elements of X in accordance with the individual's preferences. This is stated more precisely in Definition 1.B.2.

Definition 1.B.2: A function $u: X \to \mathbb{R}$ is a *utility function representing preference relation* $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Note that a utility function that represents a preference relation $\succsim$ is not unique. For any strictly increasing function $f: \mathbb{R} \to \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function representing the same preferences as $u(\cdot)$; see Exercise 1.B.3. It is only the ranking of alternatives that matters. Properties of utility functions that are invariant for any strictly increasing transformation are called *ordinal*. *Cardinal* properties are those not preserved under all such transformations. Thus, the preference relation associated with a utility function is an ordinal property. On the other hand, the numerical values associated with the alternatives in X, and hence the magnitude of any differences in the utility measure between alternatives, are cardinal properties.

The ability to represent preferences by a utility function is closely linked to the assumption of rationality. In particular, we have the result shown in Proposition 1.B.2.

Proposition 1.B.2: A preference relation $\succsim$ can be represented by a utility function only if it is rational.

Proof: To prove this proposition, we show that if there is a utility function that represents preferences $\succsim$, then $\succsim$ must be complete and transitive.

Completeness. Because $u(\cdot)$ is a real-valued function defined on X, it must be that for any $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. But because $u(\cdot)$ is a utility function representing $\succsim$, this implies either that $x \succsim y$ or that $y \succsim x$ (recall Definition 1.B.2). Hence, $\succsim$ must be complete.

Transitivity. Suppose that $x \succsim y$ and $y \succsim z$. Because $u(\cdot)$ represents $\succsim$, we must have $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Therefore, $u(x) \geq u(z)$. Because $u(\cdot)$ represents $\succsim$, this implies $x \succsim z$. Thus, we have shown that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$, and so transitivity is established. ■

At the same time, one might wonder, can *any* rational preference relation $\succsim$ be described by some utility function? It turns out that, in general, the answer is no. An example where it is not possible to do so will be discussed in Section 3.G. One case in which we can always represent a rational preference relation with a utility function arises when X is finite (see Exercise 1.B.5). More interesting utility representation results (e.g., for sets of alternatives that are not finite) will be presented in later chapters.

1.C Choice Rules

In the second approach to the theory of decision making, choice behavior itself is taken to be the primitive object of the theory. Formally, choice behavior is represented by means of a *choice structure*. A choice structure $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family (a set) of nonempty subsets of X; that is, every element of $\mathscr{B}$ is a set $B \subset X$. By analogy with the consumer theory to be developed in Chapters 2 and 3, we call the elements $B \in \mathscr{B}$ *budget sets*. The budget sets in $\mathscr{B}$ should be thought of as an exhaustive listing of all the choice experiments that the institutionally, physically, or otherwise restricted social situation can conceivably pose to the decision maker. It need not, however, include all possible subsets of X. Indeed, in the case of consumer demand studied in later chapters, it will not.

(ii) $C(\cdot)$ is a *choice rule* (technically, it is a correspondence) that assigns a nonempty set of chosen elements $C(B) \subset B$ for every budget set $B \in \mathscr{B}$. When $C(B)$ contains a single element, that element is the individual's choice from among the alternatives in B. The set $C(B)$ may, however, contain more than one element. When it does, the elements of $C(B)$ are the alternatives in B that the decision maker *might* choose; that is, they are her *acceptable alternatives* in B. In this case, the set $C(B)$ can be thought of as containing those alternatives that we would actually see chosen if the decision maker were repeatedly to face the problem of choosing an alternative from set B.

Example 1.C.1: Suppose that $X = \{x, y, z\}$ and $\mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$. One possible choice structure is $(\mathscr{B}, C_1(\cdot))$, where the choice rule $C_1(\cdot)$ is: $C_1(\{x, y\}) = \{x\}$ and $C_1(\{x, y, z\}) = \{x\}$. In this case, we see x chosen no matter what budget the decision maker faces.

