CHAPTER IV: Woodin Cardinals

WE NOW INTRODUCE the concept of a *Woodin cardinal.* We begin with a provisional definition, which we shall use in Chapter V, and then establish its equivalence to our official definition in terms of *extenders*. This official definition is the one that we shall use in Chapter VI. After a review of the principal results of the book, we then give some further properties of extenders.

1: Woodin cardinals and strong embeddings

In this chapter we work in ZFC: therefore the following definition, which uses class variables, is to be regarded as provisional.

1·0 DEFINITION θ is a *Woodin cardinal* if it is a limit ordinal and

$$\forall H \subseteq \theta\, \exists \delta < \theta\, \forall \lambda < \theta\, \exists j : V \to M\ cp(j) = \delta\ \&\ j(\delta) \geq \lambda\ \&\ H \cap \lambda = j(H \cap \delta) \cap \lambda.$$

Here j is a class variable, denoting an elementary embedding of the universe V into an inner model, M. The *critical point* of j, $cp(j)$, is by definition the first ordinal moved.

We shall seek a formulation of this definition which uses no class variables. Before doing so, let us establish some elementary properties of Woodin cardinals.

1·1 LEMMA *Let θ be Woodin. Then $cf(\theta) > \omega$.*

Proof : Suppose to the contrary that $H \subseteq \theta$ is of order type ω and its supremum is θ. Let δ_H be as in the definition, and pick λ so that $\overline{\overline{H \cap \delta}} < \overline{\overline{H \cap \lambda}} < \aleph_0$. Then no such j can exist, for $\overline{\overline{j(H \cap \delta) \cap \lambda}} \leq \overline{\overline{j(H \cap \delta)}} = \overline{\overline{H \cap \delta}} < \overline{\overline{H \cap \lambda}}$, the sets in question all being finite. ⊣ (1·1)

1·2 PROPOSITION *Let θ be Woodin. Then $\{\delta < \theta \mid \delta$ is measurable$\}$ is stationary in θ, so that in particular θ is a Mahlo cardinal.*

Proof : Let C be closed unbounded in θ. Let $\delta = \delta_C$; pick $\lambda > \delta_C$ with $\lambda < \theta$ and $\lambda \in C'$, the derived set of C. We need $cf(\theta) > \omega$ to know that C' is non-empty. Let j be such that $j(\delta) \geq \lambda$ and $j(C \cap \delta) \cap \lambda = C \cap \lambda$.

If $C \cap \delta$ were empty or bounded in δ, then $j(C \cap \delta)$ would equal $C \cap \delta$, as $\delta = cp(j)$, and thus would also be empty or bounded in δ; but $\bigcup j(C \cap \delta) \geq \lambda = \bigcup (C \cap \lambda)$. So $\bigcup (C \cap \delta) = \delta$; hence $\delta \in C$.

Thus every closed unbounded subset of θ contains a measurable cardinal, as required. ⊣ (1·2)

Our class-free definition will be in terms of *extenders*, which broadly speaking are directed systems of measures and which are precisely defined in the next section. Once the concept of a (δ, λ, H)-extender is available, our official definition of a Woodin cardinal will be this:

1·3 DEFINITION A limit ordinal θ is a *Woodin cardinal* if

$$\forall H \subseteq \theta\ \exists \delta < \theta\ \forall \lambda < \theta\ \exists \text{ a } (\delta, \lambda, H)\text{-extender}.$$

As a (δ, λ, H)-extender is, for $\delta < \lambda = \bigcup \lambda$, an object of rank λ, the above is clearly a Π^1_1 definition: thus the first Woodin cardinal is Π^1_1-describable and therefore not weakly compact.

2: Extenders

Let δ, λ, θ be limit ordinals with $\delta < \lambda < \theta$, and let $H \subseteq \theta$. In formulating the notion of a (δ, λ, H)-extender it will be convenient to use this notation:

2·0 Definition Let $s \subseteq t \in [\lambda]^{<\omega}$, and $b \in [\delta]^{<\omega}$ with $\overline{\overline{t}} = \overline{\overline{b}}$. We write $b^{t,s}$ for $\pi``s$ where π is the unique order-preserving isomorphism between t and b.

2·1 Definition A (δ, λ, H)-*extender* is a family $E = \langle E(s) \mid s \in [\lambda]^{<\omega} \rangle$ satisfying five conditions, (*EXT 1*) to *EXT 5*. We shall introduce these in stages.

The first condition is a natural one to those familiar with the theory of measurable cardinals.

The reader will recall the link between elementary embeddings and ultrafilters. [R1]

Briefly, if $j : V \to M$ is an elementary embedding of the universe into an inner model, and $cp(j)$ is the first ordinal moved by j, then

$$U(j) \;=_{\text{df}}\; \{X \subseteq cp(j) \mid cp(j) \in j(X)\}$$

is a $cp(j)$-complete ultrafilter on $cp(j)$ which is *normal* in the usual sense that given $0 \notin X \in U(j)$ and $f : X \longrightarrow cp(j)$ such that $\forall \zeta :\in X\; f(\zeta) < \zeta$, f will be constant on some subset Y of X with $Y \in U(j)$.

Conversely, given a κ complete non-principal ultrafilter on κ, we may, following Scott, build an ultrapower of the universe, and it will be extensional and well-founded, and hence isomorphic to a transitive inner model M, say. We may prove a version of Łoś's theorem, and thus obtain an elementary embedding of V into M with critical point κ.

We may also do this when we are taking ultrapowers of an inner model using an ultrafilter not in the model: provided AC is true in that model, the quantifier step of the induction on length of formulæ that proves Łoś's theorem will still work.

The critical point of the resulting embedding j_U will be κ; every ordinal ν less than κ will be represented by the constant function with value ν in the ultrapower; and if in addition the ultrafilter is normal, the identity function represents the ordinal κ.

The two processes are not exact counterparts: j will factor through $j_{U(j)}$ but need not equal it, since some elementary embeddings are much more interesting than a single ultrafilter can achieve.

The concept of an extender is an attempt to reduce these stronger embeddings to single sets, which may be thought of as directed systems of measures.

With those preliminary remarks, we give the first property of a (δ, λ, H) extender:

EXT 1: *for all s, $E(s)$ is a δ-complete ultrafilter on $[\delta]^{\overline{\overline{s}}}$.*

Let us (temporarily) write $M(s)$ for the ultraproduct of the universe by $E(s)$.

We wish to have canonical elementary embeddings $\pi_{s,t}$ from $M(s)$ to $M(t)$ whenever $s \subseteq t$, and the natural definition is this:

$$\pi_{s,t}(f)(v) = f(v^{s,t})$$

where $f : [\delta]^{\overline{\overline{s}}} \to V$.

For this to be well-defined and 1-1, we need

$$\{u \mid f(u) = g(u)\} \in E(s) \Longleftrightarrow \{v \mid \pi_{s,t}(f)(v) = \pi_{s,t}(g)(v)\} \in E(t)$$

but this second condition is equivalent to saying that $\{v \mid f(v^{s,t}) = g(v^{s,t})\} \in E(t)$, or, more suggestively that $\{v \mid v^{s,t} \in \{u \mid f(u) = g(u)\}\} \in E(t)$ and thus will be achieved if E satisfies our second condition, the *coherence condition*:

EXT 2: $s \subseteq t \implies \forall A \left[A \in E(s) \implies \{v \mid v^{s,t} \in A\} \in E(t)\right]$.

Plainly this requirement is saying that $E(s)$ lies below $E(t)$ under the Rudin-Keisler ordering, using the map $v \mapsto v^{s,t}$.

[R1] books: Jech, Drake, Lévy, Dodd. articles: Kunen *Some applications of iterated ultrapowers in set theory*, Mitchell, Mitchell–Baldwin

2·2 PROPOSITION *The above map induces an elementary embedding of $M(s)$ into $M(t)$.*

We now have a directed system of embeddings, and may form their direct limit $M(E)$, with associated embeddings $\pi_{s,E}$. Our next condition is the *well-foundedness condition*:

EXT 3': $M(E)$ *is well-founded*

EXT 3', however, is about classes, and we wish therefore to find a formulation which is only about sets. Given that $M(E)$ is a model of ZF set theory, and therefore admits a rank function, it will be well-founded if and only if its ordinals are well founded; and if they are, M will be set-like, as it will have V_ζ's. Hence we shall be able to transitise it.

Using DC, ill-foundedness is equivalent to the existence of a descending chain of "ordinals"; using that fact, it may be shown that *EXT 3'* is equivalent to

EXT 3: given $A_n \in E(s_n)$ *where* $s_n \subseteq s_{n+1}$ *there is a sequence* u_n *with each* $u_n \in A_n$ *and* $u_n = u_{n+1}^{s_n,s_{n+1}}$.

Given the well-foundedness condition, we may identify $M(E)$ with its transitive collapse, and now may ask how certain ordinals are represented in the model.

We explore a possible representation, and will be led to formulating a *normality condition*, *EXT 4*, on the extender, which will ensure that our representation works.

2·3 DEFINITION For $\nu \in s$ we define $f^s_\nu(u) = \bigcup(u^{\{\nu\},s})$.

2·4 LEMMA *If $\nu \in s \subseteq t$, then $\pi_{s,t}(f^s_\nu) = f^t_\nu$.*

Proof : $\pi_{s,t}(f^s_\nu)(v) = f^s_\nu(v^{s,t}) = (v^{s,t})^{\{\nu\},s} = v^{\{\nu\},t} = f^t_\nu(v)$. ⊣ (2·4)

We wish the following to be true:

2·5 PROPOSITION $\forall\nu < \lambda\, \forall s \ni \nu\ [\![f^s_\nu]\!]_E = \nu$.

