tf's talk at NF-in-The-Bay-Area

Thomas Forster

June 6, 2008

Contents

The background to this meeting is my annual migration to New Zealand: I estivate there every northern summer, and—like any self-respecting migrating animal—I stop *en route* to refresh myself and decrease my chances of deep vein thrombosis. Since the Bay Area contains a number of people who have an interest in NF and know—about a number of central topics in Mathematical Logic—rather more than I do, it seemed sensible to me to try and grab them so I could pick their brains. I am very grateful to you all for coming, and to Wes Phoa for funding this meeting, and Sergei for organising it.

Since my original motivation was to pick your brains, you will not be surprised to find that I am presenting new results only to the extent that those new results motivate the questions I am asking you. My presentation will be in the style of what Mirna Dzamonja used to organise under the title ‘Problem Sessions’. A problem session presentation would be along the lines: here is a problem: this is why i find it interesting; here is an obvious way of tackling it that doesn’t work for this interesting reason; i’ve tried this cute trick but it hasn’t worked ...

I’ve hived off a certain amount of material into appendices, in order to focus on the problems not the results.

Terminology: KF is the system of Kaye-Forster 1992: MacLane restricted to stratified Δ_0 separation.

1 Questions about TNT

I suppose it is now too late to say that the theory of negative types (with types ordered like the integers) should be called 'TZT' so that we can leave 'TNT' to denote the typed theory of sets whose types are organised like the *negative* integers?

That's a pity, because Marcel Crabbé has an interesting question about this second theory: is it the case that every model of it admits an upward extension?

Natural questions.

1. Can every model of TST (with T_0 infinite) be extended downwards?
2. Can every model of TNT be extended upwards into a model of TNT?
3. Are there ω-standard models of TZT?
4. Are there symmetric models of TZT?
5. Are there ambiguous models of TZT?
6. Are there standard models of TZT?
7. Does TZT decide all $\forall^*\exists^*$ sentences?

There are no very-nice models (standard or countably complete) and all the moderately-nice models (3,4 and 5) contradict choice, but the failures of choice seem to be unrelated! It would be nice to know why.

Holmes has pointed out to me that any model of TZT that has V_ω as a member is ω-standard.

It's an old result that TZT has no standard (second-order) models: that is to say, no models in which every subset of T_n is coded by an element of T_{n+1}. Indeed it doesn't even have any models in which every *countable* subset of T_n is coded by an element of T_{n+1}. This underlines just how bizarre this theory is.[1]

[1]Notice that this doesn't rely on AC_ω to ensure that total orders that are not worders must have descending ω-sequences: the descending ω-sequence we need can be constructed as $\Omega > T\Omega > T^2\Omega \ldots$.

What's the difference between countably complete and 3? Subtle but important. We are considering structures for the language of set theory (typed or untyped, it makes no difference). If $\mathcal{M}$ is **ω-standard**, it means that every subclass of $\mathbb{N}^{(\mathcal{M})}$ (= the natural-numbers-from-$\mathcal{M}$) is a set of $\mathcal{M}$; if $\mathcal{M}$ is **countably complete** it means that every surjective image $f``(\mathbb{N}^{(\mathcal{M})})$ that is a subset of $\mathcal{M}$ is a set of $\mathcal{M}$. This second condition is clearly likely to be stronger.

However TST is well-behaved in other ways. For each concrete n there is a sentence ϕ_n such that any model $\mathcal{M}$ of TST has an initial extension with n new types added on the bottom iff $\mathcal{M} \models \phi_n$. And there is even a sentence ϕ_∞ such that any model $\mathcal{M}$ of TST has arbitrarily long initial extension iff $\mathcal{M} \models \phi_\infty$.

Symmetric Models

It's natural to wonder if TZT can have a countable model where every element is the denotation of a closed term. Every element denoted by a closed term is symmetric, and this is an easier notion to consier.

It's not known if TZT has any symmetric models. Symmetric models feel very much like Fraenkel-Mostowski models and choice fails very badly in them.

If x is n-symmetric then $\bigcup^n x = \emptyset \vee \bigcup^n x = V$. Therefore any model of TZT in which every set is symmetric believes **.

$$\text{For every } x \text{ there is an } n \text{ such that } \bigcup^n x = \emptyset \vee \bigcup^n x = V. \qquad **$$

It is not hard to use Omitting Types to construct models of TZT satisfying **.

Let us say a set x at level k is **almost-n-symmetric** if the group of permutations of level $k-n$ that fix x extends the symmetric group on a cofinite subset of that level (or, equivalently, extends the pointwise stabiliser of a finite set). A set is **almost-symmetric** if it is almost-n-symmetric for some n. The almost-symmetric sets seem to have all the pathologies of symmetric sets: for example there is no almost-symmetric choice function on the set of all pairs. A word of warning: almost-n-symmetric does not seem to imply almost-k-symmetric for $k > n$. After all, every finite set is almost-1-symmetric.

I mention this because it may well be easier to find models of TZT where every set is almost-symmetric than it is to find models where every set is symmetric, and it might give us a way in.

Models satisfying $\in$-determinacy

In the "$\in$ game" G_x each player picks a member of the other player's previous choice, `I` starting. Any player attempting to pick a member of the empty set loses. If x is n-symmetric then one of the two players has a strategy to win in at most $n+1$ moves. By using omitting types we can find models of TZT satisfying $\in$-determinacy

Finally: is there a connection with Yablo's paradox?

Questions concerning NF and related (untyped) theories

1.1 Universal-Existential

I have a long-standing conjecture that every universal-existential formula of $\mathcal{L}(TZT)$ is decided by TZT, and that every universal-existential formula ϕ of $\mathcal{L}(\in)$ that is consistent with NF is true in some (Rieger-Bernays) permutation model in the strong sense that every model of NF thinks it contains a permutation that makes ϕ true. Progress with the second half of this conjecture awaits an understanding of the structure of the family of permutation models. This seems hard. Well, either it's very hard or Esser and I were looking at it the wrong way—which is entirely possible. Esser was in Cambridge in 2005 and we spent a lot of time on it and got nowhere.

