Back to the Roots

We mean roots in the sense of fundaments or foundations

Would the development of early twentieth century metamathematics have taken a completely different direction if computer aided proof verification had been available? Is the modern algebra notion of a function a category mistake? We consider a simple formal system in which, as in set theory and type theory, it is possible to implement all of mathematics (at least all the mathematics that can be written down) and in which at least some mathematical notions have a very simple and natural implementation. We consider varieties of axioms of extensionality and axioms of existence. We introduce a useful abstraction notation, make many definitions and prove a few easy theorems. We outline a programme of further work in the theory itself, in category theory and in computing.

The Elementary Theory of Relations

The language of an elementary theory of relations is the language of the lower predicate calculus with just one predicate symbol, a three-place predicate symbol which we shall call " Cog ".

The idea is that $\mathrm{Cog}(x,y,z)$ may be interpreted as

" y relates x to z " .

Notice we write our relations in the middle, but also be aware that in this theory everything is a relation.

So the signature of the language is $< \mathrm{Cog} >$ [1]

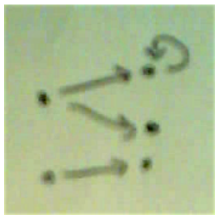
Thus the models of the theory are structures

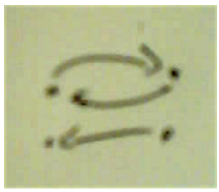
$$< \mathrm{U}, \mathrm{C} > \text{ where } \mathrm{C} \subseteq \mathrm{U} \times \mathrm{U} \times \mathrm{U}$$

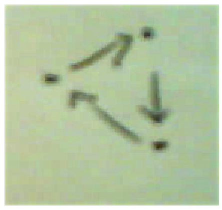
in exactly the same why that the models of a set theory are structures

$$< \mathrm{U}, \mathrm{E} > \text{ where } \mathrm{E} \subseteq \mathrm{U} \times \mathrm{U}$$

We can find plenty of finite models for this system (especially as we have not specified any axioms yet), for example, directed graphs where $\mathrm{Cog}(x,y,z)$ means there is a directed edge y from x to z or algebras with a binary operation * (e.g. groups, monoids) where $\mathrm{Cog}(x,y,z)$ is interpreted as $x*z=y$.







some small relations

[1] Or $< \mathrm{Cog}, = >$ if the underlying logic does not have equality built-in.

As with set theory you can encode the entire of mathematics in this language (at least all of it that can be written down).

Of course, we now have to give some axioms for the theory, but before doing that it is worth noting some of the things we can do without them. [2]

Firstly, we can make the usual definitions: [3]

x is transitive

$$\mathrm{Trans}\ (x) =_{\mathrm{df}} \mathrm{Cog}(a,x,b) \wedge \mathrm{Cog}(b,x,c) \rightarrow \mathrm{Cog}(a,x,c)$$

x is symmetric

$$\mathrm{Sym}\ (x) =_{\mathrm{df}} \mathrm{Cog}(a,x,b) \rightarrow \mathrm{Cog}(b,x,a)$$

x is antisymmetric

$$\mathrm{Antisym}\ (x) =_{\mathrm{df}} \mathrm{Cog}(a,x,b) \wedge \mathrm{Cog}(b,x,a) \rightarrow a = b$$

The cumbrous phrasing of these definitions leads us to introduce some more easily readable notation.

" Cog " being the only predicate symbol, we can without danger of ambiguity from here on write

xyz for $\mathrm{Cog}(x,y,z)$ [4]

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[2] A word about the underlying logic: this can be whatever you like, but given that only a few axioms of extensionality and existence are sufficient to force the law of excluded middle, it is simply masochistic not to work in classical logic. This does not imply any view on intuitionism or constructivism in our system though - the problems of the proof theory and the problems of proving existence remain open to analysis. We shall also assume equality is built-in.

[3] We are using letters $a,b,c, \ldots ,y,z$ to denote variables

[4] Also we shall omit more brackets and leave it to you decide where they should be (according to the usual conventions), but we may give you a clue by using dots (a notation attributed to Peano, but with much earlier precedent), foo**.**bar is meant to suggest ((foo)(bar)), which is also conventional, as in e.g., " λx.x+1". Furthermore, of course, we have already been omitting the universal quantifiers in sentences.

