CHAPTER III: FUNDAMENTAL CONSEQUENCES OF AD

THE TIME HAS COME to introduce the concept of a game and to examine the immediate consequences of the Axiom of Determinacy, which is the assumption that all games of a certain natural kind are determined. Among the most important of these consequences is the *Coding Lemma* of Moschovakis, which we shall prove in various forms. We apply the Uniform form of it to give a good coding of functions which we shall use in Chapter V, and apply it in this to establish the existence under AD of cardinals with a strong partition property.

1: Gaming Trees

We review the general context of games that we shall use.

Our games will be played in trees of a certain kind. As this particular notion varies slightly from other notions of trees we might consider, we shall call this sort a gaming tree.

1·0 DEFINITION A *gaming tree* is a non-empty set T of finite sequences closed under initial segments and such that each element of T has at least one proper extension. The nature of the elements of the sequences is not specified. We write $s\,\mathbf{in}\,t$ to mean that s is a proper initial segment of t. By a *subtree* of such a tree we shall mean a subset which is also a gaming tree. For example, the tree

$$T_s =_{df} \{t \in T \mid s\,\mathbf{in}\,t \text{ or } s = t \text{ or } t\,\mathbf{in}\,s\}$$

is a subtree of T for each $s \in T$.

If $t \in T$, a *move at* t is any x such that the concatenation $t^\frown x$ is in T and of length one more than t. A *path* through T is any function f with domain ω such that each $f\restriction n$ is in T. The set of paths through T is denoted by $[T]$.

We are interested in *games* in the tree T played between two players, whom we call Adam and Eve.[N 1] They play alternately, Adam generally moving first. We shall need the notions of *strategy* and *policy*. Informally, a *strategy* for one of the players is a function selecting a move for him at each position, and a *policy* is a function supplying at each position a non-empty set of possible moves. One may refine any given policy to a strategy if the underlying tree has a well-ordering or if the Axiom of Dependent Choice is available.

Formally, the definitions are such that strategies and policies are subtrees of the original tree. Thus

1·1 DEFINITION a *policy for Adam in* T is a subtree S of T such that for each $s \in S$ of odd length — so that it is Eve's turn to move — each move in T at s is also a move in S at s.

a *strategy for Adam in* T is a policy for him in T such that at each $s \in S$ of even length — so that it is his turn to move — there is exactly one move in S.

a *policy for Eve in* T is a subtree S of T such that for each $s \in S$ of even length — so that it is Adam's turn to move — each move in T at s is also a move in S at s.

a *strategy for Eve in* T is a policy for her in T such that at each $s \in S$ of odd length — so that it is her turn — there is exactly one move in S.

However, it is sometimes convenient to consider a strategy to be not a subtree of the tree of play but a function defined on positions and furnishing moves.

1·2 DEFINITION Let $A \subseteq [T]$. A policy S for Adam in T is said to be a *winning policy for Adam in the game* $G(A,T)$ if $[S] \subseteq A$. Similarly a policy S for Eve in T is said to be a *winning policy for Eve in the game* $G(A,T)$ if $[S] \cap A$ is empty.

$G(A,T)$ is called *determined* if one of the players has a winning strategy for $G(A,T)$.

[N 1] Naming the players has a purpose: in Chapter VI we shall consider games where Adam chooses another game and decides whether to play first or second in it.

1·3 DEFINITION The *Axiom of Determinacy*, *AD*, is the hypothesis that for the particular gaming tree $T = {}^{<\omega}\omega$, $G(A,T)$ is determined for every $A \subseteq [T]$.

1·4 REMARK Informally one speaks of a set A being determined when one should with more accuracy speak of a game $G(A,T)$ being determined. The custom is not without danger: the reader may like as an exercise to assume that there are undetermined games in the tree $T = {}^{<\omega}\omega$ and to find A, B, S, with S a sub-gaming tree of T, $A \cup B \subseteq [S]$, $G(A,S)$ and $G(B,T)$ determined and $G(A,T)$ and $G(B,S)$ not determined.

1·5 CONVENTION We shall customarily use the following notation. We call a player's infinite sequence of moves during a run of a game his *play*, and use α for Adam's play and ε for Eve's play. The sequence $(\alpha(0), \varepsilon(0), \alpha(1), \varepsilon(1), \ldots$ of all moves during the run is called the *outcome* of the game, and will often be denoted by γ. We use the letter σ for a strategy for the first player, and τ for a strategy for the second player.

We sometimes use Moschovakis' notation, who writes $\sigma \star [\varepsilon]$ to denote the first player's play when he uses strategy σ against the play ε of his opponent, and $[\alpha] \star \tau$ for the second player's play using strategy τ against the play of α by the first player. With our convention on letters, our notation would also permit us to write $\alpha(\sigma, \varepsilon)$ or $\varepsilon(\alpha, \tau)$ in those two cases. We shall not, however, follow Moschovakis when he writes $\sigma \star \tau$ for the outcome when the first player uses strategy σ and the second uses strategy τ, but in such a case would write $\gamma(\sigma, \tau)$.

Actually in the course of the book we shall consider many different games, and will find it convenient to preserve a certain flexibility in our ways of presenting them.

AD is known to imply some of the consequences of the axiom of choice:

1·6 PROPOSITION (Mycielski ?[R 1]) *AD implies $AC_{\aleph_0,\mathcal{N}}$, the Axiom of Choice for countable families of subsets of $\mathcal{N}$.*

Proof : Let $\langle A_n \mid n \in \omega \rangle$ be a family of non-empty subsets of $\mathcal{N}$. Consider the game in which Adam plays the sequence a, Eve the sequence e, and Eve wins iff $e \in A_{a(0)}$. Eve can defeat any putative strategy of Adam; so by *AD* she has a winning strategy, τ say. Let c_n be the sequence with constant value n. Then the map $n \mapsto [c_n] \star \tau$ is a choice function for the family. ⊣ (1·6)

1·7 COROLLARY *AD implies that ω_1 is regular.*

Proof : the weak form of Choice that we have just established is sufficient to prove that the union of a countable set of countable ordinals is countable. ⊣ (1·7)

1·8 EXERCISE Show that $AC_{\aleph_0,\mathcal{N}}$ implies that every infinite subset of $\mathcal{N}$ has a coutably infinite subset.

