Imagine you are given the task of finding a natural deduction proof of the tautology

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

Obviously the first thing you do is to attack the principal connective, and claim that $(p \to q) \to (p \to r)$ is obtained by an $\to$-introduction as follows:

$$\cfrac{\begin{array}{c} p \to (q \to r) \\ \vdots \end{array}}{(p \to q) \to (p \to r)} \to\text{-int} \tag{1}$$

in the hope that we can fill the dots in later. Notice that we don't know at this stage how many lines or how much space to leave .... At the second stage the obvious thing to do is try $\to$-introduction again, since '$\to$' is the principal connective of '$(p \to q) \to (p \to r)$'. This time my proof sketch has a conclusion which looks like

$$\cfrac{\cfrac{\vdots}{p \to r} \to\text{-int}}{(p \to q) \to (p \to r)} \to\text{-int} \tag{2}$$

and I also know that floating up above this—somewhere—are the two premisses $p \to (q \to r)$ and $p \to q$. But i don't know where on the page to put them!

This motivates a new notation. Record the endeavour to prove

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

by writing

$$\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

using the new symbol '$\vdash$'.[1] Then stage two (which was formula 1) can be described by the formula

$$p \to (q \to r) \vdash ((p \to q) \to (p \to r)).$$

which says that $(p \to q) \to (p \to r)$ can be deduced from $p \to (q \to r)$. Then the third stage (which i couldn't write down and which was formula 2) which says that $p \to r$ can be deduced from $p \to q$ and $p \to (q \to r)$ comes out as

$$p \to (q \to r),\ p \to q\ \vdash\ p \to r$$

This motivates the following gadgetry.

A **sequent** is a formula $\Gamma \vdash \psi$ where Γ is a set of formulæ and ψ is a formula. This says that there is a deduction of ψ from Γ. In sequent calculus one reasons

[1]For some reason this symbol is called 'turnstile'.

not about formulæ—as one did with natural deduction—but instead about sequents, which are assertions about deductions between formulæ. Programme: sequent calculus is natural deduction with control structures! A sequent proof is a program that computes a datural deduction proof.

Capital Greek letters denote sets of formulæ and lower-case Greek letters denote formulæ.

We accept any sequent that has a formula appearing on both sides. Such sequents are called **initial sequents**. Clearly the allegation made by an initial sequent is correct!

There are some obvious rules for reasoning about these sequents. Our endeavour to find a nice way of thinking about finding a natural deduction proof of

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

gives us something that looks in part like

$$\cfrac{\cfrac{\cfrac{p \to (q \to r), (p \to q), p \vdash r}{p \to (q \to r), (p \to q) \vdash (p \to r)}}{p \to (q \to r) \vdash (p \to q) \to (p \to r)}}{\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r))}$$

and this means we are using a rule

$$\frac{\Gamma, A \;\vdash\; B}{\Gamma \;\vdash\; A \to B} \to \mathrm{R} \tag{3}$$

Of course there are lots of other rules, and here is a summary of them:

$$\vee L: \frac{\Gamma, \psi \vdash \Delta \quad \Gamma', \phi \vdash \Delta'}{\Gamma \cup \Gamma', \underline{\psi \vee \phi} \vdash \Delta \cup \Delta'} \qquad \vee R: \frac{\Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, \underline{\psi \vee \phi}}$$

$$\vee R: \frac{\Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \underline{\psi \vee \phi}}$$

$$\wedge L: \frac{\Gamma, \psi, \phi \vdash \Delta}{\Gamma, \underline{\psi \wedge \phi} \vdash \Delta} \qquad \wedge R: \frac{\Gamma \vdash \Delta, \psi \quad \Gamma' \vdash \Delta', \phi}{\Gamma \cup \Gamma' \vdash \Delta \cup \Delta', \underline{\psi \wedge \phi}}$$

