Recall Chang's conjecture (CC): $\mathcal{P}^{\omega_1}(\omega_2) = \{x \subseteq \omega_2 : ot(x) = \omega_1\}$ is stationary.

One method to obtain consistency of CC is to get the consistency of:

There is a generic embedding $j : V \to M \subseteq V^{\mathbb{P}}$ with $cp(j) = \omega_1$, $j(\omega_1) = \omega_2$ and $j''\omega_2 \in M$.

Notice: If there is a Woodin card., then CC iff
$\exists j : V \to M \subseteq V^{\mathbb{P}}$, etc ..., where $\mathbb{P} = \mathbb{P}_{<\kappa}$ below a, $a = \mathcal{P}^{\omega_1}(\omega_2)$.

Remarks: Suppose $G \subseteq \mathbb{P}_{<\kappa}$ is generic and κ is a Woodin card. Let $j : V \to M \subseteq V[G]$ be the induced embedding. Then, $j(\kappa) = \kappa$.
$\kappa = \sup\{\delta : j(\delta) = \delta\}$

Reason: κ is the sup of Ramsey cardinals, hence is the sup of completely Jonsson cards.

Interesting fact: κ is still a Woodin card. in $V[G]$.
Also, stationary subsets of κ are preserved in $V[G]$.

Question:
Suppose κ is Woodin. Suppose $G \subseteq \mathbb{Q}_{<\kappa}$ is V-generic. Let $j : V \to M \subseteq V[G]$ be the induced embedding.
Then, $cp(j) = \omega_1$, $j(\omega_1) = \kappa$, $M^{\omega} = M^{<\kappa} \subseteq M$ in $V[G]$

<u>Application</u>: Suppose κ is Woodin and there exists a measurable card. above κ. Then, $\underset{\sim}{\Sigma}^1_3$ sets are Lebesgue measurable.

Pf: Let $G \subseteq \mathbb{Q}_{<\kappa}$ be V-generic. Let $j: V \to M \subseteq V[G]$ be the induced embedding.

It suffices to show that in $V[G]$, the Σ^1_3 sets defined with parameters from V are Lebesgue measurable. (since $\mathbb{R}^M = \mathbb{R}^{V[G]}$ and $\mathbb{R}^V \prec \mathbb{R}^M$)

So, fix a Σ^1_3 formula $\varphi(x,y)$. Fix $r \in V$.

Let $A^{V[G]} = \{s : V[G] \models \varphi[r,s]\}$

In $V[G]$, let $R = \{s : s$ is random over $V\}$

(work in V): In $V[G]$, R is measurable and has measure one.

But for $s \in R$, $V[s] \models \varphi[r,s]$ iff $V[G] \models \varphi[r,s]$

(this is because $V[G] = V[s][G/s]$, where G/s is $V[s]$-generic for $\mathbb{Q}_{<\kappa}/s$ which has card. $\kappa < \delta$ in $V[s]$. But then, Σ^1_3 statements are absolute between $V[s]$ and $V[G]$ (by Martin-Solovay)

<u>Martin-Solovay thm</u>: Suppose δ is measurable and $\mathbb{P}$ is a poset of card. $<\delta$. Suppose $G \subseteq \mathbb{P}$ is V-generic. Then V is $\underset{\sim}{\Sigma}^1_3$-$\mathbb{P}$-absolute. (actually use only #'s).

This argument generalizes:

<u>Thm</u> (Shelah-Woodin): If there are n Woodin cardinals with a measurable above, then $\underset{\sim}{\Sigma}^1_{n+2}$ sets of

reals are Lebesgue measurable.

Next goal: Theorem: Suppose there are infinitely many Woodin cards with a measurable above them all. Then $L(\mathbb{R}) \models$ "all sets are Lebesgue measurable".

Main goal: Theorem: $\text{con}(ZFC + \exists^\infty \text{ Woodin cards}) \Rightarrow \text{con}(ZF + DC + AD)$

Theorem: The following are equiconsistent:

① $ZFC + \exists^\infty$ Woodin cards, with a meas. above

② $ZFC + DC + AD +$ There is a meas. card. above Θ

(Recall: $\Theta = \sup\{\alpha : \exists \pi : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}$)

---

② is much stronger than $ZFC + DC + AD$:

$AD + \exists$ meas. above

$\vdots$

$AD + \mathbb{R}^{\#\#}$ exists

$AD + \mathbb{R}^{\#}$ exists

$AD + V = L(\mathbb{R})$

First thm. we will prove:

Theorem: Suppose κ is Woodin and $T \subseteq (\omega \times \lambda)^{<\omega}$ is a tree. Then, $\exists \delta < \kappa$ s.t. if $g \subseteq \text{Coll}(\omega, \delta)$ is V-generic, then in $V[g]$, T is $<\kappa$ weakly-homogeneous.

Theorem: Suppose κ is Woodin. Suppose S, T are trees contained in $(\omega\times\lambda)^{<\omega}$ and that $p[S]=\omega^\omega \setminus p[T]$ in $V[g]$, where $g \subseteq \mathrm{Coll}(\omega,\kappa)$ is generic.

Then, S, T are $<\kappa$ weakly-homogeneous.

Martin-Steel Thm: Suppose κ is Woodin and $T \subseteq (\omega\times\lambda)^{<\omega}$ is a κ^+-weakly homogeneous tree. Fix a witness for the κ^+-weak-homogeneity (i.e., a countable set of measures). Let S be the tree for the complement obtained from the witness. Then S is $<\kappa$ homogeneous.

How to produce trees: (projective case)
Suppose κ is Woodin and $\underset{\sim}{\Sigma}^1_n$ truth is absolute for forcing extensions by partial orders $\mathbb{P}$ with $|\mathbb{P}|<\kappa$. Then, there are trees $S, T \subseteq (\omega\times\kappa)^{<\omega}$ s.t.:

① $p[S]=\omega^\omega \setminus p[T]$ in $V[g]$, for all g generic over V for partial orders of card. $<\kappa$

② $p[S]$ = "complete" $\underset{\sim}{\Sigma}^1_n$ set in any such $V[g]$.