I suspect that it might also be worth in this connection investigating whether or not TZT decides every universal-existential sentence where the two quantifiers are read as "for all but finitely many ..." and "there are infinitely many ...". My guess is that this question is easier. In fact i wouldn't be surprised if *every* model of TZT satisfied complete ambiguity for this language.

The best i can do is:

> **Every countable model of TST is a direct limit of copies of the canonical model**[2]

See appendix p. 13

2 Stratified $\in$-induction

This comes in two versions, with and without parameters.

[2]The model with T_0 empty.

SI: stratified induction. This is the scheme of all **stratified** instances of

$$\frac{\forall x((\forall y \in x)\phi(y) \to \phi(x))}{\forall x \phi(x)}$$

WSI: Weakly stratified induction is the scheme of all **weakly stratified** instances of

$$\frac{\forall x((\forall y \in x)\phi(y) \to \phi(x))}{\forall x \phi(x)}$$

This is the same as requiring ϕ to be stratified while allowing it to contain parameters.

It is easy to check that if no member of x is the universal set then neither is x. This is a very significant proof, because it is a theorem of first-order logic, not of set theory, so it has a cut-free proof. It must also (for model-theoretic reasons) have a purely stratified proof. However it has no stratified cut-free proof.

Either way the proof uses no parameters, so if we are allowed stratified parameter-free $\in$-induction we infer that there is no universal set. I have been unable to find any other consequence of stratified parameter-free $\in$-induction, so I am wondering if there might be a converse.

> **Question: Does stratified parameter-free $\in$-induction hold if there is no universal set?**

(The version with parameters is not interestingly weak, because stratified $\in$-induction implies Γ $\in$-induction for any class Γ for which we have comprehension. The stratification constraint sort-of disappears; it's not interesting.)

3 Constructive NF

The Kuratowski-finite ("Kfinite") sets are defined inductively by: $\emptyset$ is kfinite; if x is kfinite then so is $x \cup \{y\}$. Constructive NF proves that not every set is Kuratowski-finite. If every set were Kuratowski-finite then Ω, the algebra of truth-values would be Kfinite, which would imply excluded middle, and enable us to execute Specker's proof of infinity. However, the proof that **if** not-every-set-is-Kfinite **then** there-is-an-implementation-of-arithmetic is strongly classical, so we cannot deduce that there is an interpretation of arithmetic in constructive NF. Might constructive NF be very weak?

4 The Baltimore Model and “Every set is the same size as a set of singletons”

IO is the assertion “Every set is the same size as a set of singletons” $\exists NO$ is the assertion that there is a set NO of wellorderings such that every wellordering is isomorphic to a membernof NO.

The point about IO is that there are stratified fragments T of NF such that we can interpret ZF in T + IO. So IO is strong. It seems natural to expect that KF + not-IO is consistent. If it were, its models might give us a pointer to what models of KF + $\exists NO$ might be like, and eventually a pointer to models of NF.

The rôle played by IO in these interpretations is the way it enables us to make disjoint copies of things—at least if we have Quine pairs. If X and Y are two carrier sets for two structures, then there are sets $\iota``X'$ and $\iota``Y'$ that are the same size as X and Y respectively. But then $\iota``X' \times \{X'\}$ and $\iota``Y' \times \{Y'\}$ are isomorphic copies of X and Y respectively and are disjoint. In particular this enables us to show that the BFEXTs model the power set axiom.

Explain the interpretation of ZF in strZF + IO. Then $\exists NO$

Nathan Bowler has recently proved that HS believes IO, but this relies on AC holding in the original model.

Miscellaneous other open problems

1. Holmes and Solovay have—independently, as I understand it—devised models of TST in which every object of type > 0 has a unique description using objects of type 0 as parameters, what I call a **term model**. Crabbé asks if there is a term model for the theory of a terminal segment of a model of TZT. This is an axiomatisable theory: it is TST plus “from ϕ^+ infer ϕ”. Every model of this theory is elementarily equivalent to a terminal segment of a model of TZT. It's not clear whether or not the Holmes/Solovay construction of a term model for TST would work for this theory nor whether—even if it did—one could get a term model of TZT out of it.

2. When $\mathcal{M} \models TST$ and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then $\mathcal{M}^{\mathcal{U}}$ is a model of TST as well. The types of this model are indexed by the naturals of the nonstandard model $\mathbb{N}^{\mathcal{U}}$ of arithmetic. Naturally it consists of one model of TST plus a family $\mathcal{F}$ of uncountably many disjoint models

of TZT. Which complete extensions of TZT find models in this family $\mathcal{F}$?

3. It would be nice to have methods of making models of TZT other than by compactness. I have a suggestion on this score.

   Let $\mathcal{M}$ be the canonical model for TST: the structure with empty bottom type. From this we obtain $\mathcal{M}_n$ for each concrete n where $\mathcal{M}_n$ is exactly the same structure but relabelled so as to have types extending down to $-n$. Let i_n be a $\mathcal{L}_{TZT}$-map from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$. Then we have a direct limit, which is a structure for $\mathcal{L}_{TZT}$. There are lots of ways of choosing i_n of course (not all of them sensible) but we get a huge family of direct limits. Some of them will be models for TZT. The family has an obvious topological structure. Is this structure any use?

4. Can we have a model of TZT which admits a function f such that $(\forall x)(f(x) = \mathcal{P}(f(\bigcup x)))$? Can we have a model of TZT in which the graph of this function is locally a set? In a nice model like that of item 11 there is such a function f: its values are V_ns and V. However its graph is not a set because if it were then $\{x : |f(x)| < \aleph_0\}$ would be a set, and this is V_ω which is not a set in nice models.

5. Every model of TST admits an antimorphism, so there are lots of models of TZT with antimorphisms. Can there be models of TZT with antimorphisms whose graphs are locally sets?

6. If V exists every sentence is Δ_2. This is because any formula ϕ is equivalent to both

   $$(\exists x)(\forall y)(y \in x \wedge \phi^x)$$

   and

   $$(\forall x)(\exists y)(y \notin x \vee \phi^x)$$

   where ϕ^x is obtained from ϕ by restricting all quantifiers to x.

