CHAPTER I: Prerequisites from set theory

We extend a warm welcome to all our readers and hope that they have some familiarity with ZF set theory and its weak subsystems such as Kripke–Platek, and with the rudiments of constructibility and of forcing.

A theme, possibly the characteristic feature, of descriptive set theory is the relationship between two ways of defining sets of reals: on the one hand when one permits oneself to use sets of all possible ranks; on the other when one confines oneself to what can be said within the framework of numbers, reals, and relations on sets of reals.

We begin with a notion with which we shall be concerned throughout the book, that of a *pre-well-ordering* of a set of reals, and in keeping with this theme, we give both a set-theoretical and an analytical definition.

1: Vopěnka algebras

1·0 Historical Note Gödel in his Princeton Bicentenary Lecture in 1946 hinted at the notion of ordinal definability. The concept was developed explicitly in a celebrated paper of Scott and Myhill[R 1]. These ideas were combined with the theory of forcing by Vopěnka to prove the result that every set of ordinals is generic over the inner model HOD of all sets hereditarily ordinal-definable.

⊛ ⊛ ⊛ In this section we begin by proving an easy version of Vopěnka's result, and then generalise that in various ways, first by allowing ordinal-definability relative to a fixed set Q of ordinals, then by considering finite sequences of reals instead of single real, and finally (generic) infinite sequences of reals. Two results from this section will have many applications: the first is the genericity result for reals over many models, the second is that in a transitive class M in which the statement $V = L(Q, \mathbb{R}, \sigma)$ is true for some set Q of ordinals and set σ of reals, $(HOD_{Q,\sigma})^M$ is itself of the form $L(T)$ where T is a set of ordinals; in the case $Q = \varnothing = \sigma$, T may be taken to be a subset of Θ.

We shall distinguish typographically between OD and OD, the former being the class and the latter the predicate of being ordinal definable. Similarly we shall distinguish between HOD and HOD. Thus we might say "A being OD is in OD".

Crude and fine definitions of ordinal-definability

1·1 Remark A delicate point: Scott and Myhill show that each set definable over some V_ζ permitting ordinal parameters (less than ζ) is definable over some other V_ξ with no parameters required. However that ξ might be much larger than the original ζ: for example one might define something using some indiscernibles for V_ζ, and arrange for them to be definable by coding in some much larger ξ.

In the closing paragraph of this section we shall want to say that, assuming $V = L(\mathbb{R})$, every OD set of reals has an OD name less than Θ. I am not sure whether that is true if no parameters are permitted; and in any case it would be tedious to prove that they could be dropped, so I shall permit the parameters. To preserve the simplicity of an OD-name as one ordinal and one formula, let us agree that we use some constructible enumeration of finite sequences of ordinals and let the single ordinal refer to that. Implicitly therefore there is some agreement between the length of that sequence of ordinals and the number of spare places in the formula.

1·2 Remark The Scott-Myhill predicate of being OD is Σ_2^{ZF}, but not Σ_1^{ZF}: start from L and add a Cohen real a. In $L[a]$ a is but one member of a perfect set of Cohen reals, and very far from definable. Now, taking care to add no new hereditarily countable sets, make a further extension so that for some large κ,

[R 1] Scott, Myhill

$a = \{n \in \omega \mid 2^{\aleph_{\kappa+n}} = \aleph_{\kappa+n+1}\}$, as done by McAloon[R 2]. In that larger model, a is OD , but HC is $L_{\omega_1}[a]$, in which a is not OD. Under DC, HC is always a Σ_1-submodel of V so the predicate OD is not Σ_1.

Thus in ZF alone HOD is an unruly notion, but inside a hierarchical structure such as $L[\alpha]$ or $L[\mathbb{R}]$ it is better-behaved, for the OD sets are then precisely those OD inside some $J_\eta[\alpha]$ or $J_\eta(\mathbb{R})$ as the case may be. In one direction that follows by applying Lévy reflection for the (relative-)constructible hierarchy to the formula expressing ordinal definability; in the other direction, it is enough to note that if something is definable over $J_\zeta(\mathbb{R})$, then it is definable over $V_{\omega\zeta}$ since the sequence $\langle J_\nu(\mathbb{R}) | \nu < \zeta\rangle$ is $\Sigma_1(\mathbb{R})$ over $V_{\omega\zeta}$.

1·3 Remark Indeed, the predicate of being $(OD)^{L(\mathbb{R})}$ is always $\Sigma_1(\mathbb{R})$. A particularly elegant way of seeing that is to note that in $L(\mathbb{R})$ the OD sets are those that are rudimentary words in $\mathbb{R}$ and the $J_\xi(\mathbb{R})$'s, but without any parameters for individual reals. So without assuming $V = L(\mathbb{R})$, one has

$$x \in (OD)^{L(\mathbb{R})} \iff \exists \text{ rudimentary } h\ \exists \vec{\xi}\ x = h(\mathbb{R}, J_{\xi_0}(\mathbb{R}), \dots J_{\xi_{k-1}}(\mathbb{R})).$$

1·4 Remark Thus in $L(\mathbb{R})$, the definition of OD may be considerably simplified. We have seen how every element of $L(\mathbb{R})$ is a rudimentary word in $\mathbb{R}$, some individual reals, and some $J_\xi(\mathbb{R})$'s. The OD sets are precisely those which admit such representation without using any individual reals. This may be seen by noting that anything ordinal-definable over a V_η is definable over what some $J_\xi(\mathbb{R})$ thinks is V_η, and hence over $J_\xi(\mathbb{R})$; but therefore it is Δ_0 in the parameter $J_\xi(\mathbb{R})$ and hence a rudimentary word in $J_\xi(\mathbb{R})$ plus whatever smaller ordinal parameters plus $\mathbb{R}$ were used on the way.[C 1]

However, we shall in this section retain the traditional definition of ordinal-definable, but one note of caution should be sounded. In §4 we shall be particularly interested in OD pre-well-orderings of sets of reals, and in the earliest stage η at which they are detected to be ordinal definable. Thus we shall be looking at ordinals η such that $J_\eta(\mathbb{R})$ believes that for each ordinal ν the class $\{x \mid \varrho_\in(x)\}$ is a set, where $\varrho_\in$ is the usual rank function defined for all sets by recursion on the $\in$-relation. In such $J_\eta(\mathbb{R})$ there is no problem with the concept of ordinal definability.

We shall at the same time be interested in certain ordinals ξ such that in $J_\xi(\mathbb{R})$ $\{x \mid \varrho_\in(x) < \omega + 2\}$ is not a set. In such $J_\xi(\mathbb{R})$ the usual concept of ordinal-definability breaks down, and we shall not attempt to replace it by another. [N 1]

In section 4, we prove a $\diamondsuit$-like principle for Σ_1 statements in $L(\mathbb{R})$: therefore it would have been natural to switch to this Σ_1 definition, though we have not done so.

Genericity over *HOD*

⊛ ⊛ Our aim now is to prove the following instance of Vopěnka's result.

1·5 Proposition (Vopěnka) *Let κ be an infinite ordinal and $a \subseteq \kappa$. There is a complete Boolean algebra $\mathbb{K} = \mathbb{K}^1(\kappa)$ in HOD and a $(HOD, \mathbb{K})$-generic filter G such that $a \in M[G]$.*

Proof : Let $\mathcal{B} = \{C \mid C \in OD \,\&\, C \subseteq \mathcal{P}(\kappa)\}$. $\mathcal{B}$ is not in HOD, but a copy of it is: every set in OD has an OD-name (ζ, ϕ) where ζ is an ordinal and ϕ is a formula with one free variable such that ϕ defines the given set over V_ζ. All OD-names are constructible. Given a OD-name N we write N^v for the set it names — an evaluation that is highly dependent on which universe we are in.

Indeed every OD set has a class of such names. Let λ be large enough so that V_λ contains at least one name for every element of $\mathcal{B}$, and let X be the set of such OD-names in V_λ.

1·6 Definition On X we define two relations $\sim_o$ and $\leq_o$ by

$$N_1 \sim_o N_2 \iff_{\text{df}} (N_1)^v = (N_2)^v$$
$$N_1 \leq_o N_2 \iff_{\text{df}} (N_1)^v \subseteq (N_2)^v$$

Note that these relations are OD, and hence are in HOD. The first is a congruence with respect to the second, so the second induces a partial ordering on the set of equivalence classes of names, which we shall also denote by $\leq_o$.

[R 2] Mc Aloon

[C 1] We strictly exclude $\dot{H}$ from our ordinal-definitions.

[N 1] I intend to include a brief treatment of Scott–Myhill at the end of Appendix One.

Consider the map which sends each set $C \in \mathcal{B}$ to the equivalence class of its names. This map, call it ^{o}, is OD. We write K for the set $X/\sim_o$.

Let $\mathbb{K}^1(\kappa) = \langle K, \leq_o \rangle$. Then $\mathbb{K} = \mathbb{K}^1(\kappa)$ is in HOD and is the complete Boolean algebra we are seeking. What are its operations ? The Boolean supremum of an OD sequence of OD-names is a OD-name for their union; such a name may easily be constructed since the sequence is OD. The complement of a OD-name is a ODname for its complement.

Given $a \subseteq \kappa$ let $G_a^{\mathcal{B}} = \{C \in \mathcal{B} \mid a \in C\}$ and $G_a = G_a^{\mathbb{K}} = \{C^o \mid a \in C\}$.

We should check that G_a is $(HOD, \mathbb{K})$-generic. But a dense set is one the union of the sets named in which is the whole of $\mathcal{P}\kappa$: if $\Delta \in HOD$ is dense in $\mathbb{K}^1(\kappa)$, $\bigcup\{C \mid C^o \in \Delta\}$, which is OD, equals the whole of $\mathcal{P}(\kappa)$, and so G_a will meet each dense open set.

For each $\alpha \in \kappa$, the set $S(\alpha) \ =_{\mathrm{df}} \ \{b \subseteq \kappa \mid \alpha \in b\}$ is in $\mathcal{B}$; so

$$\begin{aligned} \alpha \in a &\iff S(\alpha) \in G_a^{\mathcal{B}} \\ &\iff S(\alpha)^o \in G_a \end{aligned}$$

Thus $a = \{\alpha \mid S(\alpha) \in G_a\}$. Now the map $\alpha \mapsto S(\alpha)^o$ is OD and thus in HOD. Hence $a \in HOD[G_a]$ and a is thus generic over HOD. $\dashv$ (1·5)

1·7 DEFINITION We write $OD_{\{a\}}$ for the class of sets ordinal definable using the set a as a parameter; and $HOD_{\{a\}}$ for the class of those sets which are hereditarily in $OD_{\{a\}}$.

We gather some facts about a, G_a and $HOD_{\{a\}}$ in the following

1·8 PROPOSITION *(i) $a \in HOD[G_a]$ and so a is generic over HOD.*

(ii) $HOD[G_a] = HOD_{\{a\}}$

(iii) $HOD[a] \subseteq HOD_{\{a\}}$.

Proof : (i) has been established.

(ii): G_a is visibly definable from the parameter a and the OD map π, and so $HOD[G_a] \subseteq HOD_{\{a\}}$.

Since $HOD_{\{a\}}$ models AC, it is enough to show that every set of ordinals in it is also in $HOD[G_a]$. So let $x \in HOD_{\{a\}}$, where x is a set of ordinals.

Suppose that $\eta > \mathrm{rank}(x) \cup \kappa$ is such that $\forall \zeta\ \zeta \in x \iff \zeta < \eta \ \&\ V_\eta \models \phi(\zeta, a)$. We seek a name $\dot{x}$ that evaluates to x under ϕ_{G_a}. Write $N(\zeta)$ for $\{Y \subseteq \kappa \mid V_\eta \models \phi(\zeta, Y)\}$ and note that $N(\zeta)$ is OD and a subset of $\mathcal{P}(\kappa)$; hence

$$\begin{aligned} \zeta \in x &\iff a \in N(\zeta) \\ &\iff N(\zeta) \in G_a^{\mathcal{B}} \\ &\iff N(\zeta)^o \in G_a \end{aligned}$$

The map $\zeta \mapsto N(\zeta)^o$ is HOD, and thus we may build a $\mathbb{K}$-name $\dot{x}$ such that $[\![\hat{\zeta} \ \epsilon \ \dot{x}]\!]^{\mathbb{K}} = N(\zeta)^o$. Then $\phi_{G_a}(\dot{x}) = x$, as required.

(iii) follows trivially from (i) and (ii) $\dashv$ (1·8)

$\lessdot$ 1·9 REMARK Thus, perhaps surprisingly, $HOD_{\{a\}}$ is a set-generic extension of HOD. It is natural to ask whether for certain a we may expect $HOD[a] = HOD_{\{a\}}$ There are times when that is known to happen. To describe them, we must introduce the notion of an ∞-Borel set of reals, which we shall do in Chapter II.

Definability relative to a given set of ordinals.

1·10 REMARK We may relativise the concept of definability to a given set of ordinals Q. We write $HOD_{\{Q\}}$ for the universe of sets hereditarily definable from ordinals and the parameter Q. We may find, uniformly in κ and Q a Boolean algebra $\mathbb{K}^1(\kappa, Q)$ in $HOD_{\{Q\}}$, complete inside that model, with respect to which each subset of κ is generic over $HOD_{\{Q\}}$.

The argument begins by considering the set $\{C \subseteq \mathcal{P}(\kappa) \mid C \in OD_Q\}$, ordered under reverse inclusion, and then mapping each such C to a representative set of names for it: an OD-Q name for C will be a pair

(φ, ζ) where $\zeta > \bigcup Q$ and $C = \{x \mid V_\zeta \models \varphi(x, Q)\}$. The further details follow the argument already given for the case $Q = \varnothing$.

$\lessdot$ 1·11 REMARK We shall use this more general form in section 8 of Chapter III, when for a fixed set of ordinals S and an arbitrary countable set σ of reals, we shall wish to say that each real in $L(S, \sigma)$ is generic over $HOD^{L(S,\sigma)}_{\{S\}}$ with respect to the relevant Vopěnka algebra.

The two-dimensional case

⊛ ⊛ We discuss the genericity of pairs of reals, and relate the two Vopěnka algebras.

1·12 The above discussion may easily be generalised in a second way: we may show that for each ordinal κ and each $k < \omega$, there is a notion of forcing $\mathbb{K}^k(\kappa)$ in *HOD* with respect to which each sequence $\langle a_0, \ldots a_{k-1}\rangle$ of subsets of κ is generic over *HOD*. Further the definitions are uniform, so that the sequence $\langle \mathbb{K}^k(\kappa) \mid k < \omega\rangle$ is in *HOD*.

We give the details for the case $\kappa = \omega$ — which is the case we shall apply heavily in section 8 of Chapter III — and write $\mathbb{K}^k$ for $\mathbb{K}^k(\omega)$.

First consider the case of $k = 2$. Let (α, β) be an ordered pair of reals. To see that the pair is generic over *HOD*, we start from the collection $\{C \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega) \mid C \text{ is OD }\}$. As before we copy that into *HOD*, by some system of names; call the resulting complete Boolean algebra $\mathbb{K}^2$. Then we can define an ultrafilter $G^2_{\alpha,\beta}$ as the set of names of those C of which (α, β) is a member. The argument is much as before.

Now we wish to relate that algebra to our previous one, $\mathbb{K}^1$. We know that α is $(HOD, \mathbb{K}^1)$ generic, and we know from Remark 3·5 that β is $(HOD_\alpha, \mathbb{K}^{1,\alpha})$-generic, where $\mathbb{K}^{1,\alpha}$ is the algebra in HOD_α created by names of OD_α sets of reals. It will calm our nerves if we can show that those two forcings compose to give the forcing $\mathbb{K}^2$.

1·13 DEFINITION Let $C \subseteq \mathcal{P}\omega \times \mathcal{P}\omega$. A *section* of C is a subset of $\mathcal{P}(\omega)$ of the form $C_\alpha =_{\text{df}} \{y \mid (\alpha, y) \in C\}$ for some $\alpha \in \mathcal{P}\omega$.

1·14 PROPOSITION *(i) If C is OD, C_α is OD in α. (ii) Every OD_α subset D of $\mathcal{P}\omega$ is of the form C_α for some OD set $C \subseteq \mathcal{P}\omega \times \mathcal{P}\omega$.*

Proof : (i) is plain. For (ii), suppose $D = \{y \mid \phi(y, \kappa, \alpha)\}$. Let C be $\{(x, y) \mid \phi(y, \kappa, x)\}$. ⊣ (1·14)

1·15 PROPOSITION *For given α the map ς, that associates to each name C^o of an OD subset C of the plane a name $\varsigma(C^o, \alpha)$ for the section C_α, is definable from α and therefore in the model $HOD[G^1_\alpha]$, which equals $HOD_{\{\alpha\}}$.* ⊣

1·16 DEFINITION $C_{\downarrow 1} =_{\text{df}} \{x \mid \exists y(x, y) \in C\}$ and $D^{\uparrow 2} = \{(x, y) \mid x \in D \;\&\; y \in \mathcal{P}(\omega)\}$.

1·17 REMARK Note the existential quantifier in the definition of $C_{\downarrow 1}$.

1·18 LEMMA *There are maps $\pi : \mathbb{K}^2 \longrightarrow \mathbb{K}^1$ and $\iota : \mathbb{K}^1 \longrightarrow \mathbb{K}^2$ with ι an injection, π a surjection, both order-preserving, and $\pi(\iota(D)) = D$, and $\iota(\pi(C)) \supseteq C$.*

Proof : using C for 2-dimensional and D for a 1-dimensional set, we put $\pi(C) = C_{\downarrow 1}$ and $\iota(D) = D^{\uparrow 2}$. Then ι is 1-1 and order-preserving; π is order-preserving but not 1-1. Note that if $C \subseteq D^{\uparrow 2}$, then $C_{\downarrow 1} \subseteq (D^{\uparrow 2})_{\downarrow 1} = D$, so $C \subseteq (C_{\downarrow 1})^{\uparrow 2} \subseteq D^{\uparrow 2}$; thus $(C_{\downarrow 1})^{\uparrow 2}$ is the inf of all $\pi(D) \geqslant C$.

