Discussion-answers of Old Tripos Questions

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These notes on old tripos questions are the result of a long process of accretion and reworking, rather like North America west of the Rockies. In consequence of this fairly random process, they are of highly variable nature. Think of these notes as *discussions* (as in the subsection titles) rather than model answers.

1995

95108

Let P, Q, R be three pairwise-disjoint sets of primitive propositions; for any set S, let $\mathcal{L}(S)$ denote the set of propositional formulae whose primitive propositions are all in S. Let $s \in \mathcal{L}(P \cup Q)$ and $t \in \mathcal{L}(Q \cup R)$ be formulæ such that $(s \to t)$ is a theorem of the propositional calculus. Show that there is a formula $u \in \mathcal{L}(Q)$ such that both $(s \to u)$ and $(u \to t)$ are theorems.

Discussion

PTJ says: consider the set $U = \{u \in \mathcal{L}(Q) : \vdash (s \to u)\}$. Suppose $U \cup \{\neg t\}$ is consistent: take a suitable valuation v of $Q \cup R$, and show that the set

$$\{s\} \cup \{q : q \in Q, v(q) = 1\} \cup \{\neg q : q \in Q, v(q) = 0\}$$

is consistent. Deduce that v can be extended to a valuation of $P \cup Q \cup R$ making s true and t false, and obtain a contradiction.

Boo to that, I say. (This is Craig's Interpolation Theorem). There is a detailed constructive proof in *Logic, Induction and Sets* pp 91–92.

(Here is another proof—that i nicked from Sam Buss.

Think of the literals (and let there be k of them) from Q that occur in s, and of the valuations of those k literals which can be extended to valuations that make s true. Let $v_1 \ldots v_n$ be those valuations. Now let u be the disjunction This might ne

$$\bigvee_{1 \leq i \leq n} ({q_1}^{(i)} \wedge {q_2}^{(i)} \ldots {q_k}^{(i)})$$

where ${q_j}^{(i)}$ is q_j or $\neg q_j$ depending on whether or not v_i believes q_j. $s \vdash u$ is immediate. To show $u \vdash t$ notice that a valuation on Q that satisfies u can be extended to a valuation on $P \cup Q$ that satisfies s. Any such valuation also satisfies t since $s \vdash t$. Hence $u \vdash t$ as desired.)

2002

2002 IIA paper 1 Q 7J

`http://www.maths.cam.ac.uk/undergrad/pastpapers/2002/Part_2/index.html`

The only part i'm going to answer is (ii)(b), since the earlier parts are bookwork.

I have to confess that I don't know what the subtext of this question is: usually I know this kind of thing but I'm not omniscient.[1] I'm going to have to fall back on appealing to background knowledge and sniffing around suspicious features. So what do I spot?

The first thing that catches my eye is the use of the symbol '$\Rightarrow$' for this operation on sets. "Might it", one wonders, "be anything to do with implication?" Can we prove the inference

$$\frac{T \subseteq B; \qquad T \subseteq (B \Rightarrow C)}{T \subseteq C} \tag{1}$$

?

You will find it easy to prove this once your suspicions have been aroused.

The second unfamiliar—and therefore eyecatching—part of this question is the condition '$(\forall t \in T)(\exists s \in S)(t \leq s)$' in clause (ii) of the definition. In the literature on quasiorders this is sometimes written '$T \leq^+ S$". I shall use it here, as it saves space: once you spot that this '+' is useful you can prove some trivial but useful facts like:

$$T \leq^+ S \ \wedge \ S \subseteq (B \Rightarrow C) \ \to \ \ T \subseteq (B \Rightarrow C) \tag{2}$$

Now that we are armed with these I think it is safe to approach the question. A work of warning before we jump in: overloading of '$\rightarrow$' for implication and the operation on sets defined in the question: read formulæ carefully! In fact, i think i'll doctor this file by using '$\Rightarrow$' for the set operation. It's obviously got something to do with Heyting algebras, after all.

To show that $B \Rightarrow C$ is R-closed whenever B and C are we have to show that if $S \subseteq (B \Rightarrow C)$ and $\langle S, a\rangle \in R$ then $a \in (B \Rightarrow C)$.

So let $S \subseteq (B \Rightarrow C)$ and a s.t. $\langle S, a\rangle \in R$ be arbitrary. We will show that $a \in (B \Rightarrow C)$.

$S \subseteq B \Rightarrow C$ is
$(\forall a \in S)(\forall b \leq a)(b \in B \to b \in C)$
We have assumed $\langle S, a\rangle \in R$. Then we have to show that $a \in (B \Rightarrow C)$. To do this we have to show

$$(\forall b \leq a)(b \in B \to B \in C)$$

$$(\forall b \leq a)(\exists T \subseteq A)(\langle T, b\rangle \in R \ \wedge \ T \leq^+ S \ \wedge \ (\forall t \in t)(\langle \{b\}, t\rangle \in R)) \tag{3}$$

Now suppose $b \leq a$. By formula 3 we have

$$(\exists T \subseteq A)(\langle T, b\rangle \in R \ \wedge \ T \leq^+ S \ \wedge \ (\forall t \in t)(\langle \{b\}, t\rangle \in R))$$

Assume further that $b \in B$; we want $b \in C$. We know that $b \in T$ and that $T \leq^+ S$ from formula 3; formula 2 tells us that $T \subseteq (B \Rightarrow C)$, and then formula 1 tells us that $T \subseteq C$. But then $b \in C$ as desired. But b was arbitrary. This establishes $(\forall b \leq a)(b \in B \to B \in C)$. This is precisely the condition for a to be in $B \Rightarrow C$.