Corollary: Suppose λ is a limit of Woodin cards. Then, every projective set is $<\lambda$ weakly-homo-Suslin.

Thm: Suppose there exist n Woodin cards. with a measurable above. Then, $\underset{\sim}{\Sigma}^1_{n+2}$ sets are weakly-homo-Suslin.

Thm: Assume λ is a limit of Woodin cards and there exists a measurable card above λ. Then, every set $A \subseteq R$, $A \in L(R)$ is $<\lambda$ weakly-homo-Suslin

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<u>Theorem</u>: Suppose κ is Woodin and $T \subseteq (\omega\times\lambda)^{<\omega}$ is a tree. Then, there exists $\delta<\kappa$ s.t. if g is generic for $\mathrm{Coll}(\omega,\delta)$, then T is $<\kappa$ weakly-homogeneous in $V[g]$.

(<u>Note</u>: for $\lambda<\kappa$, trivial.)
since T is countable in $V[g]$.

<u>Pf</u>: We can assume, WLOG, that $\lambda=\kappa$. (Why?: Choose a subtree $S \subseteq T$, $|S|\leq\kappa$, with $p[S]=p[T]$ in $V[G]$, where $G \subseteq \mathrm{Coll}(\omega,<\kappa)$ is V-generic. (use the κ-cc of $\mathrm{Coll}(\omega,<\kappa)$)). Suppose the theorem holds for S. We get $\delta_S<\kappa$ s.t. S is $<\kappa$ weakly-homo. in $V[g_S]$, where $g_S \subseteq \mathrm{Coll}(\omega,\delta_S)$ is V-generic. But $S\subseteq T$ and $p[S]=p[T]$ in $V[g_S]$. So, it follows that T is $<\kappa$ weakly-homogeneous in $V[g_S]$ (use the same measures).

(view $S \subseteq (\omega\times\kappa)^{<\omega}$)

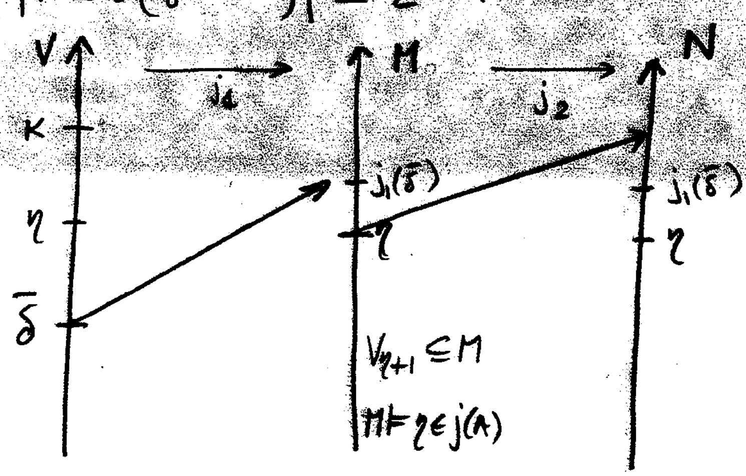
So, let $A=\{\delta<\kappa : \delta \text{ is } <\kappa\ V_\kappa\times T\text{-strong}\}$

Choose $\bar\delta$ s.t. $\bar\delta$ is $<\kappa$ $A\times V_\kappa\times T$-strong.

(Since κ is Woodin, for every $H\subseteq V_\kappa$ there is $\delta_H<\kappa$ which is $<\kappa$ H-strong)

$\delta = |\mathrm{meas}(\bar\delta^{<\omega})| \leq 2^{2^{\bar\delta}}$.

$\eta\in A$



<u>Assumptions</u>:

$j_1\in V$, $j_2\in M$

where:

1. $\eta\in A$
2. $j_1(T)\restriction\eta = T\restriction\eta$
3. $V_{\eta+1}\subseteq M$, $\eta\in j_1(A)$
4. $\mathrm{cp}(j_2)=\eta$

⑤ $M_{j_1(\bar\delta)+\omega} \subseteq N$

⑥ $j_2(j_1(T)) \upharpoonright j_1(\bar\delta) = j_1(T) \upharpoonright j_1(\bar\delta)$

⑦ $p[T]^{V^{Coll(\omega, 2^{2^{\bar\delta}})}} = p[T \upharpoonright \eta]^{V^{Coll(\omega, 2^{2^{\bar\delta}})}}$

WMA, $j_1(\kappa) = j_2(\kappa) = \kappa$.

Suppose $(s,t) \in T \upharpoonright \eta$. Let $\mu_{(s,t)}$ be the measure:

$\mu_{(s,t)}(X) = 1$ iff $t \in j_1(X)$.

Thus, $\mu_{(s,t)}(T_s) = 1$ (by ② above)

Let $\bar\sigma = \{\mu_{(s,t)} : (s,t) \in T \upharpoonright \eta\}$

So, $\bar\sigma \subseteq meas(\bar\delta^{<\omega})$, $|\bar\sigma| \leq 2^{2^{\bar\delta}}$.

For $\mu \in \bar\sigma$, define ν_μ as follows:

$\nu_\mu \in meas(\eta^{<\omega})$

Note that, by ⑤, $j_1(\mu) \in N$, $(j_1(\mu) \in M_{j_1(\bar\delta)+2})$

and $M_{j_1(\bar\delta)+\omega} \subseteq N$.

Define ν_μ in M by: $\nu_\mu(X) = 1$ iff $j_2(X) \cap j_1(\bar\delta)^{<\omega} \in j_1(\mu)$

So, $\nu_\mu(\eta^{<\omega}) = 1$.