On the other hand, *AD* is known to contradict *AC* in numerous ways. For example, one may easily by transfinite induction using a well-ordering of $\mathcal{N}$ build A and B that form counterexamples to the dichotomy expressed in the following striking application of *AD*:

1·9 PROPOSITION (Wadge) *Assume AD. Let A and B be subsets of $\mathcal{N}$. Then either B is continuously reducible to A or A is to $\neg B$.*

Proof : Consider the game where Adam builds α in his play, Eve builds ε, and Adam wins iff $\alpha \in A \iff \varepsilon \in B$.

If σ is a winning strategy for Adam, then

$$\forall \varepsilon\ \varepsilon \in B \iff \sigma \star [\varepsilon] \in A$$

and B is continuously reducible to A; if on the other hand τ is a winning strategy for Eve, then

$$\forall \alpha\ \alpha \in A \iff [\alpha] \star \tau \notin B$$

and A is continuously reducible to $\neg B$. ⊣ (1·9)

The continuity is of a strong kind, and is known as *Lipshitz reducibility*: in symbols, we have proved that $A \leqslant_\ell \neg B$ or $B \leqslant_\ell A$.

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Here is another application of Wadge's Lemma: call a pointclass Γ a *bold-face class* if it contains all clopen ($\mathbf{\Delta}^0_1$) sets and is closed under continuous pre-images.

1·10 PROPOSITION *(AD) A bold-face pointclass has universal sets iff it is not closed under complements.*

Proof : first note that if $U(\cdot,\cdot)$ is universal for Γ, then $\{x \mid \neg U(x,x)\}$ is not in Γ, but $\{x \mid U(x,x)\}$ is.

Now let $A \in \Gamma$ but $\neg A \notin \Gamma$. Let B be any set in Γ: by Wadge, either $B \leq_\ell A$ or $A \leq_\ell \neg B$; but the second alternative would imply that $\neg A \leq_W B$ and hence $\neg A \in \Gamma$, contrary to assumption. So each B in Γ is Lipshitz reducible to A.

Put

$$C = \{(x,\sigma) | \sigma \star [x] \in A\}$$

Then C is in Γ and each B in Γ is of the form C_σ for some σ. $\dashv$ (1·10)

1·11 HISTORICAL NOTE Wadge's lemma is in his thesis [R2]. For more on this subject see [R3] [R4] [R5].

Of particular interest to us is the fact that *AD* implies that all ultrafilters on ω are principal. To prove that, we define a particular kind of colouring which changes rapidly:

1·12 DEFINITION Let $k \in \omega$. A *k-chameleon* is a colouring $\pi : [\omega]^\omega \longrightarrow k$ of the set of infinite subsets of ω by k colours, with the property that for $n \in X \in [\omega]^\omega$, $\pi(X) \neq \pi(X \setminus \{n\})$.

1·13 PROPOSITION *Suppose that there is a non-principal ultrafilter U on ω. Then for each $k \geqslant 2$ there is a k-chameleon.*

Proof : Fix such k. For X an infinite subset of ω, enumerate it in increasing order as x_i; let s_0 be the closed interval $[0, x_0]$ of all natural numbers up to and including x_0 and thereafter let s_{i+1} be the half-open interval $(x_i, x_{i+1}]$. For $0 \leqslant j < k$ let X^k_j be $\bigcup_{n\in\omega} s_{kn+j}$. Then ω being the disjoint union of the k infinite sets $X^k_0, X^k_1, \ldots X^k_{k-1}$, exactly one of them, say X^k_j, is in U. We define $\pi^U_k(X)$ to be that j. $\dashv$ (1·13)

1·14 REMARK The chameleons just constructed have the further property that as one successively removes points from a set X the colours change in a fixed cycle: one might call them *cyclic* chameleons. Of course every 2-chameleon is cyclic.

1·15 PROPOSITION (AD) *There is no* 2*-chameleon.*

Proof : Let π be a 2-chameleon. We consider the following game, played in the tree of strictly increasing finite sequences of natural numbers. Let α be Adam's play, and ε Eve's. The image of α is an infinite set of integers, which we also denote by α.

Rule: Adam wins if $\pi(\alpha) = \pi(\varepsilon)$.

Let σ be a strategy for Adam. We describe two runs of the game played against σ which cannot both be wins for Adam.

Let a_0 be $\sigma(\emptyset)$, which will be Adam's opening move in both runs. In the first run, Eve responds by playing $a_0 + 1$; let $a_1 = \sigma(a_0, a_0 + 1)$. Eve plays a_1, in the second run; let $a_2 = \sigma(a_0, a_1)$. She now plays a_2 as her second move in the first run. When Adam replies with a_3, she uses that as her next move in the second run; his reply to that will be used as her next move in the first run, and so on. The following diagram illustrates the play.

$$\begin{array}{ccccc}
\textit{First} & \textit{Run} & & \textit{Second} & \textit{Run} \\
& & & & \\
\sigma & & & \sigma & \\
a_0 & a_0+1 & (\leftarrow) & a_0 & a_1 \\
a_1 & a_2 & \leftarrow & a_2 & a_3 \\
a_3 & a_4 & \leftarrow & a_4 & a_5 \\
& & & & \\
A & E & & B & F
\end{array}$$

[R2] Wadge
[R3] Louveau
[R4] Kechris
[R5] Moschovakis

We write A, B for the plays of Adam in the two runs following σ, and E, F for the corresponding plays of Eve.

Note that $A = \{a_0\} \cup F$, so that $\pi(A) \neq \pi(F)$, whereas, writing $C = B \smallsetminus \{a_0 + 1\}$, $E = C \cup \{a_0\}$, so that $\pi(E) = 1 - \pi(C) = \pi(B)$. Hence Adam cannot have won both runs, and therefore σ cannot be a winning strategy for him.

