CHAPTER VI: Getting One Woodin Cardinal from Δ^1_2 Determinacy

OUR TASK IN THIS CHAPTER is to refine the argument of the previous one. On the one hand we shall show that the existence of an inner model of $ZFC+$ "there exists a Woodin cardinal "follows from a much weaker form of Determinacy:

0·0 THEOREM *Assume that $V = L[x]$ for some real x and that Δ^1_2 Determinacy holds. Then Θ is Woodin in HOD.*

DEFINITION It will be convenient to write DD for (light-face) Δ^1_2 determinacy, and if S is a class of ordinals, to write AD^-_S for $\forall p :\in \mathcal{N}\ Det(OD_S(p))$. A lot of the time, it seems that $Det(OD_S)$ plus the existence of the sharp of every real suffices. But we'll check that at the end. The point is that AD^-_S is compatible with the Axiom of Choice, but is equivalent to AD if $V = L(S, \mathbb{R})$.

0·1 METACOROLLARY *If $ZFC + DD$ is consistent, so is ZFC + "there is a Woodin cardinal ".*

On the other hand we shall show that under AD there are very many transitive models of that theory:

0·2 THEOREM *Assume AD, and that S is a class of ordinals. Then there is a real y such that whenever y is recursive in x, it is true in $L[x;S]$ that Θ is Woodin in HOD_S.*

The first half of the chapter is devoted to developing and applying the concept of a *pre-strategy* which goes back to a paper of Solovay and Kechris.[R 1]. We shall introduce a game in which the moves are essentially countable ordinals or real numbers; we call this game the S-game, and we prove the following, from which the above theorems may be derived.

0·3 THEOREM *Let S be a class of ordinals. Suppose that there is a real x such that $V = L[x;S]$, that $Det(OD_S)$ holds, that the second player wins the S-game and that* **either** $S \subseteq \Theta$ **or** $2^{2^{\aleph_0}} = (2^{\aleph_0})^+$. *Then Θ is Woodin in HOD_S.*

The latter half of the Chapter is devoted to the proof of that theorem; the proof consists of adapting the arguments of Chapter V to constructing extenders in HOD.

[R 1] on OD determinacy, TAMS somewhere

1: Schemes of games and pre-strategies

Let $\mathcal{I}$ be a Turing-closed set of reals. [N 1]

1·0 DEFINITION An *$\mathcal{I}$-scheme of games* is a system $\mathfrak{S} = \big\langle \langle M_x \mid x \in \mathcal{I}\rangle, \langle A_x \mid x \in \mathcal{I}\rangle \big\rangle$ where for each $x \in \mathcal{I}$,

SG 1: $M_x \subseteq \mathcal{N}$

SG 2: $x \in M_x$

SG 3: $\forall a, b{:}{\in} M_x \; \forall c{:}{\in} \mathcal{N} \; \big(c \leqslant_{\text{Turing}} a \oplus b \implies c \in M_x\big)$

SG 4: $A_x \subseteq \mathcal{N}$

The above contains all that is formally necessary, but the underlying intuition is that $M_x = \mathcal{N} \cap \mathfrak{M}_x$ for some transitive model $\mathfrak{M}_x$ of a (possibly weak) set theory, for which we might have $A_x \cap M_x \in \mathfrak{M}_x$. The following two formulæ express without reference to $\mathfrak{M}_x$ the intuitive idea of there existing in $\mathfrak{M}_x$ a winning strategy for either player in the game $A_x \cap M_x$.

1·1 DEFINITION $\mathit{Astrat}(\mathfrak{S}, x) \iff_{\text{df}} \exists \sigma{:}{\in} M_x \; \forall \varepsilon{:}{\in} M_x \;\; \mathrm{oc}(\sigma, [\varepsilon]) \in A_x$

1·2 DEFINITION $\mathit{Estrat}(\mathfrak{S}, x) \iff_{\text{df}} \exists \tau{:}{\in} M_x \; \forall \alpha{:}{\in} M_x \;\; \mathrm{oc}([\alpha], \tau) \notin A_x$.

Note that by *SG 3* the outcomes $\mathrm{oc}(\sigma, [\varepsilon])$ and $\mathrm{oc}([\alpha], \tau)$ will be in M_x.

What, roughly, is going to happen is given a scheme we can define a global game part of which produces an x which defines a local game played in a (small) model $\mathfrak{M}_x$, with the local pay-off set for the first player being $A_x \cap M_x$. In fact we shall look at two kinds of global game.

Our examples of schemes are, broadly, of two kinds. Instances of the first will be repeatedly used in this section and the next. For the definition of the second, instances of which will recur in the following three sections, we shall use a concept introduced by Toulmin:[R 2]

1·3 DEFINITION An ordinal ζ is *indecomposable* if $\nu < \zeta \,\&\, \mu < \zeta \implies \nu + \mu < \zeta$.

Such ordinals have the following agreeable property:

1·4 PROPOSITION *Let S be a class of ordinals, ζ indecomposable and x a real: then*

$$\forall y{:}{\in} \mathcal{N} \; \big[x \leqslant_{\text{Turing}} y \in J_\zeta[x; S] \implies J_\zeta[x; S] = J_\zeta[y; S]\big].$$

Proof : see Appendix Two . The proof relies on the fact that the sequences $\langle S_\nu[z; S] \mid \nu < \omega\zeta\rangle$ and $\langle J_\nu[z; S] \mid \nu < \zeta\rangle$ are uniformly Σ_1^S over $J_\zeta[x; S]$. ⊣ (1·4)

1·5 EXAMPLE Let S be a class of ordinals; let $\Phi(A; S)$ be some set-theoretic formula in the single free variable A, distinct from the variable x and intended to range over sets of reals, in which S is treated as a unary predicate of ordinals. We emphasize that the variable x is to have no occurrence in Φ.

Let

$$\mathcal{I}_{\text{cls}}(\Phi; S) \;=_{\text{df}}\; \Big\{x \in \mathcal{N} \mid \big[\exists A{:}{\subseteq} \mathcal{N} \; \big(A \text{ is } \mathrm{OD}_S \;\&\; \Phi(A; S)\big)\big]^{L[x;S]}\Big\}.$$

That is plainly Turing closed.

For $x \in \mathcal{I}$, let $\mathfrak{M}_x$ be the inner model $L[x; S]$, $M_x = \mathcal{N} \cap \mathfrak{M}_x$ and A_x be in $\mathfrak{M}_x$ the $<_{OD(S)}$-first set of reals with property Φ. Write $\mathfrak{S}_{\text{cls}}(\Phi; S)$ for this scheme.

In this example, $x \in \mathcal{I} \;\&\; x \leqslant_{\text{Turing}} y \in L[x; S] \implies L[x; S] = L(y; S]$, and so $y \in \mathcal{I}$, $M_x = M_y$ and $A_x = A_y$; moreover, A_x is always a subset of M_x.

In the present section and the next we shall concentrate on examples of the above type. But in the study of DD and the proof of Theorem 0·0, examples of a different type will be important: in this second type, the models $\mathfrak{M}_x$ might well be countable transitive sets, typically of the form $J_\xi[x; S]$ where all that is required of ξ is that it be indecomposable.

We shall be interested as before in such models in which in some sense there is an ordinal-definable failure of some principle; as these models will satisfy only a weak set theory, ordinal definability in full generality may not be conveniently definable in them; but, these models being sets, we may express the

[N 1] $\oplus$ is recursive union; $\mathcal{C}_\chi \;=_{\text{df}}\; \{x \mid \chi \leqslant_{\text{Turing}} x\}$.

[R 2] G: Toulmin,

existence of an ordinal definable failure by quantifying over formulae of the appropriate object language; this quantification is therefore outside the model under discussion. We also quantify over ordinals of the model, which is not problematical; we do not bother to quantify over finite sequences of ordinals of the model;

More generally, we may quantify in this way over classes of reals definable in the model from S, a class of ordinals.

In Example 1·5, Φ could be an arbitrary set-theoretical formula; here we must face the possibility that A might not be a set of our model, and treat instead those Φ in which A appears as a class variable, rather than as a set. The cases we have in mind are these: Φ says that the game A has no winning strategy: which is a statement of analysis in which A only occurs as a class; and Φ says that there is no pre-strategy (a concept still waiting to be introduced) for the game A. $\dot\Phi$ is the translation of Φ into an object language. [Should it be $\mathit{\Phi}$?]

Again it is important that (a name for) x should have no occurrence in $\dot\Phi$.

1·6 Example Let

$$\mathcal{I}_{\rm set}(\Phi; S) =_{\rm df} \{x \in \mathcal{N} \mid \exists\zeta\ \zeta \text{ is indecomposable and } \exists\varphi\exists\eta{:}{<}\omega\zeta\ J_\zeta[x;S] \models \dot\Phi(\hat{\mathfrak{x}}\varphi(\mathfrak{x}, \underline{\eta}; \dot S); \dot S)\}.$$

Here $\dot S$ is a unary predicate interpreted in $J_\zeta[x;S]$ as membership of $S \cap \omega\zeta$, $\underline{\eta}$ is a constant interpreted as η, and $\hat{\mathfrak{x}}\varphi$ is the formal expression denoting the class of those elements of the model which the model believes to have the property φ.

For $x \in \mathcal{I}_{\rm set}(\Phi; S)$, let $\zeta(x)$ be the least such ζ, and let $\mathfrak{M}_x = J_{\zeta(x)}[x;S]$. Let $M_x = \mathcal{N} \cap \mathfrak{M}_x$; let φ_x and η_x be the first pair of such φ and η, in some reasonable well-ordering (independent of x); and A_x be the set of those reals a in M_x for which $\mathfrak{M}_x \models \varphi_x[a, \eta_x]$. Loosely we may think of A_x as the $<_{OD(S)}$-first $\mathfrak{M}$-class of reals with property Φ. Write $\mathfrak{S}_{\rm set}(\Phi; S)$ for this scheme.

Again we shall have $x \leqslant_{\rm Turing} y \in M_x \implies y \in \mathcal{I}$.

Given a $\mathcal{I}$-scheme $\mathfrak{S}$, we shall be particularly interested in reals with the following properties:

1·7 Definition

$$\begin{aligned} \mathfrak{s}_1(x, \mathfrak{S}) &\iff_{\rm df} x \in \mathcal{I}\ \&\ \forall y{:}{\in}\mathcal{N}\ (x \leqslant_{\rm Turing} y \in M_x \implies y \in \mathcal{I}) \\ \mathfrak{s}_2(x, \mathfrak{S}) &\iff_{\rm df} x \in \mathcal{I}\ \&\ \forall y{:}{\in}\mathcal{N}\ (x \leqslant_{\rm Turing} y \in M_x \implies A_y = A_x) \\ \mathfrak{s}(x, \mathfrak{S}) &\iff_{\rm df} \mathfrak{s}_1(x, \mathfrak{S})\ \&\ \mathfrak{s}_2(x, \mathfrak{S}). \end{aligned}$$

We shall say x is $\mathfrak{S}$-*smooth* if $\mathfrak{s}(x, \mathfrak{S})$. Smooth reals play an important role in our discussion.

For the next few exercises fix some S and Φ as above.

1·8 Exercise Show that every x in $\mathcal{I}_{\rm cls}$ is $\mathfrak{S}_{\rm cls}$-smooth.

1·9 Exercise Show that if $x \in \mathcal{I}_{\rm set}$ then

$$J_{\zeta(x)}[y;S] = J_{\zeta(x)}[x;S] \implies y \in \mathcal{I}_{\rm set}\ \&\ \zeta(y) \le \zeta(x).$$

1·10 Exercise Infer that $\forall x{:}{\in}\mathcal{I}_{\rm set}\ \mathfrak{s}_1(x, \mathfrak{S}_{\rm set})$.

1·11 Exercise Let $\mathcal{B}$ be a set of reals such that $\mathcal{B} \cap \mathcal{I}_{\rm set} \neq \varnothing$ and $z \geqslant_{\rm Turing} y \in \mathcal{B} \implies z \in \mathcal{B}$. Among the members of $\mathcal{B} \cap \mathcal{I}_{\rm set}$, let x be one with $\zeta(x)$ minimal. Show that $x \leqslant_{\rm Turing} y \in M_x \implies M_x = M_y$. Deduce that A_x is a subset of M_x and that x is $\mathfrak{S}_{\rm set}$-smooth.

The importance of smoothness lies in the study of two games associated with a $\mathcal{I}$-scheme $\mathfrak{S}$; in one of these a real parameter p will occur. We shall call the two games $\mathcal{G}(\mathcal{I}, \mathfrak{S})$ and $\mathcal{H}(\mathcal{I}, \mathfrak{S}, p)$.

The game $\mathcal{G}(\mathcal{I}, \mathfrak{S})$.

Given a $\mathcal{I}$-scheme $\mathfrak{S}$, we define a game $\mathcal{G}(\mathcal{I}, \mathfrak{S})$:

$$\text{Adam} \begin{cases} a \\ b \end{cases} \qquad \left.\begin{matrix} d \\ e \end{matrix}\right\} \text{Eve}$$

PACE: we specify that in round n Adam plays $(a(n),\, b(n))$, and then Eve plays $(d(n),\, e(n))$. [C 1]

We put c for the coded quartet $\langle a, b, d, e\rangle$.

Rule $\mathcal{GIS}$1: $c \in \mathcal{I}$, otherwise Adam loses

Rule $\mathcal{GIS}$2: if $\mathrm{ocl}(a, e) \in A_c$, Adam wins.

1·12 Proposition *Let $\mathfrak{S}$ be a $\mathcal{I}$-scheme of games. From a winning strategy ψ for either player in the game $\mathcal{G}(\mathcal{I}, \mathfrak{S})$, and an $\mathfrak{S}$-smooth real x with $\psi \leqslant_{\text{Turing}} x$ we may construct a winning strategy for the corresponding player for the game $A_x \cap \mathfrak{M}_x$ played in $\mathfrak{M}_x$. More formally:*

(i) if σ is a winning strategy for Adam in the game $\mathcal{G}(\mathcal{I}, \mathfrak{S})$ and $d \in \mathcal{I} \cap \mathcal{C}_\sigma$ is $\mathfrak{S}$-smooth, then $Astrat(\mathfrak{S}, d)$;

(ii) if τ is a winning strategy for Eve in the game $\mathcal{G}(\mathcal{I}, \mathfrak{S})$, and $b \in \mathcal{I} \cap \mathcal{C}_\tau$ is $\mathfrak{S}$-smooth then $Estrat(\mathfrak{S}, b)$.

Proof: Given d, σ as in (i), so that $\sigma \leqslant_{\text{Turing}} d$, let $e \in M_d$, and let a and b be Adam's response using σ; then $c = \langle a, b, d, e\rangle \in M_d$; $d \leqslant_{\text{Turing}} c$ so by property $\mathfrak{s}_1(d)$, $c \in \mathcal{I}$; Adam has won this run of $\mathcal{G}$, so $\mathrm{ocl}(a, e) \in A_c$. $A_c = A_d$, by $\mathfrak{s}_2(d)$ since $d \leqslant_{\text{Turing}} c \in M_d$.

Similarly let b be as in (ii), and a an arbitrary member of M_b; let Adam play a and b against τ, with d, e being Eve's response. Again $c = \langle a, b, d, e\rangle \in M_b$ so as $\mathfrak{s}_1(b)$, $c \in \mathcal{I}$. Rule 1 has been kept; Eve has won this run of $\mathcal{G}$, and so $\mathrm{ocl}(a, e) \notin A_c$. As $\mathfrak{s}_2(b)$, $A_b = A_c$. ⊣ (1·12)

As a notation for these derived strategies, we write $a = (\sigma \otimes_I d) \star [e]$ in the first case, and $e = [a] \star (\tau \otimes_{II} b)$ in the second. Here is an application:

1·13 Proposition *Let S be a class of ordinals. Suppose that $Det(OD_S)$ holds. Then*

$$\exists g \forall x \mathbin{:\geqslant_{\text{Turing}}} g \big(Det(OD_S)\big)^{L[x;S]}.$$

Proof: let $\Phi(A; S)$ say that A is not determined. Put $\mathcal{I}_{\text{cls}} = \mathcal{I}_{\text{cls}}(\Phi; S)$, $\mathfrak{S}_{\text{cls}} = \mathfrak{S}_{\text{cls}}(\Phi; S)$. By Exercise 1·8, every $x \in \mathcal{I}_{\text{cls}}$ is $\mathfrak{S}_{\text{cls}}$-smooth. Consider the game $\mathcal{G}(\mathcal{I}_{\text{cls}}, \mathfrak{S}_{\text{cls}})$: it is OD_S, and so determined. By Proposition 1·12, if Adam has a winning strategy σ then $\mathcal{I}_{\text{cls}} \cap \mathcal{C}_\sigma = \varnothing$; if Eve has a winning strategy τ, $\mathcal{I}_{\text{cls}} \cap \mathcal{C}_\tau = \varnothing$; in either case $\mathcal{I}_{\text{cls}}$ is disjoint from a cone, and hence its complement, which equals $\{x \mid (Det(OD_S))^{L[x;S]}\}$, contains one. ⊣ (1·13)

1·14 Corollary (AD^-_S) *Let S be a class of ordinals. Then*

$$\forall p \exists g \mathbin{:\geqslant_{\text{Turing}}} p \forall x \mathbin{:\in} \mathcal{N} \Big[g \in L[x;S] \implies \big(Det(OD(p;S))\big)^{L[x;S]}\Big]$$

We leave the proof as an exercise.

Note that $\mathrm{Det}(OD)$ is not strong enough for this result: suppose $V = L[c]$ and models DD; then $\mathrm{Det}(OD_c)$ is false on a cone.

[C 1] There is some room for flexibility here, but I know no use for it.

The game $\mathcal{H}(\mathcal{I}, \mathfrak{S}, p)$

We have seen in the above how in certain circumstances a strategy for a "global" game yields in a continuous fashion strategies for every, or almost every, game in a scheme.

Consider the following variant of our former game, which we call $\mathcal{H}(\mathcal{I}, \mathfrak{S}, p)$, where p is a real. The basic idea is that Adam picks a side in a game to be played and then tries to win.

Let $p \in \mathcal{N}$ and let $\mathcal{I}$ be a (Turing-closed) set of reals.

$$\text{Adam} \begin{cases} \chi = \pm 1 & \\ \quad a & \\ \quad b & \end{cases} \qquad \left.\begin{matrix} d \\ e \end{matrix}\right\} \text{Eve}$$

PACE: Adam's first move is to be $\chi, a(0), b(0)$, and thereafter he plays $(a(n), b(n))$; Eve at each move plays $(d(n), e(n))$.

Put $c = \langle \chi, a, b, d, e \rangle$. Eve is to win if any of the following three rules is broken:

Rule $\mathcal{HIS}1$: $(b)^0 = p$;

Rule $\mathcal{HI}\mathfrak{S}2$: $c \in \mathcal{I}$;

Rule $\mathcal{HIS}3$: **either** $\chi = +1$ and $\mathrm{ocl}(a, e) \in A_c$ **or** $\chi = -1$ and $\mathrm{ocl}(e, a^\star) \notin A_c$.

We shall examine the consequences of this game being determined, on the supposition that $\mathcal{I}$ meets every Turing cone, with the consequent existence of sufficiently many $\mathfrak{S}$-smooth reals.

1·15 PROPOSITION *If Eve has a winning strategy τ in the game $\mathcal{H}(\mathcal{I}, \mathfrak{S}, p)$, then for $\mathfrak{S}$-smooth $x \in \mathcal{C}_{\tau \oplus p}$, $A_x \cap M_x$ is not determined in $\mathfrak{M}_x$: formally, neither $Astrat(\mathfrak{S}, x)$ nor $Estrat(\mathfrak{S}, x)$.*

Proof : if $\pi \in M_x$ were a winning strategy for the first player in the game $A_x \cap M_x$ played in $\mathfrak{M}_x$, then the following would defeat τ

$$\text{Adam} \begin{cases} \chi = +1 \\ a = \pi \star [e] \\ b = (p, x) \end{cases} \qquad \left.\begin{matrix} d \\ e \end{matrix}\right\} \text{Eve, using } \tau$$

For put $c = \langle +1, a, b, d, e \rangle$; then $x \leqslant_{\text{Turing}} c \leqslant_{\text{Turing}} x \oplus \pi \in M_x$; $c \in \mathcal{I}$ by $\mathfrak{s}_1(x)$ and so by $\mathfrak{s}_2(x)$, $A_c = A_x$. Adam has observed Rules 1 and 2, and has won under Rule 3, since $\varepsilon = +1$ and $\mathrm{ocl}(a, e) \in A_x = A_c$.

On the other hand, if $\rho \in M_x$ were a winning strategy for the second player in the game $A_x \cap M_x$ played in $\mathfrak{M}_x$, Adam can defeat τ again by playing

$$\text{Adam} \begin{cases} \chi = -1 \\ a(0) = 0 \\ a^\star = [e] \star \rho \\ b = (p, x) \end{cases} \qquad \left.\begin{matrix} d \\ e \end{matrix}\right\} \text{Eve, using } \tau$$

Put $c = \langle -1, a, b, d, e \rangle$; then $x \leqslant_{\text{Turing}} c \leqslant_{\text{Turing}} x \oplus \rho \in M_x$, and so $c \in \mathcal{I}$ and $A_c = A_x$. Again, Adam has observed Rules 1 and 2, and since $\mathrm{ocl}(e, a^\star) \notin A_x$, he has won under Rule 3. $\dashv$ (1·15)

We shall generally use the contrapositive of that: in applications we shall know that $A_c \cap M_c$ is usually determined in $\mathfrak{M}_c$ and conclude that Eve has no winning strategy in the main game.

Now let us look at the consequence of Adam having a winning strategy ψ: the concept of a *pre-strategy* emerges from this proof, the definition of which will be given at the end of our discussion.

We shall write χ_ψ for the value of χ given by ψ: note that χ_ψ is independent of any play by the opponent since this information is available from Adam's first move, before any play by Eve.

1·16 PROPOSITION *Suppose that Adam has a winning strategy, ψ in $\mathcal{H}(\mathcal{I}, \mathfrak{S}, p)$. Then*

i) p is recursive in ψ;

ii) if $\chi_\psi = +1$, then for each $\mathfrak{S}$-smooth $x \in \mathcal{I} \cap \mathcal{C}_\psi$, there is a winning strategy in M_x for the first player in the game $A_x \cap M_x$ played in $\mathfrak{M}_x$; formally, $Astrat(\mathfrak{S}, x)$;

iii) if on the other hand $\chi_\psi = -1$, then for each such x, there is a winning strategy for the second player in the game $A_x \cap M_x$ played in $\mathfrak{M}_x$; formally, $Estrat(\mathfrak{S}, x)$;

iv) these winning strategies are uniformly recursive in ψ and x.

Proof : Note first that $p \leqslant_{\text{Turing}} \psi$: for let Eve play $d = e = \vec{0}$, the real $\lambda n.0$ with $\forall n\, e(n) = 0$. Then $p \leqslant_{\text{Turing}} b \leqslant_{\text{Turing}} \text{ocl}(\psi, \vec{0})$.

Suppose now that $\chi_\psi = +1$, let $x \in \mathcal{I} \cap \mathcal{C}_\psi$ be smooth, and let e be any real in M_x. Consider the following run of the game:

$$\text{Adam, using } \psi\text{: } \left\{\begin{array}{ll} \varepsilon = +1 & \\ \quad a & d = x \\ & \quad e \\ \quad b & \end{array}\right\}\text{: Eve}$$

The referee (us) puts $c = \langle +1, a, b, x, e\rangle$. Then $x \leqslant_{\text{Turing}} c \in M_x$, and so $c \in \mathcal{I}$, and $A_c = A_x$. Since Adam has won, $\text{ocl}(a, e) \in A_c = A_x$.

We again introduce a notation for the derived strategy: we write $a = (\psi \otimes x) \star [e]$: the strategy $\psi \otimes x$ is recursive in x (since ψ is), for in a loose but suggestive notation, for finite sequences s, $(\psi \otimes x)(s) = \psi(x \restriction \ell h(s), s)$; and hence $\psi \otimes x$ is in M_x. Since $\psi \leqslant_{\text{Turing}} x$, $a \in M_x$ by *SG3.*

Turning to the case $\chi_\psi = -1$, we may again build a strategy for the second player in $A_x \cap M_x$ played in M_x by considering for any real e in M_x, the play:

$$\text{Adam, using } \psi\text{: } \left\{\begin{array}{ll} \chi = -1 & \\ \quad a & d = x \\ & \quad e \\ \quad b & \end{array}\right\}\text{: Eve}$$

As before, putting $c = \langle -1, a, b, x, e\rangle$, $x \leqslant_{\text{Turing}} c \in M_x$ and so $c \in \mathcal{I}$ and $A_c = A_x$. This time $\text{ocl}(e, a^\star) \notin A_c = A_x$, and so the second player PERSISTENTLY wins by playing $a^\star$ which we may write as $[e] \star (\psi \otimes x)$. As before $\psi \otimes x \in M_x$, and $a^\star \in M_x$ by *SG3.* $\dashv$ (1·16)

There is no ambiguity in the two uses of $\psi \otimes x$ for the first definition applies when $\chi_\psi = +1$ and the second applies when it is -1. We may combine the two results by saying that for $x \geqslant_{\text{Turing}} \psi$ and smooth, $\psi \otimes x$ is *a winning strategy in M_x for the indicated player* for the game $A_x \cap M_x$ played in M_x.

Now consider the particular case of $\mathfrak{S}_{\text{cls}}(\Phi; S)$ for some Φ, as defined in Example 1·5: in this case, if x is smooth and $y \in M_x$ is such that $L[y; S] = L[x; S]$, then putting $y' = (\psi \oplus y)$, we have $\mathfrak{M}_{y'} = \mathfrak{M}_x$ and hence $A_{y'} = A_x$; but also $\psi \leqslant_{\text{Turing}} y'$; and so $(\psi \otimes (\psi \oplus y))$ is a winning strategy for the indicated player for A_x in $\mathfrak{M}_x$.

We may now formulate our first definition of a *pre-strategy.*

1·17 DEFINITION We shall write χ_ψ for the first bit, $I'LL\_GO\_FIRST$ or $I'LL\_GO\_SECOND$, which is independent of the input from the opponent; and denote the rest of the input given d, e by $a_\psi(d, e)$ and $b_\psi(d, e)$.

$$\begin{aligned} WIP(\psi, x; A) \iff_{\text{df}} & \chi_\psi = I'LL\_GO\_FIRST \;\&\; \forall e \,{:}{\in} \mathcal{N}\ \text{ocl}(a_\psi(x, e), e) \in A \\ & OR\ \ \chi_\psi = I'LL\_GO\_SECOND \;\&\; \forall e \,{:}{\in} \mathcal{N}\ \text{ocl}(e, (a_\psi(x, e))^\star) \notin A. \end{aligned}$$

$$PS_{\text{cls}}(\psi, A; S) \iff_{\text{df}} \forall y \,{:}{\in} \mathcal{N}\ \big(V = L[y; S] \implies WIP(\psi, \psi \oplus y; A)\big).$$

Thus

$$\begin{aligned} PS_{\text{cls}}(\psi, A; S) \iff_{\text{df}} \forall y \,{:}{\in} \mathcal{N}\ (V = L[y; S] \implies & \psi \otimes (\psi \oplus y) \\ & \text{is a winning strategy for the indicated player in } A). \end{aligned}$$

In terms of this definition, we have proved the following

1·18 PROPOSITION *Let S be a class of ordinals, and p a real, and $\Phi(A; S)$ some predicate of S and a set of reals A.*

Suppose that Adam has a winning strategy, ψ in $\mathcal{H}(\mathcal{I}, \mathfrak{S}, p)$. Then

i) p is recursive in ψ;
ii) if $x \geqslant_{\rm Turing} \psi$ is in $\mathcal{I}_{\rm cls}$, then $(PS_{\rm cls}(\psi, A_x; S))^{L[x;S]}$;
iii) $\forall z {:\in} \mathcal{I}_{\rm cls} \bigl(\psi \in L[z;S] \implies (PS_{\rm cls}(\psi, A_z; S))^{L[z;S]}\bigr)$.

Proof : to see part ii), remember that every $x \in \mathcal{I}_{\rm cls}(\mathfrak{S};\Phi)$ is smooth; for part iii) take $x = \psi \oplus z$ and apply part ii), bearing in mind that $L[x;S] = L[z;S]$ and $A_z = A_x$. ⊣ (1·18)

1·19 Definition We call such a ψ a *pre-strategy* for the scheme of games $\mathfrak{S}_{\rm cls}(\Phi;S)$.

As an application we have the following result:

1·20 Proposition (AD^-_S) *Let S be a class of ordinals and p a real. Then*

$$\exists b\ p \leqslant_{\rm Turing} b\ \&\ \forall x \geqslant_{\rm Turing} b \bigl[\forall A {:\in} OD_S\ \exists\psi {:\geqslant_{\rm Turing}} p\ \bigl(PS_{\rm cls}(\psi, A; S)\ \&\ Det(OD(\psi;S))\bigr)\bigr]^{L[x;S]}.$$

Proof : Take $\Phi(A,S)$ to say that A is determined but $\neg\exists\psi\bigl(PS_{\rm cls}(\psi,A;S)\ \&\ p \leqslant_{\rm Turing} \psi\ \&\ Det(OD(S,\psi))\bigr)$, put $\mathcal{I} = \mathcal{I}_{\rm cls}(\Phi;S)$, and $\mathfrak{S} = \mathfrak{S}_{\rm cls}(\Phi;S)$.

We must show that $\mathcal{I}$ is disjoint from a cone. As every $x \in \mathcal{I}$ is $\mathfrak{S}$-smooth, Proposition 1·15 shows that if Eve has the winning strategy τ in the game $\mathcal{H}(\mathcal{I},\mathfrak{S},p)$, $\mathcal{I}$ must be disjoint from the intersection of $\mathcal{C}_\tau$ and a cone on which $\mathrm{Det}(OD(S))^{L[S,x]}$ holds, which will exist by Prop 1.14 or 1.15.

Suppose therefore that ψ is a winning strategy for Adam in the game $\mathcal{H}(\mathcal{I},\mathfrak{S},p)$. By Corollary 1·13, there is an a such that $Det(OD(S,\psi)$ holds in $L[x;S]$ for $x \geqslant_{\rm Turing} a$. Take b to be $a \oplus \psi$.

Then $\mathcal{I} \cap \mathcal{C}_b = \varnothing$ by Proposition 1·18.

⊣ (1·20)

Remark We shall call such ψ a *good pre-strategy.* We shall in later sections refine this concept. "good" refers to the fact that games definable from ψ are determined; "pre-strategy" has just been defined.

Definition $GPS_{\rm cls}(\psi,A;S) \iff_{\rm df} PS_{\rm cls}(\psi,A;S)\ \&\ Det(OD(S,\psi))$.

Problem GPS is intended to be used in an $L[x;S]$ context, so the definition to be given of OD_S might well be of $h(L_\varsigma(\mathfrak{R})$ type.

$\lessdot$ 1·21 Remark A comment on the importance of establishing results such as 1·14 and 1·20 for the case that $\sup S > \Theta$. In chapter VII, when we come to build models with infinitely many Woodin cardinals, we shall wish to consider models N of the form $L[A, S_0, y]$ where y is a real, A is a subset of an ordinal countable in V and S_0 is the canonical subset of Θ with $L[S_0] = HOD$, and shall want it to be true that Θ^N is Woodin in HOD^N_{A,S_0}; Θ^N will be often be quite small, say $(\omega_2)^N$ and very much less than $\sup S_0$ which will be strongly inaccessible in N.

Residual truth

Given a class S of ordinals S, we know from $\mathrm{Det}(OD_S)$ that every statement is true in $L[S,x]$ or false for a cone of x's, or, as we shall say, is *residually S-true* or *residually S-false.*

In some cases we may determine which:

1·22 Proposition (AD^-_S) *CH is residually S-true.*

That will follow from

1·23 Proposition (AD^-_S) *$\diamondsuit$ is residually S-true.*

Proof : We rely on the fact that the true ω_1 is strongly inaccessible[C 2] in every inner model of AC of the form $L[p;S]$. Given x, let δ be the first uncountable ordinal of $L[S,x]$. Let D be a Cohen subset of δ generic over $L[S,x]$: conditions here are countable subsets of δ: it is well known that in $L[S,x][D]$, $\diamondsuit$ holds. Now do almost-disjoint forcing over $L[S,x][D]$ to add a real y coding D over $L[S,x]$, and use the fact that $\diamondsuit$ cannot be destroyed by a c.c.c. forcing of size $\aleph_1$. ⊣ (1·23)

Now we see that once we are in the cone on which CH is true, GCH will hold up to the true ω_1:

[C 2] actually, Mahlo, a fact that will be used in Chapter VII. Indeed I think Ω is measurable in HOD_S though it might not be in $L[S]$. What does this imply for the $HOD[x] = HOD_x$ principle ?