π is onto, since given $D \in \mathbb{K}^1$, $D = (D^{\uparrow 2})_{\downarrow 1}$. ⊣ (1·18)

1·19 PROPOSITION *Let α and β be two reals, possibly distinct. Let G^1_α be the generic filter in $\mathbb{K}^1$ defined by α, let $G^{1,\alpha}_\beta$ be the filter in $\mathbb{K}^{1,\alpha}$ defined by β, and let $G^2_{\alpha,\beta}$ be the generic filter in $\mathbb{K}^2$ defined by the pair (α, β). Then*

$$HOD[G^1_\alpha][G^{1,\alpha}_\beta] = HOD[G^2_{\alpha,\beta}].$$

Proof : We use D for a typical OD set of dimension 1, C for one of dimension 2.

We shall use special cases of definitions that will be given below in full generality. In one direction, given G^1_α (which makes the map ς available) and $G^{1,\alpha}_\beta$, we have

$$\begin{aligned} C^o \in G^2_{\alpha,\beta} &\Longleftrightarrow (\alpha,\, \beta) \in C \\ &\Longleftrightarrow \beta \in C_\alpha \\ &\Longleftrightarrow \varsigma(C^o) \in G^{1,\alpha}_\beta \end{aligned}$$

In the other direction, suppose we are given $G^2_{\alpha,\beta}$: then first we may recover G^1_α by the equivalence

$$D^o \in G^1_\alpha \Longleftrightarrow (D^{\uparrow 2})^o \in G^2_{\alpha,\beta}.$$

We know that the map ς is in $HOD[G^1_\alpha]$, and we know that every OD_α set is a section of an OD set, so we may recover $G^{1,\alpha}_\beta$ by the equivalence

$$\varsigma(C^o) \in G^{1,\alpha}_\beta \Longleftrightarrow C^o \in G^2_{\alpha,\beta}. \qquad \dashv\ (1{\cdot}19)$$

1·20 REMARK The heart of the matter is that $(\alpha,\, \beta) \in C \Longleftrightarrow \beta \in C_\alpha$.

1·21 The relationship between $\mathbb{K}^1$ and $\mathbb{K}^2$ is worth examining. Suppose we make a generic extension of M for $\mathbb{K}^1$ by adding G. Then in $M[G]$ we consider $\mathbb{Q} = \{C \in \mathbb{K}^2 \mid \pi(C) \in G\}$. That will be a partial ordering of some kind, and if we force with it, adding a generic H, then $M[G][H]$ will be of the form $M[K]$ where K is $(M, \mathbb{K}^2)$-generic. But that partial ordering is far from separative. G will be defined by some real α; so in essence we are looking at $\{C \mid \alpha \in C_{\downarrow 1}\}$, that is, those C's whose α-section is non-empty. Now on closer inspection we find that two elements of $\mathbb{Q}$ are inseparable if and only if their α-sections are equal; so when we factor by the relationship of inseparability, we obtain a partial ordering isomorphic to the set of non-empty α-sections.

Higher finite dimensions

⊛ ⊛ Now we extend that discussion to finite sequences of arbitrary length.

1·22 DEFINITION We write x^n to denote an arbitrary n-tuple of subsets of ω. If D is an OD subset of $(\mathcal{P}(\omega))^m$, we shall say that D is of *dimension* m.

Let $n \geq m$, and let C and D be of dimension n and m respectively.

1·23 REMARK We should comment on our finite sequences of reals. When defining the algebras $\mathbb{K}^k$ we permitted members of the C's of dimension k not to be 1-1: in this way a finite sequence of reals which had repetitions would be still be generic over HOD. But now we aim to simulate a forcing, $\mathbb{E}$, not yet introduced, for which conditions are 1-1 finite sequences of reals, and so **we shall assume henceforth that all our OD sets C of dimension n are subsets of $\{x^n \mid i < j < n \implies x_i \neq x_j\}$.**

1·24 DEFINITION
$$\begin{aligned} D^{\uparrow n} &=_{\mathrm{df}} \{x^n \mid x^n \restriction m \in D\} \\ C_{\downarrow m} &=_{\mathrm{df}} \{x^m \mid \exists y^n (y^n \in C \;\&\; y^n \restriction m = x^m)\} \end{aligned}$$

1·25 REMARK Note the existential quantifier in the definition of $C_{\downarrow m}$: in the context of OD sets it is harmless, but we shall have trouble with it in the later context of ∞-Borel sets.

1·26 LEMMA *Let $m \leq n \leq p$, and D, E OD sets of dimension m and p respectively.*

(i) $D^{\uparrow n}$ *and* $E_{\downarrow n}$ *are* OD;

(i) $D^{\uparrow m} = D_{\downarrow m} = D$;

(ii) $(D^{\uparrow p})_{\downarrow n} = D^{\uparrow n}$;

(iii) $(D^{\uparrow p})_{\downarrow m} = D$;

(iv) $(E_{\downarrow m})^{\uparrow n} \supseteq E_{\downarrow n}$;

(v) $(E_{\downarrow m})^{\uparrow p} \supseteq E$.

(vi) *The operation* $D \mapsto D^{\uparrow p}$ *is monotonic in the sense that* $D \subseteq D' \implies D^{\uparrow p} \subseteq D'^{\uparrow p}$; *the map* $E \mapsto E_{\downarrow m}$ *is monotonic in the same sense.*

(vii) *The following are equivalent:*

$$\text{(a)}\quad E_{\downarrow m} \subseteq D; \qquad \text{(b)}\quad E \subseteq D^{\uparrow p}; \qquad \text{(c)}\quad E_{\downarrow n} \subseteq D^{\uparrow n}.$$

Proof : trivial. ⊣

The infinite-dimensional case

Our official definition is this: $\mathbb{K}^\omega$ is the complete Boolean algebra associated to the notion of forcing in *HOD* whose members are pairs (n, C^o) where $C^o \in \mathbb{K}^n$ and the intended meaning is that the first n terms of $\vec{\alpha}$ lie in C. To achieve that, the partial ordering $\leq^*$ is defined thus:

$$(n, C^o) \leq^* (m, D^o) \iff_{\text{df}} n \geq m \;\&\; C_{\downarrow m} \subseteq D$$

1·27 Remark Note the use of $C_{\downarrow m}$, and therefore of an existential quantifier ranging over subsets of $\mathcal{N}$.

1·28 Remark By Lemma 3·16, $(n, C^o) \leq^* (m, D^o) \iff_{\text{df}} n \geq m \;\&\; C \subseteq D^{\uparrow n}$. Moreover the Lemma easily implies that the partial ordering $\leq^*$ is not separative:

1·29 Lemma *If* $m \leq n$, *then*

(i) $(m, D) \geq^* (n, D^{\uparrow n})$;

(ii) *there is no condition* $(p, E) \leq^* (m, D)$ *which is incompatible with* $(n, D^{\uparrow n})$.

Proof : exercise. ⊣

Thus in the complete Boolean algebra $\mathbb{K}^\omega$, the conditions (m, D) and $(n, D^{\uparrow n})$ are identified for each $n \geq m$. In particular $\mathbb{K}^\omega$ may be construed as being, for any n, generated by conditions of dimension $\geq n$. This point will be used in proving the weak homogeneity of $\mathbb{K}^\omega$ below.

1·30 Lemma *Let* p *and* q *be two elements of* $\mathbb{K}^\omega$ *different from* $\mathbb{0}^{\mathbb{K}^\omega}$. *Then there is an automorphism* $\tilde{\pi}$ *of* $\mathbb{K}^\omega$ *such that* $\tilde{\pi}(q)$ *is compatible with* p.

Proof : Let $p = (n, C^o)$ and $q = (m, D^o)$. Let $k = m + n$. Let π be the permutation of ω defined by

$$\pi(p) = \begin{cases} p + n & \text{if } p < m \\ p - m & \text{if } m \leqslant p < k \\ p & \text{if } k \leqslant p \end{cases}$$

$\mathbb{K}^\omega$ is generated by $\{(\ell,\, E^o) \mid \ell \geq k\}$, and hence π induces an automorphism $\tilde{\pi}$ of $\mathbb{K}^\omega$ by setting for E of dimension ℓ

$$\pi^v(E) = \{\langle x_{\pi(i)} | i < \ell\rangle \mid x^\ell \in E\} \quad\text{and}\quad \tilde{\pi}(\ell,\, E^o) = (\ell,\, (\pi^v(E))^o).$$

Then $\pi^v(D^{\uparrow k}) \cap C^{\uparrow k}$ is non-empty, and so the conditions $\tilde{\pi}((k,\, (D^{\uparrow k})^o))$ and $(k,\, (C^{\uparrow k})^o)$ are compatible; but the latter is inseparable from $(n,\, C^o)$ and $(k,\, (D^{\uparrow k})^o)$ from $(m,\, D^o)$, and so $\tilde{\pi}$ is the desired automorphism. ⊣ (1·30)

1·31 Remark That is what Jech[R3] calls *weak homogeneity.*

1·32 The (0, 1) Law: *Let* $\dot{\Phi}$ *be any sentence of the forcing language for* $\mathbb{K} = \mathbb{K}^\omega$ *with constants* $\hat{x}$, *where* x *is in the ground model, a constant (or rather, 1-place predicate)* $\hat{V}$ *identifying members of the ground model, and* $\dot{J}$, *the* $\mathbb{K}$*-name for the set* $\{\alpha_i \mid i < \omega\}$. *Then the Boolean truth value* $[\![\dot{\Phi}]\!]^{\mathbb{K}}$ *is either* $\mathbb{0}^{\mathbb{K}}$ *or* $\mathbb{1}^{\mathbb{K}}$.

[R3] *Set Theory, page*

Proof : if not, let p and q be conditions with $p \Vdash^{\mathbb{K}} \dot\Phi$ and $q \Vdash^{\mathbb{K}} \neg\dot\Phi$. By the Lemma, there is an automorphism $\tilde\pi$ of $\mathbb{K}$ such that $\tilde\pi(q)$ is compatible with p. $\tilde\pi$ extends to an automorphism of the Boolean-valued model associated with the forcing $\mathbb{K}$. The name $\hat V$ for the ground model also preserves its meaning under $\tilde\pi$, as the functions $\hat x$ representing the sets in the ground model take only $\mathbb{0}^{\mathbb{K}}$ and $\mathbf{1}^{\mathbb{K}}$ as values, and $\hat V$ is defined in terms of them in an invariant way.

But as $q \Vdash^{\mathbb{K}} \neg\dot\Phi$, and $\hat x$ and $\dot J$ preserve their meaning under $\tilde\pi$, $\tilde\pi(q) \Vdash^{\mathbb{K}} \neg\dot\Phi$, and therefore cannot be compatible with p. $\dashv$ (1·32)

Chain conditions

1·33 PROPOSITION *In HOD, the algebras* $\mathbb{K}^k$, *for* $k \geqslant 1$, *have the* Θ*-chain condition.*

Proof : for $1 \leqslant k < \omega$, the algebra $\mathbb{K}^k$ will have the Θ chain condition because any sequence $\langle C^o_\nu \mid \nu < \eta\langle$ defining a maximal antichain corresponds in V to a partition of $(\mathcal{P}(\omega))^k$ into a sequence of pairwise disjoint non-empty subsets, yielding a map of $\mathbb{R}$ onto η, which must therefore be less than Θ. $\dashv$ (1·33)

1·34 PROPOSITION *If* $cf^{HOD}(\Theta) > \omega$, *the algebra* $\mathbb{K}^\omega$ *has the* Θ*-chain condition in HOD.*

Proof : conditions in the partial ordering $\mathbb{K}^\omega$ are pairs (n, C^o); if the proposition is false, some n will the first co-ordinate of Θ conditions, contradicting the previous Proposition for $k = n$. $\dashv$ (1·34)

1·35 PROPOSITION *If* $V = L(\mathbb{R})$, *every initial ordinal of HOD exceeding* Θ *is an initial ordinal in* V.

Proof : if not, then there are ordinals $\eta < \zeta$ with ζ coded by some subset A of η. A is definable from ordinals and some real a, and hence by Proposition 3·8, is in $HOD[G_a]$, and ζ will be singular in that model though regular in HOD. But G_a is a generic for an algebra which in HOD has the Θ chain condition, and therefore preserves all cardinalities and cofinalities from Θ onwards. $\dashv$ (1·35)

Proof that inside $L[\mathbb{R}]$, $\exists S :\subseteq \Theta$ $HOD = L[S]$

⊛ ⊛ Our aim now is to express HOD as constructible from a set of ordinals, by representing $L(\mathbb{R})$ as a co-generic extension of it. We shall prove the result twice, first in the form that suffices for the immediate application, and then we shall repeat the argument to obtain a more intricate case which we shall use in section 8 of Chapter III.

1·36 THEOREM *Let* Q *be a set of ordinals, and suppose that* $V = L(Q, \mathbb{R})$. *Then there is a set of ordinals* S *such that* $HOD_{\{Q\}} = L(Q, S)$. *If* $Q = \varnothing$, S *may be taken to be a subset of* Θ.

1·37 PROBLEM Is DC used in that proof ?

Let $\mathbb{R}$ be the set of reals. We shall define two notions of forcing, one in $L(Q, \mathbb{R})$ and one in HOD_Q, and show that the two Boolean extensions coincide.

1·38 First, in $L(\mathbb{R})$ let $\mathbb{E}$ be the forcing which adds an enumeration of $\mathbb{R}$. Conditions are finite sequences $f : n \xrightarrow{1-1} \mathbb{R}$, and the partial ordering is by extension: $p \leq q$ iff $q = p \restriction lh(q)$, where, as always, lower conditions are stronger. This forcing is straightforward: it adds a sequence $\vec\alpha = \langle \alpha_i | i < \omega\rangle$ where $\mathbb{R} =$ the $\mathbb{R}$ of the ground model $= \{\alpha_i | i < \omega\}$. Hence $L(\mathbb{R}) \subseteq L[\vec\alpha]$ and $L[\vec\alpha]$ is a model of AC. When discussing this forcing we shall let $\dot{\vec\alpha}$ be an $\mathbb{E}$-name for the generic sequence added, and $\dot I$ an $\mathbb{E}$-name for the image of that sequence, so that if H is $(L(\mathbb{R}), \mathbb{E})$ generic, the evaluation $\phi_H(\dot I)$ equals $\mathbb{R}$.

Now because in $L(\mathbb{R})$ each finite sequence of reals is generic over HOD with respect to a Vopěnka algebra, the conditions of $\mathbb{E}$ can in some sense be described from within HOD.[C 2] Hence we have a hope that $L[\vec\alpha]$ is a generic extension of HOD.

We aim now to build and examine the following diagram:

[C 2] "in some sense" is used to mean that the argument being immediately presented is heuristic in the sense of physicists: it is meant to supply the intuitive clue that has inspired the formal argument that will then follow. In no case (our authors hope) will the reader will be left in the lurch.

$$\begin{array}{ccc} L(\mathbb{R}) & \xrightarrow{\mathbb{E}} & L[\vec{\alpha}] \\ & & \uparrow \mathbb{K}^\omega \\ & & HOD^{L(\mathbb{R})} \end{array}$$

Here $\mathbb{E}$ is the forcing that adds an enumeration of $\mathbb{R}$, and $\mathbb{K}^\omega$ is the forcing formed from the forcings $\mathbb{K}^k(\omega)$ (for $k \in \omega$) with respect to which each k-sequence of subsets of ω reals is generic over HOD.

1·39 Theorem *Let $H = \langle \alpha_i \mid i < \omega \rangle$ be $(L(\mathbb{R}), \mathbb{E})$-generic. Let*

$$G = \big\{ (n, C^o) \bigm| C^o \in \mathbb{K}^n \ \&\ (\vec{\alpha} \restriction n \in C)^{L(\{\alpha_i \mid i<\omega\})} \big\}.$$

Then G is in $L[H]$, G is $(HOD, \mathbb{K}^\omega)$-generic and $HOD[G] = L(\mathbb{R})[H] = L[H]$.

Proof : We write M for $(HOD)^{L(\mathbb{R})}$. Let $\Delta^o \in M$ be dense in $\mathbb{K}^\omega$. Define

$$\Lambda = \{ q \in \mathbb{E} \mid \exists C\ (\ell h(q), C^o) \in \Delta^o \,\&\, q \in C \}$$

We shall show that Λ is dense in $\mathbb{E}$.

Given $p \in \mathbb{E}$, of length k say, let $Z = \{ (C_{\downarrow k})^o \mid \exists n\ n \geq k \ \&\ (n, C^o) \in \Delta^o \}$. Z is dense in $\mathbb{K}^k$, since given $E^o \in \mathbb{K}^k$, $(k, E^o) \in \mathbb{K}^\omega$, and so there is an $(m, D^o) \in \Delta^o$ with $(m, D^o) \leq^* (k, E^o)$, so $m \geq k$ and $D_{\downarrow k} \subseteq E$; then $(D_{\downarrow k})^o \in Z$. But p is $(M, \mathbb{K}^k)$-generic (as is every sequence of reals of length k) and so meets the dense set Z: thus there is an (n, C^o) in Δ^o with $n \geq k$ and a $q \in C$ such that $p = q \restriction k$. This q is in Λ.

Hence the generic sequence $\vec{\alpha}$ meets Λ. So there is an m with $\vec{\alpha} \restriction m \in \Lambda$. This tells us that there is a C with $(m, C^o) \in \Delta^o$ and $\vec{\alpha} \restriction m \in C$. But then this (m, C^o) is in G.

Δ^o being arbitrary, G meets all dense sets and so is $(M, \mathbb{K}^\omega)$-generic.

To see that $M[G] = L(\mathbb{R})[H] = L[\vec{\alpha}]$, note that G yields a generic object for each $\mathbb{K}^n$, namely the set $\{ C^o \mid (n, C^o) \in G \}$, which will evaluate to $\vec{\alpha} \restriction n$. Hence $\vec{\alpha} \in M[G]$. Note also that from $\vec{\alpha}$ we may form $L[\{\alpha_i \mid i < \omega\}]$ (which will be $L(\mathbb{R})$) and hence its HOD which will be M. But G, from its definition, is constructible from $\vec{\alpha}$, and so $M[G] \subseteq L[\vec{\alpha}]$. So $M[G] = L[\vec{\alpha}]$. ⊣ (1·39)

1·40 Remark The above argument turns on an apparently simple point, namely that the members of Z are indeed conditions: we need to know that if C (of dimension $n \geq k$) is OD so is $C_{\downarrow k}$. When, in Chapter III, we seek to repeat the argument using ∞-Borel sets, as defined in Chapter II, instead of OD sets, this step is no longer trivial, and (so far as is known) we must invoke the Axiom of Determinacy to show that the projection of an ∞-Borel subset of $\mathbb{R}^n$ to $\mathbb{R}^k$ is also ∞-Borel.