$(j_2(\eta) > j_1(\bar\delta)$. So, $j_2(\eta^{<\omega}) \cap j_1(\bar\delta)^{<\omega} = j_1(\bar\delta)^{<\omega})$

But $V_{\eta+1} \subseteq M$. So, $\nu_\mu \in meas(\eta^{<\omega})$ in V.

Let $\sigma = \{\nu_\mu : \mu \in \bar\sigma\}$. Thus, $|\sigma| \le 2^{2^{\bar\delta}}$.

Notice that if $(s,t) \in T \restriction \eta$, then $\nu_{\mu_{(s,t)}}(T_s) = 1$.

This follows from 6 :

$\mu_{(s,t)}(T_s) = 1$. So, $j_1(\mu_{(s,t)})(j_1(T_s)) = 1$

But $j_1(T)\restriction\eta = j_2(j_1(T))\restriction\eta$. So, $\nu_{\mu_{(s,t)}}(j_1(T_s)\restriction\eta) = 1$.

(Why?: $\nu_{\mu_{(s,t)}}(j_1(T)\restriction\eta) = 1$ iff $j_2(j_1(T)\restriction\eta) \cap j_1(\bar\delta)^{<\omega} \in j_1(\mu_{(s,t)}$

But $j_2(j_1(T_s)\restriction\eta) \cap j_1(\bar\delta)^{<\omega} = j_2(j_1(T_s))\restriction j_1(\bar\delta)$)

(What's going on: we start with a measure on $\bar\delta^{<\omega}$, we move it by j_1 to a meas. on $j_1(\bar\delta)^{<\omega}$, which is also a measure in N. Now, since $j_2(\eta) > j_1(\bar\delta)$, we pull this measure back using j_2 to a measure on $\eta^{<\omega}$)

Further, each measure in σ is η-complete. (Use same argument)

Claim: σ witnesses the η-weak-homogeneity of T in $V^{Coll(\omega,\, 2^{2^{\bar\delta}})}$

Let $g \subseteq Coll(\omega, 2^{2^{\bar\delta}})$ be generic over V.

Suppose $x \in p[T]^{V[g]}$.

Each $\nu \in \sigma$ is η-complete. Also, $2^{2^{\bar\delta}} < \eta$. So, each $\nu \in \sigma$ canonically defines an η-complete measure in $V[g]$.

Call it ν.

$x \in p[T]^{V[g]}$. So, $x \in p[T\restriction\eta]$.

Choose $y \in \eta^\omega$ with $(x,y) \in [T\restriction\eta]$.

For each $s<\omega$, let $\nu_s = \nu_{\mu(x\restriction s,\, y\restriction s)}$.

Thus, $\nu_s(T_{x\restriction s}) = 1$, all $s<\omega$.

We must verify:

① $\langle \nu_s : s<\omega\rangle$ is a tower.

② $\langle \nu_s : s<\omega\rangle$ is a countably complete tower.

①: let $\mu_s = \mu_{(x\restriction s,\, y\restriction s)}$. Let's verify $\langle \mu_s : s<\omega\rangle$ is a tower.
(Caution: the sequence $\langle \mu_s : s<\omega\rangle$ is not in V)

We must verify: if $s_1 < s_2$, then μ_{s_2} projects to μ_{s_1}.

But $y\restriction s_1$ is an initial segment of $y\restriction s_2$, and

$\mu_{s_1}(X) = 1$ iff $y\restriction s_1 \in j_1(X)$

$\mu_{s_2}(X) = 1$ iff $y\restriction s_2 \in j_1(X)$

Thus, $\langle j_1(\mu_s) : s<\omega\rangle$ $(\in M[g])$ is a tower.

Since the pullback preserves towers, it follows that $\langle \nu_s : s<\omega\rangle$ is a tower in $V[g]$.

②: To show that the tower is complete, it suffices to show that if $\vec{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_s, \ldots : s<\omega\rangle \in (\eta^\omega)^{V[g]}$, then the associated tower $\langle \nu_{\vec{\alpha}_0}, \nu_{\vec{\alpha}_1}, \ldots, \nu_{\vec{\alpha}_s}, \ldots : s<\omega\rangle$ is countable complete, where $\nu_{\vec{\alpha}_s}$ is obtained as above from $\mu_{\vec{\alpha}_s}$ and $\mu_{\vec{\alpha}_s}(X) = 1$ iff $\langle \alpha_k : k<s\rangle \in j_1(X)$.

(i) for each $\nu \in \mathrm{meas}(\eta^{<\omega})$ with ν η-complete,

$$\left(\mathrm{OR}^{\eta^{<\omega}}/\nu\right)^{V[g]} \sim \left(\mathrm{OR}^{\eta^{<\omega}}/\nu\right)^{V} \quad i.e.,$$

i.e., if $F: \eta^{<\omega} \to OR$, $F \in V[g]$, then $\exists H : \eta^{<\omega} \to OR$, $H \in V$ with $F \sim_V H$.

Thus, for each $s_1 < s_2$, $j_{s_1,s_2} \restriction OR$ is the same computed in V or $V[g]$, where $j_{s_1,s_2} : V^{\eta^{<\omega}}/\nu_{s_1} \longrightarrow V^{\eta^{<\omega}}/\nu_{s_2}$.

So, the tower $\langle \nu_s : s < \omega \rangle$ is not countably complete iff there is a sequence $\langle \beta_s : s < \omega \rangle \in V[g]$ with

$j^V_{s_1,s_2}(\beta_{s_1}) > \beta_{s_2}$, all $s_1 < s_2$.

