# *DRAFT OF* **Chapter III: The Logic of Godement's** *Cours d'Algèbre*

Roger Godement is listed by Cartier as a member of the second generation of Bourbaki, in the same group as Dixmier, Eilenberg, Koszul, Samuel, Schwartz, and Serre. In 1963 he published a *Cours d'Algèbre*. An English translation appeared in 1968, which runs to some 600 pages, of which the first hundred are devoted to an account of logic and set theory.

I used the last 500 pages of this text for many years during the time that I was teaching algebra to Cambridge undergraduates, and found it excellent. I did not then read the first part, on Set Theory, consisting of §§0-5, but did so in preparation for my Danish visit, and found, unhappily, that the account offered, although containing many tart remarks to delight the reader, is flawed in various ways: one finds errors of metamathematics, mis-statements of the results of Gödel and Cohen, and an accumulation of negative messages about logic and set theory.

In this chapter I shall present my findings by combing quickly through his account and commenting as I go. In the main he follows Bourbaki's *Théorie des Ensembles*. I have rearranged this material and simplified it in one or two places. Quotations, from the English 1968 translation of his book, are given in *slanted type*; the pagination follows that edition. I have occasionally added the corresponding passages from the French revised edition of 1966. Following Godement, I use the sign § to indicate chapter and N$^o$ to indicate section within a chapter.

## 12: Godement's formal system

Godement states that the opening chapter is "an introduction to mathematical logic", and then adds, somewhat alarmingly, that "in it we have tried to give a rough idea of the way mathematicians conceive of the objects they work with." That opening chapter begins on page 20, where he writes:

> *In mathematics there are three processes: construction of mathematical objects, formation of relations between these objects, and the proof that certain of these relations are true, or, as usually said, are theorems.*
>
> *Examples of mathematical objects are numbers, functions, geometrical figures, and countless other things which mathematicians handle; strictly speaking these objects do not exist in Nature but are abstract models of physical objects, which may be complicated or simple, visible or not. Relations are assertions (true or not) which may be made about these objects, and which correspond to hypothetical properties of natural objects of which the mathematical objects are models. The true relations, as far as the mathematician is concerned, are those which may be logically deduced from a small number of axioms laid down once and for all. These axioms translate into mathematical language the most "self-evident" properties ..... [a] sequence of syllogisms .. constitutes a proof.*

He then startles the reader by attacking what he has just written:

> *Explanations of this sort .. have long since ceased to satisfy mathematicians: not only because mathematicians have small patience with vague phrases, but also and especially because mathematics itself has forced them to consider carefully the foundations of their science and to replace generalities by formulæ whose meaning should be altogether free from ambiguity, and such that it should be possible to decide in a quasi-mechanical fashion whether they are true or not; and whether they make sense or not.*

12·0 COMMENT Note the residual faith in the completeness and decidability of mathematics implicit in that last sentence.

After some further discussion he begins the presentation of his formal system. Broadly he uses the syntax summarised in Chapter One, but where Bourbaki speaks of terms, he speaks of mathematical objects, which I abbreviate to MOs; otherwise he seeks to follow Bourbaki's development, with, as in Bourbaki's first edition, a primitive sign for the ordered pair of two objects. For that sign he uses a backwards C, which I again represent by •.

He departs from Bourbaki by seeking to soften the austerity of the formalism, which leads him to make mistakes of logic, for he blurs the distinction between a uninterpreted formal language and its interpretations; and by a strange reluctance to state the axioms of set theory.