Recall that we denote by $\dot{J}$ the $\mathbb{K}^\omega$-name for the set of reals added and that for the avoidance of superscripts we sometimes denote $\mathbb{K}^\omega$ by $\mathbb{K}$.

1·41 Proposition $[\![\hat{V} \mathrel{\dot=} (H\dot{O}D)^{\dot{L}(\dot{J})}]\!]^{\mathbb{K}} = \mathbf{1}^{\mathbb{K}}$.

Proof : we apply the (0,1) Law; as we know that in the case when the generic for $\mathbb{K}$ is obtained from one for $\mathbb{E}$ the statement is true in the resulting model, the Boolean truth value cannot be $\mathbb{0}^{\mathbb{K}}$ and therefore must be $\mathbf{1}^{\mathbb{K}}$. ⊣ (1·41)

We are now able to prove that $\exists S \subseteq ON\ M = L[S]$. In M we may code the forcing relation $\Vdash^{\mathbb{K}}$ and a $\mathbb{K}$-name $\dot{\vec{\alpha}}$ for the sequence $\vec{\alpha}$ by a set S of ordinals.

S has to code $\Vdash^{\mathbb{K}}$ (for $\mathbb{K} = \mathbb{K}^\omega$, and the $\mathbb{K}^\omega$-name $\dot{\vec{\alpha}}$ for $\vec{\alpha}$. $\mathbb{K}^\omega$ is built from equivalence classes of OD names for sets of reals, and we have seen that we can find such names below Θ. So the set of such equivalence classes can be enumerated in order type Θ. The canonical name for $\vec{\alpha}$ will cause no additional difficulty, being essentially a countable object, so by the Lemma we may take S to be a subset of Θ; indeed the $<_{OD}$ first such subset.

We assert that for any $S \in M$ coding $\mathbb{K}^\omega$ and $\dot{\vec{\alpha}}$, $M = L[S]$. Since $L[S] \subseteq M$, we need only show that $M \subseteq L[S]$.

To see that, note that if G is $(M, \mathbb{K}^\omega)$ generic, $M \subseteq L[S][G]$, because from S and G we may evaluate the sequence $\bar{\beta}$; from which we may compute $L[R]$ and take its HOD, which by Proposition 3·24 will be the ground model M.

Let the pair $G_1 \times G_2$ be $(M, \mathbb{K}^\omega \times \mathbb{K}^\omega)$-generic; then it is also $(L[S], \mathbb{K}^\omega \times \mathbb{K}^\omega)$-generic, as $L[S] \subseteq M$; and both G_1 and G_2 will be $(M, \mathbb{K}^\omega)$-generic. Hence

$$M \subseteq L[S][G_1] \cap L[S][G_2] = L[S]$$

as required. $\dashv$ (1·41)

1·42 Remark Given such a pair $G_1 \times G_2$, G_1 and G_2 cannot both arise from some H_1 and H_2 each (separately) $(L(\mathbb{R}), \mathbb{E})$ generic, for then $M[G_i] = L[H_i]$; but, the H's enumerating the same set of reals in two different ways, $\{\alpha_i \mid i \in \omega\} = \mathbb{R} \in M[G_1] \cap M[G_2]$. $\dashv$ (1·42)

1·43 Remark Readers may prefer the following alternative argument that $M \subseteq L[S]$ which avoids appeal to facts about product forcing.

It is sufficient, since we are discussing models of AC, to show that every set of ordinals which is in M is in $L[S]$. Let A be such a set, say $A = \{\nu < \eta \mid (\varphi(\nu, \xi))^{L(\mathbb{R})}\}$. In $L[S]$, put $B = \{\nu < \eta \mid [\![(\varphi(\nu, \xi))^{\dot{L}(\dot{I})}]\!]^{\mathbb{K}^\omega} = \mathbf{1}\}$. Let G arise from a generic H as in Theorem 3·39. Then $\mathrm{val}_G(\dot{I}) = \mathbb{R}$ and hence $A = B$. $\dashv$ (1·43)

Finally we must show that in the case $Q = \varnothing$, we may take S to be a subset of Θ.

1·44 Lemma $(V = L(\mathbb{R}))$ *Every* OD *set of reals has an* OD *name in* J_Θ.

Proof : we apply the reflection properties of $J_\Theta(\mathbb{R})$ in $J(\mathbb{R})$ developed in the last two sections. However, a problem, discussed further in the next section, is that $V_{\omega+2}$ is not a member of $J_\Theta(\mathbb{R})$, and the relation $x = V_\xi$ is Π_1 but not Σ_1, so that a subterfuge is necessary.

Every set of reals is in $J_\Theta(\mathbb{R})$, and $J_\Theta(\mathbb{R}) \preceq_{\Sigma_1} J(\mathbb{R})$, so let X be an OD set of reals and $(\eta, \dot{\phi})$ an OD name for it such that

$$X = \{r \mid V_\eta \models \phi[r]\}$$

But now

$$\exists(\eta, \dot{\phi})\exists\xi J_\xi(\mathbb{R}) \models \dot{KP} \wedge V_\eta \,\dot{\epsilon}\, \dot{V} \wedge X \dot{=} \{r \mid V_\eta \models \dot{\phi}[r]\},$$

that is a Σ_1 statement, and so there are such η and ξ less than Θ. Let $\eta \cup \xi < \theta < \Theta$, with θ $\mathbb{R}$-admissible, say; then the sequence constructing $J_\xi(\mathbb{R})$ is in V_θ, so that

$$X = \{r \mid V_\theta \models J_\xi(\dot{\mathbb{R}}) \dot{\models} V_\eta \,\dot{\epsilon}\, \dot{V} \wedge V_\eta \models \dot{\phi}[r]\},$$

which defines X over V_η using ξ and η as parameters.

So X has an OD name in $J_\Theta(\mathbb{R})$, hence in J_Θ, OD names being constructible. $\dashv$ (1·44)

1·45 Corollary $(V = L(\mathbb{R}))$ *There is an* OD *listing of* $\{X \subseteq \mathcal{P}(\omega) \mid X \in OD\}$ *in order type* Θ.

That Corollary will generalise to OD subsets of $(\mathcal{P}(\omega))^k$ and hence a subset of Θ will code the forcing relation $\mathbb{K}^\omega$ and the term $\dot{\vec{a}}$, as required to complete the proof of Theorem 3·36.

1·46 Remark The Corollary may also be proved directly: every set of reals is in $J_\Theta(\mathbb{R})$; for $\nu < \Theta$, $(J_{\nu+1}(\mathbb{R}) \setminus J_\nu(\mathbb{R})) \cap OD$ has a definable well-ordering, of which the order-type must be less than Θ since there is a surjection of $\mathcal{N}$ onto $J_\nu(\mathbb{R}) \cap OD$. These combine to give a listing of $\{X \subseteq \mathcal{P}(\omega) \mid X \in OD\}$ in order-type at most Θ; but by Proposition 1·10 the order-type is at least Θ.

1·47 Remark Some hypothesis is necessary for the corollary, since by McAloon, it is consistent to suppose that $V = HOD$, $\mathfrak{c} = \aleph_1$, and $2^{\aleph_1} = \aleph_3$. In such a universe, there are more than Θ OD subsets of $\mathcal{P}\omega$.

The argument repeated

We run that argument again in the refined form in which we shall use it in Chapter III. We give it now since so many of the details will be fresh in the reader's mind.

The context there will be the following. Q is some fixed set of ordinals, and σ an arbitrary countable set of reals. We shall form the model $N_\sigma = L(Q, \sigma)$.

Now that model does not necessarily model $V = L(I\!R)$ nor even $V = L(I\!R, Q)$. Let $\mathbb{R}$ be the set of reals in N_σ: then the model might more easily be thought of as $L(\mathbb{R}, Q; \sigma)$. Note that σ might not be countable in $L(\mathbb{R}, Q)$, nor even a member of it.

Nevertheless we want to find, uniformly definable from σ, a submodel M_σ of N_σ over which every real may be considered as generic — the Vopěnka phenomenon — and which may be shown to be of the form $L(Q, T_\sigma)$ for some set T_σ of ordinals, T_σ again being definable uniformly from σ.

To meet both these requirements, we find that we must take the model M_σ to be $HOD_{\{Q,\sigma\}}^{L(Q,\sigma)}$.

Some comments on that definition. At a casual glance M_σ might be thought to equal N_σ but the difference is that elements of N_σ are definable there from Q, the set σ and the individual members of σ; whereas the members of M_σ are definable solely from Q and σ *without* using members of σ.

We shall work entirely inside $L(Q, \sigma)$ and therefore present our result in the following form, where one might take $I\!R$ to be the set of reals in $L(Q, \sigma)$.

1·48 THEOREM *Let Q be a set of ordinals, σ a set of reals, and suppose that $V = L(I\!R, Q; \sigma)$: so that σ is functioning as a predicate. Then there is a set of ordinals S such that $HOD_{\{Q,\sigma\}} = L(Q, S)$.*

Broadly our argument follows the same lines as previously. However, the presence of σ entails an alteration to our Vopěnka conditions. We must take them to be copies of $\{C \mid C \subseteq \mathcal{P}(\omega) \ \&\ C \in OD_{Q,\sigma}\}$.

Recall that we use these conditions to add one, or finitely many or eventually with $\mathbb{K}^\omega$ an infinite sequence of reals. We define the predicate $\dot\sigma$ in the forcing language in M_σ, the intended meaning being that the real in question will be a member of σ; so the name of the 1-dimensional set $\{x \mid x \in \sigma\}$ will be a condition forcing $\dot x \,\epsilon\, \dot\sigma$ ($\dot x$ being our canonical name for the real being added), and the name of $\{x \mid x \notin \sigma\}$ will force $\neg \dot x \,\epsilon\, \dot\sigma$. Both those sets of reals being definable from σ will have names in the Vopěnka algebra that concerns us.

Similarly for higher dimensional cases, there will be conditions which decide for each real in the sequence whether it is to be in σ or not.

Let $\mathbb{R}$ be the set of reals. We shall define two notions of forcing, one in $L(\mathbb{R}, Q; \sigma)$ and one in $HOD_{\{Q,\sigma\}}$, and show that the two Boolean extensions coincide.

1·49 First, in $L(\mathbb{R}, Q; \sigma)$ let $\mathbb{E}$ as before be the forcing which adds a sequence $\vec\alpha = \langle \alpha_i | i < \omega \rangle$ where $\mathbb{R} = \{\alpha_i | i < \omega\}$.

Hence $L(\mathbb{R}, Q; \sigma) \subseteq L(Q, \sigma)[\vec\alpha]$ and $L(\vec\alpha, Q; \sigma)$ is a model of AC. When discussing this forcing we shall let $\dot{\vec\alpha}$ be an $\mathbb{E}$-name for the generic sequence added, and $\dot I$ an $\mathbb{E}$-name for the image of that sequence, so that if H is $(L(\mathbb{R}, Q; \sigma), \mathbb{E})$ generic, the evaluation $\phi_H(\dot I)$ equals $\mathbb{R}$.

We aim now to build the following diagram:

$$\begin{array}{ccc} L(\mathbb{R}, Q; \sigma) & \xrightarrow{\mathbb{E}} & L(Q; \sigma)[\vec\alpha] \\ & & \uparrow \ \mathbb{K}^\omega(Q, \sigma) \\ & & HOD_{\{Q,\sigma\}}^{L(\mathbb{R},Q;\sigma)} \end{array}$$

Here $\mathbb{E}$ is the forcing that adds an enumeration of $\mathbb{R}$, and $\mathbb{K}^\omega(Q, \sigma)$ is the forcing formed from the forcings $\mathbb{K}^k(Q, \sigma)$ (for $k \in \omega$) with respect to which each k-sequence of subsets of ω reals is generic over $HOD_{\{Q,\sigma\}}^{L(\mathbb{R},Q;\sigma)}$.

We write M_σ for $HOD_{\{Q,\sigma\}}^{L(\mathbb{R},Q;\sigma)}$, $\mathbb{K}^k_\sigma$ for $(\mathbb{K}^k(Q, \sigma))^{L(\mathbb{R},Q;\sigma)}$. and $\mathbb{K}^\omega_\sigma$ for $(\mathbb{K}^\omega(Q, \sigma))^{L(\mathbb{R},Q;\sigma)}$.

1·50 THEOREM *Let $H = \langle \alpha_i \mid i < \omega \rangle$ be $(L[Q, \sigma, I\!R], \mathbb{E})$-generic. Let $G = \{(n, C^o) | C^o \in \mathbb{K}^n(Q, \sigma) \ \&\ \vec\alpha \restriction n \in C\}$. Then G is in $L(Q, H; \sigma)$, G is $(M_\sigma, \mathbb{K}^\omega_\sigma$-generic, and $M_\sigma[G] = L(Q, \mathbb{R}; \sigma)[H] = L(Q, H; \sigma)$.*

Proof : The previous argument goes through: let $\Delta^o \in M_\sigma$ be dense in $\mathbb{K}^\omega_\sigma$. Define

$$\Lambda = \{q \in \mathbb{E} \mid \exists C\ (\ell h(q), C^o) \in \Delta^o \,\&\, q \in C\}$$

We shall show that Λ is dense in $\mathbb{E}$.

Given $p \in \mathbb{E}$, of length k say, let $Z = \{(C_{\downarrow k})^o \mid \exists n\ n \geq k\ \&\ (n, C^o) \in \Delta^o\}$. Z is dense in $\mathbb{K}^k_\sigma$, since given $E^o \in \mathbb{K}^k_\sigma$, $(k, E^o) \in \mathbb{K}^\omega$, and so there is an $(m, D^o) \in \Delta^o$ with $(m, D^o) \leq^* (k, E^o)$, so $m \geq k$ and $D_{\downarrow k} \subseteq E$; then $(D_{\downarrow k})^o \in Z$. But p is $(M_\sigma, \mathbb{K}^k_\sigma)$-generic (as is every sequence of reals of length k) and so meets the dense set Z: thus there is an (n, C^o) in Δ^o with $n \geq k$ and a $q \in C$ such that $p = q \restriction k$. This q is in Λ.

Hence the generic sequence $\vec{\alpha}$ meets Λ. So there is an m with $\vec{\alpha} \restriction m \in \Lambda$. This tells us that there is a C with $(m, C^o) \in \Delta^o$ and $\vec{\alpha} \restriction m \in C$. But then this (m, C^o) is in G.

Δ^o being arbitrary, G meets all dense sets and so is $(M_\sigma, \mathbb{K}^\omega)$-generic.

Where the argument has novel features is here: to see that $M_\sigma[G] \supseteq L[\vec{\alpha}, Q; \sigma]$, note that G yields a generic object for each $\mathbb{K}^n_\sigma$, namely the set $\{C^o \mid (n, C^o) \in G\}$, which will evaluate to $\vec{\alpha} \restriction n$. Hence $\vec{\alpha} \in M_\sigma[G]$. Further, $\sigma \in M_\sigma[G]$, for given $\gamma \in \mathbb{R}$, we find the n with $\gamma = \alpha_n$; then we know that if γ is in σ, the name, S_n, say, of $\mathcal{N}^n \times \sigma$ will be in G, and otherwise the name T_n of $\mathcal{N}^n \times (\mathcal{N} \smallsetminus \sigma)$ will be in G; and the sequences $(S_n)_n$, $(T_n)_n$ are both in M_σ. Finally $Q \in M_\sigma$.

Conversely, to see that $M_\sigma[G] \subseteq L[\vec{\alpha}, Q; \sigma]$, note that from $\vec{\alpha}$ we may form $\{\alpha_i \mid i < \omega\}$, (which equals $\mathbb{R}$); hence in $L[\vec{\alpha}, Q; \sigma]$, we may form $L(Q, \mathbb{R}; \sigma)$, our starting model, and hence $(HOD_{Q,\sigma})^{L(Q,\mathbb{R};\sigma)}$, which is M_σ.

But G, from its definition, is constructible from $\vec{\alpha}$, and so $M_\sigma[G] \subseteq L[Q, H, \sigma]$. So $M_\sigma[G] = L[Q, \vec{\alpha}, \sigma]$. $\dashv$ (1·50)

In the following, $\mathbb{K}$ is now $\mathbb{K}^\omega_\sigma$, and $\dot{J}$ is a $\mathbb{K}$-name for the set of reals added.

1·51 Proposition $[\![\hat{V} = (HO\dot{D}_{\hat{Q},\dot{\sigma}})^{\dot{L}(\dot{J},\hat{Q};\dot{\sigma})}]\!]^{\mathbb{K}} = \mathbf{1}^{\mathbb{K}}$.

Proof : we apply the (0,1) Law; as we know that in the case when the generic for $\mathbb{K}$ is obtained from one for $\mathbb{E}$ the statement is true in the resulting model, the Boolean truth value cannot be $\mathbb{0}^{\mathbb{K}}$ and therefore must be $\mathbf{1}^{\mathbb{K}}$. $\dashv$ (1·51)

We may now show that $\exists S \subseteq ON\ M_\sigma = L[Q, S]$. In M_σ we may code three things — the forcing relation $\Vdash^{\mathbb{K}}$, a $\mathbb{K}$-name $\dot{\vec{\alpha}}$ for the sequence $\vec{\alpha}$, and a term $\dot{\sigma}$ as described above — by a set S of ordinals.

We assert that for any such $S \in M_\sigma$, $M_\sigma = L[Q, S]$. Since $L[Q, S] \subseteq M_\sigma$, we need only show that $M_\sigma \subseteq L[Q, S]$.

To see that, note that if G is $(M_\sigma, \mathbb{K}^\omega_\sigma)$ generic, $M_\sigma \subseteq L(Q, S)[G]$, because from S and G we may evaluate the sequence $\dot{\vec{\alpha}}$ as a sequence $\bar{\beta}$ of reals, and the term $\dot{\sigma}$ as a subset $\bar{\sigma}$ of those reals; using those and Q we may form $L(Q, \bar{\beta}_i; \bar{\sigma})$ and take its $HOD_{\{Q,\bar{\sigma}\}}$ which by Proposition 3·45 will be the ground model M_σ.

