CHAPTER II: Prerequisites from recursion theory

THIS CHAPTER reviews some fundamental aspects of the post-Kleene approach to descriptive set theory *via* recursion theory. We urge the reader to lay a foundation by mastering Chapter III of Moschovakis' excellent treatise *Descriptive Set Theory* and in our review we summarise material from that and other chapters. References such as 3B.2 are to Moschovakis' book. We defer to our next chapter consideration of the consequences of *AD* that we shall require.

1: The effective theory

We assume a knowledge of recursion theory on the natural numbers, or rather a familiarity with the general feel of recursiveness, such as that to be obtained by reading §3A of Moschovakis. The higher flights such as priority arguments will not be needed.[C 1]

We consider *pointspaces* $\mathcal{X}$: those of *type 0* are ω^k for $k = 1, 2, 3 \ldots$, where ω has the discrete topology; those of *type 1* are the Baire space, $\mathcal{N}$ which is ${}^\omega\omega$, equipped with the product topology, and other products of a finite number of copies of $\mathcal{N}$ and a finite number of copies of ω, *e.g.* $\mathcal{N} \times \omega \times \omega \times \mathcal{N} \times \mathcal{N}$. All other point spaces considered are Polish spaces; fundamentally we are only interested in spaces of type 0 and 1, which have the property that finite sequences of elements may be coded as single elements.

For each space $\mathcal{X}$ that we consider we may define a basis of neighbourhoods $(N(\mathcal{X}, s) \mid s \in \omega)$, with $N(\mathcal{X}, 0)$ always being the empty set, in an effective way which is uniform in the sense that

[3B.1] there are recursive functions f, g, h such that

$$N(\mathcal{X},\, s) \times N(\mathcal{Y},\, t) = N(\mathcal{X} \times \mathcal{Y},\, f(s,t)),$$
$$N(\mathcal{X} \times \mathcal{Y},\, s) = N(\mathcal{X},\, g(s)) \times N(\mathcal{Y},\, h(s));$$

[3B.2] there are recursive functions f and g such that

$$N(\mathcal{X},\, s) \cap N(\mathcal{Y},\, t) = \bigcup_n N(\mathcal{X},\, f(s,t,n))$$
$$\bigcap_{i \le m} N(\mathcal{X},\, (u)_i) = \bigcup_n N(\mathcal{X},\, g(u,m,n))$$

1·0 DEFINITION $G \subseteq \mathcal{X}$ is *semi-recursive* if there is a recursive $\varepsilon : \omega \to \omega$ such that $G = \bigcup_n N(\mathcal{X}, \varepsilon(n))$.

We write Σ^0_1 for the class of semirecursive pointsets. It should be remarked [3C.12] that Σ^0_1 is actually independent of the effective enumeration chosen. Further, the notion of *semi-recursive* coincides with the notion of *recursively enumerable* for $\mathcal{X} = \omega$.

1·1 PROPOSITION *Σ^0_1 contains the empty set, every product space $\mathcal{X}$, every recursive relation on ω^k, every basic neighbourhood $N(\mathcal{X}, s)$ and the basic neighbourhood relation*

$$\{(x, s) \mid x \in N(\mathcal{X}, s)\}$$

for each $\mathcal{X}$; moreover it is closed under substitution of trivial functions, and under &, $\vee$, $\exists^{\le}$, $\forall^{\le}$, *and* $\exists^\omega$.

Here a *trivial function* is a combination of projections, a typical one being $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_1)$; and $\exists^{\le}$ and $\forall^{\le}$ are the bounded numerical quantifiers $\exists k{:}\le n$ and $\forall k{:}\le n$.

[C 1] Their influence, though, is apparent in such arguments as Martin's proof of Borel Determinacy.

1·2 Definition $P \subseteq \mathcal{X}$ is *recursive* iff both P and $\mathcal{X} \setminus P$ are semirecursive.

1·3 Remark For $\mathcal{X} = \omega$ this coincides with the classical notion of a recursive set.

1·4 Proposition *The recursive sets are closed under* $\neg, \&, \vee, \exists^{\leq}, \forall^{\leq}$.

We now define the notion of a *recursive total function* from one pointspace to another: this extends the classical notion of a recursive function of natural numbers. We shall shortly extend the definition to that of a partial recursive function.

1·5 Definition (Moschovakis §3D) $f : \mathcal{X} \to \mathcal{Y}$ is *recursive* if G^f is semirecursive; more generally we say that f is Γ-*recursive* if its neighbourhood diagram G^f is in Γ.

Here the *neighbourhood diagram* $G^f \subseteq \mathcal{X} \times \omega$ is defined by

$$G^f = \{(x, s) \mid f(x) \in N(\mathcal{Y}, s)\}$$

1·6 Definition Γ is a Σ-*pointclass* if it contains all semirecursive pointsets, is closed under trivial substitution and under $\&$, $\vee$, $\exists^{\leq}$, $\forall^{\leq}$, $\exists^{\omega}$.

The smallest Σ pointclass is Σ^0_1.

1·7 Lemma (Dellacherie) *Let Γ be a Σ-pointclass. Then $f : \mathcal{X} \to \mathcal{Y}$ is Γ-recursive if and only if for every semirecursive $P \subseteq \omega \times \mathcal{Y}$, P^f is in Γ, where*

$$P^f(n, x) \iff_{\mathrm{df}} P(n, f(x))$$

1·8 Proposition ([3D.2]) *Let Γ be a Σ-pointclass.*

(i) A function $f : \mathcal{X} \to \omega$ is Γ-recursive if and only if Graph(f) $= \{(x, n) : f(x) = n\}$ is in Γ.

(ii) If $\mathcal{X}$ is of type 0 and $f : \mathcal{X} \to \omega$, then f is Σ^0_1-recursive exactly when it is recursive in the sense of Moschovakis §3A.

(iii) Suppose $Q \subseteq Y_1 \times \ldots \times Y_\ell$ and

$$P(x) \iff Q(f_1(x), \ldots f_\ell(x)),$$

where each f_i is trivial or Γ-recursive into ω. If Q is in Γ, so is P.

To each $f : \mathcal{X} \to \mathcal{N}$ we may associate the function $f^* : \mathcal{X} \times \omega \to \omega$ defined by

$$f^*(x, n) = f(x)(n).$$

1·9 Proposition ([3D.3]) *Let Γ be a Σ pointclass.*

(i) f is Γ-recursive iff f^* is;

(ii) $f : \mathcal{X} \to \mathcal{Y} = Y_1 \times \ldots \times Y_\ell$ is Γ-recursive if and only if

$$f(x) = (f_1(x), \ldots f_\ell(x))$$

with suitable Γ-recursive functions $f_1, \ldots, f_\ell$.

1·10 Remark It is not always true that the Γ-recursive functions are closed under composition.

1·11 Remark Σ pointclasses need not be closed under recursive substitution: see [3G.3] for a counterexample. But the class Σ^0_1 is: see [3D.5].

1·12 Definition [Moschovakis §3E] Γ is *adequate* if it contains all recursive pointsets and is closed under Σ^0_1-recursive substitution, $\&$, $\vee$, $\exists^{\leq}$, and $\forall^{\leq}$.

1·13 Example Σ^0_1 is adequate, and closed under $\exists^{\omega}$. Δ^0_1, to be defined next, is also adequate, and closed under $\neg$.

Adequacy is relevant to the study of *hierarchies.* The following is the standard notation for the Kleene pointclasses beyond Σ^0_1, the class of all semirecursive pointsets:

$$\begin{array}{r} \text{the} \\ \text{arithmetical} \\ \text{classes} \end{array} \left\{ \begin{array}{rl} \Sigma^0_1 &= \text{semirecursive} \\ \Pi^0_n &= \neg\Sigma^0_n, \\ \Sigma^0_{n+1} &= \exists^\omega \Pi^0_n \\ \Delta^0_n &= \Sigma^0_n \cap \Pi^0_n \end{array} \qquad \begin{array}{rl} \Sigma^1_1 &= \exists^{\mathcal{N}} \Pi^0_1 \\ \Pi^1_n &= \neg\Sigma^1_n \\ \Sigma^1_{n+1} &= \exists^{\mathcal{N}} \Pi^1_n \\ \Delta^1_n &= \Sigma^1_n \cap \Pi^1_n \end{array} \right\} \begin{array}{l} \text{the} \\ \text{analytical} \\ \text{classes} \end{array}$$

1·14 PROPOSITION ([3E.2]) *All Kleene pointclasses are adequate.*

Here we are beginning to use notation for generating new pointclasses from old ones. For $w \in \mathcal{W}$, $\Gamma(w)$ is the class of all sets of the form $\{y \in \mathcal{Y} | A(w, y)\}$ where $A \subseteq \mathcal{W} \times \mathcal{Y}$ is in Γ. The bold-face class $\boldsymbol{\Gamma}$ is the union of all $\Gamma(\alpha)$'s with $\alpha \in \mathcal{N}$. Pointset operations such as $\exists^\omega$ have a natural meaning applied to pointsets: for example, $\exists^\omega A$ will be defined when $A \subseteq \mathcal{X} \times \omega$ for some $\mathcal{X}$ and then is the set $\{x \in \mathcal{X} \mid \exists n\ A(x, n)\}$: where following the custom in descriptive set theory we write in such contexts $A(x, n)$ interchangeably with $(x, n) \in A$. Such operations will also be applied to pointclasses: $\exists^\omega\Gamma$ will be the set of all pointsets of the form $\exists^\omega A$ where for some $\mathcal{X}$, A is in $\Gamma \restriction (\mathcal{X} \times \omega)$. We use the negation sign of logic, $\neg$ for complementation: so that if $P \subseteq \mathcal{X}$ is a pointset, $\neg P = \mathcal{X} \setminus P$; and for a pointclass Γ, we write $\neg\Gamma$ for the pointclass $\{\neg P \mid P \in \Gamma\}$. Good book-keeping would in this context require us to distinguish the empty subset of one space from the empty subset of another: if driven to it, we will write $\varnothing^{\mathcal{X}}$.

Another notation in being for $\neg\Gamma$ is $\breve{\Gamma}$.

1·15 PROPOSITION ([3E.1]) *If Λ is adequate, so are $\neg\Lambda$, $\exists^\omega\Lambda$, $\forall^\omega\Lambda$, $\exists^{\mathcal{N}}\Lambda$, $\forall^{\mathcal{Y}}\Lambda$. Moreover $\exists^{\mathcal{N}}\Lambda$ is closed under $\exists^{\mathcal{Y}}$ and $\forall^{\mathcal{N}}\Lambda$ under $\forall^{\mathcal{Y}}$, for all product spaces $\mathcal{Y}$.*

The light-face, bold-face notation was introduced in a paper of Addison written in Warsaw in the fifties with a view to unifying the pre- and post-Kleene approaches to descriptive set theory. In the hands of some printers, the visual distinction between Σ^1_2 and $\boldsymbol{\Sigma^1_2}$ is not great, and the custom is developing of using the traditional manuscript indication of a wiggly underline, but stylised into a tilde, to mark the bold-face classes: for example $\underset{\sim}{\Sigma}{}^1_2$.

The custom of writing Δ for the ambiguous class $\Gamma \cap \neg\Gamma$, which goes back to Addison, leads to ambiguity when combined with the bold-face notation, since the two operations do not commute in full generality. For example, it is easy to construct a Γ such that its ambiguous class $\Delta(\Gamma)$ is empty whereas the ambiguous class of the corresponding bold-face class $\boldsymbol{\Gamma}$ is not. Hence we emphasize that in this context $\boldsymbol{\Delta}$ is defined to be $\boldsymbol{\Gamma} \cap \neg\boldsymbol{\Gamma}$; or in our tilde notation, to be $\Delta(\underset{\sim}{\Gamma})$ and not $\underset{\sim}{\Delta(\Gamma)}$.

Universal Sets.