   Is the converse true?

5 Appendices

5.1 Stratified induction

Proposition 1. *$KF \vdash$ if every set is the same size as a set of singletons then SI holds.*

Proof:

What we will need is that every set should be the size of a set-of-singletons^n for each n, but this follows easily in *KF* from the assumption of the theorem.

We note *en passant* that if every set is the same size as a set of singletons then the universe is not a set.

The idea of this proof is that if $(\forall y \in x)(\phi(y)) \to \phi(x)$ is provable where x is of type n in ϕ then a simple use of Boffa's permutation lemma shows that every singleton^n is ϕ. (This is the step that doesn't work for WSI). Then every set of singletons^n is ϕ too. But IO implies that every set is n-equivalent to such a set.

Let X be an arbitrary set and suppose that there is a set Y and a permutation π of V mapping $\bigcup^n X$ onto $\iota^n``Y$. Every member of $\iota^n``Y$ is ϕ because every singleton^n is ϕ and $(\forall x)((\forall y \in x\ \phi(y)) \to \phi(x))$. By Boffa's permutation lemma we have

$$(\forall z)(\phi(z) \longleftrightarrow \phi((j^n`\pi)`z)$$

so in particular we have

$$(\phi(X) \longleftrightarrow \phi((j^n`\pi)`X).$$

Now everything in $\bigcup^n((j^n`\pi)`X)$ is in $\iota^n``Y$ and so is ϕ, so everything in $(j^n`\pi)`X$ and $(j^n`\pi)`X$ itself will be ϕ by $\in$-induction.

All we need to do now is to check that there is such a π and such a Y. We have assumed that every set is the same size as a set of singletons, so there is certainly Y such that X is the same size as $\iota^n``Y$. What we need to show is that there is actually a permutation of V mapping one onto the other. We need the following theoremelette.

Lemma 2. *($KF + \neg\exists V$)*

If A and B are two sets the same size then there is a permutation π of V, moving only a set of things, such that $\pi``A = B$.

Proof:

This seems to have affinities with theorem **??** which says that whenever $\alpha + \beta = \alpha + \gamma$ there are α', β' and γ' such that $\beta = \beta' + \delta$, $\gamma = \gamma' + \delta$ and $\alpha = \alpha' + \aleph_0 \cdot (\beta' + \gamma')$. (The proof can be carried out in *KF*.)

However it seems simplest to reason as follows.

1. If $A \setminus B$ and $B \setminus A$ are the same size, the bijection between them is the permutation we need.

2. If one is bigger than the other—suppose without loss of generality that $A \setminus B$ is bigger than $B \setminus A$—then there is a proper subset $X \subset A \setminus B$ the same size as $B \setminus A$. Then A is the same size as $(A \cap B) \cup X$, which is a proper subset of A, so A is dedekind-infinite.

3. If $A \setminus B$ and $B \setminus A$ are incomparable in size then the axiom of infinity holds, for the following reasons. Consider the set of those subsets of $A \setminus B$ that are the same size as a member of $B \setminus A$. This set has no $\subseteq$-maximal member, and so $A \setminus B$ has finite subsets of unbounded size.

In cases (2) and (3) we can assume the axiom of infinity and can argue as below:

Obviously what we have to do is extend $A \cup B$ to something of the right size by adding $\aleph_0$ copies of $A \setminus B$ and $\aleph_0$ copies of $B \setminus A$. As long as there is no universal set we can be sure that there is something not in the first few levels of $\langle\langle A \cup B \rangle\rangle$. Pick one such object a and let $\mathbb{N}$ be the natural numbers relative to $\mathcal{M}_a$. Then $(A \setminus B) \times \mathbb{N}$ and $(B \setminus A) \times \mathbb{N}$ are going to be disjoint from each other and from $A \cup B$. Furthermore $(A \setminus B) \times \{n\}$ will be the same size as $A \setminus B$ because we can have local type-level pairs.

The set we are after is therefore $((A \setminus B) \times \mathbb{N}) \cup (A \cup B) \cup ((B \setminus A) \times \mathbb{N})$.

The map π that we want is defined as follows

1. when $x \in A \setminus B$ $\pi(\langle x, n+1 \rangle) = \langle x, n \rangle$;

2. when $x \in A \setminus B$ $\pi(\langle x, 0 \rangle) = x$;

3. when $x \in A \cup B$ $\pi(x) = f(x)$;

4. when $x \in B \setminus A$ $\pi(x) = \langle x, 0 \rangle)$;

5. when $x \in B \setminus A$ $\pi(\langle x, n \rangle) = \langle x, n+1 \rangle$ when $n > 0$;

(Notice that we do not need to treat separately the case where one of $A \setminus B$ and $B \setminus A$ is empty!)

■

Next we need to know that this π gives rise to a setlike permutation of the universe. But it is easy to show that every permutation of a set extends to a setlike permutation of V, because for any z, $(j^n`\pi)`z$ is definable inside some M_w. So we do at least know that $\forall xy(\phi(\iota^n`x) \longleftrightarrow \phi(\iota^n`y))$.

Much easier in ZF of course!

Notice that this works for SI only, and not for WSI, because a parameter t would be skewed to $(j^n`\sigma)`t$.

So we have the following implications:

Every set is the same size as a set of singletons $\to SI \to V$ is not a set. It would be nice to reverse these arrows.

Irvine, 2001. While we're about it, let's prove another consequence of the nonexistence of V.

Lemma 3. *For all x and y there are x' and y' the same size as x and y which are both disjoint from x and y and from each other.*

Proof: There must be two things a and b that aren't members of anything in x or anything in y, beco's o/w $\bigcup(x \cup y)$ would be V or the complement of a singleton, in which case V would be a set. x' is now $\{w \cup \{a\} : w \in x\}$ and y' is $\{w \cup \{b\} : w \in y\}$. Both of these sets exist by $\Delta_0^{\mathcal{P}}$ separation. ■

So we can make disjoint copies of two things simultaneously.