Let $T^* = \langle ((\alpha_0, \beta_0), \ldots, (\alpha_s, \beta_s)) : s < \omega,\ \alpha_0, \ldots, \alpha_s < \eta$ and $j^V_{s_1,s_2}(\beta_{s_1}) > \beta_{s_2}$, where ν_{s_1}, ν_{s_2} are defined from $\langle \alpha_k : k < s_1 \rangle$, $\langle \alpha_k : k < s_2 \rangle \}$

$T^* \in V$.
If things go bad, T^* has an infinite branch in $V[g]$.
But then, T^* has an infinite branch in V.
So, WMA $\langle \alpha_s : s < \omega \rangle \in V$. $(\rightarrow\leftarrow)$.

(the point is that using small forcing extensions (i.e., $< \eta$), the ultrapowers do not change, so countable completeness is preserved.)

(General fact: if μ is κ-complete and F is a function on ordinals and we force with $\mathbb{P}$, $|\mathbb{P}| < \kappa$, then $F^{V^{\mathbb{P}}} \cap V \sim F$ (mod μ)

Theorem: Suppose κ is Woodin and S, T are trees on $(\omega \times \lambda)^{<\omega}$ with $p[S] = \omega^\omega \setminus p[T]$ in $V^{Col(\omega, \kappa)}$

Then, S, T are $<\kappa$ weakly-homo.

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Thm: Suppose κ is a Woodin card. and T, T^* are trees with $p[T] = \omega^\omega \setminus p[T^*]$ in $V^{Coll(\omega, \kappa)}$. Then, T, T^* are $<\kappa$ weakly-homo.

Pf: Claim: $\exists S \subseteq T$, $S^* \subseteq T^*$, $|S|, |S^*| \leq \kappa$ such that if $G \subseteq \mathbb{Q}_{<\kappa}$ is V-generic, then $p[S] = p[T]$, $p[S^*] = p[T^*]$ in $V[G]$.

Pf. of Claim: There is a set of terms $\mathcal{J}$ for reals in $V^{\mathbb{Q}_{<\kappa}}$ which has card. κ and is complete (i.e., if $p \in \mathbb{Q}_{<\kappa}$, $\tau \in V^{\mathbb{Q}_{<\kappa}}$, $p \Vdash \tau \in \mathbb{R}$, then $\exists q \leq p$, $\exists \tau^* \in \mathcal{J}$ s.t. $q \Vdash \tau = \tau^*$)

$\{f : \cup a \to \mathbb{R} : a \in \mathbb{Q}_{<\kappa}\}$ generate all reals in the generic ultrapower, so these generate all reals of $V[G]$

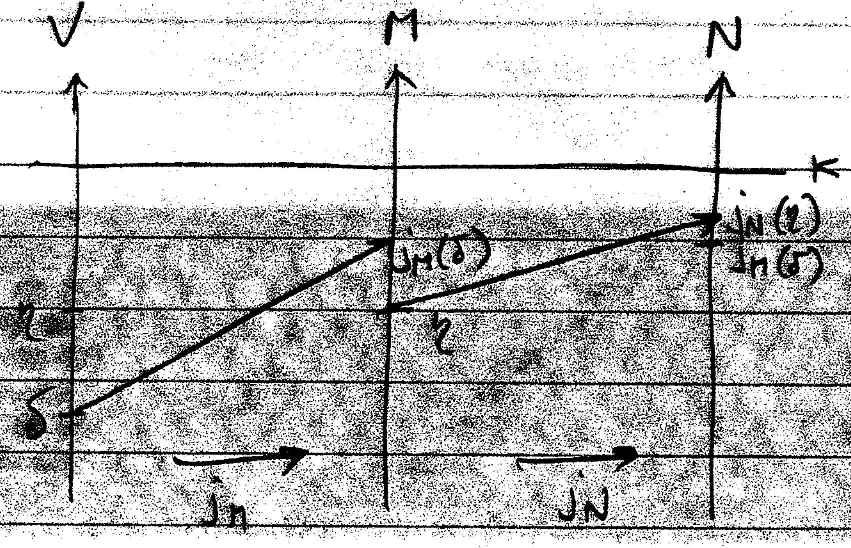
So, let $\mathcal{J} = \{\tau_f : \tau_f$ is the term for the real f gives$\}$

For each $\tau \in \mathcal{J}$, let τ^* be a term for the branch in T or T^* given by τ. This gives S, S^*. □

We can assume $T, T^* \subseteq (\omega \times \kappa)^{<\omega}$, where we now only assume $p[T] = \omega^\omega \setminus p[T^*]$ in $V^{\mathbb{Q}_{<\kappa}}$.

Repeat the proof of the previous theorem applied to T to get $\delta < \kappa$ a cofinal set $H \subseteq \kappa$ s.t. if $\eta \in H$, then there exits $\rho : T \upharpoonright \eta \longrightarrow \text{meas}(\eta^{<\omega})$

$V_{\eta+1} \subseteq M \qquad M_{j_M(\delta)+2} \subseteq N$



⓪ $\rho((s,t))$ is η-complete

such that: ① $\rho((s,t))(T_s) = 1$

② if $s \subseteq \bar{s}$, $t \subseteq \bar{t}$
then $\rho((\bar{s},\bar{t}))$ projects to $\rho((s,t))$

③ if $(x,f) \in [T \restriction \eta]$, then the tower $\langle \rho(x\restriction n, f\restriction n) : n<\omega \rangle$ is countably complete.

④ $|\text{range}(\rho)| \leq 2^{2^\delta}$

$\rho(s,t) = \nu_M(t)$, $t \in T_s$ (using last's time notation)

last time we showed ③ is absolute for small ($<\eta$) size forcing. We will not need this now.

Let $\sigma = \text{range}\ \rho$, $|\sigma| \leq 2^{2^\delta}$.

<u>Lemma</u>: There exists $\bar{\sigma} \subseteq \sigma$ such that $\bar{\sigma}$ is countable and $\bar{\sigma}$ witnes the η-weak-hom. of T.