12·1 To introduce the axioms of logic, he writes:

> *Once the list of fundamental signs has been fixed, and the list of criteria of formation of mathematical objects and relations, it remains to state the axioms. Some will be purely logical, others of a strictly mathematical nature.*

On page 25, he says that

> **true** *relations or* **theorems** ... *are those one which can be obtained by repeated application of the two rules:*
>
> *TR1: Every relation obtained by applying an axiom is true*
>
> *TR2: if $R$ and $S$ are relations, if the relation ($R \implies S$) is true and if the relation $R$ is true, then the relation $S$ is true.*

and on page 26, he says that

> *A relation is said to be* **false** *if its negation is true.*

and then immediately in Remark 1 that:

> *what characterizes true relations is that they can be proved.*

12·2 COMMENT This equation of truth and provability is the standard Bourbachiste position, found in the papers of Cartan and of Dieudonné of fifty years ago cited in *The Ignorance of Bourbaki*, and found again in Dieudonné's last book, on *Mathematics: the Music of Reason*, published in 1992. But Godement then, in *Remark 2* cautions the reader with the symbol for a *tournant dangereux*:

> *There is a natural tendency to think that a relation which is "not true" must necessarily be "false". ... Unfortunately there is every reason to believe that in principle this is not so.*

We shall return to the rest of *Remark 2* later.

12·3 REMARK Thus words like "true" and "satisfies", as for example in the sentence

> *The mathematical object $A$ is said to satisfy the relation $R$ if the relation $(A|x)R$ is true,*

are being defined in terms of a provability relation that Godement is developing, which we write as $\vdash_{\mathbf{Bou}}$. Note, too, his comment in *Remark 4* of §0, on page 28:

> *from TL2, the relation $R$* **ou** ( **non** *$R$) is true. It does not follow that at least one of the relations $R$,* **non** *$R$ is true: this is precisely the question of whether there exist undecidable relations !*

12·4 Of (TL7), which says that if $R(x)$ is true so is $(A|x)R$, Godement comments at the top of page 31 that

> *[its] purpose ... is precisely to justify this interpretation of letters as "undetermined objects";*

12·5 COMMENT We were told on page 21 that assemblies are built up from fundamental signs and letters, which suggests that a letter is a symbol, but at the top of page 30 we read

> *Let $R$ be a relation, $A$ a mathematical object, and $x$ a letter (i.e. a "totally indeterminate" mathematical object),*

so by page 30 a letter is not a symbol: Godement is slipping away from treating his system as an uninterpreted calculus, and moving towards an informal Platonism.

12·6 Quantifiers are introduced on page 31 exactly as in Bourbaki, and need little comment beyond our remarks in Chapter II. He remarks on page 37 that

> *we shall now be able, by using the Hilbert operation, to introduce them as simple abbreviations*

— a phrase which in the light of the calculations of Chapter I may strike the reader as a bit rich; and he ends the Chapter with this pleasing remark:

> *Like the God of the philosophers, the Hilbert operation is incomprehensible and invisible; but it governs everything, and its visible manifestations are everywhere.*

## 13: Godement's set-theoretic axioms

Early in the next chapter, §1, he says

*... if $a$ and $b$ are mathematical objects* (or **sets**— *the two terms are synonymous) ...* :

we shall see that that phrase causes trouble: a footnote on page 41 describes the less formal style he now wishes to adopt; as he for the formalized language

He introduces the axioms of equality in Theorem 1 on pages 41/42:

> *... intuitively this relation, when it is true, means that the concrete objects which $a$ and $b$ are thought of as representing are "identical". We do not enter into a philosophical discussion of the meaning of "identity" ...*
>
> *THEOREM 1:* a) *The relation $x = x$ is true for all $x$.*
>
> *b) The relations $x = y$ and $y = x$ are equivalent, for all $x$ and all $y$.*
>
> *c) For all $x$, $y$, $z$, the relations $x = y$ and $y = z$ imply the relation $x = z$.*
>
> *d) Let $u$, $v$ be objects such that $u = v$ and let $R\{x\}$ be a relation containing a letter $x$. Then the relations $R\{u\}$ and $R\{v\}$ ... are equivalent.*

He comments that part d) is an axiom, while parts a) b) c) can be deduced from

> *a single much more complicated axiom, (which the beginner should not attempty to understand) namely that if $R$ and $S$ are equivalent relations and $x$ is a letter then the relation $\tau_x(R) = \tau_x(S)$ is true.*

13·0 COMMENT In Part d), formal system and informal interpretation are confused: otherwise "*such that* $u = v$" has no meaning. His definitions, taken *au pied de la lettre*, would imply that he means that if $\vdash_{\mathbf{Bou}} u = v$ then $\vdash_{\mathbf{Bou}} R\{u\} \iff R\{v\}$, which is weaker than what I suspect he intended, namely that $\vdash_{\mathbf{Bou}} u = v \implies (R\{u\} \iff R\{v\})$.

