinite subset of $\mathbb{N}$. Indeed we could have embarked on our journey already armed with this generalisation, by defining wellfounded relations in terms of descending sequences indexed not by $\mathbb{N}$ but by some infinite subset of $\mathbb{N}$.

Exercise 4 *Look back at exercise* **??** *on page* **??**. *Extend the results of that exercise from n-blocks to arbitrary blocks.*

Remark 20 *Every block, considered as a set of increasing finite sequences, is wellfounded in the lexicographic order.*

Proof: Suppose we have an infinite descending chain. One thinks of something like (1, 4), (1, 3, 9), (1, 3, 8, 16), (1, 3, 8, 15, 25) ...

The first elements of the sequences in the this chain must be eventually constant. So must the second. And so on. Consider the infinite sequence consisting of those eventually constant places (as it might be (1, 3, 8, 24, 35 ...). This infinite sequence has a unique initial segment in the block. As it might be: (1, 3, 8, 15). But then the block contains both (1, 3, 8, 15) and (1, 3, 8, 15, 25), with the result that any infinite sequence beginning (1, 3, 8, 15, 25 ...) has more than one initial segment in the block.

Then we say

Definition 21 *$|B|$ is the length of the block B in the lexicographic order.*

Now we can give the combinatorial definition of BQO, the one that uses blocks:

Definition 22 *Let $\langle Q, \leq_Q\rangle$ be a quasiorder and B a block. A map $f : B \to Q$ is an* **array**. *An array is* **good** *if there are $s \lhd t \in B$ such that $f(s) \leq_Q f(t)$.*

Then $\langle Q, \leq_Q\rangle$ is a **better-quasiorder** *(hereafter "***BQO***") iff for every block B every array $f : B \to Q$ is good.*

THEOREM 23 *The two definitions of BQO—definitions 18 and 22—are equivalent.*

Proof: The definition of block was cooked up precisely to make this true. ■

In fact we can strengthen theorem **??** further.

THEOREM 24 *If $\langle Q, \leq_Q \rangle$ is BQO, so is $\langle V(Q), \leq_\infty \rangle$.*

Proof:

Suppose there is a bad array over $V(Q)$. We will show how to refine it into a bad array on on Q. This is merely a more developed version of the process we applied to $\mathcal{P}^n(Q)$ earlier on.

Let $\{X_s : s \in B\}$ be a bad array over $V(Q)$. For each pair s, t in B with $s \lhd t$ we have $X_s \not\leq_\infty X_t$. Player `false` has a winning strategy $\sigma_{X_s \not\leq_\infty X_t}$ in the game $G_{X_s \leq_\infty X_t}$.

All the games $G_{X_s \leq_\infty X_t}$ will be played simultaneously. Indeed many *plays* of these games will be going on simultaneously. To be precise, there is a play for each infinite ascending $\lhd$-sequence, so that at time t the set of plays-in-progress is indexed by $\mathbb{N}^{<\omega} \restriction t$.

It is convenient to describe what happens in terms of an ω-sequence of what one might as well call *passes*.

At the first pass, in each game $G_{X_s \leq_\infty X_t}$, `false` uses his strategy to pick a member of X_s. This will become $X_{s;t}$. At the second pass (and all subsequent passes) each play of $G_{X_s \leq_\infty X_t}$ multifurcates. At the first pass there was only one play of each game. For `false` to decide what to do as his second move in $G_{X_s \leq_\infty X_t}$ he deems `true`'s move in this game to be `false`'s move in $G_{X_t \leq_\infty X_u}$, for $t \lhd u$. Thus he deems `true` to have played $X_{t;u}$. Since he does this for *each* u such that $t \lhd u$, the one play of $G_{X_s \leq_\infty X_t}$ which was proceeding at pass one has become infinitely many. In each play he continues to use $\sigma_{X_s \not\leq_\infty X_t}$ and—since this strategy is winning—each play will terminate with a win for player `false`. This tells us that after ω passes every play of every game will have terminated in a win for player `false`.

Of course, since there is an entire bad array out there, we must expect to have to deal with $X_t \not\leq_\infty X_u$ for various u as well. For each game $G_{X_t \leq_\infty X_u}$, where $t \lhd u$, player `false` in that game uses his winning strategy to pick $X_{t;u}$. Player `false` in the game $G_{X_s \leq_\infty X_t}$ now has infinitely many replies to contend with, but he uses $\sigma_{X_s \not\leq_\infty X_t}$ to reply to each, and the play multifurcates, but `false` can continue to use $\sigma_{X_s \not\leq_\infty X_t}$ in each.

Since all the strategies $\sigma_{X_s \not\leq X_t}$ are winning for `false`, this process must halt with player `true` picking elements of Q. This gives us a bad array on Q.

■

This implies that $\langle Q, \leq_Q \rangle$ is BQO iff $\langle Q, \leq_Q \rangle$ belongs to the largest class of WQOs closed under the operation taking $\langle X, \leq \rangle$ to $\langle H_{\aleph_1}(X), \leq_\infty \rangle$. (Or, equivalently, to $\langle V(X), \leq_\infty \rangle$ or $\langle V_\Omega(X), \leq_\infty \rangle$.) Indeed, since $\langle Q, \leq_Q \rangle$ is WQO iff $\langle \mathcal{P}(Q), \leq^+ \rangle$ is wellfounded we can strengthen this to the remarkable

Corollary 25 *$\langle Q, \leq_Q \rangle$ is BQO iff $\langle Q, \leq_Q \rangle$ belongs to the largest class of wellfounded quasiorders closed under the operation taking $\langle X, \leq \rangle$ to $\langle H_{\aleph_1}(X), \leq_\infty \rangle$. (Or, equivalently, to $\langle V(X), \leq_\infty \rangle$ or $\langle V_\Omega(X), \leq_\infty \rangle$.)*

5 FFF

Consider the one-point WQO, and suppose there is a natural number k such that for all n there is a bad sequence of length n:

$$T_1^n, T_2^n, T_3^n \ldots T_n^n$$

where T_i^n is a finite tree (over the one-point WQO) with $k + i$ nodes. Then there will be an infinite triangular matrix of trees (one row for each n):

Figure 1: An infinite triangle of trees: I

$$\begin{array}{llll}
T_1^1 & & & \\
T_1^2 & T_2^2 & & \\
T_1^3 & T_2^3 & T_3^3 & \\
T_1^4 & T_2^4 & T_3^4 & T_4^4 \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Consider the first column of figure 1, the sequence $\langle T_1^n : n \in \mathbb{N} \rangle$. Each tree in this sequence has only $k + 1$ nodes, so only finitely many of them can be distinct. So some of them must be present with infinite multiplicity, and let us call 'T_1' the one that appears first. Now discard all the rows that do *not* begin with T_1. Now consider the second column and obtain T_2 analogously. Iterate with all subsequent columns. Eventually we will have constructed an infinite bad sequence of trees. But this would contradict theorem 13. Therefore the initial assumption was wrong, so there is no such k, and we have proved

Theorem 26 *$\forall k \exists n$ if $T_1 \ldots T_n$ is a list of trees where T_i has $k + i$ nodes, then there are $j < l \leq n$ s.t. $T_j \leq T_l$.*

∎