The idea of the proof will be to show first that for $\nu < \xi < \lambda$ and $s \supseteq \{\nu, \xi\}$, $[\![f^s_\nu]\!]_E < [\![f^s_\xi]\!]_E$ and then that if $[\![g]\!]_E < [\![f^s_\nu]\!]_E$ then for some $\alpha < \nu$, and $t = s \cup \{\alpha\}$, $[\![g]\!]_E = [\![f^t_\alpha]\!]_E$.

We see that this will work provided that the extender satisfies the following *normality condition*:

EXT 4: *Let* $\{u \mid g(u) < \bigcup u\} \in E(s)$*: then there is an* $\alpha < \bigcup s$ *such that setting* $t = s \cup \{\alpha\}$, $\{v \mid g(v^{s,t}) = \bigcup(v^{\{\alpha\},t})\} \in E(t)$.

Note here that both $t = s$ and $t \supset s$ are possible, for various g's.

Proof of Proposition 2·5:

First it is plain that for $\{\nu, \xi\} \subseteq s \in [\lambda]^{<\omega}$, $[\![f^s_\nu]\!]_E < [\![f^s_\xi]\!]_E$, since for each u $u^{\{\nu\},s} < u^{\{\xi\},s}$. Hence the map $\nu \mapsto [\![f^{\{\nu\}}_\nu]\!]_E$ is order-preserving.

Now suppose that $\{u \mid g(u) < f^s_\nu(u)\} \in E(s)$. Since $f^s_\nu(u) \le \cup u$, $g(u) < \cup u$ almost everywhere, and so the hypothesis of the normality condition is fulfilled, and therefore there is an $\alpha < \cup s$ such that setting $t = s \cup \{\alpha\}$,

$$\{v \mid g(v^{s,t}) = v^{\{\alpha\},t}\} \in E(t).$$

But $\alpha \in t$, and $v^{\{\alpha\},t} = f^t_\alpha(v)$ so $\pi_{s,t}(g) = f^t_\alpha \bmod E(t)$; thus $[\![f^t_\alpha]\!]_E = [\![g]\!]_E$.

Note finally that $\alpha < \nu$ since $v^{\{\alpha\},t} < v^{\{\nu\},t}$ as $\pi_{s,t}(g) < \pi_{s,t}(f^s_\nu) = f^t_\nu \bmod E(t)$. ⊣ (2·5)

2·6 COROLLARY $j_E(\delta) \ge \lambda$.
[C 1]

Finally, the purpose of the last condition is to ensure that for the indicated subset H of θ, $j_E(H) \cap \lambda = H \cap \lambda$.

Let us try to prove that in the light of our representation of ordinals less than θ.

Let $\nu < \lambda$. Suppose that $\nu \in j_E(H)$: then for $s = \{\nu\}$, $\{u \mid f^s_\nu(u) \in H\} \in E(s)$: but this is just saying that $\{\xi < \delta \mid \xi \in H\} \in E(\{\nu\})$, or $H \cap \delta \in E(\{\nu\})$. [N 1]

Thus we are led to the *H-ness condition*:

EXT 5: *For* $\nu < \lambda$, $H \cap \delta \in E(\{\nu\}) \iff \nu \in H$.

[C 1] The Mathias horror may be used to ensure that $j(\delta)$ is always strictly greater than λ.

[N 1] Not quite right: we are muddling ordinals and singletons of ordinals, but harmlessly.

Plainly the foregoing discussion shows that *EXT 5* implies that $j_E(H) \cap \lambda = H \cap \lambda$.

Collecting the various clauses together, we have the following:

2·7 DEFINITION Let δ, λ, θ be limit ordinals with $\delta < \lambda < \theta$, and let $H \subseteq \theta$. A (δ, λ, H)-*extender* is a family $E = \langle E(s) \mid s \in [\lambda]^{<\omega} \rangle$ satisfying the following five conditions:

EXT 1: For each $s \in [\lambda]^{<\omega}$, $E(s)$ is a δ-complete ultrafilter on $[\delta]^{\overline{\overline{s}}}$;

EXT 2: for $s \subseteq t \in [\lambda]^{<\omega}$, $E(s) = \{X \subseteq [\delta]^{\overline{\overline{s}}} \mid \{v \in [\delta]^{\overline{\overline{t}}} \mid v^{s,t} \in X\} \in E(t)\}$;

EXT 3: whenever $\langle A_i | i \in \omega \rangle$ and $\langle s_i | i \in \omega \rangle$ are sequences such that for each $i \in \omega$, $A_i \in E(s_i)$ and $s_i \subseteq s_{i+1}$, there is a sequence $\langle v_i | i \in \omega \rangle$ such that for each i, $v_i \in A_i$ and $v_i = v_{i+1}^{s_i, s_{i+1}}$;

EXT 4: whenever $\{u \mid g(u) < \bigcup u\} \in E(s)$, there is an $\alpha < \bigcup s$ such that, setting $t = \{\alpha\} \cup s$, $\{v \mid \pi_{s,t}(g)(v) = \bigcup v^{\{\alpha\},t}\} \in E(t)$;

EXT 5: for each $\nu < \lambda$, $\{\{\xi\} \mid \xi \in H \cap \delta\} \in E(\{\nu\}) \iff \nu \in H$.

An E satisfying the first three clauses will be called a (δ, λ)-*extender*. It will be called *normal* if it also satisfies the fourth.

3: Results of the book

Now that the concept of a Woodin cardinal is to hand we may state the principal results of this book. Those of Part I are:

3·0 THEOREM *Assuming $V = L(\mathbb{R}) + AD$, Θ is Woodin in $HOD^{L(\mathbb{R})}$.*

3·1 THEOREM *Assuming $\exists x :\in \Re \ \ V = L[x]$ and Δ^1_2-determinacy, ω_2 is Woodin in HOD.*

3·2 THEOREM *Assuming again that $V = L(\mathbb{R})$ and AD, there is an inner model with ω Woodin cardinals.*

In Part II we shall prove these results in the other direction:

3·3 THEOREM *Assuming ZFC plus the existence of a measurable cardinal greater than infinitely many Woodin cardinals, AD is true in $L(\mathbb{R})$.*

3·4 THEOREM *Assuming ZFC plus the existence of infinitely many Woodin cardinals, AD is true in some submodel of a generic extension of the universe.*

3·5 THEOREM *Assuming ZFC, if a Woodin cardinal is collapsed to ω in the standard way, Δ^1_2-determinacy is true in the generic extension.*

Taken together, these results yield the following equiconsistency results:

3·6 METATHEOREM *Consis(ZF + DC + AD) iff Consis(ZFC + there are ω Woodin cardinals)*

3·7 METATHEOREM *Consis(ZFC + Δ^1_2-determinacy) iff Consis(ZFC + there is a Woodin cardinal).*

The methods used for the above theorems yield the following result

3·8 THEOREM *Suppose that ω_1 is inaccessible to reals. Then for any real α the following two statements are equivalent:*

(3·8·0) $\Delta^1_2(\alpha)$*–determinacy*

(3·8·1) there is a transitive model M of ZFC containing α and all ordinals, and a countable ordinal ζ such that in M, ζ is a Woodin cardinal.

3·9 COROLLARY *The following two statements are equivalent:*

(3·9·0) $\boldsymbol{\Delta^1_2}$*–determinacy*

(3·9·1) $\forall \alpha :\subseteq \omega \ \exists \zeta :< \omega_1 \ \exists A :\subseteq \zeta \ \ \alpha \in L[A] \ \& \ \big(\zeta \text{ is a Woodin cardinal}\big)^{L[A]}$.

Note the striking fact that (3·8·0) is a statement solely about real numbers, whereas (3·8·1) involves large cardinal concepts.

4: Self-improvement of the definition of Woodin

Show first that our properties defined so far are self-improving to (i) arbitrarily large δ and (ii) coding; (iii) adding $V_\lambda \subseteq M$ or even $\mathcal{P}(\lambda) \subseteq M$.

Note further that by Kunen, saying that $j(\delta) \geq \lambda$ is "equivalent" to saying $V_\lambda \subseteq M$, in the following sense: suppose that j is such that for every $n \in \omega$, $j^n(\delta) < \lambda$ but that $V_\lambda \subseteq M$ then we should have a contradiction to Kunen's limitation theorem.

We summarise the above observations, and include a radically different definition of Woodin cardinal, in the following portmanteau theorem:

4·0 THEOREM *The following are equivalent:*

(i) $\forall f : \theta \to \theta\ \exists \delta < \theta\ \exists j : V \to M\ cp(j) = \delta\ \&\ V_{j(f)(\delta)} \subseteq M$

(ii) $\forall H \subseteq V_\theta\ \exists$ *arbitrarily large* $\delta < \theta\ \forall \lambda < \theta\ \exists j\ cp(j) = \delta\ \&\ V_\lambda \subseteq M\ \&\ H \cap V_\lambda = j(H) \cap V_\lambda$.

(iii) $\forall H \subseteq \theta\ \exists$ *arbitrarily large* $\delta < \theta\ \forall \lambda < \theta\ \exists j\ cp(j) = \delta\ \&\ V_\lambda \subseteq M\ \&\ H \cap \lambda = j(H) \cap \lambda$.

(iv) $\forall H \subseteq \theta\ \exists \delta < \theta\ \forall \lambda {:}{\in} (\ \delta, \theta) \exists E$ *E is a* (δ, λ, H)*-extender.*

Here as before j is a class variable, denoting an elementary embedding of the universe into an inner model, and the *critical point* of an embedding j, $cp(j)$, is by definition the first ordinal moved.

