CHAPTER VII: GETTING ω WOODIN CARDINALS FROM AD

WE ARE GOING TO COMBINE the techniques developed in the previous chapters to obtain a model containing infinitely many Woodin cardinals. Indeed we shall present three ways of doing so; with the first we find an inner model of HOD of that kind, with the second we find a generic extension. The third method, of which the first is a simplification, was the original approach. It involves the iteration of a construction of a generic elementary embedding to yield an inner model of HOD.

In the first section we introduce the concept of a stout set of ordinals and use the results of Chapters V and VI to show that stout sets encoding models with Woodin cardinals abound. In section 2 we use Martin's measure to integrate sequences of countable stout sets, leading to our first model with ω Woodin cardinals. In section 3 we give a version of Prikry forcing such that a Prikry sequence of countable stout sets will yield a set T of countable ordinals such that in $L[T]$ the ω_1 of our ground model is the supremum of an ω sequence of Woodin cardinals. In showing that the reals of the ground model can be regarded as the reals of a symmetric collapse over $L[T]$ we use a characterisation of such sets of reals which is proved in section 4. In section 5 we develop and study an infinite commuting diagram of elementary embeddings, some of which lie in generic extensions of the universe. In section 6 we identify certain submodels associated to that sequence which contain infinitely many Woodin cardinals. Finally in section 7 we apply the analysis of sections 5 and 6 to giving a characterisation of HOD_S for an arbitrary set S of ordinals.

1: Ultraproducts using the Martin measure

We wish to form ultraproducts of sets using the Martin measure $\mathfrak{m}$ on the set $\mathcal{D}$ of Turing degrees.

An instance where we would fail to get Łoś's theorem would be if we tried to form the ultraproduct, over all Turing degrees $\mathfrak{x}$, of the inner model $L(\mathfrak{x})$. There are well-known injections of the continuum into the Turing degrees — for example take continuum many almost disjoint subsets of a subset of ω that is Mathias generic over L — but the cardinality of $\mathcal{D}$ cannot, under AD, equal $2^{\aleph_0}$ as the former carries a non-trivial countably additive two-valued measure and the latter cannot. Hence, under AD, there is no function selecting a representative from each Turing degree, or even from each Turing degree in some cone. We cannot expect that such an attempt to form an ultraproduct of all $L[x]$, for $x \subset \omega$, will yield an inner model of the form $L[a]$ for $a \subseteq \omega$: there is no function available to represent that a.

However suppose that to each Turing degree $\mathfrak{x}$ we assign an inner model $M_\mathfrak{x}$ and a (class) well-ordering of the same, $<_\mathfrak{x}$. Then we shall be able to form the ultraproduct $\prod_{\mathfrak{x}\in\mathcal{D}} M_\mathfrak{x}/\mathfrak{m}$, and prove the theorem of Łoś, because we now have a canonical method of choosing witnesses in the various models to an existential statement.

This would be the case if $M_\mathfrak{x} = (HOD_S)^{L[x,S]}$ where S is a class of ordinals, and $<_\mathfrak{x}$ is the natural OD_S well-ordering defined from S in $L[x, S]$, and $x \subset \omega$ is any member of the Turing degree $\mathfrak{x}$.

The difference between the two cases is that in the case of $L[x]$, while the model $L[x]$ depends only on the Turing degree of x, its well-ordering depends on the specific x chosen, and replacing x by another y with $L[x] = L[y]$, even one of the same Turing degree, will perturb that well-ordering. So actually there is no way of choosing a well-ordering of each $L[x]$ — if there were we could choose a representative of the Turing degree of x. But in the case of $M_\mathfrak{x} = (HOD_S)^{L[x,S]}$, the well-ordering **is** independent of the exact x chosen and depends only on its Turing degree $\mathfrak{x}$, or indeed on much less.

Another case where we shall have no difficulty is in forming ultra*powers*: for example we may form the ultrapower $L[S_0]^\mathcal{D}/\mathfrak{m}$. This ultrapower will be well-founded, as $\mathfrak{m}$ is countably complete and we have DC; let N be its transitive collapse, and let $j_\mathfrak{m} : L[S_0] \to N$ be the canonical elementary embedding. Similarly we may define $j_\mathfrak{m}(\eta)$ for any ordinal η, or indeed $j_\mathfrak{m}(A)$ for A any set of ordinals.

1·0 LEMMA *Let U be an ultrafilter on an ordinal $\xi < \Theta$. Let $\eta < \Theta$. Then $j_U(\eta) < \Theta$.*

Proof : there is a map of $\mathcal{N}$ onto $\mathcal{P}(\eta \times \xi)$ and of that set onto $j_U(\eta)$. ⊣ (1·0)

1·1 PROPOSITION $(AD + V = L(\mathbb{R}))$ $\quad j_\mathfrak{m}(\delta^2_1) \geq \Theta$.

Proof : We have seen in Chapter III Kunen's argument that the set of ultrafilters on δ^2_1 is well-ordered, since each is the projection of the Martin measure $\mathfrak{m}$ by some function $f : \mathcal{D} \to \delta^2_1$, and for different ultrafilters the resulting functions will be different mod $\mathfrak{m}$. We have seen in Chapter V that there are at least Θ different ultrafilters on δ^2_1, since by Lemma 1·0, for each U, $j_U(\delta^2_1) < \Theta$ and for each $\lambda < \Theta$ there is a U with $j_U(\delta^2_1) > \lambda$. ⊣ (1·1)

1·2 COROLLARY $j_{\mathfrak{m}}(\Theta) > \Theta$.

Working in $L(I\!\!R)$ and assuming AD, we may define a sequence of sets of ordinals, starting from S_0, the subset of Θ with $L[S_0] = HOD$:

$$S_{k+1} =_{\text{df}} j_{\mathfrak{m}}(S_k)$$

That definition works because S_0 and $j_{\mathfrak{m}}$ are definable, and therefore so in turn is each S_k, which therefore lies in HOD and thus in the domain of $j_{\mathfrak{m}}$. But slightly more holds:

1·3 PROPOSITION $(AD + V = L(I\!\!R))$ *For each k, $S_{k+1} \in L[S_k]$.*

Proof : $S_1 \in L[S_0]$, being ordinal definable: now apply $j_{\mathfrak{m}}$ repeatedly to that statement. ⊣ (1·3)

We return now to consider the case of $M_{\mathfrak{x}} = (HOD_S)^{L[x,S]}$ where S is a class of ordinals, $<_{\mathfrak{x}}$ is the natural OD_S well-ordering, and $x \in \mathfrak{x}$.

Freezing on a cone

The next result illustrates the principle of stabilisation on a Turing cone that follows from AD, or even from TD, the assertion that all Turing–closed games are determined. We shall call this principle *Freeze.* First, a lemma:

1·4 LEMMA $(TD + DC)$ *(i) there is no uncountable well-ordered set of distinct reals;*
(ii) for no countable ordinal ζ is there an uncountable well-ordered set of distinct subsets of ζ;
(iii) let M be an inner model of AC. Then Ω, the true ω_1, is strongly inaccessible in M.

Proof : (i) holds because TD implies the measurability of $\mathcal{D}$ which in turn implies the measurability of ω_1, as we saw in Chapter III. $2^{\aleph_0}$ cannot be measurable, and hence there is no injection of ω_1 into $\mathcal{P}(\omega)$. (ii) follows from (i). For (iii), Ω is regular, by DC, and is therefore regular in each inner model. It must be a strong limit cardinal in each inner model of AC by (ii). ⊣ (1·4)

1·5 THE FREEZE PRINCIPLE *Assume TD+$AC_{\aleph_0}$. Let ξ be a countable ordinal. Suppose we have a function assigning to each Turing degree $\mathfrak{x}$ a subset $P_{\mathfrak{x}}$ of $\mathcal{P}(\xi)$ and a well-ordering $<_{\mathfrak{x}}$ of $P_{\mathfrak{x}}$ in order-type $\lambda_{\mathfrak{x}}$, so that under this well-ordering $P_{\mathfrak{x}}$ may be listed as $\langle A^{\mathfrak{x}}_\nu \mid \nu < \lambda_{\mathfrak{x}}\rangle$.*

Then there is a $\lambda < \omega_1$, a sequence $\langle A_\nu \mid \nu < \lambda\rangle$, definable from that function, of subsets of ξ and a Turing degree $\mathfrak{a}$ such that

$$\forall \mathfrak{x} {:}\geqslant_{\text{Turing}} \mathfrak{a}\ \langle A^{\mathfrak{x}}_\nu \mid \nu < \lambda_{\mathfrak{x}}\rangle = \langle A_\nu \mid \nu < \lambda\rangle.$$

Proof : note first that each $\lambda_{\mathfrak{x}}$ must be countable as there can be no uncountable well-ordered subset of $\mathcal{P}(\xi)$

For each countable η such that $\eta < \lambda_{\mathfrak{x}}$ on a cone, and each $\nu < \xi$ either $\nu \in A^{\mathfrak{x}}_\eta$ on a cone or $\nu \notin A^{\mathfrak{x}}_\eta$ on a cone; we may define $A_\eta \subseteq \xi$ by

$$A_\eta =_{\text{df}} \{\nu < \xi \mid \text{ on a cone } \nu \in A^{\mathfrak{x}}_\eta\}$$

and then by using $AC_{\aleph_0}$ to intersect countably many cones we see that on a cone, $A^{\mathfrak{x}}_\eta = A_\eta$. Further the map $\eta \mapsto A_\eta$ is 1-1, and so it can only be defined for a countable initial segment of the countable ordinals, say for $\eta < \lambda$. Then, again intersecting countably many cones, we see that $\langle A_\eta \mid \eta < \lambda\rangle$ is as required. ⊣ (1·5)

Now let S be a class of ordinals, let $M_{\mathfrak{x}} = (HOD_S)^{L[\mathfrak{x};S]}$ and let N_S be the transitive collapse of the ultrapower $\prod M_{\mathfrak{x}}/\mathfrak{m}$, which will exist provided the ultrapower is well-founded, as indeed it will be under DC. We write $\mathbf{0}$ for the Turing degree of recursive sets.

1·6 PROPOSITION $(TD + DC)$ *(i) $N_S \subseteq HOD_S$; (ii) for each $\mathfrak{x}$ $L[S] \subseteq M_{\mathfrak{x}}$. (iii) $L[S] = M_{\mathbf{0}}$.*

Proof : for part (i) note that N_S will have a well-ordering definable from S, namely the ultraproduct of the well-orderings $<_S$ as defined in each $L[\mathfrak{x};S]$: hence each member of N_S can be identified by the ordinal that is its position in this well-ordering. Parts (ii) and (iii) are trivial from the definition of $M_\mathfrak{x}$. $\dashv$ (1·6)

In our statement of the next few results we again rely on Kechris's theorem that DC is a consequence of $AD + V = L(I\!R)$.

1·7 PROPOSITION $(AD + V = L(\mathbb{R}))$ *Suppose that S is a class such that $HOD_S = L[S]$. Let $\xi < \Omega$, and let A be a subset of ξ. Then the following are equivalent:*

(i) $A \in N_S$;
(ii) $A \in L[S]$;
(iii) $A \in M_\mathfrak{x}$ for every $\mathfrak{x}$;
(iv) $A \in M_\mathfrak{x}$ for a cone of $\mathfrak{x}$'s.

Proof : the containments proved in Proposition 1·6 show that (i) implies (ii) and that (ii) implies (iii). That (iii) implies (iv) is trivial. It remains to see that (iv) implies (i). Let $f(\mathfrak{x}) = A$ if $A \in M_\mathfrak{x}$ and $= \varnothing$ otherwise. Then $[f]_\mathfrak{m}$ denotes a member of N_S: but that member is A, by the countable additivity of the measure. $\dashv$ (1·7)

1·8 PROPOSITION *Let N be an inner model of ZF and M an inner model of ZFC. Suppose that an ordinal κ is strongly inaccessible in M and that for each $\xi < \kappa$, $\mathcal{P}(\xi) \cap M = \mathcal{P}(\xi) \cap N$. Then $V_\kappa \cap M = V_\kappa \cap N$.*

Proof : Let $\xi < \kappa$. In M, V_ξ is of cardinality λ, say, less than κ and therefore there is a subset A of λ in M which codes $(V_\xi)^M = V_\xi \cap M$. By assumption $A \in N$. But then A can be decoded in N, showing that $V_\xi \cap M \subseteq N$. As ξ was arbitrary, we have $V_\kappa \cap M \subseteq V_\kappa \cap N$.

For the other direction, suppose $V_\kappa \cap (N \setminus M)$ is non-empty and let x be a member of minimal rank $\zeta < \kappa$. Then $x \subseteq V_\zeta$, and $V_\zeta \cap M = V_\zeta \cap N$. But $(V_\zeta)^M$, as we have seen, may be coded by a relation $A \in M$ on some $\lambda < \kappa$, and $A \in N$; moreover x can then be coded in N using A by a subset of λ; that subset is in M, as was A, and hence we may in M recover x from its code. $\dashv$ (1·8)

We have stated that proposition in a sharp form in which only one of the models is assumed to satisfy AC, but shall often apply it to a context where both models do, as in the proof of this

1·9 COROLLARY $(AD + V = L(\mathbb{R}))$ *Suppose that $L[S] = HOD_S$. Then*

$$V_\Omega \cap L[S] = V_\Omega \cap N_S; \tag{1·9·0}$$

$$\forall \xi {:<} \Omega\ \exists \mathfrak{a} {:\in} \mathcal{D}\ \forall \mathfrak{x} {:\geqslant_{\rm Turing}}\ \mathfrak{a} \qquad V_\xi \cap M_\mathfrak{x} = V_\xi \cap N_S. \tag{1·9·1}$$

Proof : the first equation holds by 1·7 and 1·8. We prove the second part by induction on ξ. For $\xi \leqslant \omega$ we may take $\mathfrak{a} = \mathbf{0}$. Suppose it true for a given ξ, with a cone defined by $\mathfrak{a}(\xi)$; list $V_\xi \cap N_S$ in N_S as $\langle y_\zeta \mid \zeta < \kappa \rangle$, where by the strong inaccessibility of Ω in N_S, κ is countable. List each $V_{\xi+1} \cap M_\mathfrak{x}$ for $\mathfrak{x} \geqslant_{\rm Turing} \mathfrak{a}(\xi)$ as $\langle A^\mathfrak{x}_\nu \mid \nu < \lambda_\mathfrak{x} \rangle$. Each $A^\mathfrak{x}_\nu$ is a subset of the fixed countable set $V_\xi \cap N_S$, so as in the proof of 1·5, we show first that each $\lambda_\mathfrak{x}$ is countable, then for each η such that $\eta < \lambda_\mathfrak{x}$ on a cone, we may freeze the value of $A^\mathfrak{x}_\eta$ as A_η say, and finally remark that there can only be countably many such η. So we find $\mathfrak{a}(\xi + 1)$ by intersecting countably many cones.