In fact we can use this to bootstrap a stronger version.

Lemma 4. *Let A be a set the same size as a set of singletons of singletons. We will make copies of everything in A and they will all be disjoint.*

Proof: We need a set B of size $T^{-2}|A|$ that is disjoint from $\bigcup^2 A$, and we can do this by lemma 3. Then there is $f : A \longleftrightarrow \iota^2``B$, and for each $a \in A$, let D_a be $\{x \cup \{\bigcup^2(f(a))\} : x \in a\}$. Then $|D_a| = |a|$ and $a \neq b \to D_a\ capD_b = \emptyset$. Finally $a \cap D_a = \emptyset$. ■

Remark 5. *The proof in the type-theoretic version of KF that $(\forall y \in x)(y \neq V) \to x \neq V$ needs an ambiguity axiom. The KF version is a stratified theorem with no stratified proof.*

Remark 6. *$SI \nvdash$ no $\in$-cycles.*

Proof: Use permutation models to make some set self-membered. SI is preserved because it is stratified. Therefore $SI \nvdash WSI$ beco's $WSI \vdash$ no $\in$-cycles. ■

(Notice also that SI must hold in the silly principal model of $KF + TC$ that we get from a set of Quine atoms.) There is a conjecture below that says that if $\mathcal{M} \models KF+$ some instance of stratified induction fails, then $\mathcal{M} \models NF$. This may be a bit strong. Here is a sketch of an approach. Now suppose $KF \vdash (\forall x)[(\forall y \in x)(\phi(y)) \to \phi(x)]$ for some stratified ϕ in which the free variable is of type n and that we have a model containing an x such that $\neg\phi(x)$. Consider $\bigcup^n x$. We know that it must contain some members that are not ϕ o/w by induction x would be ϕ too. If we can find a collection Y of things that are ϕ and a setlike permutation π that maps $Y \longleftrightarrow \bigcup^n x$ then we can infer that $\phi((j^n`\pi)`x)$ (by $\in$-induction, since it is a subset of $\mathcal{P}^n`Y$) and thence $\phi(x)$. If x really is a counterexample, then there can be no such Y. Assuming lemma **??** we infer:

If SI fails for ϕ then there is an X such that $(\forall Y)[(\forall z \in Y)(\phi(z)) \to |X| \not\leq |Y|]$

(X is secretly $\bigcup^n x$, where x is the counterexample).

This actually doesn't seem to do any more for us than the existence of a set not the same size as any set of singletons. We don't want *that* to imply that there is a universal set because if that is the case then $\exists NO \to \exists V$, which we *don't* want.

It should be possible to prove the relative consistency of KF+IO+$\neg AxInf$. There is also the question of what extra assumptions we need to show that if all members of x are the size of some set of singletons, so is x. Obviously some replacement and some form of AC at least. It seems we will have to make heavy use of the fact that $KF \vdash (\forall x)[(\forall y \in x)(\phi(y)) \to \phi(x)]$. Think about AC and TC and TCo in this connection.

There is the following argument. Suppose there is no universal set. Then we can prove by induction that every finite set is the size of a set of singletons. So we infer

$$\exists V \vee \text{AxInf} \vee \text{IO}$$

Not sure if this works in *KF*.

Remark 7. *SI follows from those instances of*

$$\frac{\forall x((\forall y \in x)\phi(y) \to \phi(x))}{\forall x \phi(x)}$$

where 'ϕ' contains no bound variables of higher type than its free vaviable. (i.e., ϕ is predicative)

Proof: Suppose we can prove $\forall x((\forall y \in x)\phi(y) \to \phi(x))$ Then we can also prove $\forall x((\forall y \in x)\psi(y) \to \psi(x))$ where $\psi(y)$ is $(\forall w \in y)\phi(w)$ as follows.

Clearly $\psi(x) \to \phi(x)$. Suppose $(\forall y \in x)\psi(y)$. Then $(\forall y \in x)\phi(y)$ which is to say $\psi(x)$. Now we invoke SI with ψ substituted for ϕ: the antecedent is provable, so we infer the conclusion, namely $\forall x\psi(x)$. But $\psi(x) \to \phi(x)$ whence $\forall x\phi(x)$ as desired. So we can obtain the instance

$$\frac{\forall x((\forall y \in x)\phi(y) \to \phi(x))}{\forall x\phi(x)}$$

of SI by using the instance

$$\frac{\forall x((\forall y \in x)[(\forall w \in \phi(w))] \to \phi(x))}{\forall x[(\forall w \in x)\phi(w)]}$$

in which the number of types occupied by bound variables that are higher than the type of the free variable in the inducted formula has been decreased by one. Clearly we can keep on doing this until the only formulæ for which we need to do induction are predicative ones.

[*HOLE Can we show that predicative stratified properties are equivalent to things with very few unrestricted quantifiers?*]

Think about SI as something that can be expressed in simple type theory. What does this do for us?

∎

Appendix 2

Theorem 8. *Every $\forall_2$ sentence is either true in all sufficiently large models of* TST *in which the bottom type is finite, or false in all sufficiently large models of* TST *in which the bottom type is finite.*

Let us call these *finitely generated* models. First we show that every countable model of TST is a direct limit of finitely generated models of TST. Despite appearances, this is not at all obvious: the obvious embedding produces a direct limit where each type is not a boolean algebra but a copy of V_ω. To prove this we will need a sublemma.

Sublemma 9. *Let* $\mathbf{B}$ *be a countable atomic boolean algebra which is a direct limit of* $\langle \mathbf{B}_n : n < \omega \rangle$*, a sequence of finite subalgebras of* $\mathbf{B}$*. Let* $\mathbf{B}'$ *be a countable atomic subalgebra of* $\mathcal{P}(\mathbf{B}$ *containing all singletons. Then there is a sequence* $\langle \pi_n : n < \omega \rangle$ *of partitions of* $\mathbf{B}$ *such that, for each* i*,*

π_i *refines* π_{i-1}

$\pi_i \subseteq \mathbf{B}'$

$\mathbf{B}_i$ *is a selection set for* π_i.