A similar argument demolishes Eve's hopes. Here is the corresponding diagram, where τ is a candidate for a winning strategy of Eve.

$$\begin{array}{cccccc} \textit{First} & \textit{Run} & & \textit{Second} & \textit{Run} \\ & & & & \\ & \tau & & & \tau \\ 0 & a_0 & \longrightarrow & a_0 & a_1 \\ a_1 & a_2 & \longrightarrow & a_2 & a_3 \\ a_3 & a_4 & \longrightarrow & a_4 & a_5 \\ & & & & \\ A & E & & B & F \end{array}$$

Here Adam plays 0 for his first move in the first run against τ. At each round he now copies Eve's moves in the first run as his move in the second run, and her response in the second run as his next move in the first run. Thus $E = B$ and $A = \{0\} \cup F$, so that $\pi(E) = \pi(B)$ and $\pi(A) \neq \pi(F)$. Eve thus cannot have won both runs, and as a winning strategy τ has failed. $\dashv$ (1·15)

1·16 COROLLARY (AD) *There is no non-principal ultrafilter on ω: hence any ultrafilter on any set will be countably complete.*

Proof : if not, we can arrange a failure of countable completeness of an ultrafilter $\mathcal{U}$ on I to yield a partition of I into disjoint sets X_n for $n \in \omega$, none in $\mathcal{U}$, and then obtain a free ultrafilter on ω by defining

$$\mathcal{V} = \Big\{ A \subseteq \omega \;\Big|\; \bigcup_{n \in A} X_n \in \mathcal{U} \Big\} \qquad \dashv (1·16)$$

1·17 HISTORICAL NOTE The above remarks draw on ideas from many people, among them Scott Johnson and Jané.

1·18 REMARK Cyclic k-chameleons can also be constructed using AC_k, the Axiom of Choice for families of sets of size k.

Open Determinacy

1·19 We may topologise the set of paths through a general gaming tree T by taking as basic open sets collections of the form

$$N_s^T =_{\rm df} \{f \in [T] \mid f \restriction lh(t) = s\}.$$

Where the tree T is clear from the context, we may write that set more simply as N_t. As usual, the *Borel sets* are those generated from the open sets by operations of countable union and intersection.

1·20 REMARK In Lévy's model mentioned in Chapter II, in which the real line, and therefore every set of reals, is a countable union of countable sets. it might be argued that every set of reals is Borel. However, AD fails in this model, ω_1 is singular there, and therefore Borel determinacy (in this degenerate sense of Borel) fails too. Thus Martin's celebrated result that all Borel games are determined requires some form of the axiom of choice: for games on ω, $AC_{\aleph_0, \mathcal{N}}$ is sufficient for then each Borel set $A \subseteq \mathcal{N}$ has a Borel code, α say, and a strategy for A can be built within $L[\alpha]$, where of course full AC is available, and then proved to be equally successful as a strategy in the universe.

Let us now prove a version of the *Gale-Stewart theorem* that all open games with respect to the above topology are determined.

1·21 DEFINITION Let T be a gaming tree, and X any subset of T closed under end-extension. We define a sequence of subsets B_ν of T by recursion on the ordinals thus:

$$\begin{aligned} & B_0 = X \\ \text{for } \nu > 0 \qquad & B_\nu = \{s \in T \mid lh(s) \text{ is even and for some move } x \text{ at } s,\ s^\frown x \in \bigcup_{\theta<\nu} B_\theta \\ & \qquad\qquad \text{or } lh(s) \text{ is odd and for all moves } x \text{ at } s,\ s^\frown x \in \bigcup_{\theta<\nu} B_\theta\}. \end{aligned}$$

and then define a rank function $\rho_{T,X}(s) : T \to ON \cup \{\infty\}$ by setting

$$\rho_{T,X}(s) \ =_{\mathrm{df}} \begin{cases} \text{the least } \nu \text{ with } s \in B_\nu & \text{if such } \nu \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

A feature of this definition is that if a position s has rank 0, it is because of the past; but if it has rank between 0 and ∞ it is because of the future.

1·22 LEMMA *If $\rho_{T,X}(s) = \infty$ and $lh(s)$ is even, then for every move x at s, $\rho_{T,S}(s^\frown x) = \infty$; if $\rho_{T,X}(s) = \infty$ and $lh(s)$ is odd, then for some move x at s, $\rho_{T,S}(s^\frown x) = \infty$.*

Proof: from the above definitions by contraposition. ⊣

Hence the winner is determined by the rank of the starting position, the empty sequence $\emptyset$:

1·23 PROPOSITION *If $\rho_{T,X}(\emptyset) < \infty$, there is a policy for the first player to reach a position s with $\rho_{T,X}(s) = 0$; if $\rho_{T,X}(\emptyset) = \infty$, there is a policy for the second player to play so that for all s, $\rho_{T,X}(s) = \infty$.*

Proof: in the first case, Adam has the *rank-reducing policy*

$$\{s \mid \forall n\ 2n+1 < lh(s)\ \&\ \rho(s \restriction 2n) > 0 \implies \rho(s \restriction (2n+1)) < \rho(s \restriction 2n)\};$$

in the second case, Eve has the *stay-on-top policy*

$$\{s \mid \forall n\ 2n+2 < lh(s)\ \&\ \rho(s \restriction (2n+1)) = \infty \implies \rho(s \restriction (2n+2)) = \infty\}.$$

1·24 LEMMA *If $N^T_s \subseteq \bigcup\{N^T_t \mid t \in X\}$, then $\rho_{T,X}(s) < \infty$.*

Proof: If not, play a game from s with the second player following the "stay-on-top" policy. The outcome will be a path $f \in [T]$ such that for all n, $\rho_X(f \restriction n) = \infty$; but $f \in N_s$ and so in some N_t with $t \in X$; for $n = lh(t)$, $\rho_X(f \restriction n) = 0$, a contradiction. ⊣

1·25 LEMMA *Let X and Y be subsets of T. If for all s, $\rho_X(s) = 0$ always implies $\rho_Y(s) < \infty$, then for all s, $\rho_X(s) < \infty$ always implies $\rho_Y(s) < \infty$.*

Proof: by induction on $\rho_X(s)$.

$\rho_X(s) = 0$: by hypothesis, $\rho_Y(s) < \infty$.