2006

Paper 4 Question

In the first part they tell you to take your wellfounded relation r and define on it by recursion the obvious rank function to the ordinals. This refines the wellfounded relation to a **prewellorder** (a wellordered partition). That was a bit cryptic, so let me be clearer. You have defined a rank function ρ from the domain of r

[1] I now (2012) realise it's all about how to recover a Heyting algebra from a possible world model. (At least, that's what i tho'rt! PTJ says: "As far as I was concerned, it was actually about constructing the subobject classifier in a topos of sheaves." Well!!–he set the question and he should know)

to the ordinals. This process partitions the domain into pieces $\rho^{-1}``\{\alpha\}$ for each ordinal α. The relation $\{\langle x, y\rangle : \rho(x) < \rho(y)\}$ is wellfounded and contains every ordered pair in r. It's not a wellorder, beco's two things in the same piece are not related to each other, tho' it is what we call a *prewellorder*. AC tells us that everything can wordered, so in particular each piece of the partition can be wellordered, and we use AC to pick such a wellorder for each piece. Then we concatenate the wellorderings

That's the way the examiners expect you to do it. However there is the clever way. Consider the collection of wellorders that are compatible with r. Partially order them by **end-extension**. Then use Zorn. The point is that a union of a chain **under end-extension** of wellorderings is another wellordering. (Look at question 2009-3-16G below where end-extensions reappear.)

2007

2007-3-16G

Observe that if $\gamma \geq \omega \cdot \beta$ then $\beta + \gamma = \gamma$. Contraposing we infer that if γ does *not* "absorb β on the left" then $\neg(\gamma \geq \omega \cdot \beta)$ whence $\gamma < \omega \cdot \beta$. Analogously if β does *not* "absorb γ on the left" then $\beta < \omega \cdot \gamma$. So, if β and γ are *commensurable* (never heard this word used this way, but never mind) we have

$\gamma < \omega \cdot \beta$ and $\beta < \omega \cdot \gamma$.

Now we use the division algorithm to find the largest power of $\omega \leq \beta$.

2007-4-II-16G

We say (for the purposes of this question) that a function $f : A \to \mathcal{P}(A)$ is **recursive** if the relation $\{\langle a, b\rangle : a \in f(b)\}$ is wellfounded. Observe that any binary relation on a set A can be thought of as a map $f : A \to \mathcal{P}(A)$.

(That is the first of the two points made by this question. The other point is that not only is wellfoundedness of a relation R a sufficient condition for definitions given by recursion-over-R to have unque solutions, but it is necessary as well. If every definition given by recursion over R has a unique solution then R is indeed wellfounded. It is disconcerting—even if only slightly—that you are being invited to prove this converse while thinking of binary relations as functions-into-power-sets [as above].)

Suppose $g : \mathcal{P}(B) \to B$. We can attempt to define a function h recursively by:

$$h(a) := g(\{h(a') : a' \in f(a)\}).$$

Clearly we are going to be able to show (by an appeal to the recursion theorem) that this recursion has a unique solution—as long as the relation $\{\langle a, b\rangle : a \in f(b)\}$ is wellfounded. But what about a converse?

Suppose $\{\langle a, b\rangle : a \in f(b)\}$ is not wellfounded. We want to find a B and $g : \mathcal{P}(B) \to B$ such that there is more than one h satisfying

$$(\forall a \in A)(h(a) = g(\{h(a') : a' \in f(a)\})).$$

Let B be a set with at least two members, and b_1 and b_2 be two members of B, and define $g : \mathcal{P}(B) \to B$ by

$g(B') = \texttt{ if } (B' = \emptyset \vee B' = \{b_1\}) \texttt{ then } b_1 \texttt{ else } b_2$

Suppose now that A' is a subset of A with no minimal member under the relation $\{\langle a, b\rangle : a \in f(b)\}$. Notice that both

$h_1(a) := b_1$

and

$h_2(a) := \texttt{if } a \in A' \texttt{ then } b_2 \texttt{ else } b_1$

are solutions to

$$h(a) = g(\{h(a') : a' \in f(a)\}).$$

2008

2008-3-16G

Obviously the rank of $\mathcal{P}(x)$ has to be one greater than the rank of x. The rank function is $\subseteq$-monotone ($u \subseteq v \to \rho(u) \leq \rho(v)$) so the rank of x is maximal among ranks of members of $\mathcal{P}(x)$. So $\rho(\mathcal{P}(x))$ must be $\rho(x) + 1$, so it is successor.