A version of *extender* with support an arbitrary transitive set, rather than an ordinal, may be defined, and used to state an equivalent form of (ii). This is done, for example, in the Martin-Steel paper.

Proof : We already know that *ii iii* and *iv* are equivalent; to see that (ii) implies (i), given f, let H be (graph of f)$\times V_\theta$, and let $j : V \to M$ be a (δ, λ, H) strong embedding, so that $j(\delta) \geq \lambda$, $V_\lambda \subseteq M$, and $j(H \cap V_\delta) \cap V_\lambda = H \cap V_\lambda$. [Proof to be finished].

In the other direction, assume (i): we shall prove (iv). Let H be a subset of θ, and suppose for a contradiction that for each δ there is a $\lambda \geq \delta$ with no (δ, λ, H) extender. Let $f(\delta) =$ the least such $\lambda \geq \delta + 2$ + 1, if δ is measurable, and $\delta + 2$ otherwise. Let $j : V \to M$ work for f: so that putting $\delta = cp(f)$, δ is closed under f, and $V_{j(f)(\delta)} \subseteq M$.

$j(f)(\delta) \geq \delta + 2$ and so δ will be measurable in M, since measures on δ are in $V_{\delta+2}$; hence $j(f)(\delta)$ will be $\lambda + 2$ where $M \models$"There is no $(\delta, \lambda, j(H))$-extender."

Let E be the (δ, λ)-extender derived from j. $E \in V_{\lambda+1}$, and so $E \in M$. We now reach a contradiction by proving that in M E is a $(\delta, \lambda, j(H))$-extender after all.

E will certainly be a (δ, λ) extender and will be normal; hence we may within M build the ultralimit of M using E and the elementary embedding j_E^M. The only problem will be to show that $j_E^M(j(H) \cap \delta) \cap \lambda = j(H) \cap \lambda$. We shall do so by establishing the following chain of equalities:

$$\begin{aligned} j_E^M(j(H) \cap \delta) \cap \lambda &= j_E^M(H \cap \delta) \cap \lambda \\ &= j_E(H \cap \delta) \cap \lambda \\ &= j(H) \cap \lambda \end{aligned}$$

Now the first equality holds because $j(H) \cap \delta = H \cap \delta$, since δ is the critical point of j. The next will hold because j_E and j_E^M agree at low levels; and the last will hold because j_E and j agree at low levels. So we only have to justify those last two statements; they are of sufficient generality to merit statement as lemmata.

4·1 LEMMA j_E *and* j_E^M *agree.*

4·2 LEMMA j_E *and* j *agree.*

4·3 PROBLEM (Krawczyk) Can f can be defined more positively.

Finally, note the persistence of Woodinry:

4·4 LEMMA *Suppose* M *and* N *are inner models of ZFC and* $P(\theta) \cap M = P(\theta) \cap N$. *If* θ *is strongly inaccessible in* M *then* $V_\theta \cap M = V_\theta \cap N$*; hence if* θ *is Woodin in one then it is in the other.*

Proof : Let $\nu = \xi + 1$ be minimal such that $V_{\xi+1} \cap M \neq V_{\xi+1} \cap N$; so there is an $A \subseteq V_\xi$ where they differ. But in M, V_ξ is of size κ say less than θ, and so coded by a subset X of κ, which is in both models; A, if in one of the models, may be coded by a subset of Y and therefore must be in the other model. So M and N have the same subsets of θ and the same extenders. The conclusion follows. $\dashv$

5: Proof that if E is an extender, $E \notin \mathrm{Ult}(V, E)$

The material of this and the next section is not used in Part I but is included as a source of further exercises that might develop the reader's concept of an extender.

5·0 Let Y be a stationary subset of κ.

Define, recursively,

$$X = \{\xi \in Y \mid \xi \cap X \text{ is non-stationary in } \xi\}$$

Then X is a stationary subset of Y non-stationary in each $\xi \in X$.

5·1 We use this idea* to give a new proof that no κ-complete non-principal ultrafilter U on $[\kappa]^k$ can be a member of its own ultrapower.

First, the case $k = 1$: let $[f]_U$ be a κ complete ultrafilter on κ in $Ult(V, U)$. Let $\kappa = [g]_U$: certainly $g \leq id$ (mod U); thus there is a $Z \in U$ such that

$$\forall \nu{:}{\in} Z \;\; f(\nu) \text{ is a } g(\nu)\text{-complete ultrafilter on } g(\nu) \;\&\; g(\nu) \leq \nu.$$

Any subset A of κ is represented in the ultrapower by f_A where $f_A(\nu) = g(\nu) \cap A$.

Now define, recursively,

$$X = \{\xi \in Z \mid g(\xi) \cap X \notin f(\xi)\}$$

Then $X \in U \iff X \notin [f]$, and so $[f] \neq U$.

5·2 For $k > 1$, let $g : [\kappa]^k \to \kappa$ represent κ, and $f : [\kappa]^k \to V$ represent an ultrafilter on $[\kappa]^k$. Each $A \subseteq [\kappa]^k$ is represented by f_A where $f_A(s) = [g(s)]^k \cap A$. If the map $s \mapsto \bigcap s$ represents an ordinal $\geqslant \kappa$, $g(s) \leq \bigcap s$ for all s in some $Z \in U$,and we may define recursively

$$X = \{s \in Z \mid [g(s)]^k \cap X \notin f(s)\}$$

and argue as before that $X \in U \iff X \notin [f]$.

Otherwise $s \mapsto \bigcap s$ represents an ordinal $\nu < \kappa$, so $X \;=_{\mathrm{df}}\; \{s \mid \bigcap s = \nu\} \in U$ by the κ-completeness of U; hence we may define an ultrafilter W on $Y \;=_{\mathrm{df}}\; \{t \in [\kappa]^{k-1} \mid \nu < \bigcap t\}$ by $A \in W \iff \{\{\nu\} \cup t \mid t \in A\} \in U$. W and the restriction of U to $\mathcal{P}(X)$ are isomorphic, so $\mathrm{Ult}(V, W) = \mathrm{Ult}(V, U)$; $W \notin \mathrm{Ult}(V, W)$ by the inductive hypothesis, so $U \notin \mathrm{Ult}(V, U)$.

5·3 Finally we show that if E is a normal (κ, λ) extender, then $E \notin Ult(V, E)$.

Suppose that $E = i_{a,\infty}(F)$ where $1 \leq k < \omega$, $a \in [\lambda]^k$ and WLOG we may assume that $\kappa \in a$. Write $i_{a,\infty}$ for the canonical elementary embedding of $Ult(V, E(a))$ into $Ult(V, E)$: $E(a)$ is here of course a κ-complete non-principal ultrafilter on $[\kappa]^k$.

Now $a = i_{a,\infty}(b)$ where $b = [id]_{E(a)}$. Hence $E(a) = i_{a,\infty}(F(b))$. However $\kappa = i_{a,\infty}(\kappa)$,‡ and so $i_{a,\infty}(F(b)) = F(b)$. Hence $E(a) = F(b) \in Ult(V, E(a))$, contradicting 5·2.

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* implicit in a proof by Jensen of Solovay's splitting theorem; does it go back earlier ?

‡ since $\kappa = i_{a,\infty}(c)$ where $c = [f^a_\kappa]_{E(a)}$ and $f^a_\kappa(s) = s^{\{\kappa\},a}$; if $c < \kappa$, $i_{a\infty}(c) = c < \kappa$; if $c > \kappa$, $i_{a,\infty}(c) \geqslant c > \kappa$, so c must equal κ.

6: Application of an extender to another model

We have seen how, given an extender E in an inner model M we may form the ultrapower $\mathrm{Ult}(M, E)$. An important idea in later chapters will be the possibility of a like construction when the extender is not itself a member of the model on which it acts.

6·0 DEFINITION We shall call E a *local* (κ, λ) *extender* if there is some inner model M with $E \in M$ and $M \models E$ `is a` (κ, λ) `extender`. We shall say that such an M *houses* E.

It is readily checked that if $M \models E$ `is a` (κ, λ)`-extender` and $N \models E$ `is a` (κ', λ')`-extender` then $\kappa = \kappa'$ and $\lambda = \lambda'$. [C 2]

6·1 DEFINITION We shall say that a local (κ, λ)-extender E *acts on* an inner model N if each $E(a)$ is an ultrafilter in the field of sets $\mathcal{P}([\kappa]^{\overline{\overline{a}}}) \cap N$.

That is equivalent to saying that $\mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap M$ or that $V_{\kappa+1} \cap N = V_{\kappa+1} \cap M$ for some (or for every) inner model M housing E.

6·2 Suppose now that E is a local (κ, λ) extender housed by M and that N is another inner model on which E acts. We are then able to define the ultrapower $\mathcal{N} = \mathrm{Ult}(N, E)$: it will be built from functions $f \in N$ with $\mathrm{Dom}(f) = [\kappa]^k$ for some $k \in \omega$. Such functions need not lie in M, but all the sets of the form $\{s \in [\kappa]^k \mid f(s) = g(s)\}$ will lie in M and be measured by each $E(a)$ with $\overline{\overline{a}} = k$, since $M \cap V_{\kappa+1} = N \cap V_{\kappa+1}$. The same will hold of sets of the form $\{s \in [\kappa]^k \mid f(s) \in g(s)\}$. Thus we can define the equivalence of functions and the membership relation $\mathcal{E}$ on the collection of Scott equivalence classes.

What we cannot guarantee in general is that $\mathcal{N}$ is well-founded, but we shall be able to establish the full Łoś theorem as a schema, as for the case of $\mathrm{Ult}(M, E)$.