1·24 PROPOSITION *Suppose the true ω_1 is strongly inaccessible in every inner model of AC. Suppose x is a real such that whenever $x \leq_T y$, CH is true in $L[S; y]$. Let $x \leq_T y$, and κ a cardinal of $L[S; y]$ that is countable in V. Then $2^\kappa = \kappa^+$ holds in $L[S; y]$.*

Proof : otherwise add a real z to $L[S; y]$ which codes a generic for $\mathrm{Coll}(\omega, \kappa)$. That forcing is of size κ, so κ^+ and κ^{++} are both preserved. Hence $L[S; y \oplus z]$ contains an injection of its ω_2 into its continuum, contrary to CH being true in the cone above x. ⊣ (1·24)

1·25 PROPOSITION (AD_S^-) *Let S be a class of ordinals. Then $Det(OD(S))$ is residually S-true.*

1·26 COROLLARY (AD_S^-) *For each S and each real p, $Det(OD(S, p))$ is residually true.*

Proof : replace S by $\langle S, p\rangle$ in the above argument. ⊣

Our eventual goal is to show that "Θ `is Woodin in` HOD_S" is residually true.

2: The S-game

Let S be a class of ordinals. We shall play a game, called the S-game, in each $L[x; S]$. This game is such that the first player's moves are (capable of being construed as) countable ordinals; the second player's moves are reals; the game, from the point of view of the first player, is open; it is therefore determined, and one of the players has a canonical winning policy, either the rank-reducing one or the "stay-on-top" one. If that player is the first player, his possible moves are canonically well-ordered and hence he will have a canonical winning strategy, which, when it exists, we shall denote by σ_x. This strategy will be to play the first ordinal that reduces the rank of the current position.

Our aim in this section is to prove the following

2·0 THEOREM (AD_S^-) *The set*

$$\{x \mid \text{ in } L[x; S] \text{ the second player has a winning policy in the } S\text{-game}\}$$

contains a Turing cone.

Definition of the S-game

$$\text{Adam: }\left\{\begin{array}{ll} A_0 & \\ A_1 & \psi_0 \\ A_2 & \psi_1 \\ \vdots & \vdots \end{array}\right\}\text{: Eve}$$

Rules:

Rule SG 0: A_0 is an $\mathrm{OD}(S)$ determined subset of $\mathcal{N}$;
Rule SG 1: $GPS_{\mathrm{cls}}(\psi_0, A_0; S)$ (that is, if $V = L[y; S]$, then $\psi \otimes (\psi \oplus y)$ is a winning strategy for a named player in A_0 and further $Det(OD(\psi_0; S))$);
Rule SG 2: A_1 is an $\mathrm{OD}(\psi_0; S)$ determined subset of $\mathcal{N}$;
Rule SG 3: $GPS_{\mathrm{cls}}(\psi_1, A_1; S)$ and $\psi_1 \geqslant_{\mathrm{Turing}} \psi_0$;
Rule SG 4: A_2 is an $\mathrm{OD}(\psi_1; S)$ determined subset of $\mathcal{N}$;
Rule SG 5: $GPS_{\mathrm{cls}}(\psi_2, A_2; S)$ and $\psi_2 \geqslant_{\mathrm{Turing}} \psi_1$;
Rule SG 6: *and so on*

The first person to break a rule loses; if all the rules are kept, Eve wins.

Proof of 2·0: if false, then for the given S, $\{x \mid$ the **first** player has a winning strategy in the S-game$\}$ will contain a Turing cone, say $\mathcal{C}_{\bar{a}}$. For each $x \geqslant_{\mathrm{Turing}} \bar{a}$, we write σ_x for the canonical winning strategy in the

S-game played in $L[x;S]$. We set $c_{-1} = \bar{a}$, and for each $k \in \omega$ we shall define a "universal move" $\bar{\psi}_k$ for Eve and a real c_k with $c_{k-1} \leqslant_{\text{Turing}} \bar{\psi}_k \leqslant_{\text{Turing}} c_k$, so that we shall have

$$\bar{a} = c_{-1} \leqslant_{\text{Turing}} \bar{\psi}_0 \leqslant_{\text{Turing}} c_0 \leqslant_{\text{Turing}} \bar{\psi}_1 \leqslant_{\text{Turing}} c_1 \leqslant_{\text{Turing}} \bar{\psi}_2 \ldots,$$

and further, and most importantly, for all $k \in \omega$ the following statement UM_k will hold:

$\text{UM}_k(\bar{\psi}_0, \ldots, \bar{\psi}_k; c_k)$: for any x with $c_k \leqslant_{\text{Turing}} x$, if in $L[x;S]$ Eve plays $\bar{\psi}_0, \ldots \bar{\psi}_k$ against σ_x, she will not have lost by the end of round k.

Note that the existence of such a sequence at once yields the contradiction that proves the theorem: for if all c_i are recursive in x, then in $L[x;S]$, Eve can defeat σ_x by playing the entire sequence of universal moves.

We define the universal moves inductively: suppose that we already have $\bar{\psi}_0$, …, $\bar{\psi}_{k-1}$, c_{k-1} with UM_{k-1} true of $\bar{\psi}_0, \ldots \bar{\psi}_{k-1}, c_{k-1}$. We must choose $\bar{\psi}_k$ and c_k.

Let $\mathcal{I}^k = \mathcal{C}_{c_{k-1}}$, where (in the case $k = 0$) we have $c_{-1} = \bar{a}$, and let $\mathfrak{M}^k_x = L[x;S]$, $M^k_x = \mathcal{N} \cap \mathfrak{M}^k_x$ and A^k_x to be in $\mathfrak{M}^k_x$ the game named by the strategy σ_x in round k of the S-game when Eve has played $\bar{\psi}_0$, …, $\bar{\psi}_{k-1}$ in the first k rounds (that is, in rounds 0, …, $k-1$.)

In $\mathfrak{M}^k_x$, therefore, this game will therefore be determined, and in the class $OD(S; \bar{\psi}_0, \bar{\psi}_1, \ldots, \bar{\psi}_{k-1})$ if $k > 0$, and in the class $OD(S)$ if $k = 0$.

Let $\mathfrak{S}^k = \big\langle \langle M^k_x \mid x \in \mathcal{I}^k \rangle, \langle A^k_x \mid x \in \mathcal{I}^k \rangle \big\rangle$, and consider the game $\mathcal{H}(\mathcal{I}^k, \mathfrak{S}^k, c_{k-1})$. By Proposition 1·19, Eve has no winning strategy, and hence let $\bar{\psi}_k$ be a winning strategy for Adam. Then $c_{k-1} \leqslant_{\text{Turing}} \bar{\psi}_k$, and for $x \geqslant_{\text{Turing}} \bar{\psi}_k$, $PS_{\text{cls}}(\psi_k, A^k_x)$ holds in $L[x;S]$.

Choose $c_k \geqslant_{\text{Turing}} \bar{\psi}_k$ with $\forall x{:}\geqslant_{\text{Turing}} c_k L[x;S] \models Det(OD(S, \bar{\psi}_0, \ldots, \bar{\psi}_k))$. c_k works.

⊣ (2·0)

2·1 Remark We shall see later that under DD plus $V = L[x]$ plus $S = \varnothing$, Eve has an "easy" win.

3: Bootstrapping Determinacy

Our aim in this section is to derive counterparts to the earlier results using DD rather than AD.

Kechris and Solovay made the remarkable discovery that Σ^1_2 determinacy implies OD determinacy, provided the universe is constructible from some real. That followed a result of Martin that Δ^1_2 determinacy implies Σ^1_2 determinacy in ZF.

We begin with an examination of these results. First some elementary observations about Δ^1_2 determinacy.

3·0 Proposition *If Δ^1_2 determinacy holds, then there is a real x such that in $L[x]$, Δ^1_2 determinacy holds.*

Proof : We make heavy use of Shoenfield's theorem, discussed in Chapter II, concerning the absoluteness of Σ^1_2 predicates.

There are only countably many Δ^1_2 sets. A difficulty is that for a set to be Δ^1_2, a Σ^1_2 set and a Π^1_2 set must coincide: this property if true stays true on moving to a smaller model, but in that smaller model more coincidences might occur. Thus we should choose x so that (i) if A is Σ^1_2 and B is Π^1_2 and in V, $A \neq B$, then there is a $y \leq_T x$ with $y \in A \triangle B$ (ii) if, in V, $A = B$, then there is recursive in x a strategy for the game A.

This x works: for suppose in $L[x]$, $A = B$. Then in V, $A = B$, and so there is a winning strategy, ρ, say, for the game A, with $\rho \in L[x]$. If ρ is a strategy for Adam, then $\forall e\; \rho * [e] \in B$: this is a Π^1_2 statement about ρ and therefore is true in $L[x]$. If ρ is a strategy for Eve, $\forall a\; [a] * \rho \notin A$: this is again Π^1_2 and so true in $L[x]$. ⊣

3·1 Remark We could also handle Π^1_2 games won by Adam or Σ^1_2 games won by Eve.

3·2 Proposition *If M is an inner model of V, and Δ^1_2 determinacy is true in M, then it is true in V.*

Proof : Let A be Σ^1_2, B Π^1_2, and suppose $A = B$ in V. Then in M, $A = B$; for if not, let $a \in M$ with $a \in A \triangle B$: say $a \in A$ and $a \notin B$: then in V, $a \in A$ and $a \notin B$.

Hence there is a strategy, ρ say for the game A in M. We finish the proof as in the previous proposition. ⊣

3·3 Proposition (Martin) *$Det(\Delta^1_2)$ implies $Det(\Sigma^1_2)$*

Let A be a Σ^1_2 set. We suppose that Eve has no winning strategy for the game A, and, using Δ^1_2 determinacy build one for Adam.

The statement, Φ, that Eve has no winning strategy is Π^1_3: it says that $\forall\tau\exists\alpha\,[\alpha]*\tau\in A$. Hence Φ will by Shoenfield hold in every inner model; in particular in each $L[x]$, x a real; and thus by reflection within each $L[x]$, we conclude that for every real x there is a least indecomposable ordinal, $\zeta(x)$, such that

$$J_{\zeta(x)}[x]\models\dot\Phi\wedge\texttt{every well-ordering is isomorphic to an ordinal}.$$

We set $\mathcal{I}=\mathcal{N}$, and for each real x, put $\mathfrak{M}_x=J_{\zeta(x)}[x]$, $M_x=\mathcal{N}\cap\mathfrak{M}_x$ and $A_x=A^{\mathfrak{M}_x}$. Note that $A_x\subseteq A$, but that we do not know that $A_x=A\cap M_x$. Let $\mathfrak{S}=\big\langle\langle M_x\mid x\in\mathcal{I}\rangle,\langle A_x\mid x\in\mathcal{I}\rangle\big\rangle$, and consider the game $\mathcal{G}(\mathcal{I},\mathfrak{S})$.

This game is Δ^1_2 and hence determined.

Were τ a winning strategy for Eve, then for x in $\mathcal{C}_\tau$ with $\zeta(x)$ minimal, x will be $\mathfrak{S}$-smooth and hence A_x would be determined in Eve's favour in $\mathfrak{M}_x$, contrary to our choice of $\zeta(x)$.

Hence Adam has a winning strategy, σ, say. But then for each $e\in\mathcal{N}$, *play* $((\sigma\otimes_I\sigma)*[e],\,e)\in A_{c(\sigma,e)}\subseteq A$, so that $\sigma\otimes_I\sigma$ is a winning strategy for Adam in A. $\dashv$ (3·3)

4: Reduction of OD games to Σ^1_2 ones

Let S be a class of ordinals, and let $\Phi(A;S)$ say that A is not determined.

Consider the scheme $\mathfrak{S}=\mathfrak{S}_{\rm set}(\Phi;S)$, defined for each x in the class $\mathcal{I}=\mathcal{I}_{\rm set}(\Phi;S)$. CHECK ITS DEFINITION.

Exercises 1·7 – 1·9 show, in effect, that if $\mathcal{I}$ meets every cone, then

$$\mathcal{B}\ =_{\rm df}\ \{b\in\mathcal{I}\mid\forall y{:}{\in}\mathcal{N}\ b\leqslant_{\rm Turing}y\in M_b\implies A_b=A_y\}$$

meets every cone.

Assume $Det(OD_S)$: then $\mathcal{I}$, being OD_S, either contains or is disjoint from a cone. If it contains a cone, then Proposition 1·12 yields a contradiction, for if τ is a winning strategy for Eve, then $b\in\mathcal{B}\cap\mathcal{C}_\tau\implies A_b$ is determined in M_b; and similarly for Adam. Thus $\mathcal{I}$ is disjoint from a cone, and we have proved the following

4·0 PROPOSITION *Let S be a class of ordinals, and assume $Det(OD_S)$. Then there is a real g such that*

$$\forall y{:}{\in}\mathcal{C}_g\ \forall\ \textit{indecomposable}\ \zeta\ J_\zeta[y;S]\models\texttt{"Det(OD}_S\texttt{-C)"}.$$

Here $J_\zeta[y;S]\models\texttt{"Det(OD}_S\texttt{-C)"}$ is shorthand for $\forall\varphi\forall\eta:<\omega\zeta\ \ J_\zeta[x;S]\models\neg\dot\Phi(\hat{\mathfrak{x}}\varphi(\mathfrak{x},\underline{\eta};\dot S);\dot S)\}$, for the particular Φ used in this section; expressing the notion that all OD_S classes of reals are determined.

Such a g will be said to *guarantee* the truth of $Det(OD_S)$, or to be an *S-guarantor*. Note that if g is an S-guarantor and $g\leqslant_{\rm Turing}h$ then h is also an S-guarantor.

Consider now the special case that S is a subset of ω coding the (graph of the) real p: then $\mathcal{I}$ is a $\Sigma^1_2(p)$ pointclass; thus we have proved

4·1 COROLLARY *If $Det(\Sigma^1_2(p))$ then there is a real g such that $p\leqslant_{\rm Turing}g$ and*

$$\forall y{:}{\in}\mathcal{C}_g\ \forall\ \textit{indecomposable}\ \zeta\ J_\zeta(y)\models\texttt{"Det(OD(}p\texttt{)-C)"}.$$

Again we should stress that y is not to be mentioned; in this context $J_\zeta(y)=J_\zeta((y,\,p))$.

4·2 DEFINITION Let $p\in\mathcal{N}$: g is a *p-guarantor* if $p\leqslant_{\rm Turing}g$ and $\forall y{:}{\in}\mathcal{C}_g\ \forall$ indecomposable ζ

$$J_\zeta(y)\models\texttt{"Det(OD(}p\texttt{)-C)"}.$$

4·3 REMARK Note that we have taken the convenient step of modifying the definition of S-guarantor to make p recursive in its guarantors.

4·4 EXERCISE Show that the relation "g is a p-guarantor" is a Π^1_2 predicate of g and p.

4·5 EXERCISE Show that if g is an S-guarantor, then

$$g \in J_\zeta[y;S] \implies J_\zeta[y;S] \models \texttt{"Det(OD(}S\texttt{)-C)"}.$$

4·6 PROPOSITION *If g is an S-guarantor then whenever $g \leqslant_{\rm Turing} x$, $(Det(OD_S))^{L[x;S]}$ holds.*

Proof : in that model, ON is indecomposable, so if $Det(OD_S)$ failed then by reflection to some $J_\zeta[x;S]$, with ζ indecomposable, we would have a contradiction. ⊣ (4·6)

Now consider the case $S = \varnothing$: Corollary 4·1 tells us that if $\mathrm{Det}(\Sigma^1_2)$ then a 0-guarantor exists, and therefore on a cone of x's $Det(OD)$ holds in $L[x]$; Proposition 3·3 reduces the determinacy hypothesis to DD; if $\exists y {:\in} \mathcal{N}\ V = L[y]$ then on a cone of x's $L[x] = V$; and so we have proved the

4·7 THEOREM (Kechris–Solovay) *Suppose that $\exists y {:\in} \mathcal{N}\ V = L[y]$. If DD then $Det(OD)$.*

5: Good prestrategies

We wish to imitate our previous result that in the right context, if $Det(OD(p))$ holds then for each OD_p game A there is a pre-strategy ψ for which $Det(OD(\psi))$ holds. Our proof using AD relied on the fact that for any real a, on a cone of x's $Det(OD(a;S))$ would be true in $L[a;S]$. We have just seen a similar result, proved starting from DD, and using the concept of a guarantor.

In adapting this result to the DD context, a difficulty will be the problem of knowing that the model $\mathfrak{M}_x$ correctly recognises ψ-guarantors; the property of being a ψ-guarantor being Π^1_2, and these models being often countable, misinterpretations of Π^1_2 statements might arise.

5·0 DEFINITION In a transitive model M, set or class, we shall say that a real z *is of top S-degree* if $(V = L[z;S])^M$. When $S = \varnothing$, we shall simply say that z is *of top degree* in M.

5·1 DEFINITION We denote by `SKP` the system of Kripke–Platek strengthened by the addition of the axiom of Σ_1-separation, as discussed in APPENDIX ONE.

5·2 REMARK `SKP` is finitely axiomatisable and is sufficiently strong to prove that every well-ordering is isomorphic to an ordinal.

5·3 PROPOSITION *Let x, g, ψ be reals, and ξ an ordinal. Let $M = J_\xi(x)$ where x is a ψ-guarantor, and suppose that $M \models$ `SKP`, that $g, \psi \in M$, and that g is of top degree in M. Then ξ will be indecomposable and*

$$g \text{ is a } \psi\text{-guarantor} \iff (g \text{ is a } \psi\text{-guarantor})^M.$$

Proof : The forward implication is immediate as M will be absolute for Π^1_1 statements about reals of M, and the property of being a guarantor is Π^1_2 and hence downwards absolute.

Now for the converse: let ζ be an indecomposable ordinal. $x \in J_\xi(g)$, so if $\xi \le \zeta$, then for any y with $g \in J_\zeta(y)$, $J_\xi(g) \subseteq J_\zeta(g)$, and so $x \in J_\zeta(y)$. As x is a ψ-guarantor, we shall have $J_\zeta(y) \models \texttt{Det(OD(}\psi\texttt{)-C)}$.

So suppose that $\zeta < \xi$, and let g be recursive in y. We distinguish two cases.

Suppose that ζ is uncountable in M. Since $x \in J_\xi(g) = M$, x will be, by Gödel condensation provable in `SKP`, in $J_\Omega(g)$, where Ω denotes the first ordinal of M; and therefore $x \in J_\zeta(g) \subseteq J_\zeta(y)$; but x is a true ψ-guarantor and therefore $J_\zeta(y) \models \texttt{Det(OD(}\psi\texttt{)-C)}$.

If on the other hand ζ is countable in M, then let $c \in M$ be a real that codes ζ. By hypothesis $M \models \hat{g}\ \texttt{is a}\ \hat{\psi}\texttt{-guarantor}$, and is absolute for Π^1_1 statements about reals in M. In particular, the statement that $\forall y {:\in} \mathcal{N}\ g \in J_\zeta(y) \implies J_\zeta(y) \models Det(OD_\psi)$ is a Π^1_1 statement about c, g and ψ which is true in M; it is therefore true in V. ⊣ (5·3)

Our modification of the property of being a *good pre-strategy* is the following:

5·4 DEFINITION $GPS_{\rm set}(\psi, A) \iff_{\rm df}$

$$Det(FD_{\mathcal{N}}(\psi))\ \&\ \forall y \big[V = L[y]\ \&\ y \text{ is a } \psi\text{-guarantor} \implies WIP(\psi, \psi \oplus y; A) \big].$$

Our aim now is to prove the following

5·5 THEOREM *Let p be a real, and suppose that $Det(OD(p))$ holds. Then*

$$\exists a{:}{\in}\mathcal{N}\ \forall y{:}{\in}\mathcal{C}_a\ \big[\forall A{:}{\in}\mathcal{P}(\mathcal{N})\cap OD(p)\ \exists\psi\ p\leqslant_{\text{Turing}}\psi\ \&\ GPS_{\text{set}}(\psi,A)\big]^{L[y]}.$$

REMARK The assumption is a little too strong; what we need is $\Sigma^1_2(\psi)$ determinacy for ψ Δ^1_3 in p.
Proof : let $\Phi(A,p) \iff_{\text{df}}$ `SKP` $\&\ \neg\exists\psi\,[p\leqslant_{\text{Turing}}\psi\ \&\ GPS_{\text{set}}(\psi,A)]$.

Let $\mathcal{I}=\mathcal{I}_{\text{set}}(\Phi;p)$, and $\mathfrak{S}=\mathfrak{S}_{\text{set}}(\Phi;p)$, and consider the game $\mathcal{H}(\mathcal{I},\mathfrak{S},p)$. Note that $\mathcal{I}$ is $\textsc{od}(p)$, and so will be decided by a cone.

If τ is a winning strategy for Eve in this game then, by Proposition 1·15, $\mathcal{I}$ is disjoint from a cone.

Leave that on one side for the moment and now suppose that Adam has a winning strategy in the game $\mathcal{H}$. As the game $\mathcal{H}$ is $\Sigma^1_2(p)$ the results of Section 7 of Chapter III tell us that he has a winning strategy ψ that is $\Delta^1_3(p)$; hence $Det(OD(\psi))$ holds, and there are ψ-guarantors.

Let $\mathcal{B}=\{x\mid x\text{ is a }\psi\text{-guarantor}\}$. $\mathcal{B}$ is upwards closed, so, assuming that $\mathcal{I}$ contains a cone, let $x\in\mathcal{B}\cap\mathcal{I}$ with $\zeta(x)$ minimal. Then x is $\mathfrak{S}$-smooth. We assert that $\mathfrak{M}_x\models GPS_{\text{set}}(\psi,A_x)$.

Proposition 1·16 tells us that $p\leqslant_{\text{Turing}}\psi$; as $\psi\leqslant_{\text{Turing}}x\in M_x$, $\psi\in M_x$ and $p\in M_x$. Now let $y\in M_x$ be of top degree and a ψ-guarantor in $\mathfrak{M}_x$. By Proposition 5·6, y is a ψ-guarantor in V. Hence $y\in\mathcal{B}$. Put $y'=\psi\oplus y$: so y' is in $\mathcal{B}$. By Exercise 1·10 $y'\in\mathcal{I}$, and $\zeta(x)=\zeta(y')$; hence $\mathfrak{M}_x=\mathfrak{M}_{y'}$ and $A_x=A_{y'}$. As $\psi\leqslant_{\text{Turing}}y'$, we know that $\psi\otimes y'$ is a winning strategy for the indicated player in $\mathfrak{M}_{y'}=\mathfrak{M}_x$ for the game $A_{y'}=A_x$. Thus in $\mathfrak{M}_x$, $PS_{\text{set}}(\psi,A_x)$ holds.[C 3] Since $x\in\mathcal{B}$, $Det(OD(\psi))$ holds in $\mathfrak{M}_x$ too.[N 2] We have now checked all the clauses of GPS_{set} and proved that they hold in $\mathfrak{M}_x$.

But from the definition of Φ and $\mathcal{I}$, that is a manifest contradiction: hence $\mathcal{B}\cap\mathcal{I}$ is empty.

So whether Adam or Eve wins the game $\mathcal{H}$, $\mathcal{I}$ has been shown to be disjoint from a cone. Hence its complement contains one, and so

$$\exists a\forall y{:}{\in}\mathcal{C}_a\ \forall\text{ indecomposable }\zeta\ J_\zeta(y)\models\bigwedge A\big(A\dot{\subseteq}\dot{\mathcal{N}}\wedge A\ \epsilon\ OD(p)\ \longrightarrow\ \neg\dot{\Phi}(A,p)\big).$$

By reflection, therefore, we have

$$\exists a\forall y{:}{\in}\mathcal{C}_a\ \Big(\forall A\big(A\subseteq\mathcal{N}\ \&\ A\in OD(p)\implies\neg\Phi(A,p)\big)\Big)^{L[y]},$$

from which a trivial argument using the definition of Φ yields the theorem. ⊣ (5·5)

REMARK We have to check that the “class” form reflects to something contradicting the negation of the “set” form.

5·6 COROLLARY *If $\exists x{:}{\in}\mathcal{N}\ V=L[x]$, then*

$$\forall p{:}{\in}\mathcal{N}\ Det(OD(p))\implies\forall A{:}{\in}\mathcal{P}(\mathcal{N})\cap OD(p)\ \exists\psi\ GPS_{\text{set}}(\psi,A,p).$$

It is thus of interest to modify the definition of the S-game by replacing PS_{cls} by GPS_{set} throughout its rules.

Definition of the modified S-game

$$\text{Adam:}\left\{\begin{array}{cc} A_0 & \\ & \psi_0 \\ A_1 & \\ & \psi_1 \\ A_2 & \\ & \vdots \\ \vdots & \end{array}\right\}:\ \text{Eve}$$

Rules:

[C 3] **Where is PS$_{set}$ defined ?**
[N 2] More exactly, $\mathfrak{M}_x\models$ `"Det(OD`$_\psi$`-C)"`.

Rule MSG 0: A_0 is an OD determined subset of $\mathcal{N}$;
Rule MSG 1: $GPS_{\text{set}}(\psi_0, A_0)$ (that is, $Det(OD(\psi_0)$, and if $V = L[y]$ and y is a ψ_0-guarantor then $\psi \otimes (\psi \oplus y)$ is a winning strategy for a named player in A_0);
Rule MSG 2: A_1 is an OD(ψ_0) determined subset of $\mathcal{N}$;
Rule MSG 3: $GPS_{\text{set}}(\psi_1, A_1)$ and $\psi_1 \geqslant_{\text{Turing}} \psi_0$;
Rule MSG 4: A_2 is an OD(ψ_1) determined subset of $\mathcal{N}$;
Rule MSG 5: $GPS_{\text{set}}(\psi_2, A_2)$ and $\psi_2 \geqslant_{\text{Turing}} \psi_1$;
Rule MSG 6: and so on

The first person to break a rule loses; if all the rules are kept, Eve wins.

With the above modification, we have the following:

5·7 THEOREM *If $\exists x{:}{\in}\mathcal{N}\ V = L[x]$ and DD and $S = \varnothing$ then Eve wins the modified S-game.*

Proof : Corollary 5·10 shows that she will never lack for moves. ⊣ (5·7)

5·8 DEFINITION If need be, we shall allude to the two versions of the S-game as the *class S-game* and the *set S-game* respectively.

6: Interlude: stability of theories

REMARK Note that in the scheme defined in this section M_x seems not to be Turing invariant. We must check the notation: $\mathfrak{M}$, $\mathfrak{N}$, M N seem to be used indiscriminately.

We continue to write DD for light-face Δ^1_2 determinacy.

6·0 DEFINITION We write $\mathcal{A}$ for the class of sets of reals of the form

$$\{\gamma \in \mathcal{N} \mid J_\kappa(\gamma) \models \varphi(\gamma, \zeta)\}$$

where $\zeta < \kappa$ are ordinals and φ is a set-theoretic formula with two free variables.

6·1 THEOREM (Kechris, Woodin) *DD implies $Det(\mathcal{A})$.*

By Shoenfield, this implies, of course, Martin's earlier result that LD implies $Det(\Sigma^1_2)$; but covers much more. The conclusion is a special case of the theorem of Kechris and Solovay, but from a weaker hypothesis.

Proof of 6·1: deny. Then for each real x there is by Lévy reflection a transitive set $\mathfrak{N}$ with $x \in \mathfrak{N}$ and $\mathfrak{N} \models \mathtt{T} + \neg\mathtt{Det}(\mathcal{A})$, where $\mathtt{T}$ is an appropriate subsystem of ZF; we may assume by Löwenheim–Skolem that $\mathfrak{N}$ is countable, and by Shoenfield that $\mathfrak{N} \in L[x]$. $\mathtt{T}$ need not be very strong: KP + "$\mathcal{N}$ is a set" will suffice, [N 3] for from that it will follow that for a transitive model $\mathfrak{M}$ of $\mathtt{T}$ and an ordinal $\kappa \in M$, each class of reals of the form $\{x \mid J_\kappa(x) \models \varphi(x, \zeta)\}$ is a set, and for γ a real in $\mathfrak{M}$, $(J_\kappa[\gamma])^M = J_\kappa[\gamma]$. Furthermore such $\mathfrak{M}$ will be closed under Turing reducibility and recursive join.

Define $\mathfrak{N}_x$ to be the $<_{L[x]}$-first $\mathfrak{N}$ as above in $L[x]$.

$\mathcal{A}$-games are identified by the triple $\langle \kappa, \varphi, \zeta \rangle$, which in turn may be coded as a single ordinal: we shall in this context speak of the *triple code* of the given game. We write η_x for the least triple code of an $\mathcal{A}$-game $\mathfrak{N}_x$ considers to be undetermined.

We now define

$$\xi(x) \ =_{\text{df}}\ \inf\{\eta_y \mid y \leqslant_{\text{Turing}} x \in \mathfrak{N}_y\}.$$

The set over which that infimum is taken is non-empty, since x is in it. Note that $\xi(x)$ depends only on the Turing degree of x.

With a view to applying 1·12, we define a scheme of games. We take $\mathcal{I}$ to be $\mathcal{N}$. For each x, let $y(x)$ be the real y recursive in x with least possible index such that $\eta_y = \xi(x)$ and $y \leqslant_{\text{Turing}} x \in N_y$, and define $\mathfrak{M}_x = \mathfrak{N}_{y(x)}$ and $M_x = \mathcal{N} \cap \mathfrak{N}_{y(x)}$: certainly $x \in M_x$. Finally, define A_x to be the game specified by $\xi(x)$, namely $\{\gamma \in \mathcal{N} \mid J_\kappa[\gamma] \models \phi(\gamma, \zeta)\}$. By our choice of $\mathtt{T}$, $A_x \cap M_x \in \mathfrak{M}_x$ and $A_x \cap M_x = (A_x)^{\mathfrak{M}_x}$.

[N 3] Indeed if we used the Solovay-Kechris device for quantifying over OD classes of reals, we could avoid assuming that $\mathcal{N}$ is a set; but the effort hardly seems worth it.

6·2 LEMMA $x \leqslant_{\text{Turing}} w \in M_x \implies \xi(w) \le \xi(x)$

Proof : $\xi(x) = \eta_{y(x)}$; $y(x) \leqslant_{\text{Turing}} x \in \mathfrak{N}_{y(x)}$, so if $w \in N_{y(x)}$ and $x \leqslant_{\text{Turing}} w$ then $y(x) \leqslant_{\text{Turing}} w$; and so $\xi(w) \leqslant \eta_{y(x)} = \xi(x)$. ⊣ (6·2)

6·3 LEMMA *Let $\mathcal{B}$ be an upwards Turing closed set of reals and let $x \in \mathcal{B}$ have $\xi(x)$ minimal. Then $x \leqslant_{\text{Turing}} w \in M_x \implies A_x = A_w$*

Proof : $\xi(w) \le \xi(x)$ by 6·2. $w \in \mathcal{B}$ since x is and $\mathcal{B}$ is upwards Turing closed. So by minimality of $\xi(x)$, $\xi(x) \le \xi(w)$. Hence $\xi(x) = \xi(w)$, and therefore $A_x = A_w$. ⊣

Write $\mathfrak{S}$ for the scheme we have defined. The game $\mathcal{G}(\mathcal{N}, \mathfrak{S})$ is Δ^1_2, on our assumption that for each $x \in \mathcal{N}$ a model N_x exists; hence this game is determined; let ψ be a winning strategy for the lucky player. By 6·3, taking $\mathcal{B} = \mathcal{C}_\psi$, an $\mathfrak{S}$-smooth real x exists with $\psi \leqslant_{\text{Turing}} x$; and by 1·12, the same player will have a winning strategy for the game $A_x \cap \mathfrak{M}_x$ played in $\mathfrak{M}_x$, manifestly contradicting our choice of these games and the absoluteness of their definition. ⊣ (6·3)

7: The grand plan and the settings

7·0 DEFINITION For S a set or class of ordinals, write $\mathfrak{C}(S)$ for the set $\{x \subseteq \omega \mid V = L[x, S]\}$. Note that $\mathfrak{C}(S)$ is definable from S. We refer to the members of $\mathfrak{C}(S)$ as *S-constructors.*

In the rest of this chapter we seek to prove some theorems ressembling the main result of Chapter V but arguing from weaker hypotheses:

7·1 THEOREM *If there is an $\varnothing$-constructor and DD holds, then ω_2 is Woodin in HOD.*

7·2 THEOREM *If S is a subset of Θ, and there is an S-constructor, $Det(OD_S)$ holds and the second player has a winning strategy for the (class) S-game, then Θ is Woodin in HOD_S.*

7·3 THEOREM *If S is a class of ordinals, there is an S-constructor, and further $2^{\mathfrak{c}} = \mathfrak{c}^+$ for $\mathfrak{c} = 2^{\aleph_0}$, $Det(OD_S)$ holds and the second player has a winning strategy for the S-game, then Θ is Woodin in HOD_S.*

In our discussion we shall refer those as Case I, Case II and Case III respectively. Case III proves to reduce not to case II but to a more general version of case II, which we shall call case IV, and which we now state. It involves a further variant of the S-game, which is explained after the statement of the theorem.

7·4 THEOREM *Suppose that S is a subset of Θ, that there is an S-constructor, and that $Det(OD_S)$ holds. Suppose further that there is a non-empty subset $\mathfrak{B}$ of $\mathfrak{C}(S)$ which is* OD_S, *such that the second player has a winning strategy for the $(S, \mathfrak{B})$-game: then Θ is Woodin in HOD_S.*

7·5 DEFINITION For S, $\mathfrak{B}$ as above, the $(S, \mathfrak{B})$ game is the result of modifying the S game by weakening the notion of pre-strategy. Hitherto, in the class form, a pre-strategy ψ has had to succeed in producing a strategy from input any element of $\mathfrak{C}(S)$; we now ask it only to succeed for input an element of $\mathfrak{B}$.