Let the pair $G_1 \times G_2$ be $(M_\sigma, \mathbb{K}^\omega_\sigma \times \mathbb{K}^\omega_\sigma)$-generic; then it is also $(L(Q, S), \mathbb{K}^\omega_\sigma \times \mathbb{K}^\omega_\sigma)$-generic, as $L(Q, S) \subseteq M_\sigma$; and both G_1 and G_2 will be $(M_\sigma, \mathbb{K}^\omega_\sigma)$-generic. Hence

$$M_\sigma \subseteq L(Q, S)[G_1] \cap L(Q, S)[G_2] = L(Q, S)$$

as required. $\dashv$ (1·51)

2: The ordinal Θ

2·0 DEFINITION A *norm* on a set X is a map $\phi : X \to ON$. The order type of the image of ϕ will be called its *length*, and written $|\phi|$. If the image of ϕ is an initial segment of ON, then the norm ϕ is termed *regular*.

Given a norm ϕ on X, consider the relation $\leq_\phi$ on X defined by the rule that

$$x \leq_\phi y \iff_{\text{df}} \phi(x) \leq \phi(y).$$

2·1 DEFINITION A pair $P = (X_P, \preceq_P)$, is a *pre-well-ordering* if for some norm ϕ on X_P, $\preceq_P = \leq_\phi$.

2·2 EXERCISE If ϕ is a norm on X with I its image, then the composition $\psi = \varpi_I \circ \phi$ is a regular norm on X, $|\psi| = |\phi|$, and $\leq_\phi = \leq_\psi$.

Now for a definition not mentioning ordinals:

2·3 DEFINITION A *pre-well-ordering* of a set is a pair $P = (X_P, \preceq_P)$, where $\preceq_P$ is a transitive, reflexive, connex and well-founded relation on the set X_P; so that these axioms hold for all x, y, z in X_P:

PWO 1: $x \preceq_P y \preceq_P z \implies x \preceq_P z$

PWO 2: $x \preceq_P x$

PWO 3: $x \preceq_P y$ or $y \preceq_P x$

PWO 4': $\varnothing \neq Y \subseteq X_P \implies \exists y{:}{\in} Y\ \forall y'{:}{\in} Y\ y \preceq_P y'$.

We may remove the quantification over subsets of X_P by writing $x \prec_P y \iff_{\text{df}} x \preceq_P y \,\&\, y \not\preceq_P x$, and invoking the axiom of dependent choice and then reformulating the last condition as:

PWO 4: $\neg\exists\langle x_i | i < \omega\rangle\ \forall i{:}{<}\omega\ x_{i+1} \prec_P x_i$.

For relations on the reals, of course, *DCR* suffices, and in this case, since any such sequence may be coded as a single sequence of reals, PWO 4 involves only quantification over reals and integers.

It is clear that any relation arising as in Definition 1·1 satisfies the axioms PWO 1–4. To see that any such $\preceq_P$ is conversely determined by some regular norm, factor X_P by the equivalence relation $x =_\phi y \iff_{\text{df}} x \leq_\phi y \,\&\, y \leq_\phi x$, then the first three properties show that the relation $\preceq_P$ induces a linear ordering on the set of equivalence classes, which by the fourth property is a well-ordering: the unique mapping of that well-ordering onto an ordinal composed with the factoring gives a regular norm ϕ_P with $\leq_{\phi_P} = \preceq_P$.

The associated regular norm, which we shall denote by ϱ_P, though the notation $|\cdot|_P$ is also in use, may also be defined directly by recursion:

$$\varrho_P(x) = \sup\{\varrho_P(y) + 1 \mid y \prec_P x\}.$$

Strong KP, by which is meant the Kripke–Platek axioms plus the Σ_1 separation scheme, suffices to prove that the function $x \mapsto \varrho_P(x)$ restricted to X_P is a regular norm on X_P.

Some further notation for pre-well-orderings:

2·4 DEFINITION Let $P = (X_P, \preceq_P)$ be a pre-well-ordering. We write $x =_P y$ for $x \preceq_P y \,\&\, y \preceq_P x$, and $|P|$ for the length of P, that is, the image of ϱ_P. For $\xi < |P|$, we shall call the set $\{x \in X_P \mid \varrho_P(x) = \xi\}$ the ξ^{th} *component* of P, and denote it by $(P)_\xi$.

The following argument, which goes back to Hartogs and Tarski shows using the axiom of replacement that the class of ordinals onto which the continuum can be mapped is a set, and thereby render the following non-vacuous.

2·5 DEFINITION We write Θ for the term "the least ordinal > 0 onto which $\mathbb{R}$ cannot be mapped".

2·6 REMARK $\Theta = \bigcup\{\delta \mid \exists f : \mathbb{R} \overset{\text{onto}}{\longrightarrow} \delta\}$ = the supremum of lengths of pre-well-orderings of $\mathbb{R}$.

2·7 PROPOSITION *Θ is an initial ordinal $> \aleph_1$.*

Proof : That Θ is not of the same cardinal as any smaller ordinal follows from composition of mappings. To map $\mathbb{R}$ onto ω_1, send each real coding [C 3] a countable ordinal to that ordinal, and other reals to 0. ⊣ (2·7)

[C 3] The simple coding required for this proof will be found in the discussion of Π^1_1 sets in Chapter II.

2·8 PROPOSITION ($AC_{\aleph_0,\mathcal{N}}$) $\operatorname{cof}(\Theta) > \omega$.

Proof : Here $AC_{\aleph_0,\mathcal{N}}$ is the Axiom of Choice for countable families of sets of reals. Given $f : \omega \longrightarrow \Theta$, pick for each n a pre-well-ordering of the reals coding $1 + f(n)$, and combine your choices with an a surjection of $\mathbb{R}$ onto $\omega \times \mathbb{R}$ to give a map of $\mathbb{R}$ onto the supremum of the image of f, which must therefore be less than Θ. ⊣ (2·8)

The ordinal Θ in $L(\mathbb{R})$

We shall principally be interested in the meaning of Θ assuming that $V = L(\mathbb{R})$. Without this assumption even the regularity of Θ becomes a strong assumption in the presence of AD. [R4]

2·9 LEMMA *(i) For each ordinal η, every element of $J_\eta(\mathbb{R})$ is definable in $J_\eta(\mathbb{R})$ from reals and ordinals less than η; (ii) if $V = L(\mathbb{R})$, every set is* OD *from a real; (iii) if $\nu < \Theta$ there is a map of $\mathbb{R}$ onto $J_\nu(\mathbb{R})$.*

Proof : Part (i) is covered by the discussion of Appendix Two. Part (ii) follows easily. For part (iii), we know that there is a (definable) mapping of $\mathcal{N}$ onto the set of all finite sequences of members of $\mathcal{N}$, and thus that if there is a map of $\mathbb{R}$ onto ν there is one onto ${}^{<\omega}\nu$: now compose these mappings in the light of Part (i). ⊣ (2·9)

2·10 PROPOSITION *There is an* OD *function assigning to each $\nu < \Theta^{L(\mathbb{R})}$ a pre-well-ordering of length ν.*

Proof : Every such λ is the length of some pre-wellordering in $L(\mathbb{R})$ of $\mathcal{N}$. Every member of $L(\mathbb{R})$ is ordinal definable from some real. For each real x let P_x be the first (in a natural well-ordering of ordinal parameters) pre-well-ordering of $\mathcal{N}$ of length λ definable from ordinals and the parameter x, if such exists; otherwise let P_x be the trivial pre-well-ordering of $\mathcal{N}$ of length 1. Then the map $(a, x) \mapsto \varrho_{P_x}(a)$ from $\mathcal{N} \times \mathcal{N}$ to λ is onto λ and when composed with a definable bijection between $\mathcal{N} \times \mathcal{N}$ and $\mathcal{N}$, yields an OD prewellordering of $\mathcal{N}$ of length λ. ⊣ (2·10)

2·11 PROPOSITION *If $V = L(\mathbb{R})$ then Θ is regular.*

Proof : let $\langle \theta_\nu \mid \nu < \lambda \rangle$ be a sequence, of length $\lambda < \Theta$, of non-zero ordinals less than Θ. Let $f : \mathcal{N} \xrightarrow{\text{onto}} \lambda$. By Proposition 1·10 there is a sequence $\langle \pi_\nu \mid \nu \rangle$ such that for each ν, $\pi_\nu : \mathcal{N} \xrightarrow{\text{onto}} \theta_\nu$. Define a surjection $\chi : \mathcal{N} \times \mathcal{N} \xrightarrow{\text{onto}} \sup_\nu \theta_\nu$ by

$$\chi(a, b) = \pi_{f(a)}(b).$$

That supremum is therefore less than Θ. ⊣ (2·11)

Before our next lemma, we shall require the concept from model theory of the Σ_n-*hull* of a subset $Z \cap M$ of a (usually transitive) set M:

2·12 DEFINITION $\mathrm{Hull}_n(M, Z) \;=_{\mathrm{df}}\; \{x \mid$ for some Σ_n formula φ with parameters $\vec{z}$ from $Z \cap M$, $\{x\} = \{y \in M \mid M \models \varphi[y, \vec{z}]\}\}$.

We shall chiefly be concerned with the case $n = 1$, and when $Z = \mathbb{R}$, but we shall have a few uses for other cases.

Because we are working in a world where AC fails, we must take care with the concept of a Skolem function. When M is of the form $J_\xi(\mathbb{R})$, then everything in it is definable over it from reals and ordinals less than ξ, and all the reals are in each $J_\xi(\mathbb{R})$, so our method will be to show that for each real that works with some ordinal in a definition, the least ordinal that works for that real is in the hull under discussion. That method leads to a proof of the following

2·13 LEMMA *$\mathit{Hull}_n(M, Z) \preceq_{\Sigma_n} M$.*

We should explain our notation: $A \preceq_{\Sigma_n} B$ means that for every Σ_n formula ϑ with all parameters in the smaller model A, $A \models \vartheta \iff B \models \vartheta$. Where, as later in this chapter, we have added to the language a one-place predicate $\dot{H}$ we shall add a superscript to the Σ, for example $J_\tau(\mathbb{R}; H) \preceq_{\Sigma_1^{\dot{H}}} J_\Theta(\mathbb{R}; H)$. $\preceq$ denotes the full elementary submodel relation. The superscript R added to $\preceq$: $A \preceq^{\mathbf{R}}_{\Sigma_n} B$ will mean that the preservation of truth for Σ_n formulæ is only asserted for statements about real numbers: thus the only parameters permitted in ϑ are names of reals, and the ever-present constant $\dot{\mathbb{R}}$.

[R4] Solovay, Cabal Volume I, paper on DC.

We shall give a detailed proof in the special case $n = 1$ in section 2. Similar arguments cover the following amalgamated case:

2·14 DEFINITION $\text{Hull}(M,Z) =_{\text{df}} \bigcup_{n<\omega} \text{Hull}_n(M,Z)$.

2·15 LEMMA *Hull*$(M,Z) \preceq M$.

2·16 LEMMA $J_\Theta(I\!\!R) = \{x \mid \exists f \; f : I\!\!R \xrightarrow{\text{onto}} TC(\{x\})\}$

Here TC denotes the *transitive closure* of a set: a definition is given in Appendix One.

Proof : suppose first that $x \in J_\Theta(I\!\!R)$: then for some $\nu < \Theta$, $x \in J_\nu(I\!\!R)$. So $TC(\{x\}) \subseteq J_\nu(I\!\!R)$, and hence is a surjective image of $I\!\!R$ by Part (iii) of Lemma 1·9.

Suppose secondly that there is a surjection of $I\!\!R$ onto $A = TC(\{x\})$. Since $V = L(I\!\!R)$, there is some λ with $x \in J_\lambda(I\!\!R)$: so $A \subseteq J_\lambda(I\!\!R)$. Form $N = \text{Hull}(J_\lambda(I\!\!R), A \cup I\!\!R^+)$. By condensation, [N 2] there is an ordinal ξ such that under the collapsing map ϖ_N, N is isomorphic to $J_\xi(I\!\!R)$. But then $\varpi_N(x) = x$, as A cannot be collapsed, being transitive; so $x \in J_\xi(I\!\!R)$. All members of N are definable in $J_\lambda(I\!\!R)$ from reals and members of A; hence N is a surjective image of $I\!\!R$ because A is; and so therefore is ξ, which is thus less than Θ. Thus $x \in J_\Theta(I\!\!R)$ as required. $\dashv$ (2·16)

2·17 COROLLARY *(i) Every subset of a member of* $J_\Theta(I\!\!R)$ *is in* $J_\Theta(I\!\!R)$;

(ii) every set of reals is in $J_\Theta(I\!\!R)$;

(iii) $J_\Theta(I\!\!R)$ *models all axioms of* $ZF - \mathcal{P}$.

Here $ZF - \mathcal{P}$ is the system ZF with the power set axiom omitted.

Proof : (i), and therefore (ii), are immediate from the Lemma, and in turn (i) implies that all instances of the Separation scheme are true in $J_\Theta(I\!\!R)$. To see that all instances of the Collection scheme are true, suppose that in $J_\Theta(I\!\!R)$, $\forall x{:}{\in} u \; \exists y \; \Phi(x,y)$ is true, where $u \in J_\Theta(I\!\!R)$. Define $f : u \to \Theta$ by setting $f(x)$ to be the least ξ such that $\exists y{:}{\in} J_\xi(I\!\!R) \; \Phi(x,y)$. The order type of the image of f is less than Θ (since we know by Lemma 1·9 that, unless u is empty, when we have nothing to prove, there is a surjection of $\mathcal{N}$ onto u) and so the image is bounded in Θ, since we have seen that if $V = L(I\!\!R)$, Θ is regular. Let η be such a bound. Then $\forall u{:}{\in} x \; \exists y{:}{\in} J_\eta(I\!\!R) \; \Phi(x,y)$, as required. $\dashv$ (2·17)

2·18 REMARK We shall see shortly that $J_\Theta(I\!\!R)$ is **not** a model of the Power Set Axiom.

2·19 PROPOSITION *If* $V = L(I\!\!R)$, Θ *is a limit of smaller ordinals* ζ *with* $J_\zeta(I\!\!R) \models \mathtt{ZF} - \mathtt{P}$.

Proof : Θ is regular, and by our previous remarks there can be no (even partial) map of a member of $J_\Theta(I\!\!R)$ cofinal in Θ. For each $\xi < \Theta$ consider $N =_{\text{df}} \text{Hull}(J_\Theta(I\!\!R), \xi + 1 \cup I\!\!R^+)$. By condensation, N collapses to $J_\eta(I\!\!R)$, say; every element of N is definable from reals and ordinals $< \xi + 1 < \Theta$, so there is a map of $I\!\!R$ onto N, and so $\xi < \eta < \Theta$. $J_\eta(I\!\!R) \models \mathtt{ZF} - \mathtt{P}$. $\dashv$ (2·19)

2·20 PROPOSITION *If* $V = L(I\!\!R)$, Θ *is the least* ν *with* $\mathcal{P}(I\!\!R) \subseteq J_\nu(I\!\!R)$.

Proof : We have already seen that $\mathcal{P}(I\!\!R) \subseteq J_\Theta(I\!\!R)$. Let $\zeta < \Theta$ be strongly $I\!\!R$-admissible, let $f : I\!\!R \xrightarrow{\text{onto}} \zeta + 1$, and consider $\{\langle x,\, y\rangle \mid f(x) < f(y)\}$. This subset of $I\!\!R \times I\!\!R$ cannot be in $J_\zeta(I\!\!R)$ since it is a well-ordering of order type greater than ζ, and a (strongly) admissible set reconstructs the order-type of an element which is a well-ordering. Hence some subset of $I\!\!R$ is also missing from J_ζ.[N 3] $\dashv$ (2·20)

2·21 COROLLARY $\mathcal{P}(I\!\!R) \notin J_\Theta(I\!\!R)$.

2·22 PROPOSITION (DC) *Suppose that*

$$J_\zeta(I\!\!R) \models \dot\Theta = \dot{ON} \wedge \texttt{every pre-well-ordering of a set of reals has length an ordinal.}$$

Suppose that $I\!\!R \subseteq M$, $I\!\!R \in M$ *and* $M \preceq_{\Sigma_1} J_\zeta(I\!\!R)$. *Then* M *is transitive and is of the form* $J_\xi(I\!\!R)$ *for some* $\xi \le \zeta$.

Proof : the hypothesis means, first that each ordinal less than $\omega\zeta$ is the length of a pre-well-ordering of the reals, and secondly that every such in $J_\zeta(I\!\!R)$ has a rank function on it in $J_\zeta(I\!\!R)$.

[N 2] to be covered in Appendix Two

[N 3] J_2 contains a bijection between $I\!\!R$ and $I\!\!R \times I\!\!R$. — obvious if $I\!\!R$ is $\mathcal{N}$.

Let $\lambda \in M \cap ON$. Then $\lambda < \zeta$ and so there is a pre-well-ordering of $I\!\!R$ of length λ. But saying that X is a pre-wellordering of $I\!\!R$ is a Δ_0 statement about $I\!\!R$ and X: most of the clauses are trivially Δ_0: and by DC, to say that X is a pre-well-ordering it is enough to say that there is no sequence of reals strictly descending in the ordering; and since such a sequence may be coded by a single real, that is tantamount to saying that no $a \in I\!\!R$ detects a descending sequence. Hence

$$J_\zeta(I\!\!R) \models \bigvee X\ X \texttt{ is a pre-wellordering of } I\!\!R \texttt{ of length } \lambda$$

and so

$$M \models \bigvee X\ X \texttt{ is a pre-wellordering of } I\!\!R \texttt{ of length } \lambda$$

Let therefore $X \in M$ be what M believes to be a pre-well-ordering of length λ: but then X genuinely is a pre-well-ordering, with a map, f, say, in M, of the field of X, $I\!\!R$, onto λ. Now for each ordinal $\zeta < \lambda$, there is a real a with $f(a) = \zeta$: each a is in M and so ζ is in M. Hence every ordinal less than λ is in M, and so $M \cap ON$ is an ordinal, ξ, say.

