CMU talk about NFU

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http://math.boisestate.edu/~holmes/holmes/nf.html

1 Definitions: mainly concerning type theory

We will have two set-theoretic languages in mind permanently. They are intimately related, and similarly expressive, but since both proofs I am going to give you rely on the relationship *between* these languages, you must be sure to keep them separate!

The language of set theory

The first is the language of set theory, with $\in$ and $=$, which you all know and love. NF and NFU are theories expressed in this language.

The language of simple typed set theory

The other is the language of simple type theory. (Readers for whom the expression 'type theory' connotes Church [1940] should be warned by this that we are here dealing with something much simpler!) It would be more correct to call it the language of simply typed set theory.

This language has a type for each nonnegative integer, an equality relation at each type, and between each pair of consecutive types n and $n+1$ a relation $\in_n$.

The axioms of simple typed set theory

The theory of simple types, TST, has the following axioms. An axiom of extensionality at each type

$$\forall x_{n+1} \forall y_{n+1} \ (x_{n+1} = y_{n+1} \longleftrightarrow \forall z_n (z_n \in_n x_{n+1} \longleftrightarrow z_n \in_n y_{n+1}))$$

and (at each type) an axiom scheme of comprehension

$$\forall \vec{x} \ \exists y_{n+1} \forall z_n \ (z_n \in_n y_{n+1} \longleftrightarrow \phi(\vec{x}, z_n))$$

with 'y_{n+1}' not free in 'ϕ'.

Note that these subscripts are an integral part of the variables they're attached to.

TSTU is like TST, except it allows for distinct nonempty sets. The 'U' is short of *urelemente*.

No AC or GCH. (Explain why we could add GCH)

The axioms of NF

I've told you the axioms of TST, but not the axioms of NF.. That's beco's i couldn't tell you the axioms of NF until after i had explained the language of simple typed set theory.

The axioms of NF are those formulae of the language of set theory which could become axioms of TST if we were to decorate them with type subscripts.

A formula of set theory is *stratified* iff by assigning type subscripts to its variables we can turn it into a wff (well-formed formula) of simple type theory. That is to say, a wff ϕ is stratified iff we can find a *stratification assignment* (henceforth "stratification" for short) for it, namely a map f from its variables (after relettering where appropriate) to $\mathbb{N}$ such that if the atomic wff '$x = y$' occurs in ϕ then $f(`x') = f(`y')$, and if '$x \in y$' occurs in ϕ then $f(`y') = f(`x') + 1$. Variables receiving the same integer in a stratification are said to be of the same *type*. If n successive integers are used, the formula is said to be *n-stratified*. There is a notion of a *canonical stratification* which assigns each variable the lowest possible type. A formula with one free variable, and that being assigned type n in the canonical stratification, is an *n-formula*. If Φ and Ψ are two closed stratified formulae, then we can assign integers to their variables independently, and so the canonical stratification for $\ulcorner \Psi \wedge \Phi \urcorner$

will be that function whose restrictions to the two sets of variables are the two canonical stratifications.

So the axioms of NF are extensionality plus all stratified instances of naive comprehension axioms. NFU is like NF but has extensionality for nonempty sets only, so it allows atoms. Neither of them have AC, GCH nor—obviously—foundation. $V \in V$ after all.

We can think of NF as that theory axiomatised by taking the axioms of TST and erasing the type subscripts. This is historically correct. But notice that it doesn't give us a consistency proof for NF!

2 Specker's equiconsistency lemma

So there is this syntactical relation between NF (NFU) and TST (TSTU). A great breakthrough in NF studies happened when Specker showed that this corresponds to a relation between models of NF and special kinds of models of TST (TSTU).

If Φ is an expression in the language of simple type theory let Φ^+ be the result of raising all type indices in Φ by 1, and let Φ^n be the result of doing this n times.