Rules of the $(S, \mathfrak{B})$-game

DEF $PS_{\mathfrak{B}}(\psi, A) \iff_{\text{df}} \forall y{:\in}\mathfrak{B}\; WIP(\psi, \psi \oplus y; A)$.

DEF $GPS_{\mathfrak{B}}(\psi, A; S) \iff_{\text{df}} PS_{\mathfrak{B}}(\psi, A)\ \&\ Det(OD(\psi, S))$.

$$\text{Adam:} \left\{ \begin{array}{ll} A_0 & \\ & \psi_0 \\ A_1 & \\ & \psi_1 \\ A_2 & \\ & \vdots \\ \vdots & \end{array} \right\} \text{: Eve}$$

Rules:

Rule SBG 0: A_0 is an $\text{OD}(S)$ determined subset of $\mathcal{N}$;

Rule SBG 1: $GPS_{\mathfrak{B}}(\psi_0, A_0; S)$ (that is, if $y \in \mathfrak{B}$, so that $V = L[y; S]$, then $\psi \otimes (\psi \oplus y)$ is a winning strategy for a named player in A_0, and further $Det(OD(\psi_0; S))$);

Rule SBG 2: A_1 is an $\text{OD}(\psi_0; S)$ determined subset of $\mathcal{N}$;

Rule SBG 3: $GPS_{\mathfrak{B}}(\psi_1, A_1; S)$ and $\psi_1 \geqslant_{\text{Turing}} \psi_0$;
Rule SBG 4: A_2 is an $\mathrm{OD}(\psi_1; S)$ determined subset of $\mathcal{N}$;
Rule SBG 5: $GPS_{\mathfrak{B}}(\psi_2, A_2; S)$ and $\psi_2 \geqslant_{\text{Turing}} \psi_1$;
Rule SBG 6: and so on

The first person to break a rule loses; if all the rules are kept, Eve wins.

7·6 Remark Note that a winning strategy for the second player in the S game is also a winning strategy in each $(S, \mathfrak{B})$-game where $\mathfrak{B} \subseteq \mathfrak{C}(S)$.

Our stratagem for the rest of this chapter is to prove Cases I and IV in tandem; then to say that Case II is Case IV with $\mathfrak{B} = \mathfrak{C}(S)$; and finally to show that Case IV leads to Case III.

7·7 Remark We shall actually prove something stronger, namely that the second player need only have a strategy for surviving a small number $\mathfrak{k}$ of rounds in the S-game. We shall determine $\mathfrak{k}$. Today's estimate: $\mathfrak{k} = 3$

7·8 Remark Case I is almost the case $S = \varnothing$ of Case II, for we have seen that $Det(OD)$ follows from DD if V is $L(x)$ for some real x; but the notion of S-game is slightly different in the two cases; we invoke our discussion of good parameters, and in the first theorem, the set version is used, in the second, the class version.

7·9 Remark Case III, for S a set of ordinals with supremum greatly exceeding Θ, describes a situation which will hold on a cone if $V = L(\mathbb{R})$ holds with AD, and we shall exploit this result in the next chapter to obtain in that context inner models with infinitely many Woodin cardinals.

Some general remarks

⊛ 7·10 To prove its main theorem, Chapter V used a mixture of two methods — *coding lemmata*, which rely on the determinacy of certain games, and *reflection arguments*, which rely on the hierarchical structure of the universe, $L(\mathbb{R})$ considered as $L(\mathbb{R}; H)$ — together with the fact that the relevant filters $\mathfrak{W}$ are all, under AD, ultrafilters. The reflection arguments survive more or less intact, and we start our discussion much as before.

The setting for Case I

We suppose that $V = L[x]$ and that $Det(OD)$. We are going to show that Θ, which in this context equals ω_2 is, in HOD, a Woodin cardinal.

Let $H \subseteq \Theta$, $H \in HOD$. We add a unary predicate $\dot{H}$ to the formal language of set theory, to be interpreted as membership of H.

7·11 Definition Let δ_H be the least ordinal $> \omega_1$ such that

$$\langle J_{\delta_H}(\mathbb{R}), H \cap \delta_H \rangle \preceq^{\mathbf{R}}_{\Sigma_1} \langle J_\Theta(\mathbb{R}), H \rangle.$$

We write M_H for $\langle J_{\delta_H}(\mathbb{R}), H \cap \delta_H \rangle$.

Note that we do not name any particular $\varnothing$-constructor x, but instead regard this as the same case as Chapter V, for $V = L(\mathbb{R})$ is still true. The difference is that we no longer assume AD, but instead DD plus $\exists x {:\in} \mathcal{N}\ V = L[x]$

We may now define a $\diamondsuit$-like object, which we call $\diamondsuit$, as before, by least counterexample, such that for any Λ which is a prewellordering that is ordinal definable in $\langle J_\Theta(\mathbb{R}), H \rangle$, and any real a, any Σ_1 statement about $\delta_H, \Lambda, \diamondsuit, a; H$ holds of $\delta, \diamondsuit(\delta), \diamondsuit \restriction \delta, a; H$ for some δ in the domain of $\diamondsuit$.[N 4]

The setting for Case IV

Here we start from a subset S of Θ and suppose that there exist S-constructors; and that we have a fixed non-empty OD_S subset $\mathfrak{B}$ of $\mathfrak{C}(S)$ such that the second player wins the $(S, \mathfrak{B})$ game.

We fix a subset H of Θ, and now treat the universe as $L(\mathbb{R}; S, H, \mathfrak{B})$.

[N 4] we emphasize that here H is only appearing as a one-place predicate in the language, not as a set.

Let $H \subseteq \Theta$, with $H \in HOD_S$. We have to show that HOD_S contains appropriate H-strong extenders.

We reconstruct V as generated by the sequence $J_\nu(I\!\!R; S, \mathfrak{B}, H)$. Thus we throw in all reals; and we have a predicate for the particular set $\mathfrak{B}$ of reals, and the subsets S and H of Θ.

7·12 REMARK **We henceforth assume that H codes S** so that we work in fact with $J_\nu(I\!\!R; \mathfrak{B}, H)$.

We define $\delta_{\mathfrak{B},H}$ to be the least ordinal ν such that $J_\nu(I\!\!R; \mathfrak{B}, H) \preceq_{\Sigma_1^{\dot H}} J_\Theta(I\!\!R; \mathfrak{B}, H)$. We write $M_{\mathfrak{B},H}$ for the model $J_{\delta_{\mathfrak{B},H}}(I\!\!R; \mathfrak{B}, H)$.

Some terminology

In proving Cases I and IV, we follow broadly the lines of the proof of Chapter V; but, without AD, the filters $\mathfrak{W}$ will not be ultrafilters and the coding lemmata as originally formulated will not hold.

⊛ ⊛ In §8 we discuss variations of the coding lemmata that become provable when we apply a *Simple Basis Theorem*, so called because its proof rests on the existence of pre-strategies for absurdly simple games; in section 9 we review the definition of the filters $\mathfrak{F}$ and shall find that the definition survives unchanged, but we shall have to use our modified coding lemmata to prove modified versions of completeness and normality. In section 10 we examine our former definitions of $\mathfrak{W}$ and $\mathfrak{U}$ and define the extenders; their easier properties are established in section 11 and their normality in section 12; finally section 13 gives the reduction of Case III to Case IV.

7·13 In Chapter V we considered a subset H of Θ, and in our arguments a certain admissible set M_H played a major part. We shall continue to use our former terminology: sets which are members of $M_{\mathfrak{B},H}$ will be called *tame*, or *H-tame* if we need to specify H. Then when we have selected a pre-well-ordering Λ in HOD or in HOD_S, and are considering, say, some object R which is projective in Λ, we shall use the $\diamondsuit$-sequence to define, for many ν, reflected versions $R(\nu)$ from $\diamondsuit(\nu)$ as R was defined from Λ. We shall call such objects *reflectible*, or H-Λ-reflectible, though really this is an intensional concept: it is a particular definition of an object which reflects, rather than the object itself.

Many of the arguments in Chapter V took the following form: given a pre-well-ordering Z we built a set Z^* full in Z, tame or at least reflectible, and definable from H and a real parameter. As here not all games are determined — a problem that did not arise in Chapter V — we try where necessary to confine ourselves to parameters which define only determined games. Such a parameter we call *good*, or *good over S* if games definable from it and S are determined; we shall use the arguments developed in the first part of this Chapter to ensure a supply of good parameters. By extension, we shall refer to objects as *good* if they have a definition from good parameters.

In Case IV, we shall use the word *Good*, with a capital letter, to refer to parameters which are not only good but have the property that the $(S, \mathfrak{B})$ game starting from them is a win for the second player: in other words, a position of rank ∞ in our canonical computation of ranks of positions in open games. [N 5] In Case I, that is the same as being good, by Corollary 5·6. In Case II, it appears to be stronger.

Since we are also interested in lowering the hypothesis of the second player winning the S-game, or $(S, \mathfrak{B})$ game, to that of merely surviving for a small number of rounds, we shall refer to a position as k-good if the second player can survive k rounds starting from that position. Thus if play is at a $k+1$-position, it is liable, one round later, to be at only a k-position.

[There appears to be some merit in considering Eve to survive a round if she can play a ψ that is a pre-strategy for Adam's challenge, and to consider that if ψ is not good, that she loses in the next round, when Adam will play an indeterminate game definable from ψ, for which no pre-strategy will exist. For the moment, therefore, we will keep the definition of a round and of 0-position and 1-position slightly vague.]

[N 5] This appears not to be quite the same as saying it is a winning position in the $(S, \mathfrak{B})$-game, for that suggests that there is a strategy which would lead to that position.

8: The Simple Basis Theorem and the Coding Lemmata

The Simple Basis Theorem uses the existence of good pre-strategies to show that we can in a uniform way find non-empty good and tame subsets of non-empty good sets of reals.

8·0 THE SIMPLE BASIS THEOREM, Case I: *For a certain Σ^1_4 predicate $B_I(\alpha,\beta)$ of real variables α and β, we may prove the following: let $A \subseteq \mathcal{N}$ be* OD(γ)*, where $\gamma \in \mathcal{N}$ and $Det(OD(\gamma))$ holds, and suppose that there is a $\varnothing$-constructor. Then there is a $\psi \geqslant_{\text{Turing}} \gamma$ such that $Det(OD(\psi))$, $B_1(\psi) =_{\text{df}} \{\beta \in \mathcal{N} \mid B_I(\psi,\beta)\}$ is non-empty, and if A is non-empty, $B_1(\psi) \subseteq A$.*

Proof : Let $\bar{\mathcal{G}}_A$ be the game where Adam plays a, Eve plays e, and

Rule: Adam wins iff $a \in A$.

This game is plainly determined, Adam winning if A is non-empty, and Eve otherwise. Let ψ be a good pre-strategy for A and γ: more precisely, we apply Corollary 5·10 to infer that there is a ψ such that $GPS_{set}(\psi, A, \gamma)$. Then $\gamma \leq_T \psi$, and for non-empty A, and x a $\varnothing$-constructor,

$$\forall v \forall t \big[L[x] = L[v] \;\&\; v \text{ is a } \psi\text{-guarantor} \implies (\psi \otimes (\psi \oplus v)) \star [t] \in A\big].$$

Thus we have a mechanism for generating a non-empty subset of A. However, we need not run through every t for a given v — we could take $t = v$ or take t to be some fixed function, for instance the function that is constantly 0 — and there seems to be no virtue in making our non-empty subset of A fatter than necessary; so let us save a quantifier by considering the set

$$C = \big\{c \mid \exists v \big[L[x] = L[v] \;\&\; v \text{ is a } \psi\text{-guarantor} \;\&\; c = (\psi \otimes (\psi \oplus v)) \star [v]\big]\big\}.$$

C is non-empty by Corollary 4·1 and Definition 4·2. If A is non-empty, then $C \subseteq A$, since $\psi \otimes (\psi \oplus v)$ will be a winning strategy for Adam in that case.[N6]

C is $\Sigma^1_3(\psi, x)$, since for reals x and v, the clause "$x \in L[v]$" is $\Sigma^1_2(x, v)$, it saying that there is a countable well-founded model of $KP + V = L[v]$ containing x, and $L[x] = L[v] \iff x \in L[v] \,\&\, v \in L[x]$. To remove the reference to the $\varnothing$-constructor x, re-write the definition of C thus:

8·1 DEFINITION $\quad B_I(\alpha,\beta) \iff_{\text{df}} \exists v \big[\underbrace{\forall z\, z \in L[v]}_{\Pi^1_3} \;\&\; \underbrace{v \text{ is a } \alpha\text{-guarantor}}_{\Pi^1_2} \;\&\; \beta = (\alpha \otimes (\alpha \oplus v)) \star [v]\,\big].$

We see that B_I is a Σ^1_4 predicate and that $C = B_1(\psi) = \{\beta \in \mathcal{N} \mid B_I(\psi,\beta)\}$. $\dashv$ (8·0)

8·2 REMARK Note that for any ψ, $B_1(\psi)$, being projective, will be a member of M_H for any H that we consider under Case I, since $\mathbb{R} \in M_H$ and M_H is admissible. Thus $B_1(\psi)$ is always tame, and good if ψ is.

8·3 THE SIMPLE BASIS THEOREM, Case IV: *There is a certain predicate $B_{IV}(\mathfrak{Z},\alpha,\beta)$ of third-order number theory which is positive Σ^1_1 in the real variables α and β and the third-order variable $\mathfrak{Z}$ denoting a set of reals, with the following property: whenever S is a subset of Θ for which there is an S-constructor and $Det(OD_S)$ holds, and whenever $\mathfrak{B}$ is a non-empty OD_S set of S-constructors, such that the second player has a strategy for surviving one round in the $(S, \mathfrak{B})$-game, we may, for any non-empty* OD$_S$ *set A of reals, choose ψ so that $B_4(\mathfrak{B},\psi) =_{\text{df}} \{\beta \in \mathcal{N} \mid B_{IV}(\mathfrak{B},\psi,\beta)\}$ is a non-empty subset of A. Further*

(i) if the second player has a $k+1$-survival strategy for the $(S,\mathfrak{B})$-game, ψ may be chosen to be k-good;

(ii) if the second player has a winning strategy for the $(S,\mathfrak{B})$-game, ψ may be chosen to be Good.

Proof : again consider the game where Adam plays a, Eve plays e and Adam wins iff $a \in A$. Plainly Eve has a winning strategy if A is empty, and Adam has one otherwise. Suppose that $PS_{cls}(\psi, A; S)$, where if the second player has a $k+1$-survival strategy for the $(S,\mathfrak{B})$-game, we may suppose ψ to be k-good, and if the second player has a winning strategy for the $(S,\mathfrak{B})$-game, we may suppose ψ to be Good.

8·4 DEFINITION $\quad B_{IV}(\mathfrak{Z},\alpha,\beta) \iff_{\text{df}} \exists y \!:\in \mathfrak{Z}\; \beta = (\alpha \otimes (\alpha \oplus y)) \star [y].$

Then $B_4(\mathfrak{B},\psi) = \{\beta \in \mathcal{N} \mid B_{IV}(\mathfrak{B},\psi,\beta)\}$ is as required. $\dashv$ (8·3)

[N6] pedantry: if A is empty, $\psi \otimes (\psi \oplus v)$ will be an Eve strategy, and we therefore ought to write $[v] \star (\psi \otimes (\psi \oplus v))$ to get something sensible; we could re-write the definition of C to cover this case, but we won't.

8·5 REMARK Again the set $B_4(\mathfrak{B}, \psi)$ will be a member of each $M_{\mathfrak{B},H}$ considered under Case IV, for the set $\mathfrak{B}$ is arranged to be in each such $M_{\mathfrak{B},H}$.

8·6 DEFINITION In discussions that depend little on which Case we are in, we shall write $B_?(\psi)$ to indicate a (good, tame) set produced by the Simple Basis Theorem.

An application

We suppose $\mathrm{Det}(OD_S)$ and consider a OD_S pre-wellordering $\Lambda = (\mathrm{Field}(\Lambda), <_\Lambda)$ of a subset of $\mathcal{N}$. Let $a \in \mathrm{Field}(\Lambda)$. What can we say about $<_\Lambda \restriction a$? Specifically, $<_\Lambda$ is good, for it is definable from S; and it will be reflectible using $\diamondsuit$. Is the same true of the initial segment $I = <_\Lambda \restriction a$?

The definition of I as $<_\Lambda \restriction a$ is certainly reflectible, but liable not to be good as a need not be good. We can find another definition of I which is good, by setting $\eta = |a|_\Lambda$, and using the ordinal η in this second definition, but then this definition is liable not to be reflectible since η might well be greater than δ_H.

We proceed as follows. Let $T = \{b \in \mathrm{Field}(\Lambda) \mid \varrho_\Lambda(b) = \eta\}$. That is OD_S. Hence by the Simple Basis Theorem it contains, at the cost, in Case IV, of one round in the S game, [N 7] some set of the form $B_?(\psi)$, defined from the S-good parameter ψ, and which will be both tame and good. Now consider

$$\{c \mid \exists x :\in B_?(\psi)\ \varrho_\Lambda(c) < \varrho_\Lambda(x)\}$$

That defines the required initial segment I using only S and the S-good parameter ψ, and this third definition of I is both good and reflectible.

8·7 In particular this discussion shows that given any ordinal η less than the length of the pre-well-ordering Λ, we can find, at the cost, in Case IV, of one round, a non-empty good and tame set of the form $B(\psi)$ which is a subset of the component $(\Lambda)_\eta$; that in turn gives us a way of reflecting the ordinal η, since for many ν, $B(\psi)$ will be a subset of a single component of $\diamondsuit(\nu)$, and we then set $\eta(\nu)$ to be the rank in $\diamondsuit(\nu)$ of the points of $B(\psi)$. Two different ways of doing that will agree almost everywhere.

Further we can handle higher dimensions:

8·8 DEFINITION For $\varnothing \neq s \subseteq \lambda$, we write $(\Lambda)_s$ for the product $(\Lambda)_{\zeta_0} \times (\Lambda)_{\zeta_1} \times (\Lambda)_{\zeta_2} \times \dots (\Lambda)_{\zeta_{k-1}}$, where the members of s in increasing order are $\zeta_0, \dots \zeta_{k-1}$.

8·9 REMARK That notation is not unambiguous as finite ordinals are themselves finite sets of ordinals; in case of serious doubt we can write $\vec{s}$ and $\{\zeta\}$.

8·10 Let D be any non-empty tame subset of $(\Lambda)_s$. We can use that to define a map r_D which associates to each ν in some set in $\mathfrak{F}$ a subset of $|\diamondsuit(\nu)|$ of size k, the size of s, as follows.

The statement "D `is a subset of a single component of the product ordering` Λ^k" is reflectible: we define $r_D(\nu)$ to be that unique finite subset s' of $\diamondsuit(\nu)$ with $D \subseteq (\diamondsuit(\nu))_{s'}$, when such s exists, and undefined otherwise. Here we use the fact that tame objects reflect to themselves.

If we have two different tame subsets D_1 and D_2 of $(\Lambda)_s$, the statements "C `and` D `are subsets of the same component of` $(\Lambda)^k$" will also reflect, and therefore the functions r_C and r_D will agree modulo a set in $\mathfrak{F}$.

8·11 Our problem, of course, will be to show that any such tame subsets exist. We consider a variant of our previous game: let $k = \overline{\overline{s}}$, for s a finite subset of λ. Adam plays a and Eve plays e and Adam is to win if $((a)_0, \dots, (a)_k) \in \Lambda_s$.

Given a pre-strategy ψ for that game, then in Case I, we define

$$B_1^k(\psi) = \{((z)_0, \dots, (z)_{k-1}) \mid B_I(z, \psi)\}$$

and in Case IV, we define

$$B_4^k(\mathfrak{B}, \psi) = \{((z)_0, \dots, (z)_{k-1}) \mid B_{IV}(\mathfrak{B}, \psi, z)\}.$$

[N 7] "At the cost" is slightly misleading, since the costs are not cumulative. It would be better to say "by playing one round in an $(S, \mathfrak{B})$-game".

Then in either case $B^k_?(\psi)$ is a tame non-empty subset of $(\Lambda)_s$, which is good to the extent that ψ is.

8·12 REMARK $B^1_1(\psi)$ is not the same as $B_1(\psi)$: either will do when we are working solely in dimension 1; but the former is preferable when we are considering several dimensions simultaneously, as we do, for example, in establishing that our extenders are coherent.

$\lessdot$ 8·13 REMARK In sections 11 and 12, we shall extend the notation $u^{s,t}$ introduced in Chapter IV for finite sets of ordinals to finite sequences. Thus when $\eta \in t$ and $s = t \setminus \{\eta\}$, we shall mean by $u^{s,t}$ the sequence u with the term corresponding to η deleted. For s of size 1, the definition of r_D yields the singleton of some ordinal, and must be prefaced by $\bigcup$ in order to obtain the ordinal itself.

A review of the coding lemmata

$\circledast\circledast$ 8·14 With these remarks in mind, we examine the coding lemmata: we shall find that we shall have to treat them as being in two stages. In the first we have a pre-well-ordering Z and a background pre-well-ordering Λ; typically Z is good but not tame; we find a real a such that G^Λ_a is full in Z in the sense that it meets every component met by Z. This set G^Λ_a is tame, or at least reflectible, but probably not good.

The Simple Basis Theorem will then supply a non-empty, good and tame subset $B_?(\psi)$ of $\{a \mid G_a \subseteq Z \text{ and } G_a \text{ full in } Z\}$, and then instead of taking Z^* to be G^Λ_a, we shall take Z^* to be $\bigcup\{G_a \mid a \in B_?(\psi)\}$. Then Z^* will be good, full and either tame or reflectible; the existence of such Z^* forms the second stage of the coding lemma.

Thus the first stage contains the argument much as before, and we simply keep track of the amount of determinacy being used. In the second we use the Simple Basis Theorem and must monitor the goodness of parameters.

8·15 PROBLEM TERMINOLOGY: we should perhaps distinguish between "dense in Z" and "full in Z"; there is a slight difference in that we are trying to meet every component versus every selected component ?

The Coding Lemma revisited

8·16 THE CODING LEMMA, first stage: *let T be a set or class of ordinals. Let $\prec$ be a pre-well-ordering that is* OD*(T). Let $Z \subseteq Field(\prec) \times \mathcal{N}$, Z* OD *from T. Let G be universal for $pos\mathbf{\Sigma^1_1}(\prec) \restriction \mathcal{N} \times \mathcal{N}$. If $Det(OD_T)$ holds, then for some a, $G_a \subseteq Z$ and is full in Z.*

Here *full in Z* refers as before to the familiar pre-well-ordering of Z by applying $\prec$ to the first co-ordinate of its members.

Proof : we play the game as in section III.4: the notion of a being acceptable is definable from G and Z, and hence from $\prec$ and Z; similarly the concept of first failure is definable from Z and G, and thus from $\prec$ and Z. So the game is determined provided that games definable from $\prec$ and Z are determined. $\dashv$ (8·16)

8·17 THE CODING LEMMA, second stage, Case I: *Let $<_\Lambda$ be a pre-wellordering of $\Lambda \subseteq \mathcal{N}$ ordinal definable from the real β; let $Z \subseteq \Lambda \times \mathcal{N}$ be ordinal definable from the real α. Suppose that $Det(OD(\alpha, \beta))$ holds and that there is a $\varnothing$-constructor. Then there is a ψ such that (i) $Det(OD(\psi))$; (ii) $\langle\alpha, \beta\rangle \leqslant_{\text{Turing}} \psi$; (iii) there is a full $Z^* \subseteq Z$ which is $pos\Sigma^1_1(<_\Lambda, B_1(\psi))$.*

Proof : Consider $F =_{\text{df}} \{\gamma \mid G^\Lambda_\gamma \subseteq Z \text{ and is full}\}$, where as in Chapter III, G indexes $pos\Sigma^1_1(<_\Lambda)$. F is definable from α and β and is non-empty, by the First Stage of the Coding Lemma.

So by the Simple Basis Theorem, there is a $\psi \geq \langle\alpha, \beta\rangle$ such that $Det(OD(\psi))$ and $\varnothing \neq B_1(\psi) \subseteq F$.

Take for Z^* the set

$$\big\{a \bigm| \exists\gamma\big(\gamma \in B_1(\psi) \;\&\; a \in G^\Lambda_\gamma\big)\big\}.$$

This set is plainly definable from ψ alone, and is full. It is at worst $pos\mathbf{\Sigma^1_4}(\Lambda)$; but if G is sufficiently complicated so that it indexes the predicate B_I, then Z^* will be $pos\mathbf{\Sigma^1_1}(\Lambda)$. $\dashv$ (8·17)

8·18 REMARK If we wish to represent Z^* as G_u for some G and u, we must start our proof by assuming that G is universal for $pos\Sigma^1_4(\Lambda)$ sets, not just $pos\Sigma^1_1(\Lambda)$ ones. But for our purposes, that is not important in itself; what is important is that Z^* be both good and reflectible or even, if Λ is shorter than δ_H, tame.

8·19 The Coding Lemma, second stage, Case IV: *Let Z and $\prec$ be* OD *from S and p where S is a subset of Θ for which Det(OD_S) holds and p is a real parameter. Suppose that $\mathfrak{B}$ is a non-empty* OD_S *subset of $\mathfrak{C}(S)$.*

(i) If p is $(k+1, S, \mathfrak{B})$-good, there is a $(k, S, \mathfrak{B})$-good parameter ψ and a set Z^* that is both full in Z and pos$\Sigma^1_1(\prec, \mathfrak{B}, \psi)$;

(ii) if p is $(S, \mathfrak{B})$-Good, there is a $(S, \mathfrak{B})$-Good parameter ψ and a set Z^* that is both full in Z and pos$\Sigma^1_1(\prec, \mathfrak{B}, \psi)$.

Proof : let G be the appropriate $pos\boldsymbol{\Sigma}^1_1(\prec)$ universal set; let $B = \{a \mid G_a \subseteq Fld(\prec) \;\&\; G_a \text{ full in } Z\}$. B is non-empty; so there is a ψ with $B_4(\mathfrak{B}, \psi) \subseteq B$: that costs us one round; so the $(S, \mathfrak{B})$-goodness of ψ will be one less than that of p. We take

$$Z^* = \bigcup_{a \in B_4(\mathfrak{B}, \psi)} G_a. \qquad \dashv (8{\cdot}19)$$

8·20 Problem ARE WE HOPING to get $p \leqslant_{\text{Turing}} \psi$?

8·21 Remark Z^* is almost projective, that is, projective in $\prec$ and $\mathfrak{B}$; so if $\Lambda \in M_H$, so is Z^*.

8·22 Remark Again, if we take G to index $pos\boldsymbol{\Sigma}^1_1(\prec, \mathfrak{B})$ from the outset, our final Z^* will also be of the form G_e; we can find e recursive in ψ and therefore good. But the essential aspect is that the Z^* we find be both good and reflectible or even tame.

The Uniform Coding Lemma revisited

For $P = (X, \prec_P)$ a pre-well-ordering of a subset X of $\mathcal{N}$ of length λ, we write, as in III.5, $(P)_\xi$ for $\{x \in \text{Field}(P) \mid \rho_P(x) = \xi$, and $P \restriction \zeta$ for the restriction of $\prec_P$ to the set $\bigcup_{\xi<\zeta}(P)_\xi$. Thus $P \restriction 0$ is empty, $P \restriction \lambda = P$, for $\xi < \zeta$, $(P \restriction \zeta)_\xi = (P)_\xi$, and $X = \bigcup_{\xi<\lambda}(P)_\xi$.

8·23 The Uniform Coding Lemma, first stage: *Let P be a pre-well-ordering of length λ, and $K \subseteq \lambda$. Suppose that Det($OD_{\{P,\,K\}}$) holds. Let G^P_ε be a parametrisation of pos$\Sigma^1_1(\prec_P)$ that is good in the sense of §II.3. Then there is an index a such that, writing G^ζ_a for $G^{P\restriction\zeta}_a$,*

$$\forall\zeta \leq \lambda \forall\xi < \zeta \big[G^\zeta_a \cap (P)_\xi \neq \varnothing \iff \xi \in K\big]$$

Proof : the monotonicity remarked in (II·3·1) tells us that for $0 \leq \xi \leq \zeta \leq \lambda$ and any index a, $G^\xi_a \subseteq G^\zeta_a$. We now follow the proof in Chapter III, noting that *acceptability* is definable from the pwo P and $K \subseteq \lambda$, and that *first failure* is definable from K and P. So the game is determined provided games definable from P and K are. $\dashv$ (8·23)

8·24 Problem IN CHAPTER THREE, we want to define G^Q_ε; and for coding by fill, a parametrisation of pos$\Sigma^1_1(\prec_Q, \preceq_Q)$. That should perhaps be in Chapter TWO.

For the second stage, we consider the set $\mathcal{W}$ of all such successful indices a; invoking the aforementioned monotonicity, we may define that set thus:

$$\mathcal{W} = \big\{a \;\big|\; \forall\xi{:}{<}\lambda \;\big[(\xi \in K \implies G^{\xi+1}_a \cap (P)_\xi \neq \varnothing) \,\&\, (\xi \notin K \implies G^\lambda_a \cap (P)_\xi = \varnothing)\big]\big\},$$

and we know by the first stage that $\mathcal{W}$ is non-empty. $\mathcal{W}$ is definable from P and K.

In Case I, for an arbitrary relation Q and real β we define

8·25 Definition $\qquad W_1(Q, \beta) \;=_{\text{df}}\; \bigcup\{G^Q_a \mid a \in B_1(\beta)\}$.

8·26 Remark $W_1(Q, \beta)$ is Σ^1_4 in Q and β, and is monotonic in Q.

Then we apply the Simple Basis theorem, Case I, to obtain ψ with $B_1(\psi) \subseteq \mathcal{W}$. The set $W_1(P, \psi) = \bigcup\{G^P_a \mid a \in B_1(\psi)\}$ is pos$\Sigma^1_1(P, B_1(\psi))$, and pos$\Sigma^1_4(P, \psi\;)$ and good and also reflectible, or even tame, if P is tame. We have proved:

8·27 The Uniform Coding Lemma, second stage, Case I:

Let P be a pre-well-ordering of length λ that is OD *in α, and K a subset of λ that is* OD *in β. Suppose that Det($OD_{\{\alpha,\,\beta\}}$) holds and that there is a $\varnothing$-constructor. Then there is a real ψ such that*

$$\forall\xi{:}{<}\lambda \;\big[(\xi \in K \implies W_1(P \restriction (\xi+1), \psi) \cap (P)_\xi \neq \varnothing) \,\&\, (\xi \notin K \implies W_1(P, \psi) \cap (P)_\xi = \varnothing)\big]\big\},$$

$\langle\alpha,\, \beta\rangle \leqslant_{\text{Turing}} \psi$ and Det(OD_ψ) holds.

8·28 Remark SHOULD WE GIVE the more local coding ?

$$\forall\xi{:<}\lambda\ \big[(\xi \in K \iff W_1(P\restriction(\xi+1),\psi)\cap(P)_\xi \neq \varnothing)\big]\big\},$$

In Case IV, for an arbitrary relation Q, real β and set of reals $\mathfrak{Z}$, we define

8·29 Definition $\qquad W_4(Q,\mathfrak{Z},\beta) =_{\mathrm{df}} \bigcup\{G_a^Q \mid a \in B_4(\mathfrak{Z},\beta)\}.$

8·30 Remark $W_4(Q,\mathfrak{Z},\beta)$ is Σ^1_1 in Q, $B_4(\mathfrak{Z},\beta)$ and β, and is monotonic in Q; and hence will be tame for tame Q and $\mathfrak{Z}$.

Then we apply the Simple Basis theorem, Case IV, to obtain ψ with $B_4(\mathfrak{B},\psi) \subseteq \mathcal{W}$. The set $W_4(P,\mathfrak{B},\psi) = \bigcup\{G_a^P \mid a \in B_4(\mathfrak{B},\psi)\}$ is $pos\Sigma^1_1(P, B_4(\mathfrak{B},\psi))$, and good and also reflectible, or even tame, if P is tame. We have proved:

8·31 The Uniform Coding Lemma, second stage, Case IV: *Let P be a pre-well-ordering of length λ, and $K \subseteq \lambda$, both of them* OD *from S and p where S is a subset of Θ for which Det(OD_S) holds and p is a real parameter. Suppose that $\mathfrak{B}$ is a non-empty* OD_S *subset of $\mathfrak{C}(S)$. Then there is a real ψ such that*

$$\forall\xi{:<}\lambda\ \big[(\xi \in K \implies W_4(P\restriction\xi+1,\mathfrak{B},\psi)\cap(P)_\xi \neq \varnothing)\ \&\ (\xi \notin K \implies W_4(P,\mathfrak{B},\psi)\cap(P)_\xi = \varnothing)\big]\big\}.$$

Further, if p is $(k+1,S,\mathfrak{B})$-good, ψ may be chosen to be $(k,S,\mathfrak{B})$-good; and if p is $(S,\mathfrak{B})$-Good, ψ may be chosen to be $(S,\mathfrak{B})$-Good.

8·32 Problem Is the sole use of uniform coding to obtain the admissible coding about to come ?

8·33 Problem Tameness of the above.

8·34 Remark The cost in Case IV is, I BELIEVE, one round.

8·35 Remark Again, SHOULD WE GIVE the more local coding ?

8·36 Problem SHOULD WE define W_I and W_{IV} ?