Now we assert that $M = J_\xi(I\!\!R)$. For $J_\zeta(I\!\!R) \models \bigwedge x \bigvee \zeta\ \zeta \in ON \wedge x \in J_\zeta(I\!\!R)$; that is a Π_2 sentence, and so is true in M. But it is easily checked that for $\eta \in M$, $(J_\eta(I\!\!R))^M = J_\eta(I\!\!R)$. ⊣ (2·22)

2·23 Remark We should perhaps draw attention to the point used in the above proof, that our models $J_\xi(I\!\!R)$, since they contain all reals and model DC, are correct in deciding whether a pre-linear-ordering belonging to them is a pre-well-ordering, whatever ξ may be.

2·24 Proposition $J_\Theta(I\!\!R) \preceq_{\Sigma_1} J(I\!\!R)$.

Proof: let $a \in J_\Theta(I\!\!R)$, say $a \in J_\zeta(I\!\!R)$, where $\zeta < \Theta$, and suppose that in $L(I\!\!R)$ there is a b such that $\models^0 \dot\Phi[a,b]$, where $\dot\Phi$ is Δ_0. Let ξ be minimal such that $J_{\xi+1}(I\!\!R) \models \bigvee b\, \dot\Phi[a,b]$. Form $H =_{\rm df} \mathrm{Hull}_1(J_{\xi+1}(I\!\!R), I\!\!R^+ \cup J_\zeta(I\!\!R))$, and let π collapse it to $J_\gamma(I\!\!R)$, say. Then there is a map of $I\!\!R$ onto $J_\gamma(I\!\!R)$, so $\gamma < \Theta$; $a = \pi(a) \in J_\gamma(I\!\!R)$; $J_\gamma(I\!\!R) \models \bigvee b\, \dot\Phi[a,b]$, since H does, being a Σ_1-elementary submodel[C 4] of $J_{\xi+1}(I\!\!R)$, and so $J_\Theta(I\!\!R) \models \bigvee b\, \dot\Phi[a,b]$ as required. ⊣

2·25 Remark We shall see shortly that Θ is not the least such ordinal.

[C 4] we could take the full hull of course; taking the Σ_1 hull is "less expensive", but suggests, falsely in the present case, that there is a reason not to take the full hull.

3: Stable ordinals

We continue to assume that $V = L[\mathbb{R}]$. Let $H \subseteq \Theta$: we now reconstruct the universe allowing an extra unary predicate $\dot{H}$ interpreted as denoting membership of H. We write $J_\nu(\mathbb{R}; H)$ for the levels of this new hierarchy. For the J-minded, the reconstruction means adding to the generators of the class of rudimentary functions the function $x \mapsto H \cap x$: then each $(J_\nu(\mathbb{R}; H), H \cap \omega\nu)$ will model $\Delta_0^{\dot{H}}$ separation. For the L-minded, the change in the definition is that $L^H_{\nu+1}(\mathbb{R})$ will be $\mathrm{Def}^{\dot{H}}(L^H_\nu(\mathbb{R}), H \cap \nu)$, those subsets of the previous stage definable in the enlarged language with the one-place predicate $\dot{H}$, interpreted as $H \cap \nu$.

3·0 Definition We shall say that ζ *is an $\mathbb{R}$-H ordinal* if $J_\zeta(\mathbb{R}; H)$ models all axioms of ZF except possibly the Power Set Axiom but including all instances of the separation and collection schemes in the language augmented by a predicate for H.

We call H *M-amenable*, or simply say that (M, H) *is amenable* when for every $\nu \in M$, $H \cap \nu \in M$.

3·1 Exercise $(J_\Theta(\mathbb{R}), H)$ is amenable.

3·2 Exercise Let ν be an $\mathbb{R}$-admissible ordinal of cofinality ω, and let H be a subset of ν of order type ω with $\bigcup H = \nu$. Show that while $(J_\nu(\mathbb{R}), H \cap \nu)$ is amenable, it is not H-admissible.

While we cannot expect $J_\nu(\mathbb{R}; H) = J_\nu(\mathbb{R})$ for every ν, we do have this equality in one important case:

3·3 Proposition $J_\Theta(\mathbb{R}; H) = J_\Theta(\mathbb{R})$

Proof : to see that $J_\Theta(\mathbb{R}) \subseteq J_\Theta(\mathbb{R}; H)$, note that the sequence $\langle J_\nu(\mathbb{R}) \mid \nu < \gamma\rangle$ is uniformly Σ_1 over each $J_\gamma(\mathbb{R}; H)$, as shown in Appendix 2. In the other direction, there will be for $\nu < \Theta$ a map of $\mathcal{N}$ onto $J_\nu(\mathbb{R}; H)$, (by the same reasoning that showed there was a map of $\mathcal{N}$ onto $J_\nu(\mathbb{R})$); hence, by Lemma 1·16, $J_\Theta(\mathbb{R}; H) \subseteq J_\Theta(\mathbb{R})$. ⊣ (3·3)

3·4 Remark If, say, $H = \Theta$ then $J_\Theta(\mathbb{R}; H)$ is **not** a $\Sigma_1^{\dot{H}}$-elementary submodel of $L[\mathbb{R}]$.

3·5 Exercise Show, however, that for every $H \subseteq \Theta$, $J_\Theta(\mathbb{R}; H)$ is an elementary submodel of $L[\mathbb{R}]$ with respect to all Σ_1^H formulæ in which H has only positive occurrences.

3·6 Proposition *(i) Θ is an $\mathbb{R}$-H ordinal;*

(ii) $\forall \nu \,{:<}\, \Theta\ \exists \xi \,{:\in}\, (\nu, \Theta)\ \ \xi$ is an $\mathbb{R}$-H ordinal.

Proof : the argument for Part (i) follows closely that for 1·17, and is left to the reader. For Part (ii), we must avoid our previous condensation argument. Given $\nu_0 < \Theta$, we define ω-sequences $\langle M_n \mid n < \omega\rangle$ and $\langle \nu_n \mid n < \omega\rangle$ of $\preceq_{\Sigma^{\dot{H}}}$-submodels of $J_\Theta(\mathbb{R}; H)$ and of ordinals less than Θ: given $\nu_k < \Theta$, set $M_k = \mathrm{Hull}(J_\Theta(\mathbb{R}; H), \nu_k + 1 \cup \mathbb{R} \cup \{\mathbb{R}\})$; then there is a map of $\mathbb{R}$ onto M_k, and so $\nu_{k+1} =_{\mathrm{df}} \sup(ON \cap M_0)$ will be less than Θ, which we know to be regular.

Each $M_k \preceq_{\Sigma^{\dot{H}}} J_\Theta(\mathbb{R}; H)$, and $\forall k \,{:<}\, \omega\ M_k \subseteq M_{k+1}$, so by easy model theory, $\forall k \,{:<}\, \omega\ M_k \preceq_{\Sigma^{\dot{H}}} M_{k+1}$, and $M_\omega =_{\mathrm{df}} \bigcup_{k<\omega} M_k$ will be a $\preceq_{\Sigma^{\dot{H}}}$-elementary submodel of $J_\Theta(\mathbb{R}; H)$ with $M_\omega \cap ON = \bigcup_{k<\omega} \nu_k = \nu_\omega$, say. M_ω will think that $V = L(\mathbb{R})$, and hence, as in the proof of 1·23 we shall have $M_\omega = J_{\nu_\omega}(\mathbb{R}; H)$; further it will be an $\mathbb{R}$-H-ordinal. ⊣ (3·6)

There is a certain important ordinal less than Θ associated with H. We shall give five definitions in all of this ordinal, three now and two in the next chapter, and when we have established their equivalence, we shall call the ordinal δ_H. It will transpire that for each ν in a closed unbounded subset of δ_H, $J_\nu(\mathbb{R}; H) = J_\nu(\mathbb{R})$, so that one may consider that the perturbation in the order of construction introduced by the addition of the predicate for H has settled down well before stage δ_H.

We work towards our first three definitions of δ_H.

H-safe ordinals

3·7 Definition Call an ordinal ξ *H-safe* if the following three conditions hold:

HS 1 $\qquad J_\xi(\mathbb{R}; H) = J_\xi(\mathbb{R})$

HS 2 $\qquad J_\xi(\mathbb{R}; H) \models \dot{\Theta} = \dot{ON}$

HS 3 $\qquad \xi$ is a limit of $\mathbb{R}$-H ordinals.

Note that these are all Δ_1^H conditions on ξ: for the first clause may be reformulated as $J_\xi(\mathbb{R};H) \models \dot V = J(\mathbb{R})$, and $\models$ is a Δ_1 relation; and the last clause also may be recast in terms of the relation $J_\xi(\mathbb{R};H) \models$. Property HS 2 says that every ordinal is the length of some prewellordering of the reals.

3·8 LEMMA *(i) Θ is the largest H-safe ordinal;*

(ii) the class C_H of H-safe ordinals is closed, and unbounded below Θ.

Proof : (i) follows from properties already established for Θ: e.g., HS1 is a consequence of Θ being admissible. The others are left to the reader to verify. (ii) holds as each HS property defines a closed class of ordinals, which various properties already established of Θ show to be unbounded below Θ. $\dashv$ (3·8)

3·9 REMARK It would have been simpler to list admissibility among the HS properties, but the class of admissible ordinals is not closed: the limit for example of the first ω admissibles not being admissible. Hence we have formulated the third condition as we have.

3·10 EXERCISE Show that if ζ is H-safe, then $\zeta \le \Theta$, $\omega^\zeta = \zeta$,

$$J_\zeta(\mathbb{R};H) \models \texttt{every well-ordering is isomorphic to an ordinal}$$

and

$$J_\zeta(\mathbb{R};H) \models \texttt{every pre-well-ordering has length an ordinal.}$$

3·11 EXERCISE Show that there is a single Π_2 sentence $\Phi_{\rm safe}$ of the language $\mathcal{L}(\dot H, \dot{\mathbb{R}})$ such that for any ζ,

$$\zeta \text{ is } H\text{-safe} \iff J_\zeta(\mathbb{R};H) \models \Phi_{\rm safe}.$$

We establish some properties of the $\Sigma_1^{\dot H}$ hull of the reals in certain models.

3·12 DEFINITION $\underset{\sim}{\rm Hull}{}_1^H(M) =_{\rm df} \big\{x \in M \big| \{x\} = \{y \in M \mid (M, H\cap M) \models \bigvee\mathfrak{w}\vartheta(\mathfrak{w}, x, a)\}$ for some $\Delta_0^{\dot H}$ formula ϑ and some real $a \in M\big\}$.

3·13 LEMMA *Suppose $M = J_\xi(\mathbb{R};H)$, where $\xi = \omega^\xi$. Then*

$$\underset{\sim}{\rm Hull}{}_1^H(M) \preceq_{\Sigma_1^{\dot H}} M.$$

Proof : let $N = \underset{\sim}{\rm Hull}{}_1^H(M)$. We shall show that if $x \in N$ and $\exists y{:}{\in} M\ M \models \varphi(x,y)$, where φ is a $\Delta_0^{\dot H}$ formula then $\exists y{:}{\in} N\ M \models \varphi(x,y)$. That will imply the same thing for $\Sigma_1^{\dot H}$ φ, since we may use the pairing functions of M to coalesce the further existential quantifier with $\exists y$; and then the result will follow by Tarski's criterion.

Now we know that each y is of the form $\hbar(i, \vec a, \vec\zeta)$ where $\hbar : \omega \times {}^{<\omega}\mathbb{R} \times {}^{<\omega}ON$ is a Σ_1 total function, and each ζ is less than ξ. We may with ease treat $\vec a$ as a single real a (regarding i as including the information that a is to be read as a k-tuple), and if $\xi^\omega = \xi$ then we may likewise easily treat $\vec\zeta$ as a single ordinal $\zeta < \xi$.

We suppose therefore that there is a witness y of the form $h(i,a,\zeta)$ where $i \in \omega$, $a \in \mathbb{R}$ and $\zeta < \xi$. Fix such i, a, and let $\bar\zeta$ be the least such ζ for those i, a, h. Consider

$$\{\zeta \mid M \models \varphi(x, h(i,a,\zeta)) \wedge \forall\eta{:}{<}\zeta\ \neg\varphi(x, h(i,a,\eta))\}$$

That is of course equal to $\{\bar\zeta\}$: but we must remove the reference to x. Since x is in N, it is (over M) the unique solution of a $\Sigma_1^{\dot H}$ predicate in a real parameter: so there is a $\Delta_0^{\dot H}$ formula ϑ and a real b with

$$x = \iota z\ M \models \bigvee w\, \vartheta(w,z,b)$$
$$\text{and}\quad \{\bar\zeta\} = \iota\zeta\, M \models \zeta \,\epsilon\, ON \wedge \bigvee z\big[\bigvee w\, \vartheta(w,z,b) \wedge \varphi(z, h(i,a,\zeta)) \wedge \bigwedge\xi{:}{<}\zeta\ \neg\varphi(z, h(i,a,\xi))\big].$$

Thus $\bar\zeta$ is in N. Similarly $h(i,a,\bar\zeta)$ will be in N, as required, being the unique $y \in M$ such that

$$M \models \bigvee\zeta\,\Big[\zeta \,\epsilon\, ON \wedge \bigvee z\big[\bigvee w\, \vartheta(w,z,b) \wedge \varphi(z, h(i,a,\zeta)) \wedge \bigwedge\xi{:}{<}\zeta\ \neg\varphi(z, h(i,a,\xi))\big] \wedge y = h(i,a,\zeta)\Big],$$

where of course we replace $\hbar$ by its Δ_1 definition. $\dashv$ (3·13)

3·14 Proposition *(i) If ζ is H-safe and $J_\xi(\mathbb{R};H) \preceq_{\Sigma_1^{\dot H}} J_\zeta(\mathbb{R};H)$ or $J_\xi(\mathbb{R}) \preceq_{\Sigma_1^{\dot H}} J_\zeta(\mathbb{R})$ then ξ is H-safe.*

(ii) If ζ is H-safe then $\underset{\sim}{\mathrm{Hull}}(J_\zeta(\mathbb{R};H)) \preceq_{\Sigma_1^{\dot H}} J_\zeta(\mathbb{R};H)$

(iii) If ζ is H-safe and $N \preceq_{\Sigma_1^{\dot H}} J_\zeta(\mathbb{R};H)$ then N is transitive and of the form $J_\eta(\mathbb{R};H)$ where $\eta \le \zeta$ and η is H-safe.

Proof : by combining earlier results. $\dashv$

We call an ordinal χ *H-characterisable* when there is a $\dot H$-formula $\varphi(\mathfrak{v})$ in exactly one free variable and a real a such that (i) $J_\chi(\mathbb{R};H) \models \varphi[a] + \Phi_{\text{safe}}$ and (ii) χ is the least ordinal for which (i) holds.

3·15 Definition $\sigma_H =_{\text{df}} \sup\{\chi < \Theta \mid \chi \text{ is } H\text{–characterisable}\}$.

3·16 Definition $\tau_H =_{\text{df}} \inf\{\tau \mid J_\tau(\mathbb{R};H) \preceq_{\Sigma_1^{\dot H}} J_\Theta(\mathbb{R};H)\}$

3·17 Definition $\rho_H =_{\text{df}} \inf\{\rho \mid J_\rho(\mathbb{R};H) \preceq^{\mathbb{R}}_{\Sigma_1^{\dot H}} J_\Theta(\mathbb{R};H)\}$

3·18 Proposition $\rho_H = \sigma_H = \tau_H$

We divide the proof into steps.

3·19 Lemma *(i) τ_H is H-safe; (ii) $\sup(C_H \cap \tau_H) = \tau_H$*

Proof : (i) follows from 2·8 (i) because any Π_2 sentence true in $J_\Theta(\mathbb{R};H)$ will be true in $J_{\tau_H}(\mathbb{R};H)$.

(ii) holds because for each $\zeta < \tau_H$, $J_\Theta(\mathbb{R};H) \models \bigvee\eta\,(\zeta < \eta \wedge \eta \ \texttt{is}\ H\texttt{-safe})$ (witnessed by $\eta = \tau_H$) and so, that assertion about ζ being Σ_1^H, $J_\tau(\mathbb{R};H)$ also models it. $\dashv$ (3·19)

3·20 Lemma *(i) τ_H is $\mathbb{R}$-H-admissible. (ii) $\forall\xi{:<}\tau_H\ \omega^\xi < \tau_H$ (iii) $\omega^{\tau_H} = \tau_H$*

Proof : Part (i) holds because $J_{\tau_H}(\mathbb{R};H)$ is a $\Sigma_1^{\dot H}$-elementary submodel of $J_\Theta(\mathbb{R};H)$, which is $\mathbb{R}$-H-admissible. (ii) is an immediate consequence of (i) and the recursive definability of ordinal exponentiation; and (iii) follows from (ii). $\dashv$ (3·20)

3·21 Definition $M_H =_{\text{df}} J_{\tau_H}(\mathbb{R};H)$

3·22 Remark It is always to be understood that the predicate $\dot H$ is interpreted in M_H as denoting membership of $H \cap (ON \cap M)$.

3·23 Proposition $M_H = \underset{\sim}{\mathrm{Hull}}_1^H(M_H)$.

Proof : put $N = \underset{\sim}{\mathrm{Hull}}_1^H(M_H)$: by N is a Σ_1^H elementary submodel of $J_\tau(\mathbb{R};H)_H$ and therefore of $J_\Theta(\mathbb{R};H)$. By N is transitive and of the form $J_\tau(\mathbb{R};H)$ for some $\tau \le \tau_H$. But then $\tau = \tau_H$ by the minimality of τ_H. $\dashv$ (3·23)

3·24 Corollary *M_H is not strongly admissible.*

3·25 Lemma $\rho_H = \tau_H$

Proof : Plainly $\rho_H \le \tau_H$, since it is the least ordinal satisfying a less restrictive condition than that imposed on τ_H. Let $x \in M_H$. By the last lemma, $x \in \underset{\sim}{\mathrm{Hull}}_1^H(M_H)$ and so there is a real a and a $\Delta_0^{\dot H}$ formula ϑ such that

$$x = \iota y\, M \models \bigvee w\, \vartheta(w, y, a).$$

Write N for $J_{\rho_H}(\mathbb{R};H)$: by definition of ρ_H,

$$\exists x'{:\in}N\ \exists w'{:\in}N\ N \models \vartheta(w', x', a).$$

But $N \subseteq M_H$, so by the absoluteness of Δ_0 truth, x' has in M_H the defining property of x and so $x = x'$: thus $x \in N$ as required. $\dashv$ (3·25)

Now we turn to H-characterisable ordinals. Suppose that $\chi < \Theta$ is such, characterised by the pair (φ, a). Then

$$J_\Theta(\mathbb{R};H) \models \bigvee\chi\ [J_\chi(\mathbb{R};H) \dot\models \Phi_{\text{safe}} + \varphi(a)];$$

the relation $\models$ is Δ_1^{KP}, and the function $\nu \mapsto J_\nu(\mathbb{R};H)$ is $\Delta_1^{\mathbb{R}}$, and so the above is a $\Sigma_1^{H,\mathbb{R}}$ statement about a and so holds in M_H. Hence $\chi < \tau_H$, and therefore $\sigma_H \le \tau_H$.