There is a corresponding operation on models: Suppose $\mathcal{M} \models TST$, or $TSTU$. Form a new model $\mathcal{M}^*$ by deleting the bottom type (type 0) from $\mathcal{M}$ and relabelling the old type 1 as type 0, with the others consequently. ϕ^+ is the result of raising all type indices in ϕ by 1, and ϕ^n is the result of raising all type indices in ϕ by n. So we have

$$\mathcal{M} \models \phi^+ \text{ iff } \mathcal{M}^* \models \phi$$

Now let Γ be a sensible class of closed formulæ, like a quantifier class, or formulæ using only two types or only three types or something like that, but at all events closed under this + operation. This last condition means that we have a sensible notion of an *ambiguity scheme for* Γ, which is $\{\Phi \longleftrightarrow \Phi^+ : \Phi \in \Gamma\}$, written "$Amb(\Gamma)$". Full ambiguity is the scheme of all $\Phi \longleftrightarrow \Phi^+$. A "tsau" (type-shifting automophism) of a model $\mathcal{M} \models TST$ is an isomorphism $\mathcal{M} \to M^*$. Thus, if σ is a tsau then, for all objects x_n of type n,[1] $\sigma(x_n)$ is an object of type $n+1$.

Theorem 1 Specker [1962]. *Given a model $\mathcal{M}$ of TST or TSTU plus full ambiguity, there is $\mathcal{M}'$ elementarily equivalent to $\mathcal{M}$ with a tsau σ.*

Proof come from general model-theoretic nonsense involving saturated models. ■

The quotient over σ

Now we see how to recover a one-sorted structure from a many-sorted structure.

[1] Strictly these maps are automorphisms only if $\mathcal{M}$ is a model of TNT.

If $\mathcal{M}$ is a structure for the language of TST, and σ is a type-shifting automorphism of $\mathcal{M}$, then we can construct a new structure with domain the set of things in $\mathcal{M}$ of type 0, equality in the sense of $\mathcal{M}$, and $x \in y$ iff $\mathcal{M} \models x \in \sigma(y)$.

This new structure is a structure for the one sorted language, the language of set theory, and if $\mathcal{M}$ was a model of TST (or TSTU) then the new structure is a model for NF or NFU.

This tells us that if TST (or TSTU) plus complete ambiguity is consistent, then so is *NF* (or *NFU*).

Corollary 2 Specker's Equiconsistency Lemma.
NF is equiconsistent with TST plus full ambiguity.

The converse is even easier, since if $\mathcal{M}$ is a model of *NF* we obtain a model of *TNT* plus complete ambiguity by making Z copies of it $(\langle M \times \{z\} : z \in \mathrm{Z}\rangle)$ and saying $\langle x, n\rangle$ is "in" $\langle y, n+1\rangle$ iff $\mathcal{M} \models x \in y$.

Nobody has yet produced a model of TST (with full extensionality) with complete ambiguity, but there is a construction of a model of TSTU with complete ambiguity, and therefore indirectly of a model of NFU, which is due to Jensen.

3 Jensen's Proof

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Now for Jensen's extracted models.

Let $\mathcal{M} = \langle M_i : i \in \mathbb{N} \rangle$ be a model of TST. For $I \subseteq \mathbb{N}$, let the *extracted model* $\mathcal{M}_I$ be $\langle M_i : i \in I \rangle$ with a new $\in$ relation. We say $x_{i_n} \in y_{i_{n+1}}$ iff y is a set of singletons$^{i_{n+1}-i_n-1}$ (otherwise y is an *urelement*) (The superscript is the number of times that ι is to be iterated) and $\iota^{i_{n+1}-i_n-1}(x)$ is a member of y in the sense of $\mathcal{M}$. We check that $\mathcal{M}_I$ is a model of TST with *urelemente*.

I claim that all models extracted in this way are $\mathcal{P}$-extensions of the model we start with. What is the $\mathcal{P}$-embedding in this case? It's defined by recursion in exactly the same way as in the FM case, the only complication being that the number of curly brackets that we have to wrap round $e``x$ at each stage depends on the number of types that have been discarded between the type of x and the type of its members. Otherwise it's exactly the same.

This extracted model will be a model of TSTU for the same reasons that the extracted model in the Fraenkel-Mostowski case gave us a model of ZF(C)U*.

Notice that a model obtained by extraction from a model that has itself been obtained by extraction can be obtained by a single extraction. So we can iterate extractions smoothly.

