Permutation Models for Set Theory

and undefinability of cardinality

Yuan Yuan Zheng

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INTRODUCTION

0.1 Permutation Models

In the 1920s, Fraenkel suggested a method to show that the Axiom of Choice is unprovable. Then, in the 1930s, Mostowski worked out Fraenkel's ideas, and introduced a construction of models known as Fraenkel-Mostowski models, or *permutation models*. This Fraenkel-Mostowski method was then modified by Specker, to get the so-called Fraenkel-Mostowski-Specker method.

However, the method does not solve the independence problem of the Axiom of Choice in Zermelo-Fraenkel set theory (ZF). It gives models in which the Axiom of Choice fails, but in a universe which allows *atoms* (empty sets which are not $\emptyset$), or *Quine atoms* ($x = \{x\}$) in the Specker case. Clearly, the existence of atoms contradicts the Axiom of Extensionality, and the existence of Quine atoms contradicts the Axiom of Foundation.

In spite of this, this method is a particularly simple tool for proving consistency results involving the Axiom of Choice. It also inspires a method using forcing, which does give the independence of the Axiom of Choice in ZF. After the advent of forcing, most results obtained in the modified set theory can be transferred straight into ZF.

0.2 Undefinability of Cardinality

We know that cardinal numbers are definable in ZF with or without the Axiom of Choice. But what if we don not have the Axiom of Foundation? With the help of a permutation model, we can prove that in absence of both the Axiom of Choice and the Axiom of Foundation, there is no adequate definition of cardinality.

0.3 Outline

We shall concentrate on the Fraenkel-Mostowski-Specker version of the method. That is, we sacrifice the Axiom of Foundation to have Quine atoms in our universe.

In Chapter 1, we give some relevant definitions and axioms; then we prove the consistency of the negation of Foundation with other axioms in ZF. So we are within our rights to sacrifice the Axiom of Foundation. You are welcome to skip this chapter if you have decided to believe that the universes we will be working in are all consistent.

In Chapter 2, we introduce the surrounding universe. We construct a general Fraenkel-Mostowski-Specker model in it, and show that the model satisfies all the axioms in ZF except the Axiom of Foundation. We also have two examples, in which the Axiom of Choice fails, and this will shed some light on the next Chapter.

Finally in Chapter 3, we recall what we mean by cardinality. Using a permutation model very similar to one in Chapter 2, we prove that cardinality is undefinable in ZF without the Axiom of Foundation. We work in a slightly different universe, whose validity is justified in Chapter 1.

In the last chapter, we briefly discuss the independence of the Axiom of Choice from ZF, and other results given by permutation models.

0.4 Acknowledgement

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CONSISTENCY RESULTS

1.1 Preliminary

First of all, let us agree on the axioms of ZF:

(Empty) $\exists x \forall y (y \notin x)$

(Ext) $\forall x, y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$

(Pair) $\forall x, y \exists z [\forall u (u \in z \leftrightarrow u = x \vee u = y)]$

(Union) $\forall x \exists y \forall z [z \in y \leftrightarrow \exists u (z \in u \wedge u \in x)]$

(Inf) $\exists x, y [y \in x \wedge \forall z (z \in x \rightarrow \exists u (u \in x \wedge u \neq z \wedge \forall v (v \in z \rightarrow v \in u)))]$

(Power) $\forall x \exists y \forall z [z \in y \leftrightarrow \forall u (u \in z \rightarrow u \in x)]$

(Rep) $\forall x_1, ..., x_n [\forall x \exists! y \phi(x, y) \rightarrow \forall a \exists b \forall y (y \in b \leftrightarrow \exists x (x \in a \wedge \phi(x, y)))]$
for each formula ϕ with free variables $x, y, x_1, ...x_n$

(Found) $\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \forall z (z \notin y \vee z \notin x))]$

These are the axioms of ZF.

The Axiom of Choice says:

(AC) For every family F of nonempty sets, there is a function f such taht $f(S) \in S$ for each set S in the family F.

If the above holds, then we say that f is a *choice function* on F.

Recall that an ordering $<$ of a set S is a *well-ordering* iff every non-empty $x \subseteq S$ has a least element in the ordering $<$. The Well-ordering Principle says:

> Every set can be well-ordered.

(AC) is equivalent to the Well-ordering Principle.(See [6] for a proof.) So we will use them interchangeably.

x is a *Quine atom* iff $x = \{x\}$. Note that this is equivalent to $\forall y(y \in x \leftrightarrow y = x)$. So the existence of Quine atoms contradicts (Found). By ZF$^{\circ}$we mean ZF without (Found).

We assume the consistency of ZF. $\langle V, \in \rangle$ is a universe V with membership relation $\in$.

1.2 The Consistency of ¬(Found) with ZF$^{\circ}$

Before working in a universe with Quine atoms, we would like to know if the existence of Quine atoms is relatively consistent. In this section we prove the consistency of the negation of the Axiom of Foundation ¬(Found) with ZF$^{\circ}$.

Let us make precise what a permutation of the universe is.

Definition. Let $\phi(x, y)$ be a ZF-formula with free variables x, y. ϕ defines a *permutation* of the universe if it satisfies the following:

(P1) ϕ is a function: $ZF \vdash \forall x, y, z[\phi(x, y) \wedge \phi(x, z) \rightarrow y = z]$

(P2) ϕ is injective: $ZF \vdash \forall x, y, z[\phi(x, y) \wedge \phi(z, y) \rightarrow x = z]$

(P3) ϕ is surjective and defined on the universe:$ZF \vdash \forall x \exists y \phi(x, y) \wedge \forall y \exists x \phi(x, y)$

A permutation ϕ gives a corresponding class

$$F = \{\langle x, y \rangle : \phi(x, y)\}$$

and we say

$$F(x) = y \text{ iff } \langle x, y \rangle \in F$$

We may say that F defines a permutation if its corresponding ϕ does.

Note that, by symmetry of each of (P1)-(P3), we have that ϕ^{-1} is a permutation iff ϕ is a permutation, where ϕ^{-1} is defined such that $\phi^{-1}(x, y) = \phi(y, x)$. Hence we can define F^{-1} correspondingly, with $F^{-1}(x) = y$ iff $F(y) = x$.

Definition. Let ϕ be a permutation of the universe, and let F be its corresponding class.

(a) Define a binary predicate $\in_F$ by
$$x \in_F y \text{ iff } F(x) \in y.$$

(b) For a ZF-formula ψ, let $Rep(F, \psi)$ be the formula obtained from ψ by replacing the symbol $\in$ by $\in_F$ at all places of occurence. Write $Rep(F, \psi)$ as ψ_F for short.

Theorem 1.1. *Suppose that F defines a permutation of the universe $\langle V, \in\rangle$. Then $Rep(F, -)$ is a* syntactic model *of ZF^o in ZF. i.e.*
$$ZF \vdash Rep(F, \psi) \text{ for every axiom } \psi \text{ of } ZF^o$$

Proof. We consider each of the axiom of ZF^o.

(Empty) $\exists x \forall y (y \notin_F x)$, i.e. $\exists x \forall y (F(y) \notin x)$
Let $x = \emptyset$, the real empty set. Then the axiom holds.

(Ext) $\forall x, y[\forall z(F(z) \in x \leftrightarrow F(z) \in y) \to x = y]$
We assume that
$$\forall z(F(z) \in x \leftrightarrow F(z) \in y)$$
Since F is surjective by (P3), for all $b \in x$ there exists c such that $F(c) = b \in x$. Hence $b = F(c) \in y$. This gives
$$\forall b(b \in x \to b \in y)$$
Similarly we have $\forall b(b \in y \to b \in x)$. Then by (Ext) in $\langle V, \in\rangle$, we have $x = y$ as required.

(Pair) $\forall x, y \exists z[\forall u(F(u) \in z \leftrightarrow u = x \vee u = y)]$
Let $z = \{F(x), F(y)\}$. z is a set by (Pair) in $\langle V, \in\rangle$. Since F is a permutation, we have
$$F(u) \in z \text{ iff } u = x \vee u = y$$

(Union) $\forall x \exists y \forall z[F(z) \in y \leftrightarrow \exists u(F(z) \in u \wedge F(u) \in x)]$
Let $y = \{F(z) : \exists u(F(z) \in u \wedge F(u) \in x)\}$. y is a set by (Rep) and (Union) in $\langle V, \in\rangle$. Since F is injective by (P2), we have
$$F(z) \in y \text{ iff } \exists u(F(z) \in u \wedge F(u) \in x)$$

(Inf) $\exists x, y[F(y) \in x \wedge \forall z(F(z) \in x \to \exists u(F(u) \in x \wedge u \neq z \wedge \forall v(F(v) \in z \to F(v) \in u)))]$
Define in $\langle V, \in\rangle$ a function f with domain ω:
$$f(0) = \emptyset$$
$$f(n+1) = \{F(f(n))\} \cup f(n)$$
(c.f. the definition of finite ordinals $0 = \emptyset$ and $n + 1 = \{n\} \cup n$.)
Let $x = \{F(f(n)) : n \in \omega\}$. x is a set by (Rep) in $\langle V, \in\rangle$. Then the axiom holds with $y = \emptyset$.

(Power) $\forall x \exists y \forall z[F(z) \in y \leftrightarrow \forall u(F(u) \in z \to F(u) \in x)]$

Given x, let $y = \{F(t) : t \subseteq x\}$. y is a set by (Power) and (Rep) in $\langle V, \in \rangle$. Since F is a permutation on V, for all z,

$$\forall u(F(u) \in z \to F(u) \in x) \leftrightarrow z \subseteq x \leftrightarrow F(z) \in y$$

So the axiom holds.

(Rep) $\forall x_1, ..., x_n[\forall x \exists! y \psi_F(x, y) \to \forall a \exists b \forall y(F(y) \in b \leftrightarrow \exists x(F(x) \in a \wedge \psi_F(x, y)))]$

Suppose $\forall x \exists! y \psi_F(x, y)$ holds. Let a be a set. Then $a^* := \{F^{-1}(x) : x \in a\}$ is a set by (Rep) in $\langle V, \in \rangle$. Let b^* be the image of a^* under ψ_F, i.e.

$$\forall y[y \in b^* \leftrightarrow \exists u(u \in a^* \wedge \psi_F(u, y))]$$

b^* is a set by (Rep) in $\langle V, \in \rangle$. $b := \{F(y) : y \in b^*\}$ is also a set by (Rep) in $\langle V, \in \rangle$. Now

$$\begin{aligned} F(y) \in b \text{ iff } & y \in b^* \\ \text{iff } & \exists u(u \in a^* \wedge \psi_F(u, y)) \\ \text{iff } & \exists u(F(u) \in a \wedge \psi_F(u, y)) \end{aligned}$$

□

We denote "with respect to $\in_F$" by adding a prefix F-, or a suffix F, e.g. F*-empty* means empty with respect to F, and $x \cap_F y = \{z : z \in_F x \wedge z \in_F y\}$.