Admissible coding revisited

We recall our definition of admissible coding in section III.6, which consisted of an application of the uniform coding lemma. We wish to obtain a variant of that available in our present contexts, which are these: we have an admissible set M which is M_H in Case I and $M_{\mathfrak{B},H}$ in Case IV, of height $\delta = \delta_H$ or $\delta_{\mathfrak{B},H}$ respectively, and equipped with a pre-well-ordering Υ_H or $\Upsilon_{\mathfrak{B},H}$, Σ_1 over the model concerned and of length the height of the model.

As before, we shall use the function χ, which is primitive recursive, and therefore always available. Then, given a partial function $B : \delta \longrightarrow \delta$, we define $E_B = \{\chi(\alpha,\beta) \mid B(\beta)\downarrow= \alpha\}$; the first stage is to find a c such that G_c codes E_B: that follows from the existence of strategies for determined games.

8·37 Admissible Coding, first stage: *Let P be a pre-well-ordering of length δ, and $K \subseteq \delta$. Let G_ε be a parametrisation of $pos\Sigma^1_1(\prec_P)$ that is good in the sense of §II.3. Let $B : \delta \longrightarrow \delta$ be a partial function. Suppose that Det($OD_{\{P,K,B\}}$) holds. Then there is an index c that codes the function B in this sense:*

$$B(\eta)\downarrow= \beta \iff_{\mathrm{df}} G_c^{\chi(\eta,\beta)+1} \cap (P)_{\chi(\eta,\beta)} \neq \varnothing\ \&\ \forall\beta'{:<}\beta\ G_c^{\chi(\eta,\beta')+1}\cap(P)_{\chi(\eta,\beta')} = \varnothing$$

Proof : much as before. By the Uniform Coding Lemma, first stage, there is an index c such that

$$\forall\xi{:<}\lambda\ \ G_c^{\xi+1} = \bigcup\{(\Lambda)_\nu \mid \nu \in E_B \cap (\xi+1)\}$$

Then $$\forall\eta{:<}\lambda\ B(\eta) \simeq c(\eta)$$

where $\simeq$ is used in Kleene's sense that if either side is defined so is the other and they are equal. HERE WE SEEM TO USE CODING BY FILL. $\dashv$ (8·37)

For the second stage, we proceed as with the Uniform Coding Lemma: we first form the set of successful indices c, which we define thus:

$$\mathcal{S} = \big\{c \mid \forall\xi{:<}\lambda\ \big[(\xi \in K \implies G_c^{\xi+1} \cap (P)_\xi \neq \varnothing)\,\&\,(\xi \notin K \implies G_c^{\lambda} \cap (P)_\xi = \varnothing)\big]\big\},$$

and we know by the first stage that $\mathcal{S}$ is non-empty. $\mathcal{S}$ is definable from P and K.

We then use the appropriate case of the Simple Basis Theorem to find a subset W of $\mathcal{S}$. It is easier now to reformulate our coding directly in terms of W. What we get in the two cases is:

8·38 Admissible Coding, second stage, Case I: *Suppose that there is a $\varnothing$-constructor. Let H be an* OD *subset of Θ, and let $f : \delta_H \to \delta_H$ be a partial function that is* OD *in β. where Det($OD_{\{\beta\}}$) holds. Then there is a ψ, which codes f in this sense:*

$$f(\eta)\downarrow = \beta \iff_{\mathrm{df}} W_1(\Upsilon\restriction(\chi(\eta,\beta)+1),\psi)\cap(\Upsilon)_{\chi(\eta,\beta)} \neq \varnothing\ \&$$
$$\&\ \forall\beta'{:<}\beta\ W_1(\Upsilon\restriction(\chi(\eta,\beta')+1),\psi)\cap(\Upsilon)_{\chi(\eta,\beta')} = \varnothing;$$

Further ψ may be chosen with $\beta \leqslant_{\mathrm{Turing}} \psi$ and such that Det($OD_{\{\psi\}}$) holds.

8·39 Admissible Coding, second stage, Case IV: *Let P be a pre-well-ordering of length λ, and $K \subseteq \lambda$, both of them* OD *from S and p where S is a subset of Θ for which Det(OD_S) holds and p is a real parameter. Suppose that $\mathfrak{B}$ is a non-empty* OD_S *subset of $\mathfrak{C}(S)$. Let $f : \delta \to \delta$ be partial and definable from S and p. If actually Det($OD_{S,p}$) holds, then there is a ψ, which codes f in this sense:*

$$f(\eta)\downarrow = \beta \iff_{\mathrm{df}} W_4(\Upsilon\restriction(\chi(\eta,\beta)+1),\mathfrak{B},\psi)\cap(\Upsilon)_{\chi(\eta,\beta)} \neq \varnothing\ \&$$
$$\&\ \forall\beta'{:<}\beta\ W_4(\Upsilon\restriction(\chi(\eta,\beta')+1),\mathfrak{B}\psi)\cap(\Upsilon)_{\chi(\eta,\beta')} = \varnothing$$

Further if p is $(k+1,S,\mathfrak{B})$-good, ψ may be chosen to be $(k,S,\mathfrak{B})$-good; and if p is $(S,\mathfrak{B})$-Good, ψ may be chosen to be $(S,\mathfrak{B})$-Good.

Roughly, we have shown this:

8·40 Proposition *Let $f : \delta \longrightarrow \delta$ which is a partial function and definable from S and some good parameter p, then with Det($OD_{\{S,\,p\}}$) we get an admissible code, of goodness one less than that of p.*

8·41 Definition For a partial function g or a subset X of δ, we may denote such codes by $\psi_g^{(1)}$ or $\psi_X^{(1)}$ in Case I, by $\psi_g^{(4,\mathfrak{B})}$ or $\psi_X^{(4,\mathfrak{B})}$ in Case IV, and by $\psi_g^{(?)}$ or $\psi_X^{(?)}$ in a discussion where the Case is left ambiguous.

8·42 Exercise Show that every initial segment of a definable such f is in M.

8·43 Problem ψ good over the definition of f ? $p \leqslant_{\mathrm{Turing}} f$?

8·44 Remark As before, we note that to determine whether $B(\nu)\downarrow = \zeta$ or not, we need only use $P\restriction\chi(\nu,\zeta)+1$; so when P is Υ, the coding will be as admissible as before, NOT THAT WE SEEM TO USE THAT, except possibly for the following

8·45 Problem There must be some strong partition result going somewhere.

The non-monotonic coding lemma revisited

[We shall use the version proved in §7 of Chapter V, but with a different proof that might be easier to follow and which certainly gives further information. The game is entirely definable from the sequence of $Q(\nu)$'s; so the First Stage is readily obtained. Mercifully we shall not need a Second Stage.]

9: The filter $\mathfrak{F}$

Case I

We have fixed an OD $H \subseteq \Theta$, and defined δ_H; we have defined the diamond sequence, which we call $\diamondsuit$ without a subscript H. Let $\delta_H < \lambda < \Theta$ and let Λ be an OD pwo of length λ. We proceed to define the filter $\mathfrak{F}_{H,\Lambda}$

We take a universal set

$$\Upsilon_{H,\Lambda} =_{\mathrm{df}} \{a \mid J_\Theta(x) \models \Phi(\delta_H, \Lambda, \diamondsuit, H, a)\};$$

We define the sets S_z for $z \in \Upsilon_{H,\Lambda}$ as before, by

$$S_z = \{\delta < \delta_H \mid J_{\delta_H}[x] \models \Phi(\delta, \diamondsuit(\delta), \diamondsuit \restriction \delta, H, z)\}$$

and we take $\mathfrak{F}_{H,\Lambda}$ to be the filter generated by the S_z's for $z \in \Upsilon_{H,\Lambda}$.

As before, each S_z is non-empty, by the reflection property of the $\diamondsuit$ sequence $\diamondsuit$; and syntactical manipulations show that there is a recursive π such that $S_a \cap S_b = S_{\pi(a,b)}$, so that $\mathfrak{F}_{H,\Lambda}$ is a proper filter. Further, we may show that $\mathfrak{F}_{H,\Lambda}$ is countably complete.

It would be natural to ask if it is δ_H complete, but δ_H lies strictly between ω_1 and ω_2 and therefore is not a cardinal. However our previous arguments, in Chapter V, may be applied to show that if $\eta < \delta_H$ and $\langle A_\nu | \nu < \eta\rangle$ is a **definable** sequence of elements of $\mathfrak{F}$, then $\bigcap_{\nu<\eta} A_\nu \in \mathfrak{F}$; so in particular, δ_H is, in HOD, a regular cardinal.

Case IV

We have fixed H, a subset of Θ in HOD_S, and which, we have agreed to suppose, codes S. We proceed to define $\delta_{\mathfrak{B},H}$, and find a $\diamondsuit$ sequence, which again we call $\diamondsuit$, reflecting all Σ_1 statements of an appropriate kind, mentioning $\mathfrak{B}$, H, and an arbitrary OD_S pre-well-ordering P.

We fix a universal such formula Φ_1.

Let $\delta < \lambda = \bigcup \lambda < \Theta$, and let Λ be an OD_S pre-well-ordering of $\mathcal{N}$ of length λ.

Write $\Upsilon_{\mathfrak{B},H,\Lambda}$ for the universal set:

$$\{a \in \mathcal{N} \mid J_\Theta(\mathbb{R}; \mathfrak{B}, H) \models \Phi_1(a, \Lambda, \diamondsuit; \mathfrak{B}, H)\}$$

For $a \in \Upsilon_{\mathfrak{B},H,\Lambda}$, define

$$S_a =_{\mathrm{df}} \{\nu \in \mathrm{Dom}(\diamondsuit) \mid J_{\delta_{\mathfrak{B},H}}(\mathbb{R}; \mathfrak{B}, H) \models \Phi_1(a, \diamondsuit(\nu), \diamondsuit \restriction \nu; \mathfrak{B}, H)\}$$

Let $\mathfrak{F}_{\mathfrak{B},H,\Lambda}$ be the filter on $\delta_{\mathfrak{B},H}$ generated by the family $\{S_a \mid a \in \Upsilon_{\mathfrak{B},H,\Lambda}\}$.

9·0 Remark In Case IV, WE SHOULD HAVE SHOWN IN CHAPTER ONE that all subsets of $\mathcal{N}$ are in $J_\Theta(\mathbb{R}, \mathfrak{B}; H)$.

We shall use the unadorned letter $\mathfrak{F}$ when the arguments do not require the Case to be specified.

Apart from the appearance of S and $\mathfrak{B}$, these definitions are as in Chapter V, and we may check that $\mathfrak{F}$ is a countably complete filter. In Chapter V, where we had full determinacy, we proved further that $\mathfrak{F}$ is a δ_H-complete filter in the universe; here we have only limited determinacy and shall prove weaker properties, such as that $\mathfrak{F} \cap HOD_S$ is in HOD_S δ_H-complete, that δ_H is regular in HOD_S, and, most importantly, that a version of normality holds for this $\mathfrak{F}$.

9·1 Proposition *$\mathfrak{F}$ is a countably complete free filter.*

HOW MUCH CHOICE IS NEEDED FOR THAT ???

The discussion leading to Lemma (V·5·6) still holds, for all that relies solely on reflection arguments, which are still valid. So we have these useful properties of $\mathcal{F}$.

9·2 Lemma *Let Q and R be pre-well-orderings projective in Λ.*

(i) If $R = Q$ *then* $\{\nu \in \mathit{Dom}(\diamondsuit) \mid R(\nu) = Q(\nu)\} \in \mathfrak{F}$*;*
(ii) if $b \in \mathit{Field}(Q)$*, then* $C(b) \ =_{\mathrm{df}} \ \{\nu \mid b \in \mathit{Field}(Q(\nu))\} \in \mathfrak{F}$*;*
(iii) for $\nu \in C(b)$*,* $|(Q \restriction b)(\nu)| = |b|_{Q(\nu)}$*;*
(iv) if $c <_Q b$*,* $F(c, b) \ =_{\mathrm{df}} \ \{\nu \mid c <_{Q(\nu)} b\} \in \mathfrak{F}$*;*
(v) for $\nu \in C(b) \cap F(c, b)$*,* $|c|_{(Q \restriction b)(\nu)} = |c|_{Q(\nu)}$.

==

HERE WE MIGHT INTERPOLATE first stage versions of the Coding Lemmata.

==

9·2 Proposition (Case I: $\mathrm{Det}(OD)$ and $\mathfrak{C}(\varnothing) \neq \varnothing$) *$\mathfrak{F} \cap HOD$ is a δ_H-filter in that model; hence δ_H is regular in HOD.*

9·3 Proposition (Case IV: $\mathrm{Det}(OD_S)$ and $\mathfrak{C}(S) \neq \varnothing$) *$\mathfrak{F} \cap HOD_S$ is a $\delta_{\mathfrak{B},H}$-filter in that model; hence $\delta_{\mathfrak{B},H}$ is regular in HOD_S.*

Proof : Let $\langle A_\nu \mid \nu < \xi \rangle \in HOD$; we find a pre-well-ordering Ξ in M_H of length ξ, and OD_S (definable in M_H ???) and with field $\mathcal{N}$. We set

$$Z = \{(z, a) \mid a \in \Upsilon_{H,\Lambda} \ \& \ S_a \subseteq A_{|z|_\Xi}\}$$

9·4 Remark Z is OD_S.

By the FIRST STAGE of the Coding Lemma, we find a p such that G_p^Ξ is TAME and FULL in Z. Let $Q = \{a \mid \exists z (a, z) \in G_p\}$. Then Q is TAME. So the following is a Σ_1 statement:

$$\forall a \in Q \Phi(a, \diamondsuit, \ldots).$$

Therefore there is an $\bar{a}$ with

$$S_{\bar{a}} \subseteq \bigcap_{a \in Q} S_a \subseteq \bigcap_{\nu < \xi} A_\nu,$$

which latter set is therefore in $HOD_S \cap \mathfrak{F}$. $\dashv$ (9·4)

==

Everything below here is to go to the new section on the normality of $\mathfrak{F}$. Hence we start a new page.

Beginning the discussion of the normality of $\mathfrak{F}$

⊛ Here is another application of the simple basis theorem: its significance is that it permits us to assume that members of $\mathfrak{F}$ introduced in the course of a proof are of the form S_a with a a good parameter. in its formulation, we speak of *START*, by which we mean the starting position of the $(S, \mathfrak{B})$-game.

9·5 LEMMA *In Case I, let $a \in \Upsilon_{H,\Lambda}$. Then there is a $d \in \Upsilon_{H,\Lambda}$ such that $S_d \subseteq S_a$ and d is good.*

In Case IV, let $a \in \Upsilon_{\mathfrak{B},H,\Lambda}$. If START is $k+1$-good, then there is a $d \in \Upsilon_{\mathfrak{B},H,\Lambda}$ such that $S_d \subseteq S_a$ and d is k-good; if START is Good, then d may be chosen to be Good.

Proof , *Case I:* let $BAD =_{\rm df} \{b \in \Upsilon_{H,\Lambda} \mid \neg\exists$ good b^* with $S_{b^*} \subseteq S_b\}$. The set BAD is OD. Suppose it is non-empty: then there is a good ψ such that

$$\varnothing \neq B_1(\psi) \subseteq BAD \subseteq \Upsilon_{H,\Lambda},$$

with $B_1(\psi) \in M_H$. We know that

$$\Upsilon_{H,\Lambda} = \{b \mid \Phi(\delta_H, b, \ldots)\}$$

where Φ is Σ_1, and hence that

$$\forall b{:}{\in} B_1(\psi)\ \Phi(\delta_H, b, \ldots)\}.$$

Since $B_1(\psi) \in M_H$ the above is equivalent to some Σ^H_1 statement about ψ, and hence by logic there is a real c, recursive in ψ and therefore good, such that

$$S_c = \{\zeta \mid \forall b{:}{\in} B_1(\psi)\ \Phi(\zeta, b, \ldots)\}.$$

Let $b \in B_1(\psi)$: then $S_c \subseteq S_b$, c is good and $b \in BAD$, a contradiction.

Proof , *Case IV:* a very similar argument. Let $BAD_k =_{\rm df} \{b \in \Upsilon_{\mathfrak{B},H,\Lambda} \mid \neg\exists$ k-good b^* with $S_{b^*} \subseteq S_b\}$. The set BAD_k is OD_S. Suppose it is non-empty and START is $k+1$-good then there is a k-good ψ such that

$$\varnothing \neq B_4(\mathfrak{B}, \psi) \subseteq BAD_k \subseteq \Upsilon_{\mathfrak{B},H,\Lambda},$$

with $B_4(\mathfrak{B}, \psi) \in M_{\mathfrak{B},H}$. As before, one finds a c recursive in ψ and therefore k-good, such that

$$S_c = \{\zeta \mid \forall b{:}{\in} B_4(\mathfrak{B}, \psi)\ \Phi(\zeta, b, \ldots)\}.$$

Let $b \in B_4(\mathfrak{B}, \psi)$: then $S_c \subseteq S_b$, c is k-good and $b \in BAD_k$, a contradiction.

If START is Good and the theorem false, one may find a Good ψ such that

$$\varnothing \neq B_4(\mathfrak{B}, \psi) \subseteq BAD_\omega \subseteq \Upsilon_{\mathfrak{B},H,\Lambda},$$

and $B(\mathfrak{B}, \psi) \in M_{\mathfrak{B},H}$, where $BAD_\omega =_{\rm df} \{b \in \Upsilon_{\mathfrak{B},H,\Lambda} \mid \neg\exists$ Good b^* with $S_{b^*} \subseteq S_b\}$, and the c found later will be Good, being recursive in the Good real ψ; leading again to a contradiction. ⊣ (9·5)

9·6 COROLLARY *Thus in Case I, $\mathfrak{F}$ is generated by sets S_d with d good; and in Case IV, if START is $k+1$-good; $\mathfrak{F}$ is generated by sets S_d with d k-good; and if START is Good, $\mathfrak{F}$ is generated by sets S_d with d Good.*

The normality of F

We follow the proof mentioned in Chapter V that avoids appeal to an induction on the length of the pre-well-ordering in question. We now formulate the normality statement in two cases.

9·7 THEOREM (Case I) *Let Q be a pre-well-ordering Σ^1_1 in Λ and a good parameter p_0, so that Q is definable from S and p_0. Let g be a partial function from δ_H to ON which is also definable from S and p_0, and such that $X_0 =_{\rm df} \{\nu \mid \nu \in Dom(\Diamond) \cap Dom(g)\ \&\ g(\nu) < |Q(\nu)|\}$ is in $\mathfrak{F}^+$. SUPPOSE THAT HOW MUCH DETERMINACY HOLDS ? Then there is a real $a \in Field(Q)$ such that*

$$\{\nu \mid \nu \in X_0\ \&\ a \in Field(Q(\nu))\ \&\ g(\nu) = |a|_{Q(\nu)}\} \in \mathfrak{F}^+.$$

9·8 REMARK Note that X_0 is definable from g and Q.

9·9 THEOREM (Case IV) *Let $\mathfrak{B}$ be a non-empty* ODset *of S-constructors. Let Q be a pre-well-ordering Σ^1_1 in Λ, $\mathfrak{B}$ and a Good parameter p_0, so that Q is definable from S and p_0. Let g be a partial function from δ_H to ON which is also definable from S and p_0, and such that $X_0 =_{\mathrm{df}} \{\nu \mid \nu \in Dom(\diamondsuit) \cap Dom(g) \;\&\; g(\nu) < |Q(\nu)|\}$ is in $\mathfrak{F}^+$. Then there is a real $a \in Field(Q)$ such that*

$$\{\nu \mid \nu \in X_0 \;\&\; a \in Field(Q(\nu)) \;\&\; g(\nu) = |a|_{Q(\nu)}\} \in \mathfrak{F}^+.$$

9·10 REMARK Note that X_0 is again definable from g and Q.

9·11 PROBLEM Might we want a to have some GOODNESS ?

Reduction to a quasi-disjoint sequence

Step 1: g is defined on a set X_0 in $\mathfrak{F}^+$. We may assume that for each $b \in \mathrm{Field}(Q)$, $D(b) \in \widetilde{\mathfrak{F}}$, where

$$D(b) =_{\mathrm{df}} \{\nu \mid \nu \in X_0 \,\&\, b \in \mathrm{Field}(Q(\nu)) \;\&\; g(\nu) = |b|_{Q(\nu)}\};$$

since otherwise we would have reached the conclusion of the theorem.

9·13 REMARK Goodness is not needed here.

Step 2: Define

$$Z =_{\mathrm{df}} \{(b,\, c) \mid b \in \mathrm{Field}(Q) \;\&\; c \in \Upsilon_{\mathfrak{B},H,\Lambda} \;\&\; S_c \cap D(b) = \varnothing\}.$$

By the Coding Lemma, first stage, there is a $Z^* \subseteq Z$, in $pos\boldsymbol{\Sigma}^1_1(<_Q)$, and hence projective in Λ, which meets every component of Q in the sense that

$$\forall b' {:\in} \mathrm{Field}(Q)\; \exists (b,\, c) {:\in} Z^*\; b =_Q b'.$$

9·14 PROBLEM DO WE NEED Z^* to have any goodness ? SHOULD WE HAVE another notation, if this is only the first stage ? The NEXT STRETCH needs only reflection arguments and therefore is still valid.

9·15 PROBLEM WHICH IS TRUE AND RELEVANT ? — Z^* is definable from Q and Z; Z is definable from Q, $\mathfrak{B}$, H and Λ.

Note that we can turn Z^* into the field of a prewellordering Q^* by setting

$$(b, d) \leq_{Q^*} (b', d') \iff b \leq_Q b'.$$

Then $|Q^*| = |Q|$, and $|(b,d)|_{Q^*} = |b|_Q$. Since Z^* is of the form $G^{<Q}_{a^*}$, we may set, for $\delta \in \mathrm{Dom}(\diamondsuit)$, $Z^*(\delta) = G^{<Q(\delta)}_{a^*}$ and reflect Q^* correspondingly to $Q^*(\delta)$, which for each δ in a set in $\mathfrak{F}$ will be a pre-well-ordering.

Reflection tells us, of course, that since Z^* meets every component of Q, for almost all[C 4] δ, $Z^*(\delta)$ will meet every component of $Q(\delta)$; our aim though is to show that Z^* itself meets every component of $Q(\delta)$, for many δ.

Now $\qquad \forall (b,\, c) {:\in} Z^*\; \Phi(c, \Lambda, \diamondsuit, \delta_H; \mathfrak{B}, H);$

so for δ in some $X_1 \in \mathfrak{F}$, $\qquad \forall (b,\, c) {:\in} Z^*(\delta)\; \Phi(c, \diamondsuit(\delta), \diamondsuit \restriction \delta, \delta; \mathfrak{B}, H);$

in other words,

(9·15·1) $$\delta \in X_1 \;\&\; (b,\, c) \in Z^*(\delta) \implies \delta \in S_c.$$

[C 4] by which we mean 'for all δ in a set in $\mathfrak{F}$'.

Suppose that $(b, c) \in Z^* \cap Z^*(\delta)$; then $(b, c) \in Z$, so $S_c \cap D(b) = \varnothing$; but also $\delta \in S_c$; so $\delta \notin D(b)$; and so we have

$$\delta \in X_0 \cap X_1 \ \&\ (b, c) \in Z^* \cap Z^*(\delta) \implies g(\delta) \neq |(Q \restriction b)(\delta)| = |b|_{Q(\delta)};$$

or, put another way,

$$\delta \in X_0 \cap X_1 \ \&\ g(\delta) = \alpha < |Q^*(\delta)| \implies (Q^*(\delta))_\alpha \cap Z^* = \varnothing \tag{9·15·2}$$

DETERMINACY IS NOW USED:

But (9·15·2) is a Σ_1 statement about Λ; or at least it would be if we could code $X_0 \cap X_1$ and g appropriately. An admissible coding is to hand, as we saw in §6 of Chapter III, starting from the pre-wellordering $\prec_H$ of length δ_H of the universal set Υ_H for Γ_H, at a cost of at least one round. WHAT IF WE WANT the codes to have some goodness.

So let $\psi_{01}^{(?)}$ be an admissible code of the characteristic function of $X_0 \cap X_1$, and let $\psi_g^{(?)}$ be an admissible code of the function g. Then we may re-write (9·15·2) as the following Σ_1^H statement about Λ, $\psi_{01}^{(?)}$, $\psi_g^{(?)}$ and δ_H:

$$\forall\delta{:<}\delta_H \left(\psi_{01}^{(?)}(\delta) \simeq 1 \implies \forall\alpha{:<}|Q^*(\delta)| \left(\psi_g^{(?)}(\delta)\downarrow = \alpha \implies (Q^*(\delta))_\alpha \cap Z^* = \varnothing\right)\right)$$

There will be an $X_2 \in \mathfrak{F}$ with the corresponding reflected statement holding for all $\zeta \in X_2$. In other words,

$$\forall\delta{:<}\zeta \left(\psi_{01}^{(?)}(\delta) \simeq 1 \implies \forall\alpha{:<}|Q^*(\delta)| \left(\psi_g^{(?)}(\delta)\downarrow = \alpha \implies (Q^*(\delta))_\alpha \cap Z^*(\zeta) = \varnothing\right)\right)$$

Let us check that that is indeed the correct reflected statement. First, note that

$$\forall\delta{:<}\delta_H\ Q^*(\delta) \in J_{\delta_H}(\mathbb{R}; H) \ \&\ |Q^*(\delta)| < \delta_H,$$

because the pre-well-orderings $Q^*(\delta)$ are all members of M_H; Q^* is projective in Λ, and so $Q^*(\delta)$ is projective in $\diamondsuit(\delta)$; the $\diamondsuit$ function is very local in its Σ_1^H recursive definition, and thus the definition of $Q^*(\delta)$ will retain its meaning in $J_\zeta(\mathbb{R}; H)$; hence we may assume that for all $\zeta \in X_1$,

$$\forall\delta{:<}\zeta\ Q^*(\delta) \in J_\zeta(\mathbb{R}; H) \ \&\ |Q^*(\delta)| < \zeta.$$

Secondly, every H-safe **OH ! OH !** ordinal is a limit of ordinals closed under the Gödel pairing function for ordinals that we used in establishing the admissible coding, and the initial segments of the canonical pre-well-ordering of U^H used in computing the admissible coding can be recovered in a Δ_1 way, so the Δ_1 statements $\psi_{01}^{(?)}(\nu) \simeq 1$ and $\psi_g^{(?)}(\delta)\downarrow \leq \alpha$ are going to preserve their meaning on reflection.

Thus taking $X = X_0 \cap X_1 \cap X_2$ and translating back to the original terms, we obtain this statement:

$$\exists X{:\in}\mathfrak{F}^+\ \forall\zeta{:\in}X\ \forall\delta{:\in}\zeta \cap X\ \left[g(\delta) = \alpha < |Q^*(\delta)| \implies (Q^*(\delta))_\alpha \cap Z^*(\zeta) = \varnothing\right] \tag{9·15·3}$$

9·17 REMARK X is the result of intersecting the original X_0 by a set in $\mathfrak{F}$; which we may choose to be of goodness only one less than that of START. SEE 8·22 ????

9·18 REMARK At this point: we have not yet needed $\psi_g^{(?)}$ and $\psi_{01}^{(?)}$ to be good; and the indices c of the S_c involved in Z have not been required to be good.

[The non-mono proof has been moved to "section 15"]

Here are some remarks relevant to the application of the non-mono theorem.

REMARK ONE: (needing to be initialised)

9·19 REMARK *Note that* TRACK *is* $\Sigma_1^H(\psi_g^{(?)})$ *since it is defined from the function g and the sequence $\langle R(\delta) \mid \delta < \delta_H\rangle$ which in turn is defined uniformly from the sequence $\langle\diamondsuit(\delta) \mid \delta < \delta_H\rangle$, which, as seen in Chapter*

I §4, is Σ^H_1. IS X INVOLVED ? The definability of X seems not to be so important since we can work if necessary from a better subset of it.

NOTE: we shall have suppressed X in the treatment of the non-mono:

We define a sequence of pre-well-orderings:

REMARK TWO:

9·20 REMARK the bound on the length of search in treatment of non-mono *ensures that the functions* ADD *and* CHOP *are* $\Delta^H_1(\psi_g^{(?)}, \psi_{01}^{(?)})$.

REMARK THREE

TO NORMALITY OF F:

9·21 REMARK We stress that the computation of the sets $\text{TRACK}_{\leqslant b}$ and $\text{TRACK}_{<b}$ only occurs in contexts where we have already checked that $b \in \text{TRACK}$ with $\delta_b < |T|$; thus the length of T is being used as a bound on the length of search. GAMMA ?

REMARK FOUR

9·22 REMARK Each of those relations $F_i(b, T)$ is $\Delta^H_1(T, \vec{R}, g)$. MOVE TO NORMALITY OF $\mathcal{F}$?

Normality of $\mathfrak{F}_{\Lambda,H}$

We apply the non-monotonic theorem taking $R(\delta)$ to be $Q^*(\delta)$ and X to be $X_0 \cap X_1 \cap X_2$. **HO-HO!**

For a real a in the field of Q we define $K_{\infty,a}$ to be $\text{TRACK} \cup (Q)_a$, extending the pre-well-ordering of TRACK by a single component placed at the top, namely the component $(Q)_a$ of a in the pre-well-ordering Q.

We apply the index s that we have found with $ff(s) = \infty$ to the pre-well-orderings $(K_{\infty,a})$: we assert that for the a's in exactly one component of Q, the intersection $U_s(K_{\infty,a}) \cap (Q)_a$ is non-empty; for to say that there are two Q-inequivalent such a's is to make a Σ^H_1 statement that cannot reflect to a set in $\mathfrak{F}_{\Lambda,H}$ without meeting a $\nu \in X$ for which we know the statement to be false; and to say that there is no such a at all is to make another Σ^H_1 statement that cannot reflect.

Let us see why the latter is true. $\{\text{TRACK} \cup (Q)_a \mid a \in \mathcal{N}\}$ is a member of $J_\Theta(\mathbb{R}; H)$, which is the union of an increasing sequence of transitive models of $ZF - \mathcal{P}$, each containing every real. Thus we can say that there is a single such model which considers that for every real the intersection $U_s(K_{\infty,a} \cap (Q)_a) = \varnothing$. That reflects to saying that for ν in a set in $\mathfrak{F}_{\Lambda,H}$ there is a transitive model M of $ZF - \mathcal{P}$ containing all reals which believes that $\forall a \mathpunct{:}\in \mathcal{N}\ U_s(K_{\nu,a} \cap (Q(\nu))_a) = \varnothing$. The computations being absolute, that statement is true; but then it would be true for some ν in the set in $\mathfrak{F}^+_{\Lambda,H}$ that we have reached, for which we know that the statement is false.

We consider an a in that unique component of Q. The reflected statement ensures that $|a|_{Q(\nu)}$ must equal $g(\nu)$ for all ν in a set in $\mathfrak{F}^+_{\Lambda,H}$.

NEED WE TWO CASES FOR THE FOLLOWING: $\mathrm{Det}(OD_{R,A})$?

9·23 COROLLARY (Case I) *Let R be a pre-well-ordering projective in Λ, and let A be a set that meets each component of R: then*

$$\big\{\nu \bigm| \forall\alpha \mathpunct{:}< |R(\nu)|\ A \cap (R(\nu))_\alpha \neq \varnothing\big\} \in \mathfrak{F}.$$

9·24 COROLLARY (Case IV) *Let R be a pre-well-ordering projective in Λ, and let A be a set that meets each component of R: then*

$$\big\{\nu \bigm| \forall\alpha \mathpunct{:}< |R(\nu)|\ A \cap (R(\nu))_\alpha \neq \varnothing\big\} \in \mathfrak{F}.$$

Proof : let $g(\nu) =$ the least $\alpha < |R(\nu)|$ such that $(R(\nu))_\alpha \cap A = \varnothing$. If the corollary is false, g will be defined on a set X in $\mathfrak{F}^+$, and therefore there is an $b \in Field(R)$ such that on a set $Y \subseteq X$ in $\mathfrak{F}^+$, $g(\nu) = |b|_{R(\nu)}$. By assumption, $\exists a \mathpunct{:}\in A\ a =_R b$, so $T =_{\mathrm{df}} \{\nu \mid a =_{R(\nu)} b\}$ is in $\mathfrak{F}(\Lambda)$ and so meets Y. But then for $\nu \in Y \cap T$, $a \in (R(\nu))_{g(\nu)} \cap A$, a contradiction. ⊣ (9·24)

============== below here is possibly junk:

9·25 REMARK If A were (say) in $pos\mathbf{\Sigma}^1_1(\Lambda)$, so that we had reflected versions A_ν to hand, then of course A_ν would meet each component of $R(\nu)$, for almost all ν. However the corollary states that A itself meets all components of $R(\nu)$, for many ν, and A itself is not required to be reflectible. In the next section we apply this corollary to establish the normality of $\mathfrak{U}_{\Lambda,H}$: in our application there will be a set, projective in A, of strategies for certain games; the corresponding sets, projective in $A(\nu)$, will certainly be of strategies for something, but we would not know that they were strategies for the same games. Hence the importance of the corollary as stated.

9·26 REMARK The conclusion of the theorem may be derived from the corollary, which therefore may be considered as an alternative formulation of normality.

We must check the application.

We examine the proof of Corollary (V·5·9).

