Computer Science Tripos 2015 Paper 2 Question 7 part (c)
A Discussion Answer

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The Question

Let U be a set and let $F : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(U)$ be a function such that, for all $i, i', j, j' \in \mathbb{N}$, if $i \leq i'$ and $j \leq j'$, then $F(i,j) \subseteq F(i'j')$.

Prove that

$$\bigcup_{i \in \mathbb{N}} (\bigcup_{j \in \mathbb{N}} F(i,j)) = \bigcup_{k \in \mathbb{N}} F(k,k) \tag{1}$$

Discussion

The condition on F can be snappily expressed (in terminology that you might one day need) by saying that "F is an order-preserving function from the pointwise product $\langle \mathbb{N}, \leq_{\mathbb{N}} \rangle \times \langle \mathbb{N}, \leq_{\mathbb{N}} \rangle$ to $\langle \mathcal{P}(U), \subseteq \rangle$".

The first thing to say about this question is: `DON'T PANIC`. Read it through very carefully to see what it says. Then think about whether or not you believe it. Thirdly think about *why* you believe it. Then—*finally*—you can start thinking about what to write down.

The thing on the LHS of (1) contains everything that F will ever put into $\mathcal{P}(U)$. The thing on the RHS of (1) contains everything that, for some k, gets put in by stage $\langle k, k \rangle$. Now anything that gets put in at all gets put in at stage $\langle i, j \rangle$ for some $i, j \in \mathbb{N}$. But F is order-preserving, so anything that gets put in at stage $\langle i, j \rangle$ is still there at stage $\langle max(i,j), max(i,j) \rangle$ so there is the k you want: $k = max(i,j)$.

Now to write this out slightly more formally.

How do you prove equations between two sets? (This is a rhetorical question, beco's you know the answer!!) You appeal to the principle of extensionality for sets: two sets are identical iff they have the same members. So when you want to prove $A = B$ you argue by **universal generalisation**: let $x \in A$ be arbitrary. You then rewrite '$x \in A$' by a chain of biconditionals until you reach

$x \in B$, at which point you have proved $x \in A \longleftrightarrow x \in B$ and then you appeal to universal generalisation to conclude $(\forall x)(x \in A \longleftrightarrow x \in B)$ which—by extensionality (since A and B are sets)—is equivalent to $A = B$.

Well, that's not the *only* way of doing it. Sometimes it may be more convenient ["Plan B"] to prove $A \subseteq B$ and $B \subseteq A$. I suspect that that will turn out to be the favoured option in this case.

So let's go!

Actually i think Plan B is the way to go.

So suppose x is a member of the LHS. This is

$$x \in \bigcup_{i \in \mathbb{N}} (\bigcup_{j \in \mathbb{N}} F(i,j))$$

which is equivalent to

$$(\exists i \in \mathbb{N})(x \in \bigcup_{j \in \mathbb{N}} F(i,j))$$

which is equivalent to

$$(\exists i \in \mathbb{N})(\exists j \in \mathbb{N})(x \in F(i,j)) \tag{2}$$

And by the same token we can process the RHS of (1) into

$$(\exists k \in \mathbb{N})(x \in F(k,k)) \tag{3}$$

Evidently (3) implies (2), so it's the other direction we have to worry about. This is where we have to use the fact that F is order-preserving. (We haven't had to use it so far!) So let's go back to (2). Suppose we have a pair i, j that witnesses the truth of (2). Since F is order-preserving any pair i', j' with $i \leq i'$ and $j \leq j'$ will also witness the truth of (2). So let k be something $\geq$ both i and j. Then the pair k, k witnesses the truth of (2). But this fact is precisely the content of (3).

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