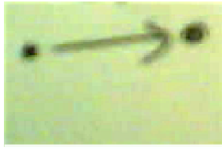
It is also useful to introduce the notation

$$xy \quad \text{for} \quad yxy$$

So to carry on with the definitions:

x is asymmetric

$$\mathrm{Asym}\,x \;=_{\mathrm{df}}\; axb \to \neg\, bxa$$



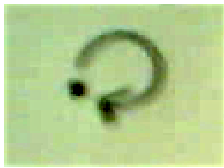
an ordered pair

x is reflexive

$$\mathrm{Refl}\,x \;=_{\mathrm{df}}\; axb \to axa \wedge bxb$$ [5]

x is an ordered pair

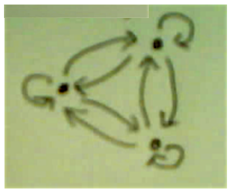
$$\mathrm{Op}\,x \;=_{\mathrm{df}}\; axb \wedge cxd \to a=c \wedge b=d$$



a singleton

x is a singleton

$$\mathrm{Sing}\,x \;=_{\mathrm{df}}\; axb \wedge cxd \to a=b \wedge b=c$$



a small class

x is primordial

$$\mathrm{Prim}\,x \;=_{\mathrm{df}}\; axb \to \neg\, axb$$ [6]

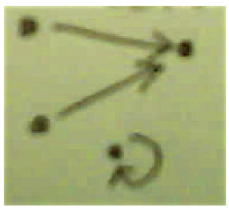
x is a class

$$\mathrm{Cls}\,x \;=_{\mathrm{df}}\; axb \wedge cxd \to bxc$$

A serious typo from the April talk corrected here

x is a jection

$$\mathrm{Jec}\,x \;=_{\mathrm{df}}\; axb \wedge axc \to b=c$$



a small jection

x is a sub-relation of y

$$x \subseteq y \;=_{\mathrm{df}}\; axb \to ayb$$

---

[5] Which can be written $axb \to xa \wedge xb$

[6] We write the definition thus to emphasize it is not necessary to search the entire universe to determine if something is primordial.

It is easy to prove:

$\mathrm{Sing}\,x \rightarrow \mathrm{Cls}\,x$

$\mathrm{Op}\,x \rightarrow \mathrm{Jec}\,x$

$\mathrm{Cls}\,x \rightarrow \mathrm{Trans}\,x \wedge \mathrm{Sym}\,x \wedge \mathrm{Refl}\,x$

$\mathrm{Prim}\,x \rightarrow \mathrm{Cls}\,x$ [... $\wedge$ $\mathrm{Trans}\,x$, .. etc.]

$\mathrm{Jec}\,x \wedge \mathrm{Cls}\,x \rightarrow \mathrm{Sing}\,x \vee \mathrm{Prim}\,x$

Which can be improved to
$\mathrm{Jec}\,x \wedge \mathrm{Cls}\,x \rightarrow \mathrm{Sing}\,x$
because $\mathrm{Prim}\,x \rightarrow \mathrm{Sing}\,x$
by absurdity

Example: $\mathrm{Cls}\,x \rightarrow \mathrm{Trans}\,x \wedge \mathrm{Sym}\,x \wedge \mathrm{Refl}\,x$:

	$axb \rightarrow axb \wedge axb$, thus $\rightarrow bxa$	[$\Rightarrow$ Sym]
and so	$axb \rightarrow bxa \wedge axb$, thus $\rightarrow axa$,	
and	$axb \rightarrow axb \wedge bxa$, thus $\rightarrow bxb$	[$\Rightarrow$ Refl]
also	$axb \wedge bxc \rightarrow bxa \wedge cxb$, thus $\rightarrow axc$	[$\Rightarrow$ Trans]

A typo corrected here

Axioms of Extensionality

Strong (or classic) extensionality is

$$\forall xy(xay \leftrightarrow xby) \rightarrow a=b$$

[two relations are equal if they relate the same relations]

Strong extensionality may not be desirable as in

```
x := copy_of y
```

Where a new thing `x` is created relating things in exactly the same way as `y` but distinct from it.

A typo corrected here

Upwards extensionality is

$$\forall xy\ ((axy \leftrightarrow bxy) \wedge (xya \leftrightarrow xyb)) \rightarrow\ a{=}b$$

[two relations are equal if they are related by the same relations to the same relations] [7]

There are many other simple forms of extensionality.

