PETOs

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February 11, 2010

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A **PETO** (a *Profoundly Evil Total Order*) is an infinite total order with no injective endomorphisms.

I started off thinking that ther should be no such things. After all, a perfectly satisfactory definition of infinite *set* (at least if you believe countable choice) is aset with no injection into a proper subset opf itself. However, i was wrong! However, to enter into the spirit of the Impossible Imre Question (which is to give the victim a sleepless night) I shall start off by proving that there are no countable PETOs.

Remark 1 *The collection of countable total orders is wellfounded under the relation $A \leq B$ if there is an injective homomorphism $A \hookrightarrow B$.*

(it doesn't even have any infinite antichains)

So if there are any PETOs at all there is one that is minimal wrt this $\leq$.

Does it have any infinite substructures? If it does, it injects into all of them homomorphically, by minimality, and therefore isn't a PETO.

But what if it doesn't? Then all its proper subsets are finite. But in that case it, too, is finite—and therefore isn't a PETO.

$\blacksquare$

So there are no countable PETOs

Nevertheless there *are* PETOs. The clue came to me from a theorem of Sierpinski that says that the collection of uncountable subsets of $\mathbb{R}$, ordered (well, quasiordered) by the relation $A \leq B$ if there is a cts injective $f : \mathbb{R} \to \mathbb{R}$ s.t $f``A \subseteq B$ in not wellfounded. Specifically if A is any uncountable set of reals there is $B \subset A$ s.t. A cannot be embedded into B in an order-preserving way.

Lemma 2 *(Sierpinski [1950]) (AC) If $E \subseteq \mathbb{R}$ and $|E| = 2^{\aleph_0}$ then $\exists H \subseteq E$ $|H| = 2^{\aleph_0}$ and for all strictly increasing $f : E \to E$ $(f``E) \setminus H$ is nonempty.*

Consequently the order type of H is strictly less than the order-type of E. (because there is an order-embedding $H \hookrightarrow E$ but not conversely.)

Proof:

Let us suppose the continuum has a wellordering $<_c$ of length ω_α. The significance of this is that every initial segment of this wellordering will be of size *less than* $\aleph_\alpha$. Since $|E| = 2^{\aleph_0}$ the family of increasing functions $E \to E$ is also of size $2^{\aleph_0}$. (This crucial fact depends on E being a uncountable subset of the *reals*—it doesn't work for the rationals for example! The proof is left as an exercise.) This means there is a wellordering $<_f$ of these strictly increasing functions $E \to E$ of order type ω_α. Now we define sequences $\langle p_i : i < \omega_\alpha \rangle$ and $\langle q_i : i < \omega_\alpha \rangle$ as follows.

p_1 is the $<_c$-first real in E.

q_1 is the $<_c$-first thing in $(f_1 \text{“} E) \setminus \{p_1\}$, where f_1 is the $<_f$ first strictly increasing function $E \to E$.

Thereafter for $\beta < \omega_\alpha$ we make the following recursive definition: given $A_\beta = \{p_i : i < \beta\} \cup \{q_i : i < \beta\}$, set

(A) p_β is the $<_c$-first real in $E \setminus A_\beta$. (There is such a thing because $|A_\beta| < \aleph_\alpha = 2^{\aleph_0}$ and $|E| = 2^{\aleph_0}$.)

(B) q_β is the $<_c$-first thing in $(f_\beta \text{“} E) \setminus \{p_i : i \leq \beta\}$, where f_β is the βth strictly increasing function $E \to E$ (in the sense of $<_f$). (There is such a thing because $|f_\beta \text{“} E| = 2^{\aleph_0}$ and $|\{p_i : i < \beta\}| = |\beta| < \aleph_\alpha = 2^{\aleph_0}$).

Now set $H = \{p_\beta : \beta < \omega_\alpha\}$

By construction the p_β are all distinct so $|H| = \aleph_\alpha = 2^{\aleph_0}$ as desired. Each q_β is in the range of f_β, so we want to know that $q_\beta \notin H$. Can q_β be a p_ζ? It cannot be a p_ζ with $\zeta > \beta$ because of (A). It cannot be a p_ζ with $\zeta < \beta$ because of (B). Therefore any strictly increasing $f : E \to E$ takes at least one value outside H. ■

It now requires only a slight modification of this proof to obtain a construction of a PETO. I shall probably update this file to provide it—when I have time.

References

[1] Sierpinski, W Sur les types d'ordres des ensembles linéaires. Fundamenta mathematica **37** (1950) pp 253–264.