13·1 COMMENT Notice the way in which Godement appears to grudge the reader each axiom. It may be for pedagogical reasons that he introduces the axioms one by one, but it is regrettable that there is no signal to the reader when the presentation is complete.

Nothing is said about the origins of the axioms: they are presented as oracular pronouncements.

So far as I can tell, searching the first hundred pages and assuming that no further axioms will be introduced once he starts on the algebra, his set-theoretic axioms are these:

First, the axiom of **extensionality**, which appears on page 42:

> *... In fact there is only one axiom governing the use of $\in$, namely:*
>
> *THEOREM 2: Let $A$ and $B$ be two sets. Then we have $A = B$ if and only if the relations $x \in A$ and $x \in B$ are equivalent.*

I take the preamble to Theorem 2 to mean that he considers it to be an axiom: thus we have had one set-theoretic axiom so far, that of extensionality, but stated as a theorem.

13·2 COMMENT Note the further slipping between languages: "*the relations are equivalent*" means that a certain formula is provable in the system **Bou** we are building up. That is a clear assertion, and is said to hold iff $A = B$; but $A = B$ is an (uninterpreted) relation. Does he mean "$A = B$ is true iff the relations are equivalent", or does he mean "it is provable in **Bou** that $\big(A = B \iff (x \in A \iff x \in B)\big)$" ? That he is aware of the difference is evident from part (2) of Remark 7 on page 34.

13·3 COMMENT Theorem 2 as stated is false. Let $x$ and $y$ be distinct letters. Looking ahead to pages 46–7, where Godement adds the axiom of pairing, let $A$ be $\{x\}$ and let $B$ be $\{x, y\}$. I believe that $A$ and $B$ are mathematical objects; indeed a definition using $\tau$ can be given, though Godement does not do so: I assume that formally he would put $\{x\} =_{\mathrm{df}} \tau_y((\forall z)((z \in y) \iff z = x))$ and $\{x, y\} =_{\mathrm{df}} \tau_z((\forall w)((w \in z) \iff w = x \textbf{ ou } w = y))$. On page 41 he says that the terms *mathematical object* and *set* are synonymous. Very well, $A$ and $B$ are sets, and the following is provable in **Bou**: $x \in A \iff x \in B$, so that the relations $x \in A$ and $x \in B$ are equivalent. If Theorem 2 were provable, we could infer that $A = B$ and thus that $x = y$; so we would have proved that any two sets are equal !

Of course, what is lacking is a requirement that the letter $x$ has no occurrence in $A$ or in $B$. That slip is strange in that in the statement of Theorem 4 on page 44 (a version of the scheme of separation) he is careful to say that the relation $R$ is to contain a variable $x$: though there the theorem would still be true if it did not. On the other hand the further statement of theorem 4, that *for every set $X$ there exists a unique subset $A$ of $X$, ...* shows further confusion of language; "*for every set $X$*" might mean "for every mathematical object", that is "for every term ", but "*there exists a unique subset*" certainly is using "there exists" mathematically; $A$ is a variable here not a term, and will be the subject of various assertions.

13·4 Of his version of the scheme of **separation** given in Theorem 4 of §1, on page 44, Godement says in Remark 4 on that page that

> *Mathematically, Theorem 4 cannot be proved without using axioms which are far less self-evident, and the beginner is therefore advised to assume Theorem 4 as an axiom.*

so, so far, we have had extensionality and separation.