Remark. A syntactic model $Rep(F, -)$ obtained from the universe $\langle V, \in \rangle$ has the same elements as those in V, but with a different membership relation, i.e. it is $\langle V, \in_F \rangle$. By the proof of Theorem 1.1 above, we see that:

- $\emptyset_F = \emptyset$
- $\{x, y\}_F = \{F(x), F(y)\}$
- $\cup_F x = \{F(z) : \exists u(F(z) \in u \wedge F(u) \in x)\} = \{z : \exists u(z \in_F u \wedge u \in_F x)\}_F$
- the natural numbers in $\langle V, \in_F \rangle$ are

  $$f(0), f(1), f(2), \cdots \text{ where } f(0) = \emptyset \text{ and } f(n+1) = \{f(n)\}_F \cup f(n)$$

  and $\omega_F = \{f(n) : n \in \omega\}_F = \{F(f(n)) : n \in \omega\}$
- For the power set of x with respect to $\in_F$,

  $$\begin{aligned} \mathcal{P}_F x &= \{F(t) : t \subseteq x\} \\ &= \{t : t \subseteq x\}_F \\ &= F''(\mathcal{P}x) \end{aligned}$$

  Note that $z \subseteq_F u$ iff $z \subseteq u$.

Theorem 1.2. *Suppose that F defines a permutation of the universe V. If $\langle V, \in\rangle$ satisfies (AC), then so does $\langle V, \in_F\rangle$. i.e.*

$$ZF+(AC) \vdash Rep(F, (AC))$$

Proof. Let a be a nonempty set such that its F-elements are nonemepty. Define

$$\begin{aligned} a^* &= \{\{x : F(x) \in y\} : F(y) \in a\} \\ &= \{\{x : x \in_F y\} : y \in_F a\} \end{aligned}$$

Since $\emptyset_F = \emptyset$, a^* is a set of nonempty sets. By (AC) in $\langle V, \in\rangle$, there is a set b^* such that $b^* \cap y$ is a singleton for each $y \in a^*$. i.e. b^* makes a choice for a^*.
Then consider $b := \{F(x) : x \in b^*\}$. For $y \in_F a$, $b \cap_F y = \{F(x) : x \in b^* \wedge x \in y\}$ which is a singleton. So b makes a choice for a in $\langle V, \in_F\rangle$, i.e.

$$\forall y[y \in_F a \rightarrow \exists! z(z \in_F b \wedge z \in_F y)]$$

□

We state, without proof, a theorem useful for proving consistency results. (See [2] Chapter I Section D for a proof.)

Theorem 1.3. *Let T_1 and T_2 be first order theories. Let T_2 be consistent. If there exists a syntactic model τ of T_1 in T_2, then T_1 is consistent.*

Now we are ready to prove that it is consistent to have precisely one Quine atom.

Theorem 1.4. *If ZF^o is consistent, then so is ZF^o+$\exists! x(x = \{x\})$.*

Proof. By Theorem 1.1 if we have a permutation F such that

$$\exists! x(x = \{x\}_F) \text{ in } \langle V, \in_F\rangle$$

then we have a syntactic model $\langle V, \in_F\rangle$ of ZF^o+$\exists! x(x = \{x\})$ in ZF. Then apply Theorem 1.3 with

$$T_1 = \text{ZF}^o + \exists! x(x = \{x\}) \text{ and } T_2 = \text{ZF}$$

We will have that ZF^o+$\exists! x(x = \{x\})$ is consistent as required.

The following F will do: F interchanges 0 and 1, while fixing all others.

Now $1 \in_F 1$ since $F(1) = 0 \in 1$; $1 = \{1\}_F$ since $\{1\}_F = \{F(1)\} = \{0\} = 1$ □

Corollary 1.5. *If ZF is consistent, then ZF^o+$\neg$(Found) is consistent, and ZF^o+(AC)+$\neg$(Found) is consistent.*

Proof. The consistency of $\mathrm{ZF}^{\mathrm{o}}+\neg(\mathrm{Found})$ follows trivially from Theorem 1.4.

For the consistency of $\mathrm{ZF}^{\mathrm{o}}+(\mathrm{AC})+\neg(\mathrm{Found})$, we can have a universe $\langle V,\in\rangle$ satisfying (AC). Then by Theorem 1.2, $\langle V,\in_F\rangle$ also satisfies (AC). So we can proceed as in the proof of Theorem 1.4, with $\langle V,\in_F\rangle$ a syntactic model of $\mathrm{ZF}^{\mathrm{o}}+(\mathrm{AC})+\neg(\mathrm{Found})$ in ZF. $\square$

1.3 The Consistency of Having Infinitely Many Quine Atoms

We have proved in the previous section that the Axiom of Foundation can be violated with there being precisely one Quine atom. However, in Chapter 2 we would like to work in a universe with a countable set of Quine atoms; in Chapter 3, we will work in a universe with a proper class of Quine atoms, which is in 1-1 correspondence with the class of all ordinals, On.

Let us prove the consistency of $\mathrm{ZF}^{\mathrm{o}}$with there being such a set or class of Quine atoms.

Theorem 1.6. *If ZF is consistent, then so is the theory ZF^{ω}, which is*

ZF^{o} + " there exists a set A of all the Quine atoms such that there is a 1-1 correspondence between A and ω."

Proof. The proof is similar to that of Theorem 1.4. We would like a permutation F of V. We require $\langle V,\in_F\rangle$ to be a model of ZF^{ω}. It should have a countable set of Quine atoms. We will give a correspondence, in $\langle V,\in_F\rangle$, between the Quine atoms and ω_F.

Idea: In the proof of Theorem 1.4 where there is only one Quine atom, we construct an F swapping 0 and $\{0\}=1$ while fixing others. So, here we consider F swapping elements of ω^* and s^* as follows. Define

$$\omega^* = \omega - \{1\} = \{0,2,3,4,\dots\}$$
$$s^* = \{\{0\},\{2\},\{3\},\{4\},\dots\} = \{\{n\} : n\in\omega^*\}$$
$$F(x) = \begin{cases} \{x\} & \text{if } x\in\omega^*, \\ y & \text{if } x=y \text{ some } y\in\omega^*\text{, i.e. } x\in s^*, \\ x & \text{otherwise.}\end{cases}$$

$F(x)$ is well-defined since if $y\in\omega^*$, then $\{y\}\notin\omega^*$. So $s^*\cap\omega^*=\emptyset$. Let us try some examples:

$$F(0)=\{0\}=1, \qquad F(\{0\})=0;$$
$$F(2)=\{2\}, \qquad F(\{2\})=2;$$
$$F(n)=\{n\}, \qquad F(\{n\})=n.$$

Note that 1 is the only element in ω that is the singleton of another element of ω. We have removed it from ω^* to make the definition work. Thus we have

$$F(\{0\})=0\in\{0\}, \qquad \text{so } \{0\}\in_F\{0\};$$
$$F(\{2\})=2\in\{2\}, \qquad \text{so } \{2\}\in_F\{2\};$$
$$F(\{n\})=n\in\{n\}, \qquad \text{so } \{n\}\in_F\{n\}.$$

Note that, by definition of F, $x=\{F(x)\}$ iff $x\in s^*$:

if $x \in \omega^*$, then $\{F(x)\} = \{\{x\}\} \neq x$;

if $x \in s^*$, then $\{F(x)\} = \{y\} = x$;

otherwise, $\{F(x)\} = \{x\} \neq x$

Now we have a lot of Quine atoms in $\langle V, \in_F \rangle$. Let us find an F-correspondence between the Quine atoms in $\langle V, \in_F \rangle$ and ω_F. Recall from the Remark after Theorem 1.1 that $\{x, y\}_F = \{F(x), F(y)\}$. We know that an *ordered pair* in $\langle V, \in \rangle$ is

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}$$

Therefore an ordered pair in $\langle V, \in_F \rangle$ must be

$$\begin{aligned} \langle x, y \rangle_F &= \{\{x\}_F, \{x, y\}_F\}_F \\ &= \{F(\{F(x)\}), F(\{F(x), F(y)\})\} \end{aligned}$$

Again recall from the Remark after Theorem 1.1 that the natural numbers in $\langle V, \in_F \rangle$ are

$$f(0), f(1), f(2), \ldots, \text{ where } f(0) = \emptyset, f(n+1) = \{f(n)\}_F \cup f(n)$$

and $\omega_F = \{f(n) : n \in \omega\}_F$.

Consider a function g defined as

$$g = \{\langle f(n), \{n+1\} \rangle_F : n \in \omega, n \geq 1\}_F \cup \{\langle f(0), \{0\} \rangle_F\}_F$$

Clearly g is a bijective function in $\langle V, \in_F \rangle$ from ω_F to the set

$$R := \{\{n+1\} : n \in \omega, n \geq 1\}_F \cup \{\{0\}\}_F$$

of all the Quine atoms in $\langle V, \in_F \rangle$ as required. $\square$

For the case where there is a proper class R of Quine atoms which is in 1-1 correspondence with On, the proof is very similar. We need to know what ordinals in $\langle V, \in_F \rangle$ are, for a permutation F of V.