Left extensionality is

$$\forall xy\ (axy \leftrightarrow bxy)\ \rightarrow\ a{=}b$$

Left extensionality is really the same as strong extensionality, writing relations on the left instead of in the middle.

If xyz is interpreted as $x{*}z{=}y$ for some binary operation *, then the axiom of associativity, which can be written as

$$ayx \wedge bxc \wedge azb \rightarrow\ zyc$$

is of the nature of an axiom of extensionality.

In general, Horn clauses will have the nature of axioms of extensionality. Finite collections [8] of these will, by Ramsey's theorem, have finite models or be inconsistent.

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[7] In, for example, ZF or NF set theory if you substitute upwards for strong extensionality then you get the same theory. but not so in general.

[8] One might say types or lists

A Useful Abstraction Notation

We introduce a useful abstraction notation for relations, which we call “2-hat abstraction”, “bi-hat abstraction”, or simply “***hat abstraction***”: a hat abstraction term has the form,

$$\hat{x}\phi\hat{y} \ , \qquad \text{where } \phi \text{ is some formula.}$$

The idea is that in general we desire

$$a.\hat{x}\phi\hat{y}.b \quad \leftrightarrow \quad \phi[a/x, b/y] \qquad \textbf{[H]}.$$

Set theoretic abstraction, λ abstraction, μ (minimal element) abstraction, Russell’s ι abstraction and Hilbert’s ε abstraction (the τ abstraction of the later editions of Bourbaki) are all special cases of hat abstraction.

Having made this tribute to early twentieth century mathematics and logic, from now on we shall write

$$<x|\phi|y> \qquad \text{for} \qquad \hat{x}\phi\hat{y}$$ [9]

It is important to note that hat abstraction is ***<u>just a notation</u>***; if we were setting it up as as component of a formal meta-language we would need to specify quite a few conversion schemata: we would need to know what to do with

$$\phi[\mathbf{s}/x] \qquad \text{for an abstraction term } \mathbf{s}.$$

The traditional approach to this would involve looking at the 21 quasi-atomic formulae,

$$x{=}\mathbf{s},\ \mathbf{s}{=}x,\ \mathbf{s}{=}\mathbf{s},\ \mathbf{s}{=}\mathbf{t},\ xx\mathbf{s},\ x\mathbf{s}x,\ \mathbf{s}xx,$$
$$xy\mathbf{s},\ x\mathbf{s}y,\ \mathbf{s}xy,\ \mathbf{ss}x,\ \mathbf{s}x\mathbf{s},\ x\mathbf{ss},\ \mathbf{st}x,\ \mathbf{s}x\mathbf{t},$$
$$x\mathbf{st},\ \mathbf{sss},\ \mathbf{sst},\ \mathbf{sts},\ \mathbf{tss} \text{ and } \mathbf{stu},$$

---

[9] This is not only because $\hat{x}\phi\hat{y}$ is not so easy to type on a twenty-first century computer as it is to type on a twentieth century typewriter, but also for other reasons which we shall get to later. However we shall continue to refer to it as “hat abstraction”.

(with **s**, **t** and **u** being abstraction terms) and of course involves careful treatment of the variables free and bound in these terms and the order (that is to say, sequencing) of substitutions.

This is old hat and we leave to you to make the best interpretation of what we have to say next. However, it is worth recalling that it is a pipedream to expect $\exists x.\, x=\mathbf{s}$ in general, which would immediately give rise to Russell's paradox. Furthermore, it may well be desirable that in general it should ***not*** be the case that $\mathbf{s}=\mathbf{t} \wedge \mathbf{t}=\mathbf{u} \rightarrow \mathbf{s}=\mathbf{u}$.

