This house does not accept the axiom of choice

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When i say you shouldn't accept the axiom of choice, i don't mean that you should *never* accept it—AC is easily provable in the finite world. What i mean is that there are circumstances in which you should not accept it.

It's no accident that AC came to the attention of mathematicians at the same time that we started to take seriously the actually-infinite. Taking the actually-infinite seriously posed a real challenge to the correct use of our intuitions, since those intuitions arose from our aquaintance with finite objects. Infinite objects behave differently from finite objects, and intuitions developed for finite objects can lead one astray. In the finite realm ordinals and cardinals are in some sense the same: all total orderings of a given finite set are isomorphic. In the infinite world cardinals and ordinals are different things. Facts about finite sets do not easily port to infinite sets. For example: If A and B are finite sets and there is an injection $A \hookrightarrow B$ that is not a surjection [or, for that matter, a surjection $B \twoheadrightarrow A$ that is not an injection] then there is no injection $B \to A$: $|A| < |B|$. But clearly this does not work for infinite sets. Take A to be the evens and B to be $\mathbb{N}$. It took us quite a while to get our heads round this. Galileo worried about it, and that was a long time ago. He took it to mean that there was no sense to be made of infinite collections. Some guff about the whole being commensurable (or some such word) with one of its parts.

We've come to terms with that now, but some of us have still not drawn the appropriate conclusions about the axiom of choice. Crudely, the argument about AC is whether AC is one of those cute facts about finite sets that holds for infinite sets, or whether it is one of the things that fails, and this is something that needs to be thought about, not casually assumed. It is clear that every finite set can be wellordered: $|\{m : m < n\}| = n$ easily provable by induction. But $\{m : m < n\}$ is clearly wellordered, and if we can wellorder even one set of size n we can wellorder them all. The fact that this argument doesn't generalise to infinite sets doesn't mean that AC fails for [some] infinite sets, but it does flag up a warning that AC *might* fail, just as the fact that if-there-is-a-proper-injection-from-A-into-B-then-$|A| < |B|$ doesn't work if A and B are infinite.

So, given a mathematical principle that works for finite objects, how can we tell whether or not it holds for infinite objects as well? Since our intuitions are moulded in the first instance by acquaintance with the finite, the danger we have to guard against is a hasty "yes it works". If we are to accept, for infinite objects, a principle of which we know only that it holds for finite objects, we need a better story than that. I suppose there may be an kind of default assumption that principles that hold in the finite world should be assumed for the infinite world as well unless something goes obviously wrong

And it is my contention that for the axiom of choice we don't really have a good enough story to be entirely confident that it holds. Certainly not good enough for us to just straightforwardly accept it. And that is why i am proposing the motion tonight.

Let me dissect one consideration that leads people to think that AC is obviously true.

[usual fallacy of equivocation]

There is another thing that causes people to think they need AC in circumstances when in fact they don't, and that is a mistaken belief that you need AC to make a *single* choice. Every partition into pairs of a set X that can be embedded into the plane has a choice function. By equipping the plane with a pair of axes one can choose uniformly between any two points—one can choose from infinitely many pairs. Choosing once-for-all a pair of axes does not need choice.

Finally AC an invitation to laziness. It enables you to state things with more generality than you actually need. EVERY set of pairs has a choice function. Well, obviously! Have you ever come across a vector space that didn't have a basis? Neither have i. The problem with asserting more generality than you actually need is that you reach for the axiom of choice to secure that extra generality and this launches you on a massive cascade of error: the proofs you come up with use the axiom of choice, you then think that the axiom of choice must be part fo the reason why these things-you-have-just-proved are true, and you then reason backwards by inference-to-the-best-explanation to the conclusion that the axiom of choice must be true too, or at least that you need to adopt it.