The fun starts when you ask about the ordinals that can be ranks of $\mathcal{P}_{\aleph_0}(x)$ (the set of finite subsets of x) or $\mathcal{P}_{\aleph_1}(x)$ (the set of countable subsets of x). Either of those could be a fair tripos question, i think...

2009

2009-2-16G

Suppose there were an axiomatisation T of the theory of groups all of whose elements have finite order. Add, for each n, the axiom $(\forall x)(x^n = 1 \to x = 1)$. This theory has no models, but all its finite fragments do.

2009-3-16G

Let $x \subseteq y \subseteq$ the ordinals below α. x and y both inherit an ordering. Why do we know that the length of x in the inherited ordering is $\leq$ the length of y? It looks obvious, but we have to be careful. The ordering on ordinals arises from "A is iso to an an initial segment of B" rather than from "A injects into B in an order preserving fashion", and it is the second one that we want here. However these two relations are the same! This was an example sheet question in the year this question was set, so i'll provide a proof:

Remark 1. *A total order is a well order iff every subordering is iso to an initial segment.*

Proof: Let $\langle A, <_A\rangle$ be a total order where every subordering is iso to an initial segment. Consider a one-point subordering. It is iso to an initial segment so $\langle A, <_A\rangle$ has a bottom element. But now every subordering is iso to an initial segment, and so has a least element, so $\langle A, <_A\rangle$ is a well order.

For the other direction, every suborder of a well order is a well order. So if $\langle A, <_A\rangle$ is a well order and $\langle A', <_A\rangle$ a suborder of it, then one of $\langle A, <_A\rangle$ and $\langle A', <_A\rangle$ is iso to an initial segment of the other. (The two might be iso of course). We must establish that $\langle A, <_A\rangle$ cannot be iso to a proper initial segment of $\langle A', <_A\rangle$. But this is easy: such an isomorphism would be a $<_A$-decreasing injective function $A \to A$, so that there will be sets $\{f^n(a) : n \in \mathbb{N}\} \subseteq A$ with no $<_A$-least member, contradicting wellfoundedness of $<_A$. ■

The second paragraph requires care. Observe that, at any rate, the length of the union of the x_n must be at least the sup of the lengths; that much is obvious. It's the other direction that can fail.

The operation that sends each $x \subseteq \alpha$ to an ordinal $\mu(x)$ does its work by sending each member of x to an ordinal; $\mu(x)$ is then the least ordinal not used. A member a of α can appear in lots of x's. **The point of the end-extension condition is that it ensures that a gets sent to the same ordinal whatever x we are considering**; that ordinal is then the ordinal it is given in the computation of $\mu(\bigcup_n x_n)$. To illustrate the importance of the end-extension condition consider the following family of subsets of the ordinals below $\omega + 1$:

$$\{[0, n] \cup \{\omega\} : n < \omega\}$$

so that x_n is the ordinals below n with ω whacked on the end. Each subordering is of finite length, and the lengths are unbounded so the sup of the lengths is ω. However the union of the subsets is the set of ordinals below $\omega+1$ which of course is of length $\omega+1$.

Observe that in this case the ordinal that the point ω gets sent to in the calculation of $\mu(x_n)$ is not constant but depends on n.

It may be worth noting that *any countable linear order type whatever* can be obtained as a union of an ω-chain of finite total orders. Come to think of it that might make a good exercise for next year.

For the final part let x_n be $\mathbb{N} \setminus [0, n]$. Each x_n is a terminal segment of the finite ordinals and has length ω, but their intersection is empty.

2010

2010-2-16G

Show that an ordinal is indecomposible iff it is a power of ω.

One direction is easy: if α is not a power of ω then it is decomposed by applying the division algorithm to the function $\beta \mapsto \omega^\beta$, which give us the largest power of $\omega \leq \alpha$.

The other direction is easy if $\beta + \gamma = \omega^\lambda$ with λ limit. Then β and γ are both $< \omega^\alpha$ with $\alpha < \lambda$. But then

$$\beta + \gamma \leq \omega^\alpha \cdot 2 < \omega^{\alpha+1} < \omega^\lambda$$

We are left with the successor case. So suppose $\beta + \gamma = \omega^{\alpha+1} = \omega^\alpha \cdot \omega$.
Use the division algorithm to obtain $\beta = \omega^\alpha \cdot \beta_1 + \beta_2$ with $\beta_2 < \omega^\alpha$ and
$\gamma = \omega^\alpha \cdot \gamma_1 + \gamma_2$ with $\gamma_2 < \omega^\alpha$.
Then

$$\omega^\alpha \cdot \beta_1 + \beta_2 + \omega^\alpha \cdot \gamma_1 + \gamma_2 = \omega^\alpha \cdot \omega$$

Now $\beta_2 < \omega^\alpha$ so we can simplify

$$\omega^\alpha \cdot \beta_1 + \omega^\alpha \cdot \gamma_1 + \gamma_2 = \omega^\alpha \cdot \omega$$

Clearly $\gamma_2 = 0$ so we simplify again

$$\omega^\alpha \cdot \beta_1 + \omega^\alpha \cdot \gamma_1 = \omega^\alpha \cdot \omega$$

This tells us that $\beta_1 + \gamma_1 = \omega$ so one of them is ω, from which the result follows.