We know that the finite ordinals in $\langle V, \in_F \rangle$ are

$$\begin{aligned} f(0) &= \emptyset & &\text{c.f. } 0 = \emptyset \\ f(n+1) &= \{f(n)\}_F \cup f(n) & &\text{c.f. } n+1 = \{n\} \cup n \end{aligned}$$

A limit ordinal in $\langle V, \in \rangle$ is $\lambda = \cup\{\beta : \beta < \lambda\}$, so, as one would expect,

$$\begin{aligned} f(\lambda) &= \cup_F \{f(\beta) : \beta < \lambda\}_F \\ &= \cup_F \{F(f(\beta)) : \beta < \lambda\} \\ &= \{F(z) : \exists u (F(z) \in u \wedge F(u) \in \{F(f(\beta)) : \beta < \lambda\})\} \text{ by Remark after Theorem 1.1} \\ &= \{F(z) : F(z) \in f(\beta) \text{some } \beta < \lambda\} \\ &= \cup\{f(\beta) : \beta < \lambda\} \end{aligned}$$

And for infinite successor ordinals,

$$f(\alpha+1) = \{f(\alpha)\}_F \cup f(\alpha)$$

Corollary 1.7. *If ZF is consistent, then so is ZF^{On}, which is*

ZF^o + " there is a proper class A of all the Quine atoms such that there is a 1-1 correspondence between the class On and A"

Proof. Let $\mathrm{On}^* = \mathrm{On} - \{1\}$. Define

$$F(x) = \begin{cases} \{x\} & \text{if } x \in \mathrm{On}^*, \\ y & \text{if } x = y \text{ some } y \in \mathrm{On}^*, \\ x & \text{otherwise.} \end{cases}$$

Then we proceed as in the proof of Theorem 1.6, with On_F as discussed above. □

In the proofs of Theorem 1.6 and its Corollary 1.7, we constructed suitable $\langle V, \in_F \rangle$. Recall that Theorem 1.2 says that:

If $\langle V, \in \rangle$ satisfies (AC), then $\langle V, \in_F \rangle$ satisfies (AC).

So, as in Corollary 1.5, we can add (AC) to ZF^{o}in Theorem 1.6 and 1.7. We then obtain the consistency of both of ZF$^\omega$+(AC) and ZF$^{\mathrm{On}}$+(AC).

2

FRAENKEL-MOSTOWSKI-SPECKER MODELS

2.1 The Universe in Which We Are Working

In Chapter 1, we proved the consistency of ZF$^{\circ}$+(AC)+$\neg$(Found). In particular, we showed that we can have the Axiom of Foundation violated by the existence of a countable set of Quine atoms. Thus, we can legally have a universe which is a model of ZF$^{\omega}$+(AC). Let us make it precise.

Let A be a countable infinite set of Quine atoms. Define

$$\begin{aligned}
\mathcal{P}^0(A) &= A \\
\mathcal{P}^{\alpha+1}(A) &= \mathcal{P}(\mathcal{P}^{\alpha}(A)) \\
\mathcal{P}^{\lambda}(A) &= \bigcup_{\beta<\lambda} \mathcal{P}^{\beta}(A) \text{ for } \lambda \text{ a limit ordinal} \\
\mathcal{P}^{\infty}(A) &= \bigcup_{\alpha\in \mathrm{On}} \mathcal{P}^{\alpha}(A)
\end{aligned}$$

It is not provable in the universe V that $\exists A(V = \mathcal{P}^{\infty}(A))$, but this is consistent with ZF$^{\circ}$as proved in Chapter 1. So we shall assume this.

Obviously, the Axiom of Foundation does not hold in V, but we have the Axiom of Weak Foundation, defined as follows

(WF) $\exists A\forall x[x \neq \emptyset \rightarrow \exists y(y \in x \wedge (y \cap x = \emptyset \vee (\{y\} = y \wedge y \in A)))]$

(WF) holds in V. Also, we assume that (AC) holds in V. $\mathcal{P}^{\infty}(0)$ is called the *kernel* of V.

$\mathcal{P}^\infty(0)$ is the well-founded part of V, and it is a model of ZF, containing all ordinals.

2.2 Construction of a Model

To construct a model in our universe V, we need a group of automorphisms of V and a normal filter on this group. We will explain what these mean in this section. Let us talk about the idea first. Recall that the Fraenkel-Mostowski method was to make (AC) fail. This is done as follows:

- Fix a group G of automorphisms on V and a normal filter $\mathcal{F}$ on G.
- Construct a model M containing elements of V which are "symmetric enough" with respect to G and $\mathcal{F}$.
- By suitable choice of G and $\mathcal{F}$, we can make sure that any choice function that satisfies (AC) is not "symmetric enough" and hence not contained in M.

An *($\in$-)automorphism* of V is a bijective mapping $\tau : V \to V$ such that for all $x, y \in V$

$$x \in y \text{ iff } \tau(x) \in \tau(y)$$

We may write τx for $\tau(x)$ where there is no confusion.

It can be easily proved using $\in$-induction that the only automorphism in $\mathcal{P}^\infty(0)$ is the identity. But now we have Quine atoms in $\mathcal{P}^\infty(A)$. We will see that every permutation (i.e.bijective mapping) π on A can be extended to an automorphism of V.

Lemma 2.1. *The symmetry group of A, consisting of all permutations of A, is isomorphic to the group of all automorphisms on $\langle V, \in\rangle$, i.e.*

$$Sym(A) \cong Aut(V, \in)$$

Proof. We assume the knowledge that these two sets are indeed groups. First, we find a 1-1 correspondence between elements of the two sets. We consider the *rank* ρ of a set,

$$\rho(x) = min\{\alpha : x \subseteq \mathcal{P}^\alpha(A)\}$$

- Given an automorphism τ of V, since $y \in x \leftrightarrow \tau(y) \in \tau(x)$, we must have $\rho(\tau x) = \rho(x)$. This can be proved by induction:

$$\begin{aligned}
\rho(\tau x) &= min\{\alpha : \tau x \subseteq \mathcal{P}^\alpha(A)\} \\
&= min\{\alpha : (\forall y)(y \in \tau x \to y \in \mathcal{P}^\alpha(A))\} \\
&= min\{\alpha : (\forall \tau^{-1} y)(\tau^{-1} y \in x \to \tau^{-1} y \in \mathcal{P}^\alpha(A))\} \\
&\quad \text{(by induction hypothesis } y \in \mathcal{P}^\alpha(A) \leftrightarrow \tau^{-1} y \in \mathcal{P}^\alpha(A)) \\
&= min\{\alpha : (\forall y)(y \in x \to y \in \mathcal{P}^\alpha(A))\} = \rho(x)
\end{aligned}$$

So an automorphism preserves rank.

Note that for all Quine atoms $a \in A$, $\rho(a) = 0$. The only other element of rank 0 is the empty set $\emptyset$, which is fixed by automorphisms. Therefore an automorphism of V must map elements of A to elements of A. So its restriction on A, $\tau\restriction_A$, gives a permutation of A.

- Now given a permutation π of A, we define $\pi(\emptyset) = \emptyset$ and extend it inductively. Suppose we have π defined on all sets of rank $\alpha < \lambda$. For x of rank λ, the only way to extend π is

$$\pi(x) = \{\pi(y) : y \in x\}$$

Thus, the mapping θ from π to its extension π^*

$$\theta : Sym(A) \to Aut(V, \in)$$
$$\pi \mapsto \pi^*$$

is a bijection with inverse $\tau^* \mapsto \tau^*\restriction_A$

Secondly, we require θ to be a group homomorphism. It is easy to see by $\in$-induction that,

$$\theta(\pi_1\pi_2) = \theta(\pi_1)\theta(\pi_2)$$

where π_1, π_2 are permutations of A. $\square$

From now on, we do not distinguish between a permutation of A and its corresponding automorphism on $\langle V, \in\rangle$.

We keep in mind the following properties of automorphisms. Some of them have already been proved or claimed in the above. The proof is done by induction on rank, or on the complexity of the formula.

Lemma 2.2. *For $\pi, \pi_1, \pi_2 \in Aut(V, \in)$, the following properties hold:*

(a) $x \in y \leftrightarrow \pi x \in \pi y$*, and hence* $\pi x = \pi'' x$

(b) $\phi(x_1, \ldots, x_n) \leftrightarrow \phi(\pi x_1, \ldots, \pi x_n)$

(c) $\rho(x) = \rho(\pi x)$

(d) $\pi\{x, y\} = \{\pi x, \pi y\}$, $\pi\langle x, y\rangle = \langle \pi x, \pi y\rangle$

(e) If R is a relation, then πR is a relation, and $\langle x, y\rangle \in R \leftrightarrow \langle \pi x, \pi y\rangle \in \pi R$

(f) If f is a function on $X \subseteq V$, then πf is a function on πX, and $(\pi f)(\pi x) = \pi(f(x))$

(g) $\pi x = x$ *for x in the kernel*

(h) $(\pi_1\pi_2)x = \pi_1(\pi_2(x))$

Let G be a group of permutations of A. By Lemma 2.1, $Sym(A) \cong Aut(V, \in)$, so G corresponds to an automorphism of V. We let G act on V.

Definition. For $x \in V$, define the *pointwise stabiliser* of x in G to be

$$G_x = \{\tau \in G : \forall y \in x(\tau y = y)\};$$

and the *setwise stabiliser* of x in G to be

$$G_{\{x\}} = \{\tau \in G : \tau x = x\}$$

G_x and $G_{\{x\}}$ are subgroups of G.

Definition. A *normal filter* $\mathcal{F}$ on G is a set of subgroups of G such that for all $H, K \leq G$ (sub groups of G) the following holds:

(F1) $G \in \mathcal{F}$

(F2) $H \in \mathcal{F}, H \subseteq K \to K \in \mathcal{F}$ (i.e.$\mathcal{F}$ is closed under superset)

(F3) $H, K \in \mathcal{F} \to H \cap K \in \mathcal{F}$ (i.e. $\mathcal{F}$ is closed under intersection)

(F4) $\pi \in G, H \in \mathcal{F} \to \pi H \pi^{-1} \in \mathcal{F}$ (i.e. $\mathcal{F}$ is closed under conjugation)

(F5) for each $a \in A$, $G_{\{a\}} \in \mathcal{F}$

Remark. In fact, (F1)-(F3) make $\mathcal{F}$ a filter; (F4) ensures $\mathcal{F}$ is normal; (F5) is included so that A will be in the model M we are about to construct.

Now given some $G \in Sym(A)$ and a normal filter $\mathcal{F}$ on G, we are ready to define what we mean by "symmetric" and finally the Fraenkel-Mostowski-Specker model, i.e. the permutation model M.

Definition. Let x be a set. We say that x is *symmetric* iff $G_{\{x\}} \in \mathcal{F}$.

Definition. A *permutation model* M determined by G and $\mathcal{F}$ is defined as follows:

$$M = \{x : \forall y(y \in TC(x) \to G_{\{y\}} \in \mathcal{F})\}$$

where $TC(x)$ is the *transitive closure* of x, i.e.

$$TC(x) = \{x\} \cup x \cup (\bigcup x) \cup (\bigcup\bigcup x) \cup \cdots$$

Remark. The following is easy to see.

(i) M is transitive; M contains everything in the kernel of V, since G fixes everything in the kernel.

