Monday, October 29, 1990

Last time had tree production theorem

≈ Condition (1) in hypothesis can be weakened to having absoluteness for posets that can be embedded in $\mathbb{P}_{<\kappa}$.

As an application of what we have, lets prove $AD^{L(\mathbb{R})}$ from (ω+1)-Woodin cardinals

Rmk Hypothesis (1) is trivial if, e.g.

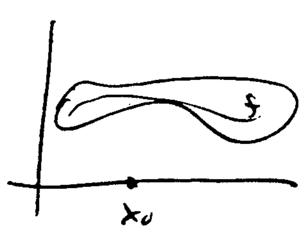
$$\varphi((b_0 b_1), s) =: b_1 \in L[s, b_0]$$

but then (2) isn't.

And if a is a real or φ is an $L(\mathbb{R})$ formula (2) is trivial but (1)

Rmk In $L(\mathbb{R}) + \neg AC$, can't uniformize "$y \notin OD(x)$". Else Uniformize it by $f \in OD(x_0)$

So else all is definable from x_0 (so have a well-ordering of reals) or f doesn't exist.



And "$y \in OD(x)$" is

$$\Sigma^2_1 : \left(\exists \text{ trans. } M \models V = L(\mathbb{R}),\ \mathbb{R} \subseteq M \text{ in which } y \in OD(x) \right)$$

So "$y \notin OD(x)$" is $\Pi^?_1$

So with AD, best thing to hope for

is Σ^1_2 is Suslin (as Suslin sets are uniformizable)

Note: having Σ^1_2 Suslin means Σ^1_2 has the scale property. Thus.

$$\underbrace{AD + \underbrace{\text{Scales}}_{\text{ie every set is suslin}}}_{\text{Can weaken to: "AD"}} \rightarrow \text{Scale}(\Sigma^1_2)$$

——— end rmks ———

Thm Assume $\exists\ \omega+1$-woodin cardinals, say $\{\kappa_i : i \leq \omega\}$, increasing. Then every set $A \subseteq \mathbb{R}$, $A \in L(\mathbb{R})$, is

$<\lambda$-weakly homogeneously Souslin where

$\lambda = \sup\{\kappa_i : \underbrace{i < \omega}_{\text{first } \omega \text{ only}}\}$

(Note: by Martin Steel, we have weak homogeneity past a Woodin cardinal, to get $AD^{L(\mathbb{R})}$

Rmk All we need is: ω-woodins + measurable above.

As seen last semester, ω-woodin cardinals alone can't work. (have $\mathbb{R}^\#$, so would get

$$\omega\text{-woodins} \to \underbrace{Con(AD)}_{\substack{AD \text{ in } L(\mathbb{R}) \\ \text{gives } Con(AD) \text{ since} \\ \mathbb{R}^\# \text{ exists}}} \to Con(\omega\text{-woodins})$$

Lemma Suppose $N \models ZFC$ and N is transitive. Suppose $\lambda \in N$ and

$N \models$ "λ is a strong limit". Suppose

$\sigma \subseteq \mathbb{R}$ and:

1) $x, y \in \sigma \Rightarrow \mathbb{R} \cap N[x, y] \subset \sigma$

2) $x \in \sigma \Rightarrow x$ is N-generic for $\mathbb{P}_x$ with

$$|\mathbb{P}_x|^N < \lambda$$

3) $\sup\{\omega_1^{N[x]} : x \in \sigma\} = \lambda$

Then $\mathbb{R} \cap N(\sigma) = \sigma$ and $N(\sigma)$ is a symmetric extension of N for $Coll(\omega, <\lambda)$

which means:

* "there exists $G \subseteq Coll(\omega, <\lambda)$", G is N-generic, and ($G \notin V$, possibly) | $\bigcup_{\alpha<\lambda} \mathbb{R} \cap N[G \restriction \alpha] = \sigma$"

Problem is: G may not exist in V. But whether G is in V or not does not affect the ontological character of $N(G)$: it is well-defined.

~~A way to look at it:~~

Point: Statement $(*)$ is absolute to
$$N' = N(\sigma)^{Coll(\omega, P(\lambda)^{N(\sigma)})}$$

Now $N' \models AC$ (have well-ordered σ

So $N' \models ZFC$. So N-genericity is simple, and can build generic bit by bit to get
$$\bigcup_{\alpha<\lambda} \mathbb{R} \cap N[G \restriction \alpha] = \sigma$$

Build a tree: enumerate dense sets & terms.
Start meeting dense sets, specifying conditions.

Every G defines a branch through this tree.

That's why in $N(\sigma)$ it makes sense to say it is symmetric extension.

Proof of lemma Pf: $N(\sigma)$ being a symmetric extension is absolute.

We find G in $V^{\mathbb{P}}$ for some $\mathbb{P}$.

$$\mathbb{P} = \left\{ g \subset Coll(\omega, \leq\alpha) \;\middle|\; \begin{array}{l} \alpha<\lambda \\ g \text{ is } N\text{-generic} \\ g \in \sigma \quad (\text{view } g \text{ as a real}) \end{array} \right\}$$

Order these by extension:

$$g_1 \le g_2 \quad \text{if} \quad \alpha_1 \ge \alpha_2$$

$\uparrow$ stronger

$$\& \quad g_1 \restriction \mathrm{Coll}(\omega, \le \alpha_2) = g_2$$

Suppose $G \subseteq \mathbb{P}$ is V-generic. We may assume

$\underbrace{G \subseteq \mathrm{Coll}(\omega, <\lambda)}_{= \mathbb{Q}}$ (Put conditions together)

Must check:

1) G is N-generic:

~~Pt~~ Point: If $D \subseteq \mathrm{Coll}(\omega, <\lambda)$, $D \in N$,

then D/g is dense in $\mathrm{Coll}(\omega, <\lambda)_g$

$$= \mathrm{Coll}(\alpha, <\lambda)$$

2) $\bigcup_{\alpha<\lambda} (\mathbb{R} \cap N[G \restriction \alpha]) = \sigma$

Suppose $\alpha < \lambda$, $G \restriction \alpha \in \sigma$. Let $g = G \restriction \alpha$.

Then $\mathbb{R} \cap N[g] \subseteq \sigma$

So proved $\subseteq$.

Must show

$$\sigma \subset \bigcup_{\alpha} N[G \restriction \alpha]$$

So suppose $g \subset Coll(\omega, \leq \alpha)$. $g \in \sigma$, and

$x \in \sigma$.

We shall find $\bar{g} \subseteq Coll(\omega, \leq \bar{\alpha})$ s.t.

$$\bar{g} \leq g \quad \text{and} \quad x \in N[\bar{g}]$$ (so by genericity we would be done)

$N[g] = N[Y_g]$.

$Y_g \in \sigma$

Let $z = \underbrace{(x, y)}_{\in \sigma}$ So z is N-generic for

some $\mathbb{P}_z \in N$, $|\mathbb{P}_z|^N < \lambda$

Have
$$N[x] \subset N[x, y] = N[z].$$

So x is $N[y]$-generic for a poset

$$\mathbb{Q}_x = \mathbb{P}_z/y \text{ in } N[y]$$

with $|\mathbb{Q}|^{N[y]} < \lambda$

So choose $\bar{\alpha} = |\mathbb{Q}|^{N[y]}$

In $N[y]$, consider

$$\mathbb{Q} \times Coll(\omega, \bar{\alpha}) \sim Coll(\omega, \bar{\alpha})$$

So choose $h \subset Coll(\omega, \bar{\alpha})$ which is

$N[x,y]$-generic for $\mathbb{Q} * \mathrm{Coll}(\omega, \delta)$

with $h \in \sigma$. To see h exists, choose $t \in \sigma$

with $w_1^{N[t]} > \underbrace{(2^{\bar\delta})^{N[x,y]}}_{<\lambda}$

Now choose $h \in N[x,y,t]$.