We are given R, a pre-well-ordering projective in Λ, and A, the latter a set that meets every component of R, and we have to show that almost everywhere A meets every component of $R(\nu)$.

We define $g(\nu)$ to be the first component of $R(\nu)$ that is not hit by A.

In the context of the application: we have reached a Z definable from ψ_0 and Λ. so Z is good. ψ_0 scores 1.

We find a Z^* which is full and flexible.

we seek an X_3 where Z^* meets every component of $Z^*(\zeta)$.

So we are pulling the stunt that Z^* meets every component of itself.

We define $g(\nu)$ to be the first component of $Z^*(\nu)$ that is not met by Z^*.

We want the pair (g, Z^*) to be jointly good. g is defined from Z^*. Therefore we need Z^* to be good, which means a further application of the Basis Theorem. So we have scored more.

We score TWO to get a ψ_0 which will work for every η, and now we need to come back with a game defined from ψ_0, namely the second stage of the Coding Lemma, to get Z^* to be good. (That doesn't require anymore of ψ_0 for the moment, it is still only one move past).

Now having found g, we must feed it into the normality theorem. Are we going round in circles ?

The main problem will be with the minimisation.

I think the theorem should say:

given Q and g jointly good there is a real.

10: The definition of $\mathfrak{U}$ and of the extenders

We choose an ordinal λ and an OD_S pre-well-ordering Λ of $\mathcal{N}$ of length λ.

We define $\Upsilon_{H,\Lambda}$ as before. That accomplished, we shall define $\mathfrak{W}_{H,\Lambda}$, a subset of $[\Upsilon_{H,\Lambda}]^{\aleph_0}$ as in Chapter III, where our previous proof applies to show that $\mathfrak{W}$ is a countably complete filter. As before, we may project that to a filter $\mathfrak{U}_{H,\Lambda}$, the intersection of which with HOD_S proves, assuming enough determinacy, to be a δ_H-complete ultrafilter in HOD_S, showing that δ_H is measurable in HOD_S. In detail:

10·0 PROPOSITION *$\mathfrak{U}$ is a countably complete filter. Its intersection with HOD_S is a member of that model.*

10·1 PROPOSITION (Det(OD_S)) *That intersection $\mathfrak{U} \cap HOD_S$ is a δ_H-complete ultrafilter in that model; δ_H is accordingly a measurable cardinal in HOD_S.*

$\mathfrak{U}_{H,\Lambda}$ will not be a countably complete ultrafilter, since there are none in $L[x]$, but we may hope that it will measure[C 5] all subsets of δ_H which are in HOD, and perhaps a few more besides. The map $f = \bigcap(\bigcap_i S_{(a)_i} \cap \bigcap_i S_{(e)_i})$ is definable, and so the games $\mathcal{G}(A, f, X)$ for $X \in HOD$ and $A = \Upsilon_{H,\Lambda}$ are determined.

10·2 PROPOSITION $\mathfrak{F} \subseteq \mathfrak{U}$.

Proof : let $e_0 \in \Upsilon_{H,\Lambda}$. Let Eve play any e with $(e)_0 = e_0$ and all $(e)_i$ in $\Upsilon_{H,\Lambda}$. Then if Adam follows all the rules, the critical ordinal will be in S_{e_0}, which therefore is in $\mathfrak{U}$. ⊣ (10·2)

In Chapter V, we were able to show that $\mathfrak{U}$ is a δ_H-complete ultrafilter in the universe, and to use it to form an ultrapower of HOD, thus generating an embedding $j_{\mathfrak{U}} : HOD \longrightarrow M$ which, being itself ordinal definable, is visible to HOD as a class. The embedding itself is H-strong, and thus yields an H-strong extender in HOD.

[C 5] that is, decide

There we could use the full Axiom of Determinacy to establish the relevant properties. Here we have only weaker forms of AD compatible with the axiom of choice, and so we shall have to build the extenders directly.

⊛ ⊛ 10·3 To discover their definition we return to the context of Chapter V and undo the tower of definitions of that construction. In the AD context, we began with an arbitrary OD subset H of Θ; that defined an ordinal δ_H, for which we could prove the existence of an appropriate $\diamondsuit$-sequence. Then we took an OD prewellordering Λ of the reals of length some limit ordinal λ between δ_H and Θ, and from that we defined a filter $\mathfrak{F}$ and an ultrafilter $\mathfrak{U}$.

The extender associated to $j_{\mathfrak{U}}$ is defined thus: for $s \in [\lambda]^{<\omega}$ and X an OD subset of $[\delta_H]^{\overline{\overline{s}}}$,

$$X \in E(s) \iff_{\rm df} s \in j_{\mathfrak{U}}(X).$$

Let us recall how ordinals less than λ were represented in the ultrapower of HOD by $\mathfrak{U}$: if $\zeta < \lambda$, take any real a with $|a|_P = \zeta$; then for ν in the domain of the $\diamondsuit$-sequence, we set $f_a(\nu) = |a|_{\diamondsuit(\nu)}$. We found that $[f_a]_{\mathfrak{U}} = \zeta$, which we proved by induction on ζ using the normality of $\mathfrak{U}$.

So to translate the above definition, let $s = (\zeta_0, \dots, \zeta_{k-1})$: pick reals a_i with $|a_i|_P = \zeta_i$; then for X as above,

$$X \in E(s) \iff \{\nu \mid (f_{a_0}(\nu), f_{a_1}(\nu), \dots f_{a_{k-1}}(\nu)) \in X\} \in \mathfrak{U}.$$

$\mathfrak{U}$ being itself defined as the projection of the ultrafilter $\mathfrak{W}$ on $[\Upsilon_{H,\Lambda}]^{\le\aleph_0}$ by a map defined using the filter $\mathfrak{F}$, and $\mathfrak{W}$ being defined in terms of games, ultimately, for $X \in HOD$, $X \in E(s) \iff$ Eve has a winning strategy in a certain game ordinal definable from a_0, …, a_{k-1}; H, X, Λ and $\mathfrak{F}$ being themselves ordinal definable.

In our present context we eliminate dependence on the reals $a_0, \dots a_{k-1}$, which might not be good, by the use of the good tames sets $B_?(\psi)$ studied in section 8.

details, Case I

We wish to define an extender in HOD with support λ. So for $s \in [\lambda]^{<\omega}$, we shall define a measure E_s on $[\delta_H]^{|s|}$ with $\langle E_s \mid s \in [\lambda]^{<\omega}\rangle \in HOD$.

Suppose that $A \in HOD$ and that $A \subseteq [\delta_H]^k$. Let $\alpha_0 < \dots \alpha_k < \lambda$ (call the set of them s). We know from our discussion in section 8 that there is a ψ — indeed we know that we can find a good such ψ — such that

$$B_1^k(\psi) \subseteq (\Lambda)_{\alpha_0} \times \dots \times (\Lambda)_{\alpha_{k-1}} = [\Lambda]_s$$

The important thing here is not so much that $B_1^k(\psi)$ is good as that it is tame.

Given such tame D we may, as seen in 8·10 define a partial map $r_D : \delta_H \longrightarrow [\delta_H]^k$. Since D is a subset of a single component, the set

$$S(D) =_{\rm df} \{\nu \mid D \text{ is a subset of a single component of } (\diamondsuit(\nu))^k\}$$

will be in $\mathfrak{F}$; r_D has domain $S(D)$ and its value at $\nu \in S(D)$ will be that member $\{\alpha_0^\nu \dots \alpha_{k-1}^\nu\}$ of $\left[|\diamondsuit(\nu)|\right]^k$ with

$$D \subseteq (\diamondsuit(\nu))_{\alpha_0^\nu} \times \dots \times (\diamondsuit(\nu))_{\alpha_{k-1}^\nu}.$$

We may use such a function r_D to project $\mathfrak{U}_{\Lambda,H}$ to a filter on $[\delta_H]^k$: we may put

$$T(A, D) = \{\nu \in S(D) \mid r_D(\nu) \in A\}.$$

To know that $\mathfrak{U}_{\Lambda,H}$ measures $T(A, D)$, we need to know that D is good. For instance, $\mathrm{Det}(OD(\psi))$ will suffice to show that $T(A, B_1^k(\psi))$ is measured by $\mathfrak{U}_{\Lambda,H}$. It is here that the existence of at least one good ψ becomes important.

We may now define the extender.

10·4 Definition For $A \in HOD_S$, we set $A \in E(s) \iff_{\rm df} \exists\psi\ B_1^k(\psi) \subseteq [\Lambda]_{\vec{s}}\ \&\ T(A, B_1^k(\psi)) \in \mathfrak{U}_{\Lambda,H}$.

10·5 Remark Goodness is not needed in the definition, for this reason: we can find a good ψ, and for such $T(A, B_1^k(\psi])$ is measured by $\mathfrak{U}_{\Lambda,H}$; and for two possible ψ's, good or otherwise, the sets $T(A, B_1^k(\psi))$ differ by a set of $\mathfrak{F}$-measure 0, and therefore if one of them is measured by $\mathfrak{U}_{\Lambda,H}$ they all are; and in the same sense.

10·6 Remark $E(s)$ is an ultrafilter since $\mathfrak{U}_{\Lambda,H} \cap HOD_S$ is an ultrafilter, and the map r_B is definable from S because it is independent, mod $\mathfrak{F}$, of the choice of B.

In Case IV we work instead with sets $B_4^k(\mathfrak{B}, \psi)$.

The rest of this chapter will be a verification that E is indeed an extender of the desired sort.

An alternative approach

The reader may prefer the following description of the measure $E(s)$.

First, an abstract discussion, in ZFC: suppose we have a set X, and in the power set of X a filter $\mathfrak{F}$ and an ultrafilter $\mathfrak{U}$, with $\mathfrak{F}$ a subset of $\mathfrak{U}$.

We have another set Y and we wish to project the ultrafilter $\mathfrak{U}$ to give an ultrafilter on Y. In our context there is no natural map from X to Y, but there are many near misses. That is, there is a natural collection of maps, r_B, each defined on a set B in $\mathfrak{F}$ and taking values in Y, and any two of these maps, r_B, r_C, say, agree on a set D in $\mathfrak{F}$ with D some subset of $B \cap C$.

Thus, given a subset W of Y, its preimage under any one of these maps r_B defines a unique member of the factor algebra $\mathcal{P}(X)/\mathfrak{F}$. So we have an injection j from $\mathcal{P}(Y)$ into $\mathcal{P}(X)/\mathfrak{F}$.

Since $\mathfrak{F} \subseteq \mathfrak{U}$, $\mathfrak{U}$ will be closed under equivalence modulo $\mathfrak{F}$, and so gives rise to an ultrafilter $\mathfrak{U}/F$ in the algebra $\mathcal{P}(X)/\mathfrak{F}$. We may now define an ultrafilter $\mathfrak{V}$ on Y as the preimage of $U/\mathfrak{F}$ under the injection j.

The above describes our context, with $\mathfrak{U} = \mathfrak{U}_{\Lambda,H} \cap HOD_S$, $\mathfrak{F} = \mathfrak{F}_{\Lambda,H} \cap HOD_S$; the maps r_B are external to HOD_S, relying in their definition as they do on the use of sets of reals; but the association of each subset of $[\delta_H]^{\overline{\overline{s}}}$ to an equivalence class mod $\mathfrak{F}_{\Lambda,H}$ is indeed definable.

11: Elementary properties of the extender

We give here our new arguments avoiding goodness; hence avoiding a division into cases.

Verification that we have an extender

11·0 Proposition *E is in HOD_S a (δ_H, λ) extender.*

We verify in turn the first three and the last of the five clauses *EXT 1* to *EXT 5* defined in Chapter IV: the verification of *EXT 4*, under, in Case IV a stronger hypothesis, will be done in Section 12.

11·1 Lemma *If $0 < \eta < \delta_H$, there is a pre-well-ordering P of $\mathcal{N}$ of length η which is both tame and OD_S.*

Proof : let Υ_H be the universal Σ_1^H set (with no mention of $\diamondsuit$) with a pre-wellordering of length δ_H. That is definable from S and H and therefore solely from S, H being OD_S. Each initial segment is tame, and so $\Upsilon_H \restriction \eta$ is nearly as required: add all reals not in its field to its 0-th component, which can be done without jeopardising its OD_S status. ⊣ (11·1)

11·2 Remark A slightly subtler argument than the one given in Chapter One.

11·3 Proposition (Det(OD_S)) *Let $s \subseteq \lambda$ and $0 < \eta < \delta_H$, and let $\langle A_\nu \mid \nu < \eta \rangle \in HOD_S$ be a sequence of members of $E(s)$. Then $\bigcap_{\nu<\eta} A_\nu$ is non-empty.*

Proof : By the lemma, there is an OD_S pre-well-ordering Λ in M_H of length η; by adding all reals not in its field to its first component, we may assume that it has field $\mathcal{N}$.

The game $\mathcal{G}_M(\{\zeta \mid r_B(\zeta) \in A_{|a|_P}\})$, is determined, being definable from S and ψ.

Consider

$$Z \ =_{\mathrm{df}} \ \big\{(a,\, \tau) \;\big|\; \tau \text{ is a winning strategy for Eve in } \mathcal{G}^*(\{\zeta \mid r_B(\zeta) \in A_{|a|_P}\})\big\},$$

which is OD from S and good parameters. Hence by Stage One of the coding lemma there is a $Z^* \subseteq Z$ which is $pos\Sigma^1_1(P, \mathfrak{B})$ and a parameter which, if we invoke stage two, may be taken to be good, [N 8] and full in Z. Let $\mathcal{E}^* = \{\tau \mid \exists a\,(a, \tau) \in Z^*\}$. Then $\mathcal{E}^* \in M_H$.

We now imitate the argument in Chapter V that $\mathfrak{U}$ is complete. [N 9]

Let $\alpha_0 \in \Upsilon_{H,\Lambda}$. Then

$$\forall \alpha (\alpha)^0 = \alpha_0 \implies \forall \tau{:}{\in}\mathcal{E}\ ([\alpha] * \tau)^0 \in \Upsilon_{H,\Lambda} :$$

that is a Σ_1 statement, and so enforced by some α_1; we may inductively pick $\alpha_k \in \Upsilon_{H,\Lambda}$ such that

$$S_{\alpha_{k+1}} \subseteq \{\zeta \mid \forall \alpha \forall i{:}{\leq} k\ (\alpha)^i = \alpha_i \implies ([\alpha] * \tau)^k \in \Upsilon_{H,\Lambda}\}.$$

Let $\bar{\alpha}$ be such that $\forall n\,(\bar{\alpha})^n = \alpha_n$, and let $\bar{\xi} = \inf \bigcap_{n<\omega} S_{\alpha_n}$. Then for any $\tau \in \mathcal{E}^*$, $\bar{\xi}$ will be the critical ordinal of the run of the game where Adam plays $\bar{\alpha}$ and Eve uses τ. Hence $\bar{\xi} \in \bigcap_{\nu<\eta} A_\nu$, as required. $\dashv$ (11·3)

11·4 REMARK It might well be tidier to prove that $\mathfrak{U}_{\Lambda,H} \cap HOD_S$ is δ_H-complete, and then to project that completeness to the $E(s)$'s.

11·5 REMARK Amount of goodness: just the START for picking a good tame thing to define a determined game; after that, I think we might not need more.

11·6 PROPOSITION *The extender is coherent.*

Proof :

$$\begin{aligned}\{y \mid y^{s,t} \in X\} \in E(t) &\iff \{\zeta \mid (r_C(\zeta))^{s,t} \in X\} \in \mathfrak{U}\\ &\iff \{\zeta \mid r_B(\zeta) \in X\} \in \mathfrak{U}\\ &\iff X \in E(s)\end{aligned}$$

provided we can choose C and B so that $(r_C(\zeta))^{s,t} = r_B(\zeta)$ *modulo* $\mathfrak{F}$. So pick $\varnothing \neq C \subseteq (\Lambda)_t$, C ordinal-definable from S and the good parameter ψ, and $C \in M_H$; take $B = \{x^{s,t} \mid x \in C\}$. Then $\varnothing \neq B \subseteq (\Lambda)_s$; B is $\Sigma^1_1(C)$, and so is both ordinal-definable from S and ψ and a member of M_H. Hence $\{\zeta \mid (r_C(\zeta))^{s,t} = r_B(t)\} \in \mathfrak{F}$, as desired. $\dashv$ (11·6)

11·7 REMARK For the above, we use simply the existence of a tame C. B inherits its tameness.

11·8 PROPOSITION *If $\langle A_i \mid i \in \omega\rangle$ and $\langle s_i \mid i \in \omega\rangle$ are sequences in HOD_S such that $\forall i\, A_i \in E(s_i)\,\&\, s_i \subseteq s_{i+1}$, then there is a sequence $\langle v_i \mid i \in \omega\rangle$ such that $\forall i\, v_i = v_{i+1}^{s_i,s_{i+1}}$ & $v_i \in A_i$.*

Proof : Let $\varsigma = \bigcup_{i<\omega} s_i$: then ς is in HOD_S and is countable there. So let $f : \omega \longleftrightarrow \varsigma$, with $f \in HOD_S$. Let $D_f = \{\alpha \in \mathcal{N} \mid \forall n (\alpha)_n \in (\Lambda)_{f(n)}\}$ — an OD_S subset of $\mathcal{N}$. D_f is non-empty, by the axiom of choice for countable families of sets of reals, which is true in all the contexts we consider.

By the Simple Basis Theorem, there is a B_f, a tame non-empty subset of D_f, definable from the good parameter ψ_f, say. For each i, B_f projects to define non-empty, tame, $B_i \subseteq (\Lambda)_{s_i}$: B_i is the set of restrictions to the appropriate co-ordinates of the elements of B_f. Each $B_i \in M_H$, and the sequence $i \mapsto f^{-1}$“s_i is (effectively) a real and so is also in M_H. So $\langle B_i \mid i \in \omega\rangle \in M_H$.

Trivially, $B_i = B_{i+1}^{s_i,s_{i+1}}$, and so there is a set $Z_i \in \mathfrak{F}$ such that

$$\zeta \in Z_i \implies r_{B_i}(\zeta) = (r_{B_{i+1}}(\zeta))^{s_i,s_{i+1}}.$$

Since $A_i \in E(s_i)$, the set $Y_i =_{\mathrm{df}} \{\zeta \mid r_{B_i} \in A_i\} \in \mathfrak{U}$. By and we know that $\mathfrak{U}$ is countably complete and extends $\mathfrak{F}$, and so we may pick $\bar{\zeta} \in \bigcap_{i<\omega}(Y_i \cap Z_i)$. Now set $v_i = r_{B_i}(\bar{\zeta})$. Then $v_i \in A_i$ as $\bar{\zeta} \in Y_i$, and $v_i = v_{i+1}^{s_i,s_{i+1}}$ as $\bar{\zeta} \in Z_i$. $\dashv$ (11·8)

11·9 REMARK Again I think we have only to find the tame subset, B_f.

11·10 PROPOSITION *For each $\nu < \lambda$, $\{\{\xi\} \mid \xi \in H \cap \delta_H\} \in E(\{\xi\}) \iff \nu \in H$.*

[N 8] I suspect this goodness is not needed as we are not going to define any further games in this proof
[N 9] are we indeed proving such completeness ?

Proof : Fix a good and tame $B \subseteq (\Lambda)_{\{\nu\}}$. Write $\mathcal{T} = \{\{\xi\} \mid \xi \in H \cap \delta_H\}$. Then

$$\mathcal{T} \in E(\{\nu\}) \iff \{\zeta \mid r_B(\zeta) \in T\} \in \mathfrak{U}.$$

$$\begin{aligned} \text{But} \qquad \nu \in H &\implies \forall x{:}{\in} B\ |x|_\Lambda \in H \\ &\implies \{\zeta \mid \forall x{:}{\in} B\ |x|_{\diamondsuit(\zeta)} \in H\} \in \mathfrak{F} \subseteq \mathfrak{U} \\ \text{and} \qquad \nu \notin H &\implies \forall x{:}{\in} B\ |x|_\Lambda \notin H \\ &\implies \{\zeta \mid \forall x{:}{\in} B\ |x|_{\diamondsuit(\zeta)} \notin H\} \in \mathfrak{F} \subseteq \mathfrak{U}, \end{aligned}$$

and for ζ in a set in $\mathfrak{F}$ and $x \in B$, $r_B(x) = \{|x|_{\diamondsuit(\zeta)}\}$. ⊣ (11·10)

11·11 Remark Here again, it is the tameness that we need. Not the goodness.

12: Derivation of the normality of the extender from the normality of $\mathfrak{F}$.

12·0 WARNING: I seem to have changed notation here: I am now writing $r(B, P)$ for the single component of $P \times \ldots \times P$ including B.

Suggestion: when B is $B_4(\mathfrak{B}, \psi)$ and P is $\diamondsuit(\zeta)$, write $r_\psi(\zeta)$. Should there be $\mathfrak{B}$'s about ?

Dislike that; shall want a better treatment of k-tuples. Shall write $r_k(B(\psi, k), P)$ for the typical thing.

EXPLAIN the notation $[B]^{s,t}$. WARNING: do we need sups ? does r_1 produce ordinals or singletons of ordinals or sequences of length one of ordinals ? HAVE WE defined r_1, r_k ?

We might have a convention that the square brackets $[B]^1$ strip things down to single elements; or that sequences of length one are single elements.

[IS FIELD $\Lambda = \mathcal{N}$?] answer yes, since Λ is any OD $_S$ pre-well-ordering of length λ and we can add the leftovers to the zeroth position, if $\lambda > 0$; quite different from canonical Spector orderings of Υ, which are of quite specific sets of reals.

12·1 Remark We may estimate the amount of survival required. The hypothesis required for the next theorem is that START be 3-good. In the proof to come, Z is defined from ψ_0 and Λ; Λ is definable from S; if ψ_0 is $\ell + 2$-good, we may find a q which is $\ell + 1$-good and which gives us a tame Z^*.

At the point where we appeal to the normality of $\mathfrak{F}$, we have X_3, X and g, all definable from q. To obtain the index that gives us a contradiction that proves normality, we must play a game and have a winning strategy for one player. Thus q must be at least 1-good; so ψ_0 must be 2-good; and therefore the start must be 3-good. This calculation assumes that the normality theorem just needs the defining parameters of the game to be 1-good.

12·2 Remark We seem to be using ψ good over ψ_0 and S to mean that $\psi_0 \leqslant_{\text{Turing}} \psi$ and Det$OD(S, \psi)$ holds. Unobjectionable; "k-good over" would be more problematical.

The Simple Extension Game

Suppose that s and t are finite subsets of λ with $s \subset t$. By the Simple Basis Theorem, we know there to exist ψ_s good over S with $B^k_?(\psi_s) \subseteq (\Lambda)_s$, where $k = \overline{\overline{s}}$ and

$$B^k_4(\psi, k) =_{\text{df}} \{((x)_0, \ldots, (x)_{k-1}) \mid \exists z{:}{\in}\mathfrak{B}\ x = (\psi \otimes (\psi \oplus z)) \star [z]\}.$$

We also know that similar ψ_t exist for t, but can we be sure of finding ψ_t with $B(\psi_t, \overline{\overline{t}})^{s,t} \subseteq B(\psi_s, \overline{\overline{s}})$? We shall find that a short S-game will do the necessary.

To handle this problem we introduce another absurdly simple game, which we call the *Simple Extension Game*, $\mathcal{SE}_{s,t}(X, Y)$. This game will be defined whenever s and t are finite sets of ordinals with $s \subseteq t$ and $Y \subseteq ((X)_t)^{s,t}$. Adam plays a and Eve plays e and Adam is to win if

Rule: $((a)_0, \ldots, (a)_{k+1}) \in (X)_t$ and $((a)_0, \ldots, (a)_{k+1})^{s,t} \in Y$.

That game is definable from X and Y and ordinal parameters: WE NEED X to be a pre-wellordering for $(X)_t$ to make sense ?; and thus is OD from S and ψ_0 and hence determined, provided ψ_0 is good over S.

Adam will win provided Y is a non-empty subset of $(X)_t$.

Our particular interest will be when $t = s \cup \{\eta\}$ for some ordinal η.

12·3 Definition Let $\eta < \lambda$, with $\eta \notin s$. We make similar definition for the two Cases I and IV.

$$SE_I(\psi, \psi_0, \eta) \iff_{\mathrm{df}} \begin{cases} \psi \text{ is good over } \psi_0, \text{ and} \\ \varnothing \neq (B_1(\psi, k+1))^{s,\{\eta\}\cup s} \subseteq B_1(\psi_0, k) \text{ and} \\ \varnothing \neq (B_1(\psi, k+1))^{\{\eta\},\{\eta\}\cup s} \subseteq (\Lambda)_{\{\eta\}}. \end{cases}$$

$$SE_{IV}(\psi, \psi_0, \eta) \iff_{\mathrm{df}} \begin{cases} \psi \text{ is good over } \psi_0 \text{ and } S, \text{ and} \\ \varnothing \neq (B_4(\psi, k+1))^{s,\{\eta\}\cup s} \subseteq B_4(\psi_0, k) \text{ and} \\ \varnothing \neq (B_4(\psi, k+1))^{\{\eta\},\{\eta\}\cup s} \subseteq (\Lambda)_{\{\eta\}}. \end{cases}$$

12·4 The Simple Extension Theorem, Case I *Let p be a good parameter. Let s be a non-empty finite subset of λ. Then there is a ψ_0 good over p with $B_1^k(\psi_0) \subseteq (\Lambda)_s$ such that for any $\eta \in \lambda \smallsetminus s$ there is a good ψ with $SE_I(\psi, \psi_0, \eta)$.*

12·5 The Simple Extension Theorem, Case IVa *Let p be an $(\ell+2)$-good parameter. Let s be a non-empty finite subset of λ. Then there is a ψ_0 $(\ell+1)$-good over p with $B_4^k(\mathfrak{B}, \psi_0) \subseteq (\Lambda)_s$ such that for any $\eta \in \lambda \smallsetminus s$ there is an ℓ-good ψ with $SE_{IV}(\psi, \psi_0, \eta)$.*

12·6 The Simple Extension Theorem, Case IVb *Suppose that the second player has a winning strategy in the $(S, \mathfrak{B})$ game. Let s be a non-empty finite subset of λ. Then there is a ψ_0 Good over S with $B_4^k(\mathfrak{B}, \psi_0) \subseteq (\Lambda)_s$ such that for any $\eta \in \lambda \smallsetminus s$ there is a Good ψ with $SE_{IV}(\psi, \psi_0, \eta)$.*

Proof : write $t = s \cup \{\eta\}$. Consider the play of the $(S, \mathfrak{B})$-game where Adam's first call is the Simple Basis $\overline{\overline{s}}$-game $\mathcal{SB}((\Lambda)_s)$; Eve replies with some ψ_0 being a pre-strategy for that game; then Adam will choose an η and as his second call takes the Simple Extension game $\mathcal{SE}((\Lambda)_t, B(\psi_0, k))$, (WHERE $k = \overline{\overline{s}}$?) and Eve replies with ψ (dependent on η).

Then ψ will be as required: if START is 2-good, such ψ exist; but if we want them to be good as well; START must be 3-good. ⊣ (12·6)

The result

WARNING: we still have not stated the determinacy assumptions for this theorem. They appear to be that the START of the S-game is 2-good (in strict rules) or 3-good (in lenient rules).

12·7 Theorem *Let $s \in [\lambda]^{<\omega}$, where $\delta_H < \lambda = \bigcup \lambda < \Theta$. Let $h \in HOD_S$, h defined on $[\delta_H]^{\overline{\overline{s}}}$, such that*

$$\{u \mid h(u) < \bigcup u\} \in E(s).$$

Then there is an $\eta < \lambda$, which might be a member of s, such that setting $t = \{\eta\} \cup s$,

$$\{v \mid h(v^{s,t}) = \bigcup v^{\{\eta\},t}\} \in E(t).$$

Moreover, $\eta < \bigcup s$.

Proof : we deny the existence of any such $\eta < \lambda$ and derive a contradiction. Once we know such an η exists, we may easily see that it must be less than $\bigcup s$, since otherwise $\eta = \max t \geqslant \max s$, and $\bigcup v^{\{\eta\},t} = \max v \geqslant \max v^{s,t} > h(v^{s,t})$ for almost all v, a contradiction.

BLEMISH: we tacitly assume that η is not a member of s. GET RID OF THE CASE $\eta \in s$ INSTANTLY !

The proof

12·8 Definition $M(\psi, \eta) =_{\mathrm{df}} \big\{\zeta \mid \big[r_{k+1}\big(B(\psi, k+1), \Diamond(\zeta)\big)\big]^{s,\{\eta\}\cup s} = r_k\big(B(\psi_0, k), \Diamond(\zeta)\big)\big\}$.

12·9 Lemma $SE(\psi, \psi_0, \eta) \implies M(\psi, \eta) \in \mathfrak{F}$.

12·10 Lemma *If $SE(\psi, \psi_0, \eta)$, we have* $[r_{k+1}(B(\psi, k+1), \Lambda)]^{\{\eta\},\{\eta\}\cup s} = r_1\big([B(\psi, k+1)]^{\{\eta\},\{\eta\}\cup s}, \Lambda\big)$.

12·11 Definition $N(\psi,\eta) \ =_{\rm df}\ \{\zeta \mid [r_{k+1}(B(\psi,k+1),\diamondsuit(\zeta))]^{\{\eta\},\{\eta\}\cup s} = r_1([B(\psi,k+1)]^{\{\eta\},\{\eta\}\cup s},\diamondsuit(\zeta))\}$.

12·12 Lemma $SE(\psi,\psi_0,\eta) \implies N(\psi,\eta)\in\mathfrak{F}$.

12·13 Definition $K \ =_{\rm df}\ \{\zeta \mid h(r_k(B(\psi_0,k),\diamondsuit(\zeta))) < \bigcup r_k(B(\psi_0,k),\diamondsuit(\zeta))\}$.

12·14 Lemma $K\in\mathfrak{U}$.

12·15 Definition

$$L(\psi,\eta) \ =_{\rm df}\ \Big\{\zeta \ \Big|\ h\big([r_{k+1}(B(\psi,k+1),\diamondsuit(\zeta))]^{s,\{\eta\}\cup s}\big) = \bigcup[r_{k+1}(B(\psi,k+1),\diamondsuit(\zeta))]^{\{\eta\},\{\eta\}\cup s}\Big\}.$$

We suppose that whenever $\eta<\lambda$ and $SE(\psi,\psi_0,\eta)$, then $L(\psi,\eta)\notin\mathfrak{U}$; since to do otherwise is to admit the truth of the theorem.

12·16 Definition $J(\psi,\eta) \ =_{\rm df}\ L(\psi,\eta)\cup(\delta_H\setminus K)\cup(\delta_H\setminus M(\psi,\eta))\cup(\delta_H\setminus N(\psi,\eta))$.

Under that supposition,

12·17 Lemma $SE(\psi,\psi_0,\eta) \implies J(\psi,\eta)\notin\mathfrak{U}$.

We know that $\forall\eta{:<}\lambda\ \exists\psi\ SE(\psi,\psi_0,\eta)\ \&\ \exists b\ |b|_\Lambda=\eta$. We shall identify five sets in $\mathfrak{F}$. Here are the first two.

$\mathrm{Fld}(\Lambda)=\mathcal{N}$, so

$$X_0 \ =_{\rm df}\ \{\nu \mid \mathrm{Fld}(\diamondsuit(\nu))=\mathcal{N}\}\in\mathfrak{F}.$$

$\bigcup s<\lambda=|\Lambda|$, so

$$X_1 \ =_{\rm df}\ \{\nu \mid \bigcup r_k(B(\psi_0,k),\diamondsuit(\nu)) < |\diamondsuit(\nu)|\}\in\mathfrak{F}.$$

Now define [WHERE IS $\mathfrak{G}$ DEFINED ?]

$$Z \ =_{\rm df}\ \big\{(b,\sigma,\psi) \ \big|\ b\in\mathrm{Field}(\Lambda)\ \&\ SE(\psi,\psi_0,|b|_\Lambda)\ \&\ b\in[B(\psi,k+1)]^{\{|b|_\Lambda\},\{|b|_\Lambda\}\cup s}\ \&$$
$$\&\ \sigma \text{ is a winning strategy for Adam in } \mathfrak{G}\big(J(\psi,|b|_\Lambda)\big)\big\}.$$

HERE AGAIN we want to know if we are getting singletons or not.

Z is definable from Λ and ψ_0, so by the SECOND STAGE of the Coding Lemma, there is a $Z^*\subseteq Z$, which is full in Z, and $pos\Sigma^1_1(\Lambda,\mathfrak{B},q)$ for some Good parameter q; so that

$$\forall a\ \Big((Z^*)_{|a|_\Lambda} \text{ is non-empty and equals } \big\{(b,\sigma,\psi)\in Z^* \ \big|\ |b|_\Lambda=|a|_\Lambda\big\}\Big).$$

Hence, Z^* BEING REFLECTIBLE,

$$X_2 \ =_{\rm df}\ \Big\{\zeta \ \Big|\ \forall a\ \Big((Z^*(\zeta))_{|a|_{\diamondsuit(\zeta)}} \text{ is non-empty and equals } \big\{(b,\sigma,\psi)\in Z^*(\zeta) \ \big|\ |b|_{\diamondsuit(\zeta)}=|a|_{\diamondsuit(\zeta)}\big\}\Big)\Big\}\in\mathfrak{F}.$$

Further, CROSS REFERENCE TO 9.47:

$$X_3 \ =_{\rm df}\ \{\zeta \mid Z^* \text{ meets every component of } Z^*(\zeta)\}\in\mathfrak{F}:$$

IT IS HERE THAT WE APPEAL TO THE NORMALITY OF $\mathfrak{F}$. If X_3 were not in $\mathfrak{F}$, then $X \ =_{\rm df}\ \delta_H\setminus X_3$ would be in $\mathfrak{F}^+$, and on X we could define $g(\zeta)$ to be the first component of $Z^*(\zeta)$ not met by Z^*. We could then apply the Normality Theorem to this pair X,g, and conclude that there is a b in the field of Z^* such that for ζ in a set in $\mathfrak{F}^+$, $b\in\mathrm{Field}(Z^*(\zeta))$ and $g(\zeta)=|b|_{Z^*(\zeta)}$, contradicting the definition of g for every such ζ.