$\rho_X(s) = \xi > 0$:

if $lh(s)$ is even, then for some x, $\rho_X(s^\frown x) < \rho_X(s)$, so $\rho_Y(s^\frown x) < \infty$, so $\rho_Y(s) < \infty$;

if $lh(s)$ is odd, then for all x, $\rho_X(s^\frown x) < \rho_X(s)$, so for all x, $\rho_Y(s^\frown x) < \infty$; by the axiom of replacement, there is a ν such that for all x, $\rho_Y(s^\frown x) < \nu$, and so $\rho_Y(s) \le \nu$. ⊣

1·26 COROLLARY *If $\bigcup\{N_s \mid s \in X\} = \bigcup\{N_s \mid s \in Y\}$, then for all s, $\rho_X(s) < \infty$ iff $\rho_Y(s) < \infty$.*

Thus the rank-reducing policy for a given open set may vary according to the particular cover by basic open sets chosen; but the stay-on-top policy is unique.

It will be useful to distinguish between *winning* and *won* positions. In a game in which the first player is trying to reduce rank to 0, a position with rank $< \infty$ is *winning for the first player* and a position with rank $= \infty$ is *winning for the second player*. This notion is evidently independent of the particular policy chosen. A position is *won* for the first player if it is of rank 0: this notion depends on the particular policy and rank function being followed. Only in degenerate cases is a position *won* for the second player.

In particular, it follows from Lemma 1.8 above that any t with $N_t \subseteq \bigcup\{N_s \mid s \in X\}$ is winning.

1·27 Remark We have considered open games above: a similar analysis will work for closed games, where we must interchange "even" and "odd" in the relevant definitions.

We summarise the above discussion of strategies derived from particular covers of an open set by basic open sets in:

1·28 Special Cover *Let T be a gaming tree, X a subset of T closed under lengthening, $C = \bigcup\{N^T_t \mid t \in X\}$ and $B = [T] \setminus C$. If Adam has a winning strategy in $G(C,T)$, then he has one which in a finite time reaches a position t in X. If Eve has a winning strategy in $G(B,T)$ then she has one which in a finite time reaches a position t in X.*

The following corollary of our discussion would be needed in a proof of Borel determinacy.

1·29 Hands Tied *Suppose D is an open subset of $[T]$, that P is a policy for Adam, and that in the game $G(D \cap [P], P)$ Adam has a winning policy, Q. Then he can play within Q to reach in finite time a won position in $G(D,T)$.*

Proof : Take $X = \{s \in Q \mid N^T_s \subseteq D\}$. Since $[Q] \subseteq D$, and D is open in $[T]$, every path through Q has a finite initial segment in X: *i.e.* $[Q] = \bigcup\{N^Q_t \mid t \in X\}$. Thus Adam may fulfil the Lemma by using $\pi_{Q,X}$. ⊣

Comparison of plays in different models of the same open game

Suppose that $M \subseteq N$ are transitive models, and that $T \in M$ is in M a gaming tree. Then it will also be one in N. Let $\mathcal{X} \in M$ be a subset of T closed under lengthening, and defining the open set $A^M =_{\rm df} \bigcup_{s\in\mathcal{X}}(N^T_s)^M$ in M, and a corresponding set A^N in N: so that $A^M = M \cap A^N$. We wish to compare the game $G(A^M,T)$ played in M with the game $G(A^N,T)$ played in N. The remarks to be made being straightforward, we state them as a series of exercises.

1·30 Exercise Let ϱ^M and ϱ^N be the rank function derived from T and $\mathcal{X}$ defined in M and in N respectively. Show that for each $s \in T$, $\varrho^M(s) = \varrho^N(s)$.

We may thus write ϱ without a superscript without fear of ambiguity.

1·31 Exercise Show that $(\varrho(\emptyset))^M \iff (\varrho(\emptyset))^N$.

1·32 Exercise Show that the rank-reducing policy and the stay-on-top policy are the same in both models.

The larger model might contain more winning strategies for the winning player. However,

1·33 Exercise show that whoever has a winning strategy for the game in one model has a winning strategy in the other, and indeed that there then is a winning strategy in the smaller model which is also a winning strategy in the larger model.

The equivalence of three definitions of ∞-Borel set

Proof that every wff-Borel set is game-Borel: We are given a wff-Borel description $\{x \mid L_\theta[S,x] \models \varphi(S,x)\}$. S is a set of ordinals, say a subset of θ. We play a closed game.

We pick a large ordinal η which reflects everything for all reals x.

We associate to each real x a game G_x in which the first player has to define a model of the form $L_\zeta[S^*;x]$ and simultaneously define an elementary embedding ψ of $L_\zeta[S^*]$ into $L_\eta[S]$: note the omission of x. Further, the formula $\varphi(S^*,x)$ must be true in his model of the real x.

The second player plays ordinals less than η which the first player must include in the image of ψ.

We must arrange the definition of the game so that it is a closed game; which we shall do by organising the rules so that the first player aims to define a complete theory with Skolem functions.

Assume for the moment that we have done that: then the game G_x will determined, both in the present universe and in any extension of it, with the same player winning. We shall show that for each real x in the ground model, the first player has a winning strategy in G_x if and only if $\varphi(S,x)$ is true in $L_\eta(S,x)$. We move to a generic extension where η has become countable. The moves are the same. It is still a closed game. So the winner is the same.

Suppose first that the first player has a winning strategy σ in the ground model, of the "stay-on-top" variety. Then σ is still a winning strategy in the generic extension. Consider the run when the second player lists all ordinals less than η. Then the embedding ψ is actually a bijection, so that the first player has built a copy of $L_\eta(S,x)$ in which $\varphi(S,x)$ is true.

Conversely suppose now that $L_\eta(S,x) \models \varphi[S,x]$. Let τ be a strategy for the second player in the ground model, and use it in the extension. This time, the first player sets out consciously to build a copy of $L_\eta(S,x)$, and he need pay little heed to τ, beyond noting the order in which he might be required to include ordinals. At the end of the game, the first player has won, and therefore τ has failed. Consequently, the second player has no winning strategy, and therefore as G_x is determined, the first player has one.