For example [10]

```
> Train = Stockholm
> Train = Uppsala
> ?Train
```

We can now make some definitions (be aware - these things being defined are just notations)

$$\breve{\ } a \quad =_{\mathrm{df}} \quad < x \mid yax \mid y >$$

$$a \circ b \quad =_{\mathrm{df}} \quad < x \mid \exists y.xay \wedge ybz \mid z >$$

$$a \cap b \quad =_{\mathrm{df}} \quad < x \mid xay \wedge xby \mid y >$$

$$a \backslash b \quad =_{\mathrm{df}} \quad < x \mid xay \wedge \neg\, xby \mid y >$$

$$a \cup_n b \quad =_{\mathrm{df}} \quad < x \mid xay \vee xby \mid y >$$

The little n stands for naïve

$$<a> \quad =_{\mathrm{df}} \quad < x \mid x=a \wedge y=a \mid y >$$

$$<a,b> \quad =_{\mathrm{df}} \quad < x \mid x=a \wedge y=b \mid y >$$

[10] This is from an old relational database system.

$$[a]_b \quad =_{df} \quad <x \mid (xba \vee abx) \wedge (yba \vee aby) \mid y>$$ [11]

$$<_a \quad =_{df} \quad <x \mid xay \wedge \neg\, x{=}y \mid y>$$

$$=_a \quad =_{df} \quad <x \mid xay \wedge x{=}y \mid y>$$ [12]

$$>_a \quad =_{df} \quad <x \mid yax \wedge \neg\, x{=}y \mid y>$$

So, assuming **[H]** , we can prove, for example,

$$\mathrm{Sng}\ <a>$$

$$\mathrm{Op}\ <a,b>$$

$$\mathrm{Cls}\ \ [a]_b$$

$$\mathrm{Cls}\ x \wedge \mathrm{Cls}\ y \rightarrow \mathrm{Cls}\ x \cap y$$ [13]

$$x.(a \circ b) \circ c.y \leftrightarrow x.a \circ (b \circ c).y$$ [14]

Axioms of Existence

Axioms of existence are sentences of the form

$$\forall a_0 \ldots a_n\ \exists b\ \ \forall xy.\ xby \leftrightarrow \phi$$

where b is not free in ϕ.

The a_i are called *parameters*.

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[11] The next door neighbours or close friends of a under b; the equivalence class of a by b if b happens to be an equivalence relation

[12] $<x \mid xay \mid x>$ is not the same thing

[13] Of course we do not have either $\mathrm{Cls}\ x \wedge \mathrm{Cls}\ y \rightarrow \mathrm{Cls}\ x \cup_n y$
or $\mathrm{Cls}\ x \wedge \mathrm{Cls}\ y \rightarrow \mathrm{Cls}\ x \backslash y$

[14] Not the same as $(a \circ b) \circ c = a \circ (b \circ c)$ unless you assume strong extensionality

Using our notation we can more simply write

$$\exists b \; b = \; < x \mid \phi \mid y >$$

or just say, " $< x \mid \phi \mid y >$ exists " .

It seems eminently reasonable to assume that if a and b exist then all of $\breve{}a$, $a \circ b$, $a \cap b$, $a \backslash b$, $a \cup_n b$, $<a>$, $<a,b>$, $<_a$, $>_a$, $[a]_b$ and $=_a$ exist.

So these are some candidate axioms of existence (relative to the parameters a and b).

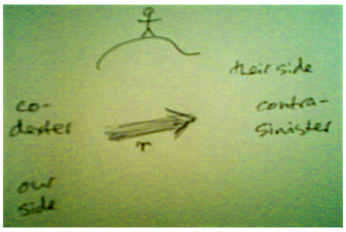
But beware, to say these things exist uniquely is stating an axiom which is of the nature of an axiom of extensionality. For example,

$$\forall a \exists ! b \; b = \breve{}a$$

is the same as strong extensionality.

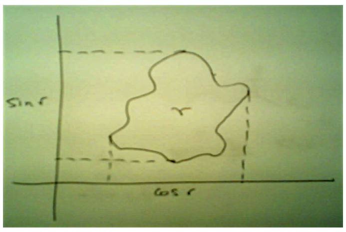
It is instructive to try and prove this - because you have to do so *without* assuming strong extensionality

Indeed it could be argued that, in the presence of strong extensionality, a and $\breve{}a$ are the same relation, viewed from different viewpoints.