Let $N(\mathcal{X}, s)$ be a recursive enumeration of open neighbourhoods, and for $\varepsilon \in \mathcal{N}$, define

$$U(\varepsilon, x) \ =_{\mathrm{df}} \bigcup_n N(\mathcal{X}, \varepsilon(n)).$$

Note that

$$U(\varepsilon, x) \iff \exists n\big(x \in N(\mathcal{X}, \varepsilon(n))\big).$$

U is open in $\mathcal{N} \times \mathcal{X}$; indeed U is in Σ^0_1, and is *universal* for $\boldsymbol{\Sigma^0_1} \restriction \mathcal{X}$ in the sense that for each pointset $O \subset \mathcal{X}$,

$$O \text{ is open } \iff \exists\alpha\forall x\big(O(x) \iff U(\alpha, x)\big),$$

for plainly each $\{x \mid U(\alpha, x)\}$ is open, and conversely, given an open O, let α enumerate (with repetitions allowed, in case O should be empty) $\{s \in \omega \mid N(\mathcal{X}, s) \subseteq O\}$. U is plainly also universal for Σ^0_1 — here we take α to be recursive — by the definition of semirecursive.

Now we shall check that the existence of universal sets propagates along hierarchies.

1·16 PROPOSITION *Let $U \in \Gamma$ be universal for $\boldsymbol{\Gamma}$. Then*

(i) $\neg U$ is in $\neg\Gamma$ and is universal for $\neg\boldsymbol{\Gamma}$;

and for $\mathcal{X} = \mathcal{N}$ or ω,

(ii) $\exists^{\mathcal{X}}U$ *is in* $\exists^{\mathcal{X}}\Gamma$ *and is universal for* $\exists^{\mathcal{X}}\mathbf{\Gamma}$*, and*
(iii) $\forall^{\mathcal{X}}U$ *is in* $\forall^{\mathcal{X}}\Gamma$ *and is universal for* $\forall^{\mathcal{X}}\mathbf{\Gamma}$.

Proof : (i) let $A(y) \iff \neg B(y)$, where $B \in \mathbf{\Gamma}$; pick β such that $\forall y \, \big(B(y) \iff U(\beta, y)\big)$; then

$$A(y) \iff \neg U(\beta, y).$$

(ii) let $A(z) \iff \exists\alpha \, C(\alpha, z)$, where $C \in \mathbf{\Gamma}$; pick γ such that $\forall\alpha\forall z \, \big(C(\alpha, z) \iff U(\gamma, \alpha, z)\big)$; then

$$A(z) \iff \exists\alpha \, U(\gamma, \alpha, z).$$

(iii) let $A(z) \iff \forall\alpha \, D(\alpha, z)$, where $D \in \mathbf{\Gamma}$; pick δ such that $\forall\alpha\forall z \, \big(D(\alpha, z) \iff U(\delta, \alpha, z)\big)$; then

$$A(z) \iff \forall\alpha \, U(\delta, \alpha, z). \qquad \dashv (1{\cdot}16)$$

1·17 Remark We have here permitted ourselves a certain vagueness about the "dimension" of U: the reader will readily supply more exact formulations.

It follows immediately that all the classes Σ^0_n, $\mathbf{\Sigma^0_n}$, Π^0_n, $\mathbf{\Pi^0_n}$ and Σ^1_n, $\mathbf{\Sigma^1_n}$, Π^1_n, $\mathbf{\Pi^1_n}$ have universal sets. The following now immediately implies that these classes are all distinct:

1·18 Proposition *If* $U \in \Gamma$ *is universal for* $\mathbf{\Gamma}$*, then*

$$\{\alpha \mid U(\alpha, \alpha)\} \in \Gamma \setminus \neg\mathbf{\Gamma}.$$

Proof : Write $P(\alpha) \iff U(\alpha, \alpha)$. Suppose $\neg P \in \mathbf{\Gamma}$; then $\exists\gamma\forall\alpha \, \big(\neg P(\alpha) \iff U(\gamma, \alpha)\big)$. But then, taking $\alpha = \gamma$, $\neg P(\gamma) \iff U(\gamma, \gamma) \iff P(\gamma)$, a contradiction. $\dashv$ (1·18)

1·19 Remark There is of course a mathematical, as distinct from a logical, interest in finding specific examples of sets in a class Γ but not in the dual class $\neg\Gamma$. For examples of Σ^1_1 sets that are not Borel, see §§4.5 and 4.6 of the expository paper of Rogers and Jayne; [R1] for examples at other levels of the projective hierarchy, see Becker. [R2]

Now we address the problem of defining for **partial** functions the property of being Γ-recursive: we follow Moschovakis §3G.

1·20 Definition We say P *computes* f *on* D (where $P \subseteq \mathcal{X} \times \omega$) if

$$x \in D \implies \forall s(f(x) \in N_s \iff P(x, s)).$$

1·21 Remark Notice that for each x there can be at most one y such that $\forall s \, (y \in N_s \iff P(x, s))$. Notice further that if P computes f and $f(x)\downarrow$ then

$$\begin{aligned} f(x) = y &\iff \forall s(y \in N_s \implies P(x, s)) \\ \text{and } f(x) \neq y &\iff \exists s(P(x, s) \;\&\; y \notin N_s). \end{aligned}$$

1·22 Definition We say that a partial function $f : \mathcal{X} \to \mathcal{Y}$ is Γ-*recursive on* D if some $P \in \Gamma$ computes f on D. We say that f is Γ-*recursive* if it is Γ-recursive on its domain and its domain is in Γ.

1·23 Remark Note that for f total, this definition agrees with that given in [3D.4].

We shall be particularly interested in the case that $D = Dom(f)$, that is, when f is Γ-*recursive on its domain*, which we shall abbreviate as Γ-*roid*.

We come now to a property of a class that enables it to carry a reasonable recursion theory.

[R1] *K-analytic sets*, pp 2 – 181 of the volume *Analytic Sets* by C.A.Rogers *et al.*, Academic Press, 1980

[R2] *Descriptive Set Theoretic Phenomena in Analysis and Topology,* pp 1 – 25 of the volume *Set Theory of the Continuum*, edited by Judah, Just and Woodin.

1·24 DEFINITION Γ *has the substitution property* if for each $Q \in \Gamma$, $Q \subseteq \mathcal{X}$, and each partial $f : \mathcal{X} \to \mathcal{Y}$ that is Γ-roid, there is a $Q^* \subseteq \mathcal{X}$ in Γ such that for each $x \in \mathcal{X}$,

$$f(x)\downarrow \implies [Q^*(x) \iff Q(f(x))].$$

By Moschovakis [3G3] Σ-pointclasses need not in general be adequate; but for those endowed with the substitution property, all is well:

1·25 PROPOSITION ([3G1]) *Suppose that Γ is a Σ pointclass with the substitution property. Then (i) the collection of partial Γ-roid functions is closed under composition; (ii) Γ is closed under the substitution of total Γ-recursive functions; so in particular, Γ is adequate.*

Here is a *resumé* of the substitution property.

(i) Σ^0_1 has it;

(ii) if Γ has it then each $\Gamma(w)$ has it, and so $\mathbf{\Gamma}$ has it;

Hence using 1·21,

(iii) if Γ is closed under $\forall^\omega$, $\exists^\omega$ (so is a Σ pointclass), and for each space $\mathcal{Y}$, either under $\exists^\mathcal{Y}$ or under $\forall^\mathcal{Y}$, then Γ has the substitution property, for then

$$\begin{aligned} f(x)\downarrow \implies &\Big(\big(Q(f(x)) \iff \exists y\,(f(x) = y \;\&\; Q(y))\big) \\ \text{and}\quad &\big(Q(f(x)) \iff \forall y\,(f(x) \neq y \;\text{ or }\; Q(y))\big)\Big)\end{aligned}$$

in particular

(iv) all Σ^1_n and Π^1_n have the substitution property.

The three results 3*H*.1, 3*H*.2, 3*H*.3 are particularly important. The first is beguilingly simple:

1·26 THE GOOD PARAMETRIZATION THEOREM, commonly called the *S-m-n* theorem

Suppose that Γ is ω-parametrized and closed under recursive substitutions. Then we can associate with each space $\mathcal{X}$ a set $G^\mathcal{X} \subseteq \mathcal{N} \times \mathcal{X}$ in Γ which is universal for $\mathbf{\Gamma}\restriction\mathcal{X}$ and so that the following properties hold:

(i) for $P \subseteq \mathcal{X}$,

$$P \in \Gamma \iff P = G^\mathcal{X}_\varepsilon \quad \textit{with a recursive } \varepsilon \in N.$$

(ii) For each space $\mathcal{X}$ of type 0 or 1 and each $\mathcal{Y}$, there is a recursive function

$$S^{\mathcal{X},\mathcal{Y}} = S : N \times \mathcal{X} \to N$$

so that

$$G^{\mathcal{X}\times\mathcal{Y}}(\varepsilon, x, y) \iff G^\mathcal{Y}(S(\varepsilon, x), y).$$

Moschovakis proves this in full generality: we shall content ourselves with proving it in a simple context, which nevertheless contains all essential ideas.

Proof of (1·26): Let $V \subseteq \omega \times \mathcal{N} \times \mathcal{N}$ be universal for $\Gamma \restriction (\mathcal{N} \times \mathcal{N})$. Set $G_1(\gamma, \beta) \leftrightarrow V(\gamma(0), \gamma^*, \beta)$ where $\gamma^*(n) = \gamma(n+1)$. Then G_1 is universal for $\mathbf{\Gamma}\restriction\mathcal{N}$: for let $Q(\beta) \in \mathbf{\Gamma}$: so for some δ and k,

$$Q(\beta) \iff R(\delta, \beta) \iff V(k, \delta, \beta)$$

Take $\gamma(0) = k$ and $\gamma^* = \delta$; if δ is recursive so will γ be. Then $G_1(\gamma, \beta) \iff Q(\beta)$.

Suppose that G_2 has likewise been defined from some $V' \subseteq \omega \times \mathcal{N} \times \mathcal{N} \times \mathcal{N}$.[C2] $G_2 \in \Gamma$. Define $P(\eta, \theta) \iff_{\mathrm{df}} G_2((\eta)^2_0, (\eta)^2_1, \theta)$. $P \in \Gamma$ as Γ is closed under unpairing and pairing functions.

[C2] Here we are using subscripts to differentiate the various functions called G; the subscript loosely indicates the "dimension", as measured by the number of variables, of the space subsets of which are being parametrised by the G in question. The subscripts will be omitted where they seem unnecessary.