6·3 Given a possibly ill-founded model $\mathcal{N}$, we can define its well-founded part, $\mathrm{wfp}(\mathcal{N})$. We look first at all its "ordinals"and see which of those are well-founded, and then, assuming that $\mathcal{N}$ satisfies enough set theory to believe that every set has a rank which is an ordinal, look at those members of $\mathcal{N}$ whose rank is a well-founded ordinal. The class or set of such members will be, by definition, $\mathrm{wfp}(\mathcal{N})$. It will be convenient to treat $\mathrm{wfp}(\mathcal{N})$ as having been replaced by the transitive set or class to which it is isomorphic.

6·4 We shall now see that in the circumstances above, the well-founded part of $\mathcal{N}$ will be of height at least $j^M_E(\kappa) + 1$.

By assumption we have that $M \cap V_{\kappa+1} = N \cap V_{\kappa+1}$.

6·5 LEMMA *Let $x \in V_{\kappa+2} \cap M \cap N$. Then $j^N_E(x)$ is in $\mathrm{wfp}(\mathcal{N})$ and (after transitisation) equals $j^M_E(x)$.*

Proof : Suppose $x \neq \varnothing$. Let $0 < k \in \omega$, $a \in [\lambda]^k$, $f \in N$ with $f : [\kappa]^k \to N$ and $\{u \mid f(u) \in x\} \in E(a)$: we may by redefining f if necessary on a set of $E(a)$-measure 0 assume that $\forall u \in [\kappa]^k,\ f(u) \in x$. Let $A_f = \{(s, z) \mid z \in f(s)\}$: then $A_f \subset V_\kappa$, so $A_f \in N \cap V_{\kappa+1}$, and hence is in M; but then $f \in M$, being recoverable from A_f by the equation $f(s) = \{z \mid (s, z) \in A_f\}$.

Similarly every such function in M will also be in N. We know that $j^M_E(x)$ is well-founded, and we have seen that the members of $j^N_E(x)$ are represented by the same functions. ⊣ (6·5)

Thus we have proved the following:

6·6 PROPOSITION *Suppose that $M \models E$ is a (κ, λ)-extender, and that $N \cap V_{\kappa+1} = M \cap V_{\kappa+1} = A$, say. Write $B = A \cup \{A\}$. Then*

(6·6·0) $j^N_E \restriction B = j^M_E \restriction B$.

In particular,

(6·6·1) $j^N_E(\kappa) = j^M_E(\kappa)$,

and, writing $j_E(x)$ for the common value of the two embeddings when $x \in B$, we have

(6·6·2) $$\mathrm{Ult}(M, E) \cap V_{j_E(\kappa)+1} = j_E(A) = \mathrm{Ult}(N, E) \cap V_{j_E(\kappa)+1},$$

[C 2] The quantification over inner models is not satisfactory, but the use we make of this definition hardly warrants the effort of avoiding that problem.

Some exercises on extenders

6·7 Exercise Let $V = L[A]$ and E be an A-strong extender: then $\lambda(E) < \sup A$.

6·8 Exercise Let E be a (κ, λ) extender, and let $\kappa \le \mu \le \lambda$: then if $D \subseteq \kappa$, $j_E(D) \cap \mu = j_{E\restriction\mu}(D) \cap \mu$.

6·9 Exercise Let E act on N: then $E \notin \mathrm{Ult}(N, E)$.

6·10 Exercise Let E act on N. Then for all $\tau < \mathrm{strength}(E)$, $E \restriction \tau \in \mathrm{Ult}(N, E)$.

6·11 Exercise If $\pi : P \longrightarrow M$ is Δ_0-elementary and E is an extender in P with $\kappa(E) < \mathrm{crit}(\pi)$ then $\pi(E)$ is an extender in M, of length $\pi(\lambda(E))$.

6·12 Exercise Let $\pi : P \longrightarrow M$ and $\rho : Q \longrightarrow M$ be two Δ_0 elementary maps; let E be an extender in P with $\kappa(E) < \mathrm{crit}(\pi)$ and less than $\mathrm{crit}(\rho)$. Then $\mathrm{Ult}(Q, E)$ is well-founded.

7: Stationary tower forcing

In this section we develop various notions of forcing which in favourable circumstances will yield generic elementary embeddings of a helpful kind. First we must introduce the notions of *club* (closed unbounded) and *stationary* sets: these definitions generalise the familiar ones for sets of ordinals (and differ slightly from them in the context of singular ordinals); where confusion might arise, we shall distinguish between, on the one hand, T-stationary and T-club (being the notions introduced here in connection with tower forcing) and, on the other hand, O-stationary and O-club, (the notions defined in elementary books on set theory concerning sets of ordinals). The context will usually make clear which is meant.

7·0 Definition For a set A write $A^{0<\omega} =_{\mathrm{df}} \bigcup_{0<k<\omega} {}^k A$.

7·1 Definition A set A is *club* if $\bigcup A$ is not empty and $\exists G : (\bigcup A)^{0<\omega} \to \bigcup A$ such that

$$A \setminus \{\emptyset\} = \big\{\sigma \subseteq \bigcup A \mid G``\sigma^{0<\omega} \subseteq \sigma \neq \emptyset\big\}.$$

Such a function G is said to *generate* A. We say that A is *club in* X if A is club and $\bigcup A = X$; so that $X \neq \emptyset$. A function $G : X^{0<\omega} \to X$ will be called an X-*function*, and if $\sigma \subseteq X$ and $G``\sigma^{0<\omega} \subseteq \sigma$, σ will be said to be *closed under* G.

7·2 Proposition *(i) If for $i < \omega$, G_i is an X-function there is an X-function H such that for each $\sigma \subset X$, σ is H-closed iff it is closed under each G_i.*

(ii) If for $i < \omega$ A_i is club in X then so is the intersection $\bigcap_{i<\omega} A_i$.

Proof : Let $\langle \cdot, \cdot \rangle$ be some reasonable pairing function. Define $H(s_0, \dots s_{\langle i, j \rangle}) = G_i(s_0, \dots, s_j)$. $\dashv$

7·3 Definition The X-function H defined as in Part (i) of that Proposition will be said to *code* the sequence $\langle G_i \mid i < \omega \rangle$ of X-functions.

7·4 Definition For sets A, X define $A_X =_{\mathrm{df}} \{a \cap X \mid a \in A\}$.

7·5 Proposition *(i) $A_X = A \iff X \supseteq \bigcup A$; $\bigcup A_X = X \cap \bigcup A$; $A_X = A_{X \cap \bigcup A}$ (ii) $(A_X)_Y = A_{X \cap Y}$ (iii) $A \subseteq B \implies A_X \subseteq B_X$; $(A \cap B)_X \subseteq A_X \cap B_X$*

7·6 Definition The *extension of A to Y*, $E(A, Y)$ is defined as

$$E(A, Y) =_{\mathrm{df}} \big\{\tau \subseteq (Y \cup \bigcup A) \mid \tau \cap \bigcup A \in A\big\}.$$

7·7 Lemma *(i) $A \subseteq E(A, Y)$; (ii) $\bigcup E(A, Y) = Y \cup \bigcup A$; (iii) $E(A, Y)_{\bigcup A} = A$; (iv) if A is club, so is $E(A, Y)$.*

Proof of (iv): let $X = \bigcup A$, and let A be generated by the X-function H.

Define a $(X \cup Y)$-function G by

$$G(s) = \begin{cases} H(s) & \text{if } s \in X^{0<\omega}; \\ \text{the first member of } s \text{ not in } X & \text{otherwise} \end{cases}.$$

Let B be the club generated by G. Then it is readily checked that $B = E(A, Y)$. $\dashv$ (7·7)

7·8 PROPOSITION *Let* $\emptyset \neq X \subseteq Y$.

(i) If C *is club in* Y *then* C_X *is club in* X.

(ii) If D *is club in* X *then the extension of* D *to* Y *is a set* C *club in* Y *with* $C_X = D$.

Proof of (i): Let C be generated by G. For each (finite) $s \in X^{0<\omega}$, the G-closure of s will be finite or countable: write $G[s]$ for this set. *A fortiori*, $G[s] \cap X$ will be finite or countable: enumerate it in order type $\leq \omega$, and for $i < \omega$ let $H_i(s)$ be the i^{th} element of $G[s] \cap X$ if one exists, and let it be the least member of $Im(s)$ otherwise.

Let H code the sequence $\langle H_i \mid i < \omega \rangle$, and let E be the subset of X generated by H. We assert that $E = C_X$.

Let $\sigma \subseteq X$ be H-closed. Let τ be the G-closure of σ. Then $\tau \in C$, and $\sigma \subseteq \tau$; but $\tau \cap X \subseteq \sigma$, as each element, x say, of $\tau \cap X$ is in $G[s] \cap X$ for some finite $s \in \sigma^{0<\omega}$; hence x is a value of H and so lies in σ. Thus $\tau \cap X = \sigma$, and so $E \subseteq C_X$.

On the other hand, let $\tau \in C$: then τ is G-closed. Put $\sigma = \tau \cap X$. Then σ is H-closed, since each value of H is a value of G and so in τ, and therefore in $\tau \cap X$, as H is an X-function. Hence $\sigma \in E$; thus $C_X \subseteq E$, and so $C_X = E$, as required.

That proves (i), and (ii) is a restatement of (iv). $\dashv$

7·9 DEFINITION A set A is *stationary* if $\bigcup A \neq \emptyset$ and whenever C is a club with $\bigcup C = \bigcup A$, there is a non-empty member of $A \cap C$.

A is *stationary in* X if A is stationary and $\bigcup A = X$.