On the other hand, let $\xi < \tau_H = \rho_H$: then by the minimality of ρ_H, there is a real a such that for some $\Delta_0^{\dot H}$ formula ϑ $\exists w :\in M_H$ $M_H \models \vartheta(w, a)$ but no such w exists in $J_\xi(I\!R; H)$. Let $\zeta < \tau_H$ be least such that $\exists w :\in J\,(I\!R; H)\zeta J_\zeta(I\!R; H) \models \vartheta(w, a) + \Phi_{\rm safe}$: that exists since ϑ is $\Delta_0^{\dot H}$ and we have already seen that τ_H is a limit of ζ's with $J_\zeta(I\!R; H) \models \Phi_{\rm safe}$. Then ζ is H-characterisable, and $\xi < \zeta < \Theta$. As ξ was arbitrary, $\tau_H \leq \sigma_H$. We have thus proved that $\sigma_H = \rho_H = \tau_H$, and so the proof of 2·18 is complete.

4: A $\diamondsuit$-like object in $L[I\!R]$

We continue to assume that $V = L(\mathbb{R}, \mathbb{B}; H, S)$ that S and H are subsets of Θ and that we have added appropriate one-place predicates $\dot H$, $\dot S$, $\dot R$, $\dot B$, to our lanbguage; INTERPRETED HOW ?

Our aim in this section is to establish a $\diamondsuit$ property similar to that of Jensen. We shall build a $\Sigma_1^{\dot H, \dot S, \dot B}$ function $\diamondsuit = \diamondsuit_H$ defined on a cofinal subset of τ_H such that $\diamondsuit(\zeta)$, when defined, is a pre-well-ordering P of a set of reals such that for some $\eta < \tau_H$, $P \in J_\zeta(I\!R)$ and $J_\eta(I\!R) \models P$ `is ordinal definable`; this function will have the property that

> *whenever a Σ_1^H assertion $\Phi(a, P, \diamondsuit, \tau_H; H)$ holds in $J_\Theta(I\!R; H)$ of a real a, an* (OD ??) *pre-well-ordering P, the partial function $\diamondsuit$ and the ordinal τ_H, then there is an ordinal $\zeta \in Dom(\diamondsuit)$ such that in $J_{\tau_H}(I\!R; H)$, $\Phi(a, \diamondsuit(\zeta), \diamondsuit{\restriction}\zeta, \zeta; H \cap \tau_H)$.*

4·0 Remark The shift from H to $H \cap \tau_H$ merely expresses the fact that the only part of H that is used in evaluating a Δ_0^H assertion about some sets is the intersection of H and the transitive closure of those sets. Having now drawn attention to this consequence of the rôle of H as a unary predicate, we shall in future not usually do so.

H-gaps

We recall our definition of *H-safe ordinal* from §2. DO WE NEED IT ?

Write C for the set of H-safe ordinals less than τ_H. We define a partition of C into disjoint intervals, which we call *H-gaps*.

4·1 Definition For $\xi \in C$, define $\phi(\xi)$ to be the least H-safe ordinal ζ with $J_\zeta(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\xi(I\!R; H)$. Then define

$$\psi(\xi) \ =_{\rm df}\ \sup\{\eta \mid \eta \in C \ \&\ J_{\phi(\xi)}(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\eta(I\!R; H)\}.$$

4·2 Remark That supremum is attained, by elementary model theory, so $J_{\phi(\xi)}(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_{\psi(\xi)}(I\!R; H)$.

4·3 Lemma *If $\zeta < \xi < \eta$ and $J_\zeta(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\eta(I\!R; H)$, then $J_\zeta(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\xi(I\!R; H)$.*

Proof : easy model theory. $\dashv$

The next exercise shows that in the context of the lemma just stated, $J_\xi(I\!R)$ may well not be a $\Sigma_1$elementary submodel of $J_\eta(I\!R)$ even when H is the empty set:

4·4 Exercise If $J_\delta(I\!R) \preceq_{\Sigma_1} J_{\delta+1}(I\!R)$ then δ is a limit ordinal and is $I\!R$-admissible.

4·5 Lemma *For η and ξ in C,*

(i) $\phi(\xi) \leq \xi \leq \psi(\xi) < \tau_H$;

(ii) $\phi(\xi) \leq \eta \leq \psi(\xi) \implies \phi(\eta) = \phi(\xi)$ & $\psi(\eta) = \psi(\xi)$.

Proof : if $\zeta < \tau_H \leq \eta$ and $J_\zeta(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\eta(I\!R; H)$, then by $J_\zeta(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_{\tau_H}(I\!R; H)$, but then $J_\zeta(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\Theta(I\!R; H)$, contradicting the minimality of τ_H: so for $\xi \in C$, $\psi(\xi) < \tau_H$. The rest of (i) is trivial.

As for (ii), $J_{\phi(\xi)}(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_\eta(I\!R; H)$ by and , and so $\phi(\eta) \leq \phi(\xi)$; so $J_{\phi(\eta)}(I\!R; H) \preceq_{\Sigma_1^{\dot H}} J_{\phi(\xi)}(I\!R; H)$ by , and hence $\phi(\eta) = \phi(\xi)$ by the minimality of $\phi(\xi)$. That $\psi(\eta) = \psi(\xi)$ is immediate. $\dashv$ (4·5)

Our gaps are therefore the intervals of the form $[\phi(\zeta), \psi(\zeta)]$, where ζ is H-safe and less than τ_H, plus the interval $[\tau_H, \Theta]$, which is the last gap.

4·6 Definition We say that ζ *starts an H-gap*, in symbols ${\rm Start}_H(\zeta)$, if ζ is H-safe and $\phi(\zeta) = \zeta$.

4·7 EXERCISE $\mathrm{Start}_H(\zeta)$ is a Δ_1^H predicate of ζ: in short it is $\Delta_1^H(\zeta)$.

4·8 EXERCISE If ζ starts an H-gap, then $\underset{\sim}{\mathrm{Hull}}_1^H(J_\zeta(\mathbb{R};H)) = J_\zeta(\mathbb{R};H)$.

4·9 EXERCISE If ζ is H-safe, then $\underset{\sim}{\mathrm{Hull}}_1^H(J_\zeta(\mathbb{R};H)) = J_{\phi(\zeta)}(\mathbb{R};H)$.

4·10 EXERCISE If $\zeta \le \xi$ are H-safe, and ζ is the least ordinal such that $J_\zeta(\mathbb{R};H) \preceq^{\mathbb{R}}_{\Sigma_1^{\dot H}} J_\xi(\mathbb{R};H)$ then in fact $J_\zeta(\mathbb{R};H) \preceq_{\Sigma_1^{\dot H}} J_\xi(\mathbb{R};H)$.

4·11 REMARK The above exercises suggest an alternative approach to the definition of gaps, which is that followed by Steel, [R 5] though he is, of course, concerned with treating all ordinals and not just safe ones.

The diamond sequence we are about to define will be a sequence of OD pre-well-orderings of sets of reals. $\diamondsuit(\zeta)$ will be defined only for certain $\zeta < \tau_H$ that start an H-gap, and, when defined, the value $\diamondsuit(\zeta)$ will be a member of $J_{\psi(\zeta)}(\mathbb{R};H)$.

The naïve adaptation of Jensen's least-counterexample argument, would be this:

> Fix a Δ_0^H formula $\Phi^0(a,x,P,F,\eta;H)$ so that $\Phi(a,P,F,\eta;H) =_{\mathrm{df}} \exists x\Phi^0(a,x,P,F,\eta;H)$ is a universal Σ_1^H formula: here a is a real variable, x a set variable, η an ordinal variable, P will range over OD prewellorderings of subsets of $\mathcal{N}$; F will be a (partial) function, and H is our ever-present unary predicate for some class of ordinals. [C 5]
>
> At stage ζ, ask if there is a real a such that
>
> $$\forall\xi :\in dom(\diamondsuit\restriction\zeta)\ \neg\Phi(a,\diamondsuit(\xi),\diamondsuit\restriction\xi,\xi;H):$$
>
> such an a is said to *seek attention* at stage ζ; if so, we ask for each such a, if there is an OD prewellordering P such that $\Phi(a,P,\diamondsuit\restriction\zeta,\zeta;H)$; such a P will be said to *serve* the real a; if so, we take $\diamondsuit(\zeta)$ to be the first (in some standard enumeration of OD sets) possible P that serves some a.

One can see fairly easily that this definition will peter out at some ordinal η below Θ; our task, in effect, is to show that $\eta = \tau_H$.

But there are several problems to be handled: with the sound analogy of Σ_1 functions of set theory to ordinary recursive functions in mind, we see that we shall be in difficulties trying to define a recursive function by asking Π_1 questions — whether there exist reals that seek attention — and Σ_1 questions — whether for such reals witnesses P exist. So we must modify the above by setting a time limit both on our testing of reals and on the duration of our search for witnesses. We shall be quite strict with the first problem; and for the second we shall use the function ψ as our clock.

NOTATION: introduce the dotted skew Delta$_0$s.

We must also be more precise about our well-ordering of OD sets. For this proof, write POD for the class of OD prewellorderings of subsets of $\mathbb{R}$.

4·12 DEFINITION We well-order POD by setting

$$Q <_{POD} P \iff \tilde\eta(Q) < \tilde\eta(P) \ \text{ or } \ \big[\tilde\eta(Q) = \tilde\eta(P)\ \&\ J_{\tilde\eta(Q)}(\mathbb{R}) \models Q <_{\dot{OD}} P\big].$$

Here $<_{\dot{OD}}$ is the formal counterpart of the (naturally definable) well-ordering of OD, $<_{OD}$: each OD set is defined by some pair (ζ,φ), and such pairs have a natural well-ordering by minimising first ζ and then the Gödel number of φ.

We shall approach the notion of an *attempt*, roughly a partial function with domain some set of ordinals and values in POD which might be an initial segment of $\diamondsuit$, through concepts that might be pronounced *pre-attempt* and *seeking attention*.

[R 5] Scales in $L[\mathbb{R}]$, The Third Cabal Volume, Cabal Seminar 79–81, Lecture Notes in Mathematics 1019, ed. A.S.Kechris, D.A.Martin, and Y.N.Moschovakis, Springer-Verlag, 1983, pp 107–156.

[C 5] universal sets are discussed in Chapter Two: in the present context we could fix a listing φ_j of all $\Delta_0^{\dot H}$ formulæ in variables a, x, P, F, η, and for $a:\omega\to\omega$ let $\Phi^0(a,\ldots)$ mean $\varphi_{a(0)}(a^\star,\ldots)$ where $\forall n:\in\omega\ a^\star(n) = a(n+1)$.

We begin the formal definition of $\diamondsuit$.

4·13 DEFINITION $pat(f,\zeta) \iff_{\rm df} Fn(f)\ \&\ \mathrm{Dom}(f)\subseteq\zeta\ \&\ \mathrm{Start}_H(\zeta)\ \&\ \forall\xi{:}{\in}\mathrm{Dom}(f)\ \mathrm{Start}_H(\xi)$

Evidently that property is $\Delta_1^H(f,\zeta)$.

4·14 DEFINITION $sa(f,\zeta,a) \iff_{\rm df} a\in\mathbb{R}\ \&\ pat(f,\zeta)\ \&\ \forall\xi{:}{\in}\mathrm{Dom}(f)\ f\restriction(\xi{+}1)\in J_\zeta(\mathbb{R};H)\ \&$
$\&\ \forall\xi{:}{\in}\mathrm{Dom}(f)\ \forall x{:}{\in}J_\zeta(\mathbb{R};H)\ \ J_\zeta(\mathbb{R};H)\models\neg\Phi^0(a,x,f(\xi),f\restriction\xi,\xi;H).$

This formula is $\Delta_1^H(f,\zeta,a)$. Notice that it exhibits a little impatience: a is said to seek attention at stage ζ if no witness to a certain Σ_1 statement has appeared by stage ζ: that differs from our "naïve" version which only paid heed to a real a if there were no such witnesses at all. That formulation of course would not be Δ_1^H. Fortunately our ultimate interest will be in what happens at the stable ordinal τ_H, where the two concepts will coïncide.

4·15 EXERCISE Show that for every $\zeta<\tau_H$ which starts an H-gap there are reals that seek attention. (Hint: consider $(\varphi,\ a)$ H-characterising an ordinal in the interval $(\zeta,\ \tau_H)$.)

4·16 DEFINITION $\mathrm{Serve}(P,\xi,f,\zeta,a) \iff_{\rm df} \big[sa(f,\zeta,a)\ \&\ \xi>\zeta\ \&\ J_\zeta(\mathbb{R};H)\preceq_{\Sigma_1^H} J_\xi(\mathbb{R};H)\ \&$
$\&\ f\in J_\xi(\mathbb{R};H)\ \&\ P\in J_\xi(\mathbb{R})\ \&\ J_\xi(\mathbb{R})\models P\ \epsilon\ \dot{POD}\ \&$
$\&\ \exists x{:}{\in}J_\xi(\mathbb{R};H)\ \big(J_\xi(\mathbb{R};H)\models\Phi^0(a,x,P,f,\zeta;H)\big)\big].$

4·17 REMARK Serve is $\Delta_1^H(P,\xi,f,\zeta,a)$. NOTE TOO THAT $\xi\leqslant\psi(\zeta)$

We shall use Kleene's symbol $\simeq$ when discussing partial functions: thus we shall write $f(x)\simeq g(y)$ to mean that $x\in\mathrm{Dom}(f)\iff y\in\mathrm{Dom}(g)$ and that when both these hold, then $f(x)=g(y)$. [C5] For a partial function f, we shall write $f(x)\downarrow$ to mean that $x\in\mathrm{Dom}(f)$, its negation being $f(x)\uparrow$.[C6]

Now we proceed to define a partial function ξ and then a partial functional F to be used in the recursion equation for the partial function $\diamondsuit_H$. We first define the domain of F and ξ.

4·18 DEFINITION $F(f,\zeta)$ is defined, in symbols $F(f,\zeta)\downarrow$, if and only if

$$\exists\xi\exists\eta{:}{<}\xi\ \exists P{:}{\in}J_\eta(\mathbb{R})\ \exists a{:}{\in}\mathbb{R}\ \ \mathrm{Serve}(P,\xi,f,\zeta,a).$$

That is plainly a $\Sigma_1^H(f,\zeta)$ condition. When it is satisfied, we shall choose the value of $F(f,\zeta)$ by minimising the place of P in $<_{OD}$, but only after minimising the moment of construction of P and the witness x.

4·19 DEFINITION $\xi(f,\zeta)\simeq_{\rm df}$ the least ξ such that $\exists\eta{:}{<}\xi\ \exists P{:}{\in}J_\eta(\mathbb{R})\ \exists a{:}{\in}\mathbb{R}\ \mathrm{Serve}(P,\xi,f,\zeta,a)$.

4·20 REMARK $F(f,\zeta)\downarrow\iff\xi(f,\zeta)\downarrow$. The assertion that $\xi(f,\zeta)\simeq\lambda$ is $\Delta_1^H(f,\zeta,\lambda)$; the assertion that $\xi(f,\zeta)\downarrow$ is $\Sigma_1^H(f,\zeta)$, being equivalent to $\exists\lambda\ \xi(f,\zeta)\simeq\lambda$.

4·21 DEFINITION When $\xi(f,\zeta)\downarrow$, we define $\eta(f,\zeta)\simeq$ the least $\eta<\xi(f,\zeta)$ such that

$$\exists P{:}{\in}J_\eta(\mathbb{R})\ \exists a{:}{\in}\mathbb{R}\ \mathrm{Serve}(P,\xi,f,\zeta,a).$$

4·22 DEFINITION $F(f,\zeta)\simeq P \iff_{\rm df} \xi(f,\zeta)\downarrow\ \&\ \eta(f,\zeta)\downarrow\ \&\ \exists a{:}{\in}\mathbb{R}\ \mathrm{Serve}(P,\xi(f,\zeta),f,\zeta,a)\ \&$
$\&\ \forall Q{:}{<}_{\eta(f,\zeta)}P\ \neg\exists a'{:}{\in}\mathbb{R}\ \mathrm{Serve}(Q,\xi(f,\zeta),f,\zeta,a').$

where we write $\forall Q{:}{<}_\eta P\ldots \iff_{\rm df} \forall Q{:}{\in}J_\eta(\mathbb{R})\ ((J_\eta(\mathbb{R})\models Q\ \epsilon\ \dot{POD}\wedge Q\dot{<}_{OD}P)\implies\ldots)$.

4·23 REMARK The statement $F(f,\zeta)\simeq P$ is $\Sigma_1^H(f,\zeta,P)$.

[C5] a pedantic point: with our system of set-theoretical definitions $f(x)$ **is** always defined, being $\varnothing$ when $x\notin\mathrm{Dom}(f)$: hence if we distinguish between being defined and being properly defined, we may express $f(x)\simeq g(y)$ as saying that if either side is properly defined then so is the other and they are equal.

[C6] A second point of pedantry: if f is a relation, $f(x)$ might be improperly defined because there are several y with $(y,\ x)\in f$.

The reader should check that there is at most one such P, and that when $\xi(f,\zeta)\downarrow$ such a P indeed exists. The purpose of minimising η is to avoid the following trap: there might be $P <_{OD} \bar{P}$ and ordinals $\bar{\eta} < \eta$ where $J_\eta(\mathbb{R})$ thinks that P should be the value and $J_{\bar{\eta}}(\mathbb{R})$ thinks that $\bar{P}$ should be. In such a case, $\bar{P}$ will be the preferred value according to our definition.