There is an e^* such that $P(\eta,\theta) \iff V(e^*,\eta,\theta)$, so

$$\begin{aligned} G_2(\varepsilon,\alpha,\beta) &\iff P(\langle\varepsilon,\,\alpha\rangle^2,\beta)\\ &\iff V(e^*,\langle\varepsilon,\alpha\rangle^2,\beta)\\ &\iff G_1(e^*{}^\frown\langle\varepsilon,\alpha\rangle^2,\beta). \end{aligned}$$

Take $S(\varepsilon,\alpha) = e^*{}^\frown\langle\varepsilon,\alpha\rangle^2$. Then S is recursive from $\mathcal{N}\times\mathcal{N}$ to $\mathcal{N}$,[C 3] and

$$G_2(\varepsilon,\alpha,\beta) \iff G_1(S(\varepsilon,\alpha),\beta). \qquad \dashv (1{\cdot}26)$$

1·27 The Uniform Closure Theorem [3H.2] *Suppose Γ is an ω-parametrized, adequate pointclass; if Γ is closed under any of the operations &, $\vee$, $\exists^{\le}$, $\forall^{\le}$, $\exists^{\mathcal{Y}}$, $\forall^{\mathcal{Y}}$, then $\mathbf{\Gamma}$ is uniformly closed under the same operation (in the codings induced by a good parametrization).*

Suppose for example that Γ is closed under &. Let G be a good parametrisation. Then there is a recursive u such that if $P(x) \iff G(\alpha,x)$ and $Q(x) \iff G(\beta,x)$ then $P\&Q \iff G(u(\alpha,\beta),x)$. For write $R(\zeta,\eta,x) \iff_{\mathrm{df}} G(\zeta,x)\&G(\beta,x)$; $R\in\Gamma$; and so there is a recursive ε^* with $R(\zeta,\eta,x) \iff G(\varepsilon^*,\zeta,\eta,x)$; hence using the *S-m-n* theorem, $R(\zeta,\eta,x) \iff G(S(\varepsilon^*,\zeta,\eta),x)$ and so we may take for u the function $(\zeta,\,\eta)\mapsto S(\varepsilon^*,\zeta,\eta)$. $\dashv$ (1·27)

1·28 Kleene's Recursion Theorem for Relations [3H.3] *If Γ is ω-parametrised, closed under Σ^0_1-recursive substitution and if $R\subseteq\mathcal{N}\times\mathcal{X}$ is in Γ then there is an ε^* such that*

$$\forall x R(\varepsilon^*,x) \iff G(\varepsilon^*,x)$$

where G is a good universal set for $\mathbf{\Gamma}\restriction\mathcal{X}$

Proof : Let $T(\alpha,x) \iff R(S(\alpha,\alpha),x)$. T is in Γ, since Γ is closed under recursive substitutions, so there is a recursive ε_0 such that $T(\alpha,x) \iff G_2(\varepsilon_0,\alpha,x) \iff G_1(S(\varepsilon_0,\alpha),x)$, so

$$\forall\alpha \quad \big[R(S(\alpha,\alpha),x) \iff G_1(S(\varepsilon_0,\alpha),x)\big].$$

Now take $\alpha=\varepsilon_0$ and $\varepsilon^*=S(\varepsilon_0,\varepsilon_0)$: then

$$\forall x \qquad R(\varepsilon^*,x) \iff G_1(\varepsilon^*,x). \qquad \dashv (1{\cdot}28)$$

This theorem is of fundamental importance: we emphasize the uniformity of its proof by stating and proving a **bold-face version**:

1·29 Theorem *Let $\mathbf{R}\in\mathbf{\Gamma}$, then there is an ε^* such that for all x, $\mathbf{R}(\varepsilon^*,x) \iff G(\varepsilon^*,x)$; and ε^* may be computed recursively from a parameter for $\mathbf{R}$.*

Proof : Let $\mathbf{R}$ be in $\Gamma(\beta)$. There is a recursive ε_1 such that

$$\begin{aligned} \forall\varepsilon\forall x\ \big[\mathbf{R}(S^{2,1}(\varepsilon,\varepsilon),x) &\iff G_3(\varepsilon_1,\beta,\varepsilon,x)\\ &\iff G_2(S^{3,2}(\varepsilon_1,\beta),\varepsilon,x)\\ &\iff G_1(S^{2,1}(S^{3,2}(\varepsilon_1,\beta),\varepsilon),x)\big]. \end{aligned}$$

Put $\bar\varepsilon = S^{3,2}(\varepsilon_1,\beta)$; then

$$\forall\varepsilon\forall x\ \big[\mathbf{R}(S^{2,1}(\varepsilon,\varepsilon),x)) \iff G_1(S^{2,1}(\bar\varepsilon,\varepsilon),x)\big].$$

Now take $\varepsilon=\bar\varepsilon$:

$$\forall x\ \big[\mathbf{R}(S^{2,1}(\bar\varepsilon,\bar\varepsilon),x) \iff G_1(S^{2,1}(\bar\varepsilon,\bar\varepsilon),x)\big],$$

so our desired ε^* is $S^{2,1}(S^{3,2}(\varepsilon_1,\beta),S^{3,2}(\varepsilon_1,\beta))$. For given (recursive) ε_1, this is plainly a recursive function of β; ε_1 itself will depend on $\mathbf{R}$ and so on the dimension of the space. $\dashv$ (1·29)

[C 3] Again we spare the reader cumbersome sub- and superscripts where possible; if we wish to specify S more closely, we may write it as $S^{2,1}$, as we do in the next proof.

2: The recursion theorem for functions

We shall in due course want a couple of results from Moschovakis, Chapter IV, but for the moment we review some material from his Chapter VII, to see how the recursion theorem for relations yields that for functions.

2·0 DEFINITION Γ is a Σ^*-*class* if it is a Σ-class, ω-parametrized and has the substitution property; and hence is adequate.

2·1 DEFINITION Fix a Σ^* class Γ and let $G \in \Gamma$ be a fixed good universal set for $\mathbf{\Gamma} \restriction \mathcal{X} \times \omega$. Here "good" means that the *S-m-n* theorem holds.

We define a "universal" partial Γ-recursive function $U : \mathcal{N} \times \mathcal{X} \to \mathcal{Y}$ by

$$U(\varepsilon, x) \downarrow = y \iff \forall s (y \in N(\mathcal{Y}, s) \iff G(\varepsilon, x, s))$$

As remarked in 1·21 above, there is at most one such y for given arguments. If such a y exists, we write $y = U(\varepsilon, y) = \{\varepsilon\}_\Gamma^{\mathcal{X},\mathcal{Y}}(x) = \{\varepsilon\}(x)$ when we can remember what $\mathcal{X}$, $\mathcal{Y}$ and Γ are. U is the largest partial Γ-recursive function computed by G on its domain.

We write '"$f \subseteq \{\varepsilon\}$"' to mean that $\{\varepsilon\}$ computes f on its domain.

2·2 PROPOSITION *(i) U is Γ-roid.*

(ii) A partial function f is $\mathbf{\Gamma}$-roid iff $\exists \varepsilon \in \mathcal{N}$ "$f \subseteq \{\varepsilon\}$"

(iii) A partial function f is Γ-roid iff there exists a recursive $\varepsilon \in \mathcal{N}$ with "$f \subseteq \{\varepsilon\}$"

(iv) For all $\mathcal{X}$, $\mathcal{W}$, $\mathcal{Y}$ there are recursive functions $S_\Gamma^{\mathcal{X},\mathcal{W},\mathcal{Y}} : \mathcal{N} \times \mathcal{X} \to \mathcal{N}$ such that for all $\varepsilon \in \mathcal{N}$,

$$\{\varepsilon\}_\Gamma^{\mathcal{X}\times\mathcal{W},\mathcal{Y}}(x, w) = \{S_\Gamma^{\mathcal{X},\mathcal{W},\mathcal{Y}}(\varepsilon, x)\}_\Gamma^{\mathcal{W},\mathcal{Y}}(w)$$

Proof : (i) is clear as $G \in \Gamma$; (ii) and (iii) are immediate from the definitions, the $\{\varepsilon\}$ coding the largest function computed by the P in question; now for (iv):

We seek an S such that $\{\varepsilon\}(x, w) \simeq \{S(\varepsilon, x)\}(w)$

Now

$$\begin{aligned} \{\varepsilon\}(x, w) \downarrow = z &\iff \forall s (z \in N_s \iff G_3(\varepsilon, x, w, s)) \\ &\iff \forall s \left(z \in N_s \iff G_2(S^{3,2}(\varepsilon, x), w, s)\right) \\ &\iff \{S^{3,2}(\varepsilon, x)\}(w) \downarrow = z \end{aligned}$$

so the S from Moschovakis Chapter III works. S is recursive. ⊣

2·3 THE RECURSION THEOREM *Let Γ be Σ^*, and let $f : \mathcal{N} \times \mathcal{X} \to \mathcal{Y}$ be $\mathbf{\Gamma}$-roid. Then there is an $\varepsilon^* \in \mathcal{N}$ such that for all $\mathcal{X}$*

$$f(\varepsilon^*, x) \downarrow \implies \left[f(\varepsilon^*, x) = \{\varepsilon^*\}(x)\right]$$

In particular, there is a recursive function ϕ such that if "$f \subseteq \{\alpha\}$" then we can take $\varepsilon^* = \phi(\alpha)$. Further, if f is Γ-roid, ε^* may be taken to be recursive.

Proof : Suppose first that f is total. Then if we set

$$R(\varepsilon, x, s) \iff f(\varepsilon, x) \in N_s,$$

$R \in \mathbf{\Gamma}$. By the recursion theorem for relations, there is an ε^* with

$$R(\varepsilon^*, x, s) \iff G(\varepsilon^*, x, s).$$

so

$$\begin{aligned} f(\varepsilon^*, x) = y &\iff \forall s\, (y \in N_s \iff R(\varepsilon^*, x, s)) \\ &\iff \forall s\, (y \in N_s \iff G(\varepsilon^*, x, s)) \\ &\iff \{\varepsilon^*\}(x) \downarrow = y, \end{aligned}$$

so in this case $\{\varepsilon^*\}$ is total and $\forall x\{\varepsilon^*\}(x) = f(\varepsilon^*, x)$.

Now drop the assumption that f is total. It is no longer clear that R as defined above is in Γ. Suppose that $\{\alpha\}$ computes f on its domain: then if $f(\varepsilon, x)\downarrow$ we should have $\{\alpha\}(\varepsilon, x)\downarrow$ and should want $R(\varepsilon, x, s) \iff f(\varepsilon, x) \in N_s \iff \{\alpha\}(\varepsilon, x) \in N_s \iff G_4(\alpha, \varepsilon, x, s)$. So *define*

$$R(\varepsilon, x, s) \iff G_4(\alpha, \varepsilon, x, s)$$

for all ε, x, s. R is in $\mathbf{\Gamma}$ as G is; so there is an ε^* as before such that

$$\forall x, s R(\varepsilon^*, x, s) \iff G_3(\varepsilon^*, x, s)$$

but then
$$\{\varepsilon^*\}(x)\downarrow = y \iff \{\alpha\}(\varepsilon^*, x)\downarrow = y$$

The bold-face version tells us that for some recursive ε_2,

$$\varepsilon^* = S(S(\varepsilon_2, \alpha), S(\varepsilon_2, \alpha)).$$

which will be recursive if α is. $\dashv$ (2·3)

3: pos $\Sigma^1_1(<_X)$ classes

The following definition will be important in later chapters.

Let Q and R be two binary relations on the reals, or more generally on a space $\mathcal{X}$ of type 0 or 1. We wish to define the class *pos* $\Sigma^1_1(Q, R)$, which will be the smallest class containing Σ^1_1 and the relations Q and R and closed under the operations &, $\vee$, $\exists^\omega$, $\forall^\omega$, and $\exists^\mathcal{N}$. We fix a set $U(\varepsilon, \alpha, \beta, \gamma, \delta, x)$ that is universal for Σ^1_1, and then take the following as our official definition:

3·0 Definition The pointclass *pos* $\Sigma^1_1(Q, R)$ consists of those pointsets which are for some recursive ε of the form

$$G^{Q,R}_\varepsilon(x) \iff_{\text{df}} \exists\alpha\beta\gamma\delta\big[\forall n\big(Q((\alpha)_n, (\beta)_n)\&R((\gamma)_n, (\delta)_n)\big) \;\&\; U(\varepsilon, \alpha, \beta, \gamma, \delta, x)\big].$$

Thus we have in effect defined this class by specifying universal sets for it.

3·1 Remark This definition is *monotonic* in Q and R in the sense that if $Q \subseteq Q'$ and $R \subseteq R'$ then $G^{Q,R}_\varepsilon \subseteq G^{Q',R'}_\varepsilon$.