13·5 Then Godement, wishing to comment on the necessity of giving a scheme of separation rather than of comprehension, writes on page 44:

> *remark 5: In spite of the dictates of common sense, it is not true that for every relation $R\{x\}$ there exists a set (in the precise sense of §0) whose elements are all the objects $x$ for which $R\{x\}$ is true.*

13·6 Comment When I turn to §0, I find that the word "set" occurs only thrice, namely on page 20 in N$^o$1, in the phrases *the development of the "theory of sets"*, *Set theory was created by Cantor*, and *set theory had given rise to genuine internal contradictions*; I can find it nowhere else in §0, although there is a lot of talk about **objects**. The word **sets** appears, in bold face, early in §1, suggesting that that is its definition, namely that a set is the same as a mathematical object. So, as I can find no precise sense of "set" in §0, whereas "mathematical object" is given a very precise sense in that section, namely that it is a term in a certain carefully specified formal language, and as I am told that a set is the same as a mathematical object, so be it: I shall take 'set' to mean 'term in Bourbaki's formal language'.

Now Remark 5 continues:

> *Suppose that there exists a set $A$ such that the relations $x \in A$ and $x \notin x$ are equivalent.*

He hopes to get a contradiction, but there is nothing wrong with that, as it stands: let $A$ be

$$\tau_y((\forall z)((z \in y) \Longleftrightarrow (z = x \;\&\; z \notin z))).$$

Then $A$ is a mathematical object, therefore a set, and the relation $x \in A \Longleftrightarrow x \notin x$ is true.

13·7 Remark Again, what is missing is the requirement that the mathematical object $A$ have no occurrence of the letter $x$; then his remark that the supposition of the existence of such an $A$ leads to contradiction would become correct.

13·8 We shall refer below to his Remark 6, on page 45, in which he says that apparently obvious assertions cease to be so simple when it is a question of effectively *proving* them.

The empty set is discussed in N$^o$4: if there is a set then the empty set exists by the scheme of separation: no axiom so far asserts the unconditional existence of any set; but the $\tau$-formalism guarantees that the existence of something is provable, as we saw in Chapter II.

Godement goes on in N$^o$5 to discuss sets of one and two elements, and then says *In the same way we can define sets of three, four ... elements. The sets so obtained are called finite sets, and all other sets are called infinite sets. These two notions will be considered afresh in §5.*

13·9 Comment Note that that is not a formal definition of "finite", the use of dots constituting an appeal to the reader's intuitive notion of finiteness.

13·10 In §1, N$^o$5, Remark 8, on page 47, he states that the existence of the **pair set** of two objects is an axiom, and goes on to say that the existence of **infinite sets** is also an axiom, and acknowledges that we have yet to define the natural numbers, which will be done in §5; so far we have had axioms of extensionality, separation, pairing and infinity, but we await a definition of *finite*.

In N$^o$6 the set of subsets of a given set is introduced thus:

> *Let $X$ be a set. Then there exists — this again is one of the axioms of mathematics — one and only one set, denoted by*
>
> $$\mathcal{P}(X)$$

*with the following property: the elements of $\mathcal{P}(X)$ are the subsets of $X$,*

so hitherto we have had extensionality, separation, pairing, infinity, and power set; we await a definition of finite.

13·11 In §2 he discusses ordered pairs and Cartesian products, no new axioms being introduced till §3. In Remark 1 of §2 on page 50, he writes:

> *The [Kuratowski method of defining] ordered pair is totally devoid of interest. ... The one and only question of mathematical importance is to know the conditions under which two ordered pairs are equal.*

13·12 COMMENT To an algebraist, that might be true. But to a set-theorist interested in doing abstract recursion theory, it is very natural to ask whether a set is closed under pairing. For that reason, an economical definition of ordered pair is desirable, such as is furnished by Kuratowski's definition.

In §2 N$^o$2 he declares that

*using the methods of §0, cartesian products can be proved to exist.*

13·13 COMMENT I deny that that can be proved from the axioms he has stated so far, given that he has refused to define ordered pair — hence we do not know where the values of the unpairing functions (projections) lie — and he has not stated a scheme of replacement.

As a curiosity, let us show that his very refusal to pick a definition of ordered pair has implications for the strength of his system.