Pf: let $G \subseteq \mathbb{Q}_{<\kappa}$ be generic and let
$j: V \to M \subseteq V[G]$ be the associated embedding.
So, $cp(j) = \omega_1$, $j(\omega_2) = \kappa$ and $M^\omega \subseteq M$ in $V[G]$.
We will show that the pointwise image of σ (which is countable), works in M.
since $|\sigma| < \kappa$, and $M^\omega \subseteq M$

Given $x \in M$, $x \in p[j(T)]^{V[G]} = p[T]^{V[G]}$, we must find a tower from $j''\sigma$, $\langle \nu_n : n<\omega \rangle$ with:
① $\nu_n(j(T)_{x \restriction n}) = 1$
② $\langle \nu_n : n<\omega \rangle$ is countably complete.

Problem: $x \in p[T \restriction \eta]$? (Not necessarily, so we need to start again)
Back to the beginning: use $\mathbb{P}_{<\kappa}$ instead of $\mathbb{Q}_{<\kappa}$.
Assume $p[T] = \omega^\omega \setminus p[T^*]$ in $V^{\mathbb{P}_{<\kappa}}$,
T, T^* are $(\omega \times \kappa)^{<\omega}$.
Get δ and cofinal subset $H \subseteq \kappa$ as before. Fix $\alpha < \kappa$.
We will show T is α-weakly homo.
For all $\eta \in H$, there exists $f_\eta : T \restriction \eta \to \text{meas}(\eta^{<\omega})$ satisfying ⓪ to ④.
Let $a_0 = \mathcal{P}_{\omega_1}(2^{2^\delta})$. Let $G \subseteq \mathbb{P}_{<\kappa}$ be generic with $a_0 \in G$. Let $j: V \to M \subseteq V[G]$ be the associated embedding.

Thus, ① $cp(j)=\omega_1$

② $(2^{\delta})^V$ is countable in $V[G]$ (since $a_0 \in G$, $j''V\alpha \in j(\alpha$

③ $j(\kappa)=\kappa$ (using ① $<\kappa$, we get $j(\omega_1)=\kappa$, which is not enough)

④ $p[T]^{V[G]} = p[j(T)]^{V[G]}$

[Pf. of ④: $p[T]^{V[G]} = \omega^\omega \setminus p[T^*]^{V[G]}$

$$p[T]^{V[G]} \subseteq p[j(T)]^{V[G]}$$

$$p[T^*]^{V[G]} \subseteq p[j(T^*)]^{V[G]}$$

$$V \models p[T] \cap p[T^*] = \emptyset$$

$$M \models p[j(T)] \cap p[j(T^*)] = \emptyset$$

So, $p[j(T)]^{V[G]} \cap p[j(T^*)]^{V[G]} = \emptyset$.

$\cancel{V \models \exists_{\beta<\kappa} \left[p[T \upharpoonright \gamma] = p[T] \right]}$ all $\gamma > \beta$

$\cancel{\text{So, } M \models \exists_{\beta < j(\kappa)} \left(p[j(T) \upharpoonright \gamma] = p[j(T)] \right)}$ all $\gamma > \beta$; $j(\kappa) = \kappa$

$\exists_\eta \in H$, $\eta > \alpha$ s.t. $p[T \upharpoonright \eta]^{V[G]} = p[j(T)]^{V[G]}$

($p[T] = p[j(T)]$ in $V[G]$. But $V[G] \models \kappa$ is strongly inaccessible.)

Choose $a \in \mathbb{P}_{<\kappa}$ s.t. $a \le a_0$, $a \Vdash p[T\restriction\eta]^{V[G]} = p[j(T)]^{V[G]}$.

<u>Lemma</u>: $a \Vdash j''\sigma$ witnesses the $j(\eta)$-weak-homo of $j(T)$ in M.

Pf: let $G \subseteq \mathbb{P}_{<\kappa}$ be V-generic, $a \in G$, let $j: V \to M \subseteq V[G]$ be the assoc. emb.
Fix $x \in p[j(T)]^M$. Thus, $x \in p[T\restriction\eta]$.
So, choose $f \in \eta^\omega$ with $(x, f) \in [T\restriction\eta]$.
let $\mu_n = \rho(x\restriction n, f\restriction n)$.
let $\nu_n = j(\mu_n)$.
Since $M^{<\kappa} \subseteq M$ in $V[G]$, $\langle \nu_n : n<\omega\rangle \in M$.
But $\nu_n = j(\rho)(j(x\restriction n), j(f\restriction n))$

$$= j(\rho)(x\restriction n, j(f\restriction n))$$

let $\bar f \in j(\eta)^\omega$ s.t. $\bar f(i) = j(f(i))$.
$\bar f \in V[G]$, so $\bar f \in M$.
$(x, f) \in [T\restriction\eta]$, so $(x, \bar f) \in [j(T)\restriction j(\eta)]$.
and $\nu_n = j(\rho)(x\restriction n, \bar f\restriction n)$.
By j applied to properties (6) – (2), $\langle \nu_n : n<\omega\rangle$ is a countably complete tower in M, $\nu_n(j(T)_{x\restriction n}) = 1$.
But $\nu_n \in j''\sigma$, $\nu_n = j(\mu_n)$. $\square$

How to generate trees?

Warmup for tree generation:

<u>Lemma</u>: Assume κ is Woodin and #'s exist (actually, only need $V_{\kappa+1}^{\#}$ exist). Then, there are trees T, T^* such that $p[T]$ = complete $\underset{\sim}{\Sigma}^1_3$ set, $p[T^*]$ = complete $\underset{\sim}{\Pi}^1_3$ set, in $V^{\mathbb{P}}$, for all $\mathbb{P}$ with $|\mathbb{P}| < \kappa$.