Have:

$$\underbrace{\mathbb{Q} * \mathrm{Coll}(\omega, \bar\delta)}_{\substack{x \quad h \\ \text{respective} \\ \text{generics}}} \sim \mathrm{Coll}(\omega, \delta) \overset{\text{isomorphic}}{=} \mathrm{Coll}(\omega, [\delta, \bar\delta]) \qquad =_{d t} \prod_{\delta \le \beta \le \bar\delta} \mathrm{Coll}(\omega, \beta)$$

Work in $N[x,y,h]$ to choose $\bar h \subset \mathrm{Coll}(\omega, [\delta, \bar\delta])$

~~with $\bar h$ $N[x]$-generic and~~

$\bar h$ is $N[y]$-generic and

$$N[y,x,h] = N[y][\bar h]$$

$$= N[\bar g], \text{ canonically, i.e.}$$

$$y \subset \mathrm{Coll}(\omega, \le \delta)$$
$$\bar h \subset \mathrm{Coll}(\omega, [\delta, \bar\delta])$$
$$\text{so } \bar g \subset \mathrm{Coll}(\omega, \le \bar\delta)$$

This $\bar g, \bar\delta$ work.

QED Lemma.

Lemma (Special Version for our case) Fix $\delta > \lambda = \sup\{\kappa_i : i < \omega\}$ κ_i's woodin cardinals as in Thm.

(Think of $\delta < \kappa_\omega$)

Then for any $a \in \mathbb{Q}_{<\kappa_0}$ letting

$$S = \{ Z \prec V_\lambda : |Z| = \omega,\ a \in Z,\ Z \cap (\cup a) \in a,\ \kappa_i \in Z \text{ for all } i$$

and if $D \subseteq \mathbb{Q}_{<\kappa_i}$ is dense and $D \in Z$, then $D \cap \cup b \in b$ for some

$$b \in D \cap Z \}$$

we have that S is stationary in $P_{\omega_1}(V_\lambda)$

Proof: An elementary chain argument as before.

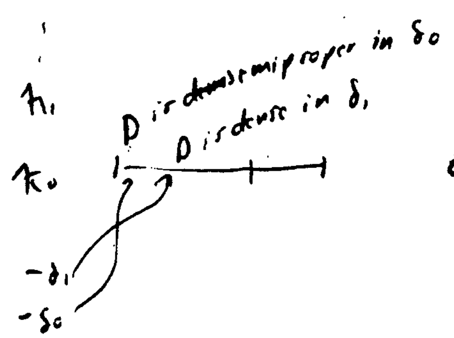
~~Want~~ Fix $H : V_\lambda^{<\omega} \to V_\lambda$, and build $Z \in S$ closed under H.

Fix $\bar\lambda \gg \lambda$, choose $Z_0 \prec V_{\bar\lambda}$, $|Z_0| = \omega$,

$\{H, \lambda, \langle \kappa_i : i < \omega \rangle, a\} \subset Z_0$

and $Z_0 \cap (\cup a) \in a$.

Will be a bookkeeping argument



← enumerate dense sets for $\mathbb{Q}_{<\kappa_0}$ with room

On ω-many steps, hit all dense sets in Z_0, but then might have have more dense sets

<u>Problem</u>: Can't pass through limits: since ω-many steps may be cofinal in κ_0, say, and can't go on.

That's why we do bookkeeping

In ω-steps construct

$Z_0 \prec Z_1 \prec V_{\bar{\lambda}}$ with

$Z_1 \cap (\bigcup b) \in b$

for some $b \in Z_1 \cap D$ whenever $D \in Z_1$, $D \subset Q$ and D is dense.

To build Z_1, build

$\langle Z_0^i : i \in \omega \rangle$, ~~where~~ and sequence $\langle \delta_i : i < \omega \rangle$ increasing, below κ_0

$\langle D_i : i < \omega \rangle$

$Z_0^0 = 0$

1) $D_i \in Z_0^i$, $\delta_i \in Z_0^i$

and $D_i \cap Q_{<\delta_i}$ is semi proper.

2) Z_0^{i+1} end-extends $Z_0^i \cap V_{\delta_i}$

3) $Z_0^{i+1} \cap b \in Ub$ for some

$$b \in Z_0^{i+1} \cap D_i \cap V_{\delta_i}$$

4) (Catch-up conditions) If $D \subset Q_{<\kappa_0}$, D is dense
(bookkeeping)
and $D \in \bigcup_i Z_0^i = Z_1$, then $D \in D_i$ for some i

This gives Z_1.

Now build Z_2, working up to κ_2, and so on.
Build

$$Z_0 \prec Z_1 \prec Z_2 \prec \cdots \prec Z_i \prec \cdots$$

where for all $D \in Q_{<\kappa_i}$, D dense, $D \in Z_{i+1}$,

$$Z_{i+1} \cap Ub$$

for some $b \in Z_{i+1} \cap D$

and (to avoid messing up what we did before)

Z_{i+2} end extends $Z_{i+1} \cap \underbrace{V_{(\kappa_{i+1})+\omega}}$, $i \geq 0$
and ~~Z_1 end extends~~

$Z_1 \cap Va = Z_0 \cap Va$

(i.e. add no new dense sets to i^{th} stage, so Z_{i+1} still works after defining Z_{i+2}.)

That proves that S is stationary. QED

Wednesday, October 31, 1990

(Proof of Thm cont.)

Proved last time that S is stationary, S built for some $a_0 \in \mathbb{Q}_{<\kappa_0}$

Similarly, if $a \in \mathbb{Q}_{<\kappa_n}$, then

can define S_a like S except we only capture dense sets ~~of~~ of $\mathbb{Q}_{<\kappa_i}$, $i \geq n$.

We don't care what a is ~~at this~~ for this theorem, so, let, say, $a = \omega$

Let $G \subset \mathbb{P}_{<\kappa_\omega}$ be generic with $S \in G$

(S from lemma last time)

Get $j : V \to M \subset V[G]$

$cp(j) = \omega_1$ $\quad\quad\quad\quad$ $(\cup S = V_\lambda)$

($j'' V_\lambda \in j(S)$, so $j'' V_\lambda$ is countable in M).

So V_λ [redacted] is countable in M.

Since we forcing with $\mathbb{P}_{<\kappa_\omega}$ (not $\mathbb{Q}_{<\kappa_\omega}$), get

$j(\kappa_\omega) = \kappa_\omega$.