Let "$F$ defines a possible pairing function", where $F$ is a three-place Bourbaki relation, abbreviate the conjunction of these statements:

(13·13·0) $\forall x \forall y \exists$ exactly one $z$ with $F(x, y, z)$

(13·13·1) $\forall z \forall u \forall v \forall x \forall y \big(F(u, v, z) \,\&\, F(x, y, z) \implies [u = x \,\&\, v = y]\big)$.

Let "$X \times_F Y \in V$" denote the formula $\exists W \forall w (w \in W \iff \exists x \exists y (x \in X \,\&\, y \in Y \,\&\, F(x, y, w)))$.

Write $\langle x,\, y\rangle_K$ for the Kuratowski ordered pair $\{\{x\},\, \{x,\, y\}\}$, and consider the following argument, which I present in $ZF$-style set theory.

13·14 LEMMA *Suppose $x \mapsto G(x)$ is a one-place function with domain $V$. Define $F(x, y, z) \iff_{\mathrm{df}} z = \langle G(x),\, \langle x,\, y\rangle_K\rangle_K$. Then $F$ defines a possible pairing function.*

*Proof :* Write $(x,\, y)_F$ for $\langle G(x),\, \langle x,\, y\rangle_K\rangle_K$. We have only to check that the crucial property $(x,\, y)_F = (z,\, w)_F \implies x = z \,\&\, y = w$ is provable. But that is immediate from the properties of the Kuratowski ordered pair. ⊣ (13·14)

13·15 LEMMA *Now let $A$ be a set, and let $B = \{\varnothing\}$; let $G$ and $F$ be as above; then if $A \times_F B \in V$, then the image $G``A$ of the set of points in $A$ under $G$ is a set.*

*Proof :* by the scheme of separation, which we have been advised to take as an axiom, and by the axiom of union, which will be become available to us on a second reading of §3 N$^o$2, the lemma follows from the fact that $G``A \subseteq \bigcup\bigcup(A \times_F B)e$. ⊣ (13·15)

Those lemmata yield the following

13·16 METATHEOREM *The scheme of replacement is provable in the theory whose axioms are those of extensionality, pairing, and union, plus the scheme of separation and the scheme that for each formula $F$ with three free variables, the sentence expressing "if $F$ defines a possible pairing function then for each $A$ and $B$, $A \times_F B$ is a set" is an axiom.*

In Remark 4 on page 56, the existence of the set of natural numbers is assumed (and referred to the existence of infinite sets, stated to be an axiom, but not formulated: so far there has been no proper definition of *finite*).

13·17 In §3 he turns to unions and intersections. In Remark 1 on page 70 it is mentioned that we need an axiom of union to form $X \cup Y$, with forward reference to section 2. Of N$^o$2 it is said that it may be omitted at a first reading. It is stated that the existence of the **union** of an arbitrary family is an axiom of mathematics. But ZF-istes must beware ! for Godement's axiom of union is really a scheme which is much stronger than the simple axiom of union, and therefore I shall speak about it as Godement's scheme of union.

With that point in mind, we have now had the axioms of extensionality, pairing, infinity, and power set, and the schemes of separation and union; we continue to await a definition of *finite*.

13·18 COMMENT His axiom of union is deceptive, for actually it is used as an axiom scheme of replacement. All turns on the concept of a family.

That his system is equivalent to Bourbaki's follows from an examination of Bourbaki's *schéma de sélection et réunion* found as S8 of the *Théorie des Ensembles*, on page E II.4:

*Soient $R$ une relation, $x$ et $y$ des lettres distinctes, $X$ et $Y$ des lettres distinctes de $x$ et $y$ et ne figurant pas dans $R$. La relation*

$$(\forall y)(\exists X)(\forall x)(R \implies (x \in X)) \implies (\forall Y)\mathrm{Coll}_x((\exists y)((y \in Y) \textit{ et } R))$$

*est un axiome.*

Bourbaki is aware of the the power of S8, drawing attention to the difference between the union of family of subsets of a given set and the union of a family of sets where no containing set is known. Thus the ZF-ists' axiom of union together with schemes of separation and replacement is equivalent to Godement's scheme of union and Bourbaki's scheme of selection and union.