To continue the induction at limit ordinals requires merely $AC_{\aleph_0}$ to show that the intersection of countably many cones contains a cone. $\dashv$ (1·9)

1·10 COROLLARY $(AD + V = L(\mathbb{R}))$ *Let S be a class of ordinals with $HOD_S = L[S]$. If the countable ordinal ξ is strongly inaccessible in N_S then it is so in $L[S]$ and in $M_\mathfrak{x}$ on a cone of $\mathfrak{x}$'s.*

Proof : N_S and $L[S]$ have the same sets of rank at most ξ. We may relate truth in N_S to truth in $M_\mathfrak{x}$ on a cone by Łoś. $\dashv$ (1·10)

1·11 PROBLEM If we define on a cone $f(\mathfrak{x})$ to be the canonical Woodin cardinal of $M_\mathfrak{x}$, that will denote a Woodin cardinal in N_S. How large is it ?

Examples of S with $HOD_S = L[S]$ are sets of the form $S_0 \oplus A$ where $L[S_0] = HOD$ and A is a subset of a countable ordinal, and $\oplus$ denotes some simple coding of the two sets of ordinals into one, as shown by:

1·12 THE EXTENDED SLIDE PRINCIPLE *Assume $AD + V = L(\mathbb{R})$. If η is countable and $A \subseteq \eta$ then $HOD_{\{A\}} = L[S_0, A]$.*

Proof : we consider Lévy extensions of $HOD_{\{A\}}$, which is a model of AC and therefore, under AD, such Lévy extensions exist in the real world. Suppose two such, f and g, are subsets of ω, both coding A as well as η, and mutually generic over $HOD_{\{A\}}$: they will therefore be mutually generic over the smaller model $L[S_0, A]$ as well. By the *Slide* Principle established in Chapter III, we know, assuming $AD + V = L(\mathbb{R})$, that for each $x \subset \omega$ $HOD_{\{x\}} = HOD[x]$; hence

$$L[S_0, A] \subseteq HOD_{\{A\}} \subseteq HOD_{\{f\}} \cap HOD_{\{g\}} = HOD[f] \cap HOD[g] \subseteq L[S_0, A][f] \cap L[S_0, A][g] = L[S_0, A]$$

as required. ⊣ (1·12)

1·13 When $S = S_0 \oplus A$, it will be convenient to change our notation, since S_0 remains constant while we vary A. Therefore for A a subset of a countable ordinal η, and $\mathfrak{x}$ the Turing degree of the real x, we shall write $M^A_{\mathfrak{x}}$ for the proper class $(HOD_{\{S_0, A\}})^{L[x,A,S_0]}$, $<^A_{\mathfrak{x}}$ for the canonical well-ordering of that model definable in the universe from A and $\mathfrak{x}$, and N^A for the transitive collapse of $\prod_{\mathfrak{x} \in \mathcal{D}} M^A_{\mathfrak{x}}/\mathfrak{m}$. Then N^A has an (S_0, A)-definable well-ordering, namely the ultraproduct $\prod <_{\mathfrak{x}}/\mathfrak{m}$; so each element of N^A will be ordinal definable from S_0 and A, and, by 1·6, $N^A \subseteq L[S_0, A]$. Moreover $L[S_0, A] \subseteq M^A_{\mathfrak{x}}$ for each x, and so $L[S_1, A] \subseteq N^A$, where $S_1 = j_{\mathfrak{m}}(S_0)$, since A, being a subset of a countable ordinal, is left unmoved by the ultrapower construction.

With this change in notation, Corollary 1·9 will read:

1·14 Proposition $(AD + V = L(\mathbb{R}))$ *Let* $A \subseteq \eta < \omega_1$.

$$(1·14·0) \qquad V_\Omega \cap L[A, S_0] = V_\Omega \cap N^A;$$

$$(1·14·1) \qquad \forall \xi :< \Omega\ \exists \mathfrak{a} :\in \mathcal{D}\ \forall \mathfrak{x} :\geqslant_{\text{Turing}} \mathfrak{a} \qquad V_\xi \cap M^A_{\mathfrak{x}} = V_\xi \cap N^A.$$

Stout sets

We write ϱ for the set theoretic rank $\varrho_\in$.

1·15 Definition Let A be a set of ordinals with supremum $\eta = \varrho(A)$, and let S be a set of ordinals. A is *stout in* S if $L[A] \cap V_\eta = L[A; S] \cap V_\eta$ and in $L[A; S]$, η is strongly inaccessible.

1·16 Definition We write $\mathfrak{M}(A)$ for $L[A] \cap V_{\varrho(A)}$.

When A is stout, $\mathfrak{M}(A)$ is a model of ZFC, and we may think of it as the model coded by A. For arbitrary A, $\mathfrak{M}(A)$ may not be a model of anything very interesting.

Of course in a universe in which $HOD = L$, we would get trivial examples of stout sets by taking A to be the set of predecessors of any ordinal strongly inaccessible in L. To get non-trivial examples under AD we shall use the following

1·17 Lemma $(TD + DC)$ *Let M be an inner model of AC. Then Ω, the true ω_1, is Mahlo in M.*

Proof : if not, let C be in M a closed unbounded subset of Ω consisting solely of singular cardinals of M.

Ω is measurable, and so there is an elementary embedding $j : M \to N$ with critical point Ω. Łoś applies, and so $j(C)$ is in N a closed unbounded subset of $j(\Omega)$ consisting solely of ordinals singular in N. But $j(C) \cap \Omega = C$, and so $\Omega \in j(C)$, but Ω is regular in V and therefore in N, a contradiction. ⊣ (1·17)

1·18 Proposition $(AD + V = L(\mathbb{R}))$ *Let M be any inner model of AC with $HOD \subseteq M$. Let η be any countable ordinal that is in M strongly inaccessible. Then for some unbounded subset A of η in M, A is stout in S_0 and $\mathfrak{M}(A) = M \cap V_\eta$.*

Proof : As the true ω_1 is Mahlo in M, there are many such η. Fix one, and by elementary set theory find $A \in M$, $A \subseteq \eta$ such that $L[A] \cap V_\eta = M \cap V_\eta$. Then as $L[A] \subseteq L[S_0, A] \subseteq M$, where $HOD = L[S_0]$ and $S_0 \subseteq \Theta$,

$$L[A] \cap V_\eta = L[S_0, A] \cap V_\eta \ \&\ (\eta \text{ is strongly inaccessible })^{L[S_0,A]} \qquad ⊣ (1·18)$$

1·19 Definition B is a *stout extension of A in S* if both A and B are stout in S, $A = B \cap \varrho(A)$, $\mathfrak{M}(A) = \mathfrak{M}(B) \cap V_{\varrho(A)}$ and in $L[B; S]$, $\varrho(A)$ is strongly inaccessible.

We now apply the results of Chapter VI to prove the following, which is the source of the bricks with which we shall build our first model of ω Woodin cardinals.

1·20 PROPOSITION (AD + V=L(R)) *Let A be a countable set which is stout in S_0. Then from any sufficiently high Turing degree $\mathfrak{x}$ we may define a countable stout extension of A in S_0, denoted by $A^{\mathfrak{x}}$, such that for some θ with $\varrho(A) < \theta < \varrho(A^{\mathfrak{x}})$,*

$$\mathfrak{M}(A^{\mathfrak{x}}) \models \theta \texttt{ is a Woodin cardinal}.$$

Proof : Let $\eta = \varrho(A)$. We know that as A is stout in S_0, η is strongly inaccessible in $L[A, S_0]$. By *Freeze* choose $\mathfrak{a}$ so that if $\mathfrak{x} \geqslant_{\text{Turing}} \mathfrak{a}$, η is strongly inaccessible in $M^A_{\mathfrak{x}}$ and $M^A_{\mathfrak{x}} \cap V_\eta = L[A, S_0] \cap V_\eta$.

Let $\mathfrak{x}$ be any Turing degree above $\mathfrak{a}$, and let ζ be the first strong inaccessible in $M^A_{\mathfrak{x}}$ greater than $\omega_2^{L[x;A,S_0]}$: so ζ has been defined from $\mathfrak{x}$ and A. As $A \in M^A_{\mathfrak{x}}$, we may by elementary set theory pick $B \subseteq \zeta$ with $B \cap \eta = A$, $B \in M^A_{\mathfrak{x}}$ and $L[B] \cap V_\zeta = M^A_{\mathfrak{x}} \cap V_\zeta$: set $A^{\mathfrak{x}}$ to be the $<_{\mathfrak{x}}$-first such B. Notice that we have defined $A^{\mathfrak{x}}$ from $\mathfrak{x}$ and A in $L[A, \mathfrak{x}, S_0]$.

We assert that $A^{\mathfrak{x}}$ is a stout extension of A in S_0. Note first that $L[S_0, A^{\mathfrak{x}}] \subseteq M^A_{\mathfrak{x}}$ as $S_0 \in M^A_{\mathfrak{x}}$ and $A^{\mathfrak{x}} \in M^A_{\mathfrak{x}}$. Hence ζ is strongly inaccessible in $L[S_0, A^{\mathfrak{x}}]$. $L[S_0, A^{\mathfrak{x}}] \cap V_\zeta \subseteq M^A_{\mathfrak{x}} \cap V_\zeta = L[A^{\mathfrak{x}}] \cap V_\zeta \subseteq L[S_0, A^{\mathfrak{x}}] \cap V_\zeta$; so $L[S_0, A^{\mathfrak{x}}] \cap V_\zeta = L[A^{\mathfrak{x}}] \cap V_\zeta$. So $A^{\mathfrak{x}}$ is stout in S_0.

$L[A^{\mathfrak{x}}] \cap V_\eta = M^A_{\mathfrak{x}} \cap V_\eta = L[A; S_0] \cap V_\eta = L[A] \cap V_\eta$. so $\mathfrak{M}(A^{\mathfrak{x}} \cap V_\eta = \mathfrak{M}(A)$; and finally η is strongly inaccessible in $L[S_0, A^{\mathfrak{x}}]$ since $L[S_0, A^{\mathfrak{x}}] \subseteq M^A_x$ and η is strongly inaccessible in M^A_x, so $A^{\mathfrak{x}}$ is a stout extension of A.

Finally, we know from the results of Chapter VI that for $\mathfrak{x}$ a sufficiently high Turing degree, $A^{\mathfrak{x}}$ will contain a Woodin cardinal greater than $\varrho(A)$, namely the ordinal $\omega_2^{L[\mathfrak{x};A,S_0]}$, which by our choice of ζ will actually be the largest inaccessible in $\mathfrak{M}(A^{\mathfrak{x}})$. ⊣ (1·20)

1·21 REMARK In section 3 we shall use a variant of that Proposition in which we choose a larger value of ζ, but still in a manner that is uniformly definable from $\mathfrak{x}$ and A.

2: an inner model of *HOD* containing ω Woodin cardinals

Let A_0 be any countable stout set. For any (Turing degree) x we write A^x for the stout extension built starting from A_0 and x, as in Proposition 1·20. For any $y >_{\text{Turing}} x$, we then write A^{xy} for the stout extension of A^x using y. For x sufficiently high[C 1] and for y sufficiently high above x, A^{xy} will contain two Woodin cardinals greater than $\varrho(A_0)$.

We proceed to define A^s for every strictly increasing sequence s of Turing degrees. In the notation A^{xyz} it will be implicit that $x <_{\text{Turing}} y <_{\text{Turing}} z$.

Given A^{xyz} we may hold x and y fixed, and form $\mathfrak{m}$-prod A^{xyz}: we shall refer to this as integrating with respect to z, [C 2] but write $A^{xy\hat{}}$ for the result, which we might otherwise call $\int A^{xyz} dz$. We may now integrate that with respect to the uppermost variable, y in this case, obtaining $A^{x\hat{}\hat{}}$ or $\int A^{xy\hat{}} dy$; in this example there is one more variable to go, namely x, and when we have removed it, we shall write A_3 rather than $A^{\hat{}\hat{}\hat{}} = \int A^{x\hat{}\hat{}} dx$.

We may write $\varrho(B)$ for the height of B, since its height will equal its rank.

The basic fact is:

2·0 PROPOSITION *given a countable set A which is stout in S_0 there is a countable B which is a stout extension of A in S_0 such that*

$$L[S_0, B] \cap V_{\varrho(B)} = L[B] \cap V_{\varrho(B)}$$

and $$L[B] \cap V_{\varrho(A)} = L[A] \cap V_{\varrho(A)}$$

and $$A = B \cap \varrho(A)$$

and $$\varrho(A) \textit{ is strongly inaccessible in } L[B, S_0]$$

So, taking the case of three variables,

[C 1] in the Turing degrees

[C 2] there being little danger of confusion with the classical notion of integration

$$L[S_0, A^{xyz}] \cap V_{\varrho(xyz)} = L[A^{xyz}] \cap V_{\varrho(A^{xyz})}$$

so $$L[S_1, A^{xy\hat{}}] \cap V_{\varrho(xy\hat{})} = L[A^{xy\hat{}}] \cap V_{\varrho(A^{xy\hat{}})}$$

so $$L[S_2, A^{x\hat{}\hat{}}] \cap V_{\varrho(x\hat{}\hat{})} = L[A^{x\hat{}\hat{}}] \cap V_{\varrho(A^{x\hat{}\hat{}})}$$

so $$L[S_3, A^{\hat{}\hat{}\hat{}}] \cap V_{\varrho(\hat{}\hat{}\hat{})} = L[A^{\hat{}\hat{}\hat{}}] \cap V_{\varrho(A^{\hat{}\hat{}\hat{}})}$$

in other words $$L[S_3, A_3] \cap V_{\varrho(A_3)} = L[A_3] \cap V_{\varrho(A_3)}$$

and $$A^{xy} = A^{xyz} \cap \varrho(A^{xy})$$

so $$A^{xy} = A^{xy\hat{}} \cap \varrho(A^{xy});$$

— here we use the fact that A^{xy} is a countable set of countable rank —

continuing, $$A^{x\hat{}} = A^{x\hat{}\hat{}} \cap \varrho(A^{x\hat{}})$$

so $$A^{\hat{}\hat{}} = A^{\hat{}\hat{}\hat{}} \cap \varrho(A^{\hat{}\hat{}})$$

in other words $$A_2 = A_3 \cap \varrho(A_2)$$

Similarly one could show that $L[A_3] \cap V_{\varrho(A_2)} = L[A_2] \cap V_{\varrho(A_2)}$ and that in $L[S_3, A_3]$, $\varrho(A_2)$ is strongly inaccessible (by starting from the fact that in $L[S_0, A^{xyz}]$ $\varrho(A^{xy})$ is strongly inaccessible); and that in $L[A_3]$ (and in $L[S_3, A_3]$) there are three Woodin cardinals less than $\varrho(A_3)$, one less than $\varrho(A_1)$, one between $\varrho(A_1)$ and $\varrho(A_2)$ and one between $\varrho(A_2)$ and $\varrho(A_3)$.

