Quine's Set Theory NF: a Briefing in the Light of Recent Developments; A talk for The Oxford Logic Seminar February 24/5 2025

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Quine's Set Theory NF is now known (after 85 years!) to be consistent. In this talk I will outline the history leading to Holmes' proof of this fact, sketch how it works, explain what it means, and why we should care. I am aiming this talk at a general – if sophisticated – Logical audience.

NF = Extensionality + stratified comprehension.

1937 Quine [10] invents NF as a simplification of [3] Simply Typed Set Theory;
1950 Rosser and Wang [14] show that NF has no β-models[1];
1953 Specker [11] refutes Choice (and thereby proves Infinity) in NF;
1958-1960 Rôle of Typical Ambiguity clarified by Specker [12] and [13];
1969 Jensen [8] proves the consistency of NFU (= NF with *urelemente*);
2010 Holmes proves Con(NF);
2020's `Lean` verification [7] of Holmes' proof; synonymy questions broached.

The notes for this talk are at `www.dpmms.cam.ac.uk/~tef10/oxfordtalk.pdf`

The Meaning of Stratification

Quine's [10] title wasn't "A New Set of Axioms for Set Theory" but "New Foundations for Mathematical Logic"; the phrase that comes to mind nowadays is that NF is an *Alternative Set Theory* but without the ideological associations that 'alternative' had in my youth – or those it has now (particularly in Germany). Nevertheless there is the question of whether or not the foundation offered by NF is in any real sense an alternative to that offered by ZF(C). To answer that we need to think about *stratification*.

What does stratification *mean?* Read [4]. Mathematics is stratified. Even Paris-Harrington is stratified.

[1] $\mathfrak{M}$ is a β-model iff every structure in $\mathfrak{M}$ that $\mathfrak{M}$ believes to be wellfounded is in fact wellfounded

Synonymy

Synonymy has recently become fashionable. One of the references I invoke is a set of slides prepared by Albert Visser for a talk at this very university. See [15]

Church's set theory CUS best characterised by its model construction. Ackermann, also [9]. Work of Button [1] shows that CUS genuinely is just syntactic sugar for ZF. In contrast Forster-Holmes [5] show that NF is *not* synonymous with any theory of wellfounded sets, in particular not with ZF.

What's it like working in NF?

NF does not contradict ZF; it goes beyond it. It certainly contradicts foundation but there is nothing to say that the wellfounded sets of a model of NF do not satisfy ZF. Indeed it seems that the following is true: *every model of Zermelo has an end-extension that satisfies NF* and we are hopeful of strengthenings. Holmes has outlined a proof of this to me. I hope he will publish it in [7].

Lots of "big" collections are sets. The universal set, the set of all cardinals, of all ordinals, the symmetric group on the universe, the free group on the universe, the set of all groups, of all rings, of all fields. All these things have stratified definitions. The complement of a set is a set. The collection of von Neumann ordinals does not have a stratified definition so it doesn't have to be a set. In fact it can't be a set. A similar thing happens with the class of transitive sets.

Freewheeling inductive definitions. "Top Down". Easy to define (for example) $\mathbb{N}$ as the intersection of all sets containing 0 and closed under $\mathtt{succ}$, because the existence of V means that the intersection is not vacuous.

Everything has to be stratified, but then Mathematics *is* stratified so no probs. Explain why inductions have to be stratified. Since $\mathbb{N}$ is the intersection of all sets of cardinals containing 0 and closed under $\mathtt{succ}$ the way to prove that every natural number has property F is to show that $\{x : F(x)\}$ is one of the sets containing 0 and closed under $\mathtt{succ}$. For this to work $\{x : F(x)\}$ has to be a set, and this can be relied on this only when F is stratified.

Background to Holmes' Proof

The obvious first question about NF is "Is it consistent?" Rosser-Wang [14] and Specker [11] are a *huge* problem beco's the fact that NF refutes AC and has no β-models means that the consistency of NF isn't going to follow in any simple-minded way from the consistency of Simply Typed Set Theory (a theory which is consistent with AC and does have β-models).

NF is not as marginal as some people would have you believe. Over the years lots of people – big guns – have taken an interest. Quine, Rosser, Church, Harvey Friedman, Dana Scott, Feferman, Solovay, Jensen, Gandy (someone in this audience may have access to Gandy's *Nachlaß* ...) these are not the *B* team. Three of them were even Oxford people (Scott at Merton, Jensen at All Souls and Gandy at Wolfson)

There have been lots of lines of talk that NF should be consistent but practically all of them fall at the first hurdle, namely "*Why does your method not also demonstrate* *the consistency of NF + AC?*'. It gradually became clear that the only method one might think of using that doesn't fail in this way is the Fraenkel-Mostowski permutation method FM, but there didn't seem to be any way of bringing FM to bear. Randall Holmes finally found a way of using it: his key move in [6] was isolating the idea of *tangled types* from Jensen's paper [8]. The idea is there in [8] but is not identified or highlighted. But even so, it took another twenty years for Holmes to see how to apply FM methods to obtain models of Tangled Type Theory aka TTT.