3·2 Proposition ([7D.7]) *pos* $\Sigma^1_1(Q, R)$ *is closed under* &, $\vee$, $\exists^\omega$, $\forall^\omega$, $\exists^\mathcal{N}$, *contains* Q, R, *and all* Σ^1_1 *pointsets, is* ω*-parametrized and is a* Σ^* *class.*

Proof : the hardest case will be closure under $\forall^\omega$. We distinguish by subscript the two different U's that will occur in this argument Note that

$$\big(\forall t\, G^{Q,R}_\varepsilon(t, x)\big) \iff \exists\hat\alpha\hat\beta\hat\gamma\hat\delta\big[\forall m\big(Q((\hat\alpha)_m(\hat\beta)_m)) \;\&\; R((\hat\gamma)_m(\hat\delta)_m))\big) \;\&\; U_5(\hat\varepsilon, \hat\alpha, \hat\beta, \hat\gamma, \hat\delta, x)\big]$$

where $\hat\varepsilon$ is chosen so that

$$U_5(\hat\varepsilon, \hat\alpha, \hat\beta, \hat\gamma, \hat\delta, x) \iff \forall t\exists\alpha\beta\gamma\delta\,\Big[\,U_6(\varepsilon, \alpha, \beta, \gamma, \delta, t, x)\;\&$$
$$\&\;\big[\forall n\exists m\,(\alpha)_n = (\hat\alpha)_m \,\&\, \forall n\exists m\,(\beta)_n = (\hat\beta)_m \,\&\, \forall n\exists m\,(\gamma)_n = (\hat\gamma)_m \,\&\, \forall n\exists m\,(\delta)_n = (\hat\delta)_m\big]\Big]:$$

such choice is possible as this last clause is Σ^1_1. Thus $[\forall t\, G^{Q,R}_\varepsilon(t, x)] \iff G^{Q,R}_{\hat\varepsilon}(x)$. Other cases are handled similarly.

The class is plainly ω-parametrised, and closed under $\exists^\mathcal{N}$, so by the *resumé*, the substitution property follows. $\dashv$ (3·2)

If R is the entire set $\mathcal{X}\times\mathcal{X}$, the above definition yields a universal set G^Q for a class we should naturally call *pos* $\Sigma^1_1(Q)$. In terms of this notation, we state a result to be used in our discussion of the Moschovakis Coding Lemma in Chapter Three, leaving its proof to the reader.

3·3 Proposition *There is a recursive function* ϕ *such that given a binary relation* P *on* $\mathcal{X}$ *and defining a binary relation* Q *on* $\mathcal{X}\times\mathcal{Y}$ *by* $Q(x, y, x', y') \iff_{\text{df}} P(x, x')$, *we have for all* ε, x *and* y,

$$G^Q_\varepsilon(x, y) \iff G^P_{\phi(\varepsilon)}(x, y).$$

4: Spector classes

The classes Γ that will be of particular interest to us are of a kind of which the class Π^1_1 is the prototype. Thus we turn now to Moschovakis Chapter IV for a discussion of the concept of a Spector class of which this will be the definition:

4·0 DEFINITION Γ is a *Spector class* if it is Σ^*, normed and closed under $\forall^\omega$.

We have met the concept of a norm in Chapter I when we discussed pre-well-orderings of subsets of $\mathbb{R}$ in the course of defining Θ. We have yet to say what it means for a pointclass to be normed. That concerns the *definability* of the relation $\leq_\phi$ on a pointset induced by a norm.[C 4]

4·1 DEFINITION Let Γ be a pointclass. A norm ϕ on a pointset X is a Γ-*norm* if there are relations $\leq^\phi_\Gamma$ and $\leq^\phi_{\breve\Gamma}$ in Γ and $\neg\Gamma = \breve\Gamma$ respectively such that for every y,

$$\text{if } y \in X \text{ then}$$
$$x \in X \;\&\; \phi(x) \leq \phi(y) \iff x \leq^\phi_\Gamma y$$
$$\text{and}$$
$$x \in X \;\&\; \phi(x) \leq \phi(y) \iff x \leq^\phi_{\breve\Gamma} y$$

4·2 DEFINITION A pointclass Γ is *normed* or *has the pre-well-ordering property*, in symbols $PWO(\Gamma)$, if every pointset $X \in \Gamma$ has a Γ-norm.

The original instance of such a norm is the set $WORD$ which we now define.

4·3 DEFINITION For $\alpha : \omega \to \omega$ we write $RC(\alpha)$ for the relation (on ω) coded by α:

$$RC(\alpha) \;=_{\mathrm{df}}\; \{(i,j) \mid \alpha(\langle i, j\rangle) = 0\}.$$
$$LORD \;=_{\mathrm{df}}\; \{\alpha \in \mathcal{N} \mid RC(\alpha) \text{ is a linear ordering of some subset of } \omega\}$$

We write $\leq_\alpha$ for that linear ordering, the field of which is $\{n \mid n \leq_\alpha n\}$, and write $n <_\alpha m$ for $n \leq_\alpha m \,\&\, m \not\leq_\alpha n$, so that if m is not in the field of $RC(\alpha)$, the statement $n <_\alpha m$ is false.

$$WORD \;=_{\mathrm{df}}\; \{\alpha \in LORD \mid RC(_\alpha) \text{ is a well-ordering.}\}$$

For $\alpha \in WORD$, we write

$$|\alpha| \;=_{\mathrm{df}}\; \mathrm{otp}(\leq_\alpha),\; \textit{i.e.} \text{ the order type of } RC(\alpha)$$

Thus, when defined, $|\alpha|$ will be a countable ordinal; and every ordinal, if countable, will be of the form $|\alpha|$ for some $\alpha \in WORD$ (which α is by no means uniquely determined.) Now suppose that $\beta \in WORD$. Then the predicate

$$\alpha \in WORD \;\&\; |\alpha| \leq |\beta|$$

may be written in Σ^1_1 form by saying that there is an order-preserving injection of $RC(\alpha)$ into $\leq_\beta$, and may be written in Π^1_1 form by saying that $\alpha \in WORD$ but there is no isomorphism of $\leq_\beta$ with a proper initial segment of $\leq_\alpha$.

Hence the map $\alpha \mapsto |\alpha|$ defines a Π^1_1 norm on the set $WORD$.

4·4 EXERCISE Show that for $\beta \in WORD$ the relation

$$\alpha \in WORD \;\&\; |\alpha| < |\beta|$$

is expressible in both Σ^1_1 and Π^1_1 form.

Now Kleene normal form for Π^1_1 sets, shows that to each Π^1_1 set $A \subseteq \mathcal{X}$ there is a recursive[C 5] $f : \mathcal{X} \to LORD$ such that

$$x \in A \iff f(x) \in WORD.$$

[C 4] See Moschovakis [§4B] for an exhaustive discussion.

[C 5] we can only say Δ^1_1 if $\mathcal{X}$ is not of type 0 or 1

This at once gives us a Π^1_1 norm ϕ on A by setting $x \leq_\phi y \iff |f(x)| \leq |f(y)|$: we leave the reader to check there is a Σ^1_1 relation $S(x,y)$ and a Π^1_1 relation $P(x,y)$ such that for $y \in A$, $x \in A$ & $\phi(x) \leq \phi(y)$ is equivalent to both $S(x,y)$ and $P(x,y)$. The length of ϕ is not greater than ω_1, but may well be shorter: for an extreme example, suppose that f is constant.

Thus we have proved that Π^1_1 is normed, and have already seen that it has the other defining properties of being a Spector class.

4·5 EXERCISE Show that if $PWO(\Gamma)$ then $PWO(\Gamma(w))$ for each w and $PWO(\boldsymbol{\Gamma})$.

4·6 DEFINITION Given a norm ϕ on a set $A \subseteq \mathcal{X}$, we define two further relations on $\mathcal{X}$:

$$x \leq^*_\phi y \iff_{\text{df}} x \in A \;\&\; (y \in A \implies x \leq_\phi y)$$
$$x <^*_\phi y \iff_{\text{df}} x \in A \;\&\; (y \in A \implies x <_\phi y)$$

4·7 EXERCISE Show that if ϕ is a Γ norm on X then these two relations are in Γ; show conversely that if $X \in \Gamma$ and these two relations are in Γ, then ϕ is a Γ norm.

The pre-wellordering property propagates easily from Π^1_1 to Σ^1_2, and thence to each $\Sigma^1_2(w)$ and to $\boldsymbol{\Sigma^1_2}$.

4·8 PROPOSITION *Suppose that Γ has the prewellordering property. Then every set in $\exists^{\mathcal{N}}\Gamma$ admits a $\exists^{\mathcal{N}}\forall^{\mathcal{N}}\Gamma$ norm.*

Proof : let $A = \{x \mid \exists\alpha\, B(x,\alpha)\}$, and suppose that $B \in \Gamma$ admits the Γ-norm ψ. Define for x, y in A,

$$x \leq_\phi y \iff \inf\{\psi(x,\alpha) \mid B(x,\alpha)\} \leq \inf\{\psi(y,\beta) \mid B(y,\beta)\}.$$

Then if $y \in A$,

$$\begin{aligned} x \in A \;\&\; x \leq_\phi y &\iff \exists\alpha\forall\beta\big((x,\alpha) \leq^*_\psi (y,\beta)\big) \\ &\iff \forall\beta\exists\alpha\big((x,\alpha) \leq^*_\psi (y,\beta)\big). \end{aligned}$$

4·9 COROLLARY *If $\forall^{\mathcal{N}}\Gamma \subseteq \Gamma$ and $PWO(\Gamma)$ then $PWO(\exists^{\mathcal{N}}\Gamma)$.*

We shall see in the next chapter, when principles of determinacy are to hand, a dual result for Γ closed under $\exists^{\mathcal{N}}$.

Now if Γ is a Spector class and $\boldsymbol{\Delta}$ the associated ambiguous bold-face class, an important ordinal is $\delta = \delta(\Gamma)$, the supremum of pre-wellorderings in $\boldsymbol{\Delta}$. When $\Gamma = \Pi^1_1$, we know that $\delta = \omega_1$; and we know also that a norm on a Π^1_1 or indeed $\boldsymbol{\Pi^1_1}$ set will be of length ω_1 iff that set is not in $\boldsymbol{\Delta^1_1}$. Further, if ϕ is a Π^1_1 norm on A, then for each $a \in A$, the set $\{x \in A \mid \phi(x) \leq \phi(a)\}$ is $\boldsymbol{\Delta^1_1}$.

These results have generalisations to Spector classes closed under $\forall^{\mathcal{N}}$, which we now establish.

4·10 REMARK If Γ is a Spector class with $\forall^{\mathcal{N}}\Gamma \subseteq \Gamma$, then by 1.24 the hypotheses of the Recursion Theorem (in light- or bold-face form) are satisfied.

4·11 PROPOSITION [4C.14] *Let Γ be a Spector class closed under $\forall^{\mathcal{N}}$. Let $\prec$ be a strict well-founded relation on $\mathcal{X}$ with $\prec \in \neg\Gamma$. Let $G \subseteq \mathcal{N} \times \mathcal{X}$ be a good universal set in Γ, and ϕ a Γ-norm on G. Then there is a recursive function $f : \mathcal{X} \to \mathcal{N} \times \mathcal{X}$ which is an order-preserving map from $\prec$ into ϕ: i.e.,*

$$x \prec y \implies f(x) \in G \;\&\; f(y) \in G \;\&\; \phi(f(x)) < \phi(f(y))$$

Proof : We shall find an f of a particularly simple form, namely $x \mapsto (\alpha, x)$; and start by writing a predicate $Q(\alpha, y)$ to say that the map works so far. We want this to be true for y $\prec$-minimal; so take

$$Q(\alpha,y) \;=_{\text{df}}\; \forall z\Big(z \prec y \implies \big[G(\alpha,z) \;\&\; \neg[G(\alpha,y) \;\&\; \phi(\alpha,y) \leq \phi(\alpha,z)]\big]\Big)$$

The last clause is in Δ by the definition of a Γ norm, so the whole thing is in Γ. By the recursion theorem, there is a recursive ε^* such that

$$\forall x\,\big[\,Q(\varepsilon^*,x) \iff G(\varepsilon^*,x)\big]$$

Take $f(x) = (\varepsilon^*, x)$. We assert that for all x, $G(\varepsilon^*, x)$: for if not, let x be a $\prec$-minimal counterexample; then $\forall y \prec x\, G(\varepsilon^*, y)$, hence $Q(\varepsilon^*, y)$, and so

$$z \prec y \prec x \implies G(\varepsilon^*, z) \;\&\; G(\varepsilon^*, y) \;\&\; \phi(\varepsilon^*, z) < \phi(\varepsilon^*, y).$$

If $Q(\varepsilon^*, x)$ fails then $\exists z \prec x\big(G(\varepsilon^*, z) \;\&\; G(\varepsilon^*, x) \;\&\; \phi(\varepsilon^*, x) \leq \phi(\varepsilon^*, z)\big)$ but then $G(\varepsilon^*, x)$ — a contradiction ! Thus $Q(\varepsilon^*, x)$ holds.