The observer on the hill sees the relation from the opposite viewpoint from ourselves

It is useful then to define the right and left classes and the demesne of a relation



sin *and* cos

$$\cos a \;=_{df}\; < x \mid \exists z.\, xaz \wedge \exists z.\, yaz \mid y >$$

$$\sin a \;=_{df}\; < x \mid \exists z.\, zax \wedge \exists z.\, zay \mid y > \text{ [15]}$$

$$\mathrm{dem}\; a \;=_{df}\; < x \mid (\exists z.\, zax \vee xaz) \wedge (\exists z.\, zay \vee yaz) \mid y >$$

We can easily prove $\mathrm{Cls}\; \cos a$, $\mathrm{Cls}\; \sin a$ and $\mathrm{Cls}\; \mathrm{dem}\; a$.

[15] We pronounce these kōs and sĭn to contrast with the trigonometric functions, kóz and sīn. If you like you can instead call them dex (for *dexter*) and cod, but bear in mind that cod here (whilst on the same side) is not quite the same as *codomain*.

The assertions of the existence of each of these are of the nature of axioms of existence, which again are entirely reasonable.

In the presence of strong extensionality these classes are unique.

In the absence of strong extensionality, the statements

$$\left.\begin{array}{l} \forall a \exists ! b \; b = \cos a \\ \forall a \exists ! b \; b = \sin a \\ \forall a \exists ! b \; b = \operatorname{dem} a \end{array}\right\} \quad \textbf{[A]}$$

are good candidate axioms of extensionality.

We can prove, using appropriate conversion schemata, and assuming **[A]** that the following are equivalent:

(i) $\mathrm{Cls}\, x$

(ii) $x = \operatorname{dem} x$

(iii) $x = \sin x$

(iv) $x = \cos x$ [16]

[(ii), (iii) or (iv) $\Rightarrow$ (i) because $\mathrm{Cls}\operatorname{dem} x$ etc;

(i) $\Rightarrow$ (ii), (iii) and (iv) follows from **[A]** (uniqueness)]

We further define

$$\mathrm{id}_a =_{\mathrm{df}} \langle x \mid \operatorname{dem} a . x \wedge x = y \mid y \rangle$$

$$\leq_a =_{\mathrm{df}} \langle x \mid xay \vee x \, \mathrm{id}_a \, y \mid y \rangle$$

$$\geq_a =_{\mathrm{df}} \langle x \mid yax \vee x \, \mathrm{id}_a \, y \mid y \rangle$$

[16] But not $\sin x = \cos x$

Adding to our list of candidate axioms of existence it seems reasonable then to assume that if a exists then so do $\cos a$, $\sin a$, $\mathrm{dem}\, a$, $\leq_a$, $\geq_a$ and id_a.

Exercises:

Define the [17] member of a singleton and the [18] right and left members of an ordered pair, formulate the corresponding axioms of existence and extensionality, investigate their consequences.

Formulate some definitions of $\subseteq_a$ and explore them.

Investigate the relations between $<_a$, $>_a$, $=_a$, $\leq_a$, $\geq_a$ and id_a, extend this to include your notions of $\subseteq_a$.

An Aside

We can now note that the theory is implicitly strongly typed. The type of a relation a is

$$\cos a \rightarrow \sin a$$ [19]

Extending the abstraction notation

We write

$$a < x : \phi \mid \psi \mid y : \chi > b \quad \text{for} \quad \phi[a/x] \wedge \psi[a/x, b/y] \wedge \chi[b/y]$$

For example

$$< x : x > 3 \mid x + y = 2 \mid y : y < -1 >$$

and $$< x : \sin a . x \mid x = y \mid y : \cos b . y >$$

[17] not necessarily unique

[18] ditto

[19] We mean "implicitly strongly typed" in one of the senses of computer programming language theory. There are other ways of typing the theory as well

Informally, if the bound variables are obvious we might abbreviate as in re-writing the above as

$$< x > 3 \mid x + y = 2 \mid y < -1 >$$

$$< \sin a . x \mid x = y \mid \cos b . y >$$

Also if the ψ in $< x : \phi \mid \psi \mid y : \chi >$ says nothing we may leave it out as in

$$< x > 3 \mid y < -1 > \quad \text{for} \quad < x > 3 \mid x = x \mid y < -1 >$$

Also we might write

$$< \mathbf{s} \mid \phi \mid \mathbf{t} > \quad \text{for} \quad < x : x = \mathbf{s} \mid \phi \mid y : y = \mathbf{t} >$$

i.e. $\quad x < \mathbf{s} \mid \phi \mid \mathbf{t} > y \quad$ for $x = \mathbf{s} \wedge y = \mathbf{t} \wedge \phi$

when the variables happen to be obvious, as in e.g.