2010-3-16G

I do only the hard part

Suppose x is a transitive set, of rank α. We will show that, for all $\beta < \alpha$, x has a member of rank β. That is to say, $\{\beta \in On : x \text{ has a member of rank } \beta\}$ is a proper initial segment of On. It must be a set beco's it is bounded above by the rank of x.

Observe the following:

(i) If x has a member y of rank $\beta+1$ then y has a member y' of rank β. But x is transitive, so $y' \in x$ and x has a member of rank β.

(ii) If x has a member y of rank λ (where λ is limit) then $(\forall \beta < \lambda)(\exists \gamma < \lambda)(\beta < \gamma \wedge y \text{ has a member of rank } \gamma)$. But x is transitive, so the same goes for x: $(\forall \beta < \lambda)(\exists \gamma < \lambda)(\beta < \gamma \wedge x \text{ has a member of rank } \gamma)$.

(i) and (ii) together mean that the collection of ordinals β such that x does not have a member of rank β is closed under successor (beco's of (i)) and limits (beco's of (ii)), so it must be a terminal segment of On. So its complement must be a proper initial segment of On. But that is what we wanted.

The other thing one can try is to prove by $\in$-induction on x that $TC(x)$ has members of all ranks less than $\rho(x)$. Suppose that, for all $y \in x$, $TC(y)$ has members of all ranks less than $\rho(y)$. $TC(x) = x \cup \bigcup_{y \in x} TC(y)$. Now let α be an ordinal less than the rank of x. Then $\alpha \leq \rho(y)$ for some $y \in x$. If we have equality then y is the thing we are looking for that is in $TC(x)$ and has rank α. If we have $<$ then we have our witness by induction hypothesis.

2011

2011 Paper 2 Q 16H

The first two parts are bookwork. For the third part i'm assuming that each element of the space is a **finite** sum of multiples of basis-element–times–a-real. There are $\aleph_1$ basis elements and $2^{\aleph_0}$ reals, making $\aleph_1 \cdot 2^{\aleph_0}$ such pairs. How many finite sets of such pairs? The cardinality of $\mathcal{P}_{\aleph_0}(X)$ (the set of finite subsets of X) can be quite hard to compute in terms of $|X|$, whereas $|X^{<\omega}|$ (the set of finite sequences-without-repetitions of members of X) is just $|X| + |X|^2 + |X|^3 \ldots$. Evidently $X^{<\omega}$ surjects onto $\mathcal{P}_{\aleph_0}(X)$ and, since any set X of size $\aleph_1 \cdot 2^{\aleph_0}$ admits a total ordering, we can use that total order to uniformly order each finite set and thus inject $\mathcal{P}_{\aleph_0}(X)$ into $X^{<\omega}$. Evidently (whatever X is) and—certainly in this case—each of the summands in $|X^{<\omega}| = |X| + |X|^2 + |X|^3 \ldots$ is $|X|$ (which is to say $\aleph_1 \cdot 2^{\aleph_0}$) so (with the help of Cantor-Bernstein) we arrive at $\aleph_1 \cdot 2^{\aleph_0} \cdot \aleph_0 = \aleph_1 \cdot 2^{\aleph_0}$ such finite sets, and therefore $\aleph_1 \cdot 2^{\aleph_0}$ vectors.

All that was without any use of AC. If we allow ourselves even mere countable choice we can infer $\aleph_1 \leq 2^{\aleph_0}$, so there are $2^{\aleph_0}$ such pairs and $2^{\aleph_0}$ finite sets of such pairs, so there are $2^{\aleph_0}$ vectors.

That seems like a lot of work, and maybe the examiners didn't expect that much detail (you have only about half an hour per question after all) but it can't do you any harm to see it done properly. My guess is that you'd've got the brownie points simply for getting the correct answer and waving your hands artistically.

If an element of the space can be an **arbitrary** sum of multiples of basis-element–times–a-real then the sums are subtly different. There are still $2^{\aleph_0}$ pairs, but now we are interested in subsets of the collection of pairs of sizes up to $\aleph_1$, which gives us $(2^{\aleph_0})^{\aleph_1}$ elements. Now $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_0 \cdot \aleph_1}$. Now $\aleph_1 \leq \aleph_0 \cdot \aleph_1 \leq \aleph_1 \cdot \aleph_1 = \aleph_1$, so $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1}$, which won't simplify any further (in case you were wondering). The continuum hypothesis implies that $2^{\aleph_1} = 2^{2^{\aleph_0}}$ and Luzin's hypothesis [don't ask] is that $2^{\aleph_1} = 2^{\aleph_0}$. Both are consistent with ZFC—tho' of course not jointly!

