Adams*

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*Americans – most of them anyway – seem to have a speech defect that gives them difficulties with stops in certain positions, for example '*t*' after '*n*' and other liquids too, at times. They also seem to have a tendency to voice intervocalic stops. In plain language: they talk funny, the poor dears. (They're not doing it to annoy, they just can't help themselves.). When an American logician tells you that ZFU is ZF with Adams he's not saying that it comes from the Garden of Eden – the foundational claims of set theory notwithstanding. An Adam isn't an *Urmensch*, despite the meaning of the German particle *Ur*. (And it wasn't Adam who came from Ur of the Chaldees in case you were wondering, that was *Abraham*). No, an Adam is an *Urelement* – a set with no internal structure. I was a bit puzzled by this beco's the Adams i know definitely *do* have internal structure. Mind you they're both from America – and so of course talk funny – which may be something to do with it. It's all rather confusing. But it's not just Adams. I was briefly alarmed the other day – in the course of a discussion about forcing – to hear an american colleague talk about the "Cannibal Chain Condition". My clean-living vegetarian brain was immediately assailed by menacing images of chain gangs of slavering flesh-eating zombies roaming the land unchecked, preying on – well, innocent clean-living vegetarians (such as your humble correspondent). I took it personally – as i think anyone would. Of course my Babel fish was eventually able to sort it out ... but it was an awful moment while it lasted.

(Strictly speaking) what my colleague *should* of course have said was the cannibal **antichain** condition.

I found another example later. The American voice-over robot accompanying a sequence of clips from the Muppet Show wot i saw on youtube the other day spoke reverently about the assorted 'immorals' who had appeared on the Muppet Show over the years, and i don't think they meant Bill Cosby (tho' he was one of them). It wasn't their character that was the topic so much as their status as something more than mere evanescent Earthlings. They presumably meant *immortals*.

Someone should larn them to talk proper!

If you are the kind of sad missfit who thinks that set theory is a foundation of mathematics, then you might be interested in setting[1] up Set theory with atoms (non sets) for representing mathematical entities whose internal structure is not in any obvious way set-theoretiic and which accodingly might not be configurable as sets in any natural way. How are we to treat these adams ("atoms NOS[2] as it were)? There are going to be lots of them. How do we tell them apart? Are we to tell them apart set-theoretically, so they have to have different members? Or do we say that they are a different sort of thing from sets, so that they don't have to obey the rules of set-theoretic equality, so two atoms could have the same extension and yet be distinct? The second option is the usual one. If we insist that they behave like sets then the obvious way to distinguish them is to say that an atom has itself as sole member – tell them apart by their labels, and label them with themselves, as it were.

One might think that there is no difference between these two approaches, and that seems to have been the view held by Quine who was (or at least we are going to narrate it so that he was) the first person to advocate the first horn: think of an atom as something identical to its own singleton. Such atoms are called *Quine atoms*. It turns out that there is a difference. It doesn't show up in ZFU, but it does show up in NFU.

I now think i have some sort of understanding of this situation, partly as a result of a conversation with Randall Holmes, who spotted the importance of the function i below. Suppose you are working in a set theory whose name has a suffix 'U' - ZFU etc. You want to turn a model of this theory into a model with no *urelemente* so as to restore extensionality. First thought: turn the (empty) *urelemente* into Quine atoms. How do you do that? Well, you add ordered pairs to the membership relation of the model so that each *urelement* acquires some members and thereby becomes a set. Keep things in the family - turn it into its own singleton. Sadly this doesn't work beco's the resulting model is not extensional: if u was a *urelement* then in the new model u and $\{u\}$ have the same extension but are distinct. However what you can do is a kind of Rieger-Bernays construction: let i be the function that sends every *urelement* u to $\{u\}$ and sends $\{u\}$ to $\{\{u\}\}$ and so on up through all the braces, and fixes everything else. Then the new membership relation ϵ is given by $x \,\epsilon\, y \longleftrightarrow x \in i(y)$. The new model has a new membership relation but has the same carrier set as the old, and it is extensional. Observe that u has become a Quine atom in the new model – as indeed have all the iterated singletons $\iota^n(u)$ – so every old (empty) *urelement* has become a Quine atom in the new model. More precisely, for every *urelement* in the original model we have countably many Quine atoms in the new. This works in ZF or Zermelo in a completely straightforward way, just as intended. I'm not suggesting for a moment that that is what Quine had in mind, but at any rate one can tell the story that way without doing actual violence to the Mathematics.