Fix $m < \omega$. Let

$$\Sigma_m = \{ j(F)(j''(Ua)) \mid F \in V \quad a \in Q_{<\kappa_m} \}$$

$$\left(= \{ j(F)(j'' V_\delta) \mid F \in V \quad \delta < \kappa_m \} \quad \text{as } a = \mathbb{P}_{\omega_1}(V_\delta) \text{ ways} \right)$$

Have

$$\Sigma_m \prec M \quad (\text{Got choice})$$

<u>Key claim</u>: For all $m < \omega$, $G \cap Q_{<\kappa_m}$ is V-generic and

$$G \cap Q_{<\kappa_m} \sim G \cap V_{\kappa_m} \quad (\text{"}\sim\text{" means what is below})$$

(So below S, $Q_{<\kappa_m}$ is completely embedded in V_{κ_m})

Pf/ Fix m. First: If $a \in G \cap V_{\kappa_m}$, then $a \not\perp S$, so have a common refinement, and $Ua \subset US$, so there is $b \in G$ s.t. $b \leq a, S$. So for $\sigma \in b$, $\sigma \cap Ua \in a$, $\sigma \cap US \in S$, so letting $a^* = b \cap P_{\omega_1}(Ua)$, then $a \in G$ iff $a^* \in G$, and $a^* \in G \cap Q_{<\kappa_m}$. So $G \cap Q_{<\kappa_m} \sim G \cap V_{\kappa_m}$

On to the real proof.

Fix $b \leq S$. Fix $D \subset Q_{<\kappa_m}$. Suffices to find $a \leq b$, $c \in D$, $a \leq c$ (so generically meet D).

$b \leq S, s\sigma$

get $V_\lambda = \cup S \leq \cup b$

~~Since b≤S, letting~~ Letting:

$b^* = \{\sigma \in b \mid D \in \sigma\}$

get $b^* \leq b$ (b^* is cub in b)

And $\sigma \in b^* \Rightarrow \sigma \cap V_\lambda \in S$ since $\cup S = V_\lambda$

So get a regressive function:

By def. of ∇S, since $D \in \sigma$, there exists $c_\sigma \in \sigma \cap D$ with $\sigma \cap (\cup c_\sigma) \in c_\sigma$.

So letting $f(\sigma) = c_\sigma$, have

$f : b^* \to \cup b^*$ is a choice function,

so f is constant: $\exists a \subset b$ ~~st~~, and $\exists c$ ~~st~~

with $f \restriction a$ constant with value c,

$a \subset b$, a is stationary, $f(\sigma) = c$ for all $\sigma \in a$.

<u>So</u>:

1) $c \in D$

2) $\sigma \cap (\cup c) \in c$ all $\sigma \in a$, $\cup c < \cup a = \cup b$.

so $a \leq c$

QED main claim.

Let $N_m = \mathrm{Coll}(\bar{\Sigma}_m)$. Embedding splits:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "j_m"'] & M \\ & N_m \; (= \bar{N}_m) \arrow[u, "k_m"'] \end{tikzcd}$$

$= (\bar{j}_m)$

Let $G_m = G \cap Q_{<\kappa_m}$

$\bar{j}_m : V \to \bar{N}_m \subset V[G_m]$

be the associated embedding

<u>Claim</u> $j_m = \bar{j}_m$
$N_m = \bar{N}_m$

<u>Proof</u>: Recall $N_m = \mathrm{Coll}(\Sigma_m)_V$ and

$$\bar{\Sigma}_m = \{ j(F)(j''Ua) \mid F \in V, \ a \in Q_{<\kappa_m} \}$$

But in general: $b \in G$ iff $j''Ub \in j(b)$

So if $b \in \mathcal{P}(Ua)$, ~~then~~
$b \in Q_{<\kappa_m}$, ~~[illegible]~~ and $Ub = Ua$
then $b \in G_m$ iff $j''Ua \in j(b)$

So

$$N_m = \{ j_m(F)(j_m''Ua) \mid a \in Q_{<\kappa_m} \}$$

and

$$G_m = \{ a \in Q_{<\kappa_m} \mid j_m''(Ua) \in j_m(a) \}$$

But

$a \in G_m$ iff

$\bar{J}_m^{n} Va \in \bar{J}_m(a)$, and

$$\bar{N}_m = \{ \bar{j}_m(F)(\bar{J}_m^{n}(Va)) \mid a \in Q_{<\kappa_m} \}$$

(Generators are $\bar{J}_m^{n} \delta$, $\delta < \kappa_m$)

So now get a map $N_m \longrightarrow \bar{N}_m$ which is well defined and gives us the map we wanted

qed claim

So got a picture:

$m_1 < m_2$ $\quad \bar{X}_{m_1} \subset \bar{X}_{m_2}$

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[dr] \arrow[ddr] \arrow[dddr, bend right] & M & G \\
 & N_2 \arrow[u] & G_2 \\
 & N_1 \arrow[u] & G_1 \\
 & N_0 \arrow[u] & G_0
\end{tikzcd}$$

Let $N^{\infty} = \lim N_m = Coll(\Sigma)$ $\quad \Sigma = \bigcup \Sigma_m$

We work in $V[G]$

Have $\sigma = \mathbb{R} \cap N^{\infty} = \bigcup_n \mathbb{R} \cap N_m$

$$= \bigcup_n \left(\mathbb{R} \cap V[G_m]\right)$$

($j_m(\omega_1) = \kappa_m$ and $N_m^{<\kappa_m} \subset N_m$ in $V[G_m]$ so have same reals)

So look at σ:

$\lambda = \sup_n \{\kappa_m\}$

- λ is a strong limit

- σ satisfies ~~the~~ requirements to be a symmetric extension

<u>Claim</u> $V(\sigma)$ is a symmetric extension for $Coll(\omega_1, <\lambda)$

Pf/

1) $V[x,y] \cap \mathbb{R} \subset \sigma$ all x,y: $\exists m$ s/

$$V[x,y] \cap \mathbb{R} \subseteq \mathbb{R} \cap N_m$$

2) x is ~~ω~~-generic for some $\mathbb{P} \in V$, $|\mathbb{P}|^V < \lambda$;

But $x \in V[G_m]$ for some m, so x is generic for some

$$\mathbb{P} \subset V_{\kappa_m}. \checkmark$$

3) $\sup \omega_1^{V[x]} = \lambda$: But $\sup \omega_1^{V[x]} = \sup \omega_1^{V[G_m]} = \sup \kappa_m =$

So $V(\sigma)$ is a symmetric extension. qed.

Have

$$j_\infty : V \longrightarrow N^\infty = \varinjlim N_m$$

So

$$j_\infty : L(\mathbb{R})^V \longrightarrow L(\mathbb{R})^{N^\infty}$$

$$= L(\sigma)$$

Conclusion: with notation as above, ~~there is~~ then:

~~a symmetric emb~~ if $\langle \kappa_i : i < \omega \rangle$ are woodin cardinals, $\lambda = \sup\{\kappa_i : i \in \omega\}$, there is a generic elementary embedding

$$L(\mathbb{R}) \to L(\sigma) \qquad L(\sigma) = (L(\mathbb{R}) \text{ of } \underline{\text{the}} \text{ symmetric collapse}$$

(the in that its all homogeneous, doesn't matter which you take)

So $\mathrm{Th}_{\mathbb{R}^V}(L(\mathbb{R})) = \mathrm{Th}_{\mathbb{R}^V}(L(\sigma))$ real param.

ie. $L(\mathbb{R})^V \equiv_{\mathbb{R}^V} L(\mathbb{R})^{\mathrm{Sym.Coll.}(\omega, <\lambda)}$

<u>Corollary</u> Assumptions as above. Suppose $V[G_1] \subset V[G_2]$ and G_2 is V-generic for $\mathbb{P} \in V_\lambda$. Then

$$L(\mathbb{R})^{V[G_1]} \equiv_{\mathbb{R}^{V[G_1]}} L(\mathbb{R})^{V[G_2]}$$

Proof ~~In $V[G_1]$, still have Woodin cardinals, so may assume $V = V[G_1]$~~

Choose $g \subseteq \mathrm{Coll}(\omega, <\lambda)$, g is V-generic / $V[G_2]$.