Therefore IT IS HERE that we need the second stage of the Coding Lemma; the first would suffice to get Z^* to be reflectible. We now need Z^* to be definable from some q which is Good enough TO APPLY THE NORMALITY THEOREM.

Finally, $\forall a\exists(b,\sigma,\psi){:\in}Z^*\ |a|_\Lambda=|b|_\Lambda$, and so

$$X_4 \ =_{\rm df}\ \{\zeta \mid \forall a\exists(b,\sigma,\psi){:\in}Z^*(\zeta)\ |a|_{\diamondsuit(\zeta)}=|b|_{\diamondsuit(\zeta)}\}\in\mathfrak{F}.$$

Set $\mathcal{A}^* = \{\sigma \mid \exists b{:}{\in}\mathcal{N}\ \exists\psi{:}{\in}\mathcal{N}\ (b,\sigma,\psi)\in Z^*\}$; $\mathcal{A}^*$ is projective in Z^* and so we may take as its reflection $\mathcal{A}^*(\nu) = \{\sigma \mid \exists b{:}{\in}\mathcal{N}\ \exists\psi{:}{\in}\mathcal{N}\ (b,\sigma,\psi)\in Z^*(\nu)\}$; as before, if we start from two possible definitions of Z^*, the reflected definitions will agree almost everywhere.

Now, following the argument in V.6, we settle down to making statements, with a view to establishing that Eve controls various games.

Since everything in $\mathcal{A}^*$ is an Adam strategy,

$$\forall\sigma{:}{\in}\mathcal{A}^*\ \forall e{:}{\in}\mathcal{N}\ \left[\Phi\big((\sigma\star[e])^0,\Lambda,\diamondsuit,\delta_H;H\big)\right]:$$

that is a Σ^1_1 statement about Λ; hence we may choose $e_0\in\Upsilon_{H,\Lambda}$ such that

$$S_{e_0}\subseteq X_0\cap X_1\cap X_2\cap X_3\cap X_4\cap\big\{\nu\,\big|\,\forall\sigma{:}{\in}\mathcal{A}^*(\nu)\ \forall e{:}{\in}\mathcal{N}\ \left[\Phi\big((\sigma\star[e])^0,\diamondsuit(\nu),\diamondsuit\restriction\nu,\nu;H\big)\right]\big\}.$$

That last requirement implies that

$$\nu\in S_{e_0}\ \&\ \sigma\in\mathcal{A}^*(\nu)\implies\forall e{:}{\in}\mathcal{N}\ \ \nu\in S_{(\sigma\star[e])^0}.$$

$$\text{Again,}\quad\forall\sigma{:}{\in}\mathcal{A}^*\ \forall e{:}{\in}\mathcal{N}\ \left[e^0=e_0\implies\Phi\big((\sigma\star[e])^1,\Lambda,\diamondsuit,\delta_H;H\big)\right]:$$

that is a Σ^1_1 statement; hence we may choose $e_1\in\Upsilon_{H,\Lambda}$ with

$$S_{e_1}\subseteq\big\{\nu\,\big|\,\forall\sigma{:}{\in}\mathcal{A}^*(\nu)\ \forall e{:}{\in}\mathcal{N}\ \left[e^0=e_0\implies\Phi\big((\sigma\star[e])^1,\nu,\diamondsuit(\nu),\diamondsuit\restriction\nu,\nu;H\big)\right]\big\}.$$

In other words,

$$\nu\in S_{e_1}\ \&\ \sigma\in\mathcal{A}^*(\nu)\implies\forall e{:}{\in}\mathcal{N}\ (e^0=e_0\implies\nu\in S_{(\sigma\star[e])^1}).$$

Continue for ω steps, to build a sequence e_k of elements of $\Upsilon_{H,\Lambda}$ such that

$$\nu\in S_{e_k}\ \&\ \sigma\in\mathcal{A}^*(\nu)\implies\forall e{:}{\in}\mathcal{N}\ \big((\forall i{:}{<}k\ e^i=e_i)\implies\nu\in S_{(\sigma\star[e])^k}\big).$$

We remark, as in earlier chapters, that DC is not necessary in the construction of the sequence $(e_k)_k$, the latter being a sequence of indices each defined uniformly from its predecessor.

Let $\bar e$ be the real with $\forall k\,e^k=e_k$, and let $\bar\xi$ be the least element of $\bigcap_{k<\omega}S_{\bar e_k}$. As $\bar\xi\in S_{e_0}$, $\bar\xi$ will lie in each of $X_0,\dots,X_4$. So since $\bar\xi\in X_3$, Z^* meets every NON-EMPTY ! component of $Z^*(\bar\xi)$; as $\bar\xi\in X_0$, $\mathrm{Fld}(\diamondsuit(\bar\xi))=\mathcal{N}$; as $\bar\xi\in X_4$, $Z^*(\bar\xi)$ is non-empty. So let the triple (b^*,σ^*,ψ^*) be in both Z^* and $Z^*(\bar\xi)$: for the moment we do not specify a particular component of $Z^*(\bar\xi)$. Then $\bar\xi=\mathfrak{o}(\sigma^*\star[\bar e],\bar e)$: so $\bar\xi\notin J(\psi^*,|b^*|_\Lambda)$, so in particular $\bar\xi\in K$, whatever b^* might have been.

Thus $h\big(r_k\big(B(\psi_0,k),\diamondsuit(\bar\xi)\big)\big)<\bigcup r_k\big(B(\psi_0,k),\diamondsuit(\bar\xi)\big)$. As $\bar\xi\in X_1$, $\bigcup r_k\big(B(\psi_0,k),\diamondsuit(\bar\xi)\big)<|\diamondsuit(\bar\xi)|$. Since $\bar\xi\in X_2\cap X_4$, $|\diamondsuit(\bar\xi)|=|Z^*(\bar\xi)|$, so let $\bar\beta=h\big(r_k\big(B(\psi_0,k),\diamondsuit(\bar\xi)\big)\big)$. $\bar\beta<|Z^*(\bar\xi)|$.

We now pick a representative: let $(\bar b,\bar\sigma,\bar\psi)\in Z^*\cap(Z^*(\bar\xi))_{\bar\beta}$. Thus $|\bar b|_{\diamondsuit(\bar\xi)}=\bar\beta$. Further, since $Z^*\subseteq Z$, we know that $\bar b\in\left[B(\bar\psi,k+1)\right]^{\{|\bar b|_\Lambda\},\{|\bar b|_\Lambda\}\cup s}$, so if we write $\bar\eta=|\bar b|_\Lambda$, we shall have

$$\bigcup r_1\big(\left[B(\bar\psi,k+1)\right]^{\{\bar\eta\},\{\bar\eta\}\cup s},\diamondsuit(\bar\xi)\big)=|\bar b|_{\diamondsuit(\bar\xi)}=\bar\beta\tag{12·18}$$

Then as $\bar\sigma\in\mathcal{A}^*(\bar\xi)$, $\bar\xi=\mathfrak{o}(\bar\sigma\star[\bar e],\bar e)$; as $\bar\sigma\in\mathcal{A}^*$, Adam has won the play of the game $\mathfrak{G}(J(\bar\psi,\bar\eta))$, and so $\bar\xi\notin J(\bar\psi,\bar\eta)$, whence

$$\bar\xi\notin L(\bar\psi,\bar\eta),\qquad\bar\xi\in M(\bar\psi,\bar\eta),\qquad\text{and}\qquad\bar\xi\in N(\bar\psi,\bar\eta);$$

which tells us that

$$\begin{aligned}
\bar\beta&=h\big(r_k\big(B(\psi_0,k),\diamondsuit(\bar\xi)\big)\big) && \text{by definition of }\bar\beta,\\
&=h\big(\left[r_{k+1}\big(B(\bar\psi,k+1),\diamondsuit(\bar\xi)\big)\right]^{s,\{\eta\}\cup s}\big) && \text{as }\bar\xi\in M(\bar\psi,\bar\eta),\\
&\neq\bigcup\left[r_{k+1}\big(B(\bar\psi,k+1),\diamondsuit(\bar\xi)\big)\right]^{\{\bar\eta\},\{\eta\}\cup s} && \text{as }\bar\xi\notin L(\bar\psi,\bar\eta),\\
&=\bigcup r_1\big(\left[B(\bar\psi,k+1)\right]^{\{\bar\eta\},\{\eta\}\cup s},\diamondsuit(\bar\xi)\big) && \text{as }\bar\xi\in N(\bar\psi,\bar\eta),\\
&=|\bar b|_{\diamondsuit(\bar\xi)} && \text{by (12·18)},\\
&=\bar\beta,\quad\text{a contradiction !} && \dashv\ (12·7)
\end{aligned}$$

13: Reduction of Case III to Case IV

[This section requires revision][N 10]

If we try to prove Case III directly, we find that we have a problem with the crucial sequence $\diamondsuit$. $\diamondsuit$ has to reflect all OD_S properties of sets of reals. It *might* happen that there are OD_S subsets of $\mathcal{N}$ not in $J_\Theta(S;\ldots)$: for example $S \cap \Theta$ might be empty and $S \setminus \Theta$ might define many interesting sets.

So we must seek a set $S^* \subseteq \Theta$ such that each OD_S set of reals is in $J_\Theta(S^*,\ldots)$.

An enumeration of OD_S sets of reals in type Θ

To begin with, our discussion is in ZF, and makes no use of AD.

Let S be a class of ordinals — possibly a proper class; write $\mathfrak{C}(S)$ for $\{x \subseteq \omega \mid V = L[x,S]\}$ and suppose that $\mathfrak{C}(S)$ is not empty. Thus AC holds. We write $\mathfrak{c}$ for the cardinality of the continuum, and θ for $\mathfrak{c}^+$. In this context, $\theta = \Theta$. Finally we suppose that $2^{\mathfrak{c}} = \theta$.

For each ordinal λ less than θ, there is an OD_S prewellordering of $\mathcal{P}(\omega)$ of order-type λ, by reasoning similar to that of (I·1·11): for example, we might start by defining χ_x for $x \in \mathfrak{C}(S)$ to be the $<_{L[x,S]}$-first surjection of $\mathcal{P}(\omega)$ onto λ and then set $\psi : \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ by

$$\psi(x,y) = \begin{cases} 0 & \text{if } x \notin \mathfrak{C}(S), \\ \chi_x(y) & \text{otherwise.} \end{cases}$$

That shows that there are at least θ distinct OD_S subsets of $\mathcal{P}(\omega)$; and there cannot be more, by our assumption that $2^{\mathfrak{c}} = \mathfrak{c}^+ = \theta$.

Our next task is to refine that remark to showing that there is an OD_S well-ordering of the collection of OD_S subsets of $\mathcal{P}(\omega)$ in order-type exactly θ; to do so, we must compare the cardinals of HOD_S with those of V.

The ordinal θ is of course a regular cardinal in HOD_S, being one in V, though it may well be larger than $(\Theta)^{HOD_S}$, since very few reals might be OD_S.

13·0 DEFINITION Let $\mathbb{K}^1_S$ be the Vopěnka algebra in HOD_S which is built, as in Chapter I, from OD_S-names of OD_S subsets of $\mathcal{P}(\omega)$.

As in Chapter I, we may prove

13·1 LEMMA *In M, $\mathbb{K}^1_S$ has the θ-chain condition.*

Proof : suppose that in M there is a listing in order-type θ of pairwise incompatible non-zero members of $\mathbb{K}^1_S$. That in V yields, by the definition of $\mathbb{K}^1_S$ a listing in order type θ of pairwise disjoint non-empty OD_S subsets of $\mathcal{P}(\omega)$. That defines a surjection of $\mathcal{P}(\omega)$ onto θ, a contradiction. $\dashv$ (13·1)

13·2 LEMMA *Every cardinal of M from θ onwards is a cardinal in V.*

Proof : as in Chapter I, though here the context makes it slightly easier. Every real is $(HOD_S, \mathbb{K}^1_S)$-generic. $L[S] \subseteq HOD_S$, plainly, and therefore for $x \in \mathfrak{C}(S)$, $V = HOD_S[x] = HOD_S[G_x]$, G_x being the HOD_S-generic filter on $\mathbb{K}^1_S$ induced by x.[N 11] The universe thus being a generic extension of M by an algebra with the θ-chain condition, the lemma follows. $\dashv$ (13·2)

13·3 PROPOSITION *There is an OD_S-enumeration of the OD_S subsets of $\mathcal{P}(\omega)$ in order-type θ.*

Proof : there is in M a listing of equivalence classes of OD_S-names for those sets, in order type some initial ordinal of M. We know that ordinal to be at least θ and less than the successor in V of θ, by our assumption that $2^{\mathfrak{c}} = \theta$. By the previous lemma, it must be exactly θ. $\dashv$ (13·3)

Hence we can identify any OD_S set of reals by an ordinal less than Θ.

Reduction of a class to a subset of Θ

[N 10] What about Case IV for a class ? probably OK, provided my “case IV for a set” works.

[N 11] Since the universe is now the result of adding a single real to $L[S]$, the equation $M[x] = M[G_x]$, which caused us so much trouble in earlier chapters, is here a triviality.

We aim now to find a subset S^* of θ such that for $x \in \mathfrak{C}(S)$, $L[x, S^*]$ contains all reals and all OD_S sets of reals, and such that many statements true in V about S will be true in $L[x, S^*]$ about S^*; and we shall show that S^* can be arranged to code any particular OD_S subset of θ.

13·4 PROPOSITION *Let S be a class of ordinals and suppose that $\mathfrak{B} \subseteq \{x \in \mathcal{N} \mid V = L[S; x]\}$ is non-empty, so that AC is true, and is OD_S. Assume further that $2^{\mathfrak{c}} = \mathfrak{c}^+$, and therefore write Θ for $\mathfrak{c}^+$. Let H be any OD_S subset of Θ. Then there is an OD_S subset S^* of Θ such that*

(13·4·0) for x in $\mathfrak{B}$, if we write $V^* = L[S^*; x]$, every real is in V^*, so that V^* is independent of the particular x chosen from $\mathfrak{B}$ and each element of $\mathfrak{B}$ is an S^*-constructor;

(13·4·1) $H \in V^*$;

(13·4·2) every set of reals is in V^*;

(13·4·3) a set of reals is OD_S in V if and only if it is OD_{S^*} in V^*;

Further if we write $\Theta^* = (\Theta)^{V^*}$, we have

(13·4·4) $\Theta = \Theta^*$;

and if we write $M = HOD_S^{L[S;x]}$ and $M^* = (HOD_{S^*})^{L[S^*;x]}$, where x is any element of $\mathfrak{B}$,

(13·4·5) M and M^* have the same bounded subsets of Θ.

Further, we shall automatically have the following:

(13·4·6) for any set A of reals and any real p, A is OD from S and p in V if and only if it is OD from S^* and p in V^*;

(13·4·7) if Eve wins the $(S, \mathfrak{B})$-game in V, she wins the $(S^*, \mathfrak{B})$-game in V^*.

if Eve can survive ℓ rounds of the S-game in V, she can survive ℓ rounds of the $(S^*, \mathfrak{B})$-game in V^*

Proof : fix H, an OD_S subset of θ.

We pick a large γ such that for each x, $J_\gamma(S \cap \gamma; x)$ contains all reals and all sets of reals occurring in $L[S; x]$, and the set H; and moreover that this set is a model of a helpful fragment of ZF.

For each $\beta < \Theta$ we let N_β be the hull in $J_\gamma(S \cap \gamma; x)$ of the set of all ordinals up to β, and we collapse that to M_β which will be of the form $J_{\gamma_\beta}[S_\beta; x]$ for some ordinal γ_β and some subset S_β of β.

That structure is of size less than Θ

We take S^* to be a subset of Θ that codes, in some simple way, the sequences $\langle \gamma_\beta \mid \beta < \Theta\rangle$, and $\langle\langle \nu, \beta\rangle \mid \nu \in S_\beta\rangle$ and the set H.

Every OD_S set of reals will be OD_{S^*}, indeed will be in each $L[S^*, x]$: indeed, for any real b, the $OD_{S,b}$ sets of reals will be constructible from S^* and b.

Suppose that A is OD from S and a real p. For example, $A = \{x \in \mathcal{N} \mid J_\zeta(S, \mathbb{R}) \models \varphi[S, x, p, \eta]\}$.

Consider the following subset of $\mathcal{N} \times \mathcal{N}$:

$$D \;=_{\mathrm{df}}\; \{(x, y) \mid J_\zeta(S, \mathbb{R}) \models \varphi[S, x, y, \eta]\}.$$

D is OD_S; given the definable isomorphism between $\mathcal{N} \times \mathcal{N}$ and $\mathcal{N}$, we may conclude that $\mathcal{D}$ is in V^* definable from S^*; A is the section of D at p; all reals are in both V and V^*; so A is definable from S^* and p in V^*.

Now we have to check that the S-game played in V is identical with the $(S^*, \mathfrak{B})$-game played in V^*.

V and V^* have the same reals and sets of reals.

First, let G be an OD_S set of reals in V; then it is in V^* and OD_{S^*} there, and the game is the same for it is an integer game. Strategies are the same.

In the S-game, pre-strategy means: first, if a real of top S-degree is fed in, out comes a strategy for the game.

In the $(S^*, \mathfrak{B})$ game, precisely the same test is applied for pre-strategies (which was, of course, the point of introducing the set $\mathfrak{B}$.)

So the moves in the S-game for each player are the same moves as in the $(S^*, \mathfrak{B})$ game: Adam plays actual sets of reals which are OD_S, and they are the same as the OD_S^* sets of reals in V^*; and Eve plays good pre-strategies for them, and those are the same legitimate moves.

Since the S-game is an open game for the first player, the analysis that furnishes a strategy for the second player will be the same in V as in V^*. Hence if the second player wins in V, she wins in V^*; and if she can survive two moves in V then in V^*. $\dashv$ (13·4)

Proof of Case III: let S be a class of ordinals. Let H be the first counterexample to Woodinness in HOD_S^V. Form S^* coding H as in the Proposition. Take $\mathfrak{B}$ to be $\mathfrak{C}(S)$. Then $\mathfrak{B}$ is OD_S, and so in V^* is OD_{S^*}. Hence Case IV applies, and we know that Θ is Woodin in HOD_{S^*} computed in V^*. Let E be an H-strong extender in $M^* = HOD_{S^*}^{V^*}$. I assert that it works in HOD_S^V.

We know from elementary set theory,[N 12] that it is enough to prove that the two models M and M^* [N 13] have the same bounded subsets of Θ.

But if $\lambda < \Theta$, λ is the length of some OD_S pre-wellordering with field $\mathcal{N}$, and any subset of λ in M can therefore be coded by that prewellordering together with some other OD_S set of reals, both of which are in V^* and $\mathrm{OD}\ {}_{S^*}$ there, so that the subset can be reconstructed in V^* and will be in M^*. That argument will also work in the reverse direction.

Hence M and M^* have the same sets of rank less than Θ, and therefore the extender E is an extender in V. ⊣ (13·4)

13·5 Corollary $(AD + V = L(\mathbb{R}))$ *On a cone, Θ is Woodin in M.*

Proof of the Corollary: we know that on a cone $L[S, x]$ models $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, and that on a cone it is true in $L[S, x]$ that the second player wins the S-game. So, on a cone, Case IV applies. ⊣ (13·5)

14: The definition of the extender.

In this section I explore a possible simplification of the definition of the extender. I begin with a sketch from a letter to Hauser. My proposal shows that extenders exist without assuming the existence of constructors or of good pre-strategies: but need they be normal ? The main problem is in proving normality of the extender. The proof of normality reduces to the proof of normality for the filter F; and the present proof of that requires the existence of good pre-strategies.

My sketch

We are assuming S is a subset of Θ and $\mathrm{Det}(OD_S) + V = L(R; S)$. ($R$ = the reals). We have an OD_S subset H of Θ, (which I shall ignore in my notation); we get a reflection ordinal δ. $(= \delta_H)$.

We also have a Σ_1 set of reals, $\Upsilon_{H,\Lambda}$, which is universal for Σ_1 statements involving the diamond sequence $\diamondsuit$.

Using that we define a filter F.

Let k = card s. We have a subset A of $[\delta]^k$ in HOD_S, and we wish to say whether or not A is to go into the measure $E(s)$ that we are defining.

The present definition is this: we have a finite subset s of λ, an ordinal that we represent by a pwo Λ.

We use the Simple Basis game to get a GOOD set of reals (of the form B(p, k)) that is a subset of $(\Lambda)_s$. Then for many ordinals ν less than δ a finite set of ordinals is defined by reflection. (many = on a set in F).

I notate that finite set $r_k(B(p, k), \diamondsuit(\nu))$.

We play a game, in which the two contestants play a, e; we split each of a and e into infinitely many reals, a^n, $(e)^n$;

there are rules requiring each a^n, $(e)^n$ to be in $\Upsilon_{H,\Lambda}$;

then define the critical ordinal, o(a,e) of the play to be the least ordinal in the intersection of the sets S_{a^n}, $S_{(e)^n}$.

then there is a rule requiring r(p, a, e) (in full, $r_k(B(p, k), \diamondsuit(o(a, e)))$ to be defined (Eve wins if it is not), and finally Eve wins if r(p,a,e) is in A.

A goes into E(s) if Eve has a winning strategy in this game. The game is definable from S and the good parameter p, so is determined.

Call the above the game $G_W(A, p)$. It depends on p, but one can prove that the decision concerning A is independent of which p is used.

=========

From now on, p is not a good parameter, just a real variable.

[N 12] the result is proved in chapter seven as 1·8, but could be taken here
[N 13] (dependence on x ?)

My idea: change the rules of the game so that we read a and e as each giving two infinite sequences, not one; call them p^n, b^n (produced by a); q^n, d^n (produced by e).

The b's and the d's are subject to the same rules as the $(a)^n$ and $(e)^n$ in the games G_W. namely they have to be in $\Upsilon_{H,\Lambda}$, and they define a critical ordinal $o(b,d)$.

The p's and the q's are used to replace a single good real defining the component $(\Lambda)_s$.

Now I have two variants: in one, call it $G_H(A)$, the p's and the q's are to be prestrategies for the k-dimensional simple basis game, but need not be good prestrategies; so we use these to define sets $B(p^n,k)$, $B(q^n,k)$; and the rules will say that each of those has to be a non-empty subset of $(\Lambda)_s$; at the end of playing time, form the union of all those $B(p^n,k)$'s and $B(q^n,k)$'s, call the result B[p,q]; then that set is a (tame) subset of $(\Lambda)_s$, so use that to reflect; then test to see if $r_k(B[p,q],o(b,d))$ is defined and if it is a member of A, as before.

In the other variant, call it $G_M(A)$, the p's and the q's are simply elements of $(\Lambda)_s$; let the set of them be $R_k[p,q]$; then use $R_k[p,q]$ to reflect, and to attempt to define $r_k(R_k[p,q],o(b,d))$ and see if that finite set is in A.

Write $E_H(s)$ and $E_M(s)$ for the "measure" defined by each method.

Now as far as I can tell, the following will hold, provided you have the same sort of interweaving in the rules as in Solovay's theorem treated in Chapter III section 2.

1. if $\mathrm{Det}(OD_S)$ and $V=L(R;S)$, $E_M(s)$ is a $<\delta$-complete ultrafilter in HOD_S.

2. if $\mathrm{Det}(OD_S)$ and $V=L(R;S)$ and there are S-constructors, $E_H(s)$ is a $<\delta$-complete ultrafilter in HOD_S.

the present text gives:

3. if $\mathrm{Det}(OD_S)$ and $V=L(R;S)$ and there are S-constructors, and there is a one-round survival strategy in the strict S-game, then for good p, $E_W(s,p)$ is a $<\delta$-complete ultrafilter in HOD_S.

and

4. if $\mathrm{Det}(OD_S)$ and $V=L(R;S)$ and there are S-constructors, and there is a two-round survival strategy in the strict S-game, then for good p, $E_W(s,p)$ is a $<\delta$-complete ultrafilter in HOD_S, and the extender is normal.

I think in all cases you get coherence and well-foundedness, and the H-property; the problem is getting normality. Without normality the H property does not seem to be much use.

"strict" has to do with how you count the rounds.

The difference between these various definitions is how much you have to assume to get them to work. You need S-constructors to give meaning to "prestrategy".

I think you can fairly easily prove under the hypotheses of case 3 that $E_W(s,p)=E_H(s)$; and with rather more effort that under the hypotheses of case 2 $E_H(s)=E_M(s)$.

Can one show that the extender need not be normal if the hypotheses of Case 4 fail ?

So in other words you get $\mathrm{Det}(OD_S)$ plus $V=L(R;S)$ gives you that δ_H is strong up to Θ, but I don't see why the embeddings should be H-strong.

I feel slightly at a loss; I find it of interest that one can establish the *existence* of the extenders in HOD_S without assuming the existence of good prestrategies, or indeed of any prestrategies; on the other hand giving the details of all these games seems a lot of work for what seem very minor sharpenings of results.

It would be much more interesting if my idea of making the choice of B (or W) part of the game rather than being pre-given by a previous game (the simple basis game) could be carried further; could it go as far as proving the normality of the extender ?

We begin in the theory $V=L(\mathbb{R},S)$, where $S\subseteq\Theta$. Let H be an OD_S subset of Θ. We assume that H codes S. We define the Σ_1-reflection ordinal δ_H and the admissible set M_H, of height δ_H.

We recall from Chapter I, §4,that we have an universal $\Sigma_1^H(P,\diamondsuit)$ predicate, $\Phi_\diamondsuit(a,P,\diamondsuit,\delta_H;H)$. For Λ an OD_S pre-well-ordering of $\mathcal{N}$ of length $\lambda>0$, we define $\Upsilon_{H,\Lambda}=\{a\mid\Phi(a,\Lambda,\diamondsuit,\delta_H;H)\}$; we define the filter $\mathfrak{F}_{\Lambda,H}$ on δ_H to be that generated by the sets $S_a=_{\mathrm{df}}\{\nu\mid\Phi(a,\diamondsuit(\nu),\diamondsuit\restriction\nu,\nu;H)\}$ for $a\in\Upsilon_{H,\Lambda}$.

The following lemma might well be added to Chapter I, §4.

14·0 Lemma *Let $\Psi(a,\Lambda,\diamondsuit,\delta_H;H)$ be a $\Delta_1^H(\diamondsuit)$ predicate. Suppose that $\mathcal{F}\in M_H$ is a set of continuous functions, such that $\forall a{:}{\in}\mathcal{N}\ \big(\Psi(a,\Lambda,\diamondsuit,\delta_H;H)\implies\forall f{:}{\in}\mathcal{F}\ f(a)\in\Upsilon_{H,\Lambda}\big)$. Then*

$$\exists b{:}\in \Upsilon_{H,\Lambda}\ \forall a{:}\in \mathcal{N}\ \big(\Psi(a,\Lambda,\diamondsuit,\delta_H;H) \implies \forall f{:}\in \mathcal{F}\ S_b \cap \{\nu \mid \Psi(a,\diamondsuit(\nu),\diamondsuit\restriction\nu,\nu;H\cap\nu)\} \subseteq S_{f(a)}\big).$$

14·1 REMARK In the intended use of this lemma the members of $\mathcal{F}$ are winning strategies for the second player in a certain game.

14·2 Let s be a finite non-empty subset of λ. We write k for $\overline{\overline{s}}$. We shall define a set $E(s)$ of OD_S subsets of $[\delta_H]^k$ which will, assuming $\mathrm{Det}(OD_S)$ prove to be an ultrafilter in HOD_S; as s varies the family $\langle E(s) \mid s \in [\delta_H]^{<\omega}\rangle$ will prove to be a (δ_H,λ) extender in HOD_S; under a stronger hypothesis this extender will prove to be a normal H-extender.

That extender will be defined by reference to various games; in fact we have two variants of those games to consider, and we shall thus define two versions of $E(s)$, which we shall call $E_M(s)$ and $E_W(s,\psi)$; there is a third variant, intermediate between those two, which we might call $E_H(s)$; the difference is that the definition of $E_H(s)$ requires the existence of S-constructors, and the definition of $E_W(s,\psi)$ requires the existence of S-constructors, and further that the parameter ψ be a good pre-strategy for a certain simple basis game.

However we shall verify that provided the hypotheses permit those definitions to be made, the three actually define the same object, (independent, in the W case, of any particular choice of ψ) which we shall then call $E(s)$.

We begin with the definition of $E_M(s)$.

The game $\mathfrak{G}_M(A)$: no constructors needed

For $A \subseteq [\delta_H]^k$ with $A \in HOD_S$, we consider a game $\mathfrak{G}_r(A)$, in which Adam plays, bitwise, a and Eve e. In a previous chapter, we interpreted a and e as an infinite sequences $(a)^n$, $(e)^n$ of reals, each of which was required to be a member of the set $\Upsilon_{H,\Lambda}$.

Here it is only the $(a)^n$ and $(e)^n$ for n odd that will be required to lie in $\Upsilon_{H,\Lambda}$; the $(a)^n$ and $(e)^n$ for n even will have a different function, that of defining the k-dimensional component $(\Lambda)_s$ and thereby selecting components of its reflections to various ν.

We recall the definition of $(\Lambda)_s$ as the product $(\Lambda)_{\zeta_0} \times (\Lambda)_{\zeta_1} \times (\Lambda)_{\zeta_2} \times \ldots \times (\Lambda)_{\zeta_{k-1}}$, where the members of s in increasing order are $\zeta_0, \ldots, \zeta_{k-1}$.

Before giving the rules of the game $\mathfrak{G}_R(A)$, we introduce some terminology.

14·3 DEFINITION For $p \in \mathcal{N}$, we denote by $(p)^{<k}$ the k-tuple $\big((p)^0, (p)^1, \ldots, (p)^{k-1}\big)$.

14·4 DEFINITION For reals p, q and a pre-well-ordering Λ with field $\mathcal{N}$, we write $p =^k_\Lambda q$ to mean that the two, read as k-tuples, define the same k-fold component of Λ: specifically,

$$\forall i\!:<\!k\ (p)^i =_\Lambda (q)^i \ \&\ \forall i,j\!:<\!k\ (i<j \implies (p)^i <_\Lambda (p)^j).$$

14·5 REMARK Thus we are treating p and q as coding, with reference to Λ, an increasing k-tuple of ordinals less than λ.

14·6 DEFINITION *p is s-correct* if the k-tuple $(p)^{<k}$ is in $(\Lambda)_s$.

14·7 DEFINITION When C is a non-empty tame set of s-correct reals, then for many ν, every real in C will define a strictly increasing sequence of length k of components of $\diamondsuit(\nu)$, with any two reals in C defining the same sequence. For such ν we denote that sequence by $r_k(C, \diamondsuit(\nu))$ and we write $S_{a(C,k)}$ for the set $\{\nu \mid r_k(C, \diamondsuit(\nu))\downarrow\}$. Here $\downarrow$ is read "is defined".

14·8 EXERCISE When $p =^k_\Lambda q$, $\{\nu \mid p =^k_{\diamondsuit(\nu)} q\} \in \mathfrak{F}_{\Lambda,H}$. We write $S_{a(p,k,q)}$ for this set.

We are now ready to give the rules of the game $\mathfrak{G}_M(A)$.

Rule EM$_0$: $(a)^0$ is s-correct;
Rule EM$_2$: $(a)^2$ is s-correct;
Rule EM$_4$: $(a)^4$ is s-correct;

$\vdots$

The first failure in those rules entails defeat for the responsible player; if they all hold, form

$$R_k[a] \ =_{\rm df}\ \{(p)^{<k} \mid \exists i\, p = (a)^{2i}\}.$$

Note that then $R_k[a] \subseteq (\Lambda)_s$, so that the set $C_k(a) \ =_{\rm df}\ \big\{\nu \mid r_k\big(R_k[a], \diamondsuit(\nu)\big)\downarrow\big\}$ will be in $\mathfrak{F}_{\Lambda,H}$.

The next group of rules tests for membership of $\Upsilon_{H,\Lambda}$.

Rule EM$_{\omega+0}$: $(a)^1 \in \Upsilon_{H,\Lambda}$
Rule EM$_{\omega+1}$: $(e)^1 \in \Upsilon_{H,\Lambda}$;
Rule EM$_{\omega+2}$: $(a)^3 \in \Upsilon_{H,\Lambda}$;
Rule EM$_{\omega+3}$: $(e)^3 \in \Upsilon_{H,\Lambda}$;
Rule EM$_{\omega+4}$: $(a)^5 \in \Upsilon_{H,\Lambda}$;
Rule EM$_{\omega+5}$: $(e)^5 \in \Upsilon_{H,\Lambda}$;

$\vdots$

The first failure in those rules entails defeat for the responsible player; if all those rules have been kept, we define the *critical ordinal* $\mathfrak{o}_R(a,e)$ of the play to be $\inf \bigcap_{i\geqslant 0}\big(S_{(a)^{2i+1}} \cap S_{(e)^{2i+1}}\big)\big)$.