This procedure can in some sense be reversed. Let us imagine ourselves in

[1] joke! JOKE!!

[2] `https://en.wikipedia.org/wiki/Not_otherwise_specified`

a model of ZF(C) or some such theory with countably many Quine atoms. Let us suppose that this collection of Quine atoms is a set (we may not need this assumption but let's assume it for the moment to keep things simple) We can organise this set of Quine atoms into a copy of ω, so we think of it as a structure $\langle N, 0, S\rangle$ with a designated zero element and a successor relation. We then define a function (also written 'S') which behaves like S on N and fixes everything else. We want 0 to become an empty atom, $S(0)$ to become its singleton, and so on. So we declare a new membership relation by $x \,\epsilon\, y$ iff $S(x) \in y$. This has the desired effect.

> Now if your mind works like mine, you will free associate from this treatment to ideas of *Synonymy* and *Bi-Interpretability*. Any model of ZFU with a unique empty atom can be turned into a model of ZF with countably many Quine atoms and *vice versa*. Whether or not this tells us that ZF + "∃! empty atom" and ZF + "∃ precisely countably many Quine atoms" are synonymous i cannot say. (it even shows how one can turn countably many empty atoms into countably many Quine atoms and back). However at the very least it shows that there is not much at stake in choosing between empty atoms and Quine atoms, *at least when you are doing ZF(C)*. One needs to read the small print; this could be helpful for people who are trying to understand synonymy and bi-interpretabilty, such as your humble correspondent. See the appendix

Now consider the project of showing in NF/NFU that there is (in the above sense) nothing at stake in choosing between empty atoms and Quine atoms. We start with a model that has empty atoms and aspire to obtain from it a model with Quine atoms instead.

Now the injection that turns each empty atom (and – with it – its iterated singletons) into countably many Quine atoms has a flagrantly unstratified definition, so we have no reason to suppose that its graph is a set. We don't actually need it to be a set, but – since we wish the new model to satisfy as much comprehension as the old – we do need it to be at least *setlike*.

What happens if it is setlike? The collection of all *urelemente* is a set (its membership condition is stratified). When we turn the set of *urelemente* into a set of Quine atoms we find that the set is strongly cantorian in the new model; this is beco's every set of Quine atoms is strongly cantorian. This means it must have been strongly cantorian in the original model – beco's R-B constructions preserve strong cantorianness[3]. So if this construction worked reliably then it would be a theorem of NFU that the set of *urelemente* is strongly cantorian. And we know that this is not a theorem of NFU.

It might be an idea to check how the function i introduced above is related to the machinery in Benedikt Löwe's [5] proof that ZF and ZFU are synonymous.

Appendix

Consider the case of ZFU + "∃! empty atom". It's probably a good idea to expand the signature by having a name for the Adam, so let's call it 'a' for

[3]This isn't literally an R-B construction but it presumably has the same good behaviour. A detail to nail down.

Adam. This being ZFU we have unstratified replacement so we can obtain the graph of the function which in the text above i called 'i'. Jump thru' the wormhole into the model with Quine atoms. It's not clear to me whether or not the new model can tell the Quine atoms apart. I'm guessing that it can, so there is a canonical way of thinking of that family as $\langle N, 0, S\rangle$ and doing the return permutation. (This is worth checking!) So if we start with a model of ZFU + "$\exists!$ empty atom" we can go out and get back to where we were. (Tho' our signature does contain a name for the atom). What about the other direction? Suppose we start with a model $\mathfrak{M}$ of ZF + "$\exists$ precisely countably many Quine atoms?" We can obtain thereby a model of ZFU + "$\exists!$ empty atom", and i think it's pretty clear that if we do the construction with Holmes' injection i then we get back the model $\mathfrak{M}$ with which we started. So it looks as if the theory

- ZFU + "$\exists!$2 empty sets" (in the language with a name for the atom)

is synonymous with

- ZF + "$\exists$ precisely countably many Quine atoms, and the Quine atoms form a basis for the illfounded sets" in the language with a name for an enumeration of Quine atoms.

It's rather less clear to me what happens if we consider the two corresponding theories in the unexpanded language of Set Theory. That would probably be a very good thing to think about, a pædogogically helpful illustration.

For example, can we say in the language with ϵ what it is to be the Quine atom that was the unique empty atom in the original model? I'm guessing not, but i don't see how to prove it. This is reminding me of the experience i had just before COVID of Albertus Magnus making me think about the question of how one sets[4] up NFU: does one have a constant symbol for the empty set or does one not? More seems to hang on this than one would like.