2011 Paper 3 Q 16H

(i) The theory of dense linear orderings is axiomatisable, and the axioms are what you think they are.

(ii) The theory of countable dense linear orderings is not axiomatisable in the language of posets. For suppose it were, and T were an axiomatisation. We could expand the language of T by adding uncountably many constants, and add to T axioms to say that the constants all point to distinct things. That would be a theory with uncountable models—all of which would be models of T—contradicting the assumption that all models of T are countable. Note that there is a first order theory which is true in each and every countable DLO; the problem is that it has uncountable models as well.

(ii) The theory of uncountable dense linear orderings is axiomatisable *if you are allowed to add uncountably many constants*. (If the axiom of choice holds then every uncountable set has a subset of size $\aleph_1$, so we add $\aleph_1$ constants.) However, that takes you out of the language of posets, so it's not allowed; downward Skolemheim tells us that any theory of uncountable dense linear orders *in the language of posets, which is countable* will have countable models.

(iv) There cannot be a first-order theory T of wellorderings in the language of posets beco's if there were one could add constants $\langle a_i : i \in \mathbb{N} \rangle$ to the language, and axioms $a_i > a_{i+1}$ for each $i \in \mathbb{N}$. The result would be a theory all of whose models are wellorderings (since they are models of T) while not being wellorderings after all beco's of the a_i.

Here is another proof... one that you could conceivably be expected to find given the material you have been lectured, at least in a PTJ year ...

Every substructure of a wellorder is a wellorder. So, by the discussion on p **??**, the first-order theory of wellorders (if there is one) is a universal theory. Now if T is a universal theory then—by part (iv) of the question on p **??**—the union of a $\subseteq$-chain of models of T is another model of T; so a union of a $\subseteq$-chain of wellorderings would have to be a wellordering. But—for example—every terminal segment of $\mathbb{Z}$ is a wellorder and the union of all of them is not. (A union of a chain of wellorderings is a wellordering if it is a chain under end-extension.)

... but that looks quite hard to me.

Paper 4 question

This is mostly bookwork. To prove that every set belongs to a V_α you are clearly going to have to use the foundation axiom somehow, since if foundation fails you can have silly things like Quine atoms that do not belong to any V_α. The usual way to use foundation is to do an $\in$-induction. This is a case in point: you prove by $\in$-induction that every set belongs to a V_α. Suppose this is true for every member y of x. To each such y associate α_y, the least α s.t. $y \in V_\alpha$. We form the set of those α_y and take the sup. Probably worth pointing out that in so doing we are using the axiom scheme of replacement.

2012

2012 paper 3 Q 16H

All bookwork. Part (v) looks scary but it's actually easy. Let α be any ordinal at all. Use the division algorithm on the function $\beta \mapsto \omega_1 \cdot \beta$ to obtain the largest multiple of ω_1 that is less than or equal to α. If that number is α itself then α is a multiple of ω_1; if it isn't then there is a remainder. This remainder is countable and so has cofinality ω (if it is limit) or is successor (in which case there is a cofinal map within the meaning of the act.).

2013

2013-2-16G

Explain what is meant by a chain-complete poset. State the Bourbaki–Witt fixed-point theorem for such posets.

A poset P is called directed if every finite subset of P (including the empty subset) has an upper bound in P; P is called directed-complete if every subset of P which is directed (in the induced ordering) has a least upper bound in P . Show that the set of all chains in an arbitrary poset P , ordered by inclusion, is directed-complete.

Given a poset P, let $[P \to P]$ denote the set of all order-preserving maps $P \to P$, ordered pointwise (i.e. $f \leq g$ if and only if $f(x) \leq g(x)$ for all x). Show that $[P \to P]$ is directed-complete if P is.

Now suppose P is directed-complete, and that $f : P \to P$ is order-preserving and inflationary. Show that there is a unique smallest set $C \subseteq P \to P$ satisfying

(a) $f \in C$;
(b) C is closed under composition (i.e. $g, h \in C \to g \cdot h \in C$); and
(c) C is closed under joins of directed subsets.

Show that

(i) all maps in C are inflationary;
(ii) C is directed;

(iii) *if $g = \bigvee C$, then all values of g are fixed points of f ;*
(iv) *for every $x \in P$, there exists $y \in P$ with $x \leq y = f(y)$.*

Discussion

'Directed' is an important notion. Cast your mind back to the proof that the class of models for a $\forall^*\exists^*$ ("inductive") theory is closed under unions of chains. Look carefully at that proof. You will notice that you didn't really need chains: if T is inductive then a union of a directed set of models of T is another model of T.

We have to show that the chains of an arbitrary poset form a directed-complete poset. Not a *directed* poset! That's clearly not going to happen unless the poset is a toset! If i have a directed family F of chains in a poset $\langle P, \leq_P \rangle$ then that family has a sup, namely $\bigcup F$. We have to check that $\bigcup F$ is a chain. Suppose x and y are both elements of $\bigcup F$. They belong to two chains c_x and c_y, but—by directedness—there is now a chain $\supseteq c_x \cup c_y$ to which both x and y belong. So x and y are $\leq_P$-comparable.