Rule EM$_{\omega+\omega}$: Eve wins if $r_k(R_k[a,e], \diamondsuit(\mathfrak{o}(a,e))\uparrow$; (equivalently, if $\mathfrak{o}(a,e) \notin C_k(a)$.

Finally, if $r_k(R_k[a,e], \Diamond(\mathfrak{o}(a,e)) \downarrow$,

Rule $EM_{\omega+\omega+1}$*:* Eve wins if $r_k\big(R_k[a], \Diamond(\mathfrak{o}(b,d))\big) \in A$.

14·9 DEFINITION $E_M(s) = \{A \subseteq [\delta_H]^{\overline{\overline{s}}} \mid A \in HOD_S \;\&\;$ Eve has a winning strategy in $\mathfrak{G}_M(A)\}$.

Verification that $E_M(s)$ is an ultrafilter

We verify in turn the first three and the last of the five clauses *EXT 1* to *EXT 5* defined in Chapter IV: the verification of *EXT 4*, under, in Case IV a stronger hypothesis, will be done LATER.

14·10 PROPOSITION *If $A \in E(s)$ and $A \subseteq C$ then $C \in E(s)$.*

Proof : a winning strategy for Eve in $\mathfrak{G}_R(A)$ will serve her as well for $\mathfrak{G}_R(C)$. ⊣ (14·10)

14·11 PROPOSITION *$\varnothing \notin E(s)$; $[\delta_H]^k \in E(s)$.*

Proof : Eve has no chance of winning in $\mathfrak{G}_R(\varnothing)$ nor much of losing in $\mathfrak{G}_R([\delta_H]^k)$, since in our liberal rules repetitions are allowed, so, in order to win, she has only to find $d_0 \in \Upsilon_{H,\Lambda}$ and then play all $(e)^{2i+1} = d_0$. ⊣ (14·11)

We prove a lemma to show that Adam can always avoid defeat under Rule $\omega + \omega$.

14·12 LEMMA *Let τ be a winning strategy for Eve in a game $\mathfrak{G}_R(A)$. Adam can play so that the first rule he breaks is not Rule $\omega+\omega$. More generally, let $F \in M_H$ be a set of winning strategies for Eve. Then Adam has a uniform method of play that avoids such defeat whichever $\tau \in F$ is used by Eve.*

Proof : let p be s-correct.

Provided that Adam plays so that some $S_{(a)^{2n+1}} \subseteq S_{\{p\}}$, **BETTER NOTATION NEEDED**, the critical ordinal in a run against any $\tau \in F$ will be in $C_k(a)$, **PROVIDED ALSO THAT HE PLAYS** p **AS EVERY EVEN SLICE**; and hence if he has kept all Rules $2n$ and $\omega + 2n$ he will have kept Rule $\omega + \omega$. ⊣ (14·12)

14·13 PROPOSITION *If $u \in [\delta_H]^k$ and $\delta_H \leqslant \bigcup s$, $\{u\} \notin E(s)$.*

Proof : let q be a real and ξ an ordinal such that $\bigcup u < \xi = |q|_\Lambda < \delta_H$. Then on a large set $|q|_{\Diamond(\nu)} = \xi$. Let p be s-correct; then $|(p)^{k-1}|_\Lambda \geqslant \delta_H > \xi$, so on a large set $|(p)^{k-1}|_{\Diamond(\nu)} > \xi$. If Adam plays to ensure that the critical ordinal is in both those large sets, and avoids refeat by Rule $\omega + \omega$, the resulting $r(a, e, \mathfrak{o})$ cannot equal u, the supremum of which is less than ξ. ⊣ (14·13)

14·14 PROPOSITION $(V = L[R; S] + \mathrm{Det}(OD_S))$ *If $A \in HOD_S$ and $A \subseteq [\delta_H]^k$, and we put $C = [\delta_H]^k \smallsetminus A$, then Eve wins at least one of $\mathfrak{G}_M(A)$ and $\mathfrak{G}_M(C)$.*

Proof : if not, Adam has two winning strategies σ_A, σ_C. Eve plays each against the other. That is, she uses Adam's moves in each game as her moves in the other. If Adam's plays in the two games are a_A and a_C, and hers are e_A and e_C then $e_A = a_C$ and $e_C = a_A$. This is easy to arrange as Adam moves first.

NOTE THAT it is convenient here that no notice has been taken in the Rules of what $(e)^{2n}$ are. So she can simply copy Adam's moves in their entirety.

We must check that all preliminary rules are observed; then we note that the two runs define the same $\mathfrak{o}$ and the same R; by Rule $\omega+\omega$ $r_k(R\Diamond(\mathfrak{o}))\downarrow$, otherwise Eve will have won both games; but then that set of ordinals is neither in A nor in its complement, Adam having won both games; which is impossible. ⊣ (14·14)

Now we show that in HOD_S, the additivity of $E(s)$ is δ_H. Here we use both determinacy and reflection arguments.

14·15 LEMMA *If $0 < \xi < \delta_H$, there is a pre-well-ordering Ξ of $\mathcal{N}$ of length ξ which is both tame and OD_S.*

Proof : let Υ_H be the universal Σ_1^H set (with no mention of $\Diamond$) with a pre-wellordering of length δ_H. That is definable from S and H and therefore solely from S, H being OD$_S$. Each initial segment is tame, and so $\Upsilon_H \restriction \eta$ is nearly as required: add all reals not in its field to its 0-th component, which can be done without jeopardising its OD_S status. ⊣ (14·15)

14·16 EXERCISE Adapt the argument given in Chapter One to prove that.

14·17 PROPOSITION $(V = L[R; S] + \mathrm{Det}(OD_S))$ *Let $s \subseteq \lambda$ and $0 < \eta < \delta_H$, and let $\langle A_\nu \mid \nu < \eta\rangle \in HOD_S$ be a sequence of members of $E(s)$. Then $\bigcap_{\nu<\eta} A_\nu$ is non-empty.*

Proof : By the lemma, there is an OD_S pre-well-ordering , Ξ, with field $\mathcal{N}$, in M_H, of length ξ; by adding all reals not in its field to its first component, we may assume that it has field $\mathcal{N}$.

Consider

$$Z =_{\mathrm{df}} \big\{(a,\tau) \bigm| \tau \text{ is a winning strategy for Eve in } \mathcal{G}_R(A_{|a|_\Xi})\big\},$$

which is OD from S. Hence by Stage One of the Coding Lemma there is a $Z^* \subseteq Z$ which is $pos\Sigma^1_1(\Xi)$ with a parameter and full in Z. Let $\mathcal{E}^* = \{\tau \mid \exists a\,(a,\tau) \in Z^*\}$. Then $\mathcal{E}^* \in M_H$.

Seizing control in $\mathfrak{G}_R$

14·18 We seek to build a universal play $\bar{a}$ such that against any $\tau \in \mathcal{E}^*$,

(14·18·0) $\mathfrak{o}(\bar{a},[\bar{a}]\star\tau)$ will have a constant value independent of τ;

(14·18·1) $r_k(R_k[\bar{a},[\bar{a}]\star\tau],\diamondsuit(\mathfrak{o}(\bar{a},[\bar{a}]\star\tau)))\downarrow$ and will have a constant value, independent of τ.

We start by choosing some p_0 that is s-correct. Let $a_{2n} = p_0$. Let $a_1 \in \Upsilon_{H,\Lambda}$ be such that $S_{a_1} = S_{\{p_0\}}$. We shall apply Lemma 14·0.

$$\forall\tau{:}{\in}\mathcal{E}^*\ \forall a{:}{\in}\mathcal{N}\ \big([(a)^1 = a_1\ \&\ \forall n\ (a)^{2n} =^k_\Lambda p_0] \implies ([a]\star\tau)^1 \in \Upsilon_{H,\Lambda}\big)$$

Therefore

$$\exists a_3{:}{\in}\Upsilon_{H,\Lambda}\ \forall a{:}{\in}\mathcal{N}\ [a^1 = a_1\ \&\ \forall n\ (a)^{2n} =^k_\Lambda p_0] \implies \forall\tau{:}{\in}\mathcal{E}^*\ S_{a_3} \cap \{\nu \mid [\forall n\ (a)^{2n} =^k_{\diamondsuit(\nu)} p_0]\} \subseteq S_{([a]\star\tau)^1}$$

Next,

$$\forall\tau{:}{\in}\mathcal{E}^*\ \forall a{:}{\in}\mathcal{N}\ \big[[a^1 = a_1\ \&\ a^3 = a_3\ \&\ \forall n(a)^{2n} =^k_\Lambda p_0] \implies ([a]\star\tau)^3 \in \Upsilon_{H,\Lambda}\big]$$

therefore

$$\exists a_5{:}{\in}\Upsilon_{H,\Lambda}\ \forall a{:}{\in}\mathcal{N}\ \big[[a^1 = a_1\ \&\ a^3 = a_3\ \&\ \forall n(a)^{2n} =^k_\Lambda p_0] \implies \forall\tau{:}{\in}\mathcal{E}^*\ S_{a_5} \cap \{\nu \mid a^1 = a_1\ \&\ a^3 = a_3\ \&\ \forall n(a)^{2n} =^k_{\diamondsuit(\nu)} p_0\} \subseteq S_{([a]\star\tau)^3}\big];$$

and more generally, at stage $2\ell+1$, we note that

$$\forall\tau{:}{\in}\mathcal{E}^*\ \forall a{:}{\in}\mathcal{N}\ \big[[\forall i \leqslant \ell\ a^{2i+1} = a_{2i+1}\ \&\ \forall n(a)^{2n} =^k_\Lambda p_0] \implies ([a]\star\tau)^{2\ell+1} \in \Upsilon_{H,\Lambda}\big]$$

and conclude that there is a $a_{2\ell+3}$ such that for any a and any $\tau \in \mathcal{E}^*$,

$$[\forall i \leqslant \ell\ a^{2i+1} = a_{2i+1}\ \&\ \forall n(a)^{2n} =^k_\Lambda p_0] \implies S_{a_{2\ell+3}} \cap \{\nu \mid [\forall i \leqslant \ell\ a^{2i+1} = a_{2i+1}]\ \&\ [\forall n(a)^{2n} =^k_{\diamondsuit(\nu)} p_0]\} \subseteq S_{([a]\star\tau)^{2\ell+1}}\big];$$

Now let $\bar{a}$ be the real with $\forall i{:}{\in}\omega\ \big((\bar{a})^{2i} = p_0\ \&\ (\bar{a})^{2i+1} = a_{2i+1}\big)$.

If played against any strategy in $\mathcal{E}^*$, the critical ordinal will be the minimim of the intersection of the $S^{2n+1}_{\bar{a}}$: call that value $\bar{\delta}$. $\bar{\delta}$ will be in $C_k(\bar{a})$, by choice of a_1. So the **critical sequence** will be in the intersection of the A_ν's, as required. ⊣ (14·18)

14·19 Remark This $\bar{a}$ works, but the above Lemma and argument is needlessly complicated; with the Lemma as it is, one can prove something slightly more general, namely, that if one chooses a_{2i} with $a_{2i} =^k_\Lambda p_0$ and takes the $\bar{a}$ with $(\bar{a})^{2i} = a_{2i}$ and $(\bar{a})^{2i+1} = a_{2i+1}$, that too will work, provided that one chooses a_1 so that $S_{a_1} \subseteq \{\nu \mid \forall i\ a_{2i} =^k_{\diamondsuit(\nu)} p_0\}$. At present I know no use for this more general version, but it is possible that at some stage in the application of the principle that Z^* meets every component of $Z^*(\delta)$, that it is convenient to look at points b that are not equal to p_0 but in the same Z component of it. If that proves not to be the case, one can simplify the rules of G_M.

14·20 Proposition (Det(OD_S)) *In HOD_S, $E_R(s)$ is a proper non-principal ultrafilter.*

Proof : by the above. ⊣ (14·20)

14·21 PROBLEM Is $E(s)$ countably complete for sequences outside HOD_S ? This might follow from an argument about interweaving of strategies.

Verification that we have an extender

14·22 PROPOSITION *The extender is coherent.*

Proof : Fix s a subset of t. Let $Y = \{y \mid y^{s,t} \in X\}$; we wish to show that $Y \in E(t) \iff X \in E(s)$. Since both are ultrafilters, it is enough to show that if Y is in $E(t)$ then X is in $E(s)$; for that will imply that if Y is not in, then its complement is in, so the complement of X is in and so X is not.

Towards a contradiction, we fix a winning strategy τ for Eve in the game $G(Y)$ and a winning strategy σ for Adam in the game $G(X)$. We shall show that Eve can defeat σ. The main problem will be in the pace of the various games.

Fix a t-correct point $\bar{q}$ and an s-correct point $\bar{p}$, with $\bar{p} = (\bar{q})^{s,t}$. Against any e, $(\sigma \star [e])^0$ must be s-correct. The set $P = \{(\sigma \star [e])^0 \mid e \in \mathcal{N}\}$ of such is tame; so Eve can write a reflection set for that, call it $C_k(\sigma)$. She has reals given by $\bar{q}$ which when "tacked on" to any of Adam's proposed plays in P give a t-correct set; again the set of such is tame; so in the $(k+1)$ game she will want to ascribe to Adam a reflection set, call it $C_{k+1}(\sigma, \bar{q} \setminus \bar{p})$, for all that with the same extra reals added every time.

To combine all her tasks into one, let S_z be the following reflection set:

$$S_z \ =_{\text{df}} \ C_k(\sigma) \cap C_{k+1}(\sigma, \bar{q} \setminus \bar{p}) \cap \{\nu \mid (\bar{p})_{\Diamond(\nu)} = (\bar{q})^{s,t}_{\Diamond(\nu)}\}$$

(where we should in the last line replace s and t by subsets of ω standing in the same relationship.) **NOTATION ?** Do we need to make the last line more general ?

We use those to define somewhat brutal maps λ, π.

$(\lambda(a))^1 = z$; $(\lambda(a))^{i+2}$ is to be $(a)^i$ if i is odd and $(\lambda)^i$ is to be $\bar{q}$ if i is even. Then $\lambda(a)$ can be evolved as a evolves, and is t-correct at each even place.

In the other direction, we shall want $(\pi(e))^1 = z$; $(\pi(e))^{i+2}$ is to be $(e)^i$ if i is odd, and $(\pi(e)^i$ is to be $\bar{p}$ if i is even. Then $\pi(e)$ can be evolved as e evolves, and is s-correct at each even place.

Now let a be Adam's play in $G(X)$: feed $\lambda(a)$ as his play in $G(Y)$ and let $e = [\lambda(a)] \star \tau$. $\pi(e)$ is to be Eve's play in $G(X)$.

In the run of $G(X)$ if Adam breaks any of the rules $2n$ he loses; so suppose that he has kept them all. $\lambda(a)$ in any case observes all the Rules $2n$ in $G(Y)$. We pass to the second batch of rules. The first violation of those will be by Adam in both games. If he keeps them all, so does Eve.

Note that the critical ordinal will be the same in the runs of the two games.

Now we must show that there has been no failure under rule $\omega + \omega$, but that will be true by choice of z. Then what is defined in the run of Y is t' say, and what is defined in the run of X is s': these sets will correspond, by the third clause of S_z.

14·23 PROPOSITION *If $\langle A_i \mid i \in \omega\rangle$ and $\langle s_i \mid i \in \omega\rangle$ are sequences in HOD_S such that $\forall i\, A_i \in E(s_i)\, \&\, s_i \subseteq s_{i+1}$, then there is a sequence $\langle v_i \mid i \in \omega\rangle$ such that $\forall i\, v_i = v_{i+1}^{s_i, s_{i+1}}$ & $v_i \in A_i$.*

The following proof mixes two arguments; I am not yet sure which I prefer. We shall play ω games simultaneously; one method would be to interweave the strategies so that the critical ordinal is common to all; another would be to use the near-countable completeness of $\mathfrak{U}$ to find a common ordinal at which to evaluate the projections of the r_k's.

[The two arguments are not so different; one is interweaving strategies in this particular game; the other is saying that interweaving generally gives you near-countable completeness. So I suppose the second is to be preferred as being more general.]

Proof : Let $\varsigma = \bigcup_{i<\omega} s_i$: then ς is in HOD_S and is countable there. So let $f : \omega \longleftrightarrow \varsigma$, with $f \in HOD_S$. Let $C_f = \{\alpha \in \mathcal{N} \mid \forall n (\alpha)_n \in (\Lambda)_{f(n)}\}$ — an OD_S subset of $\mathcal{N}$. C_f is non-empty, by the axiom of choice for countable families of sets of reals, which is true in all the contexts we consider.

We fix a member α of C_f.

We pick a strategy winning strategy τ_i for Eve in each $G(A_i)$: $\mathrm{Det}(OD_S)$ is enough for that. Adam must play against all those simultaneously. He must take care that every $(e)^{2n+1}$ and every $(a)^{2n+1}$ that occurs in any one game is played by him in all games. This will ensure that the critical ordinal is the same in all games.

He must use as his s_i-correct reals the one naturally obtained from α for the given i. Let us denote it by α_i. He must also ensure that among the a's that he plays are those that will guarantee that each r_{k_i} is defined at the critical ordinal.

In an inexact notation, $\alpha_i = \alpha_{i+1}^{s_i,s_{i+1}}$, and so there is a set $Z_i \in \mathfrak{F}$ such that

$$\zeta \in Z_i \implies r_{\{\alpha_i\}}(\zeta) = (r_{\{\alpha_{i+1}\}}(\zeta))^{s_i,s_{i+1}}.$$

Since $A_i \in E(s_i)$, the set $Y_i =_{\mathrm{df}} \{\zeta \mid r_{B_i} \in A_i\} \in \mathfrak{U}$.

We know that $\mathfrak{U}$ extends $\mathfrak{F}$, and that the intersection of countably many members of U is non-empty, and so we may pick $\bar\zeta \in \bigcap_{i<\omega}(Y_i \cap Z_i)$. Now set $v_i = r_{\{\alpha_i\}}(\bar\zeta)$. Then $v_i \in A_i$ as $\bar\zeta \in Y_i$, and $v_i = v_{i+1}^{s_i,s_{i+1}}$ as $\bar\zeta \in Z_i$. ⊣ (14·23)

14·24 Proposition *For each* $\nu < \lambda$, $\{\{\xi\} \mid \xi \in H \cap \delta_H\} \in E(\{\nu\}) \iff \nu \in H$.

Proof : We fix ν and a point p with $|p|_\Lambda = \nu$.

Let $A = \{\{\xi\} \mid \xi \in H \cap \delta_H\}$. Suppose that ν is in H. Eve is to play a member of $\Upsilon_{H,\Lambda}$ that enforces the statement $|p|_{\diamond(\xi)} \in H$. So $|p|_{\diamond(\mathfrak{o})} \in H$. So the singleton of that ordinal is in A, and Eve has won. So $A \in E(\{\nu\})$.

On the other hand if ν is not in H, a similar argument will show that Eve wins the game for the complement of A. ⊣ (14·24)

The game $\mathfrak{G}_W(A,\psi)$

Woodin, assuming $V = L(R; S] + \mathrm{Det}(OD_S)$ and the existence of S-constructors, builds his extenders by first finding a ψ that is an S-good pre-strategy for a certain simple basis game, yielding a canonical non-empty tame S-good set $B(\psi,k) \subseteq (\Lambda)_s$. We denote by $C_k(B(\psi,k))$ the reflection set for the statement that every real in $B(\psi,k)$ is k-correct, which set will exist as $B(\psi,k)$ is tame.

The game $\mathfrak{G}_W(A,\psi)$ has the following $\omega+1$ rules. Bitwise, Adam plays a and Eve plays e.

Rule EW_0: $(a)^0 \in \Upsilon_{H,\Lambda}$;
Rule EW_1: $(e)^0 \in \Upsilon_{H,\Lambda}$;
Rule EW_2: $(a)^1 \in \Upsilon_{H,\Lambda}$;
Rule EW_3: $(e)^1 \in \Upsilon_{H,\Lambda}$;
Rule EW_4: $(a)^2 \in \Upsilon_{H,\Lambda}$;
Rule EW_5: $(e)^2 \in \Upsilon_{H,\Lambda}$;

$\vdots$

The first failure in those rules entails defeat for the responsible player; if all those rules have been kept, we define the *critical ordinal* $\mathfrak{o}_W(a,e)$ of the play to be $\inf\bigl(C_k(B(\psi,k)) \cap \bigcap_{i\geqslant 0}(S_{(a)^i} \cap S_{(e)^i})\bigr)$, and then follow

Rule EW_ω: Eve wins if $r_k\bigl(\hat{\psi}, \diamondsuit(\mathfrak{o}(a,e))\bigr) \in A$.

Then $E_W(s,\psi)$ is defined to be the set of A in HOD_S for which Eve has a winning strategy in the game $\mathfrak{G}_W(A,\psi)$. Each such game is definable from S and ψ, and so is determined, by the S-goodness of ψ. Thus $E_W(s,\psi)$ will be an ultrafilter on $[\delta_H]^k$.

In fact, once the existence of an S-good ψ as above is known, the existence of S-constructors is not needed to establish the existence of the ultrafilter $E_W(s,\psi)$, though at present it is not known whether the normality of the whole extender can be proved without assuming he existence of S-constructors.

We have defined, without supposing the existence of good ψ, $E_M(s)$ to be the set of A in HOD_S for which Eve has a winning strategy in the game $\mathfrak{G}_M(A)$, and shown it to be an ultrafilter on $[\delta_H]^k$.

14·25 PROPOSITION $(V = L(\mathbb{R},S) + \mathrm{Det}(OD_S))$ *If an S-good ψ exists as above, then $E_W(s,\psi) = E_M(s)$.*

Proof: We suppose that A is a subset in HOD_S of $[\delta_H]^k$ such that in the game $\mathfrak{G}_W(A,\psi)$ Eve has a winning strategy τ_1 and in the game $\mathfrak{G}_M(\neg A)$ she has a winning strategy τ_2, and reach a contradiction.

We play the two games simultaneously; Adam takes care by copying appropriate parts of Eve's moves from one game to the other that the critical ordinal of the runs of two games is the same; and that as his "s-correct" reals plays at least one in the set of such defined by ψ.

In greater detail: we denote the players' play in $G_W(A,\psi)$ by a and e, and in $G_M(\neg A)$ by b and d; so that $e = [a]\star\tau_1$ and $d = [b]\star\tau_2$; we suppose that S_z is the reflection set for $B(\psi,k)$; we choose p_0 in $B(\psi,k)$; and we arrange a and b so that

$$a^{2n} = z; \quad a^{2n+1} = d^{2n+1}; \qquad b^1 = z; \quad b^{2n+3} = e^n; \quad b^{2n} = p_0.$$

Then $r_k(\mathfrak{o})$ will be the same finite set of ordinals but will be in A in one game and in its complement in the other: a contradiction. $\dashv$ (14·25)

The game $\mathfrak{G}_H(A)$: constructors needed but not goodness

This is the notion of correctness we need here:

14·26 DEFINITION Let $s \in [\lambda]^k$ We say that ψ is *s-correct* if $B(\psi, k) \subseteq (\Lambda)_s$.

For such ψ a map $r_k(B(\psi,k), \Diamond(\zeta))$ will be defined for ζ in a set $D(\psi)$ in F: when $r_k(\psi, \Diamond(\zeta)) \downarrow$, its value will be the $u \in [|\Diamond(\zeta)|]^k$ such that $B(\psi, k)$ is a subset of $(\Diamond(\zeta))_u$.

14·27 REMARK I AM HERE using what we might call the first stage of the Simple Basis Game, for $\mathrm{Det}(OD_S)$ is enough to assure the existence of s-correct ψ; they might not be good.

When the p's and q's are s-correct we shall consider the points where

$$B[p,q] \ =_{\mathrm{df}} \bigcup_i B((p)^i, k) \cup B((q)^i, k) \tag{14·28}$$

reflects.

The rules are inspired by the previous ones; the interweaving of the p's and q's is necessary to get the proof that $E(s)$ is a filter to work, as is the use of infinitely p's and q's; relatively little is used to get the proof that A and $\neg A$ cannot both be won by the first player; the proof that we are defining an ultrafilter which will coincide with the one defined by a good strategy will work because of *Rule* ω that Eve wins if the projection $r_B(\mathfrak{o}(a,e))$ is not defined

Rule 0: $B((p)^0, k) \subseteq (\Lambda)_s$
Rule 1: $B((q)^0, k) \subseteq (\Lambda)_s$;
Rule 2: $(b)^0 \in \Upsilon_{H,\Lambda}$
Rule 3: $(d)^0 \in \Upsilon_{H,\Lambda}$;
Rule 4: $B((p)^1, k) \subseteq (\Lambda)_s$;
Rule 5: $B((q)^1, k) \subseteq (\Lambda)_s$;
Rule 6: $(b)^1 \in \Upsilon_{H,\Lambda}$;
Rule 7: $(d)^1 \in \Upsilon_{H,\Lambda}$;
Rule 8: $B((p)^2, k) \subseteq (\Lambda)_s$;
Rule 9: $B((q)^2, k) \subseteq (\Lambda)_s$;
Rule 10: $(b)^2 \in \Upsilon_{H,\Lambda}$;
Rule 11: $(d)^2 \in \Upsilon_{H,\Lambda}$;
$\vdots$

The first failure in those rules entails defeat for the responsible player; if all those rules have been kept, we form the set $B[p,q]$ as in 14·2 and define the *critical ordinal* $\mathfrak{o}_M(b,d)$ of the play to be $\inf\bigl(\bigcap_{i\geqslant 1}(S_{(b)^i} \cap S_{(d)^i})\bigr)$; we then first test to see whether $r(B, \mathfrak{o})$ is defined:

Rule ω*:* Eve wins if $B[p,q]$ is not a subset of a single s-size component of $\Diamond(\mathfrak{o}(b,d))$; in symbols, if $r_k(B[p,q], \Diamond(\mathfrak{o}(b,d)) \uparrow$.

Finally, if $r_k(B[p,q], \Diamond(\mathfrak{o}(b,d)) \downarrow$,

Rule $\omega + 1$*:* Eve wins if $r_k\bigl(B[p,q], \Diamond(\mathfrak{o}(b,d))\bigr) \in A$.

14·29 DEFINITION $E(s) \ =_{\mathrm{df}} \ \bigl\{A \in \mathcal{P}([\delta_H]^{\overline{\overline{s}}}) \cap HOD_S \bigm| \text{ Eve has a winning strategy in } \mathfrak{G}_M(A)\bigr\}$.

14·30 THEOREM $(\mathrm{Det}(OD_S) + V = L(\mathbb{R}))$ *$\langle E(s) \mid s \in [\lambda]^{<\omega}\rangle$ is, in HOD_S, a (δ_H, λ)-extender.*

The game $\mathfrak{G}_W(A, \psi)$

The original game: we suppose ψ to be a good pre-strategy for a simple basis game, supplying us with $B(\psi, k) \subseteq (\Lambda)_s$.

First remark that the game used in the present text, which we now call $\mathfrak{G}'_W(A, \hat\psi)$ is this:

One fixes $\hat\psi$ which is both 1-good over S and s-correct. Adam plays a, Eve plays e.

Rule 0: $(a)^0 \in \Upsilon_{H,\Lambda}$;
Rule 1: $(e)^0 \in \Upsilon_{H,\Lambda}$;
Rule 2: $(a)^1 \in \Upsilon_{H,\Lambda}$;
Rule 3: $(e)^1 \in \Upsilon_{H,\Lambda}$;
Rule 2: $(a)^2 \in \Upsilon_{H,\Lambda}$;
Rule 3: $(e)^2 \in \Upsilon_{H,\Lambda}$;

$\vdots$

The first failure in those rules entails defeat for the responsible player; if all those rules have been kept, we define the *critical ordinal* $\mathfrak{o}_W(a,e)$ of the play to be $\inf\bigl(D(\hat\psi) \cap \bigcap_{i\geqslant 0}(S_{(a)^i} \cap S_{(e)^i})\bigr)$, and then follow

Rule ω: Eve wins if $r_k\bigl(\hat\psi, \Diamond(\mathfrak{o}(a,e))\bigr) \in A$.

CHECK: the text might not have the intersection with $D(\hat\psi)$, though such intersection is helpful.

Relating MY extender to that of WOODIN

Contrast that with a game closely related to one in the present text: the game $\mathfrak{G}'_W(A,\hat\psi)$. One fixes $\hat\psi$ which is both 1-good over S and s-correct. Adam plays a, Eve plays e.

Rule 0:
Rule 1:
Rule 2: $(b)^0 \in \Upsilon_{H,\Lambda}$;
Rule 3: $(d)^0 \in \Upsilon_{H,\Lambda}$;
Rule 4:
Rule 5:
Rule 6: $(b)^1 \in \Upsilon_{H,\Lambda}$;
Rule 7: $(d)^1 \in \Upsilon_{H,\Lambda}$;
Rule 8:
Rule 9:
Rule 10: $(b)^2 \in \Upsilon_{H,\Lambda}$;
Rule 11: $(d)^2 \in \Upsilon_{H,\Lambda}$;

$\vdots$

The first failure in those rules entails defeat for the responsible player; if all those rules have been kept, we define the *critical ordinal* $\mathfrak{o}_W(a,e)$ of the play to be $\inf\bigl(D(\hat\psi) \cap \bigcap_{i\geqslant 1}(S_{(a)^i} \cap S_{(e)^i})\bigr)$, and then follow

Rule ω: Eve wins if $r_k\bigl(\hat\psi, \Diamond(\mathfrak{o}(a,e))\bigr) \in A$.

HERE the convention on D should be in line with the previous game.

If START is 1-good, one can find $\hat\psi$ for which the above game is determined.

[I see I am using *good* in two senses, in talking about a position in the S-game being 1-good and a real ψ being 1-good.]

14·31 Definition We define $E_M(s)$ to be the set of A in HOD_S for which Eve has a winning strategy in the game $\mathfrak{G}_M(A)$; and for good s-correct $\hat\psi$, we define $E_W(s,\hat\psi)$ to be the set of A in HOD_S for which Eve has a winning strategy in the game $\mathfrak{G}_W(A,\hat\psi)$.

14·32 Proposition *Assume $Det(OD_S)$. Then in HOD_S $E_M(s)$ is a δ_H-complete ultrafilter on $[\delta_H]^k$; further, if a good s-correct $\hat\psi$ exists, then $E_W(s,\hat\psi) = E_M(s)$.*

The proof

14·33 Proposition *for given good s-correct $\hat{\psi}$ and A, whoever has a winning strategy in $G'_W(A,\hat{\psi})$ will have one in $\mathfrak{G}_W(A,\hat{\psi})$ and vice-versa.*

Rather more challenging:

14·34 Proposition *for given good s-correct $\hat{\psi}$ and $A \in HOD_S$, whoever has a winning strategy in $\mathfrak{G}_W(A,\hat{\psi})$ will have one in $\mathfrak{G}_M(A)$ and vice-versa.*

14·35 Lemma *Let $\hat{\psi}$ be good and s-correct. Let $A \in HOD_S$. Suppose that Eve has a winning strategy in $\mathfrak{G}_M(A)$. Then she has one in $\mathfrak{G}_W(A,\hat{\psi})$.*

Proof : she imagines that A has played $\hat{\psi}$. ... $\dashv$ (14·35)

That shows easily that $E_M(s) \subseteq E_W(s,\hat{\psi})$; hence the two (intersected with HOD_S), being ultrafilters, are equal.

14·36 Remark I don't at present see a reason for any particular placing of the b's and d's relative to the p's and q's. So we'll try different versions of the rules.

Identity of the various extenders

Our aim is to prove the following:

14·37 Theorem *My pure game without constructors defines the BP extender when there are constructors; the BP extender with constructors is the Woodin extender when there are good prestrategies.*

14·38 Problem With just $Det(OD_S)$ and no $V = L(\mathbb{R}; S)$, we get some kind of well-founded extender existing somewhere. What is it ?

===== junk below here

WOFFLE: *These rules have the advantage of apparently defining the partial projection r_B before the a-e game gets going; but that is not true. IF WE WERE PLAYING A GAME OF LENGTH $\omega+\omega+2$ either player could take steps to ensure that $\mathfrak{o}$ is in the domain of definition of $\mathfrak{o}_M(a,e)$.*

*But I see no reason why the *rules* should not be of order type $\omega+\omega+2$.*

We divide a into p's and b's; and e into q's and d's.

=======

Let $\alpha_0 \in \Upsilon_{H,\Lambda}$. Then

$$\forall\alpha(\alpha)^0 = \alpha_0 \implies \forall\tau{:}{\in}\mathcal{E}\ ([\alpha]*\tau)^0 \in \Upsilon_{H,\Lambda} :$$

that is a Σ_1 statement, and so enforced by some α_1; we may inductively pick $\alpha_k \in \Upsilon_{H,\Lambda}$ such that

$$S_{\alpha_{k+1}} \subseteq \{\zeta \mid \forall\alpha\forall i{:}{\leq}k\ (\alpha)^i = \alpha_i \implies ([\alpha]*\tau)^k \in \Upsilon_{H,\Lambda}\}.$$

Let $\bar{\alpha}$ be such that $\forall n\, (\bar{\alpha})^n = \alpha_n$, and let $\bar{\xi} = \inf \bigcap_{n<\omega} S_{\alpha_n}$. Then for any $\tau \in \mathcal{E}^*$, $\bar{\xi}$ will be the critical ordinal of the run of the game where Adam plays $\bar{\alpha}$ and Eve uses τ. Hence $\bar{\xi} \in \bigcap_{\nu<\eta} A_\nu$, as required. $\dashv$ (14·38)

======

15: A non-monotonic coding theorem

In this section we shall use the determinacy of a certain game and also the recursion-theorem applied to a certain pointclass Γ^T, but neither reflection arguments nor goodness.