Let $\sigma = \bigcup_{\alpha < \lambda} (\mathbb{R} \cap V[G_2][g \restriction \alpha])$

$=$ reals of symmetric collapse over $V[G_2]$.

Then $V[G_2](\sigma)$ is a symmetric extension of $V[G_2]$ for $\mathrm{Coll}(\omega, <\lambda)$.

Also, $V[G_1](\sigma)$ is a symmetric extension of $V[G_1]$ for $\mathrm{Coll}(\omega, <\lambda)$ (reorganize G_2's poset $\mathbb{P}_2$ s.t. $\mathbb{P}_2 \times \mathrm{Coll}(\omega, <\lambda) = \mathrm{Coll}(\omega, <\lambda)$)

By the preceding corollary, applied to $V[G_1]$, $V[G_2]$, get

$$L(\mathbb{R})^{V[G_1]} \equiv_{\mathbb{R}^{V[G_1]}} L(\sigma) \equiv_{\mathbb{R}^{V[G_2]}} L(\mathbb{R})^{V[G_2]}$$

qed <u>Cor.</u>

So: let $\varphi(x, \vec{r})$ be a definition for a set of reals in $L(\mathbb{R})$ with parameters $\vec{r} \in \mathbb{R}$. (so $a = \vec{r}$)

Apply tree production lemma. Hypothesis are satisfied, as we'll check in a moment. Get trees

T_m, T_m^* for $m < \omega$, trees on κ_m,

$$p[T_m]^{V[g]} = \{ b \in \mathbb{R}^{V[g]} \mid L(\mathbb{R})^{V[g]} \models \varphi[b, \vec{r}] \}$$

$$p[T_m^*]^{V[g]} = \text{similar} \quad \text{- - - -} \quad \neg\varphi(b, \vec{r})$$

~~Hypothesis~~ For all g generic over V for a partial order of cardinality $< \kappa_n$.

Hypothesis of tree production lemma are clear by previous corollary.

So get T, T^* on λ s.t.

$$p[T]^{V[g]} = \{ b \in \mathbb{R}^{V[g]} \mid L(\mathbb{R})^{V[g]} \models \varphi(b, \vec{r}) \}$$

$$p[T^*]^{V[g]} = \{ b \in \mathbb{R}^{V[g]} \mid L(\mathbb{R})^{V[g]} \models \neg\varphi(b, \vec{r}) \}$$

for all g V-generic for some $\mathbb{P}$ with $|\mathbb{P}|^V < \lambda$

($T \sim \oplus T_m$, $T^* \sim \oplus T_m^*$. As before, the T_m's ; T_m^*'s don't interfere by absoluteness, coming back to V)

Thus, $p(T) = \omega^\omega \setminus p(T^*)$ in $V^{\mathrm{Coll}(\omega, \kappa_m)}$

for all m. So by weak homogeneity lemma,

T, T^* are $<\kappa_m$ weakly homogeneous for all m,

so they are $<\lambda$ weakly homogeneous for

all λ. So by Martin-Steel, $p[T]$ is determined.

So ~~it~~ proved: if $A \in L(\mathbb{R})$, $A \subset \mathbb{R}$, and

A is definable in real parameters, then

A is determined. So $L(\mathbb{R}) \models AD$ (If not,

in $L(\mathbb{R})$ there's a set definable from reals & ordinals that's undetermined. Minimize ordinals and now got an undetermined ~~rea~~ real-definable set) QED

Now we go for best possible result.

Next goal is:

Thm Suppose λ is a limit of woodin cardinals. Suppose $V(\sigma)$ is a symmetric extension of V for $\mathrm{Coll}(\omega, <\lambda)$. Then $L(\sigma) \models AD$.

So

Con (ZFC + $\exists$ ω-many woodin cards)

$\rightarrow$ Con (ZF + DC + AD)

Using stationary towers T_γ, S_γ work in $V[g]$ for all g generic over V for $\mathbb{P} \in V_\gamma$.

Let
$$T = \bigoplus_\gamma T_\gamma$$
$$S = \bigoplus_\gamma S_\gamma$$

There's no interference problem because: if $\gamma_1 < \gamma_2$

$$\begin{array}{ccc} & \gamma & \\ T_{\gamma_1} & & T_{\gamma_2} \\ & & \\ S_{\gamma_1} & & S_{\gamma_2} \end{array}$$

if $\mathbb{P} \in V_\gamma$, $\gamma_1 < \gamma < \gamma_2$, and in $V[g]$

$T_{\gamma_1}, S_{\gamma_1}$ fail poorly, then ~~since~~ $p[T_{\gamma_1}]^{V[g]} \cap p[S_{\gamma_2}] \neq \emptyset$

$$p[S_{\gamma_1}]^{V[g]} \cap p[T_{\gamma_2}]^{V[g]} \neq \emptyset$$

but if this happens in $V[g]$, by absoluteness it happens in V, but that can't be.

So T, S work. QED

11/7/90

H. WOODIN. Last time we proved:

Thm: If there are $\omega+1$ Woodin cardinals, then $AD^{L(\mathbb{R})}$.

Remark: The same proof works with ω Woodin cards. and a measurable above them all.

Theorem: Suppose λ is ~~a limit of Woodin cards.~~ the limit of $\kappa_0, \kappa_1, \kappa_2, \ldots$ (ω-many) Woodin cards. Suppose $V(\mathbb{R})$ is a symmetric extension of V for $Coll(\omega, <\lambda)$. Then $L(\mathbb{R})^{V(\mathbb{R})} \models AD$.

Corollary: $CON(ZFC + \exists\, \omega$ Woodin cards.$) \Rightarrow CON(ZFC + AD)$

Pf: Let $G \subseteq Coll(\omega, <\lambda)$ be generic over V.
Let $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha]$.

Suppose $L(\mathbb{R}^*) \not\models AD$.
Let γ be least such that:
(1) $L_\gamma(\mathbb{R}^*) \models ZF^{-\epsilon}$
(2) $L_\gamma(\mathbb{R}^*) \not\models AD$

By homogeneity, γ does not depend on G.
Fix a formula $\varphi(x,y)$, $a \in \mathbb{R}^*$ such that

$\{b \in \mathbb{R}^* : L_\gamma(\mathbb{R}^*) \models \varphi(b,a)\}$ is not determined.
We may assume $a \in V$ (by passing to $V[a]$).
Fix a limit card. $\eta > \gamma$ with $|V_\eta| = \eta$.
Consider $\mathbb{Q}_{<\eta}$.
Let $S = \{X \prec V_{\lambda+1} : |X| = \omega,\ \kappa_i \in X$, all i, (the ith Woodin card. κ_i)
and for all $i < \omega$, for all dense
$D \subseteq \mathbb{Q}_{<\kappa_i}$, if $D \in X$, then $X \cap \cup b \in b$
for some $b \in X \cap D\}$

As before, S is stationary in $\mathcal{P}_{\omega_1}(V_{\lambda+1})$.
Let $G \subseteq \mathbb{Q}_{<\eta}$ be V-generic with $S \in G$.
Let $j : V \longrightarrow (M, E)$ be the associated embedding.
(not well founded.)

