Combinatorial Set Theory

Thomas Forster

In recent years there have been two alternating courses with titles like 'Set Theory'. This course will cover at least what is common to them (namely historical background, ordinal arithmetic, some basic large cardinal theory and some WQO theory) but at this stage it has not been decided whether to go for the (fairly) pure set theory option, (i) or the combinatorial option (ii), and altho' at present I am leaning towards (ii) this is a lent term course and there is time for interested parties to sway me one way or the other.

(i) would involve some of the following: model theory background (products, elementary embeddings, categoricity, saturation, Ehrenfeucht-Mostowski theorem); inner models: the consistency of ZF relative to intuitionistic ZF, of ZF + the axiom of foundation, of ZF + AC (relative to ZF). Infinitary combinatorics: Erdös-Rado, leading to measurable cardinals and other large cardinals. Set theory in Analysis (eg Souslin Hyp). Fränkel-Mostowski models. Positive set theory. Set theories of Church and Quine. Antifoundation axioms. Forcing, Borel determinacy.

(ii) Ever since Gödel's Incompleteness theorem it has been known that there are sentences in the languages of elementary arithmetic demonstrably not provable by elementary means, but the examples known were unnatural and never made explicit. In recent decades we have found some well-motivated examples with useful connections to areas outside logic. One such is Friedman's finite form of Kruskal's theorem, and a proper presentation involves BQOs, fast-growing functions, termination proofs, and some proof theory. This material (BQOs etc) is probably enough, when added to the core material alluded to two paragraphs above, to occupy a full 24 lectures.

As in previous years I shall be distributing printed material so that students will not need to wear their wrists out taking notes. The material will contain exercises. Last year's notes are still available on the web on `www.dpmms.cam.ac.uk/~tf/partiii2001.dvi`, and other course materials can be obtained from my home page.

Pre-requisite Mathematics

Either way, this course is a sequel to the IIB Logic Computation and Set theory lectures of previous years. I am going to assume that everybody doing it is on top of all the material lectured in that course. Revision supervisions will be provided for students who feel they need them (or did not attend the course in the first place) and a later version of my lecture notes for that course will be in print as "Logic, Induction and Sets" in CUP's LMS lecture notes series.

Literature

These two cover the part II background

P. T. Johnstone: Notes on Set theory CUP

T. E. Forster: Logic, induction and sets CUP

There is no text for option (ii), tho' I am trying to write one. There are lots of books on option (i). The following texts are recommended, in order of importance.

Drake and Singh *Intermediate Set Theory* Wiley 1996 is a paperback that may be of use.

T.J. Jech. Set theory

K. Kunen. Set theory

Dales and Woodin. An introduction to independence for analysts. LMS lecture notes **115**. (They mean: forcing independence proofs)

A.J. Dodd The Core model. LMS lecture notes **61**

P. Aczel *Non-well-founded sets*, Lecture Notes, Number 14, (Center for the Study of Language and Information, 1988). (This is the most widely-read text on the Forti-Honsell antifoundation axiom)

Jon Barwise and Laurence Moss *Vicious Circles* CLSI. 1996. ISBN.1-57586-009-0 (paper) 202pp. The book is much better than its title. A very good read.

M. Randall Holmes, *Elementary Set Theory with a Universal Set*, volume 10 of the Cahiers du Centre de logique, Academia, Louvain-la-Neuve (Belgium), 241 pages, ISBN 2-87209-488-1.

Forster, T.E. Set Theory with a Universal Set. OUP 1994

Sidney Smith's Part III essay: "Hypersets" at `http://ucsu.colorado.edu/~bsid/logic/papers/Smith` This exposition of set theory with antifoundation axioms will certainly be at the right level!

Consult the lecturer for a more up-to-date reading list nearer the time.