$$< x + iy \mid x^2 + y^2 = 1 \mid x - iy >$$

and as above if the defining formula ϕ says nothing we may leave it out as in

$$< x \mid 2x >$$

and we may mix these modes of notation as in

$$< x > 3 \mid 2x >$$

Implementing common mathematical notions

We already have singletons, ordered pairs, sets (in the guise of classes) and quite a lot more. What about functions? The sort of functions we came across in high school maths are clearly there (or at least hinted at in our abstraction notation). However a function as a relation simple carries the information about its domain and image[20] but not its range. On the other hand a relation considered as a partial function carries extra (partial) information about how it does not converge - but that is to be expected - it would be difficult usually to write a function specification that does not do that. We could say implement a function as a class of ordered pairs, but we end up with the same thing, and with extra effort. We know from recursion theory and lambda calculus that the correct notion of function is a partial function. We are used to such things as $arctan$, $1/x$, implicit functions, meromorphic functions, delta functions etc etc being functions. We note that a function factors into a jection and an inclusion, which are two different sorts of things. We conclude that the *modern algebra* notion of a function is a category mistake, and do not bother about it. If we are to include extra information about one side of a relation then by fairness we ought to be allowed to include extra information about the other side.

So we have (partial) functions. Now to implement the Reals and so on.

We say a relation has **game nature** if the members of its right and left classes have game nature

$$\mathrm{Gnat}\ x \quad =_{\mathrm{df}} \quad \mathrm{dem}\ x.y \rightarrow \mathrm{Gnat}\ y$$ [21]

[20] but not of course their structure

[21] By this definition all relations might have game nature. A non-constructive axiom that there exists a relation without game nature is an easy but unattractive way out.

This way we get all of Conway's games, amongst which we find the all the surreal numbers, including the reals, complex numbers, ordinals and a great deal more besides. [22]

This definition is made by recursion outside the system.

Within the system there are two classical approaches: the first, construction as by Gödel, Gandy and Jensen, the second in one go by use of the notion of well foundedness, as in ZF and its many variants. [23]

The first approach involves looking at the atomic formulae and building on them [24]. It is more complicated than generally presented as in fact it is necessary to define tuples and the arity of a tuple is at first outside the system. No full proof has ever been published [25]. In relation theory the matter is much more complicated than in set theory in that whilst in set theory there are only the 8 atomic formulae

$x{=}x \quad x{=}a \quad a_0{=}a_1$ [26] $x \epsilon x \quad x \epsilon a \quad a \epsilon x \quad a \epsilon a \quad a_0 \epsilon a_1$

In relation theory we have to look at 38 [27]

$x{=}y \quad x{=}x \quad x{=}a \quad y{=}a \quad a_0{=}a_1$ [28] $xxy \quad xyx \quad yxx \quad xyy \quad yxy \quad yyx$

$xay \quad xya \quad axy \quad ayx \quad yax \quad yxa \quad xxx \quad yyy \quad axx \quad xax \quad xxa \quad ayy$

$yay \quad ayy \quad xaa \quad axa \quad aax \quad yaa \quad aya \quad aay$

$xa_0a_1 \quad a_0xa_1 \quad a_0a_1x \quad ya_0a_1 \quad a_0ya_1 \quad a_0a_1y \quad a_0a_1a_2$ remember the a_i are parameters

---

[22] Actually you have to tweak Conway's construction because in our system if a relation is empty on one side it is empty on the other side too. Also beware, in Conway's notation, $x = \{ x^L \mid x^R \}$, x^L and x^R may be *representatives* from our $\cos x$ and $\sin x$.

[23] Are there any others, as yet undiscovered ? (Forcing only counts if you have something to start from.)

[24] In practice this is usually done differently, by defining fundamental or rudimentary functions. Also it is often presented as building models of fragments of the theory within it. Does it require DC or Determinacy? - No.

[25] A call for computer verification.