But then $\forall x Q(\varepsilon^*, x)$; so

$$z \prec x \implies \phi(\varepsilon^*, z) < \phi(\varepsilon^*, x)$$

as required. $\dashv$

4·12 Corollary $\delta \leq |\phi|$

Proof : Let $\leq$ be a pwo in $\mathbf{\Delta}$, say in $\Delta(w)$; take $x \prec y \iff_{\text{df}} x \leq y \,\&\, \neg y \leq x$. Then $\prec$ is a strict well-founded relation in $\Delta(w)$. Relativise the above to $\Gamma(w)$, using the bold-face form of the Recursion Theorem, obtaining, for some ε^* recursive in w, an embedding $x \mapsto (\varepsilon^*, x)$ of $\prec$ into ϕ. Thus the length of $\leq$ is at most $|\phi|$, as required. $\dashv$ (4·12)

4·13 Corollary *Each initial segment of a Γ-norm is in $\mathbf{\Delta}$, so $|\phi| \leq \delta$ for any Γ-norm ϕ. Hence if ϕ is a Γ norm on a universal Γ set, its length is exactly δ.*

A property of Spector classes to be used in Chapter III is this:

4·14 Corollary *Let Γ be a Spector class closed under $\forall^{\mathcal{N}}$, let $A \in \Gamma \setminus \neg\Gamma$, and let ϕ be a Γ-norm on A. Let $B \subset A$ with $B \in \neg\Gamma$. Then $\{\phi(b) \mid b \in B\}$ is bounded below $|\phi|$.*

Proof : If not, we would have $\forall x\; x \in A \iff \exists y(y \in B \;\&\; x \leq_\phi y)$, giving A in $\Sigma^1_1\neg\Gamma$ and so in $\neg\Gamma$, a contradiction. $\dashv$ (4·14)

4·15 Corollary (Kleene) *If $A \subseteq WORD$ and $A \in \Sigma^1_1(\beta)$ then there is some ordinal η recursive in β such that $|\alpha| < \eta$ for each $\alpha \in A$.*

4·16 Remark Actually the above is not quite a corollary: but the method of argument is similar, and is left to the reader. For general Γ we must appeal to a Stage Comparison Theorem.

With the above discussion of Spector classes and our review of constructibility from Chapter One firmly in mind, we mention a particular family of classes that will of central importance in Chapter V and elsewhere.

4·17 Definition For H a subset of the ordinal Θ, we write Γ_H for the pointclass $\Sigma_1(J_\Theta(\mathbb{R}; H), H, \{\mathbb{R}\})$.

Note that Γ_H has these properties:

i	it is a Σ-pointclass
ii	it is ω-parametrized
iii	it has the substitution property
iv	it is closed under $\forall^\omega$
v	and it is normed.

in short, it is a Spector Class.

Facts i and iv are evident; indeed Γ_H is closed under both $\forall^N$ and $\exists^N$, so that iii follows from our *resumé* of the substitution property.

For a parametrization, enumerate the Σ_1 wffs in some recursive fashion as φ_i, and let

$$\Upsilon_H = \{\langle i, x\rangle \mid J_\Theta(\mathbb{R}) \models \varphi_i(x, H, R)\}.$$

To norm a given Σ_1 set $X = \{x | J_\Theta(\mathbb{R}) \models \varphi_i(x, H, R)\}$, we rank the members of X according to how soon this fact φ_i is verified. More exactly, note that

$$x \in X \implies \exists \xi :< \Theta\; J_{\xi+1}(\mathbb{R}) \models \varphi_i(x, H, R) \;\&\; J_\xi(\mathbb{R}) \models \neg\varphi_i(x, H, R).$$

Put $\pi(x) =$ this uniquely determined ξ, and say that

$$x \preccurlyeq_H y \iff_{\text{df}} \pi(x) \leqslant \pi(y).$$

Then for $y \in X$,

$$x \in X \,\&\, x \preccurlyeq_H y \iff \forall\zeta J_\zeta(\mathbb{R}) \models [\varphi_i(y,H,R) \longrightarrow \varphi_i(x,H,R)]$$
$$\iff \exists\xi[J_{\xi+1}(\mathbb{R}) \models \varphi_i(x,H,R) \;\&\; J_\xi(\mathbb{R}) \models \neg\varphi_i(y,H,R)]$$

so this is indeed a Γ-norm.

4·18 Definition We write $\delta = \delta_H$ for the ordinal of the Spector class Γ_H, namely the supremum of prewellorderings in $\boldsymbol{\Delta}_H$.

4·19 Definition Let U in Γ_H be a good universal set for $\boldsymbol{\Gamma}_H$, and let ϕ be any Γ-norm on it. Write $\upsilon_H^{U,\phi}$ for the length of ϕ.

Our aim now is to complete the list, begun in Chapter One, of five possible definitions of a certain ordinal. First we give alternative characterisations of the point class Γ_H:

4·20 Proposition ($V = L(\mathbb{R})$) *Let $H \subseteq \Theta$, let $\Gamma_H = \Sigma_1^{\dot{H}}(J_\Theta(\mathbb{R};H), H, \{\mathbb{R}\})$ and put M_H for $J_{\tau_H}(\mathbb{R};H)$. Then*

i) $\Gamma_H = \Sigma_1(M_H, H \cap \tau_H, \{\mathbb{R}\})$

ii) $\boldsymbol{\Delta}_H = \Delta_1(M_H, H \cap \tau_H, \{\mathbb{R}\} \cup \mathbb{R})$

iii) A set of reals is in M_H iff it is in $\boldsymbol{\Delta}_H$.

Proof : The first two parts are immediate from the definitions, since $M_H \preceq_{\Sigma_1^{\dot{H}}} J_\Theta(\mathbb{R};H)$. Part iii) takes some argument.

Suppose first that $X \in Power(\mathbb{R}) \cap \boldsymbol{\Delta}_H$. Then by (ii) there are $\Delta_0^{H,\mathbb{R}}$ formulæ ϑ, φ and reals a, b, such that

$$x \in X \iff M_H \models \bigvee w\, \vartheta(w,x,a)$$
$$x \notin X \iff M_H \models \bigvee v\, \varphi(v,x,b)$$

$$\text{Thus} \quad M_H \models \bigwedge x\colon \epsilon\, \dot{\mathbb{R}} \bigvee u\, [\vartheta(u,x,a) \vee \varphi(u,x,b)].$$

But M_H is admissible, and so as $\mathbb{R} \in M_H$, we know that for some $z \in M_H$,

$$M_H \models \bigwedge x\colon \epsilon\, \dot{\mathbb{R}} \bigvee u\colon \epsilon\, z[\vartheta(u,x,a) \vee \varphi(u,x,b)].$$

Hence $X = \{x \in \mathbb{R} \mid \exists w\colon\in z\; M_H \models \vartheta(w,x,a)\}$: that is a $\Delta_0^{H,\mathbb{R}}$ definition of X as a subset of $\mathbb{R}$ and so $X \in M_H$.

Conversely, suppose that $X \in M_H \cap Power(\mathbb{R})$. Since $M_H = \utilde{\mathrm{Hull}}_1^H(M_H)$, there is a real b and a Σ_1 formula φ such that $X = \iota z M_H \models \varphi(z,b)$. Then

$$x \in X \iff M_H \models \bigvee z\, (\varphi(z,b) \wedge x \,\epsilon\, z)$$
$$\iff M_H \models \bigwedge z\, (\varphi(z,b) \longrightarrow x \,\epsilon\, z)$$

which shows that $X \in \boldsymbol{\Delta}_\Gamma$. ⊣ (4·20)

4·21 Theorem ($V = L(\mathbb{R})$) *Let $H \subseteq \Theta$. Then*

$$\delta_H = \rho_H = \sigma_H = \tau_H = \upsilon_H^{U,\phi}$$

for any U and ϕ as above.

Proof : since Γ_H is a Spector class, we may infer that $\delta_H = \upsilon_H^{U,\phi}$ from Corollary 4·13 above.

(4·20·iii) easily extends from sets of reals to relations on the reals. So let X be a pre-well-ordering of $\mathbb{R}$ in $\boldsymbol{\Delta}_H$. By (iii), $X \in M_H$. Hence, by Lemma (I·2·19), the length of X is an ordinal in M_H and thus less than τ_H. So $\delta_H \leq \tau_H$.

On the other hand, each ordinal $\lambda < \tau_H$ is the length of a pre-well-ordering X in M_H: by (4·20·iii) again, $X \in \boldsymbol{\Delta}$, and so $\lambda \leq \delta_H$; and so $\tau_H \leq \delta_H$.

The remaining equalities have already been established. $\dashv$ (4·21)

Ordinals of the form δ_H are called *reflection ordinals* or *stable ordinals* in view of their characterisation as the least ordinal τ such that $J_\tau(I\!R;H) \preceq_{\Sigma_1}^{\mathbf{R},H} J_\Theta(I\!R;H)$.

We shall see in later chapters that reflection ordinals in $J(I\!R)$ are measurable, are strong partition cardinals, and, most important of all, when $H \in HOD$, δ_H will prove to be H-strong in HOD.

We conclude this section with recording a further characterisation which is available only for certain H, such as the empty set, or which are given by a Σ_1 recursion. Because this definition is not available for arbitrary H, it will be hardly be used in this book.

4·22 Definition $\mathbf{\Sigma^2_1}$ predicates are those of the form "$\exists X :\subseteq I\!R\ \Phi(X,\vec{x},a)$", where a is a real parameter and Φ is Π^1_∞ in the (second-order) predicate X. $\mathbf{\Pi^2_1}$ and $\mathbf{\Delta^2_1}$ are now defined as expected.

4·23 Definition δ^2_1 is the least ordinal not the order type of a $\mathbf{\Delta^2_1}$ pre-well-ordering of R.

4·24 Theorem *Let $\tau_\varnothing$ be the least ordinal τ such that $J_\tau(I\!R) \preceq_{\Sigma_1}^{\mathbf{R}} J(I\!R)$. Then*

(a) $(\mathbf{\Sigma^2_1})^{L[\mathbb{R}]} = \mathbf{\Sigma_1}(J_{\tau_\varnothing}(I\!R)) \cap Power(I\!R) = \Sigma_1(J_{\tau_\varnothing}(I\!R), R) \cap Power(I\!R)$;

(b) $(\mathbf{\Delta^2_1})^{L[\mathbb{R}]} = J_{\tau_\varnothing}(I\!R) \cap Power(R)$

(c) $(\delta^2_1)^{L[\mathbb{R}]} = \tau_\varnothing$.

Proof of (a): the second equation holds as any parameter that might be used is $k(a)$ for some real a. Let A be $(\mathbf{\Sigma^2_1})^{J(I\!R)}$, $A \subseteq I\!R$. $A = \{x \mid \exists B :\subseteq R\ \phi(x,B)\}$ where $a \in R$ and ϕ is Π^1_ω, and the atomic predicate $t \in B$ is allowed.