Can we do this in NFU? Is Holmes' bijection i setlike in NFU? Beco's – if it is – we can do an RB permutation to obtain a model of NF with countably many Quine atoms. This would give us another proof that very-few-atoms implies Con(NF).

H I A T U S

Starting from a model $\mathfrak{M}$ of ZFU Holmes' injection gives us a model $\mathfrak{M}^*$ of ZF with Quine atoms. $\mathfrak{M}^*$ is nice in the sense that the class of Quine atoms is a basis (see section 1 below) for the bottomless sets: every set-lacking-$\in$-minimal-members contains a Quine atom. We do not need the class of *urelemente* to be a set for this to work. This means (details to follow) that any permutation of the Quine atoms extends to an $\in$-automorphism of $\mathfrak{M}^*$. Thus $\mathfrak{M}^*$ has no definable way of getting back to $\mathfrak{M}$.

So this means that any model of ZFU with precisely two empty sets can be turned into a model of ZF with $\aleph_0$ Quine atoms and *vice versa*. So if i start

[4] joke! JOKE!!

with a model of ZFU + $\exists!2$ empty sets i can turn it into a model of ZF with $\aleph_0$ Quine atoms (which form a basis for the bottomless sets) and when i come back i get something isomorphic to the model i started with, but it might not be identical (beco's i can't tell which Quine atom was the *urelement* in the original model). Does this make sense? Consider a model of ZF + there are precisely $\aleph_0$ Quine atoms. Now we know how to kill off $\aleph_0$ Quine atoms when they form a basis for the bottomless sets to get a model of ZF. So this gives another proof of Benedikt's result in [5] (or something very like it) if a bit more circuitous.

Here's what i've been thinking. Start with a model of ZFU containing precisely $\aleph_0$ atoms. Use your injection to obtain from it a model of ZF with $\aleph_0$ Quine atoms. These atoms form a basis in the sense of section 1. This means you can extend functions defined on the Quine atoms to the whole model, by recursion. In particular every permutation of the atoms extends to an $\in$-automorphism of the model. So there is no way of telling which Quine atoms corresponded to urelements of the original model. However one can "undo" the injection and get back a model with urelemente instead of Quine atoms. It will be isomorphic to the model we started with. Now the other thing one can do, since the Quine atoms form a basis in the sense of the attached pdf, is perform an R-B permutation exercise to get rid of them all and get a model of ZF *tout court* with neither *urelemente* nor Quine atoms. And of course all this is reversible. So one gets Benedikt's result. Albeit in a roundabout manner…

1 Basis

An old file, of good stuff.

Definition 1 *Let $\langle X, R\rangle$ be a binary structure.*
An R**-bottomless set** *is a subset $Y \subseteq X$ s.t. $(\forall x \in Y)(\exists y \in Y)(yRx)$.*
We say $B \subseteq X$ is a **basis** *if it meets every R-bottomless set.*
We say X **is dense in** *A if every R-bottomless subset of A meets X.*

This is an important idea for a variety of reasons. One of them comes from Set Theory. If f is a function defined by $\in$-recursion then it has a unique extension from any basis to the whole of V:

Theorem 1 *Let $\langle X, R\rangle$ be a binary structure and*[5]

$g : X \times V \to V$ an arbitrary (total) function;
f a (total) function $B \to V$;[6]
B a basis for the R-bottomless subsets of X.

Then

there is a unique total function $f^* : X \to V$ satisfying
(1) $f^* \restriction B = f$;
(2) $(\forall x \in (X \setminus B))(f^*(x) = g(x, f^*``\{y : R(y, x)\}))$.

[5] I'm assuming that this is a standard result. However i have no idea what to cite!

[6] Here V is the universe, so that when we say "$g : X \times V \to V$" we mean only that we are not putting any constraints on what the values of g (or its second inputs) are to be.

Proof:

The idea is very simple. We obtain our best candidate for f^* by closing the graph of f under the operation that adds ordered pairs according to the clause that says that $f(x)$ is to be $g(x, f``\{y : R(y, x)\})$.

Now suppose (2) fails, so the subset Y of X on which f^* is not uniquely defined is nonempty. This set Y is bottomless because $f(x) =: g(x, f``\{y : R(y, x)\})$. So it meets B, since B is a basis. But then f^* is defined on at least some of Y.

■

If R is clear from context [and henceforth it will always be $\in$] we shall merely say "bottomless".