Now fix k. We shall show that the sequence $\langle A_k, A_{k+1}, \ldots\rangle$ is in $L[S_k, A_k]$. The pattern of argument will be clear from the first few cases.

The sequence $\langle A^x, A^{x\hat{}}, A^{x\hat{}\hat{}}, \ldots\rangle$ is definable from the countable set of countable ordinals A^x, and so lies in $HOD_{\{A^x\}} = L[S_0, A^x]$. Hence, integrating, the sequence $\langle A_1, A_2, A_3, \ldots\rangle$ lies in $L[S_1, A_1]$.

Similarly the sequence $\langle A^{xy}, A^{xy\hat{}}, A^{xy\hat{}\hat{}}, \ldots\rangle$ is definable from the countable set of countable ordinals A^{xy}, and so lies in $HOD_{\{A^{xy}\}} = L[S_0, A^{xy}]$. Hence, integrating twice, the sequence $\langle A_2, A_3, A_4, \ldots\rangle$ lies in $L[S_2, A_2]$.

Note finally that the sequence $\langle A_0, A_1, A_2, A_3, \ldots\rangle$ is definable from A_0, and so lies in $L[S_0, A_0]$.

From the above observations it follows that

$$L[S_0, A_0] \supseteq L[S_1, A_1] \supseteq L[S_2, A_2] \supseteq \ldots \supseteq L[A_2] \supseteq L[A_1] \supseteq L[A_0]$$

Suppose now that M is a class such that

$$L[S_0, A_0] \supseteq L[S_1, A_1] \supseteq L[S_2, A_2] \supseteq \ldots \supseteq M \supseteq \ldots \supseteq L[A_2] \supseteq L[A_1] \supseteq L[A_0]$$

Then for each k, $V_{\varrho(A_k)} \cap L[S_k, A_k] = V_{\varrho(A_k)} \cap M = V_{\varrho(A_k)} \cap L[A_k]$, since the outer two are equal and the middle one is sandwiched.

Thus if such an M exists which models ZFC, we have found the desired inner model containing infinitely many Woodin cardinals, for the A_k are rank-initial segments of M and (for $k > 0$) each contributes one more Woodin cardinal over its predecessor.

A possible M is $L[\bigcup_{k<\omega} V_{\varrho(A_k)}]$: it is plainly sandwiched and an inner model of ZF; what is not clear is that it will satisfy AC.

We shall instead take M to be the slightly larger model $L[A]$ where $A = \bigcup_{k<\omega} A_k$. Plainly $L[A]$ contains each $L[A_k]$; and we have seen that the sequence $\langle A_i \mid i < \omega\rangle$ is in each $L[S_k, A_k]$.

Thus $L[T]$ is an inner model of $HOD_{\{A_0\}}$ possessing infinitely many Woodin cardinals, and we have proved the following

2·1 THEOREM *Assume $V = L(\mathbb{R})$ and AD. Then every countable stout set A codes a rank initial segment of a subclass of $HOD_{\{A\}}$ which forms an inner model of AC plus "there are ω Woodin cardinals".*

2·2 Remark There is a natural extension of the Martin ultrafilter $\mathfrak{m} = \mathfrak{m}_1$ to the sets

$$\mathcal{D}_k \;=_{\rm df}\; \{s \in {}^k\mathcal{D} \mid \forall i{:<}k-1\; s(i) <_{\rm Turing} s(i+1)\}.$$

Namely, say

$$A \in \mathfrak{m}_2 \;\Longleftrightarrow_{\rm df}\; \{x \mid \{y \mid \langle x,\, y\rangle \in A\} \in \mathfrak{m}_1\} \in \mathfrak{m}_1,$$

and more generally set

$$A \in \mathfrak{m}_{k+1} \;\Longleftrightarrow_{\rm df}\; \{x \mid \{y \in \mathcal{D}_k \mid x^\frown y \in A\} \in \mathfrak{m}_k\} \in \mathfrak{m}_1.$$

Let us explore this definition with $k = 3$:

$$A \in \mathfrak{m}_3 \;\Longleftrightarrow_{\rm df}\; \Big\{x \in \mathcal{D}_1 \;\Big|\; \big\{y \in \mathcal{D}_1 \;\big|\; \{z \in \mathcal{D}_1 \mid \langle x, y, z\rangle \in A\} \in \mathfrak{m}_1\big\} \in \mathfrak{m}_1\Big\} \in \mathfrak{m}_1.$$

Note that this is equivalent to saying

$$A \in \mathfrak{m}_3 \;\Longleftrightarrow_{\rm df}\; \big\{\langle x,\, y\rangle \in \mathcal{D}_2 \;\big|\; \{z \in \mathcal{D}_1 \mid \langle x, y, z\rangle \in A\} \in \mathfrak{m}_1\big\} \in \mathfrak{m}_2$$

and to

$$A \in \mathfrak{m}_3 \;\Longleftrightarrow_{\rm df}\; \big\{x \in \mathcal{D}_1 \;\big|\; \{\langle y,\, z\rangle \in \mathcal{D}_2 \mid \langle x, y, z\rangle \in A\} \in \mathfrak{m}_2\big\} \in \mathfrak{m}_1$$

Thus, as with the associative law, we can vary the groupings but not vary the order. We leave it to the reader to find out what chaos would result if we did.

In terms of these ultrafilters our construction would read

$$A_k = \prod\{A^s \mid \ell h(s) = k \;\&\; s \text{ is fast-growing}\}/\mathfrak{m}_k.$$

3: A variant of Prikry forcing

We begin with a general version of Prikry forcing.

3·0 Historical Note Prikry forcing was in his dissertation [ ]. Mathias [ ] proved the property called by certain writers *geometric*. The proof of the Prikry property given below derives from arguments in *Happy Families* which have been developed by Blass, Louveau, Gitik and in the forthcoming book of Woodin and Cummings.

Let X be a set, let $Y = {}^{<\omega}X$, the set of finite sequences of members of X, and suppose that we have an assignment $\mathfrak{U}$ to each $s \in Y$ of a countably complete ultrafilter $\mathfrak{U}(s)$ on X. We define a notion of forcing $P_{\mathfrak{U}}$: a condition will be a pair $\langle s,\, F\rangle$ where $s \in Y$ and $F \in \prod_{s\in Y} \mathfrak{U}(s)$. The partial ordering is given by

$$\langle s,\, F\rangle \leqslant \langle t,\, G\rangle \iff_{\mathrm{df}} t = s\restriction \ell h(t) \ \&\ \forall r{:}{\in} Y\ F(r) \subseteq G(r) \ \&\ \forall i{:}{\in}[\ell h(t), \ell h(s))\ s(i) \in G(s\restriction i)$$

3·1 Definition In working with these conditions it is convenient to define $F \sqcap G$ to be the member H of $\prod \mathfrak{U}$ with $\forall s H(s) = F(s) \cap G(s)$, and $\bigsqcap_{i<\omega} F_i$ similarly.

3·2 Definition Given $c \leqslant \omega$ and a partition $\pi : Y \to c$ of Y, there will be for each $s \in Y$ a unique $i < c$ such that the set $\{x \in X \mid \pi(s^\frown(x)) = i\}$ is in the ultrafilter $\mathfrak{U}(s)$; we write $\pi'(s)$ for that i and $G_\pi(s)$ for that set. We may call π' the *derived partition* of π.

3·3 Definition For (s, F) be a condition, let $E(s, F)$ be the set of all t with $(t, F) \leqslant (s, F)$.

3·4 Definition Let $\pi_0 : E(s, F) \longrightarrow c \leqslant \omega$ We define a sequence of partitions $\pi_k : E(s, F) \longrightarrow c$ for $k \in \omega$, by setting $\pi_{k+1} = \pi'_k$, and write for simplicity G_k for G_{π_k}. Thus

$$G_k(s) = \{x \in X \mid \pi_k(s^\frown x) = \pi_{k+1}(s)\}.$$

Now set $H(t) = \bigcap_{k<\omega} G_k(t)$ for $t \in E(s, F)$, $H(t) = F(t)$ otherwise. Note that H is definable from $\langle s,\, F\rangle$, π_0 and $\mathfrak{U}$.

3·5 Lemma *If t and u are both in $E(s, H)$ and of the same length, then for each $k \in \omega$, $\pi_k(t) = \pi_k(u)$.*

Proof : Fix k and let $\ell = \ell h(t) - \ell h(s)$. Then $\pi_k(t) = \pi_{k+\ell}(s)$, since for $\ell h(s) \leqslant i < \ell h(t)$, $t(i) \in H(t\restriction i) \subseteq G_{k+\ell h(t)-(i+1)}(t\restriction i)$, and so

$$\pi_{k+\ell h(t)-(i+1)}(t\restriction(i+1)) = \pi_{k+\ell h(t)-i}(t\restriction i).$$

Similarly $\pi_k(u) = \pi_{\ell+k}(s)$, whence the lemma. ⊣ (3·5)

As a first application, we have the renowned Prikry property:

3·6 Proposition *Let Φ be a statement of the forcing language, and $\langle s,\, F_0\rangle$ a condition. Then*

$$\exists H{:}{\in}\prod \mathfrak{U}\ \langle s,\, H\rangle \leqslant \langle s,\, F_0\rangle \ \&\ \langle s,\, H\rangle || \,\Phi.$$

Moreover, H is uniformly definable from s, F_0, Φ and $\mathfrak{U}$, and for all $t \notin E(s, F_0)$, $H(t) = F_0(t)$.

Proof : We define $\pi_0 : E(s, F_0) \longrightarrow 3$ thus:

$$\pi_0(t) \ =_{\mathrm{df}} \begin{cases} 0 & \text{if } \exists F\langle t,\, F\rangle \Vdash \Phi \\ 1 & \text{if } \exists G\langle t,\, G\rangle \Vdash \neg\Phi \\ 2 & \text{otherwise} \end{cases}$$

As two conditions $\langle t,\, F\rangle$ and $\langle t,\, G\rangle$ with the same first part are compatible, the three cases are exclusive and exhaustive. Define the derived sequence of partitions and obtain the function H satisfying the lemma. We assert that $\langle s,\, H\rangle$ decides Φ: for if not, there would be some $\langle u,\, F\rangle \leqslant \langle s,\, H\rangle$ with $\pi_0(u) = 0$ and $\langle t,\, G\rangle \leqslant \langle s,\, H\rangle$ with $\pi_0(t) = 1$: by extending one of them if necessary we may assume that $\ell h(u) = \ell h(t)$. But then the lemma tells us that $\pi_0(u) = \pi_0(t)$, a contradiction. ⊣ (3·6)

Phrases such as "shrink F to H to decide Φ" allude to applications of that Proposition, which has the following:

3·7 COROLLARY *If each $\mathfrak{U}(s)$ is countably complete, then the forcing $P_{\mathfrak{U}}$ adds no new subsets of countable ordinals.*

Proof : given a countable ordinal ζ, a condition $\langle s, F\rangle$, and a P-name $\dot{x}$ for a subset of ζ we may apply the Prikry property for each $\nu < \zeta$ to obtain a condition $\langle s, F_\nu\rangle$ deciding the statement $\hat{\nu} \epsilon \dot{x}$, the uniform definability of F_ν from the data obviating any appeal to the Axiom of Choice. The condition $\langle s, \bigsqcap_{\nu<\zeta} F_\nu\rangle$ then forces $\dot{x} \in \hat{V}$, as required. ⊣ (3·7)

3·8 REMARK Note that besides the one for which we have a canonical definition, there may be other H shrinking F for which $\langle s, H\rangle$ decides Φ, but any two must decide it the same way, since two conditions with the same stem are compatible.

In our context, we know that there is no well-ordered uncountable sequence of distinct reals, or of a given countable ordinal. As an illustration of the force of that, we prove the following, where we write $\dot{G}$ for the canonical term denoting the generic being added.

3·9 PROPOSITION ($\aleph_1 \not\leq 2^{\aleph_0}$) *Suppose that $\langle s, F\rangle$ is a condition. Let C be a set of ordinals in the ground model, and ξ a countable ordinal. Write $\dot{\Lambda}$ for the term "*`the length of the canonical well-ordering of` $\mathcal{P}(\hat{\xi}) \cap HOD_{\dot{G},\hat{C}}$*". Then there is a countable ordinal $\lambda = \lambda(s, \xi)$ such that some condition, with stem s and definable from s, F, and C, forces $\hat{\lambda} = \dot{\Lambda}$.*

Proof : We keep s fixed throughout this discussion. For each countable ordinal η we shrink F to decide whether $\hat{\eta} < \dot{\Lambda}$ or not: if the former, we say that η is *affirmed*. Suppose that every countable ordinal is affirmed. Then we may shrink further to determine the η^{th} subset of ξ, obtaining a set $A_{s,\eta} \subseteq \xi$. It is enough to notice that different η's yield different $A_{s,\eta}$'s; and only one $A_{s,\eta}$ is possible for any particular η. All this being done in a definable way, we would obtain an uncountable well-ordered list $\langle A_\eta \mid \eta < \omega_1\rangle$ of distinct subsets of ξ, which we know to be impossible. Therefore some countable ordinal, which we may call $\lambda(s, \xi)$, must be forced by shrinking to be $\dot{\Lambda}$ or more; so the least such ordinal becomes forced to be the value of $\dot{\Lambda}$. ⊣ (3·9)

Once we know λ to be the length of that well-ordering, each member of it is named by a term definable from s, F, and C and an ordinal less than λ.

A generic sequence of Woodins.

We shall couple the theory of Prikry forcing to some material on definability and our notions of stout set and stout extension to prove the following.

3·10 THEOREM *Assume $V = L(\mathbb{R})$ and AD: then we can add generically a subset T of Ω (the first uncountable ordinal) such that the reals of $L(\mathbb{R})[T]$ are those of $L(\mathbb{R})$; and that $L(\mathbb{R})[T]$ may be considered as a (symmetric singular Solovay) extension of $L[T]$. Moreover, in $L[T]$, the ordinal Ω is the supremum of ω Woodin cardinals.*

Our first attempt at a proof would be this: take X to be the set $\mathcal{D}$ of Turing degrees and each $\mathfrak{U}(s)$ to be the Martin ultrafilter of those subsets of $\mathcal{D}$ that contain some Turing cone $\mathcal{C}_x$, and add a Prikry sequence $\langle \mathfrak{d}_n \mid n \in \omega\rangle$, then for each finite initial segment $s(n) = \langle \mathfrak{d}_0, \mathfrak{d}_1, \ldots \mathfrak{d}_{n-1}\rangle$ of this sequence form the stout set $A_n = A^{s(n)}$, ensuring that A_{n+1} is a stout extension of A_n. Then define $T = \bigcup_n A_n$ and show that $L[T]$ is the desired model with ω Woodins, and that adding the reals of the ground model to it counts as a symmetric extension.