7·10 PROPOSITION *If* A *is stationary in* X *and* C *is club in* X *then* $\bigcup(A \cap C) = \bigcup A$ *and* $A \cap C$ *is stationary.*

Proof : Let $a \in \bigcup A$, and let G be the $\bigcup A$-function with constant value a. Let H generate C, and let K code H and G. As A is stationary, some non-empty σ in A is closed under K; so $a \in \sigma$ and $\sigma \in C$, so $\sigma \in A \cap C$ and $a \in \bigcup(A \cap C)$. The second part now follows from the previous proposition . $\dashv$

7·11 PROPOSITION *Let* $\emptyset \neq X \subseteq Y$ *Then (i) if* A *stationary in* Y, A_X *is stationary in* X*; (ii) if* B *is stationary in* X *the extension* $E(B, X)$ *is a set* A *stationary in* Y *with* $A_X = B$.

Proof of (i): let D be club in X, then for some club C in Y, $C_X = D$; let $\alpha \in C \cap A$; then $\alpha \cap X \in D \cap A_X$ as required.

Proof of (ii): after Lemma we have only to check that A is stationary. So let C be club in Y; then C_X is club in X; let $\beta \in C_X \cap B$, and let $\gamma \in C$ be such that $\gamma \cap X = \beta$. Set $\tau = \gamma \setminus X$. Then $\gamma = \beta \cup \tau$, and so $\gamma \in A$, as required. $\dashv$

Fix a (large-ish) cardinal κ, say κ strongly inaccessible. We wish to study the following partial ordering of V_κ:

7·12 DEFINITION $A \leq B \iff_{\text{df}} \bigcup B \subseteq \bigcup A \;\&\; A_{\bigcup B} \subseteq B$.

7·13 REMARK If ever we need to distinguish this relation from another, we shall call it $\leq_{\text{Tower}}$.

7·14 PROPOSITION $A \leq B \leq A \implies A = B$.

Proof : $\bigcup B = \bigcup A$; so $A_{\bigcup B} = A$, $B_{\bigcup A} = B$; so $A \subseteq B \subseteq A$, so $A = B$. $\dashv$

7·15 PROPOSITION $A \leq B \leq C \implies A \leq C$.

Proof : $\bigcup C \subseteq \bigcup B \subseteq \bigcup A$. $A_{\bigcup C} = (A_{\bigcup B})_{\bigcup C} \subseteq B_{\bigcup C} \subseteq C$. $\dashv$

Hence $\leq_{\text{Tower}}$ is indeed a partial order. We begin its study as a notion of forcing, where, in harmony with Boolean algebra, stronger (more informative) conditions are lower in the ordering. First we record some elementary properties:

7·16 PROPOSITION *(i)* $A \leq A_X$*; (ii)* $A \leq B \iff A_{\bigcup B} \leq B$*; (iii)* $\bigcup D = \bigcup A \implies (D \leq A \iff D \subseteq A)$*; (iv)* $E(A, Y) \leq A$.

As usual, we call two elements A, B of a partial ordering *compatible* if there is a D with $D \leq A$ and $D \leq B$, and *incompatible*, written $A \perp B$, if no such D exists.

Incompatible elements certainly exist:

7·17 PROPOSITION *Suppose that* $\bigcup A = \bigcup B = Z \neq \emptyset$ *and that* $Power(Z)$ *is the disjoint union of* A *and* B*. Then* $A \perp B$.

Proof : Suppose $D \leq A$ and $D \leq B$. Then $D_Z \subseteq A \cap B$ and so D_Z is empty. But $Z \subseteq \bigcup D$, so if $\bar{z} \in Z$ there is a $\delta \in Z$ with $\bar{z} \in \delta$, and so $\bar{z} \in \delta \cap Z \in D_Z$, a contradiction. $\dashv$

On the other hand there are elements compatible with everything:

7·18 PROPOSITION *Let $Z \neq \emptyset$ and $A = Power(Z)$. Then*

$$\forall C \, \exists D \; D \leq C \;\&\; D \leq A.$$

Proof : Let $Y = Z \setminus \bigcup C$ and let $D = \{\sigma \cup Y \mid \sigma \in C\}$. $\dashv$

We shall return to this proposition later with a more careful proof; note for the moment that it has the corollary that if $\emptyset \neq W \subset Z$ and $A = Power(Z)$ and $B = Power(W)$, then $B \not\leq A$, but B cannot be strengthened to something incompatible with A, as A is compatible with everything. So from the forcing point of view this partial order is quirky.

Some of the pathologies will disappear if we confine attention to stationary sets, and this we now do. We shall be interested in two particular cases, each equipped with the Tower ordering:

7·19 DEFINITION $P_{<\kappa} =_{\mathrm{df}} \{A \in V_\kappa \mid A \text{ is stationary}\}$;
$Q_{<\kappa} =_{\mathrm{df}} \{A \in V_\kappa \mid A \text{ is stationary and each element of } A \text{ is countable}\}$.

We shall be able to develop results for these two orderings in parallel; the proofs will adapt with little difficulty to the other; but the orderings are quite different; the Q-algebra **will not be a regular subalgebra** of the P algebra. In studying the Q case it would be reasonable to modify the notion of club to mean the set of *countable* closure points of some X-function, as if an element of $Q_{<\kappa}$ has to meet a club, it is only the countable elements of the club that are in the running. Thus we are really discussing clubs relative to the stationary set $Power_{\aleph_1}(V_\kappa)$.

Call two conditions *inseparable* if neither can be strengthened to be incompatible with the other. Normally in discussing notions of forcing we deal with *separative* partial orders in which inseparable conditions are identical, and the partial order embeds in the Boolean algebra of regular open sets of the induced topology. Here that will not happen; non-trivial inseparable pairs exist, and become identified when we pass to the Boolean algebra.

Put another way, if we use P or Q as a notion of forcing, and have a generic filter G, if p is in G then any q inseparable from p will be too.

That point made, it will be convenient to continue to work with the conditions as given, rather than factoring by the equivalence relation of inseparability.

7·20 PROPOSITION *Let a stationary B be given; set $X = \bigcup B$; and let $Y \supseteq X$. Then the extension D of B to Y is inseparable from B, with $D \leq B$ and $\bigcup D = Y$.*

Proof : We take $D = \{\sigma \subseteq Y \mid \sigma \cap X \in B\}$, or in the Q case, $D = \{\sigma \subseteq Y \mid \sigma \cap X \in B \;\&\; \sigma \text{ is countable}\}$

We have already seen that $D \leq B$, D is stationary in Y, and $D_X = B$. It remains only to check inseparability. Let $A \leq B$: we have to show that A is compatible with D.

We have $X \subseteq \bigcup A$ and $A_X \subseteq B$.

Let $W = Y \cup \bigcup A$. Let F be the extension of A to W. Note that if A is club so is F.

Then $F \leq A$, so $F \leq B$ (as $A \leq B$) and $F_Y \subseteq D$, so $F \leq D$. $\dashv$

7·21 LEMMA *For given stationary A and B let $C = \{x \mid x \cap \bigcup B \in B \,\&\, x \cap \bigcup A \in A\}$. Then A and B are compatible in P [or in Q] iff $\bigcup C = \bigcup A \cup \bigcup B \;\&\; C$ is stationary; and then $C = A \wedge B$.*

Proof of $\Longleftarrow$: $C_{\bigcup A} \subseteq A$ and $C_{\bigcup B} \subseteq B$, and so if C is stationary and $\bigcup C = \bigcup A \cup \bigcup B$, C is a condition and $C \leq A$ and $C \leq B$. [For the Q case, note as well that if each element of A and of B is countable, so is each element of C.]

Proof of $\Longrightarrow$: let $D \leq A$ and $D \leq B$, (D stationary). Put $X = \bigcup A \cup \bigcup B$. Then $X \subseteq \bigcup D$; so D_X is stationary as D is. We show that $D_X \subseteq C$, from which it will follow that $\bigcup C = X$, C is stationary, and $D \leq C$; hence that when A and B are compatible, C is the largest condition stronger than both, which is the meaning of the concluding assertion.

So let $\delta \in D$, and put $a = \delta \cap \bigcup A$ and $b = \delta \cap \bigcup B$; then $a \in A$, as $D_{\bigcup A} \subseteq A$, and $b \in B$, and $a \cap \bigcup B = \delta \cap (\bigcup A \cap \bigcup B) \subseteq b$, and $B \cap \bigcup A \subseteq a$, and so $a \cup b = \delta \cap X$ is in C. $\dashv$

7·22 COROLLARY *If $F_{\bigcup A} \cap A$ is not stationary in $\bigcup A$, then $F \perp A$.*

Proof : $F \leq F_{\bigcup A}$; if $C \leq F$ and $C \leq A$, then $C_{\bigcup A} \subseteq F_{\bigcup A} \cap A$; C is stationary and $\bigcup A \subseteq \bigcup C$, so $C_{\bigcup A}$ is stationary in $\bigcup A$, contradicting the hypothesis. $\dashv$

Work in P [or in Q], and let conditions A and B be given. First consider the *special case* that $\bigcup A \subseteq \bigcup B$, and set $E = \{\delta \in B \mid \delta \cap \bigcup A \in A\}$ and $F = \{\delta \in B \mid \delta \cap \bigcup A \notin A\}$. B is the disjoint union of E and F, so at least one of them is stationary in $\bigcup B$[C 3]

If F is stationary in $\bigcup B$, $F \leq B$ and by the last corollary, $F \perp A$.

If F is non-stationary in $\bigcup B$, there is a set C club in $\bigcup B$ with $C \cap B \subseteq E$, which is therefore stationary in $\bigcup B$.

For the *general case*, when $\bigcup A \not\subseteq \bigcup B$, let $X = \bigcup B$ and $Y = \bigcup A \cup \bigcup B$. Let D be the extension of B to Y, [or, in the Q case, let D be $\{\sigma \subseteq Y \mid \sigma \cap X \in B \,\&\, \sigma \text{ countable}\}$.] Then $D \leq B$ and $\bigcup A \subseteq \bigcup D$: we are now back in the special case.

