1. Countable Ordinals, Proof theory and Fast-growing Functions . . . .

Thomas Forster

A good point of departure for this essay would be Exercise 10 on Sheet 4 of Prof Johnstone's Part II set Theory and Logic course in 2012/3, `https://www.dpmms.cam.ac.uk/study/II/Logic/` in which the student is invited to show how, for every countable ordinal, a subset of $\Re$ can be found that is wellordered to that length in the inherited order. The obvious way to do this involves induction on countable ordinals and leads swiftly to the discovery of fundamental sequences. These can be put to work immediately in the definition of hierarchies of fast-growing functions. This leads in turn to the Schmidt conditions, which the student should explain carefully. There is a wealth of material on how proofs of totality for the faster-growing functions in this hierarchy have significant—indeed *calibratable*—consistency strength. One thinks of Goodstein's function and Con(PA), or of Paris-Harrington. There is plenty here from which the student can choose what to cover.
The Doner-Tarski hierarchy of functions (addition, multiplication, exponentiation . . .) invites a transfinite generalisation and supports a generalisation of Cantor Normal form for ordinals. Nevertheless, the endeavour to notate ordinals beyond ϵ_0 does not use those ideas, but rather the enumeration of fixed points: such is the *Veblen hierarchy*. From this one is led to the impredicative Bachmann notation, with Ω and the ϑ function.
Currently it is my intention to rerun (in Lent term?) the reading group on ordinals that ran in Easter term 2014. Any student attempting this essay would be well advised to sign up for it.

Relevant Courses

Essential:
Part II Set Theory and Logic

References

Schwichtenberg-Wainer Proofs and Computations Cambridge University Press
`http://www.cambridge.org/us/academic/subjects/mathematics/logic-categories-and-sets/proofs-and-computations`
Diana Schmidt
`http://www.math.ucsb.edu/~doner/articles/Doner-Tarski.pdf`
`http://www.dpmms.cam.ac.uk/~tf/cam_only/fundamentalsequence.pdf`
`http://www.dpmms.cam.ac.uk/~tf/cam_only/TMStalk.pdf`