$$\neg L: \frac{\Gamma \vdash \Delta, \psi}{\Gamma, \underline{\neg\psi} \vdash \Delta} \qquad \neg R: \frac{\Gamma, \psi \vdash \Delta}{\Gamma \vdash \Delta, \underline{\neg\psi}}$$

$$\to L: \frac{\Gamma \vdash \Delta, \phi \quad \Gamma', \psi \vdash \Delta'}{\Gamma \cup \Gamma', \underline{\phi \to \psi} \vdash \Delta \cup \Delta'} \qquad \to R: \frac{\Gamma, \psi \vdash \Delta, \phi}{\Gamma \vdash \Delta, \underline{\psi \to \phi}}$$

Weakening: $\dfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta, B}$;

Contraction-L: $\dfrac{\Gamma, \psi, \psi \vdash \Delta}{\Gamma, \underline{\psi} \vdash \Delta}$; Contraction-R: $\dfrac{\Gamma \vdash \Delta, \psi, \psi}{\Gamma \vdash \Delta, \underline{\psi}}$;

Cut: $\dfrac{\Gamma \vdash \Delta, \underline{\psi} \quad \Gamma', \underline{\psi} \vdash \Delta'}{\Gamma \cup \Gamma' \vdash \Delta, \Delta'}$.

You might find useful the terminology of **eigenformula**. The eigenformula of an application of a rule is the formula being attacked by that application. In each rule in the box above I have underlined the eigenformula.

There is no rule for the biconditional: we think of a biconditional as a conjunction of two conditionals.

Now that we have rules for $\neg$ we no longer have to think of $\neg p$ as $p \to \bot$. (see appendix **??**.)

The two rules of $\vee R$ give rise to a derived rule which makes good sense when we are allowed more than one formula on the right. it is

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

I shall explain soon (section **??**) why this is legitimate.

A word is in order on the two rules of contraction. Whether one needs the contraction rules or not depends on whether one thinks of the left and right halves of sequents as sets or as multisets. Both courses of action can be argued for. If one thinks of them as multisets then one can keep track of the multiple times one exploits an assumption. If one thinks of them as as sets then one doesn't need the contraction rules. It's an interesting exercise in philosophy of mathematics to compare the benefits of the two ways of doing it, and to consider the sense in which they are equivalent. Since we are not hell-bent on rigour we will equivocate between the two approaches: in all the proofs we consider it will be fairly clear how to move from one approach to the other and back.

A bit of terminology you might find helpful. Since premisses and conclusion are the left and right parts of a sequent, what are we going to call the things above and below the line in a sequent rule? The terminology **precedent** and **succedent** is sometimes used. I'm not going to expect you to know it: I'm offering it to you here now because it might help to remind you that it's a different distinction from the premiss/conclusion distinction. I think it is more usual to talk about the **upper sequent** and the **lower sequent**.

You will notice that I have cheated: some of these rules allow there to be more than one formula on the right! There are various good reasons for this, but they are quite subtle and we may not get round to them. If we are to allow more than one formula on the right, then we have to think of $\Gamma \vdash \Delta$ as saying that every valuation that makes everything in Γ true also makes something in Δ true. We can't correctly think of $\Gamma \vdash \Delta$ as saying that there is a proof of something in Δ using premisses in Γ because:

$$A \vdash A$$

is an initial sequent. so we can use $\neg$–R to infer

$$\vdash A, \neg A.$$

So $\vdash A, \neg A$ is an OK sequent. Now it just isn't true that there is always a proof of A or a proof of $\neg A$, so this example shows that it similarly just isn't true that a sequent can be taken to assert that there is a proof of something on the right using only premisses found on the left—unless we restrict matters so that there is only one formula on the right. This fact illustrates how allowing two formulæ on the right can be useful: the next step is to infer the sequent

$$\vdash A \vee \neg A$$

and we can't do that unless we allow two formulæ on the right.

