THE ABSOLUTE ARITHMETIC CONTINUUM AND THE UNIFICATION OF ALL NUMBERS GREAT AND SMALL

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In his monograph *On Numbers and Games* [1976], J. H. Conway introduced a real-closed field containing the reals and the ordinals as well as a great many less familiar numbers including -ω, ω/2, 1/ω, √ω and ω-π to name only a few. Indeed, this particular real-closed field, which Conway calls *No*, is so remarkably inclusive that, subject to the proviso that numbers--construed here as members of ordered "number" fields--be individually definable in terms of sets of von Neumann-Bernays-Gödel set theory with Global Choice, henceforth NBG, it may be said to contain "All Numbers Great and Small." In this respect, *No* bears much the same relation to ordered fields that the system of real numbers bears to Archimedean ordered fields.

In a number of earlier works [Ehrlich 1987; 1989; 1992], we suggested that whereas the real number system should merely be regarded as constituting an Archimedean arithmetic continuum, the system of surreal numbers may be regarded as a sort of absolute arithmetic continuum (modulo NBG). In this paper, we will outline some of the properties of the system of surreal numbers that emerged in [Ehrlich 1988; 1992; 1994; 2001] which lend credence to this thesis, and draw attention to some important respects in which the theory of surreal numbers may be regarded as vast generalization of Cantor's theory of ordinals, a generalization which also provides a setting for Abraham Robinson's [1961] infinitesimal approach to analysis as well as for the profound and all to often overlooked non-Cantorian theories of the infinite (and infinitesimal) pioneered by Giuseppe Veronese [1891], Tullio Levi-Civita [1892; 1898], David Hilbert [1899] and Hans Hahn [1907] in connection with their work on non-Archimedean ordered algebraic and geometric systems and by Paul du Bois-Reymond [1870-71; 1882], Otto Stolz [1883], Felix Hausdorff [1907; 1908; 1909] and G. H. Hardy [1910; 1912] in connection with their work on the rate of growth of real functions.

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