Further, if we let $\mathbf{B}'_i$ *be the subalgebra of* $\mathbf{B}'$ *generated by* π_i (*so that* $\mathcal{P}(\mathbf{B}_i \simeq \mathbf{B}'_i)$ *then the union of the* $\mathbf{B}'_i$ *is* $\mathbf{B}'$.

$\mathbf{B}'$ *is then the direct limit of the* $\mathbf{B}'_i$ *where the injections are inclusion.*

Proof: The recursive construction we use involves iterating a step whose execution may consume several indices, so we must distinguish between *initial* and *transition* indices; 1 is an initial index. We have a well-ordering $\langle x_n : n < \omega \rangle$ of $\mathbf{B}'$.

Let i be an initial index. We have a partition π_{i-1} (π_0 is the $\mathbf{1}$ of $\mathbf{B}$); we want to refine it to π_i for which $\mathbf{B}_i$ will be a selection set; and we want the corresponding boolean algebra $\mathbf{B}'_i$ to contain x_j, where x_j is the first x not already a member of a $\mathbf{B}'_k$. So we want x_j to be a union of elements of π_i. That way, x_j is an element of $\mathbf{B}'_i$ and will appear in the direct limit, which will therefore be $\mathbf{B}'$ as desired. If we can do this, then we have ensured that every x_j appears in some $\mathbf{B}'_i$ or other. Let us say b *crosses* x iff $b \cap x$ and $b - x$ are both non-empty. We have to refine π_{i-1} to a partition whose every element is either included in, or disjoint from, x_j, that is to say, to a partition with no elements that cross x_j. If we can do this, we choose π_i to be such a partition. The difficulty is, every element of the refined partition must contain precisely one element of $\mathbf{B}_i$. A problem will arise if there is an element b of π_{i-1} that crosses x_j, but of the new points of $\mathbf{B}_i$ that we can use, all (and there may not be any in b anyway) belong to the same part, be it $b - x$ or $x \cap b$, that contains the representative of $\mathbf{B}_{i-1}$. If there is such a b we must continue considering $\mathbf{B}_{i+1}$, $\mathbf{B}_{i+2}$, etc., until we find a member of $\mathbf{B}_{i+n}$ which does belong in the half of b not containing the representative of $\mathbf{B}_{i-1}$. Then we can partition b into bits each of which is included in or disjoint from x, and each of which contains precisely one point of $\mathbf{B}_{i+n}$. This will happen for some finite n because, by hypothesis, everything in $\mathbf{B}$ sooner or later appears in one of the $\langle \mathbf{B}_n : n < \omega \rangle$. We can easily fill in the details of $\pi_i, \ldots, \pi_{i+n-1}$ so that the appropriate subalgebras are selection sets for them (these are the transition indices). When we have finished this, the next index is an initial index. ∎

Lemma 10. *Every countable model of* TST *is a direct limit of all finitely generated models of* TST.

Proof: Let M, a countable model of simple type theory, be a family $\langle \mathbf{B}_n : n < \omega \rangle$ of countable atomic boolean algebras, where $\mathbf{B}_{n+1}$ is a countable atomic subalgebra of $\mathcal{P}(\mathbf{B}_n$. Let $\mathbf{B}_1$ be a union of an ω-sequence

$\langle \mathbf{B}^i_1 : i < \omega\rangle$. We then invoke sublemma 9 repeatedly to obtain, for each n, families $\langle \mathbf{B}^i_n : i < \omega\rangle$ of subalgebras and $\langle \pi^i_n : i < \omega\rangle$ of partitions as above. Now consider the structures $\langle\langle \mathbf{B}^i_n : n < \omega\rangle, \in\rangle$ for $i < \omega$. We have constructed the $\mathbf{B}^i_n$ so that $\mathbf{B}^{i+1}_n$ is an atomic boolean algebra whose atoms are elements of a partition for which $\mathbf{B}^i_n$ is a selection set. Thus, if we want to turn the $\langle \mathbf{B}^i_n : n < \omega\rangle$ into a model of simple type theory the obvious membership relation to take is $\in$ itself. They are models of simple type theory without the axiom of infinity, and by construction their direct limit is pointwise the nth type of M, so the direct limit is M as desired. ■

We now return to the proof of the main theorem. Suppose Φ is $\exists_2$ and has arbitrarily large finitely generated models. Then it has a countable model. This model is a direct limit of all finitely generated models, and accordingly would satisfy $\neg\Phi$ (which is $\forall_2$) if $\neg\Phi$ had arbitrarily large finitely generated models. This is impossible so, if Φ is $\forall_2$, then Φ and $\neg\Phi$ cannot both have arbitrarily large finitely generated models. ■

Appendix: $TC(x) = V$ or $TC(x)$ is hereditarily finite

Theorem 11. *TZT locally omits the type that says there is a set x such that, for each n, the n-times sumset of x is neither the universe nor the empty set.*

Proof: Consider the 1-type Σ that says there is a set x such that, for each n, the n-times sumset of x is neither the universe nor the empty set.

Suppose ϕ realises Σ. That is to say: ϕ is a formula of $\mathcal{L}_{TZT}$ with one free variable 'x' s.t., for each k, $\phi(x)$ implies that $\bigcup^k x$ is neither V nor $\emptyset$. Suppose the free variable 'x' has type n in ϕ (so that ϕ is what we call a "n-formula"). Consider an x which has ϕ. Think about $\bigcup^n x$. It is not equal to V nor to $\emptyset$. In particular $V \notin \bigcup^n x$ since otherwise $\bigcup^{n+1} x$ would be V, contradicting the fact that x has ϕ. Let y be anything in $\bigcup^n x$, and consider the transposition $\tau =: (y, V)$. Then $j^n\tau(x)$ certainly has ϕ too, but its $n+1$-times sumset is V.

So no formula ϕ realises Σ. This means that TZT vacuously satisfies the condition for omitting Σ, namely: TZT $\vdash (\forall x)(\neg\phi(x))$ for all ϕ realising Σ—since there are no such ϕ! ■

So TZT has models that omit Σ. In any such model, for every element x, there is a concrete natural number n such that $\bigcup^n x$ is either V or $\emptyset$.