Thus the function of requiring Player I to define an elementary embedding from $L(S^*)$ to $L(S)$ is to enable the opponent of a strategy to make the model large, or at least well-founded.

Before giving the details of the game, let us remark that in our sketch, Player I is doing two almost independent things: he is defining a set S^* and a complete consistent theory extending (say) $KP + MAC + V = L(S^*,x) + \varphi(S^*,x)$; and he is defining an embedding of $L(S^*)$ into $L(S)$. Either might prove an impossible task. In doing the first, we have a constant for x, and gradually feed in the diagram of x. Should we at any time find we have written an inconsistent theory, we have only used finitely much information about x. If disaster strikes, the first player has hitherto only used partial information about x. The same play of ordinals in a game with another real y which agrees with x so far would be equally disastrous. The second task is, apart from the formation of S^*, independent of x, because it requires an embedding only of $L(S^*)$ and not of $L(S^*,x)$.

That is why the game will prove to be of the stated kind: G_x depends continuously on x to the extent that should the open player lose, he would lose all games G_y with y agreeing with x for a certain finite number of terms.

Models of the form $L(S^*,x)$ have very well organised Skolem functions: see Devlin []. Player I aims to define a complete theory with Skolem functions, and constants c_n

He aims to define a consistent complete extension of the theory $ZFC_N + V = L(\dot S^*, \dot x) + \dot S^*$ `is a set of ordinals` $+\ \dot x : \dot\omega \dot\longrightarrow \dot\omega + \varphi(\dot S^*, \dot x)$. Let us begin with a convention that c_{8n} is to represent the number n; that c_4 represents ω; that c_{8n+4} represents an infinite ordinal; that c_2 represents S^*, about which at the outset we know little; that c_{4n+6} represents a member of $L(S^*)$ in the sense of the model; that c_1 represents x and that c_{2n+3} represents a member of $L(S^*,x)$: should it prove that x is constructible from S^*, these last constants will have to be set equal to earlier ones.

We must make a list of all the formulæ in a theory with these constants, and with terms for Skolem functions — though we can always take those in the form "`the` $<_{S^*,x}$`-first object such that ...`" — and our theory will satisfy a witness condition: whenever we decide to add an existential statement to our theory, we must within a stated time declare a witness for that statement.

Then we begin. Moves by player II will simply be ordinals less than η. Moves by player I will be of various kinds: at times he will decide whether $c_i E c_j$ or not; at times he will decide whether an existential statement is to be added to his theory; at times he will pick a witness for that existential statement; at times he will respond to a move ζ_n by player II with a promise that it will be the image of constant c_k under the embedding, k being some multiple of 4: here k is allowed to be some constant that has already been discussed, but player I might like to buy time by choosing k much larger than the stage of the game; at times he will decide whether a constant denoting an ordinal is to be a member of S^* or not; at times he will extend the definition of the embedding by declaring an element e_n of $L_\eta(S)$ to be the image of c_{2n}; at times he will decide to add statements of the form $\dot x(c_{8n}) = c_{8m}$, by referring to the diagram of x.

However, we must be strict with player I: every time he promises to do something, a time limit must be set for the fulfilment of that promise, so that if he breaks the promise, we know at some finite time that he has done so. By such strictness we can be sure that if he lasts the entire game without mistake, then he has indeed defined a complete consistent theory, and has indeed defined an elementary embedding, so that his model is well-founded. Thus we require a schedule so that player I attends to all his tasks in an orderly manner at predictable times.

Every element of a model of the form $J_\eta(S)$ is of the form $h(J_{\nu_0}(S), J_{\nu_1}(S), \ldots J_{\nu_k}(S))$ for some rudimentary function h and ordinals ν_i less than η. We can use that fact if we wish to streamline the proof; it will be convenient to introduce further constants d_n to denote $J_{c_{4n}}(S^*)$.

At each round we see whether Player I has failed in any of the following ways: his theory has become inconsistent; he has failed to establish a witness to an existential statement; his map might violate the principle that $h(d_m, d_n)Eh(d_k, d_j) \iff h(J_{e_{2m}}(S)) \in h(J_{e_{2k}}(S))$ — he might violate the principle that an ordinal is to go into S^* if and only if its image is in S; he might violate the diagram of x^*.

If ever he does so, he is immediately declared to lose. If he is never declared to lose, then he has indeed won.

Proof that every game-Borel set is tree-Borel: we define a tree which reflects the computation of the ranks in the open game. Here are the details, where we use s for an initial segment of a possible α and u for a finite sequence of ordinals: let $\mathcal{X} = \{(s,\, u) \mid \neg R(s, u)\}$. Define

$$B_0^\alpha = \{u \mid (\alpha \restriction \ell h(u), u) \in \mathcal{X}\}$$

for $\nu > 0$
$$B_\nu^\alpha = \{s \in T \mid lh(s) \text{ is even and for some move } x \text{ at } s,\ s^\frown x \in \bigcup_{\theta<\nu} B_\theta^\alpha$$
$$\text{or } lh(s) \text{ is odd and for all moves } x \text{ at } s,\ s^\frown x \in \bigcup_{\theta<\nu} B_\theta^\alpha\}.$$

By replacement there is an ordinal κ such that for each α $B_\kappa^\alpha = B_{\kappa+1}^\alpha$. We may treat all ν as ranging over the ordinals less than κ.

Set $A_\nu^u = \{\alpha \mid u \in B_\nu^\alpha\}$, for each u and ν. Then A_0^u is clopen, and for $\nu > 0$, these equations hold:

$$A_\nu^u = \bigcup_\zeta \bigcup_{\xi<\nu} A_\xi^{u^\frown(\zeta)} \qquad \text{if } \ell h(u) \text{ even;}$$
$$A_\nu^u = \bigcap_\zeta \bigcup_{\xi<\nu} A_\xi^{u^\frown(\zeta)} \qquad \text{if } \ell h(u) \text{ odd.}$$

These equations permit a definition of the sets A_ν^u simultaneously for all u by induction on ν. Finally the game-Borel set $\alpha \mid \forall\zeta_0\exists\zeta_1 \ldots \forall n R(\alpha\restriction n, \vec{\zeta}\restriction n)\}$ equals $\mathcal{N} \setminus \bigcup_\nu A_\nu^\emptyset$, and thus tree-Borel.