Our discussion shows this: given B and A, there is a D with $D \leq B$ and D inseparable from B with $\bigcup D = \bigcup A \cup \bigcup B$; then either we can intersect D with a set club in $\bigcup D$ to obtain an $E \leq A$ or there is an $F \leq D$ with $F \perp A$.

So *modulo* clubs and inseparability, the tower ordering is separative.

7·23 Exercise If two conditions with the same union differ by a set non-stationary in that union, they are inseparable; conversely, if two conditions are inseparable, then their extensions to the union of their two domains differ by a set non-stationary in that union.

7·24 Remark Take the case discussed earlier when $A = Power(\bigcup A)$: the above argument yields for any B a $D \leq B$ with $\bigcup D = \bigcup B \cup \bigcup A$; $D_{\bigcup A} \subseteq A$, and so $D \leq A$. Thus A is compatible with everything.

In the Q case, take $A = Power_{\aleph_1}(\bigcup A)$, the set of all countable subsets of $\bigcup A$; then in Q A will be compatible with everything.

Now for some normality.

7·25 Lemma *Let A be stationary and $F : A \to \bigcup A$ a choice function, so that $F(\alpha) \in \alpha$ for all non-empty α in A. Then there is a $b \in \bigcup A$ with $\{\sigma \in A \mid F(\sigma) = b\}$ stationary in $\bigcup A$.*

Proof : Suppose not. Then for all such b there is a $\bigcup A$-function G_b such that for all G-closed σ, $F(\sigma) \neq b$. Let $H(b_0, b_1, \ldots, b_n) = G_{b_0}(b_1, \ldots b_n)$. Let $\sigma \in A$ be H-closed. Say $F(\sigma) = b \in \sigma$. Then σ is G_b-closed as $G_b(s_1, \ldots s_n) = H(b, s_1, \ldots s_n) \in \sigma$, so $F(\sigma) \neq b$, a contradiction. $\dashv$

[C 3] Little lemma ? — if not, pick a club avoiding each, then B avoids the intersection of the two clubs.

8: Semi-proper sets of tower conditions

Suppose G is V-generic for $P_{<\theta}$ or $Q_{<\theta}$: we may build in $V[G]$ a structure $\mathcal{M}$ and an elementary embedding from the ground model V into it. The precise definition is set out at the beginning of the next section.

Our aim will be to prove that in the special case when θ is a Woodin cardinal, $\mathcal{M}$ is well-founded and closed under sequences in $V[G]$ of length less than θ.

To do this, we shall need to study our P's and Q's rather more closely, and an important concept will be that of a *semi-proper* set of conditions. In this rather technical section we introduce and develop this notion.

In this discussion think of δ [a limit cardinal ?] and $A \subseteq V_\delta$ as fixed, and treat κ as variable but always $\geq \delta+1$. The cases of particular interest are $\kappa = \delta+1$ and κ is a limit ordinal greater than δ with $V_\kappa \models KP$.

8·0 Definition Call X a *small κ-δ-A model*, [or *small κ-model* for short] if $X \prec V_\kappa$, $\overline{\overline{X}} < \delta$, $A \in X$, and $\delta \in X$.

8·1 Definition If X and Y are small κ-δ-A models, we say that Y *neatly extends* [or *δ-neatly extends*] X if $X \subseteq Y$ and there is an $\eta < \delta$ such that $X \cap V_\delta = X \cap V_\eta = Y \cap V_\eta$, so that in particular $Y \cap V_\delta$ end-extends $X \cap V_\delta$, and any element of $(Y \setminus X) \cap V_\delta$ is of rank at least η.

8·2 Definition We say that Y *A-improves* X if Y neatly extends X and $\exists a{:}\in Y \cap A \; Y \cap \bigcup a \in a$.

One might say that such a Y is weakly compatible with some element of A.

8·3 Proposition *If Y A-improves X and Z neatly extends Y then Z A-improves X.*

Proof: Let $\eta < \delta$ be such that $Y \cap V_\delta = Y \cap V_\eta = Z_\cap V_\eta$. Then there is an $a \in Y \cap V_\eta$ with $a \in A$ & $Y \cap \bigcup a \in a$. But $\bigcup a \subseteq V_\eta$ and so $Y \cap \bigcup a = Z \cap \bigcup a$. $\dashv$

8·4 Definition $\mathrm{sp}_\delta(A) =_{\mathrm{df}} \{X \mid X$ is a small $(\delta+1)$-model which has an A-improvement$\}$

Plainly $\mathrm{sp}_\delta(A) \subseteq Power_\delta(V_{\delta+1})$. If, as we might hope, $Power_\delta(V_{\delta+1}) \setminus \mathrm{sp}_\delta(A)$ is non-stationary in $Power_\delta(V_{\delta+1})$, we may say that *almost all* small $(\delta+1)$-models may be improved.

In this terminology our proposition reads:

8·5 Proposition *Let $A \subseteq V_\delta$. Let $\kappa \geq \delta+\omega$ be such that $V_\kappa \models KP$. Then the following are equivalent:*

(i) ALMOST ALL small $(\delta+1)$-A models can be improved

(ii) EVERY small κ-A-model can be improved.

8·6 Remark I darkly suspect that with hard work, **EVERY** $\kappa \geq \delta+2$ would do.

8·7 Definition If $A \subseteq V_\delta$ fulfils the conditions of that Proposition, it will be called *semi-proper* in V_δ. The official definition is version (i).

We shall now prove that and the following proposition: the two proofs are independent of each other and may be taken in either order.

8·8 Proposition *Suppose that θ is a Woodin cardinal and that A is pre-dense in P_θ. Let δ be A-strong. Then $A \cap V_\delta$ is semi-proper in P_δ.*

Here *pre-dense* means as usual that every condition in P_θ is compatible with at least one element of A. A trivial check shows that every semi-proper A is pre-dense.

These two will be used to prove the main technical lemma from which the well-foundedness and closure of the generic ultralimit will be derived.

Proof of: Let κ be given and fix a well-ordering $<_\kappa$ of V_κ. Suppose first that A satisfies (ii)for this κ. Let $\mathcal{C} = \{X \prec V_\kappa \mid \overline{\overline{X}} < \delta \,\&\, A \in X \,\&\, \delta \in X\}$. $\mathcal{C}$ is club in $Power_\delta(V_\kappa)$, being the set of subsets closed under $\aleph_0$ Skolem functions defined using the well-ordering.

Since the rank function is definable in V_κ, [C 4] $V_{\delta+1}$ will be definable whenever δ is, and so $X \in \mathcal{C} \implies X \cap V_{\delta+1} \prec V_{\delta+1}$; so $X \cap V_{\delta+1}$ is a small $(\delta+1)$-model. Put $\mathcal{D} = \mathcal{C}_{V_{\delta+1}}$: then $\mathcal{D}$ is club in $Power_\delta(V_{\delta+1})$. Moreover each element of $\mathcal{D}$ can be improved: specifically we assert that if $X \in \mathcal{C}$ is improved by Y, then $X_0 = X \cap V_{\delta+1}$ is improved by $Y_0 = Y \cap V_{\delta+1}$; for an $a \in Y \cap A$ with $Y \cap \bigcup a \in a$ is in $V_\delta \cap Y$ as $A \subseteq V_\delta$ and so in Y_0; and if Y neatly extends X then (trivially) Y_0 does X_0.

[C 4] this is where we use the truth of KP in V_κ.

Suppose now that (i) holds. Let X be an elementary submodel of $(V_\kappa, <_\kappa)$ of cardinal less than δ. Since $\mathrm{sp}_\delta(A)$ contains a club, there is a function F such that closure under F guarantees that a small set which contains δ and A as members is in $\mathrm{sp}_\delta(A)$.

Each such F is in $V_{\delta+2}$ or at least in V_κ, and so there will one such F in X. [N 2] Naturally X will be closed under this F, so setting $X_0 = X \cap V_\delta + 1$, X_0 is closed under F and is therefore in $\mathrm{sp}_\delta(A)$. Hence there is a small $(\delta + 1)$ model Y_0 which improves X_0. Take $Y =_{\mathrm{df}} \mathrm{Hull}((V_\kappa, <_\kappa), (Y_0 \cap V_\delta) \cup X)$. We assert that Y improves X.

To see that, check first that $Y \cap V_\delta = Y_0 \cap V_\delta$. Clearly $Y_0 \cap V_\delta \subseteq Y \cap V_\delta$. Now suppose that $x \in Y \cap V_\delta$: then $x = f(\vec{b}, \vec{a})$ where each b is in $Y_0 \cap V_\delta$ and each a is in X, and where WLOG we may take f to be a $(V_\kappa, < \kappa)$ definable Skolem function of the form "the $<_\kappa$-first object of V_δ such that blah blah, or $\emptyset$ if no such object exists". Consider the map $\vec{z} \mapsto f(\vec{z}, \vec{a})$, where $\vec{z}$ varies over V_δ: this map, call it h, is a function definable from $\vec{a}$ in $(V_\kappa, <_\kappa)$, so h is in the elementary submodel X of V_κ; further, $h \in V_{\delta+1}$ so $h \in X_0 = X \cap V_{\delta+1}$, and so $h(\vec{b}, \vec{a}) \in Y_0$, as required.