Appendix: The Consistency of $\in$-determinacy

I and II play G_x as follows: each player (I starting—with x) picks a member of the other player's last choice until the game is ended by one trying to pick a member of $\emptyset$ and thereby losing. $\in$-determinacy says that one player or the other must have a winning strategy. Any symmetric model of TZT must satisfy $\in$-determinacy. (It's an old result of mine that x is n-symmetric then G_x can be wrapped up in at most $n+2$ moves). Here we are interested in something a little stronger: the possibility that, for some k one player or the other can force a win in k moves.

Theorem 12. *Let σ_k say that in G_x neither player can force a win in k moves. (Both players have a strategy to stay alive for k moves.) Let Σ be the set of all the σ_k. Then TZTI locally omits Σ.*

So TZTI has models that satisfy $\in$-determinacy.

Proof: Suppose $\phi(x)$ is a formula with only 'x' free that implies, for each k, that neither I nor II have a strategy to win in k moves. Suppose further that ϕ is an n-formula.

Now assume $\phi(x)$ and consider $\bigcup^n x$. Without loss of generality we can take n to be even. If σ is any permutation at all then if we let σ act on $\bigcup^n x$ to obtain $(j^n\sigma)(x)$ we must have $\phi((j^n\sigma)(x))$. We will design σ with care so as to obtain a contradiction.

For any set X we can consider $\bigcup^n x \cap X$ and $\bigcup^n x \setminus X$. Consider the game which is like G_x except that player I is trying to land inside $\bigcup^n x \cap X$ and II is trying to land inside $\bigcup^n x \setminus X$. This is a finite game of perfect information and one of the two players must have a winning strategy. Now suppose further that X is a moiety. (For these purposes a moiety is a set the same size as its complement. That suffices for our purposes and spares us the need to assume the axiom of infinity.)

Case (i): I has a winning strategy. For then $X \cap \bigcup^n x$ is included in a moiety and we need σ to be a permutation that swaps everything in $X \cap \bigcup^n x$ with things in $\{y : \emptyset \in y\}$. I clearly has a winning strategy in $G_{(j^n\sigma)(x)}$ contradicting $\phi((j^n\sigma)(x))$. This is easy because $\{y : \emptyset \in y\}$ is a moiety all of whose members are wins for player I.

But what if II has a winning strategy for all moieties X? The problem is that there is no obvious moiety of things that are wins for player II, since the set of those wins is a power set of the set of things that are wins for player I one level lower—and that set is too small to be a moiety unless we have the axiom of infinity. The table below (showing a hand calculation of the number of sets at each level of the canonical model that are won by I and by $\mathtt{II}$:) indicates (finite case only, admittedly) how pathetically scarce are sets are won by $\mathtt{II}$.

	$\|V\|$	$\|\mathtt{I}\|$	$\|\mathtt{II}\|$
1	1	0	1
2	2	1	1
3	4	2	2
4	16	12	4
5	2^{16}	$15 \cdot 2^{12}$	2^{12}
5	$2^{2^{16}}$	$2^{2^{16}} - 2^{15\cdot 2^{12}}$	$2^{15\cdot 2^{12}}$

However if we do have the axiom of infinity we are OK. Let $\mathtt{II}_n$ be the set of things at level n that are wins for player $\mathtt{II}$, and $\mathtt{I}_n$ similarly. Then evidently $\mathtt{II}_{n+1} = \mathcal{P}(\mathtt{I}_n)$ so as long as $|\mathtt{I}_n| = |V_n|$ we can be sure that $|\mathtt{II}_{n+1}| = |V_{n+1}|$. But $|\mathtt{I}_n| = |V_n|$ certainly follows from infinity because $\mathtt{I}_n$ extends a principal ultrafilter.

If this works (and I think it does) then TZTI locally omits the type that says there is an x s.t. for every n, neither $\mathtt{I}$ nor $\mathtt{II}$ has a strategy to win in n moves. This is actually stronger than $\in$-determinacy. ■

What i am wondering is whether $\bigcup^n x = V$ is enough to imply that G_x is determinate ... it sounds plausible but i don't know how to do it.

Appendix: IO

IO is the principle:

$$\text{Every set is the same size as a set of singletons.} \qquad \text{(IO)}$$

You will all be familiar with the idea that one can interpret one set theory in another by interpreting $x \in y$ as $\mathcal{X}\ E\ \mathcal{Y}$ where the calligraphic letters range over isomorphism types of suitable binary structures such as APGs or wellfounded extensional relations. In this connection there are two apparently fairly banal observations one can make which on close inspection turn out not be banal at all.

1. The expressions '$x \in y$' and '$\mathcal{X}\ E\ \mathcal{Y}$' are both stratified. However the second is homogeneous and the first isn't. This means that it is possible to contruct boolean combinations of formulæ like '$x \in y$' which are not stratified: '$x \in x$' is an example. However, when we write out '$\mathcal{X} E \mathcal{Y}$' in primitive notation we find that not only is it stratified but the two free variables in it receive the same type in any stratification. This has the effect that in *any* molecular combination of expressions like this

(and it is to such combinations that expressions in the language of set theory get translated) every variable—and *a fortiori* every *occurrence* of every variable—receive the same type in any stratification, so all such expressions are stratified—*even if they are the translations of unstratified formulæ of set theory.*! This in turn means that we have a method for interpreting unstratified theories in stratified theories.

2. Often in the course of manipulating models one wishes to have numerous disjoint copies of a given structure. This will happen for example, if one wishes to verify the interpretation of the power set axiom, or an instance of the scheme of replacement or collection. There is an obvious—indeed standard—way of creating $|X|$-many disjoint copies of a structure with carrier set A: for each $x \in X$ we can make a copy with carrier set $A \times \{x\}$. How large is this family of disjoint structures? Assuming we have a type-level ordered pair (the best we can do!) the formula that relates $A \times \{x\}$ to $\{x\}$ is homogeneous, and its graph will be a set. So the family is the same size as $\{\{x\} : x \in X\}$, and this is not necessarily the same size as X. To obtain a family the same size as X we have to find a set Y such that X is the same size as $\{\{y\} : y \in Y\}$, and then our family will have carrier sets in $\{A \times \{y\} : y \in Y\}$. We can form a family of $|X|$-many disjoint copies of a structure only if X is the size of a set of singletons. This leads us to IO.