Pf: By Martin-Solovay, $\underset{\sim}{\Sigma}^1_3$ statements are absolute for forcing extensions by posets of size $<\kappa$. i.e., if $G_1 \subseteq \mathbb{P}_1$, $G_2 \subseteq \mathbb{P}_2$, $|\mathbb{P}_1| < \kappa$ in V, G_1 is V-generic, G_2 is $V[G_1]$-generic and $\mathbb{P}_2 \in V[G_1]$, $|\mathbb{P}_2| \leq |\kappa|$ in $V[G_1]$, then $\mathbb{R}^{V[G_1]} \prec_{\underset{\sim}{\Sigma}^1_3} \mathbb{R}^{V[G_2]}$

Fix $\gamma < \kappa$, γ strongly inaccessible.

Key Claim: $S = \{X \prec V_\gamma : |X| \leq \omega$ and $\mathbb{R}^{M[g]} \prec_{\underset{\sim}{\Sigma}^1_3} \mathbb{R}$ for all g generic (for some $\mathbb{P} \in M$) over M, where $M = \mathrm{Coll}(X)\}$ is club in $\mathcal{P}_{\omega_1}(V_\gamma)$.

Pf: If not, $a = \mathcal{P}_{\omega_1}(V_\gamma) \setminus S$ is stationary in $\mathcal{P}_{\omega_1}(V_\gamma)$.

let $G \subseteq \mathbb{P}_{<\kappa}$ be V-generic, $a \in G$,

let $j : V \to M \subseteq V[G]$ be the ass. emb.

$a \in G$, so $j''(\cup a) \in j(a)$, $\cup a = V_\gamma$.

So, $j''V_\gamma \in j(a)$

Trans. Coll. $(j''V_\gamma) = V_\gamma$.

So, in M, there is g which is generic over V_γ for some $\mathbb{P} \in V_\gamma$ such that

$$\mathbb{R}^{V_\gamma[g]} \not\prec_{\Sigma^1_3} \mathbb{R}^M$$

But $\mathbb{R}^M = \mathbb{R}^{V[G]}$

$$\mathbb{R}^{V_\gamma[g]} = \mathbb{R}^{V[g]}$$

and g is generic over V.

$V[g] \subseteq V[G]$, so $\mathbb{R}^{V[g]} \not\prec_{\Sigma^1_3} \mathbb{R}^{V[G]}$. $(\rightarrow\leftarrow)$.

Define $T_\gamma, T^*_\gamma \subseteq (\omega\times\gamma)^{<\omega}$ which project to complete Σ^1_3, Π^1_3 sets in $V^{\mathbb{P}}$, for all $\mathbb{P}$ in V_γ.

Fix $\varphi(x,y)$ a Σ^1_3-formula which canonically defines a complete Σ^1_3 set. (We can do the same thing for any (not necessarily complete) Σ^1_3 set.)

Fix $H: V_\gamma^{<\omega} \to V_\gamma$ s.t. if $X \in \mathcal{P}_{\omega_1}(V_\gamma)$ and $H''X^{<\omega} \subset X$, then $X \prec V_\gamma$, $X \in S$.

Friday, October 25, 1990

Let's finish seeing why that tree worked:

Our situation: γ strongly inaccessible,

$$S = \{ X \prec V_\gamma \mid |X| = \omega, \quad \mathbb{R}^{M[g]} \prec_{\Sigma^1_3} \mathbb{R}$$

$$\text{for all } g \text{ generic}/M, \quad M = \text{Coll}(X) \}$$

We showed under our hypothesis that S contains a cub set in $\mathbb{P}_{\omega_1}(V_\gamma)$

So we chose $H : V_\gamma^{<\omega} \to V_\gamma$ so that if

$$X \in P_{\omega_1}(V_\gamma), \quad H''X^{<\omega} \subset X, \text{ then } X \in S$$

We used H to define a tree T_γ:

view $T \subseteq (\omega \times \omega \times \omega \times V_\gamma)^{<\omega}$ so that

$[T] = \{ (x, y, z, f) :$ y codes a pair of models $\langle \omega, E_1 \rangle = M_y$, $N_y = \langle \omega, E_2 \rangle$

N_y, M_y are ω-models of ZFC, $x \in M_y$, N_y is a generic extension of M_y, and f embedds elementarily $N_y \models \varphi(x)$,

$$f : (M_y, E_1) \hookrightarrow (V_\gamma, \in)$$

with range closed under $H \}$

Up to here this is a Σ^1_1 fact, so

Z witnesses all of this.

To see T works, using stationary tower forcing works but is an overkill.

$$p[T_\gamma] = A \subseteq \omega^\omega, \quad A \text{ is the } \Sigma^1_3 \text{ set defined by } \varphi.$$

Need to show in extension this works. Let $\mathbb{P} \subset V_\gamma$, $g \subset \mathbb{P}$, g is generic. We must check

$$p[T_\gamma]^{V[g]} = A^{V[g]}$$

If not, fix $\lambda \gg \gamma$, $Z \prec V_\lambda$, $\mathbb{P} \in Z$, $H \in Z$, $T_\gamma \in Z$, $\gamma \in Z$, Z countable. Let $M_Z = \mathrm{coll}(Z)$

$\mathbb{P}_Z$ = image of $\mathbb{P}$ under collapse

T_Z = " " T " " "

So $M_Z \models$ "forcing with $\mathbb{P}_Z$, the $p[T_Z]$ in extension isn't A",

i.e.

$$p[T_Z]^{M_Z[G]} \neq A^{M_Z[G]}$$

Let $N_Z = \mathrm{coll}(Z \cap V_\delta)$

$X = Z \cap V_\delta$. $X \in S$, since $H \in Z$, so $Z \cap V_\gamma$ is closed under H.

Also, $\mathbb{R}^{M_z[g]} = \mathbb{R}^{N_z[g]}$, sine M_z is a rank initial segment of N_z containing porct $\mathbb{P}_z$, so all terms for generic reals appear in M_z.