So we procede as follows. Enumerate the formulæ of $\mathcal{L}(TST)$ as $\langle \phi_i : i \in \mathbb{N} \rangle$. (For these purposes we disregard type suffices, so that ϕ and ϕ^+ are the same formula. Then, starting with a model $\mathcal{M}$ of TST, we obtain by repeated applications of Ramsey's theorem a succession of extracted models $\langle M_i : i \in \mathbb{N} \rangle$ where $\mathcal{M}_i \models$ ambiguity for all ϕ_j with $j \leq i$. We also have a family of $\mathcal{P}$-embeddings. (Must check that all these embeddings commute, associate etc, so that we really do have a directed system)

Now let $\mathcal{M} \models TSTU$, and let Φ be an arbitrary expression in $\mathcal{L}_{TST}$. Φ speaks of, say, five types. Let us partition $[\mathbb{N}]^5$. Let $I = \{i_1, \ldots, i_5\}$ and send $\{i_1, \ldots, i_5\}$ to 1 if $\mathcal{M}_I \models \Phi$ and to 0 otherwise. We now invoke Ramsey's theorem to find an infinite $J \subseteq \mathbb{N}$ homogeneous for this partition and consider $\mathcal{M}_J$. By homogeneity, either every model extracted from $\mathcal{M}_J \models \Phi$ or every model extracted from $\mathcal{M}_J \models \neg\Phi$, so certainly every model extracted from $\mathcal{M}_J$ satisfies $\Phi \longleftrightarrow \Phi^+$. We now repeat the process for a different

Φ. This shows that for any finite collection of formulæ $\langle \Phi_i : i \in I \rangle$, we can find an extracted model of $TSTU + \bigwedge_{i \in I} \Phi_i \longleftrightarrow \Phi_i^+$.

We now use compactness to get a model for complete ambiguity, general model theoretic nonsense to get a model with a tsau, and then the quotient is a model of NFU.

Jensen once told Adrian Mathias (his student and my Doktor-Vater) that this was his favourite among his theorems. Mind you, that was before fine structure theory!

This gives us an ω-sequence of extracted models $\langle \mathcal{M}_i : i < \omega \rangle$, with a commuting family of $\mathcal{P}$-embeddings $\mathcal{M}_i \hookrightarrow M_j$ when $i < j$. Now we consider what happens in the direct limit of this family of models. Clearly DL, the direct limit, will satisfy every $\Pi_2^{\mathcal{P}}$ formula true in arbitrarily late $\mathcal{M}_i$. The key idea now is that, as long as there is a universal set at each type, then every formula is $\Pi_2^{\mathcal{P}}$. For consider: let ϕ be any formula of $\mathcal{L}_{TST}$ whatever. Restrict every unrestricted quantifier at type n to a new variable v of the appropriate type, to obtain ϕ^v. Then consider the formula

$$(\forall \vec{v})(\exists z)(z \notin v \vee \phi^v)$$

It is $\Pi_2^{\mathcal{P}}$ and is equivalent to ϕ.

This sounds as if it ought to prove that the embeddings are all elementary! Train of tho'rt: any $\Pi_2^{\mathcal{P}}$ sentence true in cofinally many $\mathcal{M}_i$ must be true in the limit. But every sentence is $\Pi_2^{\mathcal{P}}$ so every sentence is eventually true or eventually false. This can't be true..... there must be something wrong

So what does this direct limit look like? It's a structure with ω types; it is an end-extension of an isomorphic copy of the $\mathcal{M}$ we started with; it doesn't have a universe at each level, for where would that element have come from?; what sort of ambiguity does it satisfy?

■

This proof has the advantage of showing that *NF*U is no stronger than TST without the axiom of infinity. It is possible to produce models of *NF*U directly and in bulk by cruder techniques.

4 Boffa's Proof

Proposition 3 *If $\langle V, \in \rangle$ is a model of Z (Zermelo set theory) with an (external) automorphism σ and an ordinal κ such that $\kappa > \sigma(\kappa)$, then V_κ is the domain of a model $\mathcal{M}$ of NFU whose membership relation ($\in_{\mathcal{M}}$) is*

$$\{\langle x, y \rangle : \sigma(x) \in y \wedge \rho(y) \leq \sigma(\kappa)\}$$

(ρ is set-theoretic rank)

Proof:

The true sets of $\mathcal{M}$ are the sets of rank $\leq \sigma(\kappa)$. Anything of rank $> \sigma(\kappa)$ will be an *urelement*, and $V_{\sigma(\kappa)}$ will be the universal set of $\mathcal{M}$.