Another possible choice structure is $(\mathscr{B}, C_2(\cdot))$, where the choice rule $C_2(\cdot)$ is: $C_2(\{x, y\}) = \{x\}$ and $C_2(\{x, y, z\}) = \{x, y\}$. In this case, we see x chosen whenever the decision maker faces budget $\{x, y\}$, but we may see either x or y chosen when she faces budget $\{x, y, z\}$. ■

When using choice structures to model individual behavior, we may want to impose some "reasonable" restrictions regarding an individual's choice behavior. An important assumption, the weak axiom of revealed preference [first suggested by Samuelson; see Chapter 5 in Samuelson (1947)], reflects the expectation that an individual's observed choices will display a certain amount of consistency. For example, if an individual chooses alternative x (and only that) when faced with a choice between x and y, we would be surprised to see her choose y when faced with a decision among x, y, and a third alterative z. The idea is that the choice of x when facing the alternatives $\{x, y\}$ reveals a proclivity for choosing x over y that we should expect to see reflected in the individual's behavior when faced with the alternatives $\{x, y, z\}$.[4]

The weak axiom is stated formally in Definition 1.C.1.

Definition 1.C.1: The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property holds:

If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

In words, the weak axiom says that if x is ever chosen when y is available, then there can be no budget set containing both alternatives for which y is chosen and x is not.

4. This proclivity might reflect some underlying "preference" for x over y but might also arise in other ways. It could, for example, be the result of some evolutionary process.

Note how the assumption that choice behavior satisfies the weak axiom captures the consistency idea: If $C(\{x, y\}) = \{x\}$, then the weak axiom says that we cannot have $C(\{x, y, z\}) = \{y\}$.[5]

A somewhat simpler statement of the weak axiom can be obtained by defining a *revealed preference relation* $\succsim^*$ from the observed choice behavior in $C(\cdot)$.

Definition 1.C.2: Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is defined by

$$x \succsim^* y \Leftrightarrow \text{there is some } B \in \mathscr{B} \text{ such that } x, y \in B \text{ and } x \in C(B).$$

We read $x \succsim^* y$ as "x is revealed at least as good as y." Note that the revealed preference relation $\succsim^*$ need not be either complete or transitive. In particular, for any pair of alternatives x and y to be comparable, it is necessary that, for some $B \in \mathscr{B}$, we have $x, y \in B$ and either $x \in C(B)$ or $y \in C(B)$, or both.

We might also informally say that "x is revealed preferred to y" if there is some $B \in \mathscr{B}$ such that $x, y \in B$, $x \in C(B)$, and $y \notin C(B)$, that is, if x is ever chosen over y when both are feasible.

With this terminology, we can restate the weak axiom as follows: "*If x is revealed at least as good as y, then y cannot be revealed preferred to x.*"

Example 1.C.2: Do the two choice structures considered in Example 1.C.1 satisfy the weak axiom? Consider choice structure $(\mathscr{B}, C_1(\cdot))$. With this choice structure, we have $x \succsim^* y$ and $x \succsim^* z$, but there is no revealed preference relationship that can be inferred between y and z. This choice structure satisfies the weak axiom because y and z are never chosen.

Now consider choice structure $(\mathscr{B}, C_2(\cdot))$. Because $C_2(\{x, y, z\}) = \{x, y\}$, we have $y \succsim^* x$ (as well as $x \succsim^* y$, $x \succsim^* z$, and $y \succsim^* z$). But because $C_2(\{x, y\}) = \{x\}$, x is revealed preferred to y. Therefore, the choice structure $(\mathscr{B}, C_2)$ violates the weak axiom. ∎

We should note that the weak axiom is not the only assumption concerning choice behavior that we may want to impose in any particular setting. For example, in the consumer demand setting discussed in Chapter 2, we impose further conditions that arise naturally in that context.

The weak axiom restricts choice behavior in a manner that parallels the use of the rationality assumption for preference relations. This raises a question: What is the precise relationship between the two approaches? In Section 1.D, we explore this matter.

1.D The Relationship between Preference Relations and Choice Rules

We now address two fundamental questions regarding the relationship between the two approaches discussed so far:

5. In fact, it says more: We must have $C(\{x, y, z\}) = \{x\}$, $=\{z\}$, or $=\{x, z\}$. You are asked to show this in Exercise 1.C.1. See also Exercise 1.C.2.

(i) If a decision maker has a rational preference ordering $\succsim$, do her decisions when facing choices from budget sets in $\mathscr{B}$ necessarily generate a choice structure that satisfies the weak axiom?