But now that we know that, we know (i) that Y neatly extends X, since Y_0 does X_0, and $Y_0 \cap V_\delta = Y \cap V_\delta$ and $X_0 \cap V_\delta = X \cap V_\delta$; and (ii) the a in $Y_0 \cap A$ with $Y_0 \cap \bigcup a \in a$ will work for Y as well, since $\bigcup a \subseteq V_\delta$, $A \subseteq V_\delta$, and so $Y_0 \cap \bigcup a = Y_0 \cap V_\delta \cap \bigcup a = Y \cap V_\delta \cap \bigcup a = Y \cap \bigcup a$ and $Y_0 \cap A = Y_0 \cap V_\delta \cap A = Y \cap A$. $\dashv$

8·9 Remark If S meets every set stationary in $\bigcup S$, then S contains a set club in S.
Proof : otherwise the complement of S meets every club and is thus stationary, but doesn't meet S ! $\dashv$ (8·9)

By this remark, to prove Proposition : it is sufficient to show that every stationary set meets $\mathrm{sp}_\delta(A \cap V_\delta)$.

So let b be stationary in $Power_\delta(V_{\delta+1})$, so $\bigcup b = V_{\delta+1}$. Since A is pre-dense, there is an $a \in A$ and a $c \in P_\theta$ with $c \leq a$ and $c \leq b$.

Pick λ with $\{a, c\} \in V_\lambda$, $\delta < \lambda < \theta$, and let $j : V \to M$ be such that $cp(j) = \delta$, $\lambda \leq j(\delta) < \theta$, $V_\lambda \subseteq M$ and $j(A \cap \delta) \cap V_\lambda = A \cap V_\lambda$.

We write M_η for $V_\eta \cap M$.

We are going to consider the model $\mathcal{K}$ with underlying set $M_{j(\delta)+1}$ and extra structure which we shall determine when we see what we need. We shall mark with (*) points of the argument which show what we need to put into the structure of $\mathcal{K}$.

Let $\mathcal{C}$ be the club of all elementary submodels of $\mathcal{K}$. $\bigcup \mathcal{C} = M_{j(\delta)+1} \supseteq V_\lambda \supseteq \bigcup c$. Hence $\mathcal{C}_{\bigcup c}$ is club in $\bigcup c$, and so meets the stationary set c. Thus we may pick a $Z_0 \in \mathcal{C}$ with $Z_0 \cap \bigcup c \in c$. Note that Z_0 need not be a member of M.

Since $c \leq a$, $Z_0 \cap \bigcup a \in a$; since $c \leq b$, $Z_0 \cap \bigcup b \in b$, and so $\overline{\overline{Z_0 \cap \bigcup b}} < \delta$.

Set $Z_1 = Z_0 \cap \bigcup b$: then $Z_1 = Z_0 \cap V_{\delta+1}$; $V_{\delta+1} = M_{\delta+1}$ and so is definable in $\mathcal{K}$ if δ is, (*) so we want to name δ in $\mathcal{K}$; and then Z_1 will be an elementary submodel of $V_{\delta+1}$, provided (*) we are careful with the definition of $\mathcal{K}$.

We shall show that Z_1 can be $(A \cap V_\delta)$-improved by proving that $j(Z_1)$ can in M be $j(A) \cap V_{j(\delta)}$-improved. As Z_1 is in b, which was an arbitrary stationary set, this does it.

(*) Hence we want $A \cap V_\delta \in Z_1$, which we can achieve by naming $A \cap V_\delta$ in $\mathcal{K}$ and in Z_0; so we shall also want δ in Z_0 and in Z_1.

Let Z_2 be $\mathrm{Hull}(\mathcal{K}, W)$ where W is a subset of $M_{j(\delta)+1}$ to be determined: so Z_2 is a $(j(\delta) + 1)$-model. We shall mark with (**) points where we realise what needs to go into W to enable us to prove that Z_2 is in M the desired improvement of $j(Z_1)$. For example, $W \in M$ and $\mathcal{K} \in M$ will ensure (**) that $Z_2 \in M$.

We must show that $j(A) \cap M_{j(\delta)} \in j(Z_1)$, but this will be automatic if $A \cap V_\delta \in Z_1$; and that $j(\delta) \in j(Z_1)$, which will also be automatic.

We shall want $j(Z_1) \subseteq Z_2$: this will be trivially true if (**) $j(Z_1) \subseteq W$.

We have the cardinality condition to consider; since $\overline{\overline{Z_1}} < \delta$, $j(Z_1) = j``Z_1$ and so is in M of cardinality less than δ. To keep $\overline{\overline{Z_2}} < j(\delta)$ we shall only need (**) to ensure that in M, $\overline{\overline{W}} < \delta$.

To achieve the improvement, we have to find an element d say of $Z_2 \cap [j(A) \cap V_{j(\delta)}]$ with $Z_2 \cap \bigcup d \in d$. Note here that by the A-strength of j, $j(A) \cap V_{j(\delta)} = j(A \cap V_\delta) = A \cap V_{j(\delta)}$, so we have to find a d in $Z_2 \cap A \cap V_{j(\delta)}$ with $Z_2 \cap \bigcup d \in d$. The only likely candidate is a, for $a \in A \cap V_\lambda$ and so $a \in j(A \cap V_\delta)$, so we (**) must place a in Z_2 and (*) therefore name it in $\mathcal{K}$.

[N 2] We must be able to define $\mathrm{sp}_\delta(A)$ in V_κ.

What are the chances of $Z_2 \cap \bigcup a \in a$? We already know that $Z_0 \cap \bigcup a \in a$, so it will be enough (**) to arrange that $Z_2 \cap \bigcup a = Z_0 \cap \bigcup a$. We can get the inclusion one way by having (**) $W \supseteq Z_0 \cap \bigcup a$, and the other by arranging that $Z_2 \subseteq Z_0$.

Thus we will take $W = j(Z_1) \cup (Z_0 \cap \bigcup a)$, and now we shall need to arrange that $j(Z_1) \subseteq Z_0$: this we can do (*) by adding $j \restriction V_{\delta+1}$ to $\mathcal{K}$ as a relation. Then we shall know that $x \in Z_0 \cap V_{\delta+1} \implies j(x) \in Z_0$.

We now have all our clues, so let us re-run the argument setting $J = \{\langle x, y\rangle \mid y \in V_{\delta+1} \;\&\; j(y) = x\}$ and

$$\mathcal{K} = \Big\langle M_{j(\delta)+1}, \delta, a, A \cap V_\delta, J \Big\rangle.$$

Note that $j(Z_1) \cap M_{j(\delta)} = j(Z_1 \cap V_\delta) = Z_1 \cap V_\delta \subseteq M_\delta$, as δ is the critical point of j and Z_1 is small; so the neatness of Z_2 as an extension of $j(Z_1)$ will be established if we show that $Z_2 \cap M_\delta = Z_1 \cap V_\delta$. But $Z_1 \cap V_\delta = Z_0 \cap V_\delta$ and $Z_2 \subseteq Z_0$; on the other hand $Z_0 \cap V_\delta \subseteq j(Z_1) \subseteq Z_2$. $\dashv$

9: The generic ultralimit

Given κ, we consider the class $\mathcal{L}$ of all pairs $\langle a, f\rangle$ where $a \in V_\kappa$ is stationary and $f : a \to V$. For $c \le a$ and $f : a \to V$ we define $\pi_{a,c}(f) : c \to V$ by

$$\pi_{a,c}(f)(\gamma) = f(\gamma \cap \bigcup a)$$

for $\gamma \in c$.

Now suppose that G is a V-generic filter on P_κ [or Q_κ]; then in $V[G]$ we may define two relations on the class $\mathcal{L}_G =_{\mathrm{df}} \{\langle a, f\rangle \mid a \in G \;\&\; \langle a, f\rangle \in \mathcal{L}\}$:

first, an equivalence relation:

$$\langle a, f\rangle \approx_{\mathcal{L},G} \langle b, g\rangle \iff_{\mathrm{df}} \exists c \in G\; c \le a \;\&\; c \le b \;\&\; \pi_{a,c}(f) = \pi_{b,c}(g)$$

and then on the (Scott) equivalence classes of this relation, the relation E_G:

$$\langle a, f\rangle E_G \langle b, g\rangle \iff_{\mathrm{df}} \exists c \in G\; c \le a \;\&\; c \le b \;\&\; \forall\gamma{:}{\in}c\;\; \pi_{a,c}(f)(\gamma) \in \pi_{b,c}(g)(\gamma)$$

The resulting (class) structure $\mathcal{L}_G$ will be called the *generic ultralimit* of V determined by G. We are able to prove a version of Łoś's theorem and to define an elementary embedding of V into $\mathcal{L}_G$. We are not in general able to prove that the generic ultralimit is well-founded: but it will be in the case of a Woodin cardinal.

9·0 THEOREM *Let θ be a Woodin cardinal and G V generic for P_θ [or Q_θ]. Then in $V[G]$ the generic ultralimit is well founded, locally set-like and thus isomorphic to a transitive class $M \subseteq V[G]$; further M is, in $V[G]$ closed under sequences of length less than θ and there is a generic elementary embedding j of V into M with the property that*

$$G = \Big\{a \in \hat{P}_\theta \mid j``\bigcup a \in j(a)\Big\}.$$

Our aim in this section is to prove this theorem.

9·1 PROPOSITION *Suppose that in $V[G]$, $A \subseteq M$ and $\overline{\overline{A}} < \theta$. Then $\exists x{:}{\in}M\;\; A = \{y \in M \mid yEx\}$.*

Equivalently, suppose that $\gamma < \theta$ and in V, $\langle \tau_\alpha \mid \alpha < \gamma\rangle$ is a sequence of terms such that $\forall\alpha :< \gamma\;\; \Vdash \tau_\alpha \in M$. Let $p \in P_\theta$. Then there is a $q \le p$ and a term z such that $q \Vdash z \,\epsilon\, M$ and $q \Vdash \bigwedge y{:}\,\epsilon\, M(yEz \longleftrightarrow \bigvee\alpha y = \tau_\alpha)$.