However, it does help inculcate the good habit of thinking of sequents as metaformulæ, as things that formalise facts about formulæ rather than facts of the kind formalised by the formulæ.

One thing you will need to bear in mind, but which we have no space to prove in this course, is that sequent proofs with more than formula on the right correspond to natural deduction proofs using the rule of classical negation.

Display this properly

N.B.: commas on the left of a sequent mean 'and' while commas on the right-hand side mean 'or'! This might sound odd, but it starts to look natural quite early, and you will get used to it easily.

A summary of what we have done so far with Natural Deduction and Sequent Calculus.

- A sequent calculus proof is a log of attempts to build a natural deduction proof.
- So a sequent is telling you that there is a proof of the formula on the right using as premisses the formulæ on the left.
- But we muck things up by allowing more than one formula on the right so we have to think of a sequent as saying if everything on the left is true then something on the right is true.
- Commas on the left are **and**, commas on the right are **or**.

0.0.1 Soundness of the Sequent Rules

Recall that we started off thinking of a sequent as saying that there is a proof of something on the right using only premisses found on the left. To illustrate, think about the rule $\wedge$-L:

$$\frac{A, B \ \vdash \ C}{A \wedge B \ \vdash \ C} \wedge \text{L} \tag{1}$$

It tells us we can infer "$A \wedge B \vdash C$" from "$A, B \vdash C$". Now "$A, B \vdash C$" says that there is a deduction of C from A and B. But if there is a deduction of C from A and B, then there is certainly a deduction of C from $A \wedge B$, because one can get A and B from $A \wedge B$ by two uses of $\wedge$-elim. The $\rightarrow$-L rule can benefit from some explanation as well. Assume the two sequents above the line. We want to use them to show that there is a derivation of something in Δ from $\phi \rightarrow \psi$ and all the premisses in Γ. The first sequent above the line tells us that there is either a deduction of something in Δ using premisses in Γ (in which case we are done) or there is a deduction of ϕ. But we have $\phi \rightarrow \psi$, so we now have ψ. But then the second sequent above the line tells us that we can infer something in Δ.

If we think of a sequent $\Gamma \vdash \Delta$ as an allegation that there is a natural deduction proof of something in Δ using assumptions in Γ, then we naturally want to check that all basic sequents are true and that all the sequent rules are truth-preserving. All but one of them are. In fact it is easy to check that not only are they truth-preserving they are *effective*. Consider $\wedge$-L, for example. Assume $\Gamma, A, B \vdash \Delta$. This tells us that there is a deduction $\mathcal{D}$ of some δ in Δ) assuming only assumptions in Γ A plus possibly A or B or both. We have several cases to consider.

(i) If $\mathcal{D}$ does not use A or B then it is a witness to the truth of $\Gamma, A \wedge B \vdash \Delta$;

(ii) If it uses either A or B (or both) then we can append[2] one (or two) applications of $\wedge$-elimination to it to obtain a new proof that is a witness to the

[2]The correct word is probably 'prepend'!

finish this off, with a picture

truth of $\Gamma, A \wedge B \vdash \Delta$

The one exception is $\neg$-R. ($\neg$-L is OK because of *ex falso*.)

This illustrates how

- sequent rules on the **right** correspond to natural-deduction **introduction** rules; and
- sequent rules on the **left** correspond to natural-deduction **elimination** rules.

The sequent rules are all sound. Given that the sequent $\Gamma \vdash \phi$ arose as a way of saying that there was a proof of ϕ using only assumptions in Γ it would be nice if we could show that the sequent rules we have are sound in the sense that we cannot deduce any false allegations about the existence of proofs from true allegations about the existence of proofs. However, as we have seen, this is sabotaged by our allowing multiple formulæ on the right.

However, there is a perfectly good sense in which they are sound. If we think of the sequent $\Gamma \vdash \Delta$ as saying that every valuation making everything in Γ true makes something in Δ true then all the sequent rules are truth-preserving.