§4, on equivalence relations, calls for little comment. Example 4 on page 78 speaks of the set of rational integers, though we still await a definition of ℕ.

In §5, on *Finite sets and integers* the concept of finiteness will be defined at last, on page 95 in N$^o$4. Kronecker's witticism is here attributed to Dedekind. Godement remarks that *the integers with which we are concerned here are mathematical objects, not concrete integers,* which underlines the need for a formal definition of *finite*. The cardinal of X is elegantly defined as $\tau_Y(Eq(X,Y))$, emphasizing the reliance on the identity of $\tau$-selected witnesses to equivalent propositions. In §5 N$^o$1 on page 90, discussing equipotence, he writes that the "ordinary" numbers are metaphysical ideas derived from concrete experience, whereas "Mathematical" numbers are objects defined by following the procedures of §0.

The **axiom of choice** sneaks in *via* the $\tau$-operator, about which Godement says, on page 39, §0 N$^o$9:

*It is also used nowadays in place of the axiom of choice. (§2 Remark [8]).*

Turning to §2 N$^o$8, Remark 8, on page 63, we have

*... (using $\tau$ to get AC) .. we can define a function $h$ by $h(y) = \tau_x(f(x) = y)$ ...*

section 4: Dedekind's definition of *finite* is given, (which relies on a mild form of the Axiom of Choice to be correct); a natural number is then defined to be a finite cardinal.

13·19 That appears to be all the axioms given by Godement. I find no mention of an axiom of foundation; but, as we have seen, the scheme of replacement is embedded in his scheme of union.

13·20 His treatment of cardinals in §5 N$^o$2, follows Bourbaki. Thus 0 is defined on page 90 to be the cardinal of the empty set, therefore some object equipollent to the empty set; therefore (as remarked by Bourbaki but not by Godement) the empty set itself.

1 is the cardinal of the singleton of the empty set, so is some object with exactly one element and therefore not equal to 0. We have seen in Chapter I how this definition gets out of hand.

2 is the cardinal of the von Neumann ordinal 2, so is some object with exactly two elements. The calculations of Chapter I would presumably yield even more monstrously long assemblies for this and other finite cardinals. Why not use von Neumann ordinals ? — for he gives them as examples of sets with the required number of elements.

There is a forward reference to §5 N$^o$4.

Finally he assumes without proof:

*Theorem 2: any set of cardinal numbers has a sup and an inf.*

13·21 COMMENT That too involves an appeal to replacement. Without it, curious things happen. Suppose we define $Card(n)$ to be the set $\{\aleph_k \mid k < n\}$, where we take $\aleph_k$ being an initial ordinal: a reasonable definition, by Bourbachiste standards, as it is indeed a set of cardinality $n$. But then without some version of the axiom of replacement, (for example, in Zermelo set theory) the class of finite cardinals as we have just defined them need not be a set.

13·22 COMMENT There are other unheralded uses of replacement: for example, the construction on page 97 of the set $\bigcup_{n\in N} X_n$, and in the footnote on the same page, the proof that the class of cardinals less than a given cardinal is a set.

13·23 COMMENT The errors listed in this section are easily corrected, but give an impression of tattiness unexpected in a mathematician who has small patience with vague phrases.

## 14: Misunderstandings of work of logicians

However, when Godement comes to mention the work of Gödel and other logicians, he makes rather more serious errors. For example, on page 26, remark 2, after defining the notion of an *undecidable* relation, Godement says:

> *At the present time no example is known of a relation which can be proved to be undecidable (so that the reader is unlikely to meet one in practice ... ) But on the other hand the logicians (especially K. Gödel) showed thirty years ago that there is no hope of eventually finding a "reasonable proof" of the fact that every relation is either true or false;* [**note use of word FACT** — A.R.D.M.] *and their arguments make it extremely probable that undecidable relations exist. Roughly speaking the usual axioms of mathematics are not sufficiently restrictive to prevent the manifestation of logical ambiguities.*

The French original ... *du fait que toute relation est soit vraie soit fausse ...* of those words was written in 1963, thirty three years after the incompleteness theorems were announced. What is one to make of these statements ? If he believes that theorems are deduced from a small number of axioms laid down once and for all, and if that means that the set of axioms is recursive, then if his chosen system is consistent, undecidable relations are certainly known; and given his definition of "true" as provable and "false" as refutable, it is simply not the case that every relation is either true or false.