We saw in Chapter II that every tree-Borel set is wff-Borel, and thus we have completed a proof of the equivalence of the three definitions. We should remark that in each case the transformation of a code of one type to a code of another type can be accomplished in the constructible closure of the first code.

1·34 Exercise Is the transformation always primitive recursive ?

The following three healthy exercises, which we recommend to all except ourselves, give the cycle of proofs in the reverse direction.

1·35 Exercise Guided by the principle that computation of the rank function is local, give a direct proof that every game-Borel set is wff-Borel.

1·36 Exercise Develop a method of associating games to ∞-Borel codes and thus give a direct proof that every tree-Borel set is game-Borel.

1·37 Exercise Fill in the details of the proof sketched in Chapter II that every wff-Borel set is tree-Borel.

2: Solovay's countably complete filters

Let A be an infinite subset of $\mathcal{N}$, and let $S \subseteq [A]^{\aleph_0}$, the set of countably infinite subsets of A.

2·0 DEFINITION The game $\mathcal{G}(A, S)$: Adam plays $a \in {}^\omega\omega$, which we interpret as a sequence a^n of reals by the formula

$$a^n(k) \ =_{\text{df}}\ a(k + (n+k)(n+k+1)/2);$$

Eve plays e, which we similarly interpret as a sequence e^n:

Rule 1: $a^0 \in A$;

Rule 2: $e^0 \in A$;

Rule 3: $a^1 \in A$;

$\vdots$

The first of those rules to be broken entails defeat for the corresponding player. If they are all kept, then a decision will be made under

Rule ω: Eve wins iff the set $\{a^0, e^0, a^1, \ldots\} \in S$.

2·1 REMARK Thus should the set $\{a^0, e^0, a^1, \ldots\}$ be finite, Eve loses.

2·2 DEFINITION $\mathfrak{W}(A) \ =_{\text{df}}\ \{S \subseteq [A]^{\aleph_0} \mid$ Eve has a winning strategy in $\mathcal{G}(A, S)\}$.

2·3 DEFINITION (Keisler) A filter $\mathcal{F}$ on $[A]^{\aleph_0}$ is *fine* if for each $a \in A$, $\{s \in [A]^{\aleph_0} \mid a \in s\} \in \mathcal{F}$.

2·4 REMARK If A is uncountable and we have $AC_{\aleph_0,\mathcal{N}}$, and $\mathcal{F}$ is fine, then that is certainly enough to guarantee that $\mathcal{F}$ contains no countable sets.

2·5 THEOREM (Solovay) *(i) if A has a countably infinite subset, $\mathfrak{W}(A)$ is a fine filter on $[A]^{\aleph_0}$.*

(ii) Assuming $AC_{\aleph_0,\mathcal{N}}$, $\mathfrak{W}(A)$ is a countably complete fine filter on $[A]^{\aleph_0}$.

(iii) Under AD, $\mathfrak{W}(A)$ is a countably complete fine ultrafilter on $[A]^{\aleph_0}$.

2·6 REMARK It might be thought that an upwards closed subset of $\mathcal{P}(X)$ containing for each $Y \subseteq X$ exactly one of Y and $X \setminus Y$ is necessarily an ultrafilter, but no ! $\{x \subseteq 7 \mid \overline{\overline{x}} \geq 4\}$ is a counter-example.

Proof of the Theorem: essentially by composing strategies. First it is plain that $\mathfrak{W}$ is upwards closed, that $[A]^{\aleph_0} \in \mathfrak{W}$ and that $\emptyset \notin \mathfrak{W}$. Then we shall show that if $X \in \mathfrak{W}$ and $Y \in \mathfrak{W}$ then $X \cap Y \in \mathfrak{W}$, and that $\mathfrak{W}$ is fine. We then extend the argument to show that under $AC_{\aleph_0,\mathcal{N}}$, $\mathfrak{W}$ is countably complete. Finally we show, assuming AD, that if $S \notin \mathfrak{W}$ then $[A]^{\aleph_0} \setminus S \in U$.

Suppose then that Eve has winning strategies τ_X and τ_Y in the games $G(A, X)$ and $G(A, Y)$ respectively. Let a be the sequence of Adam's moves in a play of $G(A, X \cap Y)$. Eve will answer this by the sequence e obtained as follows: we run on the side a play of G_X in which Adam plays a sequence b and Eve, following τ_X, responds with a sequence β; and also a play of G_Y in which Adam plays a sequence c to which Eve, following τ_Y, responds with a sequence γ. The sequences b, c and e are defined by the requirements that

$$\begin{array}{lll} b^{2k} = a^k, & c^{2k} = a^k, & e^{2k} = \beta^k, \\ b^{2k+1} = \gamma^k, & c^{2k+1} = \beta^k, & e^{2k+1} = \gamma^k \end{array}$$

This is a definition by simultaneous recursion which unfolds as the play of $G(A, X \cap Y)$ progresses. We leave it to the reader to verify that each value of β and γ is generated in a side game before it is required in the main game.

We must verify that each of the preliminary rules is observed by Eve. We shall show that for each ℓ, if each of $a^0, \ldots a^\ell$ is in A then so are each of $b^0, \ldots b^\ell$, $c^0, \ldots, c^\ell$, $\beta^0, \ldots \beta^\ell$, $\gamma^0, \ldots \gamma^\ell$, and finally $e^0, \ldots e^\ell$.

We shall handle the first four cases by hand, and then give an inductive argument, taking cases on the congruence class of ℓ modulo 4.

b^0 is a^0;

b^1 is γ^0, a response to $c^0 = a^0$;

b^2 is a^1;
b^3 is γ^1, a response to $c^1 = \beta^0$, itself a response to $b^0 = a^0$.

c^0 is a^0;
c^1 is β^0, a response to $b^0 = a^0$;
c^2 is a^1;
c^3 is β^1, a response to $b^1 = \gamma^0$, itself a response to $c^0 = a^0$.

b^{4k} is a^{2k};
b^{4k+1} is γ^{2k}, a response to $c^{2k} = a^k$;
b^{4k+2} is a^{2k+1};
b^{4k+3} is γ^{2k+1}, a response to $c^{2k+1} = \beta^k$, itself a response to b^k.

c^{4k} is a^{2k};
c^{4k+1} is β^{2k}, a response to $b^{2k} = a^k$;
c^{4k+2} is a^{2k+1};
c^{4k+3} is β^{2k+1}, a response to $b^{2k+1} = \gamma^k$, itself a response to c^k.