   What if we don't assume we have a type-level ordered pair? Then in the circumstances above we would want X to be the same size as a set^n of singletons for some $n > 1$. Interestingly the assertion that every set is the same size as a set^n of singletons is of course equivalent to IO. Thus we do not have to worry about implementations of ordered pairs after all.

There are two ways of using isomorphism classes of graphs to obtain models of set theories, and the important point of distinction is the treatment of equality. Do we interpret '=' as isomorphisms between the relevant binary structures (so that our variables range over binary structures) or do we interpret it genuinely as '=' between the (implemented) isomorphism types? Both course of action are legitimate. Neither is obviously better than the other, since each has advantages that the other lacks.

If we take the first horn, and do not implement our isomorphism classes, then we have to assume collection in the verification of the translation of replacement (or choice).

The second horn uses what I call the "quotient as witness" construction. We need a representative of a particular ismorphism class α. The obvious candidate is the family of isomorphism classes "below" α. This method works only if you have an implementation to hand and it cannot be employed in the implementation-free approach.

The difficulty with the second horn comes when we try to verify replacement: we wish to have for each $x \in X$ the unique y such that .... So we collect samples of the correct types. Then we form the sum set of the collection and take a quotient.

5.2 IO and $\exists NO$

$\exists NO$ is the assertion that there is a set that contains wellorderings of all lengths. This is a theorem of NF but is refuted by ZF. It would be nice to find a stratified fragment of ZF with which $\exists NO$ was consistent. We can do this if there is a wellordered counterexample to IO

THEOREM 13. *If there is a wellordered counterexample to IO then* $\exists NO$.

Proof: Suppose IO fails, and let $\langle X, <_X \rangle$ be a wellordering of minimal length not iso to a worder of a set of singletons. Consider the set of its proper initial segments. So by collection followed by stratified separation there is a set Y of wellorderings-of-sets-of-singletons such that every initial segment of $\langle X, <_X \rangle$ is iso to a member of Y and *vice versa*. Then every wellordering is iso to a member of RUSC^{-1} "Y. This is beco's every IO-compliant wellorder is iso to an initial segment of $\langle X, <_X \rangle$. (If you are IO compliant you cannot be the same length as $\langle X, <_X \rangle$ beco's o/w $\langle X, <_X \rangle$ would be IO-compliant too, so you must be shorter. So if Z is any worder, $RUSC(Z)$ is iso to an initial segment of $\langle X, <_X \rangle$ and therefore iso to a member of Y. But then Z is iso to a member of RUSC^{-1} "Y.

Appendix: IO and strZF

The interpretation of ZF in strZF + IO which we will see below also gives an interpretation of ZF into strZFU + IO.

So can we interpret strZF in strZFU? (Not obvious: "hereditarily atom-free" isn't stratified.) Randall emphasises that my proof that AC $\to$ IO relies heavily on the fact that NF + AC is inconsistent. Therefore strZFU + AC might not prove IO. So if we are to interpret strZF in strZFU it must be an interpretation of strZF + not-IO.

Don't forget that ZF minus extensionality can be interpreteted in Zermelo. (Scott [find the reference!])

StrZ + IO is not the same theory as strZF + IO since the first is a subsystem of Z and the second is as strong as ZF.

Draw a diagram displaying relations between the following twelve systems in the form of a tower of three squares

ZF, ZFU, Z, ZU

strZF+IO, strZFU+IO, strZ+IO, strZU+IO

strZF, strZFU, strZ, strZU

NZF is the theory axiomatised by all the known theorems of ZF and NF. It contains (at the very least!) pairing, power set, sumset, extensionality, existence of a Dedekind-Infinite set, transitive containment, stratified separation, and full collection. The following fact might help us orient ourselves, and may yet even be useful. (HS is the Baltimore model.)

Remark 14. $HS \models NZF$

Proof: By lemma **??** $HS \models$ full collection. It clearly satisfies stratified separation, and the other stratified axioms. The existence of the various S_α ensures transitive containment. ■

But AC and IO are not axioms of NZF!

It is highly desirable that stratified collection should be weaker than full collection. See section 9 of [**?**] which has some remarks showing that for V_ζ to be a model of strat $\Delta_0^{\mathcal{P}}$ collection, ζ must equal $\beth_\zeta$. (prop 9.4; but there are other relevant remarks scattered about).

Assuming there is a way of interpreting arithmetic in both NF and ZF we can conclude that not both ZF and NF prove Con(NZF). One would expect ZF $\vdash Con(NZF)$ but not NF $\vdash Con(NZF)$.

ABSTRACT

The set theory strZF is a subsystem of both ZF and NF. IO is the axiom that says that every set is the same size as a set of singletons. It is probable (but not yet established) that ZF $\vdash$ Con(strZF). This is in contrast to the two facts (established here) that ZF can be interpreted in both strZF + AC and in strZF + IO.

Health Warning: the strZF of this paper is not the same as the stratified fragment of ZF in [**?**]: *this* system has stratified collection and *that* one has only stratified replacement.

Introduction

Let strZF be the theory obtained from ZF by discarding the unstratified instances of the axiom schemes of replacement, collection and separation. It is the theory axiomatised by the stratified axioms of ZF-axiomatised-with-collection+replacement+separation. Notice that we do *not* mean the theory axiomatised by the stratified theorems of ZF! We can define strZ and strZFU analogously.

StrZF is a subsystem of both ZF and NF, and NF is strZF + the existence of a universal set.