This modification of the recursion theorem looks trivial, but there are cases where it is useful. For example, one can add Quine atoms to a model of ZFC by Rieger-Bernays permutation methods in a controlled way so that every $\in$-bottomless class contains a Quine atom. But then any function defined on the Quine atoms can be uniquely extended to the whole universe.

Remark 1 *Let $\mathfrak{M}$ be a model of ZF(C)-with-foundation, and a an element of $\mathfrak{M}$ about which we otherwise know nothing.*

Then, in $\mathfrak{M}^{(a,\{a\})}$, $\{a\}$ is the unique Quine atom, and it is a basis for the illfounded sets.

Proof:

Let A be a bottomless set in $\mathfrak{M}$. If nothing in A is moved then A would have been bottomless in $\mathfrak{M}$—and therefore empty. So A contains a or $\{a\}$. If it contains a we have what we want. So suppose it contains $\{a\}$ but not a. But then $A \cup \{a\}$ is bottomless in $\mathfrak{M}$, which contradicts the assumption that $\mathfrak{M} \models ZF(C)$-with-foundation.

■

Secondly, as we shall see, it is quite easy to start from a wellfounded model to obtain an illfounded model by R-B methods and retain control of the basis. Finally, certain natural conditions on bases enable us to to obtain wellfounded models from illfounded models by R-B methods. Another way to motivate the idea of a basis is the factoid that the late Jon Barwise used to call "The Solution Lemma". (See [1] chapter 6). It is a consequence of the Forti-Honsell antifoundation axiom that every system of equations in the style

$$x_1 = \{\emptyset, x_2, x_3\}; \quad x_2 = \{\{\emptyset\}, x_3\}; \quad x_3 = \{\emptyset, x_1\} \qquad ((1))$$

has a unique solution.

Let's show how to add such bad sets by permutations. Start with three sets x_1, x_2 and x_3—it won't much matter what they are, but let's take them to be von Neumann reals, or something large and remote like that—and consider the product of the three transpositions: $(x_1, \{\emptyset, x_2, x_3\})$, $(x_2, \{\{\emptyset\}, x_3\})$ and

$(x_3, \{\emptyset, x_1\})$. In the resulting Rieger-Bernays permutation model x_1, x_2, x_3 form a solution to the system of equations (1). My guess is that this relies on the transpositions being disjoint—but this is easy to arrange.

I'm guessing that this is in [1] but this should be checked

What is the feature of interest here? There are these things we're inventing, namely the xs, and we are declaring them in terms of each other and some *wellfounded* sets. In this new model the bad sets x_1, x_2, x_3 form a basis. [we should really prove this!] This basis B has the nice property that $TC(B) \setminus B$ is wellfounded: the bad sets are declared in terms of each other and wellfounded sets only. This is the key to getting rid of them later, as we shall see.

Using Bases to get rid of illfounded Sets by Permutations

There is an isomorphism between the wellfounded part of $\mathfrak{M}$ and the wellfounded part of $\mathfrak{M}^\sigma$, defined by the obvious recursion: $i_\sigma(x) =: \sigma^{-1}(i_\sigma \text{``} x)$. Clearly this is defined on all the wellfounded sets of $\mathfrak{M}$. Conversely let $x$ be an $\in_\sigma$-minimal wellfounded set of $\mathfrak{M}^\sigma$ not in the range of $i_\sigma$. $x$ is $\sigma^{-1}(i_\sigma \text{``} y)$ for some set y, but then by replacement in $\mathfrak{M}$ this thing is a set of $\mathfrak{M}$.

It is also standard in the literature that $\mathfrak{M}$ and $\mathfrak{M}^\sigma$ satisfy the same stratified formulæ.

Now let us think about what has to happen for there to be a permutation σ available in an illfounded model $\mathfrak{M}$ so that $\mathfrak{M}^\sigma$ is wellfounded.

Consider the model $\mathfrak{M}$ obtained from a wellfounded model of some theory like ZF by swapping $\emptyset$ with $\{\emptyset\}$. This model has a single Quine atom a which is a basis for the bottomless sets. Now we can get get rid of this object by means of the transposition $(a, \{\emptyset\})$—σ for short. Suppose X is a collection of sets which is bottomless in the sense of $\mathfrak{M}^\sigma$. That means X must be inhabited by at least some of the things that are moved by σ. For suppose it did not. Then $\in_\sigma \restriction X$ would be the same as $\in \restriction X$ and X would have to contain a since $\{a\}$ is a basis. But if $a \in X$, X does indeed have an $\in_\sigma$-minimal element, namely a. And if $\emptyset \in X$ then $a \in X$, since $\emptyset$ is $\{a\}$ in the sense of $\mathfrak{M}^\sigma$. The moral of this is that a single Quine atom added to a wellfounded model by a single transposition can be got rid of by a single transposition and the state of nature restored.