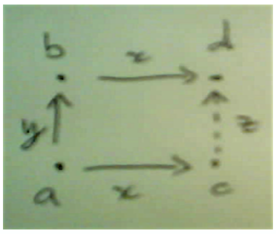
[26] $a{=}a$ is the same as $x{=}x$

[27] The induction is on the complexity of ϕ in $< x \mid \phi \mid y >$

[28] $y{=}y$ and $a{=}a$ are the same as $x{=}x$

This is one programme of work. Now to consider the second approach, well foundedness is a fundamental mathematical concept (not just a trick to enable induction). The point is that it is defined within the theory rather than the induction principle which is higher order [29]. Well foundedness generalises well order which in turn generalises finiteness. It is easy to see how to implement linear orders and trees and so on in our theory; when it comes to well foundedness should we seek minimal elements only for the sub-classes in the demesne of a relation? We seek a generalisation of the notion of well foundedness. This is another programme of work.

Correlators



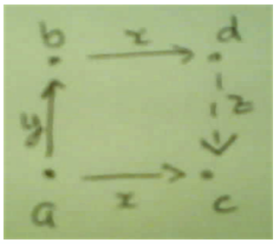
a correlator

We say x is a *correlator* between y and z if

$$ayb \wedge axc \wedge bxd \rightarrow czd$$

x is a *skew correlator* between y and z if

$$ayb \wedge axc \wedge bxd \rightarrow dzc$$



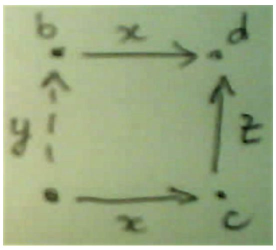
a skew correlator

x is a *contrarelator* between y and z if

$$czd \wedge axc \wedge bxd \rightarrow ayb$$

x is an *antirelator* between y and z if

$$ayb \wedge axc \wedge bxd \rightarrow \neg czd$$



a contrarelator

We can go on and define *skew contrarelators, anticontrarelators, skew anticontrarelators, anti skew correlators* etc etc, similarly.

[29] It is quantified over formulae

We can prove

> the composition of two correlators is a correlator
>
> the composition of a correlator and a skew correlator is a skew correlator
>
> the composition of two skew correlators is a correlator

$$\mathrm{Cor}(x,y,z) \quad =_{\mathrm{df}} \quad ayb \wedge axc \wedge bxd \rightarrow czd$$

We write

$$y\underline{x}z \qquad \text{for} \qquad \mathrm{Cor}(x,y,z)$$ [30]

So we have

$$x\underline{a}y \wedge y\underline{b}z \rightarrow x\underline{a \circ b}z$$

$$x \underline{=_x} x$$

$$x \underline{\mathrm{id}_x} x$$

Almost every mathematical notion we can think of is a correlator. Linear transforms, Fourier transforms, meromorphic functions, integral and differential operators, connections, implicit functions, theories, functors etc are all correlators. The question is what do they correlate and how?

And, of course, *any* relation is a correlator

$$\cos x \;\; \underline{x} \;\; \sin x$$

[30] For typographical convenience. Another notation might be better.

Further work [31]

limits, bounded quantifiers, skew axioms [32]

computing

implement the system in a theorem prover [33], experiment with different axioms, find finite models classify them

implement the abstraction notation as a formal metalanguage experiment with different conversion schemata

category theory

carry over all the notions, theorems of category theory

develop a model theory of pure category theory

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[31] If you are interested please do make contact. The e-mail is sometimes not very reliable, so if you do not get a reply then please assume the message has not got through and try again.

[32] These are statements such as, “given a ternary relation p, there exist ternary relations q and r such that p(x,y,z) iff q(x,z,y) iff r(y,z,x)”. We also call them “twist axioms”. They are important for all sorts of reasons. Ternary, indeed n-ary relations and tuples in general can be implemented as trees, by additional predicate symbols (not a very attractive method), as correlators with initial segments of the naturals, etc.

[33] Chad Brown has now done this for the basic system on Jitpro and LEO-II, see mathgate.info/elemtheoryrelns/index.php. Thanks also to Chad for pointing out the strengthening of the theorem on page 4.

Endnote: This talk was originally given at the Cameleon seminar in Cambridge in April (see www.dpmms.cam.ac.uk/~tf/cameleon.html), then again at the BLC meeting in Nottingham in September (see www.cs.nott.ac.uk/~nza/blc08/). Many thanks to Thomas Forster (Cameleon) and Natasha Alechina (BLC) for making these presentations possible. Thanks also to the members of the audiences for all their helpful comments.

basilgsmith@btinternet.com September 14 2008