> I think it is now becoming evident to people in NF studies that the idea of Tangling is as important as Typical Ambiguity – and possibly more fertile. Or perhaps one should say rather that TTT is the correct way to unfold/unravel T$\mathbb{Z}$T: if T$\mathbb{Z}$T is going to tell us anything about NF it will be through the medium of Tangled Type Theory.

Talk over the consistency of NFU, and [6]. What Jensen does in [8] can be characterised as *"Every model of T$\mathbb{Z}$T can be expanded to a model of TTTU"*. The other good idea in [8] is the application of Ramsey's theorem.

Fragments. *i*NF, NFU, NF_2, NF_3, NF0, NF^{pf} (the parameter-free version), Predicative NF: NFP and NFI

Specker 1953 has been mechanised. `https://us.metamath.org/nfeuni/nchoice.html`

Failure of cartesian-closedness. NF is finitely axiomatisable; this is beco's stratified Δ_0 separation is finitisable (stratified version of Gödel's rudimentary functions, tho' there are other proofs) So are are NF_2, NF0. Not clear what the significance is. Not known whether *i*NF, NF_3 or NF^{pf} are finitely axiomatisable.

Not known whether or not NF^{pf} refutes choice/proves infinity. Does NF refute the Prime Ideal Theorem? The P.I.T. fails in Holmes' models. Does *i*NFinterpret Heyting Arithmetic?

Most items listed below are in the *Comprehensive NF Bibliography* at `https://randall-holmes.github.io/Bibliography/setbiblio.html`; Many of the items listed there have links to actual text.

References

[1] Tim Button "Boolean Level Theory Part 3: a Boolean Algebra of Sets arranged in wellordered levels", Bulletin of Symbolic Logic **28**, Issue 1, March 2022, pp. 1 - 26; `https://doi.org/10.1017/bsl.2021.15`

[2] Nathan Bowler and Thomas Forster "Internal Automorphisms and Antimorphisms of Models of NF" Journal of Symbolic Logic *in press*.

[3] Rudolf Carnap "Abriss der Logistik" Schriften zur Wissenschaftlichen Weltauffassung **2**, Wien, Julius Springer 1929, 114pp.

[4] Thomas Forster "The Burali-Forti Paradox" The Reasoner **17** sept 2023, pp. 40–41. Also at `https://www.dpmms.cam.ac.uk/~tef10/buraliforti.pdf`

[5] Thomas Forster and Randall Holmes "Synonymy Questions concerning the Quine Systems" Journal of Symbolic Logic, *in press*.

[6] Randall Holmes "The Equivalence of NF-style Set Theories with "tangled" Type Theories; the Construction of ω-models of predicative NF (and more)". Journal of Symbolic Logic **60** (1995) pp. 178-189.

[7] Holmes and Wilshaw: "The Consistency of NF" `https://arxiv.org/html/1503.01406v22`

[8] Jensen, R.B. [1969] "On the consistency of a slight(?) modification of Quine's NF'; Synthese **19** pp. 250-263.

[9] Richard Kaye and Tin Lok Wong. "On interpretations of arithmetic and set theory". Notre Dame Journal of Formal Logic **48**, Number 4 (2007), 497–510.

[10] Quine, W.V. "New Foundations for Mathematical Logic" (1937) most easily accessible in multiple paperback editions of *From a Logical Point of View*.

[11] Specker, E.P. [1953] "The Axiom of Choice in Quine's New Foundations for Mathematical Logic" Proceedings of the National Academy of Sciences of the USA **39** pp. 972-975. `http://www.pnas.org/cgi/reprint/39/9/972`
There is a computer-verified version of this proof on
`https://us.metamath.org/nfeuni/nchoice.html`

[12] Specker, E.P. "Dualität" Dialectica **12** (1958) pp. 451-465. Annotated English translation by tf available at `https://randall-holmes.github.io/Bibliography/dualityquinevolume.pdf`

[13] Specker, E.P. "Typical Ambiguity", Logic, Methodology and Philosophy of Science (Proc 1960 International Congress.), Stanford, 1962, pp. 116–124.

[14] Wang, H. and Rosser, J.B. "Non-standard models for formal logic", Journal of Symbolic Logic **15** (1950) pp. 113–129.

[15] Visser, A. "Categories of theories and interpretations"

https://www.academia.edu/48566741/Categories_of_theories_and_interpretations