(ii) If an individual's choice behavior for a family of budget sets $\mathscr{B}$ is captured by a choice structure $(\mathscr{B}, C(\cdot))$ satisfying the weak axiom, is there necessarily a rational preference relation that is consistent with these choices?

As we shall see, the answers to these two questions are, respectively, "yes" and "maybe".

To answer the first question, suppose that an individual has a rational preference relation $\succsim$ on X. If this individual faces a nonempty subset of alternatives $B \subset X$, her preference-maximizing behavior is to choose any one of the elements in the set:

$$C^*(B, \succsim) = \{x \in B: x \succsim y \text{ for every } y \in B\}$$

The elements of set $C^*(B, \succsim)$ are the decision maker's most preferred alternatives in B. In principle, we could have $C^*(B, \succsim) = \varnothing$ for some B; but if X is finite, or if suitable (continuity) conditions hold, then $C^*(B, \succsim)$ will be nonempty.[6] From now on, we will consider only preferences $\succsim$ and families of budget sets $\mathscr{B}$ such that $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$. We say that the rational preference relation $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

The result in Proposition 1.D.1 tells us that any choice structure generated by rational preferences necessarily satisfies the weak axiom.

Proposition 1.D.1: Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$, satisfies the weak axiom.

Proof: Suppose that for some $B \in \mathscr{B}$, we have $x, y \in B$ and $x \in C^*(B, \succsim)$. By the definition of $C^*(B, \succsim)$, this implies $x \succsim y$. To check whether the weak axiom holds, suppose that for some $B' \in \mathscr{B}$ with $x, y \in B'$, we have $y \in C^*(B', \succsim)$. This implies that $y \succsim z$ for all $z \in B'$. But we already know that $x \succsim y$. Hence, by transitivity, $x \succsim z$ for all $z \in B'$, and so $x \in C^*(B', \succsim)$. This is precisely the conclusion that the weak axiom demands. ■

Proposition 1.D.1 constitutes the "yes" answer to our first question. That is, if behavior is generated by rational preferences then it satisfies the consistency requirements embodied in the weak axiom.

In the other direction (from choice to preferences), the relationship is more subtle. To answer this second question, it is useful to begin with a definition.

Definition 1.D.1: Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim)$$

for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

In words, the rational preference relation $\succsim$ rationalizes choice rule $C(\cdot)$ on $\mathscr{B}$ if the optimal choices generated by $\succsim$ (captured by $C^*(\cdot, \succsim)$) coincide with $C(\cdot)$ for

6. Exercise 1.D.2 asks you to establish the nonemptiness of $C^*(B, \succsim)$ for the case where X is finite. For general results, See Section M.F of the Mathematical Appendix and Section 3.C for a specific application.

all budget sets in $\mathscr{B}$. In a sense, preferences explain behavior; we can interpret the decision maker's choices as if she were a preference maximizer. Note that in general, there may be more than one rationalizing preference relation $\succsim$ for a given choice structure $(\mathscr{B}, C(\cdot))$ (see Exercise 1.D.1).

Proposition 1.D.1 implies that the weak axiom must be satisfied if there is to be a rationalizing preference relation. In particular, since $C^*(\cdot, \succsim)$ satisfies the weak axiom for any $\succsim$, only a choice rule that satisfies the weak axiom can be rationalized. It turns out, however, that the weak axiom is not sufficient to ensure the existence of a rationalizing preference relation.

Example 1.D.1: Suppose that $X = \{x, y, z\}$, $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. This choice structure satisfies the weak axiom (you should verify this). Nevertheless, we cannot have rationalizing preferences. To see this, note that to rationalize the choices under $\{x, y\}$ and $\{y, z\}$ it would be necessary for us to have $x \succ y$ and $y \succ z$. But, by transitivity, we would then have $x \succ z$, which contradicts the choice behavior under $\{x, z\}$. Therefore, there can be no rationalizing preference relation. ■

To understand Example 1.D.1, note that the more budget sets there are in $\mathscr{B}$, the more the weak axiom restricts choice behavior; there are simply more opportunities for the decision maker's choices to contradict one another. In Example 1.D.1, the set $\{x, y, z\}$ is not an element of $\mathscr{B}$. As it happens, this is crucial (see Exercises 1.D.3). As we now show in Proposition 1.D.2, if the family of budget sets $\mathscr{B}$ includes enough subsets of X, and if $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, then there exists a rational preference relation that rationalizes $C(\cdot)$ relative to $\mathscr{B}$ [this was first shown by Arrow (1959)].