(ii) $x \in M$ iff $x \subseteq M$ and $G_{\{x\}} \in \mathcal{F}$
However, we'd better not use this to decide whether a Quine atom $a = \{a\}$ is in M. As $a \in M$ iff $a \subseteq M$, we would get into a loop. In fact, M contains all Quine atoms $a \in A$. See (iv) below.

(iii) M consists of hereditarily symmetric sets.

(iv) M contains all Quine atoms $a \in A$ and $A \in M$.
By the definition (F5) of a normal filter, for each Quine atom $a \in A$,

$$G_{\{a\}} \in \mathcal{F}$$

Hence M contains a as a set, i.e. $a \in M$.

Recall from the proof of Lemma 2.1 that all automorphisms setwise fix A, i.e.

$$G_{\{A\}} = G \in \mathcal{F}$$

$G \in \mathcal{F}$ is by the definition (F1) of a filter. Thus we know that

$$A \subseteq M \text{ and } G_{\{A\}} \in \mathcal{F}$$

Therefore by (ii) we have $A \in M$.

Thus we have constructed M from G and $\mathcal{F}$. However, instead of giving the normal filter $\mathcal{F}$ directly, we can have it derived from a *normal ideal* on A. Sometimes a normal ideal helps make the construction clearer.

Definition. A *normal ideal* I on A is a family of subsets of A such that for all $E, F \subseteq A$, the following holds:

(I1) $\emptyset \in I$

(I2) $E \in I, F \subseteq E \to F \in I$ (i.e. I is closed under subset)

(I3) $E, F \in I \to E \cup F \in I$ (i.e. I is closed under union)

(I4) $\pi \in G, E \in I \to \pi'' E \in I$

(I5) for each $a \in A$, $\{a\} \in I$.

Remark. (I1)-(I3) make I an ideal; we will see the correspondence between (I4), (I5) and (F4), (F5) in the definition of a normal filter.

Given a normal ideal I on A, we define a filter $\mathcal{F}$ to be the filter on G generated by G_E for all $E \in I$.

We need to check that this filter $\mathcal{F}$ is normal, so it can be used to construct M.

Lemma 2.3. *$\mathcal{F}$ defined as above from I is a normal filter.*

Proof. $\mathcal{F}$ is a filter on G generated by G_E, $E \in I$. So $\mathcal{F}$ is closed under superset and intersection, with $G \in \mathcal{F}$. It is sufficient to check (F4), (F5) of the definition of a filter being normal.

(F4) says: $\pi \in G, H \in \mathcal{F} \to \pi H \pi^{-1} \in \mathcal{F}$. We prove by induction on the "complexity" of H.

- $H = G \in \mathcal{F}$:
  $\pi G \pi^{-1} = G \in \mathcal{F}$

- $H = G_E$ some $E \in I$:
  $\pi G_E \pi^{-1} = G_{\pi'' E}$, $\pi'' E \in I$ by definition (I4) of normal ideal. So $\pi H \pi^{-1} = \pi G_E \pi^{-1} \in \mathcal{F}$.

- $K \subseteq H \in \mathcal{F}$, where $\pi K \pi^{-1} \in \mathcal{F}$:
  $\pi K \pi^{-1} \subseteq \pi H \pi^{-1}$. So $\pi H \pi^{-1} \in \mathcal{F}$ since $\mathcal{F}$ is closed under supersets.

- $H = K_1 \cap K_2$, where $K_1, K_2 \in \mathcal{F}$ and $\pi K_1 \pi^{-1}, \pi K_2 \pi^{-1} \in \mathcal{F}$:
  $\pi H \pi^{-1} = \pi K_1 \pi^{-1} \cap \pi K_2 \pi^{-1} \in \mathcal{F}$ since $\mathcal{F}$ is closed under intersection.

(F5) says: for each $a \in A, G_{\{a\}} \in \mathcal{F}$. In the definition (I5) of normal ideal, we put $\{a\}$ into I for each $a \in A$ deliberately. So we must have $G_{\{a\}} \in \mathcal{F}$. $\square$

The following property will be useful.

Proposition 2.4. *For any set $S \subseteq G$,*

$$S \in \mathcal{F} \textit{ iff there exists } E \in I \textit{ such that } G_E \subseteq S.$$

Hence

$$x \textit{ is symmetric iff } G_E \subseteq G_{\{x\}} \textit{ for some } E \in I.$$

In such cases, we say that E is a support *of x.*

Proof. Suppose $G_E \subseteq S$ for some $E \in I$. Since $G_E \in \mathcal{F}$ and $\mathcal{F}$ is closed under supersets, we must have $S \in \mathcal{F}$.

Conversely, suppose $S \in \mathcal{F}$. We induct on the complexity of S.

- $S = G$: $G_E \subseteq G$ for all $E \in I$.
- $S = G_E$: trivial.
- There exists $H \in \mathcal{F}$ such that $H \subseteq S$: by induction hypothesis, there exists $E \in I$ with $G_E \subseteq H \subseteq S$.
- $S = H \cap K$ for some $H, K \in \mathcal{F}$: by induction hypothesis, there exist $E_1, E_2 \in I$ with $G_{E_1} \subseteq H, G_{E_2} \subseteq K$. So $G_{E_1} \cap G_{E_2} \subseteq H \cap K = S$. But $G_{E_1} \cap G_{E_2} = G_{E_1 \cup E_2}$, and $E_1 \cup E_2 \in I$ since by the definition (I3) of ideal, I is closed under union.

$\square$

2.3 M is a Model of ZF$^{\mathrm{o}}$

We constructed a model M in our universe $V = \mathcal{P}^{\infty}(A)$ where A is the set of Quine atoms in V. M is constructed from a group G of automorphisms of V and a normal filter $\mathcal{F}$ on G. M consists of the hereditarily symmetric sets in V. We need to prove that such M is indeed a model of ZF$^{\mathrm{o}}$.

Theorem (Specker)**.** *In ZF^{o}+($V = \mathcal{P}^{\infty}(A)$), $\langle M, \in\rangle$ is a model of ZF^{o}.*

We prove this theorem step-by-step in this section.

Lemma 2.5. *If $x \in M$ and $\tau \in G$, then $\tau(x) \in M$.*

Proof. Since $\tau \in G$ fixes the kernel $\mathcal{P}^{\infty}(0)$, it is sufficient to consider x not in the kernel.

$x \in M$, so by Remark(ii) to the definition of M, we have

$$x \subseteq M \text{ and } G_{\{x\}} \in \mathcal{F}.$$

If $x \in A$, then $\tau x \in A$ and therefore $\tau x \in M$ because M contains all Quine atoms.

If $x \notin A$, then we prove by showing that $\tau x \subseteq M$ and $G_{\{\tau x\}} \in \mathcal{F}$.

- $G_{\{\tau x\}} \in \mathcal{F}$:
  $$G_{\{\tau x\}} = \tau G_{\{x\}} \tau^{-1}$$
  But $G_{\{x\}} \in \mathcal{F}$ and $\mathcal{F}$ is closed under conjugation. So
  $$\tau G_{\{x\}} \tau^{-1} \in \mathcal{F}$$

- $\tau x \subseteq M$:
  By the definition of an automorphism, $\tau x = \{\tau y : y \in x\}$.
  By induction hypothesis, $y \in M \to \tau y \in M$, so $\tau x \subseteq M$ holds.

□

Now we begin the proof of Specker's Theorem.

Proposition 2.6. *The axioms (Empty),(Ext),(Pair),(Union),(Inf) and (Power) hold in $\langle M, \in\rangle$.*

Proof. Simply check each of them.

(Empty) The real empty set $\emptyset$ is in the kernel of V, and therefore in M.

(Ext) (Ext) holds in V, and M is transitive. So (Ext) holds in M.

(Pair) For $x, y \in M$, we have $G_{\{x\}}, G_{\{y\}} \in \mathcal{F}$. Since $\mathcal{F}$ is closed under intersection,

$$G_{\{\{x,y\}\}} = G_{\{x,y\}} = G_{\{x\}} \cap G_{\{y\}} \in \mathcal{F}$$

So, as $\{x, y\} \subseteq M$ and $\{x, y\}$ symmetric, $\{x, y\} \in M$.

(Union) For $x \in M$, we have $G_{\{x\}} \in \mathcal{F}$ and $x \subseteq M$. We claim that $G_{\{x\}} \subseteq G_{\{\cup x\}}$. If so, then

- $G_{\{\cup x\}} \in \mathcal{F}$ since $\mathcal{F}$ is closed under superset
- $\cup x \subseteq M$ since: For $z \in \cup x$, $z \in y \in x$ some y. But $x \in M$, so by transitivity of M, $z \in M$.

Thus, $\cup x$ is symmetric and $\cup x \subseteq M$, so $\cup x \in M$. Hence it is sufficient to prove $G_{\{x\}} \subseteq G_{\{\cup x\}}$.

Let $\tau \in G_{\{x\}}$. Since τ is an $\in$-automorphism,

$$z \in y \in x \to \tau z \in \tau y \in \tau x = x$$

Hence

$$z \in \cup x \to \tau z \in \cup x$$

By applying $\tau^{-1} \in G_{\{x\}}$, we get the other direction, so

$$z \in \cup x \leftrightarrow \tau z \in \cup x$$

Thus, by (Ext), $\tau(\cup x) = \cup x$, i.e. $\tau \in G_{\{\cup x\}}$.

(Inf) (Empty) shows $\emptyset \in M$. Using (Pair) and (Union), we know by induction that all ordinals are in M. Hence $\omega \in M$. Alternatively, M contains everything in the kernel of V, including all ordinals.

(Power) Let $x \in M$, so $x \subseteq M$ and $G_{\{x\}} \in \mathcal{F}$. We prove that $\mathcal{P}x \cap M \in M$.

