COMPUTER SCIENCE TRIPOS Part IA 2013 Paper 2 Q 5(b)

> *Suppose there is a surjection $f : D \twoheadrightarrow (D \to D)$. Show that this happens if and only if D has precisely one element.*

If D has one element then $D \to D$ is the singleton of the identity function $\mathbb{1}_D$ and both D and $D \to D$ are singletons so there is a surjection as desired.

If D is empty then $D \to D$ is the singleton of the empty function. There can be no surjection from the empty set to a nonempty set, so again, we get the result we want.

Now suppose D is a set with at least two members. Let us name two of them 'a' and 'b'. Suppose further that $f : D \to (D \to D)$. We will show that f is not surjective.

The challenge is to cook up a function $\delta : D \to D$ which is not in the range of f. And we have to cook up such a function using only f, a and b. ...

Observe that we *have* to use *both* a and b. After all, we saw above that if f has only one member there *is* a surjection. We should expect a diagonal construction to appear, so tinkering with $f(x)$ applied to x would be a good thing to start with. And of course we have to alter the thing on the diagonal, so something like the following would be worth trying.

Define a function $\delta : D \to D$ by

$$\texttt{if}\ (f(x))(x) = a\ \texttt{then}\ b\ \texttt{else}\ a$$

The chief effect of this definition is that

$$(\forall d \in D)(\delta(d) \neq (f(d))(d)) \tag{1}$$

We now claim that δ is not in the range of f. For suppose δ were $f(d_0)$; we obtain a contradiction by considering $\delta(d_o)$.

$\underline{\delta}(d_o) = \underline{f(d_0)}(d_0)$ (the underlined parts are identical by definition).

But we also have

$\delta(d_o) = (f(d_0))(d_0)$ by (1)

giving us the contradiction we sought.

Observe that we used only the fact that D has two distinct elements. We had not assumed that D was finite. You can try [tho' you shouldn't] to prove this result by induction on the size of D, but that only proves it for D that are finite.

For the *cognoscenti*... we have also used excluded middle on $x = a$, in the definition of δ.