$x \in A \iff J_{\tau_\varnothing}(I\!R) \models x \ \epsilon\ \dot{A}$, as all reals are in $J_{\tau_\varnothing}(I\!R)$ and there is one existential quantifier. A is $\Sigma_1(J(I\!R), I\!R)$; hence $\Sigma_1(J_{\tau_\varnothing}(I\!R), I\!R)$. If $A \in \Sigma_1(J_{\tau_\varnothing}(I\!R), I\!R)$ say

$$x \in A \iff J_{\tau_\varnothing}(I\!R) \models \phi[x,y]$$

We know $J_{\tau_\varnothing}(I\!R)$ is the surjective image of R under a map in $J(I\!R)$: so

$$\begin{aligned} x \in A \iff \exists E :\subseteq I\!R\ \exists I :\subseteq I\!R\ \big(&\text{the structure } \langle I\!R, I, E\rangle \\ &\text{is well-founded and models } V = L(I\!R)\ \big). \end{aligned}$$

Here I represents the equality relation and E the set membership relation. This is a $\mathbf{\Sigma^2_1}$ definition. $\dashv$

Proof of (b): $J_{\tau_\varnothing}(I\!R) \preceq_{\Sigma_1} J(I\!R)$, every parameter being $k(a)$ for some $a \in R$ and so $J_{\tau_\varnothing}(I\!R)$ is admissible. By part (a),A is $(\mathbf{\Delta^2_1})^{J(I\!R)}$ iff A is $\mathbf{\Delta_1}(J_{\tau_\varnothing}(I\!R))$ iff $A \in J_{\tau_\varnothing}(I\!R)$, by the admissibility of $J_{\tau_\varnothing}(I\!R)$ as $I\!R \in J_{\tau_\varnothing}(I\!R)$. $\dashv$

Proof of (c): every $\mathbf{\Delta^2_1}$ pre-well-ordering is in $J_{\tau_\varnothing}(I\!R)$ and the ordinal coded can be recovered inside $J_{\tau_\varnothing}(I\!R)$, by the admissibility of $J_{\tau_\varnothing}(I\!R)$, and so is isomorphic to an ordinal less than $\tau_\varnothing$. So $\delta^2_1 \leq \tau_\varnothing$. Conversely, let $\beta < \tau_\varnothing$. There is an $h \in J_{\tau_\varnothing}(I\!R)$ with $h : I\!R \xrightarrow{\text{onto}} \beta$, so $\{(x,y) \mid h(x) \leq h(y)\} \in J_{\tau_\varnothing}(I\!R)$, and hence is $\mathbf{\Delta^2_1}$ by (b), and so $\beta < {\delta^2_1}^{J(I\!R)}$.

We should record here two properties of the class $\mathbf{\Sigma^2_1}$, which follow easily from the results of this section.

4·25 Proposition *Every $\mathbf{\Sigma^2_1}$ predicate is witnessed by a $\mathbf{\Delta^2_1}$ set of reals.*

4·26 Proposition *Every $\mathbf{\Sigma^2_1}$ subset of $\mathcal{N} \times \mathcal{N}$ is uniformised by a $\mathbf{\Sigma^2_1}$ set.*

4·27 Remark In Section 3 of Chapter I we briefly considered models of the form $L(\mathbb{R}, Q; \mathfrak{B})$ where $\mathfrak{B}$ is some set of reals and Q is a set of ordinals. Such models will be important in Chapter VI. For the moment let us remark that without Q but with $\mathfrak{B}$, then one can quite sensibly define the concept of a $\mathbf{\Sigma^2_1}(\mathfrak{B})$ predicate, and prove an analogue of Theorem 4·24.

4·28 Remark We should emphasize the importance in Theorem 4·24 of the availability of the constant $I\!R$ in the various wffs. Without that, far fewer sets of reals would be Σ_1 definable:

4·29 Proposition *A set of reals is $\Sigma_1(J_\delta(I\!R), R)$ iff it is $\mathbf{\Sigma^1_2}$.*

Proof : Let A be a set of reals, and γ a real with

$$\forall\alpha{:}\in{}^{\omega}\omega\ \alpha \in A \iff J_\delta(\mathbb{R}) \models \phi[\alpha,\gamma],$$

where ϕ is a Σ_1 wff with no further parameters.

Then
$$\alpha \in A \iff \exists M\ \big[M \text{ is a countable well-founded model of } KP\mathbb{R}\ \& \\ \&\ \omega^M = M\ \&\ \alpha \in M\ \&\ \gamma \in M\ \&\ M \models \phi[\alpha,\gamma]\big]$$

which (as a mild exercise in coding will show) is a $\Sigma^1_2(\gamma)$ predicate of α.

4·30 Exercise Consider the following subset $\mathcal{H}$ of Θ, which is defined recursively:

$$\mathcal{H} = \{\zeta < \Theta \mid \neg\exists\xi < \zeta\big((J_\xi(\mathbb{R}), \mathcal{H}\cap\xi) \preceq_{\Sigma^{\mathcal{H}}_1} (J_\zeta(\mathbb{R}), \mathcal{H}\cap\zeta)\big)\}.$$

Define $\tau_{\mathcal{H}}$ as above. Establish the truth of the following statements:

(4·30·0) $\mathcal{H} \in HOD$;
(4·30·1) $\mathcal{H}$ consists of those ordinals less than Θ that start $\mathcal{H}$-gaps;
(4·30·2) $\tau_{\mathcal{H}} \in \mathcal{H}$;
(4·30·3) $\max \mathcal{H} = \tau_{\mathcal{H}}$;
(4·30·4) $\sup \mathcal{H}\cap\tau_{\mathcal{H}} = \tau_{\mathcal{H}}$;
(4·30·5) $\tau_{\mathcal{H}} = \tau_{\varnothing}$.

5: Souslin sets: properties of projections of trees

Let T be a tree on $\omega\times\lambda$: that is, the elements of T are finite sequences of pairs $\langle n, \zeta\rangle$ where $n\in\omega$ and $\zeta<\lambda$. It will be convenient to abuse notation and treat the members of T as pairs of finite sequences $\langle s, v\rangle$ where $s \in^{<\omega} \omega$, $v \in^{<\omega} \lambda$ and $lh(s) = lh(v)$.

A path through T can then be construed as a pair $\langle x, f\rangle$ where $x \in^{\omega} \omega$, $f \in^{\omega} \lambda$ and $\forall n\langle x{\restriction}n, f{\restriction}n\rangle \in T$. We write $[T]$ for the set of such paths.

5·0 Definition $p[T] =_{\rm df} \{x \mid \exists f\ \langle x, f\rangle \in [T]\}$

Such a set of reals is called λ-*Souslin*: a familiar instance will be the ω-Souslin sets which are precisely the $\boldsymbol{\Sigma^1_1}$ sets.

We refer to Chapter II of Moschovakis's treatise for discussion of the classical results concerning trees and their projections: we draw attention here briefly to the fact that the theorem [Souslin ?] that an uncountable $\boldsymbol{\Sigma^1_1}$ set contains a perfect subset has been refined in two directions: first, by Harrison, it has been made more effective: his result being that if a $\Sigma^1_1(\alpha)$ set contains a real not hyperarithmetic in α, then it has a perfect subset; and then Mansfield and others generalised the arguments to λ trees. For example, they proved that if a tree T is in an model M and its projection contains a real not in M, then its projection contains a perfect subset; and (in more classical terms still) the projection of a tree on $\omega\times\kappa$ which has more than κ elements must have a perfect set of elements.[N 1]

We should emphasize here the absoluteness of the property "$x \in p[T]$": this of course follows from the absoluteness of well-foundedness for models of (say) $KP+$"every well-ordering is isomorphic to an ordinal"; since $x \in p[T]$ is plainly a Σ_1 statement about x and T, namely that there is a path; and $x \notin p[T]$ is equivalent to saying that $(T(x), \preceq_{KB})$ is a well-ordering, therefore (by the axioms of the theory) that there is an isomorphism with an ordinal; and hence is also Σ_1.

These remarks of course yield Mostowski's classical result about the absoluteness of Π^1_1 predicates for even countable transitive models. We shall recall Shoenfield's generalization of this to Σ^1_2 predicates with the proviso that the transitive model concerned contains all countable ordinals.

We adopt this notation for the Kleene-Mostowski tree for a $\boldsymbol{\Pi^1_1}$ set: if the Π^1_1 predicate $P(\alpha,\gamma)$ has been expressed in Kleene normal form as $\{\langle\alpha, \gamma\rangle \mid \forall\beta\exists n\ R\big(\bar\alpha(n), \bar\beta(n)\bar\gamma(n)\big)\}$, where R is a recursive predicate, then

$$\mathcal{M}_R(\alpha,\gamma) = \{v \mid \forall n :\leq lh(v)\, \neg R\big(\bar\alpha(n), \bar v(n), \bar\gamma(n)\big)\}.$$

[N 1] the second statement follows from the first for regular cardinals; what about singular ones ?

This is the tree of *unsecured sequences*. By familiar properties of the Kleene-Brouwer ordering,

$$\begin{aligned} P(\alpha,\gamma) &\iff \mathcal{M}_R \text{ is well-founded under extension} \\ &\iff (\mathcal{M}_R, \preceq_{KB}) \text{ is a well-ordering} \\ &\iff \exists f\ f : (\mathcal{M}_R, \preceq_{KB}) \widetilde{\longrightarrow} \Omega \end{aligned}$$

where Ω is a sufficiently large ordinal.

Consider now a Σ^1_2 predicate $Q(\gamma)$: we shall show that $\{\gamma \mid Q(\gamma)\}$ is the projection of a tree which will ensure the absolute meaning of the predicate. We have the following chain of equivalences:

$$\begin{aligned} Q(\gamma) &\iff \exists\alpha\forall\beta\exists n\ R(\bar\alpha(n), \bar\beta(n), \bar\gamma(n)) \\ &\iff \exists\alpha\ \mathcal{M}_R(\alpha,\gamma) \text{ is well-founded} \\ &\iff \exists\alpha\ \langle \mathcal{M}_R(\alpha,\gamma), \preceq_{KB}\rangle \text{ is a wellordering} \\ &\iff \exists\alpha\exists f\ f : \mathcal{M}_R(\alpha,\gamma) \widetilde{\longrightarrow} \Omega \end{aligned}$$

Here Ω may be taken to be true ω_1: though as that might shift as we enlarge the universe, it is convenient to think of Ω as ON, so that then the Shoenfield tree is a proper class, of which we may take a sufficiently large initial segment when we need to.

Now consider the Shoenfield tree $\mathcal{S}_R$ of triples (u, w, t) where $u\ w\ t$ are finite sequences of the same length, u and t are sequences of finite ordinals and w of arbitrary ordinals ($< \Omega$), and we think of $u\ w\ t$ as initial segments of α, f, γ respectively.

$$(u, w, t) \in \mathcal{S}_R \iff \forall i, j :< lh(u)(s_i, s_j \subseteq T(u,t) \implies [w(i) \le w(j) \iff s_i \preceq_{KB} s_j]$$

Here $\langle s_i \mid i < \omega$ is a recursive enumeration of ${}^{<\omega}\omega$ with $s\, IN\, t \implies s$ occurs before t.

Then $Q(\gamma) \iff \exists\alpha\exists f\forall n\ (\alpha\restriction n, f\restriction n, \gamma\restriction n) \in \mathcal{S}_R$ so that $Q = p[\mathcal{S}_R]$ as required. [C 6]

5·1 Proposition *(i) If $p[S]$ is empty, it remains so in any extension of the universe.*
(ii) If $p[S]$ and $p[T]$ are disjoint, they remain so in any extension of the universe.