- $\mathcal{P}x \cap M \subseteq M$ is clear.
- $G_{\{\mathcal{P}x \cap M\}} \in \mathcal{F}$: We prove $G_{\{x\}} \subseteq G_{\{\mathcal{P}x \cap M\}}$. Hence $G_{\{\mathcal{P}x \cap M\}} \in \mathcal{F}$ as $\mathcal{F}$ is closed under superset.
  Suppose $\tau \in G_{\{x\}}$, and $y \in \mathcal{P}x \cap M$. Since τ is an $\in$-automorphism,

  $$y \subseteq x \to \tau y \subseteq \tau x = x, \text{ so } \tau y \in \mathcal{P}x$$

  $y \in M, \tau \in G$, so by Lemma 2.5, $\tau y \in M$. Thus $\tau(\mathcal{P}x \cap M) \subseteq \mathcal{P}x \cap M$. And the same holds for τ^{-1}, so $\tau(\mathcal{P}x \cap M) = \mathcal{P}x \cap M$

□

We tackle (Rep) by first proving a weaker version of it, that is, the Axiom scheme of Separation, (Sep). (Sep) is an axiom scheme in Zermelo set theory. Suppose ϕ is a formula with free variables $y, x_1, \ldots, x_n$, (Sep) $\quad (\forall x_1, \dots, x_n)\forall a \exists b \forall y[y \in b \leftrightarrow y \in a \wedge \phi(y, x_1, \dots, x_n)]$

Recall the Axiom scheme of Replacement: Suppose ψ is a formula with free variables $x, y, x_1, \dots, x_n$,

(Rep) $\quad \forall x_1, \dots, x_n[\forall x \exists! y \psi(x, y) \rightarrow \forall a \exists b \forall y(y \in b \leftrightarrow \exists x(x \in a \wedge \psi(x, y)))]$

By setting $\psi(x, y) = (x = y) \wedge \phi(x, x_1, \dots, x_n)$, we see that (Rep) implies (Sep).

Let ϕ^M denote the formula obtained from ϕ by restricting every quantifier to M. It is the *relativisation* of ϕ to M.

Lemma 2.7. *(Sep) holds in M, that is: Let $\phi(y, x_1, \dots, x_n)$ be a formula with free variables $y, x_1, \dots, x_n$. If $x_1, \dots, x_n \in M$ and $a \in M$, then there is $b \in M$ such that*

$$\forall y(y \in b \leftrightarrow y \in a \wedge \phi(y, x_1, \dots, x_n))$$

Proof. We would like to prove that the set

$$b = \{y : y \in a \wedge \phi^M(y, x_1, \dots, x_n)\}$$

is in M. Note $b \subseteq a \in M$, so by transitivity of M, $b \subseteq M$. Thus, to prove $b \in M$, it is sufficient to show $G_{\{b\}} \in \mathcal{F}$ (using Remark(ii) to the definition of M).

- For $y \in M$ and $\tau \in G$,

  $$\phi^M(y, x_1, \dots, x_n) \leftrightarrow \phi^M(\tau y, \tau x_1, \dots, \tau x_n)$$

  This can be proved by induction on the complexity of ϕ.

- Since $x_1, \dots, x_n, a \in M$, we have

  $$G_{\{x_1\}}, \dots, G_{\{x_n\}}, G_{\{a\}} \in \mathcal{F}$$

  Hence, as $\mathcal{F}$ is closed under (finite) intersection,

  $$G_0 := G_{\{x_1\}} \cap \dots \cap G_{\{x_n\}} \cap G_{\{a\}} \in \mathcal{F}$$

  Automorphisms in G_0 setwise fix each of $x_1, \dots, x_n$ and a.

- Now, if we prove $G_0 \subseteq G_{\{b\}}$, then $G_{\{b\}} \in \mathcal{F}$ as required. Take $\tau \in G_0, y \in b$.

  $y \in b$ gives $y \in a$, so $\tau y \in \tau a = a$ as $G_0 \subseteq G_{\{a\}}$
  and $\phi^M(y, x_1, \dots, x_n)$ by definition of z.

  Hence we have

  $$\phi^M(\tau y, \tau x_1, \dots, \tau x_n) = \phi^M(\tau y, x_1, \dots, x_n)$$

  as $\tau \in G_0 \subseteq G_{\{x_i\}}$ for $i \in \{1, \dots, n\}$.
  Thus, $\tau y \in a$ and $\phi^M(\tau y, x_1, \dots, x_n)$, so $\tau y \in b$. This gives $\tau b \subseteq b$. By applying τ^{-1}, altogether we get

  $$\tau b = b \text{ for } \tau \in G_0$$

  So $G_0 \subseteq G_{\{b\}}$ as required.

□

In the presence of (Sep), (Rep) is equivalent to the following: for any formula $\phi(x, y)$ with free variables $x, y, x_1, \ldots, x_n$, for all $x_1, \ldots, x_n$

$$\forall x \exists! y \phi(x, y) \to \forall a \exists b \forall x, y[x \in a \wedge \phi(x, y) \to y \in b] \tag{2.1}$$

i.e. we have a set b in M such that b contains the image under ϕ of the set a in M. Note that b may be strictly bigger than the image, but since we have (Sep), we can always extract the image from b.

Proposition 2.8. *(Rep) holds in M.*

Proof. Let $\phi(x, y)$ be a formula, with free variables $x, y, x_1, \ldots, x_n$. Assume that, for $x_1, \ldots, x_n \in M$ and $x \in M$, we have

$$(\exists! y \in M)\phi^M(x, y)$$

Let a be a set in M. Define

$$t = \{y \in M : \exists x(x \in a \wedge \phi^M(x, y))\}$$
$$b = \bigcup\{\tau(t) : \tau \in G\}$$

Note that G contains the identity automorphism, so $t \subseteq b$. Thus if b is a set in M, then (2.1) holds, and by the argument above Proposition 2.8, (Rep) holds in M.

By (Rep) in V, t is a set. Therefore b is a set in the universe V, as G is a set in V. We prove $b \in M$ by showing that $b \subseteq M$ and $G_{\{b\}} \in \mathcal{F}$ (according to Remark(ii) to the definition of M).

- $t \subseteq M$ by definition of t. So by Lemma 2.5, $\tau(t) \subseteq M$. Hence $b \subseteq M$.
- b is setwise fixed by any $\tau \in G$ by the definition of b. So $G_{\{b\}} = G \in \mathcal{F}$.

Therefore $b \in M$ as required. □

This completes the proof of Specker's Theorem.

Recall the Axiom of Weak Foundation (WF) from Section 2.1,

$$\exists A \forall x[x \neq \emptyset \to \exists y(y \in x \wedge (y \cap x = \emptyset \vee (\{y\} = y \wedge y \in A)))]$$

(WF) holds in M because it holds in the surrounding universe V and the set A of all Quine atoms is a set in M.

The following property is useful.

Proposition 2.9. *A set $x \in M$ can be well-ordered in M iff its pointwise stabiliser is in the filter $\mathcal{F}$, i.e. $G_x \in \mathcal{F}$.*

Proof. Recall that (AC) holds in the universe V. In particular, any choice function of a set of nonempty sets of $\mathcal{P}^\infty(0)$ is a set in $\mathcal{P}^\infty(0)$.

M contains everything in the kernel. So (AC) holds in the kernel in M. Therefore, any $x \in M$ can be well-ordered iff there is a 1-1 function f in M from x into the kernel.

Suppose $x \in M$. As (AC) holds in V, we have a 1-1 function f in V, $f : x \to \mathrm{On}$. We would like to see if $f \in M$. Recall that $f \in M$ iff $f \subseteq M$ and $G_{\{f\}} \in \mathcal{F}$. Actually, only $G_{\{f\}} \in \mathcal{F}$ needs to be considered, since any such f satisfies $f \subseteq M$: Wlog, we have

$$f = \{\langle y_0, 0\rangle, \langle y_1, 1\rangle, \cdots, \langle y_\alpha, \alpha\rangle, \cdots, \langle y_\lambda, \lambda\rangle\}$$

with $y_\alpha \in x$, $\alpha \in \mathrm{On}$.
$x \in M$, so by transitivity of M, $y_\alpha \in M$; α is an ordinal, so $\alpha \in M$. M satisfies (Pair), so $\langle y_\alpha, \alpha\rangle \in M$ for each α. Hence $f \subseteq M$.

<u>Claim.</u> $\pi f = f$ iff $\pi \in G_x$, i.e. $G_{\{f\}} = G_x$
Proof of the claim: Recall, from Lemma 2.2(f), that $\pi(f(y)) = (\pi f)(\pi y)$ for $y \in x$. Now $f(y) \in \mathrm{On}$ and π fixes everything in the kernel, so $(\pi f)(\pi y) = \pi(f(y)) = f(y)$.
Suppose $\pi \in G_x$. Then for each $y \in x$, $\pi y = y$, so $f(y) = (\pi f)(\pi y) = (\pi f)(y)$. Hence $f = \pi f$.
Conversely, suppose $\pi f = f$. Then for each $y \in x$, $f(y) = (\pi f)(\pi y) = f(\pi y)$. But f is 1-1, so $\pi y = y$. Hence $\pi \in G_x$. The claim is proved.

($\Leftarrow$) Suppose $G_x \in \mathcal{F}$. We prove $f \in M$ by showing that $G_{\{f\}} \in \mathcal{F}$. Then x is well-orderable in M. But it is clear that by the claim above, $G_{\{f\}} = G_x \in \mathcal{F}$.

($\Rightarrow$) Suppose x can be well-ordered in M. Then there is a 1-1 function $f : x \to \mathrm{On}$ in M. By the claim above, $G_x = G_{\{f\}} \in \mathcal{F}$ as required.

□

2.4 Examples of Permutation Models in which (AC) Fails

In the previous sections, we constructed a general permutation model M determined by a group G of permutations of A and a normal filter $\mathcal{F}$ on A, and proved that it is a model of ZF°+(WF). We also showed that a set x can be well-ordered in M iff its pointwise stabiliser is in the normal filter $\mathcal{F}$.

In this section, we construct two models M with specific G and $\mathcal{F}$. At the end of Section 2.2, we showed that a normal filter $\mathcal{F}$ can be determined from a normal ideal I. We will use normal ideals in our examples.

2.4.1 The basic Fraenkel model

Recall that A is a countably infinite set of Quine atoms such that $V = \mathcal{P}^{\infty}(A)$.

- Let G be the group of *all* permutations of A.
- Let I be the set of all *finite* subsets of A.

We need to check that I is a normal ideal on A. Then the filter generated by G_E, for $E \in I$, will be our normal filter $\mathcal{F}$ on G.

Lemma 2.10. *The set I of all finite subsets of A is a normal ideal.*

Proof. Check each of the conditions (I1)-(I5) in the definition of normal filter.

(I1) $\emptyset \in I$: $\emptyset$ is finite.

(I2) $E \in I, F \subseteq E \to F \in I$:
$E \in I$ so E must be finite. Hence a subset F of E is finite.

(I3) $E, F \in I \to E \cup F \in I$:
For E, F finite, $E \cup F$ is also finite.

(I4) $\pi \in G, E \in I \to \pi'' E \in I$:
π is a permutation on A, so $\pi'' E$ is finite if E is finite.