Why is he so reluctant to allow Gödel's discoveries to be established rather than be merely "extremely probable" ?

The last sentence quoted above is perfectly correct, more so than perhaps Godement realised. Incompleteness pervades mathematics: the phenomenon may be found in almost any branch of mathematics and is not something confined to artificial and contrived assertions on the very margin of our science. In §0 N$^o$4, on page 26, he writes

> *Remark 3: ... contradictory relations are both true and false. The efforts of the logicians to establish a priori that no such relations exist have not so far met with success.*
> — and Bourbachistes claim that they really did understand Gödel !

He follows Bourbaki in using $\tau$ to get the Axiom of Choice: for example on page 63, in Remark 8 of §2 N$^o$8, he writes *we can define a function h by* $h(y) = \tau_x(f(x) = y)$. His comment,

> *The possibility of constructing a [choice set] (which is obvious if one uses the Hilbert operation) is known as the axiom of choice. Until recent times it was regarded with suspicion by some mathematicians, but the work of Kurt Gödel (1940) has established that the axiom of choice is not in contradiction to the other axioms (which of course does not in any way prove that the latter are non-contradictory)*

is well-phrased, thought it might leave the reader wondering what the problem was that Gödel solved.

14·0 The first French edition of Godement's *Cours d'Algèbre* was written presumably just before Cohen's discoveries were announced in 1963. In Remark 9 of §5 N$^o$7, on page 98, he mentions Cohen's proof that $CH$ is undecidable, and acknowledges that his earlier assertion that the reader would not encounter an undecidable relation should now be amended; but he makes no mention of Gödel's consistency proof for $CH$.

## 15: Unease in the presence of logic

Besides the above mis-statements by Godement concerning the work of logicians, we find repeatedly an undertow of unhappiness about logic.

He writes on page 22, in §0 N$^o$2,

> *It has been calculated that if one were to write down in formalized language a mathematical object so (apparently) simple as the number 1, the result would be an assembly of several tens of thousands of signs.*

15·0 COMMENT We have seen in Chapter I that this remark goes back to Bourbaki, and that the estimate given is too small by a factor of perhaps a hundred million. But even if their estimate were correct, **what is the point of all those symbols ?** Why not adopt the von Neumann definition of 1 as $\{\varnothing\}$ ?

In a language where the class forming operator is a primitive symbol, the empty set, 0, can be defined in 6 symbols, and if, as I favour, one has primitive symbols for restricted quantifiers, the number 1 can be defined in nine symbols; otherwise with just ordinary quantifiers one can do it in thirteen.

Poincaré mocked Couturat for taking perhaps twenty symbols to define the number 1, in an attempt to reduce that arithmetical concept to one of logic; now Bourbaki is taking 4 European billions (= American trillions) of symbols to do the same thing: one million thousand-page books of densely packed symbols. Suppose an error occurred somewhere in those pages: would anyone notice ? would it matter ? That is not where the mathematics resides.

15·1 In short, the chosen formalism is ridiculous, and Godement knows that it is ridiculous, for he makes the excellent remark:

> *A mathematician who attempted to manipulate such assemblies of signs might be compared to a mountaineer who, in order to choose his footholds, first examined the rock face with an electron microscope.*

Did it occur to him to wonder whether other formalisms might be possible ? One feels that his attitude towards logic is that of the Victorian schoolboy towards Latin, who in his heart thinks that no nonsense is too absurd to be a possible translation from a Roman author.

Let us list the other symptoms:

On page 25, there is a footnote:

*it is very difficult, in practice, to use the sign $\Longrightarrow$ correctly.* *

On page 31, another footnote:

> *it is very difficult to use the signs $\exists$ and $\forall$ correctly in practice, and it is therefore preferable to write "there exists" and "for all", as has always been done.*†

What will he do, I asked myself, with the set-forming operator ? The answer astonished me: he does not use it. I have been right through the book searching, and I cannot find it at all. He introduces signs for singletons and unordered pairs; but every time he wants to introduce a set, for example a coset in a group, he writes out in words "let $F$ be the set of ....".