Thus if a_i is in A for each $i < n$, b_i and c_i, and therefore β_i and γ_i, are in A for each $i < 2n$.

e^{4k} is β^{2k}, a response to $b^{2k} = a^k$
e^{4k+1} is γ^{2k}, a response to $c^{2k} = a^k$;
e^{4k+2} is β^{2k+1}, a response to $b^{2k+1} = \gamma^k$, itself a response to c^k.
e^{4k+3} is γ^{2k+1}, a response to $c^{2k+1} = \beta^k$, itself a response to b^k.

Thus if a_i is in A for each $i < n$, e_i will be in A for each $i < 4n$.

The result of this dovetailing of strategies is that each of the three games produces the same countable set, namely the set $\{a^k, \beta^k, \gamma^k \mid k \in \omega\}$, for decision under Rule ω: and hence this set will be in $X \cap Y$. $\dashv$ (2·6)

To see that $\mathfrak{W}$ is fine, note that if $a \in A$ and A has a countably infinite subset, Eve has an easy strategy for winning the game that will show that $\{s \in [A]^{\aleph_0} \mid a \in s\} \in \mathfrak{W}$.

That every infinite $A \subseteq \mathcal{N}$ has a countably infinite subset follows from $AC_{\aleph_0,\mathcal{N}}$ was done by diligent readers as Exercise 1·8. To see that under $AC_{\aleph_0,\mathcal{N}}$ the filter $\mathfrak{W}$ is countably complete, we start by using $AC_{\aleph_0,\mathcal{N}}$ to select for each of countably many sets S_i in the filter $\mathfrak{W}$ a winning strategy τ_i for the second player, Eve.

We now describe a winning strategy for her in the game $\bigcap_{i<\omega} S_i$.

Suppose a is the sequence of Adam's moves in this game. We shall define plays α_i, β_i and ε, where α_i is a play by the first player in S_i, β_i is the response of Eve using her strategy τ_i, and ε will be composed from the β_i's and is to be her response to a in the game $\bigcap_i S_i$.

Let $\langle\cdot, \cdot\rangle : \omega \times \omega \to \omega$ be a pairing function. We shall (as above) define

$$c^k(n) = c(\langle k, n\rangle)$$

for any c.

We require our pairing function to have these properties:

$$k \leqslant \langle j, k\rangle \qquad \text{for all } j, k$$
$$k < k' \implies \langle k, n\rangle < \langle k', n\rangle \qquad \text{for each } n$$

From those it will follow that $\langle k, n\rangle \leqslant \langle\langle j, k\rangle, n\rangle$ for each j, k, n.

The following is a simultaneous definition:

$$\alpha_i(\langle\langle 0, k\rangle, n\rangle) = a(\langle k, n\rangle)$$
$$\alpha_i(\langle\langle j+1, k\rangle, n\rangle = \begin{cases} \beta_j(\langle k, n\rangle) & (j < i) \\ \beta_{j+1}(\langle k, n\rangle) & (j > i) \end{cases}$$
$$\beta_i = [\alpha_i] * \tau_i$$
$$\varepsilon(\langle\langle j, k\rangle, n\rangle) = \beta_j(\langle k, n\rangle)$$

The conditions on our pairing function ensure that this definition does not break down: at each stage when the definition of one function calls another, it is already sufficently well-defined. Timing is tight in the first few steps, but after that gets easier.

Then we shall find that in each game S_i, as in the game T, the critical set is

$$\{x \mid \exists i \exists k (x = a^k \ \text{ or } \ x = \beta_i^k)\}.$$

Finally, assume AD, which implies $AC_{\aleph_0, \mathcal{N}}$. Suppose that $X \notin \mathfrak{W}$, so that Eve has no winning strategy in the game $G(A, X)$. By AD, Adam must have one: let us call it σ. Put $Y = [A]^{\aleph_0} \setminus X$. We show that Eve has a winning strategy in $G(A, Y)$.

Let a be the sequence of Adam's moves in a play of $G(A, Y)$. We shall define Eve's responses e by considering a play of $G(A, X)$ in which Eve plays the sequence β, Adam the sequence α dictated by σ and β, and

$$\beta = a, \qquad e = \alpha.$$

To flesh out this recursive definition a little: in $G(A, Y)$ Adam plays $a(0)$; Eve replies with $\alpha(0)$, being Adam's first move in $G(A, X)$ according to σ; Adam now plays $a(1)$; Eve plays $\alpha(1)$, being Adam's move using σ at the position $\alpha(0), \beta(0)$ in $G(A, X)$, where $\beta(0) = a(0)$; and so on. There is a time lag, since Eve knows what her move $e(k)$ will be before Adam has played $a(k)$.

The outcome is that the same countable subset Z of A is produced by both plays: $Z = \{a^n,\, e^n \mid n \in \omega\}$. Since $Z \notin X$ as play in $G(A, X)$ has followed σ, $Z \in Y$, and hence Eve has won the play of $G(A, Y)$. ⊣ (2·6)

2·7 Exercise What would go wrong if in Rule ω and in the definition of $\mathfrak{W}(A)$ we replaced "Eve" by "Adam" ?

3: The measurability of ω_1

The first proof of measurability of ω_1 from AD was given by Solovay using his fine measure that we constructed in the last section.