That approach runs into the following difficulty: there are bounded subsets of $ON \cap \mathfrak{M}(A^x)$ which are definable from x but are not members of $\mathfrak{M}(A^x)$ and are therefore not definable from A^x. Hence we must reduce the power of the stems of our conditions to define things; and we shall indeed work with conditions whose stems are sequences of stout sets, and the T we obtain will be of the form $\bigcup_n A_n$ where each A_{n+1} is a stout extension of A_n.

In arranging that the reals may be seen as those of a symmetric extension, we shall apply a characterisation due to Woodin of the reals of such models, which we state now in a general form.

3·11 PROPOSITION *Let λ be a [singular] strong limit cardinal, and suppose that in some (not necessarily Boolean) extension, W, of V there is a set S of reals such that*

(i) $\forall x{:}\in S\ \exists P$ [P is in V a forcing of size less than λ and x is (V,P)-generic]
(ii) given any two members x, y of S, every real in $V[x,y]$ is in S.
(iii) each ordinal less than λ is recursive in at least one member of S.
Then in some Boolean extension of W there is a filter G that is $(V, Coll(\omega,<\lambda))$-generic and for which

$$S = S_G,$$

S_G being the set of those reals in $\bigcup_{\nu<\lambda} V[G\cap Coll(\omega,\nu)]$.

We defer the proof of that to the next section. In our context, the V of the proposition will be $L[T]$, and W will be $L(\mathbb{R})[T]$, which is certainly an extension of $L[T]$ even if we do not know it to be in any sense a Boolean one. S is $\mathbb{R}$, the reals of our world in which AD holds. λ is Ω, the ω_1 of $L(\mathbb{R})$. Properties (ii) and (iii) will therefore be easy to verify; (iii) will hold since every ordinal less than Ω is countable in $L(\mathbb{R})$, and (ii) will hold since $L[T][x,y]\subseteq L(\mathbb{R})[T]$ which has the same reals as $L(\mathbb{R})$. So the only challenge lies in ensuring the truth of property (i). To do so, we have recourse to a variant of the ideas of Vopěnka that we discussed in detail in Chapter I.

3·12 Lemma (AC) *Suppose that a real a is added to the universe by a set-forcing $\mathbb{P}$ satisfying the $<\theta$ chain condition, where θ is strongly inaccessible. Then that real is added by a complete Boolean algebra of size at most θ, with the same chain condition.*

Proof : Let $\dot a$ be a name for a in the given forcing, which we take to be a complete Boolean algebra. Starting from the countably many members $[\![\hat n\in\dot a]\!]^{\mathbb{K}}$ build a complete Boolean subalgebra by alternating adding sups for every subset of the elements so far with adding (finitary) Boolean combinations. At each stage fewer than θ things are added. Stop if ever we find we have reached a complete subalgebra. We must do so at stage θ if not before since then every sup is over a subset of size less than θ and has therefore already been realised. ⊣ (3·12)

3·13 Lemma (AC) *Let θ be strongly inaccessible and $\mathbb{P}$ a forcing of cardinality θ satisfying the $<\theta$ chain condition. Then there is a regular embedding of $\mathbb{A} =_{\rm df}$ r.o. $\mathbb{P}$ in $\mathbb{D} =_{\rm df}$ r.o. $Coll(\omega,\Theta)$.*

Proof : in these circumstances $\mathbb{A}$ is of cardinality θ and there are only θ maximal antichains of $\mathbb{A}$. Hence in $V^{\mathbb{D}}$, where θ has become countable, we can enumerate both $\mathbb{A}$ and those antichains in order type ω and build an $(N,\mathbb{A})$ generic. Let $\dot G$ be a name for that generic. Define $\pi(a) = [\![a\in\dot G]\!]^{\mathbb{D}}$.

π is a complete homomorphism; to ensure that it is 1-1, use a generic enumeration of $\mathbb{A}$ to ensure that every non-zero element of $\mathbb{A}$ has a non-zero chance of getting into $\dot G$. ⊣ (3·13)

3·14 Proposition *Let a be a real and S a set of ordinals. Let N be HOD_S. Then there is a notion of forcing $\mathbb{P}$ in N and a $(N,\mathbb{P})$ generic G such that*
(3·14·0) $a\in N[G]$
(3·14·1) there is no antichain in N of cardinality Θ (of V).
(3·14·2) if Θ is strongly inaccessible in N, $\mathbb{P}$ may be taken to be of cardinality at most Θ in N, and to be embeddable into $Coll(\omega,\Theta)$.

Proof : Vopěnka with some embellishments. The first guess at $\mathbb{P}$ is the copy $\mathbb{K}^1$ of the set of OD_S subsets of $\mathcal{P}\omega$. Its cardinality in V embeds into $2\exp 2\exp\aleph_0$. Any antichain in N is well-orderable, and defines a partition of $\mathcal{P}\omega$ in V; the cardinality of the antichain must therefore be less than Θ. Thus $\mathbb{K}$ is a complete Boolean algebra with the $<\Theta$ chain condition in N. Apply the two lemmata. ⊣ (3·14)

3·15 Now let A be a countable set stout in S_0, and consider a real y. Then any real x which is OD in y and A will be by the Extended *Slide* principle be a member of $L[A,S_0,y]$ and therefore by Vopěnka will be generic over $HOD^{L[A,S_0,y]}_{A,S_0}$, the model $M^A_{\mathfrak{y}}$ with respect to a certain algebra which is defined independently of the particular real x. We aim to show that uniformly in A and $\mathfrak{y}$ we may define a stout extension B of A such that the relevant Vopěnka algebra is a member of $\mathfrak{M}(B)$. Then we shall define our Prikry forcing so that whenever A is in the generic sequence, so will some such B be, thus ensuring by a density argument that each x is generic over $L[B]$, for B a proper initial segment of T, with respect to an algebra lying in $V_{\varrho(B)}\cap L[B]$.

We write $\Xi(A,\mathfrak{y})$ for $(2^{2^{\aleph_0}})^{L[A,S_0,\mathfrak{y}]}$, the value of the term $\overline{\overline{\mathcal{P}(\mathcal{P}(\omega))}}$ computed in $L[A,S_0,\mathfrak{y}]$. $\Xi(A,\mathfrak{y})$ will be less than Ω, since Ω is strongly inaccessible in $L[A,S_0,\mathfrak{y}]$. Remember that to form the Vopěnka algebra we began by considering, in $L[A, S_0, \mathfrak{y}]$, the collection $\mathcal{B}$ of subsets of $\mathcal{P}(\omega)$ that are $\mathrm{OD}_{\{A,S_0\}}$. Write κ for the cardinality of that collection in $L[A, S_0, \mathfrak{y}]$: then $\kappa \leqslant \Xi(A, \mathfrak{y}) < \Omega$. We then made a copy $\mathcal{B}^*$ of that collection in $M^A_{\mathfrak{y}}$, using a system of OD_{A,S_0}-names for the members of that collection. Let $\eta(A, \mathfrak{y})$ be the cardinality of that copy $\mathcal{B}^*$, computed in $M^A_{\mathfrak{y}}$. In $L[A, S_0, \mathfrak{y}]$, $\eta(A, \mathfrak{y})$ is injectible into κ, so must also be less than Ω.

Now repeat the construction of Proposition 1·20, only this time defining ζ to be the first strongly inaccessible in $M^A_{\mathfrak{y}}$ greater than both $\Xi(A, \mathfrak{y})$ and $\eta(A, \mathfrak{y})$. The definition is still uniform.

3·16 Definition Write $\pi_A(\mathfrak{y})$ for the $<^A_{\mathfrak{y}}$-first stout extension of A in $S_0 \oplus A$ of rank ζ coding $M^A_{\mathfrak{y}} \cap V_\zeta$.

Plainly we have just defined $\pi_A(\mathfrak{y})$ uniformly from A and y, and we have shown that if x is OD in y and A, then x is generic over $M^A_{\mathfrak{y}}$ with respect to an algebra lying in $\mathfrak{M}(\pi_A(\mathfrak{y}))$; further all subsets of the algebra in $M^A_{\mathfrak{y}}$ are in $\mathfrak{M}(\pi_A(\mathfrak{y}))$; and hence x will be generic over any model of which $\mathfrak{M}(\pi_A(\mathfrak{y}))$ is a rank-initial segment.

3·17 Now we are ready to choose our version of Prikry forcing. We use "stout" to mean "stout in S_0", and similarly for "stout extension". For each countable stout set A, let $Stout(A)$ be the set of countable stout extensions of A. We have just shown how to associate to each countable stout A a map $\pi_A : \mathcal{D} \longrightarrow Stout(A)$. The pre-image of no point contains a cone: for when the degree $\mathfrak{y}$ is sufficiently high, the canonical Woodin cardinal of $\mathfrak{M}(\pi_A \mathfrak{y})$ is $\omega_2^{L[A,S_0,\mathfrak{y}]}$ and hence though not recursive in $\mathfrak{y}$, will be recursive in $\mathfrak{z}$ for a cone of $\mathfrak{z}$'s. Hence π_A projects the Martin measure $\mathfrak{m}$ to a countably additive non-trivial measure $\mathfrak{m}_A$ on $Stout(A)$.

Thus we look at finite sequences s of stout sets proceeding by stout extension, and the ultrafilter we assign to s is $\mathfrak{m}_A$ where A is the last and largest element of s: we write $A = \max s$.

We shall refer to the function F_0 defined by

$$F_0(s) = \{B \mid \exists \mathfrak{y}\; B = \pi_{\max s}(\mathfrak{y}) \text{ and } \mathfrak{M}(B) \text{ has a Woodin cardinal exceeding } \sup \max s\}$$

as the *trivial function.* It is visibly definable and each $F_0(s)$ is of measure one with respect to $\mathfrak{m}_{\max s}$.

Proof of the Theorem: we work with the variant of Prikry forcing that we have just described, with one final restraint: to get started, let A_0 be a definable countable stout set: for example, let η be the first strongly inaccessible cardinal in $L[S_0]$ and let A_0 be the $<_{L[S_0]}$-first unbounded subset of η in that model for which $V_\eta \cap L[A_0] = V_\eta \cap L[A_0, S_0]$. We consider only conditions the stem of which begins with A_0.

3·18 Remark We could, more generally, start from some arbitrary countable stout set A_0 and take that to be the first element of all our conditions, but then everything below should have A_0 added to the list of defining parameters.

To recapitulate, a condition has a non-empty stem, which is a finite sequence of sets, the first being A_0, and each later one being a stout extension of the one before in S_0; to each such sequence s we assign the measure $\mathfrak{m}_A$, A being the largest term in the sequence s.

If G is generic for this forcing, there is in $L(\mathbb{R})[G]$ a sequence $(A_n)_{n\in\omega}$ such that the stem of each condition in G is some initial segment $(A_0, \dots A_n)$. We write T for $\bigcup_n A_n$, the union of the stems of conditions in the generic G, and $\dot{T}$ for the forcing name of T. As each ultrafilter is countably complete, we have added no new reals; hence Ω is preserved. To see that in $L[T]$ it has become the limit of Woodin cardinals, we shall show that $V_\Omega \cap L[T] = \bigcup_n \mathfrak{M}(A_n)$.

Proof that no new definable bounded sets are added

Each $\mathfrak{M}(A_n) \subseteq L[T]$ as A_n is an initial segment of T, so to see that $V_\Omega \cap L[T] = \bigcup_n \mathfrak{M}(A_n)$, it is enough by a suitable variant of Proposition 1·6 to show that whenever $B \subset \xi < \Omega$, and $B \in L[T]$, then B is in some $\mathfrak{M}(A_n)$. Fix such a B. B is of course in $L(\mathbb{R})$, by the fact that no new subsets of countable ordinals are added, so $\hat{B}$ is defined. But B also has a definition from T: our task is to use that definition to prove that in $L(\mathbb{R})$, B is definable from some A_n.

Let $\langle s, F\rangle$ be a condition in the generic sequence, with s sufficiently long to ensure that $\xi < \bigcup A_n$, A_n being the last entry in s, and which for some ordinal η forces the statement

$$\texttt{the } \hat{\eta}^{\text{th}} \texttt{ object constructible from } \dot{T} \texttt{ is } \hat{B}.$$

We assert that $B \in L[A_n]$. To see that, write $\dot b$ for the term `the` $\hat\eta^{\text{th}}$ `object constructible from` $\dot T$. Start from $\langle s,\, F_0\rangle$, F_0 being the trivial function and thus definable. By our proposition 3·6, shrink to H such that $\langle s,\, H\rangle$ decides the statement $\dot b \subseteq \hat\xi$ — and we know which way the decision must go — and then decides for each $\nu < \xi$ the statement $\hat\nu \mathrel{\epsilon} \dot b$; we know we may find such H OD from s and $\mathfrak{U}$ (which is itself definable) and from $\dot G$ (again definable). All those statements must be in agreement with the ones forced by $\langle s,\, F\rangle$. Hence B is definable from s as $\{\nu < \xi \mid \langle s,\, H\rangle \Vdash \hat\nu \mathrel{\epsilon} \dot b\}$.

Now the earlier terms in the sequence s are all OD from the top term, all being initial segments of it. So B is in fact OD from A_n. Hence if $\rho = \bigcup A_n$, $B \in L[A_n, S_0] \cap V_\rho = L[A_n] \cap V_\rho = \mathfrak{M}(A_n)$.

3·19 For positive n, each A_n adds at least one Woodin to those in A_{n-1}, and so Ω will be the supremum of the set of smaller Woodin cardinals in $L[T]$. What will be the order type of that set ?

A full treatment of that question must use facts from Part II of this book, but the following remarks may help. Let us introduce some predicates of limit ordinals ζ and η:

3·20 Definition $\Psi_\Omega(\zeta) \iff_{\text{df}}$ there is a notion of forcing $\mathbb{P} \in V$ and a $\mathbb{P}$-term $\dot S$ such that

$$1^{\mathbb{P}} \Vdash \dot S \texttt{ is a set of ordinals and in } \dot L[\dot S]\ \hat\zeta = \dot\omega_1 \texttt{ and } AD \texttt{ is true in } \dot L(\dot{\Re}).$$

3·21 Definition $\Psi_\Xi(\zeta, \eta) \iff_{\text{df}}$ there is a notion of forcing $\mathbb{P} \in V$ and a $\mathbb{P}$-term $\dot S$ such that

$$1^{\mathbb{P}} \Vdash \dot S \texttt{ is a set of ordinals and in } \dot L[\dot S]\ \hat\zeta \texttt{ is a limit of } \hat\eta \texttt{ Woodin cardinals}.$$

3·22 Definition $\Psi_\Lambda(\zeta) \iff_{\text{df}}$ there is a notion of forcing $\mathbb{P} \in V$ and a $\mathbb{P}$-term $\dot S$ such that

$$1^{\mathbb{P}} \Vdash \dot S \texttt{ is a set of ordinals and in } \dot L[\dot S]\ \hat\zeta \texttt{ is a limit of Woodin cardinals}.$$

3·23 Theorem (*ZFC*) $\Psi_\Omega(\zeta) \iff \Psi_\Lambda(\zeta)$.