Thus, ① $\mathrm{cp}(j) = \omega_1$
② $G = \{a \in \mathbb{Q}_{<\eta} : j''\cup a \text{ "}\in\text{" } j(a)\}$

i.e., $a \in G$ iff $\exists z \in M$ s.t.
$\{t : t E z\} = \{j(t) : t \in \cup a\}$

In particular, $j \restriction V_\beta \text{ "}\in\text{" } M$, all $\beta < \eta$.
$S \in G$. So, as before, $G_m = G \cap \mathbb{Q}_{<\kappa_m}$ is
V-generic for $\mathbb{Q}_{<\kappa_m}$ (all m)
Then, as before,

$$\begin{tikzcd}[column sep=small]
V \arrow[rrr, "j"] \arrow[d, "j_0"'] \arrow[drr, "j_1"] \arrow[drrrr, "j_2"] & & & (M,E) & & & \\
(N_0,\in) & & (N_1,\in) & & (N_2,\in), \ldots\ldots, & & (N^*,\in^*) \arrow[uulll, "k^*"']
\end{tikzcd}$$

where $j_m : V \to N_m$ is the embedding corresponding to G_m.

$$R^{N^*} = \bigcup_{m\in\omega} R_m = \bigcup\{R \cap V[G_m] : m<\omega\}$$

So, $V(R^{N^*})$ is a symmetric extension of V for $Coll(\omega, <\lambda)$.

How well-founded is M?

Clearly, $\eta \subseteq M$ $\quad (j''\delta \in M$ all $\delta < \eta)$

Fix $\delta < \eta$ (δ regular, $\delta > \lambda$). Then, $j \upharpoonright V_\delta \in M$ and $G \cap V_\delta \in M$.

So,

$$\begin{tikzcd}
V_\delta \arrow[rrr, "j\upharpoonright V_\delta"] \arrow[d, "j_0\upharpoonright V_\delta"'] \arrow[rr] \arrow[rrr, bend right] & & & (j(V_\delta), E) & \\
j_0(V_\delta) & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & & (j^*(V_\delta), E^*) \arrow[ul, "k^*\upharpoonright j^*(V_\delta)"']
\end{tikzcd}$$

this diagram "$\in$" M.

[Why? : For each m, let

$$X_m = \{ j(F)(j''\cup a) : a \in G \cap \mathbb{Q}_{<\kappa_m} \}$$

$m<n$ implies $X_m \subseteq X_n$. So,

$$N_m = Coll(X_m)$$

let $Y_m = \{ j(F)(j''(\cup a)) : a \in G \cap \mathbb{Q}_{<\kappa_m}$ and $F \in V_\delta^{V_{\kappa_m}} = V_\delta \}$

Note that $j \upharpoonright V_\delta \in M$ and $G \cap V_\delta \in M$.

So, $\langle Y_m : m<\omega \rangle \in M$

this is an elementary chain in $j(V_\delta)$

The above diagram comes from taking the transitive collapse. i.e., $j_m(V_\delta) = Coll(Y_m)$]

So, $j^*(V_\delta) \in M$ and $k^*: j^*(V_\delta) \to j(V_\delta)$ is also in M.
(Note: $j^*(V_\delta) = coll^M(\bigcup Y_m)$, k^* is the inverse of the collapse map.)

Since V_δ is an initial segment of V, $j^*(V_\delta)$ is an initial segment of N^*. Also, $k^* \restriction j^*(V_\delta) \in M$

Claim: δ "$\subseteq$" N^*.

[Pf: $j^*(V_\delta) \in M$ and $M \models j^*(V_\delta)$ is well-founded

$j^* \restriction V_\delta \in M$. So, $M \models \operatorname{rank}(j^*(V_\delta)) \geq \delta$

So, for all $\delta < \eta$, δ regular, $(\delta > \lambda)$, δ is an initial segment of N^*. So, η is an initial segment of N^*.

So, $L_\eta(\mathbb{R}^*)$ is an initial segment of $L(\mathbb{R})^{N^*}$.

But $\eta > \gamma$. So,

$$L_\gamma(\mathbb{R}^*) = (L_\gamma(\mathbb{R}))^{N^*}$$

We have that γ is least with $L_\gamma(\mathbb{R}^*) \models ZF^{-\epsilon} + \neg AD$

$a \in \mathbb{R}^V$, $\varphi(x,y)$, and

$\{b : L_\gamma(\mathbb{R}^*) \models \varphi(b,a)\}$ is not determined.

Let $\psi(x,a) \equiv x \in \mathbb{R} \wedge L_\gamma(\mathbb{R}) \models \varphi(x,a)$, where γ is least s.t. $L_\gamma(\mathbb{R}) \models ZF^{-\epsilon} + \neg AD$

Claim: ψ is absolute between forcing extensions of V of size $<\lambda$.

Pf: Suppose $V[G_1] \subset V[G_2]$ are generic extension of V with G_2 generic for $\mathbb{P}$ and $|\mathbb{P}|^V < \lambda$. Let $g \subset Coll(\omega, <\lambda)$ be generic over $V[G_2]$. Let $\mathbb{R}^* = \bigcup\{\mathbb{R} \cap V[G_2][g \upharpoonright \alpha] : \alpha < \lambda\}$. Thus, $V[G_2](\mathbb{R}^*)$ is a symmetric extension for $coll(\omega, <\lambda)$ of $V[G_2]$. But $V[G_1](\mathbb{R}^*)$ is also a symmetric ext. for $coll(\omega, <\lambda)$ of $V[G_1]$.

Repeat the previous construction over $V[G_1]$ and $V[G_2]$.

Let $\gamma_1 = $ least γ s.t. $L_\gamma(\mathbb{R}^{V[G_1]}) \models ZF^{-\epsilon} + \neg AD$

$\gamma_2 = $ least γ s.t. . . . $\mathbb{R}^{V[G_2]}$

Thus, there are generic embeddings:

$$j_1 : L_{\gamma_1}(\mathbb{R}^{V[G_1]}) \longrightarrow L_{\gamma^*}(\mathbb{R}^*)$$

$$j_2 : L_{\gamma_2}(\mathbb{R}^{V[G_2]}) \longrightarrow L_{\gamma^*}(\mathbb{R}^*) \quad \square$$

Thus, the def. of ψ is absolute.

So, by the tree production theorem, there are trees T, T^* on λ with $p[T]^{V[G]} = $ set defined by ψ, $p[T^*]^{V[G]} = $ set defined by $\neg\psi$.

where G is V-generic for some $\mathbb{P}$ with $|\mathbb{P}|^V < \lambda$

λ is limit of Woodin cards. So, T, T^* are $<\lambda$ weakly homogeneous. So, by Martin-Steel theorem, $p[T]$ is determined.

But we have a generic embedding

$$L_\gamma(\mathbb{R}) \longrightarrow L_{\gamma^*}(\mathbb{R}^*)$$

let $B = p[T] = \{b \in \mathbb{R} : L_\gamma(\mathbb{R}) \models \varphi(b, a)\}$

let $B^* = \{b \in \mathbb{R}^* : L_{\gamma^*}(\mathbb{R}^*) \models \varphi(b, a)\}$.

So, B^* is not determined. ($\rightarrow\leftarrow$), for if τ is a winning strategy for B, then τ is winning for B^*.

<u>Theorem</u>: Suppose λ is the sup of the first ω Woodin cardinals.

let $G \subseteq Coll(\omega, <\lambda)$ be generic and let $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha]$.

So, $V(\mathbb{R}^*)$ is a symmetric extension of V for $Coll(\omega, <\lambda)$.