Proof: (i) holds because $p[S]$ is empty iff the whole tree S is well-founded; (ii) follows from (i) by considering the tree

$$U =_{\text{df}} \{(s, \langle u, v\rangle) \mid (s, u) \in S \,\&\, (s, v) \in T\}$$

which has the property that $p[U] = p[S] \cap [p[T]$. $\dashv$

Now suppose that we are given a tree T and a continuous function $f : \mathcal{N} \to \mathcal{N}$, and that $\forall x\ x \in A \iff f(x) \in B = p[T]$. We shall design a tree S such that $A = p[S]$, and in the section on weakly homogenous trees in Chapter VIII shall show that if T is weakly homogeneous, so is S.

By the continuity of f,

$$\forall x\forall n\exists k = k(n, x)\forall y(y\restriction k = x\restriction k \implies f(y)\restriction n = f(x)\restriction n).$$

We want a converse to this, so define for $s \in {}^{<\omega}\omega$,

$$\psi(s) =_{\text{df}} sup\{n \le lh(s) \mid \forall x\forall y(y\restriction lh(s) = s = x\restriction lh(s) \implies f(y)\restriction n = f(x)\restriction n\}$$

We write $\phi(s)$ for this longest common segment so that $\psi(s) = lh(\phi(s))$ and $\phi(x\restriction k) = f(x)\restriction\psi(x\restriction k)$.

Now set

$$S =_{\text{df}} \{(s, u) \mid lh(s) = lh(u)\ \&\ (\phi(s), u\restriction\psi(s)) \in T\}.$$

Then

$$\begin{aligned} x \in p[S] &\iff \exists g\forall k\ (x\restriction k, g\restriction k) \in S \\ &\iff \exists g\forall k\ (\phi(x\restriction k) g\restriction(\psi(x\restriction k)) \in T \\ &\iff \exists g\forall n\ (f(x)\restriction n, g\restriction n) \in T \\ &\iff f(x) \in p[T]. \end{aligned}$$

[C 6] We emphasize that the Shoenfield tree once defined works in every larger model not rendering the starting model countable, by the absoluteness of well-foundedness.

In Chapter III we shall review the notion of a *scale*, which is a sequence of norms with certain further properties; a set admits a semi-scale iff it is the projection of a tree.

6: ∞-Borel sets

The class of *Borel sets* is commonly defined as the smallest class containing all rational intervals and closed under countable union and complementation. A problem arises with this definition in the absence of the Axiom of Choice, there being a famous model of Lévy in which the real line is the countable union of countable sets, and thus in that model all sets of reals would be Borel according to that definition. That particular problem disappears when $AC_{\aleph_0}$ is adopted.

It is natural to ask what happens to that definition if larger well-ordered cardinals λ are used in place of $\aleph_0$. Under AC the definition would again trivialise once the value $\lambda = 2^{\aleph_0}$ is permitted. But in set theories where the continuum is not well-orderable the concept retains interest even when λ is permitted to be arbitrarily large. Our aim in this section is to consider various possible definitions of this notion of ∞-Borel : we emphasize that we are working in $ZF + DC$, but are not assuming AC. In the next chapter we shall see that, assuming some consequences of AD, the concepts of OD and ∞-Borel sets of reals become almost identical.

There are three unofficial definitions:

6·0 DEFINITION The class of ∞-Borel sets of reals is the least class containing all rational intervals and closed under complementation and well-ordered union.

6·1 DEFINITION A set A of reals is ∞-Borel if there is a set of ordinals S and a formula φ such that

$$\forall\alpha{:\in}\mathcal{N}\ \alpha \in A \iff L[S,\alpha] \models \varphi[S,\alpha]$$

6·2 DEFINITION A set of reals A is ∞-Borel if there is a tree $R \subseteq {}^{<\omega}\big(\omega \times ON\big)$ such that

$$\alpha \in A \iff \forall\zeta_0\exists\zeta_1\forall\zeta_2\exists\zeta_3\ \ldots\ \forall n R(\alpha{\restriction}n, \vec{\zeta}{\restriction}n).$$

Each of these definitions is open to criticism, though they may be found intuitively helpful: in particular the second one will be our "working definition" of ∞-Borel .

An ∞-Borel *tree* is a labelled tree composed of finite sequences of ordinals. It has a single top point, namely the empty sequence, and is well-founded under end-extension. The labelling is simple: a bottom point, *i.e.* a node s with no proper extensions will bear a label naming a basic open set $N_{\lambda(s)}$: a node with exactly one immediate extension will bear a label saying 'Take complements', and a node with more than one immediate extension will bear a label saying 'Take unions'.

Thus assuming the basis to have been enumerated, the labelling λ is a function from the set of bottom points of the tree into ω.

We write $\nu_0(T)$ for the set of bottom points of T; $\nu_1(T)$ for the set of points with one immediate extension, an immediate extension of the finite sequence s being a finite sequence t extending s with $\ell h(t) = \ell h(s) + 1$; $\nu_2(T)$ for the rest, namely those nodes with more than one immediate extension. We write $\xi(T)$ for the single top node of T.

Given such a pair (T, λ), we write

$$\begin{aligned}\Psi(f,T,\lambda,\alpha) \iff_{\mathrm{df}}\ & f:\ \text{nodes of } T \longrightarrow \{0,1\}\ \&\\ & \&\ \forall s{:\in}\nu_0(T)\ f(s) = 1 \iff \alpha \in \mathcal{N}_{\lambda(s)}\ \&\\ & \&\ \forall s{:\in}\nu_1(T)\ f(s) = 1 - f^*(s)\ \&\\ & \&\ \forall s{:\in}\nu_2(T)\ f(s) = f^*(s)\end{aligned}$$

where we write $f^*(s)\ =_{\mathrm{df}}\ \sup\{f(t) \mid t \text{ is an immediate extension of } s\}$; then $B_\infty(T,\lambda)$, the ∞-Borel set coded by $\langle T,\ \lambda\rangle$, is given by

$$\alpha \in B_\infty(T,\lambda) \iff_{\mathrm{df}} \exists f\ \Psi(f,T,\lambda,\alpha)\ \&\ f(\xi(T)) = 1;$$

since such f exists and is uniquely determined by its recursive definition, T being well-founded, we shall also have

$$\alpha \in B_\infty(T, \lambda) \iff_{\rm df} \forall f\ \Psi(f, T, \lambda, \alpha) \implies f(\xi(T)) = 1;$$

6·3 Remark A rather more streamlined version of the above is the following inductive definition:

$$\langle 0, m, n\rangle \text{ codes } \{f \mid f(m) = n\};$$
$$\langle 1,\ s\rangle \text{ codes } \{f \mid f \notin B\} \text{ if } s \text{ codes } B;$$
$$\text{if } \zeta \in ON \text{ and } h : \zeta \to V \text{ and } \forall \eta < \zeta\ h(\eta) \text{ codes } A_\eta \text{ then } \langle 2,\ h\rangle \text{ codes } \bigcup_{\eta<\zeta} A_\eta.$$

The flaw in the second unofficial definition is that without sharps we may not have a truth definition for $L[S, \alpha]$: so we shall rely on Lévy reflection to show that the following legitimate definition covers all instances of the above:

6·4 Definition A set of reals is ∞-Borel if there is a set of ordinals S, a formula φ, and an ordinal η such that

$$\forall \alpha{:}{\in}\mathcal{N}\ \big(\alpha \in A \iff J_\eta[S, \alpha] \models \varphi[S, \alpha]\big)$$

That infinite string of quantifiers in the third unofficial definition is merely a picturesque way of saying that there is a game $\mathcal{G}(R, \alpha)$, open for Adam, such that

$$\alpha \in A \iff \text{ Eve has a winning strategy in the game } \mathcal{G}(R, \alpha).$$

so again our official definition must be more strictly worded:

let $R \subseteq {}^{<\omega}\big(\omega \times ON\big)$ be a set closed under shortening. The game $\mathcal{G}(R, \alpha)$ is played with ordinals as moves, and Adam wins if $\exists n \neg R(\alpha{\restriction}n, \vec{\zeta}{\restriction}n)$.

6·5 Definition We refer to sets satisfying the three definitions as *tree-Borel, wff-Borel* and *game-Borel* respectively, and to the related objects $\langle T,\ \lambda\rangle$, $\langle S, \varphi, \eta\rangle$ and R as ∞-Borel *tree-codes, wff-codes* and *game-codes* respectively. It will also be convenient when C is an ∞-Borel set to denote by C^c an ∞-Borel code for it, without specifying the kind of that code; and when B is an ∞-Borel code of one of the three kinds, to write B^v for the set it codes.

6·6 We of course wish to prove that our official definitions are equivalent, but prefer to defer the discussion of game-Borel sets till the end of the first section of Chapter III, in which games are introduced and the existence of definable strategies for open games established. Our plan therefore is to prove here that every tree-Borel set is wff-Borel, and to sketch a proof that every wff-Borel set is tree-Borel. In the next Chapter we shall replace that sketch with full proofs that every wff-Borel set is game-Borel and every game-Borel set is tree-Borel, thereby establishing the equivalence of the three definitions without, we emphasize, making any appeal to a hypothesis of determinacy such as AD.

6·7 Remark The equivalence is local in the sense that if T is a Borel code according to one of the three definitions, then in $L(T)$ there is a code according to either of the other two.

Proof that every tree-Borel set is wff-Borel: Take S to code (in some trivial manner, using a constructible enumeration of finite sequences of ordinals) the pair $(T,\ \lambda)$, and φ to express the property $\alpha \in B_\infty(T, \lambda)$. Evidently S can be chosen to lie in $L(T)$. Let η be a cardinal in V larger than $\bigcup S$. Then for each α, the unique function f with $\Psi(f, T, \lambda, \alpha)$ will be in $L_\eta[S, \alpha]$. $\dashv$

6·8 Remark Note that this last stage raises the spectre that two ways of defining the same set in this world might define distinct sets in the next, for by the time we have added a lot more reals, the ordinal chosen as the reflection point might no longer be able to cope.

Proof in outline that each wff-Borel set is tree-Borel: We keep S fixed; a wff-code $\langle S, \varphi, \eta\rangle$ will be considered *simpler* than a wff-code $\langle S, \psi, \xi\rangle$ if either $\xi = \eta$ and ψ is shorter than ξ or $\eta < \xi$. [C 7] We use an induction on the length of φ, when discussing truth in any one $J_\eta(\alpha)$, to reduce to atomic φ: these atomic cases will be of the form $h(J_\xi(\alpha), \alpha) \in g(J_\zeta(\alpha), \alpha)$ or $h(J_\xi(\alpha), \alpha) = g(J_\zeta(\alpha), \alpha)$; fine-structural techniques then serve to reduce η. We may suppose φ to be built up from atomic formulæ using only existential quantifiers, disjunction and negation, and it is plain that these reductions are embraced by the available ∞-Borel operations. In reducing η we are also, uniformly in α, taking well-ordered unions of simpler things.

Eventually we reduce to truth of atomic sentences in $J_1(\alpha)$; and these are now basic open sets.

Finally we remark that the above analysis may be done in $L(S)$. ⊣ (6·8)

6·9 EXERCISE Fill in the details of that last remark.

6·10 EXERCISE Show that 'I am ∞-Borel 'is Σ_2 but also $\Sigma_1^{I\!R}$ if $V = L(I\!R)$. What happens if V is not $L(I\!R)$?

6·11 PROPOSITION *Assume $V = L(I\!R)$. Then every ∞-Borel set has a code of size less than Θ.*

Proof : let C be ∞-Borel . The statement that there is a set of ordinals T which is a tree-code for C is $\Sigma_1^{I\!R}$ — remember to say that there is an order-preserving embedding into the ordinals — and hence is true in $J_\Theta(I\!R)$: indeed if we reconstruct Ł$(I\!R)$ using a predicate for C we see that there will be such a code $\boldsymbol{\Delta}_1^2$ in C. ⊣ (6·11)

6·12 COROLLARY *Assume $V = L(I\!R)$. Then every* HB *set has an* OD *Borel code of size less than Θ.*

Proof : reflect a similar statement which includes the promise that T is OD: that is $\Sigma_1^{I\!R}$, so may be included without further expense. ⊣ (6·12)

Differences between Souslin and ∞-Borel sets

[A short section to contain general facts relating Souslin and ∞-Borel sets. Sierpiński's equations show that all Souslin sets are ∞-Borel . If $V = L(I\!R)+$ AD, every set of reals is ∞-Borel , but not every set is Souslin. Under $ADI\!R$ every set of reals is Souslin. Point to the differences between them, particularly under AD.