Proof : we just have proved the implication from left to right; from right to left is proved in Part II. The general result is that in $L(\Re)$ of the symmetric collapse of a limit of Woodin cardinals, AD will be true. $\dashv$ (3·23)

From the theorem we get the equiconsistency of the truth of AD in $L(\mathbb{R})$ and the existence of an infinite sequence of Woodin cardinals.

We may also show that, always assuming the consistency of the systems concerned, it is not a theorem of *ZFC* plus the existence ω Woodin cardinals that AD holds in $L(\Re)$. For consider the least ζ such that $\Psi_\Lambda(\zeta)$, and go to the model $L[S]$ in which ζ is a limit of Woodin cardinals. Let κ be the ω_1 of that model; since κ is less than ζ, $\Psi_\Omega(\kappa)$ is false. Hence AD fails in $(L(\Re))^{L[S]}$, and in many other models too.

For the same minimal ζ, go to a model satisfying $V = L(\Re) + AD$ of which ζ is the ω_1. Then in this model, no countable stout set will see more than finitely many Woodin cardinals, and therefore if we add a subset T of ζ with Prikry forcing, ζ will become the limit of ω Woodin cardinals.

Suppose there is a larger ordinal ξ such that $\Psi_\Xi(\xi, \eta)$ for some $\eta > \omega$, such as $\omega + \omega$. Then in $L(\Re)$ of the symmetric collapse of ξ there will be countable stout sets which see ω Woodin cardinals and, by some oversight, we might have included one in the Prikry generic sequence, so that we cannot say in general that $L[T]$ will see exactly ω Woodin cardinals.

Treating the reals as those of a symmetric collapse.

Our final problem is to prove that the extension may be construed as a symmetric extension. But we have already seen that our choice of Prikry forcing guarantees the truth of the hypothesis of Proposition 3·11. The conclusion of that Proposition is that there is a generic E for $\mathrm{Coll}(\omega, <\Omega)$ in some further extension of $L(\mathbb{R})[T]$, and the reals in our original $L(\mathbb{R})$ are those in $\bigcup_{\kappa<\Omega} L[T][E \cap \mathrm{Coll}(\omega, \kappa)]$.

3·24 Remark The generic E cannot be in $L(\mathbb{R})[T]$, since Ω is countable in $L[T, E]$ and is uncountable in $L(\mathbb{R})[T]$.

4: The symmetric collapse

4·0 Definition For an ordinal $\kappa \geq \omega$, $\mathrm{Coll}(\omega, \kappa)$ is the partial ordering that makes κ countable: for definiteness, conditions are injections $f : n \xrightarrow{1-1} \kappa$ for some $n \in \omega$, and $f \leq g$ iff $g = f \restriction dom(g)$.

$\mathrm{Coll}(\omega, A)$ where A is a set of infinite ordinals, is the product partial ordering, with finite supports, of the partial orderings $\mathrm{Coll}(\omega, \zeta)$ for $\zeta \in A$.

4·1 Remark There is a slight ambiguity here, since an ordinal κ is also the set of ordinals less than κ; to maintain a distinction we write $Coll(\omega, < \kappa)$ in the second case. For κ singular the effects are the same, as we shall shortly see; not so for κ regular and uncountable, since $\mathrm{Coll}(\omega, \kappa)$ makes κ countable, whereas $\mathrm{Coll}(\omega, < \kappa)$ has, by a Δ-system argument, the κ-chain condition, and hence makes κ the first uncountable cardinal.

We shall need the following result of Krivine:

4·2 Proposition *Let B be a separative notion of forcing of cardinality an uncountable cardinal κ, and suppose that $\Vdash_B \hat{\kappa}$ `is countable`. Then B has a dense subset isomorphic to $Coll(\omega, \kappa)$.*

Proof : Let C be the completion of B and $\dot{G}$ the canonical C-name for the (V, C) generic filter: so that for $c \in C$, $[\![\hat{c} \mathrel{\epsilon} \dot{G}]\!]^C = c$.

Since in V^C, B and therefore $G \cap B$ are countable, there is some name $\dot{f}$ such that

$$[\![f : \omega \longleftrightarrow \dot{G} \cap B]\!]^C = 1.$$

Note that if $c \Vdash \dot{f}(\hat{n}) = \hat{b}$ then $c \Vdash b \mathrel{\epsilon} \dot{G}$ and so $c \leq b$: for if $c \not\leq b$ then $\exists d :\leq c \; d \perp b$; but then $d \Vdash b \mathrel{\epsilon} \dot{G}$ (as c does) and $d \Vdash d \mathrel{\epsilon} \dot{G}$, contradicting the fact that the generic cannot contain contradictory things. Note also that each non-zero element of B splits into κ disjoint non-zero elements, otherwise B would somewhere have the κ chain condition and therefore $[\![\hat{\kappa}$ `is countable`$]\!] < 1$.

We now define a sequence of maximal anti-chains A_n in B. Begin by setting $A_{-1} = \{1_C\}$. Then, given A_{n-1}, form A_n as follows: given $p \in A_{n-1}$, let $A'_n(p)$ be a maximal antichain in $\{q \in B \mid q \leq p\}$ such that to each $p' \in A'_n(p)$ there is a $b = b(p')$ in B with $p' \Vdash \dot{f}(\hat{n}) = \hat{b}$. $A_n(p)$ is now formed by splitting each $p' \in A'_n(p)$ into κ many disjoint non-zero pieces — possible as the κ-c.c. fails everywhere — and A_n will be $\bigcup\{A_n(p) \mid p \in A_{n-1}\}$.

Put $A = \bigcup\{A_n \mid -1 \leq n < \omega\}$.

We assert that A is the desired subset of B. To see that A is isomorphic to $\mathrm{Coll}(\omega, \kappa)$, it is enough (and easy) to check that

i for any n, any two distinct elements of A_n are incompatible;
ii for $n < m$, if $p \in A_m$ and $q \in A_n$ then either $p \leq q$ or $p \perp q$;
iii each p in A_m has exactly κ many r in A_{m+1} with $r \leq p$.

Finally we check that A is dense in C:

iv $\forall b :\in B \; \exists p :\in A \;\; p \leq b$.

Proof of iv: given b there is a $c \leq b$ and an n such that $c \Vdash \dot{f}(\hat{n}) = b$, as $\dot{f}$ is forced to be onto B. As A_n is a maximal antichain, c must be compatible with some $p \in A(n)$. By construction, there is a b' such that $p \Vdash \dot{f}(\hat{n}) = b'$; but then $b = b'$, otherwise p would be incompatible with c; so $p \Vdash \dot{f}(\hat{n}) = \hat{b}$, and so $p \leq b$. Since $p \in A$ the proof is complete. $\dashv$ (4·2)

4·3 Corollary *For λ a singular strong limit cardinal, r.o.$(Coll(\omega, < \lambda)) \cong$ r.o.$(Coll(\omega, \lambda))$*

Here r.o. stands for "the regular open algebra over", and denotes the canonical complete Boolean algebra given by a separative partial ordering.

4·4 Corollary *If B is a separative notion of forcing of cardinality at most κ, then r.o.$(B \times Coll(\omega, \kappa))$ is isomorphic to r.o.$(Coll(\omega, \kappa))$.*

4·5 Remark The case $\kappa = \omega$ also works here and in the theorem, since all countable separative partial orderings are isomorphic to the Cohen ordering.

4·6 Definition If G is $(V, \mathrm{Coll}(\omega, < \lambda))$-generic, we write G_ν for $G \restriction \mathrm{Coll}(\omega, < \nu)$ and S_G for $\bigcup_{\nu < \lambda} \mathbb{R} \cap V[G_\nu]$.

4·7 PROPOSITION *Let λ be a [singular] strong limit cardinal, and suppose that in some (not necessarily Boolean) extension, W, of V there is a set S of reals such that*

(4·7·0) $\forall x\!:\in S\ \exists P$ [*P is in V a forcing of size less than λ and x is (V,P)-generic*]

(4·7·1) *given any two members x, y of S, every real in $V[x,y]$ is in S.*

(4·7·2) *each ordinal less than λ is recursive in at least one member of S.*

Then in some Boolean extension of W there is a filter G that is $(V, Coll(\omega,<\lambda))$-generic and for which

$$S = S_G.$$

Proof : Work in W, and consider the following notion of forcing, call it $\mathbb{A}$: the conditions are objects g where $\exists x\!:\in S\ \ g\in V[x]$ and for some $\zeta=\zeta(g)<\lambda$, g is $(V, Coll(\omega,<\zeta))$ generic.

Note that (iii) implies that $\mathbb{A}$ is non-empty. We give it a natural partial ordering:

$$g_1 \le_{\mathbb{A}} g_2 \Longleftrightarrow \zeta(g_1)\ge\zeta(g_2)\ \&\ g_2 = g_1\restriction\zeta(g_2)$$

Let G be $(W,\mathbb{A})$-generic.

We may construe G as potentially a generic for $Coll(\omega,<\lambda)$, it being a collection of maps of ω onto the ordinals less than λ.

a) G is indeed V-generic for that forcing.

Proof : let $D\in V$ be dense in $Coll(\omega,<\lambda)$; we shall show that $\forall\bar g\!:\in\mathbb{A}\ \exists g\!:\in\mathbb{A}\ \ g\le_{\mathbb{A}}\bar g$ & g meets D, where by "meets" we mean that $\exists q\!:\in D\ \ g\supseteq q$.

So let $\bar g\in\mathbb{A}$ be given, and set $\bar\nu=\nu(\bar g)$.

Any condition q in $Coll(\omega,<\lambda)$ has domain a finite subset of $\omega\times\lambda$ and for any ν splits naturally into $(q)_\nu = q\restriction(\omega\times\nu)$ and $(q)^\nu = q\restriction(\omega\times[\nu,\lambda))$; so that the map $q\mapsto((q)_\nu,(q)^\nu)$ defines an isomorphism between Coll$(\omega,<\lambda)$ and Coll$(\omega,<\nu)\times$ $Coll(\omega,[\nu,\lambda))$.

Write $D/\bar g = \{(q)^{\bar\nu}\mid q\in D\ \&\ (q)_{\bar\nu}\subseteq\bar g\}$.

Then by the standard theory of two-stage iterations, $D/\bar g$ is in $V[\bar g]$ and is dense in $Coll(\omega,[\bar\nu,\lambda))$. Let $r\in D/\bar g$: pick γ with $\bar\nu<\gamma<\lambda$ and $r\in Coll(\omega,[\bar\nu,\gamma))$: possible as all conditions are finite sets.

Let $\Delta = \big(\mathcal{P}(Coll(\omega,[\bar\nu,\gamma)))\big)^{V[\bar g]}$. Pick $y\in S$ such that $\bar g\in V[y]$ and in $V[y]$, Δ is countable. This is possible since λ is a strong limit cardinal; so the members of Δ are described in V by a set of names of size less than λ, and so in $V[\bar g]$, $\overline{\overline{\Delta}}=\kappa$, say, $<\lambda$. Any y in S in which κ is recursive will do.

Inside $V[y]$ we may build a $(V[\bar g], Coll(\omega,[\bar\nu,\gamma))$-generic, h, with $r\in h$. Now put $g=\bar g\cup h$. Then g is $(V, Coll(\omega,<\gamma)$-generic, as its tail is generic over its head, and its head is generic over V; $g\le\bar g$; $r\in D_g$ so for some q in D $(q)^{\bar\nu}=r$ & $(q)_{\bar\nu}\subseteq\bar g$; hence $q\subseteq g$ and so g meets D.

b) If $x\in S_G$, there is by definition of S_G a $\nu<\lambda$ with $x\in V[G\restriction\nu]$; but $G\restriction\nu=g$, say, is in $\mathbb{A}$; hence $\exists y\!:\in S\ \ g\in V[y]$; so $x\in V[y]$, and so $x\in S$.

c) Suppose $a\in S$: we show that $\forall\bar g :\in\mathbb{A}\ \exists g:\in\mathbb{A}\ g\le\bar g$ & $a\in V[g]$. Density will then imply that $\Vdash_{\mathbb{A}}\bigwedge a\!:\epsilon\,\hat S\bigvee\nu\!:<\hat\lambda\ \ a\!:\epsilon\,\hat V[\dot G\restriction\nu]$.

To get our bearings, suppose first that $\bar g$ is the empty condition.

Let a be (V,P_a)-generic, where $\overline{\overline{P_a}}=\kappa$, say, $<\lambda$ and we may suppose that P_a is a complete Boolean algebra generated by the truth values $[\![\dot a(\hat n)=\hat m]\!]^{P_a}$. Let h be $(V[a], Coll(\omega,\kappa))$-generic. Then (a,h) is $\big(V, P_a\times Coll(\omega,\kappa)\big)$-generic. Since, by Krivine, $r.o.(P_a\times Coll(\omega,\kappa))\cong r.o.(Coll(\omega,\kappa))$, there is a g which is $Coll(\omega,\kappa)$-generic such that $V[a][h]=V[g]$.

Now since we may predict in advance how much less than λ needs to be countable to build h and find g, we may pick $y\in S$ so that the argument of the last paragraph may be carried out in $V[y]$: hence our $g\in\mathbb{A}$; since $a\in V[g]$, we are done.

Now we modify this: given $\bar g$ (non-empty) in $\mathbb{A}$, let $b\in S$ be such that $\bar g\in V[b]$, and let $c=\langle a,b\rangle$. Then $c\in S$, so c is generic over V by an algebra P_c of size less than λ; since $\bar g\in V[c]$, c is also generic over $V[\bar g]$, by an algebra Q_c, say, of cardinality less than λ in $V[\bar g]$: call this cardinal κ. Actually Q_c will be a quotient algebra of P_c, by the filter generated by truth values associated with the P_c-name of $\bar g$.