Proposition 1.D.2: If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) the weak axiom is satisfied,

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$; that is, $C(B) = C^*(B, \succsim)$ for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the *only* preference relation that does so.

Proof: The natural candidate for a rationalizing preference relation is the revealed preference relation $\succsim^*$. To prove the result, we must first show two things: (i) that $\succsim^*$ is a rational preference relation, and (ii) that $\succsim^*$ rationalizes $C(\cdot)$ on $\mathscr{B}$. We then argue, as point (iii), that $\succsim^*$ is the unique preference relation that does so.

(i) We first check that $\succsim^*$ is rational (i.e., that it satisfies completeness and transitivity).

Completeness By assumption (ii), $\{x, y\} \in \mathscr{B}$. Since either x or y must be an element of $C(\{x, y\})$, we must have $x \succsim^* y$, or $y \succsim^* x$, or both. Hence $\succsim^*$ is complete.

Transitivity Let $x \succsim^* y$ and $y \succsim^* z$. Consider the budget set $\{x, y, z\} \in \mathscr{B}$. It suffices to prove that $x \in C(\{x, y, z\})$, since this implies by the definition of $\succsim^*$ that $x \succsim^* z$. Because $C(\{x, y, z\}) \neq \varnothing$, at least one of the alternatives x, y, or z must be an element of $C(\{x, y, z\})$. Suppose that $y \in C(\{x, y, z\})$. Since $x \succsim^* y$, the weak axiom then yields $x \in C(\{x, y, z\})$, as we want. Suppose instead that $z \in C(\{x, y, z\})$; since $y \succsim^* z$, the weak axiom yields $y \in C(\{x, y, z\})$, and we are in the previous case.

(ii) We now show that $C(B) = C^*(B, \succsim^*)$ for all $B \in \mathscr{B}$; that is, the revealed preference relation $\succsim^*$ inferred from $C(\cdot)$ actually generates $C(\cdot)$. Intuitively, this seems sensible. Formally, we show this in two steps. First, suppose that $x \in C(B)$. Then $x \succsim^* y$ for all $y \in B$; so we have $x \in C^*(B, \succsim^*)$. This means that $C(B) \subset C^*(B, \succsim^*)$. Next, suppose that $x \in C^*(B, \succsim^*)$. This implies that $x \succsim^* y$ for *all* $y \in B$; and so for each $y \in B$, there must exist some set $B_y \in \mathscr{B}$ such that $x, y \in B_y$ and $x \in C(B_y)$. Because $C(B) \neq \varnothing$, the weak axiom then implies that $x \in C(B)$. Hence, $C^*(B, \succsim^*) \subset C(B)$. Together, these inclusion relations imply that $C(B) = C^*(B, \succsim^*)$.

(iii) To establish uniqueness, simply note that because $\mathscr{B}$ includes all two-element subsets of X, the choice behavior in $C(\cdot)$ completely determines the pairwise preference relations over X of any rationalizing preference.

This completes the proof. ■

We can therefore conclude from Proposition 1.D.2 that for the special case in which choice is defined for all subsets of X, a theory based on choice satisfying the weak axiom is completely equivalent to a theory of decision making based on rational preferences. Unfortunately, this special case is too special for economics. For many situations of economic interest, such as the theory of consumer demand, choice is defined only for special kinds of budget sets. In these settings, the weak axiom does not exhaust the choice implications of rational preferences. We shall see in Section 3.J, however, that a strengthening of the weak axiom (which imposes more restrictions on choice behavior) provides a necessary and sufficient condition for behavior to be capable of being rationalized by preferences.

Definition 1.D.1 defines a rationalizing preference as one for which $C(B) = C^*(B, \succsim)$. An alternative notion of a rationalizing preference that appears in the literature requires only that $C(B) \subset C^*(B, \succsim)$; that is, $\succsim$ is said to rationalize $C(\cdot)$ on $\mathscr{B}$ if $C(B)$ is a subset of the most preferred choices generated by $\succsim$, $C^*(B, \succsim)$, for every budget $B \in \mathscr{B}$.