So $\cancel{M_z[G] \prec_{\Sigma^1_3} \mathbb{R}}$

Since $x \in S$, get

$$\mathbb{R}^{M_z[g]} = \mathbb{R}^{N_z[g]} \prec_{\Sigma^1_3} \mathbb{R}^V \qquad (g \in V, g \text{ generic}/M_z)$$

So if $x \in p[T_z]^{M_z(g)}$, then since

$$p[T_z]^{M_z(g)} \subseteq p[T_\delta]^V$$

so $x \in p[T_\delta)^V = A$, so

$$p[T_z)^{M_z[g]} \subseteq A$$

and since $\mathbb{R}^{M_z[g]} \prec_{\Sigma^1_3} \mathbb{R}^V$, get

$$p[T_z)^{M_z[g]} \subseteq A^{M_z[g]}$$

So we proved $p[T_\delta)^{V[g]} \subseteq A^{V[g]}$

To get other inclusion, define S_γ for $\omega^\omega \setminus A$ in exactly the same fashion, and same argument shows

$$p[S_\gamma]^{V[g]} \subseteq ({}^\omega\omega \setminus A)^{V[g]}$$

and $p[S_\gamma]^V$, $p[T_\gamma]^V$ are complement, it follows that

$$p[S_\gamma]^{V[g]}, \quad p[T_\gamma]^{V[g]} \text{ are disjoint.}$$

... ABORT. Doesn't work. ~~Rease~~

Woodin gives up — Do stationary Forcing

An application for fun:

Thm Suppose there is a proper class of completely Jonsson cardinals. Then there is a class tree

$T \subset (\omega \times \mathrm{Ord})^{<\omega}$ s.t.

$$p[T]^{V[G]} = \text{complete } \Sigma^1_3 \text{ set for any generic } G/V.$$

(Martin & ~~Steel~~ Solovay prove this assuming all sharps exist, which is a weaker hypothesis than ours)

Rmk This proof is illustrative of what we'll do in general.

Proof

1) Σ^1_3-absoluteness, Suppose G is generic $/V$.

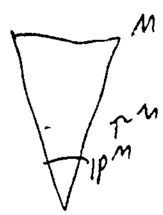
Then $\mathbb{R}^V \prec_{\Sigma^1_3} \mathbb{R}^{V[g]}$

Proof/: By Shoenfield, ~~all that could happen is that~~ suffices to show that if g is generic/V, $x \in V$, and if a Σ^1_3-statement holds about x in $V[G]$ then it holds in V

(This follows immediately from completely Jonsson: Fix γ completely Jonsson, $\mathbb{P} \in V_\gamma$. Choose a term $\tau \in V_\gamma^{\mathbb{P}}$, condition $p \in \mathbb{P}$,

$$p \Vdash \Pi^1_2(x, \tau) \qquad (\tau \text{ is witness of } \Sigma^1_3\text{-stmt})$$

Choose $\bar{X} \prec V_\gamma$, $|\bar{X}| = \gamma$, $|\bar{X} \cap \lambda| = \omega$, $\lambda = 2^{|\mathbb{P}|}$, $x, \mathbb{P}, \tau \in X$.
Let $M = \mathrm{coll}(\bar{X})$



$\mathbb{P}$, $2^{\mathbb{P}}$ are countable in V, so can build

$M[g]$ in V

But $M[g]$ has uncountable height, so we have absoluteness:

$M[g] \vDash \Pi^1_2(x, \mathrm{val}_g(\tau))$

so $V \vDash \Pi^1_2(x, \mathrm{val}_g(\tau))$)

Our other proof:

Fix $\lambda \gg |\mathbb{P}|$. Let $q_0 = P_{\omega_1}(V_\lambda)$. Let G be ~~λ completely Jonsson~~ V-generic for $P^{<\infty}$ with $q_0 \in G$

We get
$$j : V \to V[G]$$ since we have a proper class of completely Jonsson cardinals.

V_λ is countable in $V[G]$ (or $a_0 \in G$), so $\exists g \in V[G]$, g is V-generic for $\mathbb{P}$. Have

$$\begin{tikzcd} V \arrow[r, phantom, "\subset"] \arrow[rr, bend right, "j"'] & V[g] \arrow[r, phantom, "\subset"] & V[G] \end{tikzcd}$$

Σ^1_3-statement holds in $V[g]$, hence in $V[G]$ by upward absoluteness, so by elementarity $j : V \hookrightarrow V[G]$, we have Σ^1_3-fact holds <u>in</u> V.

So proved claim, that we have Σ^1_2-absoluteness

Fix a strongly inaccessible γ. Build T_γ for Σ^1_2 as before.

To ~~see why can't have tree on Π^1_3~~, build S_γ for Π^1_2 (~~to see what goes wrong~~)

works as before

<u>Claim</u> If $\mathbb{P} \in V_\gamma$, $g \subset \mathbb{P}$ is V-generic, then
$$p[T_\gamma]^{V[g]} = (\Sigma^1_3)^{V[g]}$$

ignore

(However, will not in general have that
$$p[S_\gamma]^{V[g]} = (\Pi^1_3)^{V[g]} \quad)$$

ABORT

~~Pf of claim~~:

Tree Production

Thm Suppose κ is a Woodin cardinal, $a \in V$, and $\varphi(x,y)$ is a formula s.t.:

~~i) $V \prec_{\varphi,a} V[G]$ if $G \subseteq \mathbb{P}$ is V-generic~~

① Suppose $V[G_1] \subset V[G_1][G_2]$, G_1 is generic for $\mathbb{P}_1$, $\text{card}(\mathbb{P}_1) \leq \kappa$, G_2 is generic for $\mathbb{P}_2$ $|\mathbb{P}_2|^{V[G_1]} \leq |\kappa|^{V[G_1]}$. Then for all $b \in \mathbb{R}^{V[G_1]}$,

could assume only that $V[G_1][G_2] = V^{\mathbb{P}_{<\kappa}}$

$$V[G_1] \models \varphi(b,a) \text{ iff}$$

$$V[G_2] \models \varphi(b,a)$$

② Suppose $G \subseteq \mathbb{P}_{<\kappa}$ is V-generic and let $j: V \to M \subset V[G]$ be the emb.