The set C we want is defined inductively: it's the intersection of the collection $\mathcal{C}$ of all the subsets of $P \to P$ satisfying conditions (a–c). $\mathcal{C}$ is a subset of $\mathcal{P}(P \to P)$ so it is a set, and it is nonempty, since $P \to P$ itself is such a subset. We need also to check that conditions (a–c) are *intersection-closed* (that is to say, an arbitrary intersection of sets satisfying (a–c) also satisfies (a–c)) but that is evident, beco's (a–c) are *closedness* properties. So the intersection $\bigcap \mathcal{C}$ of all such subsets satisfies (a–c) and is the set C that we want.

Clearly C is going to be the family of all iterates of f.

(i) Since C has an inductive definition it supports induction. So we should expect to be able to prove (i) by induction. If the set of all inflationary maps $P \to P$ is a member of $\mathcal{C}$ then we are home and hosed. Just check that it satisfies (a–c). It satisfies (a) beco's f is inflationary, it satisfies (b) beco's a composition of two inflationary functions is inflationary, and it satisfies (c), since the [pointwise] sup of a lot of inflationary maps is inflationary.

(ii) Suppose $D \subseteq [P \to P]$ is directed. Define $h : P \to P$ by $h(x) = \bigvee\{g(x) : g \in D\}$. For this definition to succeed we need $\{g(x) : g \in D\}$ to be directed. So let U be a finite subset of $\{g(x) : g \in D\}$. We want U to have an upper bound. Now U is $\{g(x) : g \in D'\}$ for some finite $D' \subseteq D$. But this D' has an upper bound by the assumption that D is directed.

(iii) We have to show that every value of $\bigvee C$ is a fixed point for f. For $p \in P$, $\bigvee C(p)$ is the sup of $f(p)$ for all $f \in C$. We must compute $f(\bigvee C(p))$ for arbitrary $p \in P$ and hope that it evaluates to $\bigvee C(p)$.

What is $\bigvee C$ applied to p? It's $\bigvee$ of the set $\{g(p) : g \in C\}$ of iterates of f applied to p. Now everything in C is inflationary (that was (i)) so $\bigvee\{g(p) : g \in C\} \leq_P \bigvee\{f(g(p)) : g \in C\}$. The other direction, namely $\bigvee\{f(g(p)) : g \in C\} \leq_P \bigvee\{g(p) : g \in C\}$, we get beco's $\{f(g(p)) : g \in C\} \subseteq \{g(p) : g \in C\}$, whence

$$\bigvee\{g(p) : g \in C\} = \bigvee\{f(g(p)) : g \in C\} \tag{A}$$

k this works At this point we have to use the fact (unused so far, i think) that f is order-preserving, to infer

$$f(\bigvee\{g(p) : g \in C\}) \leq \bigvee\{f(g(p)) : g \in C\}.$$

Substituting using the equation (A) we obtain

$$f(\bigvee\{g(p) : g \in C\}) \leq \bigvee\{g(p) : g \in C\}.$$

The equality in the other direction

$$\bigvee\{g(p) : g \in C\} \leq f(\bigvee\{g(p) : g \in C\})$$

follows from f being inflationary, so we infer

$$\bigvee\{g(p) : g \in C\} = f(\bigvee\{g(p) : g \in C\})$$

so $\bigvee C(p)$ is a fixed point for f as desired.

(iv) This challenge reminds me very strongly of the fact that one can infer "In every chain complete poset every element has a maximal element above it" from "Every chain-complete poset has a maximal element". That hint was enough for me to know what to do, and it should work for you too. So you run the preceding constructions on $P \uparrow x$, the substructure of $\langle P, \leq_P \rangle$ that is its restriction to the elements that are $\geq_P x$: replace 'f' by '$f \restriction (P \uparrow x)$'.

2013-3-16G

Suppose P , Q and R are pairwise disjoint sets of primitive propositions, and let $\phi \in L(P \cup Q)$ and $\psi \in L(Q \cup R)$ be propositional formulae such that $\phi \to \psi$ is a theorem of the propositional calculus. Consider the set $X = \{\chi \in L(Q) : \phi \vdash \chi\}$. Show that $X \cup \{\neg\psi\}$ is inconsistent, and deduce that there exists $\chi \in L(Q)$ such that both $\phi \to \chi$ and $\chi \to \psi$ are theorems. [Hint: assuming $X \cup \{\neg\psi\}$ is consistent, take a suitable valuation v of $Q \cup R$ and show that $\{\phi\} \cup \{q \in Q | v(q) = 1\} \cup \{\neg q : q \in Q, v(q) = 0\}$ is inconsistent. The Deduction Theorem may be assumed without proof.]