Let $\bar\nu = \nu(\bar g)$. For some $z \in S$ with $c \in V[z]$, we can build in $V[z]$ an object h which is $(V[c], Coll(\omega, [\bar\nu, \bar\nu + \kappa))$-generic. Since in $V[\bar g]$, $Q_c \times Coll\big(\omega, [\bar\nu, \bar\nu+\kappa)\big) \cong Coll(\omega, [\bar\nu, \bar\nu + \kappa))$, there is a $(V[\bar g], Coll[\omega, [\bar\nu, \bar\nu+\kappa))$-generic g' such that $V[\bar g][g'] = V[\bar g][c][h]$. Put $g = \bar g \cup g'$: then g is $(V, Coll(\omega, < \bar\nu + \kappa)$-generic; $g \in \mathbb{A}$ as $g \in V[z]$ and $z \in S$; and $a \in V[c] \subseteq V[g]$. $\dashv$ (4·7)

5: A commuting diagram

We are in $L(\mathbb{R}_0)$ and write $M_0 = HOD^{L(\mathbb{R}_0)} = L[S_0]$. We are going to build a commuting diagram

$$\begin{array}{ccccccccccccc}
M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & \dots & \xrightarrow{j_{n-1,n}} & M_n & \xrightarrow{j_{n,n+1}} & M_{n+1} & \xrightarrow{j_{n+1,n+2}} & \dots & & M_\omega \\
\cap\!| & & \cap\!| & & & & \cap\!| & & \cap\!| & & & & \cap\!| \\
L(\mathbb{R}_0) & \xrightarrow{h_{01}} & L(\mathbb{R}_1) & \xrightarrow{h_{12}} & \dots & \xrightarrow{h_{n-1,n}} & L(\mathbb{R}_n) & \xrightarrow{h_{n,n+1}} & L(\mathbb{R}_{n+1}) & \xrightarrow{h_{n+1,n+2}} & \dots & & L(\bigcup_n \mathbb{R}_n)
\end{array}$$

where the vertical relationships are inclusions, the arrows on the upper horizontal line are elementary embeddings into progressively shrinking inner models, and the arrows on the lower horizontal line are generic elementary embeddings into submodels of some large generic extension of the universe $L(\mathbb{R}_0)$ with $\mathbb{R}_0 \subseteq \mathbb{R}_1 \subseteq \mathbb{R}_2 \dots$.

First we construct the map j_{01}. This we build in $L(\mathbb{R}_0)$. Define a reducibility $\leq_0$ by

$$x \leq_0 y \iff_{\rm df} x \in M_0[y].$$

In view of the principle 1·7 that we have called *Slide* that is saying that x is (in $L(\mathbb{R}_0)$) ordinal definable from y.

Let $\mathcal{H}_0$ be the set of corresponding degrees and μ_0 the Martin measure on it. We now build the ultrapower of M_0 by μ_0: the building bricks are functions in $L(\mathbb{R}_0)$ defined on $\mathcal{H}_0$ with values in M_0.

That is straightforward: we have DC and the countable additivity of μ_0 to hand to prove well-foundedness and M_0 models AC, so we shall be able to prove Łoś's theorem.

Let M_1 be the transitive collapse of M_0, and j_{01} the usual embedding defined by the constant functions.

5·0 Lemma *The critical point of j_{01} is ω_1.*

Proof : $j_{01}(\alpha) = \alpha$ for countable ordinals by the countable additivity of μ_0. On the other hand, let $f((x)_0) = \omega_1^{HOD[x]}$; this is less than Ω as $HOD[x] \models AC$. f is less than Ω everywhere; it exceeds any countable α on a cone, so $j_{01}(\Omega) > [f] \geq \Omega$. $\dashv$ (5·0)

5·1 Lemma *(i) $M_1 \subseteq M_0$; (ii) j_{01} is a class of M_0; (iii) $M_1 \neq M_0$;*
(iv) $j_{01} \restriction V_\Omega \cap M_0 \subseteq id$; (v) $M_0 \cap V_\Omega \subseteq M_1$.

Proof of (i): As M_0 is an AC model, all sets are coded by sets of ordinals, so let $x \subseteq ON$, $x \in M_0$. x is OD. j_{01} is OD so $j_{01}(x)$ is an OD set of ordinals and therefore in HOD which is M_0. In particular, $M_0 = L[S]$, and so $M_1 \models V = L[j_{01}(S)]$; $j_{01}[S] \in M_0$ so $M_1 \subseteq M_0$. $\dashv$

Proof of (ii): We need only to know j_{01} on ordinals and sets of ordinals to compute the rest. Consider $\{\langle j_{01}(T), T\rangle | T \subseteq ON,\, T \in OD\}$. This is clearly a class of M_0 as j_{01} is OD. $\dashv$

Proof of (iii): from (i) and (ii), using Kunen's theorem[R1] that no model of ZFC can see an elementary embedding of itself into itself.

Proof of (iv): note first that if $a \subseteq \lambda < \Omega$ and $a \in \mathrm{Dom}\, j_{01}$ then $j(a) \subseteq j(\lambda) = \lambda$ by Lemma 5·0, and for $\nu < \lambda$, $\nu \in a \iff j(\nu) \in j(a) \iff \nu \in j(a)$, since $j(\nu) = \nu$; hence $j(a) = a$. Now reason much as in Proposition 1·8: since Ω is strongly inaccessible in these inner models of ZFC, each transitive set u in M_0 of rank less than Ω is coded by some extensional well-founded relation on some ordinal less than Ω; that relation will be sent to itself by j, and hence $j(u) = u$. Since each set of rank less than Ω is a member of a transitive set of rank less than Ω, (iv) follows. $\dashv$

Proof of (v): immediate from (iv). $\dashv$ (5·1)

[R1] See Kanamori, *The Higher Infinite*, page ???

5·2 REMARK It is known that in this context ω_1 is definable in M_0 as the least measurable cardinal; since it is also the critical point of the embedding, M_0 cannot be the same model as M_1.

In M_1 we may build the algebra $\mathbb{K}^\omega$, discussed in I·3, which we shall now call $\mathbb{K}_1^\omega$; then if we reify $M_1^{\mathbb{K}_1^\omega}$ we will get a model of the form $L[\vec{\alpha}]$ with $\mathbb{R}_1 = \{\alpha_i | i < \omega\}$: we want to do this carefully so that $\mathbb{R}_0 \subseteq \mathbb{R}_1$. As we shall see later on, a natural generic elementary embedding $h_{01} : L(\mathbb{R}_0) \longrightarrow L(\mathbb{R}_1)$ is in the wings waiting to be defined: but if we define it directly we shall have problems proving that it is an elementary embedding, since AC fails in the models $L(\mathbb{R}_i)$.

We proceed thus: in $L(\mathbb{R}_0)$ form the product $\mathfrak{P}$ of all models of the form $HOD[x]$, for $x \in \mathbb{R}_0$: $\mathfrak{P}$ is the class of all functions $f \in L(\mathbb{R}_0)$ with domain $\mathbb{R}_0$ and $f(x) \in HOD[x]$ for each x.

We shall add generically an ultrafilter $\tilde{G}$ to $L(\mathbb{R}_0)$: in the extension $L(\mathbb{R}_0)[\tilde{G}]$, $\tilde{G}$ will be a $\hat{V}$-complete ultrafilter in the algebra $(\mathcal{P}(\mathbb{R}_0))^{L(\mathbb{R}_0)}$. We shall use $\tilde{G}$ to factor $\mathfrak{P}$, thus obtaining a model $\mathcal{M}$, which will prove to be well-founded.

The addition of $\tilde{G}$ to $L(\mathbb{R}_0)$ adds a new real, g_1, represented in the ultraproduct by the identity function $x \mapsto x$ on $\mathbb{R}_0$, and $\mathcal{M}$ will prove to be isomorphic to $M_1[g_1]$.

We shall then reify $\mathbb{R}_1$ so that $g_1 = \alpha_0$.

The ultrafilter $\tilde{G}$ will be related to the Martin measure; indeed if $\psi : \mathbb{R}_0 \to \mathcal{H}$ is the natural map $x \mapsto (x)_0$, we shall have $\psi_*(\tilde{G}) = \mu_0$. That projection in turn induces a natural embedding of $M_1 = \mu_0$-prod M_0 into $\tilde{G}$-prod M_0, which embedding, we shall find, is actually a surjection.

Start then by putting

$$\mathcal{I} \ =_{\mathrm{df}} \ \psi_*^{-1}\{\mathcal{X} \subseteq \mathcal{H} \mid \mu_0(\mathcal{X}) = 0\}.$$

Disentangling the notation, that is $\mathcal{I} = \{A \subseteq \mathbb{R}_0 \mid \exists y \forall z \geq_0 y \ z \notin A\}$, the set of those subsets A of $\mathbb{R}_0$ such that no $\mathcal{H}$-degree in some cone is represented by any element of A.

5·3 LEMMA *$\mathcal{I}$ is a countably additive ideal in $\mathcal{P}(\mathbb{R}_0)$.*

Proof : given a sequence of sets X_n in $\mathcal{I}$, we can use countable choice to pick reals x_n with the cone $C^0_{x_n}$ of $\mathcal{H}_0$-degrees above $(x_n)_0$ disjoint from X_n; then $\bigcup_n X_n$ will be disjoint from the $\mathcal{H}_0$-cone above any real in which all the x_n are recursive. ⊣ (5·3)

Related to $\mathcal{I}$ are two other families of sets of reals,

$$\tilde{\mathcal{I}} \ =_{\mathrm{df}} \ \{A \subseteq \mathbb{R}_0 \mid \exists y \forall z \geq_0 y \ z \in A\},$$

the filter dual to $\mathcal{I}$, comprising those sets of reals the complements of which are in $\mathcal{I}$; and

$$\mathcal{I}^+ \ =_{\mathrm{df}} \ \{A \subseteq \mathbb{R}_0 \,|\, \forall y \exists z \geq_0 y \ z \in A\},$$

the complement of $\mathcal{I}$ in $\mathcal{P}\mathbb{R}_0$.

The partial ordering to add $\tilde{G}$ is $\mathcal{I}^+$, ordered under inclusion: the reader will notice the link with familiar discussions of precipitous ideals.

5·4 LEMMA *$\tilde{G}$ is an ultrafilter in $\widehat{\mathcal{P}\mathbb{R}_0}$ extending $\tilde{\mathcal{I}}$.*

Proof : plainly a filter; and given any condition A and subset B of $\mathbb{R}$ in $L(\mathbb{R}_0)$, either $A \cap B$ or $A \smallsetminus B$ will be a condition forcing $\hat{B} \in \dot{G}$ or $\hat{B} \notin \dot{G}$. ⊣ (5·4)

We use $\tilde{G}$ to factor $\mathfrak{P}$. Thus we may consider the building bricks for our generic ultrapower to be pairs (f, A) where $A \in \mathcal{K} = \mathcal{I}^+$, $\mathrm{Dom} f = A$, $f \in L(\mathbb{R}_0)$ and $\forall x{:}{\in} A \ f(x) \in HOD[x]$.

Thus in $V[\tilde{G}]$, we put $\mathcal{M} = \tilde{G}$-prod$\langle HOD[x] \mid x \in \mathbb{R}_0\rangle$.

Why should we believe that $\mathcal{M}$ is well-founded ?

5·5 LEMMA *Let $f \in \prod_{x \in \mathbb{R}_0} HOD[x]$ with $\forall x f(x) \in HOD$, dom$(f) = A \in \mathcal{I}^+$. Then $\exists B \subseteq A \ B \in \mathcal{I}^+$ and $f \restriction B$ is $=_H$ invariant.*

Proof : There is a canonical well-ordering $\leq^{HOD}$ of HOD. Let

$$B = \{x \in A \mid f(x) \text{ is } \leq^{HOD}\text{-minimal among } f(y) \text{ with } y =_H x \text{ and } y \in A\}.$$

Hence we can unambiguously define g on $B^* \ =_{\mathrm{df}} \ \{[x]_H \mid x \in B\} = \{[x]_H \mid x \in A\}$ as $g([x]_H) = f(x)$. $\mu_H(B^*) = 1$ as $B \notin I$. ⊣ (5·5)

Since $\tilde{G}$ projects to the Martin measure $\mathfrak{m}_0$, there is a natural embedding of the ultrapower by $\mathfrak{m}_0$ built from $=_0$-invariant functions into the ultrapower by $\tilde{G}$ built using arbitrary functions. The above lemma implies that as far as the ordinals of the two models are concerned, the embedding is surjective; that is, that the ordinals of $\mathcal{M}$ may be identified with those of M_1, and so we may identify M_1 with a submodel of $\mathcal{M}$; in particular the ordinals of $\mathcal{M}$ are well founded; $\mathcal{M}$ is a ZFC model as the components are and so Łoś's theorem applies; so the rank function within $\mathcal{M}$ will now guarantee that $\mathcal{M}$ itself is well-founded.

Transitise $\mathcal{M}$ as N_1, and let g_1 be the evaluation of $id\restriction\mathbb{R}_0$. By Łoś, g_1 is a real and

$$N = M_1[g_1].$$

Now each real x in $L(\mathbb{R}_0)$ is $(M_0, \mathbb{K}_0^1)$-generic, $\mathbb{K}_0^1$ being the Vopěnka algebra of dimension 1: and we can use the term "generic" more easily, knowing that in our context, $L[S_0, x] = L[S_0, G_x]$; so g_1 is $(L[S_1], j_{01}(\mathbb{K}_0^1))$-generic.

$\mathbb{K}_0^1$ is a regular subalgebra of $\mathbb{K}_0^\omega$, in symbols $\mathbb{K}_0^1 \lhd \mathbb{K}_0^\omega$; so in M_1,

$$j_{01}(\mathbb{K}_0^1) \lhd j_{01}(\mathbb{K}_0^\omega).$$

Hence there is an algebra $\mathbb{L}$ in $L[S_1][g_1]$, such that extending that model by that algebra yields a model that is a generic extension of $L[S_1]$ by $j_{01}(\mathbb{K}_0^\omega)$.

5·6 Exercise Is it true that once we have set up $\mathbb{R}_1$ we shall be able to define $\mathbb{L}$ as those (n, D) in $\mathbb{K}_1^\omega$ such that each n-tuple in D has g_1 as its first component ?

In $L[S_0]$ the algebra $\mathbb{K}_0^\omega$ has the property that forcing by it adds a sequence $\vec{\alpha}$, the range of which is denoted by the forcing term $\dot{I}$. That range is a set of reals, and indeed it is forced that $\dot{I} = \dot{\mathbb{R}}\dot{\cap}L(\dot{I})$. Further the HOD of $L(\dot{I})$ proves to be $L[S_0]$, $\mathbb{K}_0^1$ and $\mathbb{K}_0^\omega$ prove to be the algebras defined in $L(\dot{I})$ by the Vopěnka process, and S_0 to be the subset of $\Theta^{L(\dot{I})}$ that defines $HOD^{L(\dot{I})}$.

All those statements about $L(\dot{I})$ happen with truth value $\mathbb{1}$, computed in the algebra $\mathbb{K}_0^\omega$ in $L[S_0]$, as argued in Chapter I, using the weak homogeneity of $\mathbb{K}_0^\omega$ and the invariance of the term $\dot{I}$ under automorphisms of the forcing.

Hence $j_{01}(\mathbb{K}_0^\omega)$ will enjoy all those properties in M_1.

Thus if we, working in $L[S_1][g_1]$ extend by $\mathbb{L}$, we shall add a sequence of reals which, when prefaced by g_1 forms a sequence $\vec{\beta}$ that is $(L[S_1], j_{01}(\mathbb{K}_0^\omega))$-generic, with $\beta(0) = g_1$.

5·7 Definition We set $\mathbb{R}_1 \ =_{\rm df}\ \{\beta(i) \mid i \in \omega\}$.