It is easy to construct for each α a maximal antichain $A_\alpha \subseteq P_\theta$ and for each $a \in A_\alpha$ a function $g^a_\alpha : a \to V$ such that

$$a \Vdash \langle a, g^a_\alpha\rangle \text{ represents } \tau_\alpha.$$

Note the multiple occurrence of a here: no generality is lost, since if $a \Vdash \langle b, g\rangle$ represents τ, then in particular, $a \Vdash \hat{b} \in \dot{G}$, so $a \le b$, and $a \Vdash \langle a, h\rangle$ represents τ, where $h = \pi_{b,a}(g)$.

Now pick a measurable $\delta < \theta$ such that $p \in V_\delta$, $\gamma < \delta$, and with δ A_η-$<\theta$-strong for every $\eta < \gamma$: this is easy by coding all the A_η's into a single set and using the Woodin-ness of θ. We now have by Proposition that

$$\forall \eta < \gamma (A_\eta \cap V_\delta) \text{ is semi-proper in } P_\delta).$$

Now define

$$q =_{\mathrm{df}} \Big\{ X \prec V_{\delta+1} \mid X \text{ is small } \& \ X \cap \bigcup p \in p \ \& \ \gamma \in X \ \& \ \langle A_\eta \cap V_\delta \mid \eta < \gamma \rangle \in X \ \& \\ \& \ \forall \eta {:\in} X \cap \gamma \ \exists a {:\in} (A_\eta \cap V_\delta) \cap X \ \ X \cap \bigcup a \in a \Big\}$$

We assert that q *is stationary.*
Proof : Deny, and let $F : (\bigcup q)^{0<\omega} \to \bigcup q$ show that it is not. Pick a large and appropriate $\kappa > \theta$. Let M be a δ-small κ-model containing each of F, p, q, γ, δ, θ, $\langle A_\eta \mid \eta < \gamma\rangle$, and such that $M \cap \bigcup p \in p$: this last is possible as, by an argument we have seen before, the set of M's meeting the other conditions is club and p is stationary.

Let $S = M \cap \gamma$. With the second version of semi-proper in mind, we may readily define by induction on $\xi \in S$, an ascending sequence of δ-small κ-models M_ξ starting from M such that:

$$\xi \in S \implies M_\xi \ (A_\xi \cap V_\delta)\text{-improves } M \cup \bigcup_{\eta \in \xi \cap S} M_\eta.$$

Let $M^* = \bigcup_{\xi \in S} M_\xi$. By the neat extension condition,

$$\forall \xi {:\in} M^* \cap \gamma \ \exists a_\xi {:\in} M^* \cap V_\delta \cap A_\xi \ \ M^* \cap \bigcup a_\xi \in a_\xi.$$

So $M^{**} = M^* \cap V_{\delta+1} \in q$. But $F \in M^*$, $M^* \prec V_\kappa$, and so M^{**} is closed under F, and thus is not in q. This contradiction shows that no such F exists and therefore q is stationary, as required. ⊣

Now we define a function $h : q \to V$ by

$$h(X) = \Big\{ g^a_\eta (X \cap \bigcup a) \ \Big| \ \eta \in X \cap \gamma \ \& \ a \in X \cap A_\eta \cap V_\delta \ \& \ X \cap \bigcup a \in a \Big\}.$$

We now assert that

$$q \Vdash y\, E\, [\langle q,\, h \rangle] \longleftrightarrow \bigvee \eta :< \hat{\gamma}\, (y = \tau_\eta)$$

It is enough to prove the two following assertions:

$$\forall \eta :< \gamma \, \forall q_1 :\leq q \, \exists q_2 :\leq q_1 \ q_2 \Vdash \tau_\eta \, E \, h$$
$$\forall q_1 :\leq q \, [q_1 \Vdash z\, E\, h \implies \exists q_2 :\leq q_1 \, \exists \eta :< \gamma \ q_2 \Vdash z = t_\eta]$$

Proof of the first: let $\bar\eta < \gamma$ and let $q_1 \leq q$: by intersecting with a club of the form $\{x \mid \bar\eta \in x\}$ if necessary, we may assume that $\forall x {:\in} q_1 \ \bar\eta \in x$. Thus

$$\forall x {:\in} q_1 \ \exists a(x) {:\in} x \cap (A_{\bar\eta} \cap V_\delta) \ \ x \cap \bigcup a(x) \in a(x).$$

By the normality property ,

$$\exists \bar a \, \exists q_2 :\subseteq q_1 \Big[q_2 \text{ is stationary } \& \ \bigcup q_2 = \bigcup q_1 \ \& \ \forall y {:\in} q_2 \ \big[\bar a \in y \cap (A_{\bar\eta} \cap V_\delta) \ \& \ y \cap \bigcup \bar a \in \bar a\big]\Big]$$

This implies, that $(q_2)_{\cup \bar a} \subseteq \bar a$, and hence, as $\bigcup \bar a \subseteq V_\delta \subseteq \bigcup q \subseteq \bigcup q_1 = \bigcup q_2$, that $q_2 \leq a$. [Recall that $\bigcup q = V_{\delta+1}$, and that $q_2 \leq q_1 \leq q$.] As $\bar a \in A_{\bar\eta}$, $\bar a \Vdash g^{\bar a}_{\bar\eta} = \tau_{\bar\eta}$.

Define the functions $\bar g$, $\bar h$ on q_2 by $\bar h(y) = h(y \cap \bigcup q)$ and $\bar g(y) = g^{\bar a}_{\bar\eta}(y \cap \bigcup \bar a)$. We have only to check that $\forall y :\in q_2 \ \ \bar g(y) \in \bar h(y)$. But this is clear from the definition of h, since $y \cap \bigcup q \cap \bigcup \bar a = y \cap \bigcup \bar a \in \bar a$, $\bar\eta \in y \cap \bigcup q \cap \gamma$ as $\gamma < \delta$, and $\bar a \in y \cap \bigcup q \cap A_{\bar\eta} \cap V_\delta = y \cap A_{\bar\eta} \cap V_\delta$.

Proof of the second: suppose that $q_1 \leq q$ and $q_1 \Vdash z \, E \, h$, where we may assume that $\mathrm{Dom}(z) = q_1$. This means that $\forall y {:\in} q_1 \;\; z(y) \in h(y \cap \bigcup q)$, so

$$\forall y {:\in} q_1 \; \exists \eta(y) {:\in} y \;\cap \bigcup q \cap \gamma \; \exists a(y) {:\in} y \;\cap \bigcup q \cap A_\eta V_\delta \; y \cap \bigcup q \cap \bigcup a \in a \;\&\; z(y) = g^{a(y)}_{\eta(y)}(y \cap \bigcup q \cap \bigcup a).$$

By normality again (applied twice), there are $\bar\eta$ and $\bar a$ and a stationary $q_2 \leq q_1$ with $\bigcup q_2 = \bigcup q_1$ such that

$$\forall y {:\in} q_2 \;\; \bar\eta \in y \cap \bigcup q \cap \gamma \;\&\; \bar a \in y \cap \bigcup q \cap A_{\bar\eta} \cap V_\delta \;\&\; y \cap \bigcup q \cap \bigcup \bar a \in \bar a \;\&\; z(y) = g^{\bar a}_{\bar\eta}(y \cap \bigcup q \cap \bigcup \bar a)$$

Now $\bigcup \bar a \subseteq V_\delta \subseteq V_{\delta+1} = \bigcup q \subseteq \bigcup q_2$, so $\bigcup q \cap \bigcup \bar a = \bigcup \bar a$; hence tells us that $(q_2)_{\bigcup \bar a} \subseteq \bar a$, and hence that $q_2 \leq a$.

Since $q_2 \subseteq q_1$, we have only to check that for $y \in q_2$, $z(y) = g^{\bar a}_{\bar\eta}(y \cap \bigcup \bar a)$, but this is clear from above. $\dashv$

That proposition plus standard arguments establish all parts of the theorem except the last, that for $a \in V_\theta$,

$$\Vdash a \;\epsilon\; \dot G \longleftrightarrow j``\bigcup a \;\epsilon\; j(a).$$

To see that, let $a \in V_\theta$ be stationary. Let $i : a \to a$ be the identity function $i(\alpha) = \alpha$; and for any x, let $c_x : a \to \{x\}$ be the function with constant value x.

If $b \leq a$ and $b \Vdash \langle b, \, g\rangle \, E \, \langle a, \, i\rangle$, then $\forall \beta {:\in} b$, $g(\beta) \in \beta \cap \bigcup a$, so, by normality again, there is a condition $c \leq b$ (with $\bigcup c = \bigcup b$), and an $x \in \bigcup a$ such that $\forall \gamma \in c$, $g(\gamma) = x$, so that $[c, g \restriction c] = j(x)$. Conversely, for each $x \in \bigcup a$, $j(x) \, E \, [\langle a, \, i\rangle]$: thus $\langle a, \, i\rangle$ represents $j``\bigcup a$ in the model.

Now $j(a)$ is of course represented by $\langle a, \, c_a\rangle$, so

$$\begin{aligned} [\langle a, \, i\rangle] \, E \, [\langle a, \, c_a\rangle] &\Longleftrightarrow \{\alpha \in a \mid \alpha \in a\} \in G \\ &\Longleftrightarrow a \in G \end{aligned}$$

which completes the proof of the Theorem and ends our discussion of stationary tower forcing. The remainder of the chapter is devoted to further facts concerning extenders which will be required in Chapter 9 when we turn to the study of iteration trees.