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You get more ordinals every time you go up a type

That's the project, anyway. Written up to amuse Randall Holmes.

Randall sets the scene rather well:

> "The order type of the ordinals up to α is $T^2(\alpha)$.
>
> So the order type Ω of the ordinals is the first ordinal which does not have a preimage under T^2. But that means it has a preimage under T, and so the set of ordinals is the same size as a set of singletons.
>
> But there is no clear way to produce such a set of singletons: the argument is nonconstructive. If we could produce a concrete well ordering of length T^{-1} the length of the ordinals, surely we would use that set as the ordinals.
>
> Here is a reason why it is nonconstructive: in TST when we go up two types, the length of the ordinal must increase (Hartogs). But it does not have to go up when we go up one type. In TST + Amb, it must, because it must either increase at every step or not increase at every step, and we know it increases after two steps, so it must increase. But we cannot expect to find a concrete witness to this fact."

The point about *two* levels is this. Sierpinski-Hartogs tells us that $\aleph(\alpha) < \aleph(2^{2^\alpha})$. It's the double exponential that gives rise to the complications. Another version of Sierpinski-Hartogs is $\aleph^*(\alpha) \leq^* 2^{\alpha^2}$ or—to put it another way—$\aleph^*(\alpha) < \aleph(2^{\alpha^2})$. The αs of interest here are the cardinalities of local universes, and—at least as long as we have infinity—all these cardinals are idemmultiple.

It turns out that in order to make use of $\aleph^*(\alpha) < \aleph(2^{\alpha^2})$ we need to reconceptualise ordinals as arising from (isomorphism classes of) prewellorderings not wellorderings ... which we had better now define. This definition is probably not standard, co's i've just made it up. Mind you, a good idea can be had more than once.

Definition 1. *A Prewellordering is a wellfounded strict poset $\langle X, <_X\rangle$ satisfying the extra condition that the relation $u \not<_X v \not<_X u$ is an equivalence relation. The thought is that a prewellorder is an actual wellorder iff the equivalence classes are singletons. Let us write these two equivalence relations $\sim_X$ and $\sim_Y$.*

A prewellorder morphism $I : \langle X, <_X\rangle$ to $\langle Y, <_Y\rangle$ is a subset of $X \times Y$ satisfying

$I(x, y) \wedge x \sim_X x' \wedge y \sim_Y y' \rightarrow I(x', y')$

and

$I(x, y) \wedge I(x', y') \wedge R(x, x') \rightarrow S(y, y')$

I want to say out loud at this point that the preordinals (or whatever we decide to call them) behave exactly like ordinals: the obvious order relation on them is a prewellordering—in fact a wellordering.

Now some cardinal arithmetic, some *old* cardinal arithmetic. A definition: $\aleph^*(\alpha)$ is the least aleph $\not\leq^* \alpha$.

The next claim is that

THEOREM 1.

$$\aleph^*(\alpha) \leq^* 2^{\alpha^2}.$$

Proof:

This is a classical result in the mould of Sierpinski-Hartogs. God knows who proved it first; even John Truss doesn't know. Notice that it can be captured by a stratifiable expression. This fact tells us that everything is going to be all right in the end, even tho' *en route* we have problems occasioned by things cropping up at the wrong type/level.

2^{α^2} is of course the cardinality of $\mathcal{P}(A \times A)$ when $|A| = \alpha$. $\mathcal{P}(A \times A)$ is of course the set of binary relations on A. We are looking for a surjection. Let f be the following function. On being given a binary relation $R \in \mathcal{P}(A \times A)$ it sends it to 0 if it not a prewellordering. If it is a prewellordering send it to its equivalence class. Clearly any equivalence class of a prewellordering of A gets hit. The idea is that the range of this function is of size $\aleph^*(\alpha)$ but on the face of it it is at the wrong level.

For the moment i want to park the task of sorting out all these details. I don't *think* they are going to be problematic, and i want to think ahead. The point i wan to make at this stage is that the first two paragraphs of Holmes' remark, quoted above, hold also for the new style of ordinals.

There is lots of work still to do; this is a holding operation, a report on work-in-progress.

One very useful observation that should be minuted is that T of a new ordinal is always an ordinal. T of a new ordinal α is obviously the relational type of R^ι when the relational type of R is α. But notice that $T\alpha$ is also the relational type of the prewellorder defined on the quotient, the set of equivalence classes under $\sim_R$. And this prewellorder is clearly a wellordering, so $T\alpha$ is an ordinal.

So T of a new-style ordinal is an ordinal; if the partition principle (for wellordered sets) fails, is the converse true?

OK, so there are three possibilities

> (i) The ordinals and the preordinals coincide. That means that every wellorderable partition is the same size as a set of singletons. This says: $\aleph \leq^* T|V| \to \aleph \leq T|V|$.
>
> (ii) The preordinals are as long as they can be. This means that every wellorderable set is the same size as a partition $\forall\aleph)(\aleph \leq^* T|V|)$.
>
> (iii) There is a first ordinal that is not T of a preordinal. This says that there are alephs $\kappa < \lambda$ with $\kappa \leq^* T|V| \wedge \kappa \not\leq T|V|$, and $\lambda \not\leq^* T|V|$

It might help to have a proper name for the order type of the wellordering of all preordinals (it is an ordinal not a proper preordinal) Let call it Ω