Then, applying the embedding j_{01}, we find that the following holds in $L[S_1]$: the algebra $j_{01}(\mathbb{K}_0^\omega)$ has the property that forcing by it adds a sequence $\vec{\beta}$, the range of which is denoted by a forcing term which we shall denote by $\dot{J}$. That range is a set of reals, and indeed it is forced that $\dot{J} = \dot{\mathbb{R}}\dot{\cap}L(\dot{J})$. Further the HOD of $L(\dot{J})$ proves to be $L[S_1]$, $j_{01}(\mathbb{K}_0^1)$ and $j_{01}(\mathbb{K}_0^\omega)$ prove to be the algebras defined in $L(\dot{J})$ by the Vopěnka process, and S_1 to be the subset of $\Theta^{L(\dot{J})}$ that defines $HOD^{L(\dot{J})}$.

All those statements about $L(\dot{J})$ happen with truth value $\mathbb{1}$, computed in the algebra $j_{01}(\mathbb{K}_0^\omega)$ in $L[S_1]$.

So $\mathbb{R}_1 = \mathbb{R} \cap L(\mathbb{R}_1)$; $HOD^{L(\mathbb{R}_1)} = L[S_1]$; $j_{01}(\mathbb{K}_0^1)$ and $j_{01}(\mathbb{K}_0^\omega)$ are the algebras defined in $L(\mathbb{R}_1)$ by the Vopěnka process; and $j_{01}(S_0)$ is the corresponding subset of $\Theta^{L(\mathbb{R}_1)}$ that defines $HOD^{L(\mathbb{R}_1)}$.

5·8 Definition We may now feel entitled to call these objects $\mathbb{K}_1^1$, $\mathbb{K}_1^\omega$ and S_1.

5·9 Proposition *AD and DC hold in* $L(\mathbb{R}_1)$.

Proof : by elementarity of j_{01}, using the fact that the relativisation of those statements to $L(\dot{J})$ hold with truth value $\mathbf{1}^{\mathbb{K}}$.

$\dashv$ (5·9)

5·10 Proposition $\mathbb{R}_0 \subseteq \mathbb{R} \cap M_1[g_1] \subseteq \mathbb{R}_1$.

Proof : Let $y \in \mathbb{R}_0$. For almost all x, $y \in HOD[x]$; so by Łoś, $y \in M_1[g_1]$; but $M_1[g_1] \subseteq L(\mathbb{R}_1)$, as $g_1 \in \mathbb{R}_1$ and $M_1 = HOD^{L(\mathbb{R}_1)}$, so $y \in L(\mathbb{R}_1)$ and in particular $y \in \mathbb{R}_1$. $\dashv$ (5·10)

5·11 Proposition $\mathbb{R}_0 \notin L(\mathbb{R}_1)$.

Proof : if $\mathbb{R}_0$ were in $L(\mathbb{R}_1)$, it would be countable there, since it is a subset of the reals of $L[S_1, g_1]$, and would therefore be a countable member of $L[S_1, a]$ for some real $a \in \mathbb{R}_1$. But then each ordinal less than

$\Theta_0 = (\Theta)^{L(\mathbb{R}_0)}$ would be countable in $L[S_1, a]$ and so Θ_0 would be countable in $L(\mathbb{R}_1)$; hence ${S_0}^\sharp$ would be a member of $L(\mathbb{R}_1)$.

That would mean, putting all our forcing extensions together, that ${S_0}^\sharp$ can be added to $L[S_0]$ by a set forcing, $\mathbb{P}$ say; but then $\mathbb{P}\times\mathbb{P}$ will add two distinct ${S_0}^\sharp$'s, an impossibility. $\dashv$ (5·11)

5·12 Now we wish to define a map

$$h_{01} : L(\mathbb{R}_0) \longrightarrow L(\mathbb{R}_1)$$

which is elementary, extends j_{01} and is the identity on $\mathbb{R}_0$.

Every element of $L(\mathbb{R}_0)$ is definable from ordinals and reals; so our natural instinct is to say that

$$\text{if} \quad x \in L(\mathbb{R}_0),\ x = (f(a,\zeta))^{L(\mathbb{R}_0)} \qquad (a \in \mathbb{R}_0,\ \zeta \in ON)$$
$$\text{then} \quad h_{01}(x) = (f(a, j_{01}(\zeta))^{L(\mathbb{R}_1)}$$

This will be well-defined and elementary if we prove that

$$L(\mathbb{R}_0) \models \phi(\vec{a},\, \vec{\zeta}) \iff L(\mathbb{R}_1) \models \phi(\vec{a},\, \vec{\jmath}_{01}(\zeta)).$$

Let us take the case of one real and one ordinal. Let $P = \{a \in \mathbb{R}_0 | \phi(a,\zeta)\}^{L(\mathbb{R}_0)}$. In $L(\mathbb{R}_0)$, P is OD and hence ∞-Borel with code in M_0. Let (T,λ) be an ∞-Borel tree-code, which, according to our discussion in Chapter II, means that T is a well-founded tree of finite sequences of ordinals, and λ a labelling of the nodes of T; and we write $x \in B_\infty(T,\lambda)$ to mean that the real x is in the ∞-Borel set tree-coded by (T,λ).

In M_0,

$$[\![L(\mathbb{R}_0) \models \forall x \,\epsilon\, R\ \phi(x, \hat{\zeta}) \longleftrightarrow x \,\epsilon\, \dot{B}_\infty(\hat{T}, \hat{\lambda})]\!]^{C_0} = 1$$

so this is true in M_1:

$$[\![L(\mathbb{R}_1) \models \forall x \,\epsilon\, R\ \phi(x, \widehat{j_{01}(\zeta)}) \longleftrightarrow x \,\epsilon\, \dot{B}_\infty(\widehat{j_{01}(T)}, \widehat{j_{01}(\lambda)})]\!]^{C_0} = 1$$

so it will suffice to prove that $a \in B_\infty(T,\lambda) \iff a \in B_\infty(j_{01}(T), j_{01}(\lambda))$.

"$a \in B_\infty(T,\lambda)$" is computable in $L[T,\lambda,a] \subseteq M[a]$, so whenever $a \leq_H x$, $M[x]$ computes $a \in B_\infty(T,\lambda)$; so by Łoś, $M[g]$ computes $a \in B_\infty(j_{01}(T), j_{01}(\lambda))$ accordingly; and so $a \in B_\infty(j_{01}(T), j_{01}(\lambda))$. Similarly for $a \notin B_\infty(T,\lambda)$. $\dashv$

That h_{01} extends j_{01} is readily checked: it is true for ordinals by definition. Let $x \in M_0$: then for some ζ, $x = f(\zeta)$; so $h_{01} = \bigl(F(j_{01}(\zeta))\bigr)^{L(\mathbb{R}_1)}$.

We now use the weak homogeneity of C_0:

$$\begin{aligned} x = f(\zeta) &\implies M_0 \models ([\![\hat{x} = f(\hat{\zeta})]\!]^{C_0} = 1) \\ &\implies M_1 \models ([\![\widehat{j_{01}(x)} = f(\widehat{j_{01}(\zeta)})]\!]^{C_0} = 1) \\ &\implies L(\mathbb{R}_1) \models j_{01}(x) = f(j_{01}(\zeta)) \end{aligned}$$

and so $h_{01}(x) = j_{01}(x)$ as required. $\dashv$

Hence we have built a commuting diagram:

$$\begin{array}{ccc} M_0 & \xrightarrow{j_{01}} & M_1 \\ | \! \cap & & | \! \cap \\ L(\mathbb{R}_0) & \xrightarrow{h_{01}} & L(\mathbb{R}_1) \end{array}$$

5·13 We may now iterate this construction and build a sequence

$$\begin{array}{ccccccccc} M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & M_2 & \xrightarrow{j_{23}} & M_3 & \xrightarrow{j_{34}} & \ldots \\ | \! \cap & & | \! \cap & & | \! \cap & & | \! \cap & & \\ L(\mathbb{R}_0) & \xrightarrow{h_{01}} & L(\mathbb{R}_1) & \xrightarrow{h_{12}} & L(\mathbb{R}_2) & \xrightarrow{j_{23}} & L(\mathbb{R}_3) & \xrightarrow{j_{34}} & \ldots \end{array}$$

We can do this because what we need to get started was that AD is true in $L(\mathbb{R}_0)$ and that $M_0 = HOD^{L(\mathbb{R}_0)}$. After one step we reach M_1 and $L(\mathbb{R}_1)$ with $M_1 = HOD^{L(\mathbb{R}_1)}$; and AD is true in $L(\mathbb{R}_1)$ by the elementarity of h_{01}.

What lies at the end ? Notice that the top line is entirely within the ground model M_0; indeed

$$M_0 \supseteq M_1 \supseteq M_2 \ldots$$

As j_{01} is a class of M_0, we may show that the direct limit M_ω exists and is definable in M_0, and then we may apply an argument of Gaifman to show that it is well-founded, as follows:

we consider the least ordinal ζ such that the "ordinals" of M_ω below $j_{0\omega}(\zeta)$ are ill-founded. Then there will be some "ordinal" c below $j_{0\omega}(\zeta)$ such that the "ordinals" below c are ill-founded. c is of the form $j_{n\omega}(\xi)$ for some natural number n and ordinal ξ. Seen from the perspective of M_n, ξ is an ordinal such that the "ordinals" of M_ω are ill-founded below the image of ξ under what M_n thinks is the definable mapping $j_{0\omega}$ (actually the mapping $j_{n\omega}$); but if so, $j_{0n}(\zeta)$ should be less than or equal to ξ, from the definition of ζ; whereas $j_{0n}(\zeta) > \xi$, since $j_{n\omega}(\xi) = c < j_{0\omega}(\zeta) = j_{n\omega}(j_{0n}(\zeta))$.

Therefore the direct limit along the bottom line will also be well-founded since the two models have the same ordinals. Interpreting both as transitive models, we find the former will be of the form $L[S_\omega]$ for some set of ordinals $S_\omega \in \bigcap_i L[S_i]$; whereas the latter will be of the form $L(\mathbb{R}_\omega)$ for some set $\mathbb{R}_\omega$ of reals lying in some large generic extension of our original universe $L(\mathbb{R}_0)$.

5·14 PROPOSITION $\mathbb{R}_\omega = \bigcup_i \mathbb{R}_i$.

Proof : by the fact that the direct limit will be an elementary extension of each point on the chain. Anything in the limit model which thinks it is a real will be an image of some real in an intermediate model; but each such real is left untouched by all subsequent embeddings, being (say) a subset of ω, and $j_{0\omega}(\omega) = \omega$. ⊣ (5·14)

5·15 PROPOSITION $\mathbb{R}_\omega = I\!\!R \cap L(\mathbb{R}_\omega)$.

Proof : by model theory, since at each stage before the limit $\mathbb{R}_n = I\!\!R \cap L(\mathbb{R}_n)$. ⊣ (5·15)

In the next section we shall find an inner model of M_0 and an ω-sequence of Woodin cardinals in that inner model with supremum the ω_1 of $L(\mathbb{R}_\omega)$.

6: A chain of submodels

Now within the tower of models we have erected, we are going to define ordinals θ_i and λ_i for $1 \leqslant i < \omega$, with $\theta_1 < \lambda_1 < \theta_2 < \lambda_2 < \theta_3 < \ldots$; and subsets S_i of θ_i and Q_i of λ_i. The uniformity of our definitions will be important, and will be discussed after the definitions have been set up. Our definitions will have these properties:

(6·0) $$Q_i \subseteq \lambda_i < \omega_1^{L[R_i]};$$

(6·1) $$\lambda_i \text{ is strongly inaccessible in } M_i[Q_i];$$

(6·2) $$\theta_i \text{ is Woodin in } M_i[Q_i] \text{ and is less than } \lambda_i;$$

(6·3) $$Q_{i-1} = \lambda_{i-1} \cap Q_i \text{ for } i > 1$$

(6·4) $$M_i[Q_i] \cap V_{\lambda_i} = L_{\lambda_i}[Q_i] = L[Q_i] \cap V_{\lambda_i};$$

(6·5) $$\langle S_{i+1}, Q_{i+1}\rangle \text{ is definable uniformly in } L[S_i, Q_i] \text{ from } \langle S_i, Q_i\rangle.$$

Define $M_0 = HOD^{L[R_0]} = L[S_0]$. Set $x =_0 y \ =_{\mathrm{df}} \ M_0[x] = M_0[y]$ and $x \leq_0 y \ =_{\mathrm{df}} \ x \in M_0[y]$.
Form

$$\begin{aligned} N_1^x &= HOD_{S_0}^{L[S_0,x]} \\ \theta_1^x &= \omega_2^{L[S_0,x]} \\ \lambda_1^x &= \text{the first strongly inaccessible cardinal in } N_1^x \text{ above } \theta_1^x \\ Q_1^x &= \text{ the } <_{\{S_0\}}\text{-first subset of } \lambda_1^x \text{ that is stout in } N_1^x \end{aligned}$$

Here $<_{\{S_0\}}$ is the canonical well-ordering (definable in $L[S_0, x]$ from the parameter S_0) of N_1^x. λ_1^x exists and is countable as Ω is Mahlo in N_1^x.

6·6 REMARK There is some room for manœuvre in the definition of λ_i; we might take it to be the first strongly inaccessible cardinal of $L[S_0, x]$. In the original, λ_i was taken to be θ_i which creates problems that I see not how to solve. We should compare with the first two versions of the proof.

These are $=_0$-invariant; hence we may define

$$N_1 = \prod_{[x]_0 \in \mathcal{H}_0} N_1^{[x]_0}/\mu_0$$
$$\theta_1 = \prod_{[x]_0 \in \mathcal{H}_0} \theta_1^{[x]_0}/\mu_0$$
$$\lambda_1 = \prod_{[x]_0 \in \mathcal{H}_0} \lambda_1^{[x]_0}/\mu_0$$
$$Q_1 = \prod_{[x]_0 \in \mathcal{H}_0} Q_1^{[x]_0}/\mu_0\,;$$

and by Łoś we have

$$\lambda_1 \text{ is is the first strongly inaccessible} > \theta_1 \text{ in } N_1;$$
$$Q_1 \subseteq \theta_1 \text{ is stout in } N_1;$$
$$L_{\lambda_1}[Q_1] = N_1 \cap V_{\lambda_1}.$$
$$\theta_1 \text{ is Woodin in } N_1$$

The last clause holds because by the results of CHapter VI, for a cone of x's, θ_1^x is Woodin in N_1^x.

6·7 EXERCISE Which kind of cone ?

6·8 REMARK It might be more elegant to choose the Q's so that $L_{\theta_1}[Q_1 \cap \theta_1] = N_1 \cap V_{\theta_1}$.

We have built N_1 using $\mathcal{H}_0$ and μ_0, which are definable in $L[R_0]$; each N_1^x has a definable well-ordering and hence $N_1 \subseteq M_0$.

