Kleen-Post

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Contents

0.1 Kleene-Post and Friedberg-Muchnik

"Is the relation $\leq_T$ a total order?" one might ask. It turns out that the answer is 'no'. In fact, there are incomparable degrees even below $\mathbf{0}'$.

Let us start by enumerating the set $\{0,1\}^{<\omega}$ of all finite strings from $\{0,1\}$ as $\langle \eta_n : n \in \mathbb{N} \rangle$. Each string is thought of as a function from an initial segment of $\mathbb{N}$ to $\{0,1\}$. In this setting, where we are considering recursion relative to an oracle, we let $\{e\}$ be the eth member of the set of functions-in-intension–that–call–oracles. Think of $\{e\}$ as code written in a language that allows invocations of oracles. Then $\{e\}^C$ is the function computed by $\{e\}$ when given access to the oracle C. The notation '$\{e\}^C$' doesn't mean "the eth program that calls the oracle C" [which is what i used to think].

When the superscript is an η_a—which of course is *finite*—it might happen that $\{e\}$ calls for the oracle to rule on an input at which η_a is not defined. In these circumstances $\{e\}^{\eta_a}(x)\uparrow$. [It seems to me that this means that $\{e\}^{\eta_a}(x)$ is a different notation from $\{e\}^B(x)$ because If B is finite then $\{e\}^B(x)$ will diverge—if at all—only because of e; it always gets answers from B... whereas $\{e\}^{\eta_a}(x)$ might diverge because it doesn't get an answer from the oracle.]

Observe that

- if $\{e\}^{\eta_a}(x)\downarrow$ then $\{e\}^C(x)\downarrow$ for every C extending η_a. (We are thinking here of C as an infinite sequence of 0s and 1s—as a characteristic function, in fact.)
- If $\{e\}^B(x)\downarrow$ then there is $a \in \mathbb{N}$ such that $\{e\}^{\eta_a}(x)\downarrow$

We will construct two sets A and B such that $A \not\leq_T B \not\leq_T A$. There will be two sequences of binary strings $\langle \alpha_n : n \in \mathbb{N} \rangle$ such that α_{i+1} extends α_i and $\bigcup_{i \in \mathbb{N}} \alpha_i = \chi_A$; and $\langle \beta_n : n \in \mathbb{N} \rangle$ such that β_{i+1} extends β_i and $\bigcup_{i \in \mathbb{N}} \beta_i = \chi_B$.

We initialise $\alpha_0 = \beta_0 = 0$. Thereafter...

- Stage $2s+1$. Let x be the first number not in the domain of α_{2s}. [That is to say, it's length(α_{2s}) co's we start counting at 0.] If there are any η_a that are end-extensions of β_{2s} such that $\{s\}^{\eta_a}(x)\downarrow$ then use the least such a and set α_{2s+1} to be $\alpha_{2s}{::}y$, where y is the least element of $\{0,1\} \setminus \{\{s\}^{\eta_a}(x)\}$.

[Beware overloading of braces]. And β_{2s+1} is set to be β_{2s+1}::0. If there is no such η_a then set $\alpha_{2s+1} := \alpha_{2s}$::0 and $\beta_{2s+1} := \beta_{2s}$::0.

- Stage $2s + 2$. Let x be the first number not in the domain of β_{2s+1}. [That is to say, it's length(β_{2s+1}) co's we start counting at 0.] If there are any η_b that are end-extensions of α_{2s+1} such that $\{s\}^{\eta_b}(x)\downarrow$ then use the least such b and set β_{2s+2} to be β_{2s+1}::y, where y is the least element of $\{0, 1\} \setminus \{\{s\}^{\eta_b}(x)\}$. [Beware overloading of braces]. And α_{2s+2} is set to be α_{2s+1}::0. If there is no such η_b then set $\alpha_{2s+2} := \alpha_{2s+1}$::0 and $\beta_{2s+2} := \beta_{2s+1}$::0.

The idea of the construction is that at stage $2s$ (resp $2s+1$) you do something to ensure that $B \neq \{s\}^A$“$\mathbb{N}$ (resp. something to ensure that $A \neq \{s\}^B$“$\mathbb{N}$.) At the end of the construction the union of the αs is χ_A and the union of the βs is χ_B.