In Remark 5 of §1 N$^o$3, on page 44, Godement says

> *these examples show that the use of the word "set" in mathematics is subject to limitations which are not indicated by intuition.* ‡

In Remark 6 he says

> *... apparently obvious assertions cease to be so simple when it is a question of effectively proving them. The Greeks were already aware of this.* ¶

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* *Il est du reste fort difficile, dans la pratique, d'utiliser *correctement* le signe $\Longrightarrow$.*

† *Il est fort difficile d'utiliser *correctement* les signes $\exists$ and $\forall$ dans la pratique courante; il est donc préférable de se borner à écrire "il existe" et "pour tout" comme on l'a toujours fait.*

‡ *Ces exemples montrent que l'usage du mot "ensemble" est soumis en Mathématiques à des limitations que l'intuition n'enseigne pas.*

¶ *Ceci montre que des assertions en apparence évidentes cessent d'être simples lorsqu'on veut effectivement les démontrer, c'est ce que les Grecs avaient déjà remarqué.*

and on page 98, in commenting on Example 1 of §5 N$^o$5, he says:

> *it is precisely one of Cantor's greatest achievements that he disqualified the use of "common sense" in mathematics.* §

15·2 COMMENT The cumulative effect of all these comments is this: Godement tells the reader that a simple concept such as the number 1 can take thousands of signs to write out formally, that it is very difficult to use connectives correctly, and that it is very difficult to use quantifiers correctly. Coupled to this comprehensive group of negative messages about logic are some equally discouraging statements about set theory: that the concept of 'set' is counter-intuitive, that apparently obvious assertions are hard to prove, that common sense has been disqualified from set theory; further, he avoids the usual notation for forming sets and he evinces a remarkable reluctance even to state the axioms of set theory.

Can I be blamed for suspecting that Godement distrusts formalised reasoning ? I know he says, in §0 N$^o$1, on page 22, that

*formalized mathematics exists only in the imagination of mathematicians*‖

but I feel he would rather even that did not happen. He brings to mind a *mot* of Padoa:

> *Logic is not in a good state: philosophers speak of it without using it, and mathematicians use it without speaking of it, and even without desiring to hear it spoken of.*

In sum, his message is that logic and set theory are a morass of confusion: but what has happened is that Bourbaki, whom he follows, have chosen a weird formalisation, they have noticed that in their chosen system proof is very awkward, and they have concluded that the whole thing is the fault of the logicians.

15·3 Of course there are things in logic which are not yet understood, as in any living subject that tackles difficult problems, but there are ways of presenting set theory which are far better than the Bourbachiste method, reading which, to me, is like wading through hot thick volcanic mud. It is hardly surprising that a young German had to lie in a dark room for six months. *I* am used to logic and the Bourbaki presentation makes *me* scream; what will it do to some innocent hopeful who thinks he is going to master mathematics ?

Nowhere, in Bourbaki or in Godement, is there any suggestion that other formalisations are possible. Godement says "mathematicians are impatient of vague statements", he explains that formality is a good thing, and then like a sharper forcing a card, offers you a choice of exactly one formalisation, and, at that, one that is cumbersome and destructive of intuition.

15·4 If their chosen system is what the Bourbachistes think logic and set theory are like, it is no wonder that they and their disciples are against those subjects and shy away from them. But on reading through Godement one last time, I was left with the impression that he is not so much a disciple of Bourbaki as a victim: loyalty to the group has obliged him to follow the party presentation of logic and set theory, and his intelligence has rebelled against it. I would love to teach him.

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§ *C'est précisement l'une des plus grandes réussites de Cantor que d'avoir pu disqualifier d'emploi du "bon sens" en Mathématiques.*

‖ *Elles n'existent bien entendu que dans l'imagination des mathématiciens.*