Note that M_0 is a subclass of each N_1^x as $M_0 = L[S_0]$; so if we set $M_1 = M_0^{\mathcal{H}_0}/\mu_0$, we shall have $M_1 \subseteq N_1$. Inside N_1 we may form $M_1[Q_1]$; so we have $L[Q_1] \subseteq M_1[Q_1] \subseteq N_1[Q_1] = N_1$; the outer two have the same sets of rank less than θ_1; so the middle one must do as well, and hence θ_1 is Woodin in $M_1[Q_1]$ and in $L[Q_1]$ since it is so in N_1; the requisite extenders, being each of rank less than θ_1, lie in each of the smaller models, and those models have possibly fewer subsets of θ_1 to consider.

6·9 We have already seen how to set up $L(\mathbb{R}_1)$ and find a real $g_1 \in \mathbb{R}_1$ which can be used to create a commuting diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{j_{01}} & M_1 \\ \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\ L(\mathbb{R}_0) & \xrightarrow{h_{01}} & L(\mathbb{R}_1) \end{array}$$

where $S_1 = j_{01}[S_0]$, $M_1 = HOD^{L(\mathbb{R}_1)} = L[S_1]$. $\prod_{x \in \mathbb{R}_0} M_0[x]/\tilde{G}$ is $M_1[g_1]$ and $N_1 = HOD_{\{S_1\}}^{M_1[g_1]}$, though it is not clear from this definition that $N_1 \subseteq M_0$.

6·10 REMARK Note that $\theta_1 = \omega_2^{M_1[g_1]}$, so $\theta_1 < \omega_1^{L(\mathbb{R}_1)}$. $\lambda_0 = cp(j_{01}) = \omega_1^{L(\mathbb{R}_0)}$; $\mathbb{R}_0 \subseteq M_1[g_1]$; so $\lambda_0 \leq \omega_1^{M_1[g_1]} < \theta_1$.

However the M_1–$L(\mathbb{R}_1)$ relationship is exactly that of M_0 and $L(\mathbb{R}_0)$; we wish however to modify the definition of N at this stage; to ensure that $\mathfrak{M}(Q_1)$ will be a rank-initial segment of the next model, N_2.

We have $M_1 = L[S_1]$. Form

(6·11) $\quad N_2^x = HOD_{\{S_1, Q_1\}}^{L[S_1, Q_1, x]}$

(6·12) $\quad \theta_2^x = \omega_2^{L[S_1, Q_1, x]}$

(6·13) $\quad \lambda_2^x =$ the first strongly inaccessible cardinal in N_2^x above θ_2^x

(6·14) $\quad Q_2^x =$ the first subset of λ_2^x in the (S_1, Q_1, x) canonical well-ordering of N_2^x to be a stout extension of Q_1 in N_2^x, so that$L_{\lambda_2^x}[Q_2^x] = N_2^x \cap V_{\lambda_2^x}$, if such stout extensions exist; otherwise take $Q_2^x = \varnothing$.

Then in $L(\mathbb{R}_1)$ we define $x \leq_1 y \iff_{\rm df} x \in L[S_1, y]$, and define the Martin measure μ_1 on the $=_1$-equivalence classes as before; then we may define

$$N_2 = \prod N_2^{[x]}/\mu_1 \tag{6·15}$$

$$\theta_2 = \prod \theta_2^{[x]}/\mu_1 \tag{6·16}$$

$$\lambda_2 = \prod \lambda_2^{[x]}/\mu_1 \tag{6·17}$$

$$Q_2 = \prod Q_2^{[x]}/\mu_1 \tag{6·18}$$

To avert disaster we must prove the following before proceeding:

6·19 Lemma *(i) Q_1 is stout in N_2;*
(ii) For all x in some $\mathcal{H}_2$-cone, Q_1 is stout in N_2^x;
(iii) For all x in some $\mathcal{H}_2$-cone, Q_2^x is a stout extension of Q_1 in N_2^x.

Proof : by Remark 6·9 Q_1 is a countable object in $L(\mathbb{R}_1)$, and so

$$L[Q_1] \subseteq L[S_2, Q_1) \subseteq N_2 \subseteq L[S_1, Q_1] = M_1[Q_1].$$

Since the first and the last models in that sequence have the same sets of rank less than λ_1, by the stoutness of Q_1 in $M_1[Q_1]$, the same will be true of N_2; thus Q_1 is stout in N_2. Part (ii) follows by Ło ś; by our definition we will then have chosen a stout extension of Q_1 in N_2^x to be Q_2^x. ⊣ (6·19)

6·20 Proposition *θ_2 is Woodin in N_2, and Q_2 is a stout extension of Q_1 in N_2, so that $L_{\lambda_2}[Q_2] = N_2 \cap V_{\lambda_2}$.*

We note that $\prod L[S_1, Q_1]^{\mathcal{H}_1}/\mu_1 \cong L[S_2, Q_1]$, as Q_1 is a countable object in $L(\mathbb{R}_1)$ by the Remark 6·9: further each N_2^x has an $\{S_1, Q_1\}$-definable well-ordering; $=_1$ and μ_1 are definable from S_1; hence

$$M_2 \subseteq L[S_2, Q_1] \subseteq N_2 \subseteq HOD^{L(\mathbb{R}_1)}_{\{S_1,Q_1\}} = L[S_1, Q_1] = M_1[Q_1],$$

so in particular N_2 is real, that is, is contained in the ground model as a class thereof. $Q_2 \in N_2$, so

$$M_2[Q_2] \subseteq M_1[Q_1];$$

$Q_1 \in N_1$; θ_1 is Woodin in N_1 and in $M_1[Q_1]$; so Q_1 is Woodin in $M_2[Q_2]$ and $V_{\theta_1} \cap M_2[Q_2] = L_{\theta_1}[Q_1]$.

6·21 Remark We have

$$M_2 \subseteq M_2[Q_1] \subseteq N_2 \subseteq M_1[Q_1] \subseteq N_1 \subseteq M_0$$

and $M_2[Q_2] \subseteq N_2$ but $Q_1 \in M_2[Q_2]$, as $L[Q_2] \cap V_{\theta_2} = N_2 \cap V_{\theta_2}$ and $Q_1 \in V_{\theta_2}$, so the picture is actually

$$M_2 \subseteq M_2[Q_1] \subseteq M_2[Q_2] \subseteq N_2 \subseteq M_1[Q_1] \subseteq N_1 \subseteq M_0.$$

Actually $Q_1 \in L_{\theta_2}[Q_2]$ which will be relevant to the future uniformity.

From now on we may iterate as at the 2 level:

$$N^x_{k+1} = HOD^{L[S_k,Q_k,x]}_{\{S_k,Q_k\}} \tag{6·22}$$

$$\theta^x_{k+1} = \omega_2^{L[S_k,Q_k,x]} \tag{6·23}$$

$$\lambda^x_k = \text{the first strongly inaccessible cardinal in } N^x_k \text{ above } \theta^x_k \tag{6·24}$$

$$Q^x_{k+1} = \begin{array}{l}\text{the first subset of } \lambda^x_{k+1} \text{ in the } (S_k, Q_k, x) \text{ canonical well-ordering of } N^x_{k+1}\\ \text{to be a stout extension of } Q_k \text{ in } N^x_k\text{, so that} L_{\lambda^x_{k+1}}[Q^x_{k+1}] = N^x_{k+1} \cap V_{\lambda^x_{k+1}},\\ \text{if such exist; otherwise } \varnothing.\end{array} \tag{6·25}$$

In $L(\mathbb{R}_k)$ we define $x \leq_k y \iff_{\rm df} x \in L[S_k, y]$, and define the Martin measure μ_k on the $=_k$-equivalence classes as before; then we may define

$$N_{k+1} = \prod N^{[x]}_{k+1}/\mu_k \tag{6·26}$$

$$\theta_{k+1} = \prod \theta^{[x]}_{k+1}/\mu_k \tag{6·27}$$

$$\lambda_{k+1} = \prod \lambda^{[x]}_{k+1}/\mu_k \tag{6·28}$$

$$Q_{k+1} = \prod Q^{[x]}_{k+1}/\mu_k \tag{6·29}$$

In $L(\mathbb{R}_k)$ we define $x \leq_k y \iff_{\text{df}} x \in L[S_k, y]$ and μ_k as before; and define

$$N_{k+1} = \prod N^{[x]}_{k+1}/\mu_k \tag{6·30}$$

$$\theta_{k+1} = \prod \theta^{[x]}_{k+1}/\mu_k \tag{6·31}$$

$$\lambda_{k+1} = \prod \lambda^{[x]}_{k+1}/\mu_k \tag{6·32}$$

$$Q_{k+1} = \prod Q^{[x]}_{k+1}/\mu_k \tag{6·33}$$

As before, Q_{k+1} is a stout extension of Q_k in N_{k+1}.
θ_{k+1} is Woodin in N_{k+1}; so $L_{\theta_{k+1}}[Q_{k+1}] = N_{k+1} \cap V_{\theta_{k+1}}$.
Hence

$$M_{k+1} \subseteq M_{k+1}[Q_k] \subseteq M_{k+1}[Q_{k+1}] \subseteq N_{k+1} \subseteq M_k[Q_k]. \tag{6 · 34}$$

Thus we have, setting $Q_0 = \emptyset$,

$$M_0[Q_0] \supseteq N_1 \supseteq M_1[Q_1] \supseteq N_2 \supseteq M_2[Q_2] \supseteq N_3 \supseteq \ \ldots \tag{6 · 35}$$

The Woodin Cardinals

Now we wish to see that each θ_i is Woodin in $M_j[Q_j]$ for $j \geq i$ and in $M_\omega[\vec{Q}]$.
For any i, θ_i is Woodin in $L[Q_i]$ and $L_{\theta_i}[Q_i] = V_{\theta_i} \cap N_i$ so for $j \geq i$,

$$L[Q_i] \subseteq L[Q_j] \subseteq M_j[Q_j] \subseteq N_j \subseteq N_i$$

so all those models have the same sets of rank less than θ_i, and so θ_i is Woodin in each of them.
Let $Q =_{\text{df}} \bigcup_i Q_i$.

6·36 Theorem $L[Q] \models ZFC+$ *"there are infinitely many Woodin cardinals".*

Proof : For each i, $L[Q_i] \subseteq L[Q] \subseteq \bigcap_j M_j[Q_j]$, so

$$V_{\theta_i} \cap L[Q_i] \subseteq V_{\theta_i} \cap L[Q] \subseteq V_{\theta_i} \cap \bigcap_j M_j[Q_j] = \bigcap_j V_{\theta_i} \cap M_j[Q_j] = V_{\theta_i} \cap L[Q_i]$$

and thus θ_i is Woodin in $L[Q]$. $\dashv$

6·37 Remark The same argument shows that each θ_i is Woodin in $W =_{\text{df}} M_\omega[Q]$, since $L[Q] \subseteq W \subseteq \bigcap_j M_j[Q_j]$, as starting at any $M_j[Q_j]$ we may, armed with S_j, compute $\langle Q_k | k \geq j \rangle$ and the sequence $\langle M_k | k \geq j \rangle$; so $M_\omega \subseteq \bigcap_k M_k \subseteq \bigcap_k M_k[Q_k]$ $\dashv$

7: A characterisation of certain inner models

If we were simply after a model containing ω Woodin cardinals, we might have worked with each Q_i simply of height θ_i, coding the extenders and not troubling to freeze the power set of θ_i, since our θ_i is Woodin in the large model $M_i[Q_i]$ and all its requisite extenders are in the smaller models which have no subsets of θ_i not in the larger model.

However, we have chosen Q_i rather larger than that in order that the Vopěnka algebras, of cardinality θ_i should lie in $\mathfrak{M}(Q_i)$.

This will have a consequence similar to that seen in our second construction, using Prikry sequences, of a model of ω Woodins.

7·0 Proposition $\sup_i \lambda_i = \sup_i \theta_i = \omega_1^{L(\mathbb{R}_\omega)}$.

Proof : $\theta_1 < \lambda_1 < \omega_1^{L(\mathbb{R}_1)} \leqslant \theta_2 < \lambda_2 < \omega_1^{L(\mathbb{R}_2)} \leqslant \ldots$. As $\mathbb{R}_\omega$ is the set of reals of $L(\mathbb{R}_\omega)$, its ω_1 equals $\sup_i \omega_1^{L(\mathbb{R}_i)}$. ⊣ (7·0)

7·1 Definition Write η_ω for $\sup \bigcup_i Q_i$.

7·2 Proposition *Each real in $\mathbb{R}_\omega$ is generic over $M_\omega(Q_\omega)$, and also over $L(Q_\omega)$, for an algebra of size less than η_ω.*

Proof : we know that if a real a is in $\mathbb{R}_{i-1}$ it is in $L[S_i, g_i]$ and generic over N_i for an algebra of size at most θ_i computed in N_i, with the $< \theta_i$ chain condition; and thus embeddable into the collapsing algebra $\mathrm{Coll}(\omega, \theta_i)$ of that model.

Further we know that that algebra and all its subsets are in $L(Q_\omega)$.

Hence the reals of $\mathbb{R}_\omega$ may be construed as those of the symmetric collapse of η_ω over $L(Q_\omega)$ or over many models near to that, by the result proved in §4.

7·3 Proposition ($V = L(\mathbb{R})+$ AD + DC) *Let S be a set of ordinals. Then $h_\omega(S)$ is in V, even though the embedding is defined in some generic extension of V.*

Proof : Let S be definable from ordinals and the real a. Then for each $\nu \leqslant \omega$, $S \in L[S_0, a]$; $h_\nu(a) = a$ and $h_\nu(S_0) = j_\nu(S_0) = S_\nu$, so $h_\nu(S) \in L[S_\nu, a]$. ⊣ (7·3)

7·4 Theorem ($V = L(\mathbb{R})+$ AD + DC) *Let S be a set of ordinals. Then*

$$HOD_S = HOD[h_\omega(S)]$$

Proof : let S be definable from the real a. Then $h_\omega(S)$ is in $L[S_\omega, a]$, and so is in $M_\omega[Q_\omega, a]$. a is generic over $M_\omega[Q_\omega]$ for an algebra of size less than η_ω; the same is therefore true of $h_\omega(S)$, so by the absorbent nature of the Lévy collapse, we may treat the reals of $\mathbb{R}_\omega$ as those of the symmetric collapse of η_ω over $M_\omega[h_\omega(S)]$.

From that we see that any set of ordinals definable from S in $L(\mathbb{R}_0)$ is in $L[S_0, h_\omega(S)]$:

$$\begin{aligned}(\phi(\nu, S))^{L(\mathbb{R})} &\iff (\phi(j_\omega(\nu), h_\omega(S)))^{L(\mathbb{R}_\omega)} \\ &\iff L[S_\omega, Q_\omega, h_\omega(S), h_\omega \restriction \sup S] \models blah \\ &\iff L[S_0, h_\omega(S)] \models blah_2\end{aligned}$$

So $HOD_S \subseteq L[S_0, h_\omega(S)] \subseteq HOD_S$, as required. ⊣ (7·4)