Suppose $B \subseteq \mathbb{R}^*$, $B \in L(\mathbb{R}^*)$ and B is

$\Delta^2_1(a)$ definable in $L(\mathbb{R}^*)$ for some $a \in \mathbb{R}^*$.
Then, there is a tree $T \subseteq (\omega \times \lambda)^{<\omega}$ with
$T \in V[a]$, T is $<\lambda$ weakly-homo in $V[a]$
and $p[T]^{V[G]} \cap \mathbb{R}^* = B$.

Pf: Work in $V[a]$. λ is the limit of the least ω-Woodin cards. in $V[a]$.
Fix $\varphi_1(x,y)$, $\varphi_2(x,y)$, Σ^2_1 formulas s.t.
$B = \{b : L(\mathbb{R}^*) \models \varphi_1(b,a)\}$
$\mathbb{R}^* \setminus B = \{b : L(\mathbb{R}^*) \models \varphi_2(b,a)\}$
Suppose $V[a] \subseteq V[G_1] \subseteq V(\mathbb{R}^*)$
Let $\gamma^* =$ least γ s.t. $\varphi_1(x,a), \varphi_2(x,a)$ define complements in $L_\gamma(\mathbb{R}^*)$

As before, can find $j_1^* : V[G_1] \to N_1^*$,
$j_1^* : L(\mathbb{R})^{V[G_1]} \longrightarrow L(\mathbb{R})^{N_1^*}$
So, $\exists \gamma_1$ s.t. $L_{\gamma_1}(\mathbb{R}^{V[G_1]}) \models \varphi_1, \varphi_2$ define complements
and we get a generic embedding
$j_1^* : L_{\gamma_1}(\mathbb{R}^{V[G_1]}) \longrightarrow L_{\gamma^*}(\mathbb{R}^*)$
Let $\psi(b,a) \equiv L_\gamma(\mathbb{R}) \models \varphi_1(b,a)$, where γ is least s.t. $\varphi_1(x,a)$, $\varphi_1(x,a)$ define complements.

Also, get $j_2^*: L_{\gamma_2}(\mathbb{R}^{V[G_2]}) \to L_{\gamma^*}(\mathbb{R}^*)$

So, we can get these as before.

[redacted]

[redacted]

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Theorem: Suppose λ is a limit of Woodin cardinals. Let $V(\mathbb{R})$ be a symmetric extension of V for $Coll(\omega, <\lambda)$. Then $L(\mathbb{R}) \models AD$. Further, if $B \subseteq \mathbb{R}$ is $\Delta^2_1(a)$ definable, then in $V[a]$ there exists a tree T on $(\omega \times \lambda)$ which, in $V[a]$, is $<\lambda$ weakly-homogeneous and such that $p[T]^{V(\mathbb{R})} = B$.

General Theorem: Suppose λ is a limit of Woodin cardinals. Suppose $V(\mathbb{R})$ is a symmetric extension of V for $Coll(\omega, <\lambda)$. In $V(\mathbb{R})$, let

$\Gamma = \{B \subseteq \mathbb{R}$: there exists $a \in \mathbb{R}$, a tree T on $(\omega \times \lambda)$ in $V[a]$ which is $<\lambda$ weakly homo. in $V[a]$ s.t. $B = p[T]$ in $V(\mathbb{R})\}$

Note: By the properties of the symmetric collapse, every set of ordinals in $V(\mathbb{R})$ appears at some stage $V[a]$.

Then, $L(\mathbb{R}, \Gamma) \models AD$.

(Note: It never happens $\Gamma = \mathcal{P}(\mathbb{R})^{V(\mathbb{R})}$ (Why?: $\{(x,y) : y \notin V[x]\} \notin \Gamma$, for ow, it can be uniformized $\Rightarrow\Leftarrow$).)

But it can happen $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma)$ in which case we will show $L(\mathbb{R}) \models AD_{\mathbb{R}}$.

<u>Pf. of the first theorem</u>: Exactly as in the case when $\lambda = \sup$ of the first ω-many Woodin cards. except that we use

$S = \{ X \prec V_{\lambda+1} :$ X is countable and for all $\delta < \lambda$ for all $D \subseteq \mathbb{Q}_{<\delta}$, if ① $\delta \in X$ ② δ is a Woodin cardinal, ③ δ is not a limit of Woodin cardinals and ④ $D \in X$, D is dense in $\mathbb{Q}_{<\delta}$, then $X \cap (\cup b) \in b$ for some $b \in X \cap D \}$

One must show:

① S is stationary in $\mathcal{P}_{\omega_1}(V_{\lambda+1})$

[<u>Sketch</u>: Fix $\eta >> \lambda$, $H : V_{\lambda+1}^{<\omega} \longrightarrow V_{\lambda+1}$

$Z_0 \prec V_\eta$, $|Z_0| = \omega$, $H, \lambda \in Z_0$.

<u>Goal</u>: Find $Z \prec V_\eta$, $|Z| = \omega$ s.t. $Z \cap V_{\lambda+1} \in S$

<u>Basic lemma</u>: let $Y \prec V_\eta$, $|Y| = \omega$, $\delta \in Y \cap \lambda$, δ is Woodin and not limit of Woodins. Let $\alpha < \delta$, $\alpha \in Y$.

Then, $\exists Y^* \prec V_\eta$, Y^* end-extends $Y \cap V_\alpha$, and $\forall D \subseteq \mathbb{Q}_{<\delta}$, if $D \in Y^*$, D dense, then $Y^* \cap (\cup b) \in b$ for some $b \in Y^* \cap D$.]

② Fix $\eta >> \lambda$, $|V_\eta| = \eta$.

Let $G \subseteq \mathbb{Q}_{<\eta}$ be generic.

Then, if $\delta < \lambda$ and δ is a Woodin card. which is not a limit of Woodin cards., then $G \cap \mathbb{Q}_{<\delta}$ is generic.

Let $j : V \to (M, E)$ be the embedding from G.
Let $T = \{\delta < \lambda : \delta \text{ is a Woodin card. not a limit of Woodin cards}\}$
For each $\delta \in T$,

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "j_\delta"'] & & (M, E) \\ & N_\delta \arrow[ur, "k_\delta"'] & \end{tikzcd}$$

j_δ is the embedding from $G \cap \mathbb{Q}_{<\delta}$, k_δ is the inverse collapse map for $X_\delta = \{ j(F)(j''\cup a) : F \in V,\ a \in G \cap V_\delta \}$.
Proceed as before using this system:

$\delta_1 < \delta_2 \in T$

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr] \arrow[drr] & & M \\ & N_{\delta_1} \arrow[r, "j_{\delta_2}"'] & N_{\delta_2} \arrow[u, "k_{\delta_2}"'] \end{tikzcd}$$

etc....

<u>Thm</u>: $(ZF + DC + AD + V = L(\mathbb{R}))$
Then, if B is $\Delta^2_1(a)$, then $B = p[T]$ for some tree T on $(\omega \times \alpha)$ for some α with T definable from a.