In discussing partial functions with domain some set of ordinals it will be convenient to make this definition:

4·24 Definition $\lambda(f) =_{\mathrm{df}} \sup\{\nu+1 \mid \nu \in \mathrm{Dom}(f)\}$.

$\lambda(f)$ will be a successor ordinal $\rho+1$ if $dom(f)$ has a largest element ρ; 0 if f is nowhere defined; and a limit ordinal otherwise. The function λ is rudimentary.

4·25 Definition We say that f is an *attempt* if

$$pat(f,\lambda(f)) \;\&\; \forall\nu{:<}\lambda(f)\ [f(\nu) \simeq F(f\restriction\nu,\nu)].$$

$$\begin{aligned}\text{Thus}\quad f \text{ is an attempt } \iff\ & Fn(f) \;\&\; \mathrm{Dom}(f) \subseteq ON \;\& \\ & \&\ \forall\nu{:<}\lambda\,(f)\big[\big(\nu\in\mathrm{Dom}(f) \iff \eta(f\restriction\nu,\nu)\downarrow\big)\ \& \\ & \big(\nu\in\mathrm{Dom}(f)\implies f(\nu)=F(f\restriction\nu,\nu)\big)\big].\end{aligned}$$

and so being an attempt is a $\Delta_2(H)$ condition. We shall have to do better than that.

4·26 Definition We say that two partial functions f and g *agree* if for $\lambda = \inf\{\lambda(f),\lambda(g)\}$, $f\restriction\lambda = g\restriction\lambda$: in other words, whenever $\nu < min\{\lambda(f),\lambda(g)\}$, $f(\nu)\simeq g(\nu)$.

4·27 Lemma *any two attempts agree.*

Proof : for ν the least ordinal at which the two attempts disagree, $f\restriction\nu = g\restriction\nu$, and so

$$f(\nu) \simeq F(f\restriction\nu,\nu) \simeq F(g\restriction\nu,\nu) \simeq g(\nu)$$

after all. ⊣ (4·27)

4·28 Lemma *If $\zeta < \theta$ and both start H-gaps and $f \in J_\theta(\mathbb{R};H)$ then*

$$\xi(f\restriction\zeta,\zeta)\downarrow \iff J_\theta(\mathbb{R};H) \models \dot{\xi}(\dot{f\restriction}\zeta,\zeta)\dot{\downarrow}.$$

and if $F(f\restriction\zeta,\zeta)\downarrow$, then

$$F(f\restriction\zeta,\zeta) = (F(f\restriction\zeta,\zeta))_{J_\theta[\mathbb{R}]}.$$

We leave the proof as an exercise, as also that of its

4·29 Corollary *If θ starts an H-gap, $g \in J_\theta(\mathbb{R};H)$ and $J_\theta(\mathbb{R};H) \models g$ `is an attempt`, then g is an attempt with $\lambda(g) < \theta$.*

Their proofs and the next rely on the fundamental fact of constructibility that both the sequences $\langle J_\delta(\mathbb{R}) \mid \delta < \theta\rangle$ and $\langle J_\delta(\mathbb{R};H) \mid \delta < \theta\rangle$ are uniformly Σ_1 over $J_\theta(\mathbb{R};H)$.

4·30 Lemma *If θ starts an H-gap and f is an attempt, then $f\restriction\theta$ is definable over $J_\theta(\mathbb{R};H)$ and is a member of $J_{\theta+1}(\mathbb{R};H)$.*

Proof : for fixed f by induction on θ. If $f\restriction\theta = \varnothing$, the lemma is clear. $\bigcup\mathrm{Dom}(f\restriction\theta)$ is either θ or a smaller ζ that starts an H-gap. In the latter case, the induction hypothesis for ζ tells us that $f\restriction\zeta \in J_{\zeta+1}(\mathbb{R};H)$, and $f(\zeta) = F(f\restriction\zeta,\zeta) \in J_\theta(\mathbb{R};H)$, since the P chosen by F has to lie in some $J_\eta(\mathbb{R})$ where η, being in the gap started by ζ, is less than θ. In this case therefore we actually have $f\restriction\theta \in J_\theta(\mathbb{R};H)$.

When $\sup\mathrm{Dom}(f\restriction\theta) = \theta$, $f\restriction\theta$ is defined over $J_\theta(\mathbb{R};H)$ since

$$f = \Big\{(P,\zeta)\in J_\theta(\mathbb{R};H)\ \Big|\ J_\theta(\mathbb{R};H) \models \big[\textstyle\bigvee g\ g\ \texttt{is an attempt} \wedge \zeta \mathrel{\dot\epsilon} \mathrm{Dom}(g) \wedge g(\zeta) \mathrel{\texttt{=}} P\big]\Big\},$$

the induction hypothesis telling us that each $f\restriction\zeta$ is in $J_\theta(\mathbb{R};H)$ and the local character of F guaranteeing both that each such is recognised by $J_\theta(\mathbb{R};H)$ to be an attempt and that everything believed by $J_\theta(\mathbb{R};H)$ to be an attempt is indeed one. ⊣ (4·30)

4·31 COROLLARY *If g is an attempt and $\lambda(g) < \theta$ then $g \in J_\theta(I\!R; H)$.*

Proof : exercise. $\dashv$

An attempt is a partial function, and a partial function might be empty. The next definition captures the idea of an attempt being the one with the largest domain possible below a given ordinal.

4·32 DEFINITION f is *the full attempt up to θ* $\Longleftrightarrow_{\text{df}}$

$$\theta \text{ starts an } H\text{-gap } \ \& \ f = \bigcup\{g \in J_\theta(I\!R; H) \ \big| \ J_\theta(I\!R; H) \models g \texttt{ is an attempt}\}.$$

Write $\mathrm{Full}_H(f,\,\theta)$ in that case. Full_H is a Δ_1^H predicate of f and θ.

4·33 EXERCISE Show that if $\mathrm{Full}_H(f,\,\theta)$, then f is an attempt and is the union of all attempts g with $\lambda(g) < \theta$.

Hence if we make the following definition:

4·34 DEFINITION $\diamondsuit_H \ =_{\text{df}} \ \bigcup\{f \mid \exists\theta\, \mathrm{Full}_H(f,\theta)\}$

we obtain a Σ_1^H function $\diamondsuit_H$.

4·35 EXERCISE Show that $\bigcup \mathrm{Dom}(\diamondsuit_H) \geq \tau_H$, by considering H-characterisable ordinals.

4·36 EXERCISE Show that if θ starts an H-gap, then $\mathrm{Full}_H(\diamondsuit \restriction \theta, \theta)$.

Now we may approach our principal result: IT MUSTG BE CHECKED whether it holds for arbitrary P. The idea is that every P is definable from ords and reals and therefore is a section of an OD set by sme real, so picking that up should just mean rewriting, as all the reals are there anyway.

4·37 THEOREM $(V = L(I\!R)\,)$ *Let $H \subseteq \Theta$. There is a Σ_1^H partial function $\diamondsuit = \diamondsuit_H$ defined on a subset of τ_H and with values in POD such that for any real a and* OD *pre-well-ordering P, if*

$$\exists x\,{:}{\in}\, J_\Theta(I\!R; H) \ \ J_\Theta(I\!R; H) \models \Phi^0(a, x, P, \diamondsuit \restriction \tau_H, \tau_H; H)$$

then there is a $\zeta \in \mathrm{Dom}(\diamondsuit)$ such that

$$\exists x\,{:}{\in}\, J_{\tau_H}(I\!R; H) \ \ J_{\tau_H}(I\!R; H) \models \Phi^0(a, x, \diamondsuit(\zeta), \diamondsuit \restriction \zeta, \zeta; H \cap \tau_H).$$

Proof : We have defined our candidate $\diamondsuit$. Notice that $pat(\diamondsuit \restriction \tau_H, \tau_H)$, and that $\forall\theta\,{:}{\in}\,\mathrm{Dom}(f)\ f\restriction(\theta+1) \in J_{\tau_H}(I\!R; H)$. Suppose the theorem false. Then there is at least one pair (a, P) for which the conclusion fails, and so $sa(\diamondsuit \restriction \tau_H, \tau_H, a)$, but for which the hypothesis holds; thus, bearing in mind that we are told there is a witness x in $J_\Theta(I\!R; H)$, therefore in some $J_\xi(I\!R; H)$ with $\xi < \Theta$, and that $J_{\tau_H}(I\!R; H) \preceq_{\Sigma_1^H} J_\xi(I\!R; H)$ for every ξ in the interval $[\tau_H, \Theta]$, we see that $\xi(\diamondsuit \restriction \tau_H, \tau_H)$ and $\eta(\diamondsuit \restriction \tau_H, \tau_H)$ are defined, with values $\bar\xi$ and $\bar\eta$, say, with $\tau_H < \bar\eta < \bar\xi < \Theta$.

Therefore $F(\diamondsuit \restriction \tau_H, \tau_H)$ is defined: let its value be $\bar P$. Pick $\bar a \in I\!R$ such that $\mathrm{Serve}(\bar P, \bar\xi, \diamondsuit \restriction \tau_H, \tau_H, \bar a)$, and in particular that $sa(\diamondsuit \restriction \tau_H, \tau_H, \bar a)$, and consider the statement

$$\exists\zeta\exists f\ \mathrm{Full}_H(f,\zeta)\ \&\ \xi(f,\zeta)\downarrow\ \&\ \eta(f,\zeta)\downarrow\ \&\ F(f,\zeta)\downarrow\ \&\ \ \mathrm{Serve}(F(f,\zeta), \xi(f,\zeta), f, \zeta, \bar a)$$

That is a Σ_1^H statement about $\bar a$: it is true, being witnessed by τ_H and $\diamondsuit \restriction \tau_H$. Hence there are ζ^* and f^*, in $J_{\tau_H}(I\!R; H)$ witnessing it. So

$$\mathrm{Full}_H(f^*\!, \zeta^*)\ \&\ \xi(f^*\!, \zeta^*)\downarrow\ \&\ \eta(f^*\!, \zeta^*)\downarrow\ \&\ F(f^*\!, \zeta^*)\downarrow\ \&\ \ \mathrm{Serve}(F(f^*\!, \zeta^*), \xi(f^*\!, \zeta^*), f^*\!, \zeta^*\ \bar a)$$

Hence we know that ζ^* starts an H-gap, and that $f^* = \diamondsuit \restriction \zeta^*$. Write $\eta^* = \eta(f^*\!, \zeta^*)$, and $\xi^* = \xi(f^*\!, \zeta^*)$. $\eta^* < \xi^* < \tau_H$, being in the gap begun by ζ^*. By definition of $\diamondsuit$, $\diamondsuit(\zeta^*) \simeq F(\diamondsuit \restriction \zeta^*\!, \zeta^*)$ and so $\zeta^* \in \mathrm{Dom}\,\diamondsuit$, and $\mathrm{Serve}(\diamondsuit(\zeta^*),\ \xi^*\!\ \diamondsuit \restriction \zeta^*\!\ \zeta^*\!\ \bar a)$. But then

$$\exists x\,{:}{\in}\, J_{\xi^*}(I\!R; H)\ \ J_{\xi^*}(I\!R; H) \models \Phi^0(\bar a, x, \diamondsuit(\zeta^*), \diamondsuit \restriction \zeta^*, \zeta^*; H \cap \xi^*),$$

and so, easily, as $\xi^* < \tau_H$

$$\exists x :\in J_{\tau_H}(\mathbb{R};H) \ \ J_{\tau_H}(\mathbb{R};H) \models \Phi^0(\bar{a}, x, \diamondsuit(\zeta^*), \diamondsuit\restriction\zeta^*, \zeta^*; H\cap\tau_H),$$

by Δ_0 absoluteness; but this last assertion confutes our inference that $sa(\diamondsuit\restriction\tau_H, \tau_H, \bar{a})$. ⊣ (4·37)

That result is self-improving to the following:

4·38 COROLLARY ($V = L(\mathbb{R})$) *For any set $\bar{z} \in J_{\tau_H}(\mathbb{R};H)$,* OD *pre-well-ordering P, and $\Psi^0 \in \Delta_0$, if*

$$\exists x :\in J_\Theta(\mathbb{R};H) \ \ J_\Theta(\mathbb{R};H) \models \Psi^0(\bar{z}, x, P, \diamondsuit\restriction\tau_H, \tau_H; H)$$

then there is a $\zeta \in \mathrm{Dom}(\diamondsuit)$ with $\bar{z} \in J_\zeta(\mathbb{R};H)$ such that

$$\exists x :\in J_{\tau_H}(\mathbb{R};H) \ \ J_{\tau_H}(\mathbb{R};H) \models \Psi^0(\bar{z}, x, \diamondsuit(\zeta), \diamondsuit\restriction\zeta, \zeta; H\cap\tau_H).$$

Proof : For such $\bar{z}$, since we know that $\underset{\sim}{\mathrm{Hull}}{}_1^H(J_{\tau_H}(\mathbb{R};H)) = J_{\tau_H}(\mathbb{R};H)$, we may find $\vartheta \in \Sigma_1$ and $b \in \mathbb{R}$ such that

$$\{\bar{z}\} = \{v \mid J_{\tau_H}(\mathbb{R};H) \models \bigvee\mathfrak{w}\, \vartheta(v, \mathfrak{w}b)\}.$$

Now

$$\exists x :\in J_\Theta(\mathbb{R};H) \ \ J_\Theta(\mathbb{R};H) \models \Psi^0(\bar{z}, x, P, \diamondsuit\restriction\tau_H, \tau_H; H)$$

so

$$\begin{aligned}&\exists x :\in J_\Theta(\mathbb{R};H)\ \exists z :\in J_\Theta(\mathbb{R};H)\ \exists w :\in J_\Theta(\mathbb{R};H)\\ &\qquad J_\Theta(\mathbb{R};H) \models \Psi^0(z, x, P, \diamondsuit\restriction\tau_H, \tau_H; H)\ \&\ z \in J_{\tau_H}(\mathbb{R};H)\ \&\ w \in J_{\tau_H}(\mathbb{R};H)\ \&\ J_{\tau_H}(\mathbb{R};H) \models \vartheta(z, w, b).\end{aligned}$$

That asserts essentially that a certain Σ_1^H statement about b, τ_H, P and $\diamondsuit_H$ is true in $J_\Theta(\mathbb{R};H)$. Re-writing it in the form $\exists u \Phi^0(d, \ldots)$ for a suitable real d, and applying the theorem, we have

$$\begin{aligned}&\exists\theta :\in \mathrm{Dom}(\diamondsuit_H)\ \exists x :\in J_{\tau_H}(\mathbb{R};H)\ \exists z :\in J_{\tau_H}(\mathbb{R};H)\ \exists w :\in J_{\tau_H}(\mathbb{R};H)\\ &\qquad J_{\tau_H}(\mathbb{R};H) \models \Psi^0(z, x, \diamondsuit_H(\theta), \diamondsuit_H\restriction\theta, \theta; H)\ \&\ z \in J_\theta(\mathbb{R};H)\ \&\ w \in J_\theta(\mathbb{R};H)\ \&\ J_\theta(\mathbb{R};H) \models \vartheta(z, w, b).\end{aligned}$$

But if $J_\theta(\mathbb{R};H)$ models $\bigvee\mathfrak{w}\,\vartheta(z, \mathfrak{w}, b)$, so does $J_{\tau_H}(\mathbb{R};H)$, and therefore this z equals $\bar{z}$. Thus

$$\exists\theta :\in \mathrm{Dom}(\diamondsuit_H)\ \ \bar{z} \in J_\theta(\mathbb{R};H)\ \&\ J_{\tau_H}(\mathbb{R};H) \models \bigvee\mathfrak{x}\Psi^0(\bar{z}, \mathfrak{x}, \diamondsuit_H(\theta), \diamondsuit_H\restriction\theta, \theta; H)$$

as required. ⊣ (4·38)

5: A characterisation of certain inner models

If we were simply after a model containing ω Woodin cardinals, we might have worked with each Q_i simply of height θ_i, coding the extenders and not troubling to freeze the power set of θ_i, since our θ_i is Woodin in the large model $M_i[Q_i]$ and all its requisite extenders are in the smaller models which have no subsets of θ_i not in the larger model.

However, we have chosen Q_i rather larger than that in order that the Vopěnka algebras, of cardinality θ_i should lie in $\mathfrak{M}(Q_i)$.

This will have a consequence similar to that seen in our second construction, using Prikry sequences, of a model of ω Woodins.

5·0 PROPOSITION $\sup_i \lambda_i = \sup_i \theta_i = \omega_1^{L(\mathbb{R}_\omega)}$.

Proof : $\theta_1 < \lambda_1 < \omega_1^{L(\mathbb{R}_1)} \leqslant \theta_2 < \lambda_2 < \omega_1^{L(\mathbb{R}_2)} \leqslant \ldots$. As $\mathbb{R}_\omega$ is the set of reals of $L(\mathbb{R}_\omega)$, its ω_1 equals $\sup_i \omega_1^{L(\mathbb{R}_i)}$. ⊣ (5·0)

5·1 DEFINITION Write η_ω for $\sup \bigcup_i Q_i$.

5·2 PROPOSITION *Each real in $\mathbb{R}_\omega$ is generic over $M_\omega(Q_\omega)$, and also over $L(Q_\omega)$, for an algebra of size less than η_ω.*

Proof : we know that if a real a is in $\mathbb{R}_{i-1}$ it is in $L[S_i, g_i]$ and generic over N_i for an algebra of size at most θ_i computed in N_i, with the $< \theta_i$ chain condition; and thus embeddable into the collapsing algebra $\mathrm{Coll}(\omega, \theta_i)$ of that model.

Further we know that that algebra and all its subsets are in $L(Q_\omega)$.

Hence the reals of $\mathbb{R}_\omega$ may be construed as those of the symmetric collapse of η_ω over $L(Q_\omega)$ or over many models near to that, by the result proved in §4.