One significant difference is this. Let T be a tree on $\omega \times \lambda$. Then $\mathfrak{p}[T]$ is empty if and only if it is empty in $L(T)$: because that is a well-foundedness condition.

On the other hand, let B be an ∞-Borel code. Then we might well have $B^v \cap L(B)$ is empty but B^v not.

Notice that if in some extension of the universe there is a member of B then there will be one in the extension by $Coll(\omega, \overline{\overline{B}})$.]

6·13 PROBLEM

The Borel version of Vopěnka's analysis: the 1-dimensional case

If an ∞-Borel set has a code of one kind that is OD, it will have OD codes of the other two kinds. Therefore we shall define a set to be HB if it is ∞-Borel with code in HOD without needing to specify which kind of code is intended.

⊛ ⊛ We set out to imitate Vopěnka's proof that every subset of ω is generic over HOD , and shall find that we cannot sustain the analogy without help from some consequences of AD.

6·14 We consider the partial order $\mathbb{L}^1$ of OD tree-codes of ∞-Borel sets, ordered by inclusion of the sets they define. That will be a copy of the partial order ${}^*\mathbb{L}^1 = {}^*\mathbb{L}^1(\omega)$, the elements of which are ∞-Borel subsets $A \subseteq \mathcal{P}(\omega)$ ordered by inclusion.

We can tell inside HOD which sets are Borel codes, as the property of being a well-founded labelled tree is absolute to models of ZF, or indeed of weaker theories. We cannot immediately tell inside HOD when the set coded by one Borel code will be a subset of the set coded by another.

6·15 EXAMPLE Under $AD + V = L(I\!R)$ the set of reals that are not OD is (as we shall see in Chapter III) HB, but none of its members are in HOD, and so HOD might think it is a subset of the empty set.

[C 7] this won't quite do, as we replace quantifiers by rud functions in earlier $J_\theta(\alpha)$'s.

Hence we must for the moment observe that the ordering of ∞-Borel codes by inclusion of the sets they define is definable, and therefore its restriction to the ∞-Borel codes in HOD lies in HOD.

Note that as $HOD \models AC$ every union is a well-ordered union, which is an ∞-Borel operation, and so $\mathbb{L}^1$ will in HOD be complete.

6·16 PROPOSITION *To each subset a of ω there is a $(HOD, \mathbb{L}^1)$-generic G^1_a such that $HOD[a] = HOD[G^1_a]$; and to each $(HOD, \mathbb{L}^1)$-generic G there is a $b \subseteq \omega$ with $G = G^1_b$.*

Proof : We define $G^1_a = G^1_a(\mathbb{L})$ as the set of codes of HB sets of which a is a member. G will be generic, since a dense set in HOD is a set of HB codes the union of the **realisations** of which is $\mathcal{P}(\omega)$, and hence G must meet the dense set.

Note that all sets $\{x \mid n \in x\}$, $\{x \mid n \notin x\}$ are clopen and therefore ∞-Borel, and each union $\{x \mid n \in x\} \cup \{x \mid n \notin x\}$ equals $\mathcal{P}(\omega)$: hence the code of exactly one of each pair must lie in G_a.

Hence $a \in HOD[G_a]$ since a is the set of n such that the code of $\{x \mid n \in x\}$ lies in G_a.

Conversely, given a generic G, we take b to be the set of those n such that the code of $\{x \mid n \in x\}$ lies in G. But now in $HOD[b]$ we may show by an induction on tree-codes in HOD that for each tree-code T in HOD, T is in G if and only if b is a member of T^v, the ∞-Borel set coded by T, a statement which by the nature of tree codes is absolute to $L[T, b]$, and therefore to $HOD[b]$ since $T \in HOD$. Hence $G = G^1_b$ and is in $HOD[b]$. $\dashv$ (6·16)

6·17 HISTORICAL NOTE Arguments of that kind go back to Scott and to Solovay.

The 2-dimensional case

The next step would be to consider pairs (a, b) of subsets of ω. At first, all is well:

6·18 PROPOSITION *Each pair (a, b) is $\mathbb{L}^2$ generic over HOD.*

Proof : The ordering of 2-dimensional HB sets is definable, so the corresponding ordering of names is in HOD. The rest of the proof is as before. $\dashv$ (6·18)

Now we would wish to consider the relationship of b to $HOD[a]$: we know $HOD[a] = HOD[G^1_a(\mathbb{L})]$. Given a pair, we get $G^2_{a,b}$. Can we recover G^1_a from that ? Yes, as the set $\{D \mid D \times \mathcal{N} \in G^2\}$, provided we show that there is a definable map taking code-of-D to code-of-$D \times \mathcal{N}$. But that is easy: replace each leaf N_s by $N_s \times N_\emptyset$.

A more serious problem comes with trying to get a good representation of the extension from $HOD[a]$ to $HOD[b]$. In Chapter I we were able to considering the non-separative order $\{C^c \mid C_{\downarrow 1} \in G^1_a\}$. Here the problem of the existential quantifier rears its ugly head: we do not know, if C is a 2-dimensional ∞-Borel set, whether $C_{\downarrow 1}$, though certainly 1-dimensional, is ∞-Borel . If we express that set as $\{C^c \mid \exists y\ (a, y) \in C\}$, the existential quantifier evident.

$\lessdot$ 6·19 REMARK We shall overcome this difficulty in the next Chapter, using a fine measure built with the aid of AD.

For the moment, therefore, we list some properties that we can still prove. Jensen and Karp[R3] developed a notion of *primitive recursive set function* which we use in formulating the next proposition, to underline the fact that the relevant transformation of tree-codes is defined by a very simple process.

6·20 PROPOSITION *If T is a tree-code of the two-dimensional set C, and a is a real, then there is a tree-code of the section C_a primitive recursive in T and a.*

Proof : we consider a tree-code $T \in HOD$ for C, where at all the branch points we take either a well-ordered union or a well-ordered intersection, and at the **leaves**, that is, the end points, we have a basic clopen set $N_{s,t}$. The equations

$$(\bigcup_{i \in I} D^i)_a = \bigcup_{i \in I} (D^i)_a$$
$$(\bigcap_{j \in J} E^j)_a = \bigcap_{j \in J} (E^j)_a$$
$$(N_{s,t})_a = \begin{cases} N_t & \text{if } a \in N_s \\ \varnothing & \text{otherwise} \end{cases}$$

[R3] Jensen, Karp

are easily checked; so a tree-code for C_a may be obtained from one for C by retaining all the tree structure but replacing each leaf coding $N_{s,t}$ by a code of $\varnothing$ if $a \notin N_s$ and by a code of N_t otherwise. $\dashv$ (6·20)

6·21 Corollary *Let $C \subseteq \mathcal{N} \times \mathcal{N}$ be* HB *, and a a real. Then the section $C_a =_{\rm df} \{y \mid (a,\, y) \in C\}$ is ∞-Borel with code in $HOD[a]$.*

Proof : that transformation may be carried out in $L(T, a)$ and therefore, as $T \in HOD$, in $HOD[a]$. $\dashv$ (6·21)

Now for a converse, for which we use wff-codes.

6·22 Proposition *Let S be a set of ordinals and a a real. Let $D \subseteq \mathcal{N}$ be ∞-Borel with wff-code (T, φ_o, κ) in $L(S, a)$. Then D is of the form C_a where $C \subseteq \mathcal{N} \times \mathcal{N}$ is ∞-Borel with a wff code in $L(S)$.*

Proof : Let $D = \{y \mid J_\kappa(T, y) \models \varphi_0(y, T)\}$, where $T \in J_\kappa(S, a)$. We may regard T as a rudimentary word of the form $h(J_{\vec\xi}(S, a), S, a)$ with all $\vec\xi$ less than κ. Then

$$D = \{y \mid J_\kappa(S, a, y) \models \varphi(T, y)\}$$

where $\varphi(T, y)$ says that $\varphi_0(T, y)$ holds relativised to $L(T, y)$. Consider the set

$$C =_{\rm df} \big\{(x, y) \bigm| J_\kappa(S, x, y) \models \varphi\big(h(J_{\vec\xi}(S, x), S, x), y\big)\big\} :$$

that is plainly wff-Borel with code $S \oplus \vec\xi$: that is, some set S' coding both S and the finite sequence $\vec\xi$ of ordinals, and $C_a = D$. Clearly S' may be taken to be in $L(S)$. $\dashv$ (6·22)

6·23 Corollary $(V = L(\mathbb{R}))$ *Any ∞-Borel set with code in $HOD[a]$ is a section of an ∞-Borel set with code in HOD.*

Proof : we know from Chapter I that if $V = L(\mathbb{R})$, HOD is of the form $L(S)$. $\dashv$ (6·23)

6·24 Remark Thus all the would-be members of $\mathbb{L}^{1,a}$ are indeed in $HOD[a]$. It is still true that $b \in C_a \iff (a,\, b) \in C$, and every dense set in $HOD[a]$ is expressible as a union, hence as a Borel set with code in $HOD[a]$. The obstacle is that we are unable to show that the partial ordering of the elements of $\mathbb{L}^{1,a}$ is in $HOD[a]$.

6·25 Proposition $(V = L(\mathbb{R}))$ *There is a definable well-ordering of the bounded* OD *subsets of Θ in order type Θ.*

Proof : we know that every subset of every ordinal less than Θ is in $J_\Theta(\mathbb{R})$, since there is a pre-well-ordering of length κ say, so each subset of κ may be identified by its preimage which a some set of reals, and all sets of reals are in $J_\Theta(\mathbb{R})$.

Well order each $HOD \cap (J_{\xi+1}(\mathbb{R}) \smallsetminus J_\xi(\mathbb{R}))$ by $<_{OD}$: that gives us an ordinal less than Θ, as we can map $\mathcal{N}$ onto $J_\xi(\mathbb{R})$: by concatenation, we get a definable well-ordering of which each initial segment is of length less than Θ. But equally each $\{\omega\xi\}$ is listed in the $\xi^{\rm th}$ stage. $\dashv$ (6·25)

Consider $\mathcal{L}^0 =_{\rm df} \{(x, C^c, D^c) \mid C \in HB \;\&\; D \in HB \;\&\; C_x \subseteq D_x\}$, which we may consider to be a subset of $\Theta \times \mathcal{N} \times \mathcal{N}$. $\mathcal{L}^0$ is OD , certainly, and for given a, the section $(\mathcal{L}^0)_a$ is definable from a, but we want it to be in $HOD[a]$: then we could say that given a and b, b is $\mathbb{L}^{1,a}$ generic over $HOD[a]$, that being the algebra of 1-sections.

$\lessdot$ 6·26 Remark The problem then that we shall address in section 8 of Chapter III will be to consider the following variant of $\mathcal{L}^0$:

$$\mathcal{L} =_{\rm df} \{(x, \zeta) \mid x \in \mathcal{N} \;\&\; \zeta < \Theta \;\;\&\;\; \text{the section at } x \text{ of the } \zeta^{\rm th} \text{ HB set is non-empty}\}$$

and to show that it is itself HB in the sense that there is a formula $\Phi(x, \zeta, S)$ and an ordinal $\kappa > \Theta$ such that assuming $AD{+}V = L(\mathbb{R})$ and that S is a subset of Θ such that $HOD = L(S)$, then for all reals x and ordinals $\zeta < \Theta$,

$$(x,\, \zeta) \in \mathcal{L} \iff L_\kappa(S, x) \models \Phi(x, \zeta, S).$$