Discussion

Take the hint. If $X \cup \{\neg\psi\}$ is *not* inconsistent, then there is a valuation (call it 'v') making X true and ψ false. This valuation v is defined on letters in $Q \cup R$. The plan now is to extend v to a valuation v' defined on all the letters in P as well, so that v' makes ϕ true too. Now every valuation making ϕ true makes ψ true, by assumption; so $X \cup \{\neg\psi\}$ will be inconsistent as desired. So how do we extend v to a v' making ϕ true? ϕ has propositional letters in it that come from P and from Q. Doctor ϕ by setting to $\top$ every q-letter that v makes true, and setting to $\bot$ every q-letter that v makes false, and then simplify (using rewrites such as $\alpha \wedge \bot \mapsto \bot$, $\alpha \vee \top \mapsto \top$, that sort of thing) obtaining a formula in $L(P)$. If this formula hasn't simplified to $\bot$ then it has some p-letters in it and we can find a valuation making it true, and then we are happy. But what if we can't? Can that happen? That would mean that the stuff in X prevented ϕ being true. There may be stuff that prevents ϕ being true, but the stuff in X consists of **consequences** of ϕ. So ϕ is prevented from being true by some of its consequences. But that must mean that ϕ is the negation of a truth-table tautology and that case isn't interesting.

2013-1-16G

For the last part: α has to be a power of ω.... "large ordinals absorb small ordinals on the left". This absorption principle is behind the construction that shows $(\aleph_\alpha)^2 = \aleph_\alpha$.

2013-4-16G

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent in the presence of the other axioms of ZF set theory. [You may assume the existence of transitive closures.]

Given a model $\langle V, \in \rangle$ of ZF, indicate briefly how one may modify the relation $\in$ so that the resulting structure $\langle V, \in' \rangle$ fails to satisfy Foundation, but satisfies all the other axioms of ZF.

[You need not verify that all the other axioms hold in $\langle V, \in' \rangle$.]

Discussion

I omit the first part: it's bookwork.

For the second part copy out from your notes your definition of the cumulative hierarchy, for it is the R you want. Indeed that is the whole point of the definition of the cumulative hierarchy, to give a slick proof of the relative consistency of the axiom of foundation.

For the third part let τ be the transposition $(\emptyset, \{\emptyset\})$ that swaps $\emptyset$ with $\{\emptyset\}$ and fixes everything else. The relation $\in'$ that you want is $x \in' y$ iff $x \in \tau(y)$.

[You might like to do some hand-calculation to check that, in $\langle V, \in' \rangle$, the old empty set has morphed into something identical to its own singleton]

2014

Paper 2 question

Ordinals less than a power of ω are closed under $+$; ordinals less than ω^{ω^α} are closed under $\times$; ordinals below an ϵ-number are closed under exponentiation.

Discussion

Let's prove that the ordinals less than a power of ω are closed under addition. Suppose not, and let $\omega^\alpha = \beta + \gamma$. Then both β and γ are below $\omega^\delta \cdot n$ for some $\delta < \alpha$. But then $\omega^\alpha = \beta + \gamma \leq \omega^\delta \cdot n + \omega^\delta \cdot n \leq \omega^\delta \cdot n \cdot 2 < \omega^{\delta+1} \leq \omega^\alpha$.

Closure under multiplication

Suppose α and β are both less than ω^{ω^γ}. Then there is a $\delta < \omega^\gamma$ such that both α and β are below ω^δ. But then $\alpha \cdot \beta$ is below $\omega^\delta \cdot \omega^\delta = \omega^{\delta+\delta}$ and now we appeal to the fact that the ordinals below ω^γ are closed under addition!

Closure under exponentiation

Suppose $\omega^\epsilon = \epsilon$, and $\alpha, \beta < \epsilon$. Then $\alpha^\beta \leq (\omega^\alpha)^\beta =^{(1)} \omega^{\alpha\cdot\beta}$.

Now $\epsilon = \omega^{\omega^\epsilon}$ so ordinals below ϵ are closed under multiplication (see above) giving $\alpha \cdot \beta < \epsilon$, whence $\omega^{\alpha\cdot\beta} <^{(2)} \omega^\epsilon = \epsilon$.

> (1) It can't do any harm to think about why this equation holds. You could prove it from the synthetic definition of exponentiation (which you were presumably not lectured) or you could prove it by fixing α ("universal generalisation") and doing an induction on β.
>
> (2) holds beco's $\gamma \mapsto \omega^\gamma$ is normal.

2015

Paper 1, Section II 13I

State and prove the Completeness Theorem for Propositional Logic.

[You do not need to give definitions of the various terms involved. You may assume the Deduction Theorem, provided that you state it precisely.]

State the Compactness Theorem and the Decidability Theorem, and deduce them from the Completeness Theorem.

Let S consist of the propositions $p_{n+1} \to p_n$ for $n = 1, 2, 3 \ldots$. Does S prove p_1? Justify your answer. [Here p_1, p_2, p_3 … are primitive propositions.]

Bookwork

Paper 2, Section II 13I

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent. Give the inductive definitions of ordinal multiplication and ordinal exponentiation.