~~then~~ Then

$$M \models \varphi(b, j(a))$$

iff $V[G] \models \varphi(b,a)$ for all

$$b \in \mathbb{R}^M = \mathbb{R}^{V[G]}$$

(Notice that is $a \in \mathbb{R}$, then $j(a)=a$ and (2) follows (1). Also, if ~~was~~ φ defines a subset of $L(\mathbb{R})$ (2) is trivial: $(L(\mathbb{R}))^M = (L(\mathbb{R}))^{V[G]}$)

With all these assumptions, conclusion of theorem is:

There are trees $T, S \subseteq (\omega \times \kappa)^{<\omega}$ such that

$$p[T]^{V[g]} = \{ b : V[g] \models \varphi(b, a) \}$$

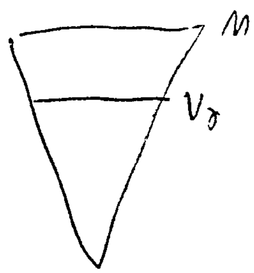
$$p[S]^{V[g]} = \omega^\omega \setminus p[T]^{V[g]}$$

where g is V-generic for $\mathbb{P} \in V_\kappa$.

Proof It's the same proof that we did for Σ^1_3. Fix $\lambda \gg \kappa$, $a \in V_\lambda$; $V_\lambda \preceq_{\Sigma_n} V$, n large ($n \gg$ complexity of φ). Fix γ strongly inaccessible, $\gamma < \kappa$. To take care of troublesome parameter a, choose

$$\bar{X} \prec V_\lambda, \quad \gamma, a, \kappa \in \bar{X}, \quad V_\gamma \subset \bar{X}, \quad |\bar{X}| = \gamma$$

Let $M = \mathrm{Coll}(\bar{X})$. M is right about Σ_n-fmlas.



Will ~~are At~~, replace V by M,

$M \in V_\kappa$

Let a_M = image of a under collapse.

So if $g \subset \mathbb{P}$ is V-generic, $\mathbb{P} \in V_\gamma$, then

$V[g] \models \varphi[b, a]$ iff

$M[g] \models \varphi[b, a_M]$.

Let

$S = \{ Z \prec M$ | 1) $|Z| = \omega$

2) $\gamma, a_M \in Z$

~~3) If g is M-generic~~ 3) Let $N = \text{coll}(Z)$.

Let a_N = image of a_M under collapse. Then if

$g \subset \mathbb{P}$ is N-generic, $\mathbb{P} \in N_{\gamma_N}$, γ_N = image of γ under collapse

then

$N[g] \models \varphi[b, a_N]$

iff $V \models \varphi[b, a]$, for all $b \in N[g]$ $\}$

<u>Claim</u>: S contains a c.u.b. set in $P_{\omega_1}(M)$

<u>Proof</u>: If not, $a = P_{\omega_1}(M) \setminus S$ is stationary in $P_{\omega_1}(M)$, so let $G \subset \mathbb{P}_{LF}$ be V-generic, $a \in G$.

Let

$j : V \to \widetilde{M} \subset V[G]$ be the embedding.

So

$j''(M) \in j(a)$ (since $\bigcup a = M$)

So $\text{Coll}(j''(M)) = M$

So in ultrapower $\widetilde{M}$, by definition of S, in $\widetilde{M}$ there exists $\mathbb{P} \in V_{\gamma_M} = V_\gamma$ (in collapse, $j(\gamma)$ in $\widetilde{M}$ goes to γ, and $j(a)$ goes to a)

and $g \in \mathbb{P}$, M-generic such that for some ~~$b \in \mathbb{R}^{\tilde{M}}$~~ $b \in \mathbb{R}^{M[g]}$,

~~$M[g]$~~ and

disagree ~~about whether $\varphi[b, a_M]$ holds or~~ not.

$$M[g] \models \varphi[b, a_M]$$

$$\text{iff} \quad \tilde{M} \models \neg\varphi[b, j(a)]$$

But $V_\gamma \subset M$, $\mathbb{P} \in V_\gamma$, so

$$M[g] \models \varphi[b, a_M] \quad \text{iff}$$

$$V[g] \models \varphi[b, a]$$

$$\text{iff} \quad V[G] \models \varphi[b, a]$$

by (1)

$$\text{iff} \quad \tilde{M} \models \varphi[b, j(a)]$$

by (2)

a contradiction.
This proves the claim that S is stationary.

Now build T_γ as before.

Choose $H : M^{<\omega} \to M$ so that if

$Z \subset M$, $|Z| = \omega$ and $H'' Z^{<\omega} \subseteq Z$, then $Z \in S$.

Now build T_γ from this.

Also build ~~[illegible]~~ S_γ for $\neg\varphi$.

Roughly: $T_\gamma \subseteq (\omega \times \omega \times M)^{<\omega}$, $S_\gamma \subseteq (\omega \times \omega \times M)^{<\omega}$ } As before, except in embedding ω-model Y into M, map a_Y into a_M (keep track of parameter)

Using stationary tower forcing, we prove that these trees project to complement in small-generic extensions. γ was arbitrarily large $\delta, \gamma < \kappa$,

So roughly,

$[T_\gamma] = (x, y, f)$

Y codes ω-models $M_Y \subset N_Y$ with $\gamma_Y \in M_Y$, $a_Y \in M_Y$

N_Y is a generic extension of M_Y for a poset in $(M_Y)_{\gamma_Y}$

x "$\in$" N_Y

$N_Y \models \varphi[x, a_Y]$

f gives an elementary embedding of

$M_Y \to M$

taking $\gamma_Y \to \gamma$ $a_Y \to a_M$ and with range closed under It