Let's think about how we might in general be able to use the existence of a basis to cook up a permutation model in which the illfounded sets have been killed off.

[At some point we have to spell out how if we start with a model of ZF + foundation, then any permutation model obtained from it has the same wellfounded sets. So any permutation either adds illfounded sets or results in something isomorphic. Are there any permutations which do not add illfounded sets? Is every illfounded model of ZF elementarily equivalent to a permutation model of a wellfounded model?]

Theorem 2
Suppose $\mathfrak{M}$ is a model with a basis B for the bottomless sets and σ is a permutation such that

$$(\forall b \in B)(\emptyset \in TC(\sigma(b))) \textit{ and}$$
$$(\forall b_1, b_2 \in B)(\sigma(b_1) \notin TC(b_2)).$$

Then $\mathfrak{M}^\sigma$ contains no illfounded sets.

Proof:
Let σ be an involution swapping every basis element with something yet to be determined, but otherwise leaving non-basis elements alone. Suppose now that A is a family of sets in that is bottomless in the sense of V^σ. We will show that A is empty$^\sigma$. It cannot consist entirely of fixed elements beco's if it did it would be a family of sets of V with no $\in$-least member and would meet the basis—and everything in the basis is moved. So it contains either a basis element or σ of a basis element. (Notice that this *reductio* argument doesn't prove that it must contain a basis element, merely that if *per impossibile* everything in it were fixed, it would contain a basis element... and i'm not sure how constructive this proof is.). To take care of the possibility that it contains a basis element (but not $\sigma(b)$ for any basis element b) we must ensure that $\emptyset \in TC(\sigma(b))$ for every basis element b. But if it contains $\sigma(b)$ for b a basis element we find that it contains everything inside the basis element and we seem to be back where we started. We can argue as before that something must be moved. If the moved thing we have found is a $\sigma(b)$ then we have made no progess. What we want to ensure is that one of the moved things that we find inside it is another basis element—not σ of a basis element. So it will suffice to ensure that whenever b_1 and b_2 are basis elements then $\sigma(b_1) \notin TC(b_2)$. ■

[looking at this later] It seems we have shown that if there is a [set] basis B such that $(\forall b_1, b_2 \in B)(\sigma(b_1) \notin TC(b_2))$, then we can swap everything in B with a wellfounded set, extend the bijection to a permutation and the resulting permutation model is wellfounded.

It seems to that we want now to do two things.

(i) Modify this result into something that holds for sets of Quine atoms not just a single Quine atom;

(ii) we've got to show that the obvious permutation model containing the new illfounded sets satisfies the (modified) condition of theorem 2. That will give us the synonymy result we want.

We can actually get by with slightly weaker conditions on σ than this. Let us say a basis element b **has rank 0** if $\sigma(b)$ is wellfounded and everything in its transitive closure is fixed. Then if basis elements of rank 0 are dense in a bottomless$^\sigma$ set A we obtain a contradiction. Indeed we can say that a basis element b has rank β if there is a set X of basis elements of lower rank s.t. $\sigma``X$ is dense in $TC(b)$. All we have to do is ensure that every basis element has a rank.

It might be worth getting a picture of where the atoms go in the wellfounded R-B model. A Quine atom q becomes the set it's swapped with, and that set becomes its singleton, and *that* singleton becomes the singleton^2 and so on as in Hilbert's Hotel.

The conditions sound quite strong, but they are satisfied in the case that really matters to us, namely the case where the illfounded sets all arise from Quine atoms. We'd better check this!

Suppose X is a collection of sets without a $\in_\sigma$-minimal element, where σ is a product of transpositions $(x, \{x\})$ over a wellfounded model. X cannot consist entirely of fixed sets, beco's all fixed sets are wellfounded. So it must contain either an x or a $\{x\}$. Clearly it has to be an x, since $\{x\}$ is really x (in the sense of $\mathfrak{M}^\sigma$). So that's OK.

Remark 2 *If B is a basis for $\mathfrak{M}$, then it is a basis for any transitive submodel of $\mathfrak{M}$.*

It may be that whenever there is a set basis for the bottomless sets then there is a permutation satisfying the conditions of theorem 2.