To show that $\mathcal{M}$ satisfies all the comprehension axioms of *NFU*, it is sufficient to show that all of them translate into expressions without any occurrences of 'σ' in them, for then we can appeal to the comprehension scheme available internally in the model of Z. We want $\mathcal{M} \models \{z : \Phi(z, \vec{a})\}$ exists. We write out $\Phi(z, \vec{a})$ in full, with '$u \in_{\mathcal{M}} v$' replaced by '$\sigma(u) \in v \wedge \rho(v) \leq \sigma(\kappa)$'. Now we want to know that the set of all things of rank $< \kappa$ satisfying this rewritten version of Φ is indeed a set and is of rank at most $\sigma(\kappa)$,

We only have to prove this in the case where Φ is stratified. Fix some stratification of Φ. Any variable in a stratified formula has some type in a stratification. Since σ is an automorphism, we have

$$(\forall x)(\forall y)(\sigma(x) \in y \longleftrightarrow \sigma^{n+1}(x) \in \sigma^n(y))$$

for any n. Indeed—which is more to the point—we also have

$$(\forall x)(\forall y)[(\sigma(x) \in y \wedge \rho(y) \leq \sigma(\kappa)) \longleftrightarrow (\sigma^{n+1}(x) \in \sigma^n(y) \wedge \rho(\sigma^n(y)) \leq \sigma^{n+1}(\kappa))]$$

and by substitutivity of the biconditional we can ensure that every occurrence of any particular variable has the same prefix (σ^n for some n) in front of it, where the 'n' depends only of the type of that variable according to the stratification. Now we find that the comprehension axiom has quantified variables x such that every occurrence of 'x' has the same prefix σ^n. Therefore we can rewrite $\forall x \ldots$ this as $\forall x \in \sigma^n$"$V_\kappa \ldots$, and delete the prefix. But σ^n"V_κ is an object inside V and therefore is allowed to appear as a parameter in a separation axiom in the model of Zermelo that we are considering. The clause "$\rho(y) \leq \sigma(\kappa)$" in the definition of the $\in$-relation of the model gives rise to occurrences of ' $\sigma^n(\kappa)$ ' for various n but all these, too, are allowed to appear as parameters. Once we do this relettering for all variables, we have reduced the *NF*U comprehension axiom to a separation axiom containing some extra parameters. Of course if Φ had bound variables of type more than (1 + type of 'z') then in the process of applying sufficiently many 'σ' to our variables to ensure that the prefix is constant for any given variable, we may find that we have replaced $V_{\sigma(\kappa)}$ by $\sigma^n(V_{\sigma(\kappa))}$; but, since σ is an automorphism, the fact that the right subset of $\sigma^n(V_{\sigma(\kappa)})$ exists is enough for the right subset of $V_{\sigma(\kappa)}$ to exist. Finally we have to check that the collection whose existence is assured by internal separation is also coded in $\mathcal{M}$. The collection (let us call it 'PHI') contains things potentially of all ranks below κ but nothing of rank as great as κ and therefore is of rank at most κ. If b was some object such that $\Phi(b, \vec{a})$ we will have $b \in$ PHI, and therefore $\sigma(b) \in \sigma$(PHI). So σ(PHI) is the object which is—in the sense of $\mathcal{M}$—the set of all things that are Φ. Fortunately σ(PHI) is of rank $\sigma(\kappa)$ at most and is therefore not an urelement!

■

Models of Z of the kind needed in proposition 3 can be obtained by use of the Ehrenfeucht−Mostowski theorem (for example). There are two points that should be noted here. (i) Successfully verifying *NF*U-style comprehension in the new model depends on being able to eliminate occurrences of 'σ' in favour of parameters. This depends crucially on the formulae in question being stratified. (ii) Once we have eliminated the occurrences of 'σ' in favour of parameters, the comprehension axiom that we need to verify in the old model has the feature that all its bound variables are restricted to parameters. This means that we need only Δ_0-separation to hold in the original model, and that accordingly systems much weaker than Z will suffice.