(I5) for each $a \in A$, $\{a\} \in I$:
Clear since $\{a\}$ is finite.

$\square$

Thus we have a model M determined by G and I. If there is a set not well-orderable in M, then (AC) fails in M. In this case, A is such a set.

Proposition 2.11. *A has no well-ordering in M.*

Proof. By Proposition 2.9,

$$M \models (x \text{ is well-orderable}) \text{ iff } G_x \in \mathcal{F}.$$

Hence it is sufficient to prove $G_A \notin \mathcal{F}$. Recall Proposition 2.4 about support,

$$G_A \in \mathcal{F} \text{ iff } G_E \subseteq G_A \text{ some } E \in I.$$

In such cases, we say E is a *finite support* of A. For every $E \in I$, i.e. finite $E \subseteq A$, there is some $\pi \in G = Sym(A)$ such that π fixes every element of E, but not all elements of A. This is because that A is infinite. Hence there is no $E \in I$ such that $G_E \subseteq G_A$. So $G_A \notin \mathcal{F}$. $\square$

digression: amorphous sets

Alternatively, we can view A as an *amorphous* set in M, i.e. A is finite, but is not the disjoint union of two infinite sets.

The existence of an amorphous set contradicts (AC):
If (AC) holds in M, then we can well-order any infinite set $X \in M$, say via a 1-1 function f in M from X onto an initial segment of On. Each ordinal α can be written as a sum

$$\alpha = \lambda + n \text{ where } \lambda \text{ is a limit ordinal and } n \in \omega.$$

We say α is *odd* if n is odd; α is *even* if n is even. Thus, consider

$$X_1 = \{x \in X : f(x) \text{ odd}\} \text{ and } X_2 = \{x \in X : f(x) \text{ even }\}$$

Clearly $X = X_1 \amalg X_2$ (disjoint union), and both X_1 and X_2 are infinite. So, if (AC) holds, then there is no amorphous set.

In the model M that we have just constructed, A is amorphous because any subset of A in M is either finite or *cofinite* (i.e. has finite complement):
Suppose $Y \subseteq A$ is infinite and $Y \in M$. Recall that M consists of hereditarily symmetric sets, so Y must be symmetric itself, i.e. $G_{\{Y\}} \in \mathcal{F}$. Therefore by Proposition 2.4, Y has a finite support: there is a finite $E \subseteq A$ such that $G_E \subseteq G_{\{Y\}}$. Since Y is inifinte, there is some $y \in Y \setminus E$. We chose G to be the group of all permutations of A, so G_E acts transitively on $A \setminus E$. Hence Y must contain the whole of $A \setminus E$. This gives that Y is cofinite.

Therefore the existence of the amorphous set A in M shows that (AC) does not hold in M.

2.4.2 The second Fraenkle model

In this model, the Axiom of Choice for countable families of pairs fails.

Since the set A of the Quine atoms is countably infinite, wlog,

$$A = \{a_0, b_0, a_1, b_1, a_2, b_2, \dots\} = \bigcup_{n\in\omega} P_n \quad \text{where } P_n = \{a_n, b_n\}$$

Let G be the group of permutations π of A such that π preserves each pair P_n, i.e.

$$\pi(\{a_n, b_n\}) = \{a_n, b_n\}$$

Let I be the set of all *finite* subsets of A. As in the previous example, it is easy to prove that I is a normal ideal on A with respect to G.

G and I determine a permutation model M.

Lemma 2.12. *x is symmetric iff there exists k such that*

$$G_{\{a_0,b_0,\dots a_k,b_k\}} \subseteq G_{\{x\}}$$

Proof. By Proposition 2.4, x is symmetric iff it has a finite support.

($\Leftarrow$) Suppose that for some k, $G_{\{a_0,b_0,\dots a_k,b_k\}} \subseteq G_{\{x\}}$. Then x has $\{a_0, b_0, \dots a_k, b_k\}$ as its finite support, so x is symmetric.

($\Rightarrow$) Suppose x is symmetric. Then x has some finite support $E \subseteq A$ with $G_E \subseteq G_{\{x\}}$. E is finite, hence there exists some finite k large enough such that $E \subseteq \{a_0, b_0, \dots a_k, b_k\}$. Thus, if $\pi \in G$ fixes each of $a_0, b_0, \dots a_k, b_k$, then π fixes each element of E, i.e.

$$G_{\{a_0,b_0,\dots a_k,b_k\}} \subseteq G_E \subseteq G_{\{x\}}$$

□

Now we are ready to prove that the Axiom of Choice for families of pairs does not hold in M.

Proposition 2.13. *The following holds for the model M described above.*

(a) Each P_n is in M.

(b) The sequence $\langle P_n : n \in \omega\rangle$ is in M; thus $\{P_n : n \in \omega\}$ is countable in M.

(c) There is no choice function for the family $\{P_n : n \in \omega\}$, i.e. there is no $f \in M$ with domain ω and

$$f(n) \in P_n \text{ for each } n \in \omega$$

Proof.

(a) $P_n = \{a_n, b_n\}$ with $a_n, b_n \in A$. Recall, from Remark(iv) to the definition of M in Section 2.2, that every Quine atom is contained in M. Therefore $a_n, b_n \in M$. Recall, from Specker's Theorem, that (Pair) holds in M. So $P_n \in M$.

(b) $\langle P_n : n \in \omega\rangle = \{\langle P_0, 0\rangle, \langle P_1, 1\rangle, \cdots, \langle P_n, n\rangle, \cdots\} = \{\langle P_n, n\rangle : n \in \omega\}$
$P_n \in M$ by (a); all natural numbers n are in the kernel, so $n \in M$. (Pair) holds in M, hence $\langle P_n, n\rangle \in M$. Thus, $\langle P_n : n \in \omega\rangle \subseteq M$. To prove it is in M, all we need is that $G_{\{\langle P_n:n\in\omega\rangle\}} \in \mathcal{F}$. But for any $\pi \in G$, $\pi'' P_n = P_n$ and $\pi n = n$. So

$$G_{\{\langle P_n:n\in\omega\rangle\}} = G \in \mathcal{F}$$

Therefore $\langle P_n : n \in \omega\rangle \in M$. It is an injection from $\{P_n : n \in \omega\}$ to ω, so $\{P_n : n \in \omega\}$ is countable in M.

(c) Suppose there is such a function $f : \omega \to P_n$ in M. f must be symmetric to be in M. By Lemma 2.12,

$$G_{\{a_0,b_0,\dots a_k,b_k\}} \subseteq G_{\{f\}} \text{ some } k$$

Let $\pi \in G_{\{a_0,b_0,\dots a_k,b_k\}}$ be such that π swaps a_{k+1} and b_{k+1}. Then we have $\pi \in G_{\{f\}}$, so $\pi f = f$.
Since $k+1$ is in the kernel, $\pi(k+1) = k+1$. Then

$$\pi(f(k+1)) = (\pi f)(\pi(k+1)) = f(k+1) \in \{a_{k+1}, b_{k+1}\}$$

This contradicts the fact that π swaps a_{k+1} and b_{k+1}.

$\square$

3

UNDEFINABILITY OF CARDINALITY IN ZF$^{\mathrm{o}}$

3.1 Cardinality in ZF

Let us first recall how we defined cardinality in ordinary set theory, e.g. ZF, or ZF+(AC).

Informally, to compare the cardinality of two sets is to ask whether one has more members than the other. We have an equivalence relation $\equiv$ where

$$(x \equiv y) \leftrightarrow (\exists f)(f : x \to y \text{ is a bijection})$$

In this case, we say that x and y are *equipollent*. But equivalence classes of this relation (except for the class $\{\emptyset\}$)) are all proper classes. We hope to represent these equivalence classes by sets.

We seek a function-class $card : V \to V$ such that

$$(\forall x, y)(card(x) = card(y) \leftrightarrow x \equiv y)$$

There are two standard ways of defining such a function, depending on whether we have (AC).

- With (AC)
  Every set can be well-ordered, so each equivalence class contains at least one ordinal. Hence we can represent each class by the least ordinal it contains
  $$card(x) = \bigcap\{\alpha \text{ an ordinal} : x \equiv \alpha\}$$
- Without (AC)
  It is unlikely that we can pick out a single element from each equivalence class; but we can pick out a nonempty sub*set* of each class using rank. We represent each class by the set of sets with least rank in that class.
  $$card(x) = \{y : y \equiv x \wedge \forall z(z \equiv x \to \rho(y) \leq \rho(z))\}$$

We write $|x|$ for $card(x)$.

Hidden Track: Sand in the Ganges

A little spice form Diamond Sutra Chapter 11:

> "Subhuti, if there were as many Ganges rivers as the number of grains of sand in the Ganges, would you say that the number of grains of sand in those Ganges rivers would be very many?"
>
> Subhuti answered,"Very many indeed, Most Honoured One. If the number of Ganges rivers were that large, how much more so would be the number of grains of sand in all those Ganges rivers."

This is the reason why ancient Chinese used *the number of grains of sand in the Ganges* for a number so large that they had not had a name for it. (But it seems that this problem was still not solved...)

3.2 Cardinality in ZF$^\mathrm{o}$

In this section, we present a theorem which tells us that, in the absence of (AC) and (Found), cardinality is not definable in ZF$^\mathrm{o}$. The proof is in the next section.

Can we define, in ZF$^\mathrm{o}$, the cardinality operation $|x|$?

- If the answer is *yes*, then the following must hold:

  (a) There is a term $t(x)$ of ZF$^\mathrm{o}$, with the only free variable x, such that

  $$\mathrm{ZF}^\mathrm{o} \vdash \forall a, b(t(a) = t(b) \leftrightarrow a \equiv b)$$

- Even if the answer is *no*, we may still have:

  (b) There is a term $t(a, x)$ of ZF$^\mathrm{o}$, with free variables a, x, such that

  $$\mathrm{ZF}^\mathrm{o} \vdash \exists x \forall a, b(t(a, x) = t(b, x) \leftrightarrow a \equiv b)$$

  i.e. $|x|$ is *relatively definable* in ZF$^\mathrm{o}$.

Clearly, (a) implies (b). We shall prove that (b) does not hold in ZF$^\mathrm{o}$without (AC) and (Found), and a fortiori (a).