<u>Need</u>: There is an inner model $H \subseteq HOD$ and a sequence $\langle \theta_i : i < \omega \rangle$ increasing, $\theta_0 = \Theta$, where
① $H \models \theta_i$ is a Woodin card. and ② there exists a generic extension of $L(\mathbb{R})$, $\mathbb{R}^*$, such that $H(\mathbb{R}^*)$ is a symmetric extension of H for $Coll(\omega, <\lambda)$, where $\lambda = \sup \theta_i$ and there exists

an elementary embedding $j : L(\mathbb{R}) \to L(\mathbb{R}^*)$
(This was proved last semester at the end)
Now, we are almost done:
Suppose B is $\Delta^2_1(a)$ definable in $L(\mathbb{R})$
So, $j(B)$ is $\Delta^2_1(a)$ definable in $L(\mathbb{R}^*)$
But $H(\mathbb{R}^*)$ is a symmetric extension of H for $Coll(\omega, <\lambda)$
So, there exists a tree T on $(\omega \times \lambda)$, $T \in H[a]$, (T is $<\lambda$ weakly-homo in $H[a]$, but don't need this) and $j(B) = p[T]^{H(\mathbb{R}^*)}$. But $H \subseteq HOD$. So, $H[a] \subseteq L(\mathbb{R})$.
Hence, $T \in L(\mathbb{R})$ and $p[T]^{L(\mathbb{R})} = B$.
(Note: $H[a] \subseteq HOD[a] \subseteq HOD_a$
in fact $=$ (done last semester)
T is ordinal definable from a. Thus, $\exists S$ which is def. from a with $p[S] = B$)

Corollary: Assume $AD + V = L(\mathbb{R}) + ZF + DC$.
Then, every $\Delta^2_1(a)$ set admits a $\Delta^2_1(a)$ scale.
(if B is $\Delta^2_1(a)$, fix Σ^2_1 formulas ψ_1, ψ_2 to define B. Let α be the least st. $L_\alpha(\mathbb{R}) \models ZF^{-\epsilon}$.
ψ_1, ψ_2 define a set and its complement in $L_\alpha(\mathbb{R})$ and in $L_\alpha(\mathbb{R})$ there is a tree definable from a which projects to the set given by ψ_1.
Use the scale from the "least" such tree (take the leftmost

branches).
This scale is $\Delta^2_1(a)$.

Thm (Martin-Steel): Assume $V=L(\mathbb{R}) + AD + ZF + DC$.
Then, every Σ^2_1 set admits a Σ^2_1 scale.
(Their proof is a typical determinacy proof)

We can get a stronger result by generalizing the previous proof.

Theorem: Assume $ZF + AD +$ Every set is Suslin $+ DC$ ($\equiv ZF+DC+AD_{\mathbb{R}}$)
Then, scale(Σ^2_1) ($\equiv ZF+DC+$ Uniformization for all sets of reals $+AD$)

Open question: ① $ZF+DC+AD \overset{?}{\Rightarrow}$ scale(Σ^2_1)

② $ZF+AD \overset{?}{\Rightarrow}$ scale(Σ^2_1)

Let $AD^* \equiv AD +$ "within scale" is
① $A \subseteq \mathbb{R} \Rightarrow \exists S \subseteq OR,\ A \in L(S,\mathbb{R})$ i.e., every set is ∞-Borel.
② ("ordinal determinacy") Suppose $\alpha < \Theta$ and $\pi : \alpha^\omega \to \omega^\omega$ is continuous. Suppose $A \subseteq \mathbb{R}$. Then, the induced game on α is determined. (see ba

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Remark: One can also prove that Σ^2_1 sets are Suslin in $L(\mathbb{R})$ using these methods. (We'll see this later)

General Theorem: Suppose λ is a limit of Woodin cardinals and that $V(\mathbb{R}^*)$ is a symmetric extension of V for $Coll(\omega, <\lambda)$. In $V(\mathbb{R}^*)$, let

$$\Gamma = \{A \subseteq \mathbb{R}^* : \exists a \in \mathbb{R}^*, \exists T \in V[a] \text{ with } V[a] \models T \text{ is } <\lambda \text{ weakly-homo. such that } p[T] = A\}$$

Then, ① $L(\mathbb{R}^*, \Gamma) \models AD$

② Reflection: For any α, there exists $\bar{\alpha}$, $\bar{\Gamma} \subseteq \Gamma$, with $L_\alpha(\mathbb{R}^*, \Gamma) \equiv L_{\bar{\alpha}}(\mathbb{R}^*, \bar{\Gamma})$ and $\mathcal{P}(\mathbb{R}^*) \cap L_{\bar{\alpha}}(\mathbb{R}^*, \bar{\Gamma}) \subseteq M$.

[Note: ② $\Rightarrow$ ①: By Martin-Steel, any $B \in \Gamma$ is determined in $V(\mathbb{R}^*)$.

Lemma: Suppose T is δ-weakly homo and τ is a winning strategy for $p[T]$.
Suppose $V[G]$ is a generic extension of V with G V-generic for some poset $\mathbb{P}$ with $|\mathbb{P}|^V < \delta$.
Then, τ is a winning strategy in the extension $V[G]$.

Pf: Let T^* be a tree for $\omega^\omega \setminus p[T]$ derived from a witness for δ-weak-homogeneity of T.
So, $p[T^*] = \omega^\omega \setminus p[T]$ in $V[G]$.
So, done, by absoluteness. □

Corollary: If T is $<\lambda$-weakly-homogeneous and τ is a winning strategy for $p[T]$. Suppose $V(\mathbb{R}^*)$ is a symmetric extension for $Coll(\omega, <\lambda)$. Then τ is a winning strategy in $V(\mathbb{R}^*)$ for $p[T]^{V(\mathbb{R}^*)}$.

Thus, we need to prove only (2) (Reflection).

Recall : (1) Wadge hierarchy

For $A, B \subseteq \mathbb{R}$, $A \leq_W B$ if A is reducible to B or to $\omega^\omega \setminus B$ via a Lipschitz(?) function.

Wadge lemma : (AD) Then, B is reducible to A or A is reducible to $\omega^\omega \setminus B$.

Pf:

$G(A,B)$

I	II
$\vdots$	$\vdots$
x	y

I wins iff ($x \in A$ iff $y \in B$)

If I wins B is reduced to A
If II wins A " $\omega^\omega \setminus B$

$\square$

Define $A \sim_W B$ iff A is reducible to B and B is " " A

$A \precsim_W B$ if A is reducible to B.

(AD) (use lemma) $A <_W B$ iff I wins both $G(B,A)$ and $G(B',A)$.

(Note: If $A \leq_W B$, then $A' \leq_W B'$.
So, if $A \leq_W B$, then either $B \leq_W A$ or I wins both $G(B,A)$ and $G(B,A')$.
Why? : $B \leq_W A$ iff $B' \leq_W A'$

But $B \leq_W A$ or I wins $G(B',A)$
$B' \leq_W A'$ or I wins $G(B,A')$)

HW: Prove that $<_w$ is well-founded.

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H. Woodin:

Monday, Rm 939, 11-12

Def: $A \leq_w B$ iff A is continuously reducible to B.
Def: $A \leq_l B$ iff A is Lipschitz reducible to B.

Def: $A \leq_l B$ iff I wins $G(B, A)$ and $G(B, A')$

Lemma: $A <_l B$ or $A \underset{l}{>} B$ or $A \sim_l B$ or $A \underset{l}{\sim} B'$

① It is easy to show that all clopen sets in ω^ω are $\sim_w$ equivalent.
② The l-degrees of the clopen sets have order type ω_1

Consider the game $G^*(A, B)$:

I	II
n_0	○
n_1	○
	○
	m_0
	○
	○
	○
	m_1
x	y

i.e., II is allowed to wait, I wins iff y is infinite and $(x \in A$ iff $y \in B)$

Def: $A <_w B$ if, I wins $G^*(B, A)$ and $G^*(B, A')$.