Despite the (putative) weakness of NZF we can prove

THEOREM 15. *Con(NZF + IO)* $\to$ *Con(ZF)*

We show this by interpreting ZF in NZF + IO. (In fact in a stratified fragment of NZF)

What we actually need is not IO but the apparently slightly stronger assertion that for all x and y there is a set of singletons the same size as x that is disjoint from y. However this assertion follows from IO as follows.

REMARK 16. $(\forall xy)(\exists z)(z \cap y = \emptyset \wedge |x| = |\iota``z|)$

Proof: Suppose x is the same size as a set of singletons, but that every set of singletons the same size as x meets y. Notice that strZF has a type-level ordered pair. Suppose $\iota``z$ is a set of ordered pairs the same size as $x$. Then so is $\iota``(z \times \{w\})$ for any z. So y meets $\iota``(z \times \{w\})$ for any w; so every w is the second component of an ordered pair in $\bigcup y$. But then the set of second components of ordered pairs in $\bigcup y$ is V. But the universal set isn't the same size as any set of singletons, contradicting our assumption that IO. ■

The main proof

A `BFEXT` is a wellfounded extensional relational structure with a top element—as usual. There is an obvious membership relation between these objects, which we will write E.

We will interpret ZF into strZF + IO. We restrict all quantifiers $(\forall x)\ldots$ to $(\forall x)(\texttt{BFEXT}(x) \to \ldots$ and all quantifiers $(\exists x)\ldots$ to $(\exists x)(\texttt{BFEXT}(x) \wedge \ldots$. We read $\in$ as E, and $=$ as $\sim$: isomorphism between `BFEXT`s.

Is every axiom of ZF interpreted by a theorem of the new system?

Clearly we expect no difficulty with Extensionality, Sumset, Infinity and Separation. The axioms that give trouble are those axioms whose instantiating `BFEXT`s can be constructed only by making copies of things. Pairing illustrates in a small way the difficulty we will have with power set and replacement. It isn't enough to take the disjoint union of the two `BFEXT`s in question and add a new top element because there are now two empty sets not one—one in each component—so the result is not extensional. We have to define an equivalence relation by recursion (each equivalence class will have at most two elements) on the union of the two carrier sets. We define a `BFEXT` on the quotient in the obvious way. However the quotient is the wrong type, so we appeal to the fact that it is the same size as a set of singletons, copy it over to this new set, throw away the outer layer of curly brackets and now we have what we want.

We can also prove pairing *without* using IO. Just make the two `BFEXT`s $\langle X, x^*, R\rangle$ and $\langle Y, y^* S\rangle$ disjoint then put together a new one. There is a canonical partial bijection between X and Y induced by the two wellfounded relations. The carrier set of the new structure consists of $X \cup$ those things in Y than are not connected by the canonical bijection to anything in X.

Power set and replacement are a bit harder.

5.3 Power set

Suppose we have a `BFEXT` $\langle X, x^*, R\rangle$. For each subset Y of $R^{-1}``\{x^*\}$ we have to add a new element $Y^R$ to the carrier set $X$, and we have to remove all directed edges going to $x^*$. Then, for each $x$ in $R^{-1}``\{x^*\}$, and each $Y \subseteq R^{-1}``\{x^*\}$ with $x \in Y$, we add a directed edge from x to Y^R. Then we add directed edges from all the Y^Rs to x^*. Exactly how are we to do this? We have to ensure that nothing we choose to be a Y^R is already in X. Simply taking Y^R to be Y is no good: not only might Y already be a member of X, even if—by a miracle—none of the Y were already in X, the definition would be unstratified. However, it is our point of departure: we do obtain the various Y^R from elements $\mathcal{P}(R^{-1}``\{x^*\})$. Here we need remark 16 to tell us that there is a set $W$ disjoint from $X$ and a bijection $f : \mathcal{P}(R^{-1}``\{x^*\}) \to \iota``W$. For $Y \subseteq R^{-1}``\{x^*\}$ we set Y^R to be $\iota^{-1}(f(Y))$. That way, for x in X and $Y \subseteq R^{-1}``\{x^*\}$, we have a directed edge going from x to Y^R iff $x \in Y$ and the collection of these new edges is defined by a stratified condition and is indeed a set.

(If we can use IO, then just take Y wlog to be a set of singletons)

5.4 Replacement

We have a `BFEXT` and a map (definable in the language of $\in$) taking its E-members to other `BFEXT`s. Since E is homogeneous, the description of this map—translated into the language of E—beomes stratified, so we can use stratified collection to obtain a set X of `BFEXT`s containing at least one representative of each of the things we want. By assumption, X is the same size as a set U of singletons, so we can replace each `BFEXT` $\langle W, w^*, R\rangle$ in X by a "U-copy" $\langle W \times \{u\}, \langle w^*, u\rangle, R^u\rangle$ where $\{u\}$ is the singleton associated with w. (The third component R^u of this triple is $\{\langle\langle x, u\rangle, \langle y, u\rangle\rangle : \langle x, y\rangle \in R\}$ of course.) !

Let us assume this done, so that X is now the set of U-copies of the set we started with. The `BFEXT`s in X are now all disjoint. What happens if we attempt to construct a new `BFEXT` whose carrier set is the union of the carrier sets of all the U-copies in X, equip it with a new top point and take the binary relation to be the union of the binary relations associated with the U-copies? Nearly there, but not quite. The result is not extensional, since there is an empty set in each U-copy. We define an equivalence relation by recursion as we did with the axiom of pairing. The quotient is now the same size as a set of singletons so we copy the relation over to the sumset of this set of singletons and then we are done. !

The need for this last manœuvre stems from the need for there to be, for each member of X, an injection from its carrier set into the carrier set of the new `BFEXT`, and the best way of ensuring that there is such an injection is for it to have a stratified description.

■

The two exclamation marks in the margin indicate the two places where we had to use IO. In the second application the set which we need to be the size of a set of singletons is a family of disjoint sets, and the assertion that every disjoint family is the size of a set of singletons is presumably much weaker. For all I know it could be consistent with NF.

Notice that to interpret *replacement* in strZF + IO we exploited the fact that strZF has stratified *collection*. Life would be a lot simpler if we could get by with using only stratified *replacement*!