Theorem 3.1 (Lévy, Gauntt)**.** *If ZF$^\mathrm{o}$is consistent, then so is ZF$^\mathrm{o}$plus the scheme*

$$\neg(*) \quad \neg\exists x \forall a \exists! y[\phi(y, a, x) \wedge \forall b(a \equiv b \leftrightarrow \phi(y, b, x))]$$

We write $(*)$ for the negation of the above scheme, i.e.

$$\text{for some formula } \phi,\ \exists x \forall a \exists! y[\phi(y, a, x) \wedge \forall b(a \equiv b \leftrightarrow \phi(y, b, x))];$$

Write (β) for the formula in (b) above, i.e.

$$\exists x \forall a, b(t(a,x) = t(b,x) \leftrightarrow a \equiv b)$$

How does Theorem 3.1 imply that (b) does not hold?

We use the Deduction Theorem, which says:

> Let ψ and ϕ be formulas, T be a theory. Suppose that either ψ has a free variable or ϕ does not, then
> $$T \cup \{\phi\} \vdash \psi \text{ iff } T \vdash (\phi \to \psi)$$

Note that $(\beta) \to (*)$: Let $\phi(y,a,x) = (t(a,x) = y)$. If (β) holds, then

$$\begin{aligned} &\exists x \forall a \exists ! y[\phi(y,a,x) \wedge \forall b(a \equiv b \leftrightarrow \phi(y,b,x))] \\ =&\exists x \forall a \exists ! y[t(a,x) = y \wedge \forall b(a \equiv b \leftrightarrow t(b,x) = y)] \text{ holds.} \end{aligned}$$

Theorem 3.1 says that ZF$^{\mathrm{o}}$and $\neg(*)$ are consistent, i.e.

$$\mathrm{ZF}^{\mathrm{o}} \cup \neg(*) \nrightarrow \perp$$

If so, then ZF$^{\mathrm{o}}\nvdash (*)$ by the Deduction Theorem. But $(\beta) \to (*)$, so ZF$^{\mathrm{o}}\nvdash (\beta)$.

3.3 Proof of Theorem 3.1

In this section, we aim to construct a permutation model M which is a model of ZF$^{\mathrm{o}}$+$\neg(*)$. This proves Theorem 3.1. The process will be very similar to what is done in Chapter 2. The first subsection is about the universe in which we are working; in the second subsection, we construct a model M; the third subsection gives a proof that M is a model of ZF$^{\mathrm{o}}$; then finally we prove that the scheme $\neg(*)$ holds in M.

3.3.1 Universe

Define the theory ZF$^{\triangledown}$as follows.

$$\begin{aligned} \mathrm{ZF}^{\triangledown} :&= \mathrm{ZF}^{\mathrm{On}} + (\mathrm{WF}) \\ &= \mathrm{ZF}^{\mathrm{o}} + \text{“ there is a proper class } A \text{ of Quine atoms equipollent with On,} \\ &\quad \text{such that for every } x \text{ there exists } y \in x \text{ with either } y \cap x = \emptyset \text{ or } y \in A\text{”} \end{aligned}$$

The consistency of ZF$^{\triangledown}$follows from Corollary 1.7 and the fact that (WF) holds there. Instead of working in $\mathcal{P}^{\infty}(A)$, we will work in a restricted universe. Some preparation is needed.

A is equipollent with On, so there is a bijective function class (proper class)

$$\mathcal{G} : \text{On} \ \to A$$

Recall from the digression about amorphous sets from the basic Fraenkel model (Section 2.4.1), the idea of an ordinal α being even or odd: $\alpha = \lambda + n$, where λ is a limit ordinal and $n \in \omega$. The parity of α is just the parity of n. For example,

$$\begin{array}{lr} 0, 2, 4, \ldots, \omega, \omega+2, \omega+4, \ldots & \text{are even;} \\ 1, 3, 5, \ldots, \omega+1, \omega+3, \omega+5, \ldots & \text{are odd.} \end{array}$$

For each ordinal α, $\{\mathcal{G}(\alpha), \mathcal{G}(\alpha+1)\}$ is a pair of Quine atoms. If α is even, then

$$\forall \beta(\beta \text{ even } \wedge \alpha \neq \beta \rightarrow \{\mathcal{G}(\alpha), \mathcal{G}(\alpha+1)\} \cap \{\mathcal{G}(\beta), \mathcal{G}(\beta+1)\} = \emptyset)$$

Definition. Define a function F on ordinals as follows.

$$F(\alpha) = \{\mathcal{G}(\beta) : (\beta \text{ even } \wedge \beta < \alpha) \vee (\beta \text{ odd } \wedge \beta \leq \alpha)\}$$

i.e. $F(\alpha)$ is the image under $\mathcal{G}$ of α, with $\mathcal{G}(\alpha)$ included if α is odd.
$F(\alpha)$ is a set of Quine atoms, i.e. $F(\alpha) \subseteq A$.

Definition. $Base(x) = TC(x) \cap A$
i.e. $Base(x)$ is the set of Quine atoms in the transitive closure of x.

Now we are ready to define the *restricted universe* V.

$$V := \bigcup_{\alpha \in \text{On}} (\cup_\gamma \mathcal{P}^\gamma(F(\alpha)))$$

That is, we restrict the universe to elements of sets built up from $F(\alpha)$'s. The restricted universe V consists of all x such that

$$\exists \alpha \exists y (x \in y \wedge Base(y) \subseteq F(\alpha))$$

Intuitively, we construct a universe from $F(\alpha)$ for each ordinal α and then glue them together. Note that V is transitive.

From now on, by universe, we mean the restricted universe V.

Each permutation f of $F(\alpha)$ is extended to an $\in$-automorphism of V as follows.

$$f(x) = \begin{cases} x & \text{for Quine atoms } x \text{ not in } F(\alpha) \text{ ,} \\ \{f(y) : y \in x\} & \text{otherwise .} \end{cases}$$

This is well-defined: If $x \in V = \bigcup_\alpha (\cup_\gamma \mathcal{P}^\gamma(F(\alpha)))$, then

$$x \in y \in \cup_\gamma \mathcal{P}^\gamma(F(\beta)) \text{ some } y, \beta$$
$$x \in y \in z \in \mathcal{P}^\delta(F(\beta)) \text{ some } y, z, \delta, \beta.$$

Then by transitivity of $\mathcal{P}^\delta(F(\beta))$,

$$x \subseteq \mathcal{P}^\delta(F(\beta))$$

By induction hypothesis, f is defined for all $u \in \mathcal{P}^\xi(F(\beta))$ for $\xi < \delta$, for all β.

3.3.2 Construction of the permutation model M

Recall, in Chapter 2, the model M is determined by G and $\mathcal{F}$, where G is a group of permutations of the set A of Quine atoms and $\mathcal{F}$ is a normal filter on G. In particular, in the second Fraenkel model (Section 2.4.2), we have $A = \{a_0, b_0, a_1, b_1, \dots\}$ and

- G is the group of permutations π of A such that

$$\pi(\{a_n, b_n\}) = \{a_n, b_n\}$$

- $\mathcal{F}$ contains all subgroups of G with a finite support.

However, we cannot do the same thing in our universe V here. A is a proper class already, so a group G of permutations of A would be beyond our reach. Thus, we make the following definitions. Note that the idea is actually the same as that in the second Fraenkel model.

Definition. A permutation f on $F(\alpha)$ is *semi-admissible* iff it preserves the class of all pairs $\{\mathcal{G}(\beta), \mathcal{G}(\beta+1)\}$, where β is even, i.e. for all even $\beta < \alpha$,

$$f(\{\mathcal{G}(\beta), \mathcal{G}(\beta+1)\}) = \{\mathcal{G}(\gamma), \mathcal{G}(\gamma+1)\} \text{ some even } \gamma < \alpha$$

Definition. A permutation f on $F(\alpha)$ is *admissible* iff it preserves all pairs $\{\mathcal{G}(\beta), \mathcal{G}(\beta+1)\}$ where β is even, i.e. for all even $\beta < \alpha$

$$f(\{\mathcal{G}(\beta), \mathcal{G}(\beta+1)\}) = \{\mathcal{G}(\beta), \mathcal{G}(\beta+1)\}$$

Admissible permutations corresponds in the second Fraenkel model to those permutations we put in G.

Recall that in Chapter 2, x is symmetric iff $G_{\{x\}} \in \mathcal{F}$ and M consists of hereditarily symmetric sets. To avoid the use of $\mathcal{F}$, we define directly the term symmetric.

Although A is a proper class, the Quine atoms moved by a permutation of A cannot be too many. They must all lie in some $F(\alpha)$. i.e. the "essential part" of the permutation is a set. Thus we can define symmetricity only using sets.

Definition. x is *symmetric* iff there is a finite set E of Quine atoms such that, for each admissible permutation τ (on some $F(\alpha)$), the following holds:

$$\tau \text{ pointwise stabilises } E \ \rightarrow \tau \text{ setwise stabilises } x.$$
$$\text{i.e. } E \text{ is a finite "support" of } x.$$

Finally, let M consist of precisely the hereditarily symmetric sets.

3.3.3 M is a model of ZF$^{\text{o}}$

Recall that in Section2.3, we proved that a model built up in $\mathcal{P}^{\infty}(A)$ with A being infinitely countable is a model of ZF$^{\text{o}}$. The proof here is very similar. Here, A is a proper class of Quine atoms, so we cannot take the group of all permutations of A, as discussed in Subsection 3.3.2. Instead, we take the group of admissible permutations on some $F(\alpha)$, which is enough for our purpose. These groups are *sets* in the universe V.

Then, with the following lemmas, we can prove that M is a model of ZF$^{\text{o}}$just as in Section 2.3.

Lemma 3.2. *In ZF$^{\triangledown}$, if f is semi-admissible and g is admissible, then $f^{-1}gf$ is admissible.*

Lemma 3.3. *$x \in M$ iff $x \subseteq M$ and x is symmetric.*

The proofs are direct from definition.

Lemma 3.4. *In ZF$^{\triangledown}$, for each x and semi-admissible permutation τ,*

$$x \in M \textit{ iff } \tau(x) \in M$$

i.e. Semi-admissible permutations preserve the model M.

The proof is very similar to that of Lemma 2.5, using Lemma 3.2 and Lemma 3.3.

The following lemma is useful for the proof of (Rep) in M.

Lemma 3.5. *For each ZF-formula $\phi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$, the following are theorems of ZF$^{\triangledown}$.*

(i) τ *semi-admissible* $\rightarrow [\phi(x_1, \ldots, x_n) \leftrightarrow \phi(\tau x_1, \ldots, \tau x_n)]$

(ii) τ *semi-admissible* $\rightarrow [\phi^M(x_1, \ldots, x_n) \leftrightarrow \phi^M(\tau x_1, \ldots, \tau x_n)]$

i.e. Semi-admissible permutations preserve ZF-formulas and their relativizations to the model M.