15·0 Suppose that we have an ordinal $\kappa < \Theta$, a sequence $\vec{R} = \langle R(\delta) \mid \delta < \kappa\rangle$ of pre-well-orderings of non-empty sets of reals, and a function $g : \kappa \longrightarrow \Theta$ such that

$$\forall\zeta{:<}\kappa\; g(\zeta) < |R(\zeta)| \tag{15·0·0}$$

$$\forall\zeta{:<}\kappa\; \forall\delta{:<}\zeta\; (R(\delta))_{g(\delta)} \cap \mathrm{Field}(R(\zeta)) = \varnothing. \tag{15·0·1}$$

15·1 Definition We define TRACK to be $\bigcup_{\delta<\kappa}(R(\delta))_{g(\delta)}$. Each $b \in$ TRACK lies in exactly one $(R(\delta))_{g(\delta)}$; we write δ_b for that δ, and we pre-well-order TRACK by the relation

$$b \leqslant_{\mathrm{TRACK}} c \iff_{\mathrm{df}} \delta_b \leqslant \delta_c.$$

15·2 Remark TRACK is a set of reals and is definable from $\vec{R}$ and g. $\delta_b = |b|_{\leqslant_{\mathrm{TRACK}}}$. The length of $\leqslant_{\mathrm{TRACK}}$ is κ, as each Field$(R(\delta))$ is non-empty.

15·3 Definition For $\delta < \kappa$ and $\alpha < |R(\delta)|$, we define a pre-wellordering $K_{\delta,\alpha}$ to have field

$$\mathcal{F}_{\delta,\alpha} =_{\mathrm{df}} \Big(\bigcup_{\gamma<\delta}(R(\gamma))_{g(\gamma)}\Big) \cup (R(\delta))_\alpha;$$

for $\alpha = g(\delta)$ the pre-well-ordering is that of the initial segment of TRACK up to and including level δ; for $\alpha \neq g(\delta)$, the pre-well-ordering is that of the initial segment of TRACK up to but excluding level δ, followed by the single component $(R(\delta))_\alpha$. Thus the length of $K_{\delta,\alpha}$ is $\delta + 1$.

15·4 Remark $\vec{\mathcal{F}}$ and $\vec{K}$ are defined from $\vec{R}$ and g.

In the following lemma, we have six cases to consider, one of which is taken in two subcases.

15·5 Lemma *Let $b \in$ TRACK. Then*

for $\gamma < \delta_b$, if $g(\gamma) \neq \alpha < \vert R(\gamma)\vert$,	$\mathrm{TRACK}_{<b} \not\subseteq \mathcal{F}_{\gamma,\alpha}$	*and*	$\mathrm{TRACK}_{\leqslant b} \not\subseteq \mathcal{F}_{\gamma,\alpha}$,
whereas, if $\gamma + 1 < \delta_b$,	$\mathrm{TRACK}_{<b} \not\subseteq \mathcal{F}_{\gamma,g(\gamma)}$	*and*	$\mathrm{TRACK}_{\leqslant b} \not\subseteq \mathcal{F}_{\gamma,g(\gamma)}$,
whilst if $\gamma + 1 = \delta_b$,	$\mathrm{TRACK}_{<b} = \mathcal{F}_{\gamma,g(\gamma)}$	*and*	$\mathrm{TRACK}_{\leqslant b} \not\subseteq \mathcal{F}_{\gamma,g(\gamma)}$;
next, if $g(\delta_b) \neq \alpha < \vert R(\delta_b)\vert$,	$\mathrm{TRACK}_{<b} \subset \mathcal{F}_{\delta_b,\alpha}$	*and*	$\mathrm{TRACK}_{\leqslant b} \not\subseteq \mathcal{F}_{\delta_b,\alpha}$,
whereas	$\mathrm{TRACK}_{<b} \subset \mathcal{F}_{\delta_b,g(\delta_b)}$	*and*	$\mathrm{TRACK}_{\leqslant b} = \mathcal{F}_{\delta_b,g(\delta_b)}$;
finally, for $\delta_b < \zeta < \kappa$,			
if $g(\zeta) \neq \alpha < \vert R(\zeta)\vert$,	$\mathrm{TRACK}_{<b} \subset \mathcal{F}_{\zeta,\alpha}$	*and*	$\mathrm{TRACK}_{\leqslant b} \subset \mathcal{F}_{\zeta,\alpha}$,
whereas	$\mathrm{TRACK}_{<b} \subset \mathcal{F}_{\zeta,g(\zeta)}$	*and*	$\mathrm{TRACK}_{\leqslant b} \subset \mathcal{F}_{\zeta,g(\zeta)}$.

Proof : straightforward. ⊣ (15·5)

15·6 For the moment, let Γ^T be the pointclass of all pointsets which are Σ_1 in $\vec{R}$, g and a third-order variable T for a binary relation which will usually be a pre-well-ordering; we index the corresponding bold-face class $\boldsymbol{\Gamma}^T$ by reals t as $U_t(T)$. The chief requirements are that Γ^T can be parametrised, that the recursion theorem will apply, and that $\Sigma^1_1\Gamma^T \subseteq \Gamma^T$. We shall see what else is required of Γ^T during the proof of the main theorem and after that shall comment on the possibility of working with a smaller pointclass.

15·7 Theorem *If the hypotheses (15·0) hold, if the family Γ^T is as above, and provided that a certain game definable from $\vec{R}$ and g is determined, then there is an index s such that for $\nu < \kappa$ and $\alpha < |R(\nu)|$,*

$$U_s(K_{\nu,\alpha}) \cap (R(\nu))_\alpha \neq \varnothing \iff \alpha = g(\nu).$$

The Proof

We seek to define two functions ADD and CHOP, inputting two reals b and t, and returning a real. The behaviour of these functions will be crucial for $b \in$ TRACK; and unimportant otherwise.

15·8 REMARK ADD and CHOP are definable from TRACK.

The general computation will be this: given a real b and a binary relation T on the reals one tests to see if T is a pre-well-ordering —in our intended applications T will be one of length strictly less than κ— and if so one asks if $b \in \bigcup_{\delta<|T|}(R(\delta))_{g(\delta)}$; if not, both $\text{ADD}(b,t)$ and $\text{CHOP}(b,t)$ are to return the value t; otherwise they both compute the sets $\text{TRACK}_{<b} =_{\text{df}} \{c \mid c <_{\text{TRACK}} b\}$, $\text{TRACK}_{\leqslant b} =_{\text{df}} \{c \mid c \leqslant_{\text{TRACK}} b\}$, and then follow the following

15·9 DEFINITION

$$U_{\text{ADD}(b,t)}(T) =_{\text{df}} \begin{cases} U_t(T) \cup \{b\} & \text{if } \text{TRACK}_{\leqslant b} \subset \text{Field}(T) \\ U_t(T) \cup \{b\} & \text{if } \text{TRACK}_{\leqslant b} = \text{Field}(T) \\ U_t(T) & \text{if } \text{TRACK}_{\leqslant b} \not\subseteq \text{Field}(T) \end{cases}$$

$$U_{\text{CHOP}(b,t)}(T) =_{\text{df}} \begin{cases} \text{TRACK}_{<b} & \text{if } \text{TRACK}_{<b} \subset \text{Field}(T) \\ U_t(T) & \text{if } \text{TRACK}_{<b} = \text{Field}(T) \\ U_t(T) & \text{if } \text{TRACK}_{<b} \not\subseteq \text{Field}(T) \end{cases}$$

15·10 REMARK We emphasize that in our notation, $Y \subset Z \iff (Y \subseteq Z \;\&\; Y \neq Z)$.

15·11 REMARK $U_t(T)$ is a set of, or, rather, an unary relation on, the reals; the functions ADD and CHOP have been arranged so that should t have the property that $U_t(T)$ is always a subset of the field of T, $\text{ADD}(b,t)$ and $\text{CHOP}(b,t)$ will also have that property.

15·12 Our definitions will be important in the context when T is one of the pre-well-orderings $K_{\delta,\alpha}$.

15·13 DEFINITION Call t *acceptable* if $\forall\delta{:}{<}\kappa\; \forall\alpha \neq g(\delta)\; U_t(K_{\delta,\alpha}) \cap (R(\delta))_\alpha = \varnothing$.

15·14 REMARK *Acceptability* is definable from $\vec{R}$ and g.

15·15 DEFINITION The g-*lexical* ordering of the set $\{(\delta,\, \alpha) \mid \delta < \kappa \;\&\; \alpha < |R(\delta)|\}$ is the usual lexical ordering perturbed by placing for given δ the pair $(\delta,\, g(\delta))$ **after** all pairs $(\delta,\, \alpha)$ with $g(\delta) \neq \alpha < |R(\delta)|$, (but before all $(\delta',\, \alpha')$ with $\delta < \delta'$).

15·16 REMARK The g-lexical ordering is definable from $\vec{R}$ and g.

15·17 DEFINITION We define the *first failure* of t, $f\!f(t)$, to be the g-lexically first pair $(\delta,\, \alpha)$ such that $0 \leq \alpha < |R(\delta)|$ and

$$\begin{aligned} &\text{either } \alpha \neq g(\delta) \;\&\; U_t(K_{\delta,\alpha}) \cap (R(\delta))_\alpha \neq \varnothing \\ &\qquad\text{or } \alpha = g(\delta) \;\&\; U_t(K_{\delta,\alpha}) \cap (R(\delta))_\alpha = \varnothing. \end{aligned}$$

If no such pair exists, we set $f\!f(t) = \infty$.

15·18 REMARK Note that for acceptable t the failure, if any, will be in the second disjunct.

15·19 REMARK The function $f\!f(t)$ is definable from $\vec{R}$ and g.

The Theorem will now follow from

15·20 PROPOSITION $(\text{Det}(OD_{\vec{R},g}))$ $\exists s\, f\!f(s) = \infty$.

Proof : We suppose that no such s exists and obtain a contradiction. For that, the determinacy of a single specific game is sufficient; in that game, which we shall call the *Non-monotonic Coding Game*: Adam plays, bitwise, a real a and Eve a real e.

Rule NM-1: a must be acceptable, otherwise Adam loses.

Rule NM-2: If a is acceptable and e not, then Eve loses.

Rule NM-3: If both a and e are acceptable, then Eve wins iff $f\!f(e) > f\!f(a)$.

15·21 REMARK That game is defined from the sequence $\vec{R}$ and g.

Let σ be a winning strategy for Adam. Let $\bar a$ be such that for any T,

$$U_{\bar a}(T) = \bigcup_{e\in\mathcal{N}} U_{\sigma\star[e]}(T).$$

It is here that we rely on the pointclass $\Sigma^1_1\Gamma^T$ being included in Γ^T.

Then $\bar a$ is acceptable, since for each e, $\sigma\star[e]$ must be acceptable and therefore for each $\alpha\neq g(\delta)$, $U_{\sigma\star[e]}(K_{\delta,\alpha})$ misses $(R(\delta))_\alpha$. Let $(\bar\delta, g(\bar\delta))$ be its first failure, and pick $b\in (R(\bar\delta))_{g(\bar\delta)}$. Then $b\in$ TRACK and $\delta_b=\bar\delta$.

Let $\widetilde{e} = \text{ADD}(b,\bar a)$. We assert that $\widetilde{e}$ is acceptable and has a higher first failure than $\bar a$, and hence its play by Eve will defeat the strategy σ.

To see the acceptability of $\widetilde{e}$, note that by the acceptability of $\bar a$, the definition of ADD$(b,\bar a)$, Lemma 15·5 and condition (15·0·1),

$$\begin{aligned}
&U_{\text{ADD}(b,\bar a)}(K_{\gamma,\alpha})\cap(R(\gamma))_\alpha = U_{\bar a}(K_{\gamma,\alpha})\cap(R(\gamma))_\alpha=\varnothing && \text{if } \gamma<\bar\delta \text{ and } g(\gamma)\neq\alpha<|R(\gamma)|,\\
&U_{\text{ADD}(b,\bar a)}(K_{\bar\delta,\alpha})\cap(R(\bar\delta))_\alpha = U_{\bar a}(K_{\bar\delta,\alpha})\cap(R(\bar\delta))_\alpha=\varnothing && \text{if } \alpha\neq g(\bar\delta),\\
\text{and}\quad &U_{\text{ADD}(b,\bar a)}(K_{\zeta,\alpha})\cap(R(\zeta))_\alpha = \big(U_{\bar a}(K_{\zeta,\alpha})\cup\{b\}\big)\cap(R(\zeta))_\alpha=\varnothing && \text{if } \zeta>\bar\delta \text{ and } \alpha\neq g(\zeta).
\end{aligned}$$

Thus ADD$(b,\bar a)$ is acceptable. Note now that by the choice of b at the first failure of $\bar a$,

$$\begin{aligned}
&U_{\text{ADD}(b,\bar a)}(K_{\gamma,g(\gamma)})\cap(R(\gamma))_{g(\gamma)} = U_{\bar a}(K_{\gamma,g(\gamma)})\cap(R(\gamma))_{g(\gamma)}\neq\varnothing && \text{if } \gamma<\bar\delta;\\
\text{and}\quad &U_{\text{ADD}(b,\bar a)}(K_{\bar\delta,g(\bar\delta)})\cap(R(\bar\delta))_{g(\bar\delta)} = \big(U_{\bar a}(K_{\bar\delta,g(\bar\delta)})\cup\{b\}\big)\cap(R(\bar\delta))_{g(\bar\delta)}\supseteq\{b\}\neq\varnothing;
\end{aligned}$$

Thus the first failure of $\tilde e$ is at a level strictly higher than $\bar\delta$; and thus Adam has no winning strategy.

Now let τ be a winning strategy for Eve. Define a function $t\mapsto t^*$, recursive in the parameter τ, of indices by

$$U_{t^*}(T) = \big\{a \;\big|\; \exists c\big[c\in\text{TRACK} \;\&\; |c|_{\text{TRACK}}+1=|T| \;\&\; a\in U_{[\text{CHOP}(c,t)]\star\tau}(T)\big]\big\}$$

By the Recursion Theorem, there is an index s such that for every T, $U_s(T)=U_{s^*}(T)$. Let $f\!f(s)=(\delta,\beta)$: we shall derive a contradiction.

Let $b\in(R(\delta))_{g(\delta)}$, so that $b\in$ TRACK and $\delta_b=\delta$. The effect of our definition of CHOP is this, where "correct" means that no failure occurs at that point in the g-lexical order:

for $\gamma<\delta_b$ & $\alpha\neq g(\gamma)$	$U_{\text{CHOP}(b,s)}(K_{\gamma,\alpha})=U_s(K_{\gamma,\alpha})$	and hence remains correct,
and for $\gamma<\delta_b$	$U_{\text{CHOP}(b,s)}(K_{\gamma,g(\gamma)})=U_s(K_{\gamma,g(\gamma)})$	and also remains correct;
for $\alpha\neq g(\delta)$,	$U_{\text{CHOP}(b,s)}(K_{\delta,\alpha})=\text{TRACK}_{<b}$	and thus becomes correct,
whereas	$U_{\text{CHOP}(b,s)}(K_{\delta,g(\delta)})=\text{TRACK}_{<b}$	and will be incorrect;
for $\zeta>\delta$ & $\alpha\neq g(\zeta)$,	$U_{\text{CHOP}(b,s)}(K_{\zeta,\alpha})=\text{TRACK}_{<b}$	and becomes correct,
while for $\zeta>\delta$,	$U_{\text{CHOP}(b,s)}(K_{\zeta,g(\zeta)})=\text{TRACK}_{<b}$	and will be incorrect.

so that whether $\beta=g(\delta)$ or not, CHOP(b,s) is acceptable: and its first failure is at $(\delta,g(\delta))$; hence $[\text{CHOP}(b,s)]\star\tau$ is acceptable and its first failure is at a higher level than δ, and so $U_{[\text{CHOP}(b,s)]\star\tau}(K_{\delta,g(\delta)})$ meets $(R(\delta))_{g(\delta)}$. Since $|b|_{\text{TRACK}}=\delta$ and $|K_{\delta,g(\delta)}|=\delta+1$,

$$U_{[\text{CHOP}(b,s)]\star\tau}(K_{\delta,g(\delta)})\subseteq U_{s^*}(K_{\delta,g(\delta)})=U_s(K_{\delta,g(\delta)}).$$

Thus the case $\beta=g(\delta)$ is ruled out: s cannot fail there.

There remains the case $\beta\neq g(\delta)$. If $U_s(K_{\delta,\beta})$ meets $(R(\delta))_\beta$, so does $U_{s^*}(K_{\delta,\beta})$, and thus there are c and a with

$$c\in\text{TRACK} \;\&\; |c|_{\text{TRACK}}+1=|K_{\delta,\beta}=\delta+1| \;\&\; a\in U_{[\text{CHOP}(c,s)]\star\tau}(K_{\delta,\beta})\cap(R(\delta))_\beta.$$

But then $c =_{R(\delta)} b$ and so $\textsc{track}_{\leqslant b} = \textsc{track}_{\leqslant c}$ and $\textsc{track}_{<b} = \textsc{track}_{<c}$; so our previous calculation with b may be repeated, and we again find that $\textsc{chop}(c,s)$ is acceptable with first failure at $(\delta,\, g(\delta))$. Hence $[\textsc{chop}(c,s)]\star\tau$ is acceptable with first failure beyond level δ, and therefore $U_{[\mathrm{CHOP}(c,s)]\star\tau}(K_{\delta,\beta})\cap(R(\delta))_\beta = \varnothing$, a contradiction. ⊣ (15·20)

⊣ (15·7)

The pointclass Γ^T

15·22 In the above proof we have taken Γ^T to be the collection of $\Sigma_1(\vec{R}, g, T)$ pointsets. Then $\boldsymbol{\Gamma}^T$ may be indexed by reals and the recursion theorem will apply.

15·23 We may also take Γ^T to be a rather smaller family. Namely, we consider the following nine relations:

$$\begin{aligned}
F_1(b,T) &\iff_{\mathrm{df}} T \text{ is a pre-well-ordering and } \exists\delta{:<}|T|\; b \in (R(\delta))_{g(\delta)}\\
F_2(b,T) &\iff_{\mathrm{df}} \neg F_1(b,T)\\
F_3(b,T) &\iff_{\mathrm{df}} T \text{ is a pre-well-ordering and } \exists\delta{:<}|T|\; \big[\delta+1=|T| \;\&\; b \in (R(\delta))_{g(\delta)}\big]\\
F_4(x,T) &\iff_{\mathrm{df}} x \in \mathrm{Field}(T)\\
F_5(x,T) &\iff_{\mathrm{df}} \neg F_4(x,T)\\
F_6(x,b,T) &\iff_{\mathrm{df}} F_1(b,T) \;\&\; x \in \textsc{track}_{\leqslant b}\\
F_7(x,b,T) &\iff_{\mathrm{df}} \neg F_6(x,b,T)\\
F_8(x,b,T) &\iff_{\mathrm{df}} F_1(b,T) \;\&\; x \in \textsc{track}_{<b}\\
F_9(x,b,T) &\iff_{\mathrm{df}} \neg F_8(x,b,T)
\end{aligned}$$

15·24 Remark $F_8(x,b,T) \iff F_6(x,b,T) \;\&\; F_7(b,x,T)$; $\quad F_9(x,b,T) \iff F_7(x,b,T)$ or $F_6(b,x,T)$.

Now take Γ^T to be the class of all relations $pos\Sigma^1_2$ in the first seven, and T. To check that this smaller Γ^T will work, we must scrutinise four things in our proof: the definition of $\textsc{add}$, the definition of $\textsc{chop}$, the definition of $\bar{a}$ in our discussion of a winning strategy σ for Adam, and the definition of the map $t \mapsto t^*$ in our discussion of a winning strategy τ for Eve.

We have the following equivalences:

$$\begin{aligned}
a \in U_{\mathrm{ADD}(b,t)}(T) \iff &\Big[F_1(b,T) \;\&\; \forall x{:\in}\mathcal{N}\; \big(F_7(x,b,T) \text{ or } F_4(x,T)\big) \;\&\; \big(a=b \text{ or } a \in U_t(T)\big)\Big]\\
&\text{or } \Big[F_1(b,T) \;\&\; \exists x{:\in}\mathcal{N}\; \big(F_6(x,b,T) \;\&\; F_5(x,T)\big) \;\&\; a \in U_t(T)\Big]\\
&\text{or } \Big[F_2(b,T) \;\&\; a \in U_t(T)\Big];
\end{aligned}$$

$$\begin{aligned}
&a \in U_{\mathrm{CHOP}(b,t)}(T) \iff\\
&\qquad\Big[F_1(b,T) \;\&\; F_8(a,t,T)\\
&\qquad\qquad \&\; \Big(\forall x{:\in}\mathcal{N}\; \big(F_9(x,b,T) \text{ or } F_4(x,T)\big) \;\&\; \exists y{:\in}\mathcal{N}\; \big(F_9(y,b,T) \;\&\; F_4(y,T)\big)\Big)\Big]\\
&\quad\text{or } \Big[F_1(b,T) \;\&\; a \in U_t(T)\\
&\qquad\qquad \&\; \Big(\exists x{:\in}\mathcal{N}\; \big(F_8(x,b,T) \;\&\; F_5(x,T)\big) \text{ or } \forall y{:\in}\mathcal{N}\; \big(F_8(y,b,T) \text{ or } F_5(y,T)\big)\Big)\Big]\\
&\quad\text{or } \Big[F_2(b,T) \;\&\; a \in U_t(T)\Big];
\end{aligned}$$

$$a \in U_{\bar{a}}(T) \iff \exists e{:\in}\mathcal{N}\; a \in U_{\sigma\star[e]}(T);$$

$$a \in U_{t^*}(T) \iff \exists c{:\in}\mathcal{N}\; \big(F_3(c,T) \;\&\; a \in U_{[\mathrm{CHOP}(c,t)]\star\tau}(T)\big).$$

From those it is plain that none of those four operations on sets will lead us out of the class of pointsets which are $pos\boldsymbol{\Sigma^1_2}(F_1,\ldots,F_7,T)$, and hence we may follow Moschovakis's discussion summarised in §§2 and 3 of Chapter II in starting with the Σ^* class $pos\Sigma^1_2(F_1,\ldots,F_7,T)$ when establishing the truth of the recursion theorem that we apply.

16: The normality of THE EXTENDER

In this first draft, we adapt the argument from Chapter Five.

Again we use the notion of *seizing control of the game.*

16·0 THEOREM *Let $s \in [\lambda]^{<\omega}$, where $\delta_H < \lambda = \bigcup \lambda < \Theta$. Let $h \in HOD_S$, h defined on $[\delta_H]^{\overline{\overline{s}}}$, such that*

$$\{u \mid h(u) < \bigcup u\} \in E(s).$$

Then there is an $\eta < \lambda$, which might be a member of s, such that setting $t = \{\eta\} \cup s$,

$$T'_\eta \;=_{\rm df}\; \{v \in [\delta_H]^{\overline{\overline{t}}} \mid h(v^{s,t}) = \bigcup v^{\{\eta\},t}\} \in E(t).$$

Moreover, $\eta < \bigcup s$.

Proof : Remark first that if $\eta \in s$ then $s = t$, and the conclusion of the theorem would then read:

$$\{v \in [\delta_H]^{\overline{\overline{s}}} \mid h(v) = \bigcup v^{\{\eta\},s}\} \in E(s).$$

If that is true of any of the finitely many members of s we need look no further; henceforth we tacitly assume that the η that we seek is not a member of s.

We shall suppose the theorem false for each $\eta \in \lambda \setminus s$ and derive a contradiction. Once we know such an η exists, we may easily see that it must be less than $\bigcup s$, since otherwise $\eta = \max t \geqslant \max s$, and $\bigcup v^{\{\eta\},t} = \max v \geqslant \max v^{s,t} > h(v^{s,t})$ for almost all v, a contradiction.

For the moment let R be projective in Λ. I suspect we can take $R = \Lambda$.

We replace η by various reals, by setting

$$T_b \;=_{\rm df}\; T'_{|b|_R}.$$

Assuming the falsity of the theorem, every such T_b is in $\widetilde{E}(t)$: then Adam has a winning strategy for every game $\mathcal{G}_M(\Upsilon_{H,\Lambda}, T_b)$ with $b \in Field(R)$. WHAT IS R ?

Let

$$Z \;=_{\rm df}\; \{(b,p,\sigma) \mid b \in \mathrm{Field}(R) \;\&\; p \text{ is } s\text{-correct } \;\&\; \sigma \text{ is a winning strategy for Adam in } \mathcal{G}_M(\Upsilon_{H,\Lambda}, T_b)\}.$$

By now familiar arguments there is a subset Z^* of Z which is $pos\Sigma^1_1(<_R)$ and such that, setting $\mathcal{A}^* = \{\sigma \mid \exists b\ (b,\sigma) \in Z^*\}$, for each b there is a $b' =_R b$, an s-correct p and a $\sigma \in \mathcal{A}^*$ with σ a winning strategy for Adam in $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_{b'})$. Z^* is projective in Λ and so for each $\delta \in \mathrm{Dom}(\diamondsuit)$, we may define $Z^*(\delta)$ from $\diamondsuit(\delta)$ as Z^* is from Λ: as before, if we start from two possible definitions of Z^*, the reflected definitions will agree almost everywhere.

$\forall (b,\,\sigma){:}{\in} Z^*\ b \in \mathrm{Field}(R)$, and so

$$X_0 \;=_{\rm df}\; \{\delta \mid \forall (b,\,\sigma){:}{\in} Z^*(\delta)\ b \in \mathrm{Field}(R(\delta))\} \in \mathfrak{F}.$$

We may treat the elements of Z as pre-well-ordered by their first co-ordinate, and this pre-well-ordering is inherited by Z^*. Since Z^* meets every component of Z^*, **WE MAY APPLY** Corollary 5·9 to conclude that

$$X_1 \;=_{\rm df}\; \{\delta \mid Z^* \text{ meets every component of } Z^*(\delta)\} \in \mathfrak{F}.$$

THE NEXT FEW LINES FIND WAYS OF SAYING THAT Z^* IS FULL IN R; but are they optimally efficient ?

Since Z^* meets every component of R , we know that for every ordinal $\eta < |R|$,

$$(Z^*)_\eta = \big\{(b,p,\sigma) \in Z^* \,\big|\, |b|_R = \eta\big\},$$

or, quantifying over reals instead of ordinals,

$$\forall a{:}{\in}\,\mathrm{Field}(R)\ \bigl[(Z^*)_{|a|_R} = \bigl\{(b,p,\sigma)\in Z^* \bigm| b\in\mathrm{Field}(R)\ \&\ |b|_R = |a|_R\bigr\}\bigr];\qquad \text{hence}$$

$$X_2 =_{\mathrm{df}} \Bigl\{\delta \Bigm| \forall a{:}{\in}\,\mathrm{Field}(R(\delta))\ \bigl[(Z^*(\delta))_{|a|_{R(\delta)}} = \bigl\{(b,p,\sigma)\in Z^*(\delta) \bigm| b\in\mathrm{Field}(R(\delta))\ \&\ |b|_{R(\delta)} = |a|_{R(\delta)}\bigr\}\bigr]\Bigr\} \in \mathfrak{F}.$$

Seizing control ...

Since everything in $\mathcal{A}^*$ is an Adam strategy,

$$\forall(b,p,\sigma){:}{\in}\,Z^*\ \forall e{:}{\in}\,\mathcal{N}\ \ (\sigma * [e])^0 =_\Lambda p:$$

Hence

$$\{\nu \mid \forall(b,p,\sigma){:}{\in}\,Z^*\ \forall e{:}{\in}\,\mathcal{N}\ \ (\sigma * [e])^0 =_{\diamondsuit(\nu)} p\} \in \mathfrak{F}$$

Since everything in $\mathcal{A}^*$ is an Adam strategy,

$$\forall(b,p,\sigma){:}{\in}\,Z^*\ \forall e{:}{\in}\,\mathcal{N}\ \ \bigl[\Phi_\diamondsuit\bigl((\sigma * [e])^1, \Lambda, \diamondsuit, \delta_H; H\bigr)\bigr]:$$

that is a Σ_1^H statement about Λ, $\diamondsuit$ and δ_H; hence we may choose $e_1\in\Upsilon_{H,\Lambda}$ such that

$$S_{e_1} \subseteq X_0\cap X_1\cap X_2\cap\bigl\{\delta \bigm| \forall(b,p,\sigma){:}{\in}\,Z^*(\delta)\ \forall e{:}{\in}\,\mathcal{N}\ \ \bigl[\,\Phi_\diamondsuit\bigl((\sigma * [e])^1, \diamondsuit(\delta), \diamondsuit\restriction\delta, \delta; H\bigr)\bigr]\bigr\}.$$

That last requirement implies that

$$\delta\in S_{e_1}\ \&\ \sigma\in\mathcal{A}^*(\delta) \implies \forall e{:}{\in}\,\mathcal{N}\ \ \delta\in S_{(\sigma*[e])^1}.$$

$$\text{Again,}\quad \forall\sigma{:}{\in}\,\mathcal{A}^*\ \forall e{:}{\in}\,\mathcal{N}\ \ \bigl[e^1 = e_1 \implies \Phi_\diamondsuit\bigl((\sigma * [e])^3, \Lambda, \diamondsuit, \delta_H; H\bigr)\bigr]:$$

that is a Σ_1^H statement; hence we may choose $e_3\in\Upsilon_{H,\Lambda}$ with

$$S_{e_3} \subseteq \bigl\{\delta \bigm| \forall\sigma{:}{\in}\,\mathcal{A}^*(\delta)\ \forall e{:}{\in}\,\mathcal{N}\ \ \bigl[e^1 = e_1 \implies \Phi_\diamondsuit\bigl((\sigma * [e])^1, \delta, \diamondsuit(\delta), \diamondsuit\restriction\delta, \delta; H\bigr)\bigr]\bigr\}.$$

In other words,

$$\delta\in S_{e_3}\ \&\ (b,p,\sigma)\in Z^*(\delta) \implies \forall e{:}{\in}\,\mathcal{N}\ \ (e^1 = e_1 \implies \delta\in S_{(\sigma*[e])^3}).$$

Continue for ω steps, to build a sequence e_{2k+1} of elements of $\Upsilon_{H,\Lambda}$ such that

$$\delta\in S_{e_{2k+1}}\ \&\ \sigma\in\mathcal{A}^*(\delta) \implies \forall e{:}{\in}\,\mathcal{N}\ \ \bigl((\forall i{:}{<}k\ e^{2i+1} = e_{2i+1}) \implies \delta\in S_{(\sigma*[e])^{2k+1}}\bigr).$$

Let $\bar e$ be a real with $\forall k\,\bar e^{2k+1} = e_{2k+1}$, let $\bar\delta$ be the least element of $\bigcap_{k<\omega} S_{\bar e^k}$, and let $(\bar b,\bar p,\bar\sigma)\in Z^*\cap(Z^*(\bar\delta))_{h(r_k(\bar p,\diamondsuit(\bar\delta)))}$: this last is possible as $\bar\delta\in X_1$.

As $(\bar b,\bar p,\bar\sigma)\in Z^*\subseteq Z$, we know that $\bar b\in\mathrm{Field}(R)$ and that $\bar\sigma$ is a winning strategy for Adam in $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_{\bar b})$. Further, as $(\bar b,\bar p,\bar\sigma)\in Z^*(\bar\delta)$, we know that $\bar\sigma\in\mathcal{A}^*(\bar\delta)$. So for each k, $\bar\delta\in S_{(\bar\sigma*[\bar e])^{2k+1}}$, and hence $\bar\delta = \mathfrak{o}(\bar\sigma * [\bar e], \bar e)$. Adam has won $\mathcal{G}^*(\Upsilon_{H,\Lambda}, T_{\bar b})$; his victory must be under Rule $\omega+\omega+1$ since for each odd k, $\bar e^k = e_k\in\Upsilon_{H,\Lambda}$, and so $\bar\delta\notin T_{\bar b}$.

But as $\bar\delta\in X_0$, and $(\bar b,\bar p,\bar\sigma)\in Z^*(\bar\delta)$, we know that $\bar b\in\mathrm{Field}(R(\bar\delta))$. As $\bar\delta\in X_2$, $h(r_k(\bar p,\diamondsuit(\bar\delta)) = |(\bar b,\bar p,\bar\sigma)|_{Z^*(\bar\delta)} = |\bar b|_{R(\bar\delta)}$. Contradiction ! $\dashv$ (16·0)

16·1 Remark In this proof, there seems to be little rôle for p (probably as a result of my dropping the requirement that Eve write s-correct things,); and we never used the possibility of reflecting the statement that every $(\sigma\star[e])^0$ is s-correct. Perhaps this slack in the argument will be useful somewhere else.