A Rieger-Bernays permutation using complementation proves immediately that the two theories NF + $(\exists x)(x = \{x\})$ and NF + $(\exists x)(x = V \setminus \{x\})$ are synonymous.

I now see—as a result of a conversation with Benedikt Löwe at NF76—that the analysis above shows that

Corollary 1 *ZF (with foundation) is synonymous with ZF + "there is a unique Quine atom, and it is a basis for the bottomless sets".*

Proof:

Let $\mathfrak{M} \models$ ZF with foundation. Then $\mathfrak{M}^{(\emptyset,\{\emptyset\})} \models$ ZF + "there is a unique Quine atom, and it is a basis for the bottomless sets". This is remark 1.

For the other direction

Let $\mathfrak{M} \models$ ZF + "there is a unique Quine atom and it is a basis for the bottomless sets", and let a be that unique Quine atom. Then $\mathfrak{M}^{(a,\{\emptyset\})} \models$ ZF with foundation. That was theorem 2. ■

So what remains to be shown is that a model that contains a solution to something like formula (1) as a result of a permutation construction outlined there can be turned back into the original model.

Let's do this properly

Consider the theory T which will be ZF (with or without choice, it doesn't matter) with the highly specific axiom that says that there is a set of Quine atoms A such that every bottomless set meets A. Observe that this does not contradict foundation, since there is nothing in it to say that A is nonempty.

The claim is going to be that $T + A = \emptyset$ and $T + A \neq \emptyset$ are synonymous.

That means we have to have two methods, one that gives us a model of $T + A = \emptyset$ from a model of $T + A \neq \emptyset$, and the other that gives us a model of $T + A \neq \emptyset$ from a model of $T + A = \emptyset$. And these two methods have to be mutually inverse.

The obvious weapon is R-B permutation models.

Actually let's use NF instead of ZF to start with. It provides a simpler context.

To obtain a model of $T + A \neq \emptyset$ from a model of $T + A = \emptyset$ we use the obvious permutation $\Pi_{i \in \mathbb{N}}(i, \{i\})$ for some suitable implementation of $\mathbb{N}$ about which we hope we don't have to be specific.

To obtain a model of $T + A = \emptyset$ from a model of $T + A \neq \emptyset$ we use the standard permutation (was it in Scott?) $\Pi_x(\iota^2(x), V \setminus \iota^2(x))$.

Suppose we start with a model $\mathfrak{M}$ of $T + A = \emptyset$. Let us identify three membership relations, (i) $\in_0$ of $\mathfrak{M}$, (ii) $\in_1$ the membership relation of the first permutation model (in which there are countably many Quine atoms) and (iii) $\in_2$ the membership relation of the second permutation, whence all the Quine atoms have been banished, and which should turn out to be the same as $\in_0$.

$$x \in_2 y. \longleftrightarrow (\exists z)(y = \iota^2(z) \vee y = V \setminus \iota^2(z)) \longleftrightarrow x \in_1 y)$$

where the iotas are taken in the sense of $\in_1$.

TO BE CONTINUED

Now $x \in_1 y$ iff

$(y \in \mathbb{N} \wedge x = y) \vee (y \in \iota``\mathbb{N} \wedge x \in \iota^{-1}(y)) \vee (y \in (V \setminus \mathbb{N}) \setminus \iota``\mathbb{N} \wedge x \in y)$.

This is not a back-of-an-envelope calculation; this is a two-sheets–of-A3 calculation!

Perhaps it is infeasible to do it by brute force; perhaps we need to be clever.

$x \in_2 y$ is the same as $x \in_1 y$ except when y is a double singleton or the complement of a double singleton in the sense of $\in_1$. So what is it to be a double singleton in the sense of $\in_1$? You might be an innocent, ordinary, double-singleton ... in which case you are a double-singleton in the sense of $\in_0$; or you might be a Quine atom, in which case you are in $\mathbb{N}$ in the sense of $\in_0$. What about the complement of a double singleton? You might be the complement of an ordinary double singleton (in which case you are a complement-of-a-double-singleton-in-the-sense-of $\in_0$) or you might be the complement of a Quine atom, and that is the same as being a complement of a member of $\mathbb{N}$ in the sense of $\in_0$.

I suspect you end up with $\in_2$ not being the same as $\in_0$ because of its effects on double-singletons. It probably isn't even isomorphic (tho' that would be enough for synonymy).

We probably really do need to tackle head-on the ZF situation this note started with. The two theories will be ZF + foundation on the one hand and, on the other, the theory ...

seems to have been truncated somehow.

References

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