Lemma: $A \le_w B$ or $B \le_w A$ or $A \sim_w B'$ or $A \sim_w B$.

Lemma (Martin): $\le_\ell$ is well-founded, assuming enough sets are Lebesgue measurable (or have Baire property).

Lemma (Martin's Proof): $<_\ell$ is well-founded for weakly homo. Suslin sets.

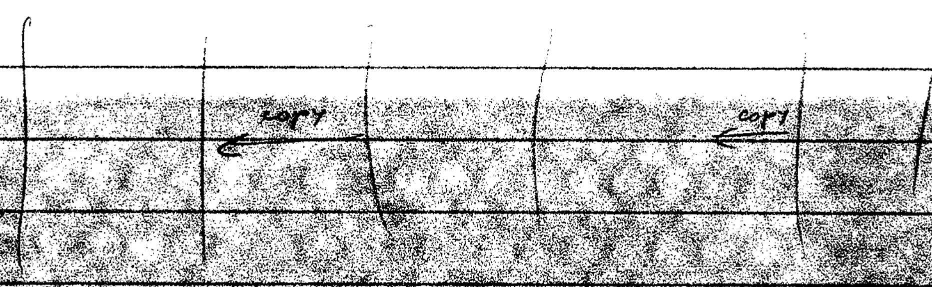
Martin's Proof: Assume $A_0 >_\ell A_1 >_\ell A_2 >_\ell \dots$

Let τ_n be a winning strategy for I in the game $G(A_n, A_{n+1})$. Let σ_n be a winning strategy for I in $G(A_n, A'_{n+1})$.

Fix $x \in 2^\omega$. $\quad \tau_n$ if $x(n)=1$

σ_n if $x(n)=0$

$A_0 \quad A_1 \quad A_2 \quad A_3 \quad \dots \quad A_n \quad A_{n+1} \quad \dots$



For each x, let y_x be the element of ω^ω generated by this matrix. given by the leftmost column.

Let $B = \{x : y_x \in A_0\}$

Claim: B does not have Baire property:

Pf: Suppose OW. Assume, WLOG, B is comeager in

some neighborhood N_s.
Choose $x \in N_s \cap B$ and $\bar{x} \in N_s \cap B$ and $x = \bar{x}$ except at one coordinate. $\square$

Note, the map $x \longmapsto y_x$ is continuous.
So, Martin's proof shows that if A has the property of Baire and all continuous preimages of A have the property of Baire, then $<_\ell$ is well-founded below A.
But if A is weakly homo. Suslin, then $f^{-1}(A)$ is also weakly-homo-Suslin for all continuous f. Further, weakly-homo-Suslin sets have Baire prop. Hence, we are done. $\square$

Theorem: Suppose λ is a limit of Woodin cardinals. Then, there exists $\delta < \lambda$ s.t., for any $A \subseteq \omega^\omega$, A is δ-weakly homo. Suslin iff A is $<\delta$ weakly homo. Suslin.
<λ?

Pf: For each $\alpha < \lambda$, let $\Gamma_\alpha = \{A : A$ is α weakly-homo-Suslin$\}$.
So, $\alpha < \beta < \lambda \Rightarrow \Gamma_\beta \subseteq \Gamma_\alpha$.
Further, if $\exists$ two Woodin cards $< \alpha$, all the games $G(A,B)$ are determined if $A, B \in \Gamma_\alpha$.
If we let $A^* = A \times \omega^\omega$, $B^* = \omega^\omega \times B$, then the payoff for $G(A,B)$ is $\underbrace{(A^* \cap B^*)}_{\in \Gamma_\alpha} \cup (\underbrace{(A^*)'}_{\in \Gamma_\alpha ?} \cap \underbrace{(B^*)'}_{\in \Gamma_\alpha ?})$.

Let $\delta_0 < \delta_1$ be Woodin cards. $< \alpha$

Then, $(A^*)' \in \Gamma_\beta$, all $\beta < \delta_1$

$(B^*)' \in \Gamma_\beta$, all $\beta < \delta_1$

Hence, the payoff set is in Γ_β for all $\beta < \delta_1$.

So, is in $\Gamma_{\alpha_0^+}$. Hence, it is determined.

So, for sufficiently large $\alpha < \lambda$, the Lipshitz games are determined in Γ_α.

Fix α_0 s.t. if $\alpha_0 < \alpha$, then these games are determined in Γ_α.

Let $\Delta_\alpha = \{A : A \in \Gamma_\alpha$ and $\omega^\omega \setminus A \in \Gamma_\alpha\}$

We first show there exists $\delta < \lambda$ with $\Delta_\alpha = \Delta_\beta$, $\lambda > \alpha$, $\beta > \delta$.

Suppose $\alpha_0 < \alpha < \beta < \lambda$, $\Delta_\beta \subsetneq \Delta_\alpha$.

Fix $B \in \Delta_\alpha \setminus \Delta_\beta$.

<u>Claim</u>: If $A \in \Delta_\beta$, then I wins $G(B,A)$, $G(B,A')$. i.e. $B \geq_\ell A$.

<u>Pf. of Claim</u>: If II wins $G(B,A)$, then $B' \leq_\ell A$ and $B \leq_\ell A'$.

But this implies that $B' \in \Gamma_\beta$ and $B \in \Gamma_\beta$

Hence $B \in \Delta_\beta$. (since weak-homo. is preserved by reduct.)

If II wins $G(B,A')$, then $B \leq_\ell A$ and $B' \leq_\ell A'$. So, $B \in \Gamma_\beta$, $B' \in \Gamma_\beta$. So, $B \in \Delta_\beta$. ⊠

Now, suppose that for some infinite $\alpha_0 < \beta_0 < \beta_1 < \cdots < \beta_i < \cdots < \lambda$ we have $\Delta_{\beta_{i+1}} \subsetneq \Delta_{\beta_i}$.

Then, choose $B_i \in \Delta_{\beta_i} \setminus \Delta_{\beta_{i+1}}$. Then, $B_0 >_W B_1 >_W B_2 >_W \cdots$ $(\rightarrow\leftarrow)$.

So, $\exists \delta < \lambda$ with $\Delta_\alpha = \Delta_\beta$, whenever $\delta < \alpha, \beta < \lambda$.

Claim: $\Gamma_\alpha = \Gamma_\beta$ whenever $\delta < \alpha, \beta < \delta$.

Pf: Choose a Woodin cardinal κ, with $\delta < \kappa < \lambda$.

Suppose $\delta < \alpha < \kappa^+ < \beta < \lambda$

Then, $\Gamma_\beta \subseteq \Delta_\alpha$. Why?: Suppose $A \in \Gamma_\beta$. Choose T β-weakly-homo. with $A = p[T]$. Let T^* be the corresponding tree for A'.

So, both T, T^* are $<\kappa$ weakly homogeneous (since $p[T] = \omega^\omega \setminus p[T^*]$ in $V^{Coll(\omega, <\kappa)}$)

So, $A, A' \in \Gamma_\alpha$. Hence, $A \in \Delta_\alpha$. $\square$

So, if $\beta > \kappa$, then $\Delta_\beta = \Gamma_\beta$.

Lemma: (Notation as above) Γ_β is closed under $\exists, \forall$ for $\beta > \kappa$.

Pf: Since $\Gamma_\beta = \Delta_\beta$, it suffice to show Γ_β is closed under $\exists$, which is immediate.

Actually, Γ_β is a σ-algebra. $\square$