5·3 PROPOSITION ($V = L(\mathbb{R})$+ AD + DC) *Let S be a set of ordinals. Then $h_\omega(S)$ is in V, even though the embedding is defined in some generic extension of V.*

Proof : Let S be definable from ordinals and the real a. Then for each $\nu \leqslant \omega$, $S \in L[S_0, a]$; $h_\nu(a) = a$ and $h_\nu(S_0) = j_\nu(S_0) = S_\nu$, so $h_\nu(S) \in L[S_\nu, a]$. ⊣ (5·3)

5·4 THEOREM ($V = L(\mathbb{R})$+ AD + DC) *Let S be a set of ordinals. Then*

$$HOD_S = HOD[h_\omega(S)]$$

Proof : let S be definable from the real a. Then $h_\omega(S)$ is in $L[S_\omega, a]$, and so is in $M_\omega[Q_\omega, a]$. a is generic over $M_\omega[Q_\omega]$ for an algebra of size less than η_ω; the same is therefore true of $h_\omega(S)$, so by the absorbent nature of the Lévy collapse, we may treat the reals of $\mathbb{R}_\omega$ as those of the symmetric collapse of η_ω over $M_\omega[h_\omega(S)]$.

From that we see that any set of ordinals definable from S in $L(\mathbb{R}_0)$ is in $L[S_0, h_\omega(S)]$:

$$\begin{aligned}(\phi(\nu, S))^{L(\mathbb{R})} &\iff (\phi(j_\omega(\nu), h_\omega(S)))^{L(\mathbb{R}_\omega)}\\ &\iff L[S_\omega, Q_\omega, h_\omega(S), h_\omega \restriction \sup S] \models blah\\ &\iff L[S_0, h_\omega(S)] \models blah_2\end{aligned}$$

So $HOD_S \subseteq L[S_0, h_\omega(S)] \subseteq HOD_S$, as required. ⊣ (5·4)

6: Random reflections

I append here, pro tem, some thoughts about the Vopěnka algebra in various contexts. First, with no additional predicates to cloud the mind.

No additional predicates

6·0 PROPOSITION *There is an* OD *function assigning to each $\nu < \Theta$ a pre-well-ordering of length ν.*

6·1 PROPOSITION *Each* OD *pre-well-ordering of length λ of $\mathcal{N}$ gives a maximal anti-chain in $\mathbb{K}^1$ of length λ. We may assume that each cell of the partition is of size $\mathfrak{c}$.*

Proof : consider the pre-well-ordering to be of $\mathcal{N} \times \mathcal{N}$, by first co-ordinate. Each cell is of size at least $\mathfrak{c}$, and is a subset of $\mathcal{N} \times \mathcal{N}$, and hence of size at most $\mathfrak{c}$. ⊣ (6·1)

6·2 PROPOSITION *Θ is regular in HOD.*

Proof : let $\langle \theta_\nu \mid \nu < \lambda \rangle$ be a definable monotonic sequence cofinal in Θ and of length $\lambda < \Theta$. Let $f : \mathcal{N} \overset{\text{onto}}{\longrightarrow} \lambda$ be OD. Let $\langle \pi_\nu \mid \nu \rangle$ be an OD sequence such that for each ν, $\pi_\nu : \mathcal{N} \overset{\text{onto}}{\longrightarrow} \theta_\nu$. Define a surjection $\chi : \mathcal{N} \times \mathcal{N} \overset{\text{onto}}{\longrightarrow} \Theta$ by

$$\chi(a, b) = \pi_{f(a)}(b).$$

Contradiction ! ⊣ (6·2)

6·3 PROPOSITION *Each $\mathbb{K}^k$ has the $< \Theta$ chain condition in HOD.*

Proof : a maximal antichain is a well-orderable set, in order type λ, of names which when interpreted gives a partition of $(\mathcal{P}\omega)^k$. But then by mapping each k-tuple of reals to the name of the set it is in, we get a surjection of $(\mathcal{P}\omega)^k$ onto λ. Since $\mathcal{P}\omega$ maps easily onto each of its finite powers, $\lambda < \Theta$. ⊣ (6·3)

6·4 REMARK The above proof makes no requirement that AC should hold outside HOD.

6·5 PROPOSITION *$\mathbb{K}^\omega$ has the $< \Theta$-chain condition in HOD.*

Proof : AC is true in HOD, and Θ is regular there. Suppose we had a subset X of $\mathbb{K}^\omega$ of size Θ. The members of the non-separative p.o. $\mathbb{K}^\omega$ are of the form (n, C^o): an excerpt from Chapter I:

Our official definition is this: $\mathbb{K}^\omega$ is the complete Boolean algebra associated to the notion of forcing in HOD whose members are pairs (n, C^o) where $C^o \in \mathbb{K}^n$ and the intended meaning is that the first n terms of $\vec{\alpha}$ lie in C. To achieve that, the partial ordering $\leq^*$ is defined thus:

$$(n, C^o) \leq^* (m, D^o) \iff_{\text{df}} n \geq m \;\&\; C_{\downarrow m} \subseteq D$$

6·6 REMARK Note the use of $C_{\downarrow m}$, and therefore of an existential quantifier ranging over subsets of $\mathcal{N}$.

Since the cofinality of θ is uncountable, Θ many members of X have the same n: but the ordering of these is as in $\mathbb{K}^n$, which we know to have the $< \Theta$-chain condition. ⊣ (6·6)

6·7 REMARK That means that the algebra $\mathbb{K}^\omega$ has the Θ chain condition, turns Θ into ω_1, is weakly homogeneous, and is (generated by) a set of size Θ. Does that make it the same as $\mathrm{Coll}(\omega, < \Theta)$? Certainly it means that we can embed $\mathrm{Coll}(\omega, < \Theta)$ into r.o. $\mathbb{K}^\omega$. Is $\mathbb{K}^\omega$ complete as it stands ? Has it become separative ? Is $\mathbb{K}^1$ separative and complete ?

6·8 PROPOSITION ($V = L(\mathbb{R})$) *Every cardinal in HOD from Θ onwards is an aleph in V.*

Proof : let μ be a counterexample, of which the cardinal in V is κ. Let $A \subseteq \kappa \times \kappa$ code μ. A is ordinal definable from some real a. But then $A \in HOD[G_a]$ a generic extension of HOD by an algebra with the Θ chain condition, so μ should still be a cardinal. Contradiction. ⊣ (6·8)

6·9 REMARK That also gives us a second proof that under $V = L(\mathbb{R})$, Θ is regular.

6·10 REMARK McAloon's techniques easily furnish a model where the conclusion is false, so some hypothesis is necessary.

6·11 PROPOSITION *There is an injection in HOD of Θ into $\mathbb{K}^1$.*

Proof : from the fact that there is for each $\nu < \Theta$ an OD pre-well-ordering of length ν. ⊣ (6·11)

6·12 PROPOSITION ($V = L(\mathbb{R})$) *There is an* OD *enumeration of the* OD *subsets of* $\mathcal{P}\omega$ *in order type* Θ.

Proof the First: from the fact, proved in Chapter I, that every OD set of reals has an OD-name in J_Θ. $\dashv$ (6·12)

Proof the Second: every set of reals is in $J_\Theta(\mathbb{R})$ so that if we enumerate $OD \cap \mathcal{P}\mathcal{P}\omega \cap J_\nu(\mathbb{R})$ for each ν that must be of length less than Θ; putting them together inside HOD, we get an enumeration of length at most and therefore exactly Θ. $\dashv$ (6·12)

Proof the Third: there is an enumeration in HOD of all the names in length some aleph at least Θ; but cardinals are absolute above Θ. So it must be of length Θ. NOT SATISFACTORY: how do we know that externally there are Θ names ?

6·13 REMARK By McAloon, we may suppose that $V = HOD$, $\mathfrak{c} = \aleph_1$, so that $\Theta = \aleph_2$, and that $2^{\aleph_1} = \aleph_3$. In such a universe, there are more than Θ OD subsets of $\mathcal{P}\omega$.

From a subset of Θ

6·14 Now we want to re-run all these arguments for OD_S, where S is a subset of Θ. Should we be writing $\mathbb{K}^k_S$ for the corresponding Vopěnka algebra ?

6·15 PROBLEM Is it true that if $V = L(\mathbb{R}; S]$ then every set of reals is in $J_\Theta(\mathbb{R}; S]$? OK provided the cofinality of Θ is regular ? uncountable ?

6·16 PROPOSITION *Each* $\mathbb{K}^k_S$ *has the* $< \Theta$ *chain condition in* HOD_S.

Proof : a maximal antichain is a well-orderable set, in order type λ, of OD_S-names which when interpreted gives a partition of $(\mathcal{P}\omega)^k$. But then by mapping each k-tuple of reals to the name of the set it is in, we get a surjection of $(\mathcal{P}\omega)^k$ onto λ. Since $\mathcal{P}\omega$ maps easily onto each of its finite powers, $\lambda < \Theta$. $\dashv$ (6·16)

6·17 PROPOSITION $\mathbb{K}^\omega_S$ *has the* $< \Theta$*-chain condition in* HOD_S.

6·18 PROPOSITION ($V = L(\mathbb{R}; S)$) *Every cardinal in* HOD_S *from* Θ *onwards is an aleph in* V, *and* Θ *is regular.*

Proof : let μ be a counterexample, of which the cardinal in V is κ. Let $A \subseteq \kappa \times \kappa$ code μ. A is ordinal definable from S and some real a. But then $A \in HOD_S[G_a]$, which is a generic extension of HOD_S by an algebra with the Θ chain condition, so μ should still be a cardinal. Contradiction. The last remark similarly. $\dashv$ (6·18)

6·19 REMARK McAloon's techniques easily furnish a model where the conclusion is false, so some hypothesis is necessary.

6·20 PROPOSITION *There is an injection in* HOD_S *of* Θ *into* $\mathbb{K}^1_S$.

Proof : from the fact that there is for each $\nu < \Theta$ an OD_S pre-well-ordering of length ν. $\dashv$ (6·20)

6·21 PROPOSITION ($V = L(\mathbb{R}; S)$) *There is an* OD_S *enumeration of the* OD_S *subsets of* $\mathcal{P}\omega$ *in order type* Θ.

Proof the Second: every set of reals is in $J_\Theta(\mathbb{R}; S)$???? ???? so that if we enumerate $OD_S \cap \mathcal{P}\mathcal{P}\omega \cap J_\nu(\mathbb{R}; S)$ for each ν that must be of length less than Θ; putting them together inside HOD_S, we get an enumeration of length at most and therefore exactly Θ. $\dashv$ (6·21)

Proof the Third: there is an enumeration in HOD of all the names in length some aleph at least Θ; but cardinals are absolute above Θ. So it must be of length Θ. NOT SATISFACTORY: how do we know that externally there are Θ names ?

6·22 REMARK By McAloon, we may suppose that $V = HOD$, $\mathfrak{c} = \aleph_1$, so that $\Theta = \aleph_2$, and that $2^{\aleph_1} = \aleph_3$. In such a universe, there are more than Θ OD subsets of $\mathcal{P}\omega$.

From a class of ordinals

6·23 Finally we want to look at the case $V = L(\mathbb{R}; S]$ where S might be a class.

6·24 PROPOSITION *Each* $\mathbb{K}^k_S$ *has the* $< \Theta$ *chain condition in* HOD_S.

Proof : a maximal antichain is a well-orderable set, in order type λ, of names which when interpreted gives a partition of $(\mathcal{P}\omega)^k$. But then by mapping each k-tuple of reals to the name of the set it is in, we get a surjection of $(\mathcal{P}\omega)^k$ onto λ. Since $\mathcal{P}\omega$ maps easily onto each of its finite powers, $\lambda < \Theta$. ⊣ (6·24)

6·25 PROPOSITION *$\mathbb{K}^\omega_S$ has the $< \Theta$-chain condition in HOD_S.*

6·26 PROPOSITION ($V = L(\mathbb{R}; S)$) *Every cardinal in HOD from Θ onwards is an aleph in V.*

Proof : let μ be a counterexample, of which the cardinal in V is κ. Let $A \subseteq \kappa \times \kappa$ code μ. A is ordinal definable from some real a. But then $A \in HOD[G_a]$ a generic extension of HOD by an algebra with the Θ chain condition, so μ should still be a cardinal. Contradiction. ⊣ (6·26)

6·27 REMARK McAloon's techniques easily furnish a model where the conclusion is false, so some hypothesis is necessary.

6·28 PROPOSITION *There is an injection in HOD of Θ into $\mathbb{K}^1$.*

Proof : from the fact that there is for each $\nu < \Theta$ an OD pre-well-ordering of length ν. ⊣ (6·28)

6·29 PROPOSITION ($V = L(\mathbb{R}; S)$) *There is an* OD *enumeration of the* OD *subsets of $\mathcal{P}\omega$ in order type Θ.*

Proof the First: from the fact, proved in Chapter I, that every OD set of reals has an OD-name in J_Θ. ⊣ (6·29)

Proof the Second: every set of reals is in $J_\Theta(\mathbb{R})$ so that if we enumerate $OD \cap \mathcal{P}\mathcal{P}\omega \cap J_\nu(\mathbb{R})$ for each ν that must be of length less than Θ; putting them together inside HOD, we get an enumeration of length at most and therefore exactly Θ. ⊣ (6·29)

Proof the Third: there is an enumeration in HOD of all the names in length some aleph at least Θ; but cardinals are absolute above Θ. So it must be of length Θ. NOT SATISFACTORY: how do we know that externally there are Θ names ?

6·30 REMARK By McAloon, we may suppose that $V = HOD$, $\mathfrak{c} = \aleph_1$, so that $\Theta = \aleph_2$, and that $2^{\aleph_1} = \aleph_3$. In such a universe, there are more than Θ OD subsets of $\mathcal{P}\omega$.

If it only takes one real to construct the universe, then that real is generic over HOD_S for a $< \Theta$-c.c. algebra; so cardinals are absolute from Θ onwards.

Does my more general argument work ? any code of an ordinal is definable from ordinals, S and a: is it still the case that $HOD_{S,a} = HOD_S[G_a]$?

In Chapter VI, as it now stands, we have $V = L[S, x]$, plus an assumption that $2^{\mathfrak{c}} = \mathfrak{c}^+$. Can we do with less ?

7: Thoughts on algebras

7·0 DEFINITION For any ordinal δ we write $\mathrm{coll}(\omega, < \delta)$ for the partial ordering with elements finite functions .... and $\mathrm{Coll}(\omega, < \delta)$ for $\mathrm{r.o.coll}(\omega, < \delta)$, the associated complete Boolean algebra.

7·1 PROPOSITION *For any limit ordinal δ, coll$(\omega, < \delta)$ is the union of coll$(\omega, < \nu)$ for $\nu < \delta$.*

7·2 PROPOSITION *Let κ be regular. Then the algebra Coll$(\omega, < \kappa)$ has the $< \kappa$ chain condition and is the union of the algebras Coll$(\omega, < \delta)$ for $\delta < \kappa$.*

Proof : The first part by a Δ-system argument. For the second, the sup of any subset of that union is the sup over a set of size less than κ and therefore is already present. Thus the union is complete. ⊣ (7·2)

7·3 DEFINITION A *smooth chain* of cBas is one such that $\mathbb{B}_0 = 2$, $\mathbb{B}_\nu \lhd \mathbb{B}_{\nu+1}$ and at limits $\bigcup_{\nu<\lambda} \mathbb{B}_\nu$ is dense in $\mathbb{B}_\lambda$.

7·4 PROPOSITION *For κ regular, the algebra Coll$(\omega, < \kappa)$ is the union of a smooth chain.*

7·5 PROPOSITION *Let $\mathbb{C}$ be a cBa which makes $\kappa = \omega_1$, and let $\mathbb{A}$ be a cBa of cardinality λ where 2^λ is less than κ. Then there is a complete embedding of $\mathbb{A}$ into $\mathbb{C}$.*

Proof : in $V^{\mathbb{C}}$ we may define a generic $\dot{H}$ for $\mathbb{A}$, by the usual Rasiowa–Sikorski definition. We use a generic permutation of λ to ensure that each non-zero element of $\mathbb{A}$ has a non-zero chance of getting into $\dot{H}$. Then we define

$$\pi(a) = [\![\hat{a} \in \dot{H}]\!]^{\mathbb{C}}. \qquad ⊣ (7·5)$$

Now we want to establish the amalgamation property.

7·6 PROPOSITION *Let $\mathbb{C}$ be a cBa which makes $\kappa = \omega_1$, and let $\mathbb{D}$ be a cBa of cardinality λ where 2^λ is less than κ. Suppose that $\mathbb{A}$ is completely embeddable in both $\mathbb{C}$ and $\mathbb{D}$. Then there is a complete embedding of $\mathbb{D}$ into $\mathbb{C}$ that makes the natural diagram commute.*

Proof : in $V^{\mathbb{A}}$ we factor $\mathbb{D}$ by the filter $F_{\mathbb{A},D}$ in $\mathbb{D}$ generated by the image under one embedding of the generic for $\mathbb{A}$; and we factor $\mathbb{C}$ by the filter $F_{\mathbb{A},C}$ generated by the image under the other embedding of the same generic.

Both $\mathbb{D}/F_{\mathbb{A},D}$ and $\mathbb{C}/F_{\mathbb{A},C}$ are complete in $V^{\mathbb{A}}$. We now apply the previous proposition in $V^{\mathbb{A}}$ to embed $\mathbb{D}/F_{\mathbb{A},D}$ into $\mathbb{C}/F_{\mathbb{A},C}$: we should say something about the condition on cardinals being preserved (which will be easily true in the case when κ is strongly inaccessible) and then general nonsense about composition and pull-backs of embeddings completes the proof. ⊣ (7·6)

7·7 THEOREM *Let κ be strongly inaccessible. Let $\mathbb{B}$ be an algebra which is the union of a smooth chain of cBas of size less than κ, with the $< \kappa$ chain condition, and which makes $\kappa = \omega_1$. Let $\mathbb{C}$ be Coll$(\omega, < \kappa)$. Then $\mathbb{B}$ and $\mathbb{C}$ are isomorphic; indeed any isomorphism between small subalgebras extends to an isomorphism of the two.*

Proof : back and forth, using the amalgamation property which both enjoy. Start from the given isomorphism between two small subalgebras; at every stage we shall have an isomorphism between two small subalgebras; at even stages look for the next algebra in the smooth chain on one side containing the current subalgebra, and use amalgamation to extend the current isomorphism to an embedding of that next algebra; at odd stages do it the other way round. At limits take unions. The union of the partial isomorphisms will be an isomorphism between the two unions. They will be Boolean algebras, though not necessarily complete; but if a subset on one side has a sup already existing on that side, the image of that sup will be the sup of the images on the other side. Then take completions on both sides.

At the final stage just take unions. ⊣ (7·7)