(i) can be proved (as we discussed in the proof of Lemma 2.7) by induction on the complexity of ϕ. The proof of (ii) is similar, using Lemma 3.4.

Now the following proposition follows just as Specker's Theorem in Section 2.3.

Proposition 3.6. *In ZF$^{\nabla}$, $\langle M, \in\rangle$ is a model of ZF^{o}.*

3.3.4 Cardinality is undefinable in M

As discussed in Section3.2, we prove that cardinality is not even relatively definable in M. This can be achieved by showing that the scheme $\neg(*)$, which is

$$\neg\exists x\forall a\exists! y[\phi(y, a, x) \wedge \forall b(a \equiv b \leftrightarrow \phi(y, b, x))]$$

holds in M. i.e. We prove Theorem 3.1(Lévy, Gauntt). M is a model of ZF$^{\text{o}}$, as proved in the previous subsection, so this will give the undefinability of cardinality in ZF$^{\text{o}}$.

We need two lemmas.

Lemma 3.7. *In ZF$^{\nabla}$, no two disjoint infinite sets of Quine atoms are equipollent in M.*

Proof. Suppose otherwise. There exist x and y, two disjoint infinite sets of Quine atoms and a bijection $g : x \to y$ in M.

x, y and g are symmetric and hence have finite "supports" a, b and c respectively, Here a, b and c are finite sets of Quine atoms and every admissible permutation τ which pointwise stabilises a(resp. b, c) setwise stabilises x(resp. y, g).

Claim. If $\mathcal{G}(\alpha) \in x - a$ and $\mathcal{G}(\alpha + 1) \notin a$ for α even, then $\mathcal{G}(\alpha + 1) \in x$.
Since neither $\mathcal{G}(\alpha)$ nor $\mathcal{G}(\alpha + 1)$ is in a, there exists some admissible permutation τ such that τ pointwise stabilises a and swaps $\mathcal{G}(\alpha)$ and $\mathcal{G}(\alpha + 1)$. But a is a "support" for x, so τ setwise fixes x. Thus $\mathcal{G}(\alpha + 1) = \tau(\mathcal{G}(\alpha)) \in x$. The claim is proved.

Because x is infinite and $a \cup b \cup c$ is finite, we can pick an even ordinal α such that

$$\mathcal{G}(\alpha) \in x - (a \cup b \cup c) \text{ and } \mathcal{G}(\alpha + 1) \notin a \cup b \cup c$$

Thus $\mathcal{G}(\alpha + 1) \in x$ by the claim above.
g is a bijection from x to y, $y \subseteq A$, and $\mathcal{G} : \text{On} \to A$ is a bijection. So every element in the image of g is some $\mathcal{G}(\beta)$. Hence, wlog,

$$g(\mathcal{G}(\alpha)) = \mathcal{G}(\beta) \in y \text{ and } g(\mathcal{G}(\alpha + 1)) = \mathcal{G}(\gamma) \in y$$

i.e. $\langle \mathcal{G}(\alpha), \mathcal{G}(\beta)\rangle,\ \langle \mathcal{G}(\alpha + 1), \mathcal{G}(\gamma)\rangle \in g$.
Take the admissible permutation π which swaps $\mathcal{G}(\alpha)$ and $\mathcal{G}(\alpha{+}1)$ while fixing others. $\mathcal{G}(\alpha),\ \mathcal{G}(\alpha + 1) \in x$ and y is disjoint from x, so π acts as identity on y.

$$\pi(\langle \mathcal{G}(\alpha), \mathcal{G}(\beta)\rangle) = \langle \pi\mathcal{G}(\alpha), \pi\mathcal{G}(\beta)\rangle = \langle \mathcal{G}(\alpha + 1), \mathcal{G}(\beta)\rangle$$

Since π also fixes every element in c,

$$\pi(g) = g$$

Thus $\langle \mathcal{G}(\alpha+1), \mathcal{G}(\beta) \rangle \in g$, contradicting $\langle \mathcal{G}(\alpha), \mathcal{G}(\beta) \rangle \in g$ and g being 1-1. $\square$

Lemma 3.8. *In ZF^{∇}, any permutations on $F(\alpha)$, which moves only finitely many Quine atoms, is in M.*

Proof. Suppose π is a permutation on $F(\alpha)$ which moves only the Quine atoms in the finite set E. E is a finite "support" of π, so π is symmetric. Then by Lemma 3.3, in order to have $\pi \in M$, it is sufficient to show that $\pi \subseteq M$. As every Quine atom is in M, and (Pair) and (Union) hold in M, we must have $\pi \subseteq M$ as required. $\square$

Theorem 3.9. *For each ZF-formula $\phi(x_1, x_2, x_3)$ with free variables x_1, x_2, x_3, the following is provable in ZF^{∇}:*

$$[\neg\, \exists x \forall a \exists! y\ (\phi(y,a,x) \wedge \forall b (a \equiv b \leftrightarrow \phi(y,b,x)))\ ]^M$$

Proof. Suppose otherwise. There is a ZF-formula $\phi(x_1, x_2, x_3)$ and a set $x \in M$ such that

$$\forall a \exists! y\ (\phi(y,a,x) \wedge \forall b (a \equiv b \leftrightarrow \phi(y,b,x)))$$

$x \in M$, so by the definition of M, there is an ordinal α such that

$$Base(x) \subseteq F(\alpha)$$

We can choose α to be even. Define

$$\begin{aligned} D_1 &= F(\alpha+\omega) - F(\alpha), \text{ i.e.} \\ D_1 &= \{\mathcal{G}(\beta) : (\beta \text{ even } \wedge \alpha \leq \beta < \alpha+\omega) \vee (\beta \text{ odd } \wedge \alpha < \beta \leq \alpha+\omega)\} \\ &= \{\mathcal{G}(\beta) : \alpha \leq \beta < \alpha+\omega\} \text{ since } \alpha \text{ and } \alpha+\omega \text{ are even} \end{aligned}$$

Thus, every admissible permutation τ fixes D_1, so D_1 is symmetric. (D_1 has finite "support" $\emptyset$.) And clearly $D_1 \subseteq M$. So $D_1 \in M$.

Suppose y is the unique set in M such that

$$\phi(y, D_1, x) \wedge \forall b (D_1 \equiv b \leftrightarrow \phi(y,b,x)) \text{ holds.}$$

i.e. y is the unique cardinality of D_1.

Then we have two cases:

Case1 $Base(y) \subseteq F(\alpha)$

Case2 $Base(y) \nsubseteq F(\alpha)$

- If Case1 holds, then define

$$\begin{aligned} D_2 :&= F(\alpha+\omega.2) - F(\alpha+\omega) \\ &= \{\mathcal{G}(\beta) : \alpha+\omega \leq \beta < \alpha+\omega.2\} \end{aligned}$$

  Let π be the semi-admissible permutation of A such that

$$\begin{aligned} &\pi(\mathcal{G}(\alpha+n)) = \mathcal{G}(\alpha+\omega+n); \\ &\pi(\mathcal{G}(\alpha+\omega+n)) = \mathcal{G}(\alpha+n); \\ &\pi(\mathcal{G}(\beta)) = \mathcal{G}(\beta) \text{ for } \beta < \alpha \text{ or } \alpha+\omega.2 \leq \beta \end{aligned}$$

  Thus, π fixes each element of $F(\alpha)$ and interchanges elements between D_1 and D_2. Since $Base(x), Base(y) \subseteq F(\alpha)$,

$$\pi(x) = x, \pi(y) = y, \pi(D_1) = D_2 \text{ and } \pi(D_2) = D_1$$

  Then by Lemma 3.5,

$$\phi^M(y, D_1, x) \leftrightarrow \phi^M(\pi y, \pi D_1, \pi x) \leftrightarrow \phi^M(y, D_2, x)$$

  Hence y is also the cardinality of D_2. This contradicts Lemma 3.7.

- If Case2 holds, i.e. $Base(y) \nsubseteq F(\alpha)$, then there is some β such that $\mathcal{G}(\beta) \in Base(y)$ but $\mathcal{G}(\beta) \notin F(\alpha)$. Pick an ordinal $\gamma > \beta$, of the same parity as β. Define a permutation τ on $D_1 \cup F(\gamma+1)$ as follows.

  If β is even, τ swaps $\mathcal{G}(\beta)$ with $\mathcal{G}(\gamma)$ and
  τ swaps $\mathcal{G}(\beta+1)$ with $\mathcal{G}(\gamma+1)$
  If β is odd, τ swaps $\mathcal{G}(\beta-1)$ with $\mathcal{G}(\gamma-1)$ and
  τ swaps $\mathcal{G}(\beta)$ with $\mathcal{G}(\gamma)$

  Since α is even, $\mathcal{G}(\xi) \in F(\alpha) \leftrightarrow \xi < \alpha$. Hence τ fixes every element of $F(\alpha)$. $Base(x) \subseteq F(\alpha)$ gives $\tau x = x$; $\mathcal{G}(\beta) \in Base(y)$, so $\tau y \neq y$.
  τ is semi-admissible, so by Lemma 3.5,

$$\phi^M(y, D_1, x) \leftrightarrow \phi^M(\tau y, \tau D_1, x)$$

  Thus τy is the cardinality of τD_1. But y is the cardinality of D_1, $y \neq \tau y$ gives that

  D_1 and D_2 are not equipollent in M

  However, by Lemma 3.8, τ is a set of M and is a bijection of M. Thus D_1 and τD_1 must be equipollent. Contradiction

$\square$

This completes the proof of Theorem 3.1(Lévy,Gauntt), so (rather surprisingly) cardinality is not definable in ZF°.

4

A FINAL WORD

The original purpose of permutation models is to prove the independence of (AC). Although it does this for a theory different from ZF, the result can be transferred into ZF with the help of forcing. Of course, there is additional complications to make the forcing work. The brief idea is that, we construct a generic extension from a Boolean-valued model, using forcing; then find a symmetric submodel (analogous to a permutation model) in which ZF+$\neg$(AC) holds. See [7] for details.

Permutation model is also a good tool for proving other consistency results. For example, the non-implication of the reverses of AC$\rightarrow$BPIT$\rightarrow$OE$\rightarrow$OP$\rightarrow$ C_ω (see [10]); the independence of (GCH) from the aleph-hypothesis (see [2] Chapter III Section D); the independence of (AC) from the Kurepa's antichain principle (see [2] Chapter III Section E).

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