There are two reasons for the possible use of this alternative notion. The first is, in a sense, philosophical. We might want to allow the decision maker to resolve her indifference in some specific manner, rather than insisting that indifference means that anything might be picked. The view embodied in Definition 1.D.1 (and implicitly in the weak axiom as well) is that if she chooses in a specific manner then she is, de facto, not indifferent.

The second reason is empirical. If we are trying to determine from data whether an individual's choice is compatible with rational preference maximization, we will in practice have only a finite number of observations on the choices made from any given budget set B. If $C(B)$ represents the set of choices made with this limited set of observations, then because these limited observations might not reveal all the decision maker's preference maximizing choices, $C(B) \subset C^*(B, \succsim)$ is the natural requirement to impose for a preference relationship to rationalize observed choice data.

Two points are worth noting about the effects of using this alternative notion. First, it is a weaker requirement. Whenever we can find a preference relation that rationalizes choice in the sense of Definition 1.D.1, we have found one that does so in this other sense, too. Second, in the abstract setting studied here, to find a rationalizing preference relation in this latter sense is actually trivial: Preferences that have the individual indifferent among all elements of X will rationalize *any* choice behavior in this sense. When this alternative notion is used in the economics literature, there is always an insistence that the rationalizing preference relation should satisfy some additional properties that are natural restrictions for the specific economic context being studied.

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EXERCISES

1.B.1^{B} Prove property (iii) of Proposition 1.B.1.

1.B.2^{A} Prove properties (i) and (ii) of Proposition 1.B.1.

1.B.3^{B} Show that if $f: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $u: X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$, then the function $v: X \to \mathbb{R}$ defined by $v(x) = f(u(x))$ is also a utility function representing preference relation $\succsim$.

1.B.4^{A} Consider a rational preference relation $\succsim$. Show that if $u(x) = u(y)$ implies $x \sim y$ and if $u(x) > u(y)$ implies $x \succ y$, then $u(\cdot)$ is a utility function representing $\succsim$.

1.B.5^{B} Show that if X is finite and $\succsim$ is a rational preference relation on X, then there is a utility function $u: X \to \mathbb{R}$ that represents $\succsim$. [*Hint*: Consider first the case in which the individual's ranking between any two elements of X is strict (i.e., there is never any indifference), and construct a utility function representing these preferences; then extend your argument to the general case.]

1.C.1^{B} Consider the choice structure $(\mathscr{B}, C(\cdot))$ with $\mathscr{B} = (\{x, y\}, \{x, y, z\})$ and $C(\{x, y\}) = \{x\}$. Show that if $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, then we must have $C(\{x, y, z\}) = \{x\}$, $= \{z\}$, or $= \{x, z\}$.

1.C.2^{B} Show that the weak axiom (Definition 1.C.1) is equivalent to the following property holding:

> Suppose that $B, B' \in \mathscr{B}$, that $x, y \in B$, and that $x, y \in B'$. Then if $x \in C(B)$ and $y \in C(B')$, we must have $\{x, y\} \subset C(B)$ and $\{x, y\} \subset C(B')$.

1.C.3^{C} Suppose that choice structure $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom. Consider the following two possible revealed preferred relations, $\succ^*$ and $\succ^{**}$:

$$x \succ^* y \Leftrightarrow \text{there is some } B \in \mathscr{B} \text{ such that } x, y \in B,\ x \in C(B), \text{ and } y \notin C(B)$$

$$x \succ^{**} y \Leftrightarrow x \succsim^* y \text{ but not } y \succsim^* x$$

where $\succsim^*$ is the revealed at-least-as-good-as relation defined in Definition 1.C.2.

(a) Show that $\succ^*$ and $\succ^{**}$ give the same relation over X; that is, for any $x, y \in X$, $x \succ^* y \Leftrightarrow x \succ^{**} y$. Is this still true if $(\mathscr{B}, C(\cdot))$ does not satisfy the weak axiom?

(b) Must $\succ^*$ be transitive?

(c) Show that if $\mathscr{B}$ includes all three-element subsets of X, then $\succ^*$ is transitive.

1.D.1^{B} Give an example of a choice structure that can be rationalized by several preference relations. Note that if the family of budgets $\mathscr{B}$ includes all the two-element subsets of X, then there can be at most one rationalizing preference relation.