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We say (for the purposes of this question) that a function $f : A \to \mathcal{P}(A)$ is **recursive** if the relation $\{\langle a, b\rangle : a \in f(b)\}$ is wellfounded. This is not standard nomenclature, but binary relations defined in this way are natural and useful: slightly surprisingly, any structure for the language of set theory can be thought of as a set A of atoms equipped with an *injective* map $f : A \to \mathcal{P}(A)$.

Suppose $g : \mathcal{P}(B) \to B$. We can attempt to define a function h recursively by:

$$h(a) =: g(\{h(a') : a' \in f(a)\}).$$

Clearly we are going to be able to show (by an appeal to the recursion theorem) that this recursion has a unique solution—as long as the relation $\{\langle a, b\rangle : a \in f(b)\}$ is wellfounded. But what about a converse?

Suppose $\{\langle a, b\rangle : a \in f(b)\}$ is not wellfounded. We want to find a B and $g : \mathcal{P}(B) \to B$ such that there is more than one h satisfying

$$(\forall a \in A)(h(a) = g(\{h(a') : a' \in f(a)\})).$$

Let B be a set with at least two members, and b_1 and b_2 be two members of B, and define $g : \mathcal{P}(B) \to B$ by

$$g(B') = \texttt{ if } (B' = \emptyset \vee B' = \{b_1\}) \texttt{ then } b_1 \texttt{ else } b_2$$

Suppose now that A' is a subset of A with no minimal member under the relation $\{\langle a, b\rangle : a \in f(b)\}$. Notice that both

$$h_1(a) =: b_1$$

and

$$h_2(a) =: \texttt{if } a \in A' \texttt{ then } b_2 \texttt{ else } b_1$$

are solutions to

$$h(a) = g(\{h(a') : a' \in f(a)\}).$$