(b) Answer, with brief justification, the following:

(i) For ordinals α, β and γ with $\alpha < \beta$, must we have $\alpha+\gamma < \beta+\gamma$? Must we have $\gamma+\alpha < \gamma+\beta$?

(ii) For ordinals α and β with $\alpha < \beta$ must we have $\alpha^\omega < \beta^\omega$?

(iii) Is there an ordinal $\alpha > 1$ such that $\alpha^\omega = \alpha$?

(iv) Show that $\omega^{\omega_1} = \omega_1$. Is ω_1 the least ordinal α such that $\omega^\alpha = \alpha$?

[You may use standard facts about ordinal arithmetic.]

This is mostly bookwork. However there is a subtlety to part (iv). The second part is obviously 'no': think about $\epsilon_0 = \sup\{\omega^n : n \in \mathbb{N}\}$. But the first part is slightly trickier.

ω^{ω_1} is of course $\sup\{\omega^\alpha : \alpha < \omega_1\}$ and we want this to be no more than ω_1. It's clearly no *less* than ω_1 because $\alpha \leq \omega^\alpha$ always and the α we are summing over are unbounded below ω_1; for it to be no *greater* than ω_1 we need ω^α to be countable whenever α is. If we try to do this by induction on α we have no problem at successor ordinals of course, co's we're just multiplying by ω, but at limit stages we are liable to find ourselves appealing to the principle that a union of countably many countable sets is countable. Why is ω^λ countable? Well, if λ is countable limit it is $\sup\langle\lambda_n : n \in \mathbb{N}\rangle$ for some suitable ("fundamental") sequence $\langle\lambda_n : n \in \mathbb{N}\rangle$ and each (von and each (von Neumann ordinal) ω^{λ_n} is [a] countable [set] by induction hypothesis, so the (von Neumann ordinal) ω^λ is [a] countable [set] by countable-union-of-countable-sets-is-countable. This use of countable choice seems to be unavoidable.

However, if we use the synthetic definition of ordinal exponentiation we obtain a set (the set of all those functions from a wellordering of length α to $\mathbb{N}$ that take the value 0 at all but finitely many arguments) equipped with a natural wellordering that is of order type ω^α. This set can be shown to be countable, as follows. Each such function can be thought of as a finite set of ordered pairs of ordinals-below-α paired with naturals. There are countably many such pairs and therefore only countably many finite sets of such pairs.

You probably don't care about such things (and i bet the examiners didn't) but i am not the Old Man of Thermopylæ who never did anything properlæ. As for whether or not you can prove the analogous result for the next operation after exponentiation (namely that the countable ordinals are closed under it) without using countable choice. . . well, i have to confess that i'd never thought about it up to now. I have an awful feeling that the answer might be 'no'.

Paper 3, Section II 13I

(i) State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.]

Where in your proof have you made use of the Axiom of Choice?

(ii) Let $<$ be a partial ordering on a set X. Prove carefully that $<$ may be extended to a total ordering of X. What does it mean to say that $<$ is well-founded? If $<$ has an extension that is a well-ordering, must $<$ be well-founded? If $<$ is well-founded, must every total ordering extending it be a well-ordering? Justify your answers.

Discussion

The first part is bookwork. However they want you to say (and *I* want to hear you say it, too) that The Axiom of Choice says that every set X has a choice function blah, and that in this case the set X that has a choice function is that set that contains, for each p in your poset, the set $\{p' \in P : p \leq p'\}$.

If you discard ordered pairs from a wellfounded relation the result is still a wellfounded relation. (If you remove ordered pairs you perforce remove descending sequences so you are certainly not going to *add* any new infinite descending sequences!) So: if $<$ has an extension that is a well-ordering, $<$ must be well-founded. The converse is clearly not going to be true. Consider the empty relation on $\mathbb{Q}$. It's wellfounded, but you can extend it to the usual order on $\mathbb{Q}$, and that is not wellfounded.

Paper 4, Section II 13I

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [You may assume the existence of transitive closures.]

Explain briefly how the Principle of $\in$-Induction implies that every set is a member of some V_α.

Find the ranks of the following sets:

(i) $\{\omega+1, \omega+2, \omega+3\}$;

(ii) the Cartesian product $\omega \times \omega$;

(iii) the set of all functions from ω to ω^2. [You may assume standard properties of rank.]

Discussion

The first part is bookwork. For the second we'll assume that the ordinals are von Neumann ordinals and that ordered pairs are Wiener-Kuratowski, and that will give the answers

(i) $\omega + 4$ (since the rank of the von Neumann ordinal α is α);

(ii) ω, because every member of $\omega \times \omega$ is an ordered pair of two things of finite rank, and W-K pairs lift rank by 2... and

(iii) $\omega^2 + 1$, beco's every function $\omega \to \omega^2$ is a subset of $\omega \times \omega^2$ and is therefore of rank at least ω (since its arguments ore of unbounded finite rank) and possibly as much as ω^2 (since its values can have any rank below ω^2) by analogy with (ii).