3·0 Proposition (Solovay) *Suppose that there is a countably complete fine measure μ on $[\mathcal{N}]^{\aleph_0}$. Then ω_1 is measurable. Hence $\aleph_1 \not\leqslant 2^{\aleph_0}$, and is strongly inaccessible in every inner model of AC.*

Proof : using a familiar idea from measure theory, we project to a measure ν on ω_1 as follows. We write ω_1^α for the least ordinal not recursive in α. That will of course be a countable ordinal. For $a \in [\mathcal{N}]^{\aleph_0}$, let $\pi(a) = \sup\{\omega_1^\alpha \mid \alpha \in a\}$. For $X \subseteq \omega_1$, set $\nu(X) = \mu(\{a \mid \pi(a) \in X\})$. Then it is easily checked that ν is a countable complete measure on ω_1; we must show that it does not concentrate on a countable set. So let η be a countable ordinal. Let $\alpha \in \mathcal{N}$ code η, so that $\eta < \omega_1^\alpha$. Then whenever $\alpha \in a \in [\mathcal{N}]^{\aleph_0}$, $\pi(a) > \eta$; but the set of such a is of μ-measure 1, by the fineness of the measure.

The last clauses follow from the impossibility of $2^{\aleph_0}$ carrying a non-trivial countably additive two-valued measure. ⊣ (3·0)

A second proof

We turn to the definition of *the Martin measure.*

3·1 Definition A *Turing cone* is a set of reals of the form $C_\alpha =_{\text{df}} \{\beta \mid \alpha \leqslant_{\text{Turing}} \beta\}$; more accurately, we should construe it as a set of Turing degrees.

3·2 Proposition (Martin 1968[R6]) *Assume AD. Let $\mathcal{X}$ be a set of Turing degrees. Then $\mathcal{X}$ contains or is disjoint from a Turing cone.*

Proof : Let $A = \{\gamma \in \mathcal{N} \mid$ the Turing degree of γ belongs to $\mathcal{X}\}$. Consider the game $G(A, \mathcal{N})$. Suppose that σ is a winning strategy for Adam. We assert that $\mathcal{X}$ then contains C_σ. For let $\sigma \leqslant_{\text{Turing}} \varepsilon$. Then

$$\varepsilon \leqslant_{\text{Turing}} \gamma(\sigma, \varepsilon) \leqslant_{\text{Turing}} \sigma \oplus \varepsilon \leqslant_{\text{Turing}} \varepsilon$$

$R6$

so the degree of ε equals the degree of $\gamma(\sigma, \varepsilon)$ which is in $\mathcal{X}$ by the rules of the game.

If τ is a winning strategy for Eve, and $\alpha \geqslant_{\text{Turing}} \tau$, then

$$\alpha \leqslant_{\text{Turing}} \gamma(\alpha, \tau) \leqslant_{\text{Turing}} \alpha \oplus \tau \leqslant_{\text{Turing}} \alpha$$

so the degree of α equals that of $\gamma(\alpha, \tau)$, which by the rules is not in $\mathcal{X}$. ⊣ (3·2)

3·3 Definition The *Martin filter*, $\mathfrak{m}_T$, on the set $\mathcal{D}$ of Turing degrees is that generated by the cones.

3·4 Proposition *(i) ($AC_{\aleph_0,\mathcal{N}}$) The Martin filter is countably complete.*

(ii) (AD) The Martin filter $\mathfrak{m}_T$ is a countably complete ultrafilter.

Proof : (i) Given a countable sequence of cones, countable choice will suffice to pick one element from each; the recursive union of these can then be formed, the cone above which is contained in the intersection of the given cones.

(ii) follows by part (i), 3·2 and 1·6. ⊣

As with Solovay's filter, we may project the Martin measure to obtain what under AD will be a countably additive ultrafilter on ω_1:

3·5 Proposition *The map $x \mapsto \omega_1^x$ projects the Martin measure to a countably additive non-trivial measure on ω_1; hence ω_1 is measurable, $\aleph_1 \not\leq 2^{\aleph_0}$, and every well-orderable set of reals is countable.*

Proof : Here ω_1^x is the least ordinal not recursive in x, and is thus countable, being the sup of a countable set of countable ordinals. Any code of ω_1^x will map to a strictly larger ordinal than does x, whence the non-triviality of the measure.

There can be no non-trivial countably additive measure on $\{f \mid f : \omega \to 2\}$: if μ is such, set A_n to be whichever of $\{f \mid f(n) = 0\}$, $\{f \mid f(n) = 1\}$ is assigned measure one by μ; then $\bigcap_n A_n$ consists of a single point, but ought to have measure one, being the intersection of countably many sets of measure one.

The last two clauses follow readily from these remarks. ⊣

3·6 Corollary (AD) *ω_1 is strongly inaccessible in every inner model satisfying AC.*

Proof : Let M be such an inner model. ω_1 will be regular in M, as it is regular in V. If ω_1 were not strongly inaccessible in M, there would be a λ less than ω_1 with an injection $f \in M$ of ω_1 into $\mathcal{P}(\lambda)$: but that combined with a bijection in V between λ and ω_1 would yield an embedding of ω_1 in $\mathcal{P}(\omega)$, which we have just shown cannot exist. ⊣

3·7 Remark The Martin measure will work for other reducibilities:

3·8 Lemma (Martin) *Given any reducibility $\leq_A$ on R such that $x \leqslant_{\text{Turing}} y \implies x \leq_A y$, let $\mathcal{E}$ be the set of A-degrees. Then every subset of $\mathcal{E}$ contains or is disjoint from a cone. This defines the Martin measure $\mathfrak{m}_A$.*

Proof : either repeat the proof for the original case $A = T$, or say that $\mathfrak{m}_A$ will be the projection of $\mathfrak{m}_T$. ⊣

Solovay's first proof re-examined

We show that closer examination of the behavious of strategies in the games associated to the definition of Solovay's filter gives further information about the structure of the measure on ω_1.

For the moment let us take the following special case: θ is to be ω_1, A is to be $WORD$, the set of reals that code countable well-orderings, and we take $f(X)$ to be the sup of the ordinals coded by members of X, where X is a countable subset of A.

Let $S \subseteq \omega_1$. We shall see that

3·9 Proposition *(AD) Eve has a winning strategy in $\mathcal{G}(WORD, f, S)$ iff S contains a closed unbounded subset of ω_1, and Adam has a winning strategy in the same game if and only if S is disjoint from a club.*

3·10 Corollary *(AD) the closed unbounded filter on ω_1 is a countably complete ultrafilter.*

3·11 Remark In fact it is the unique normal ultrafilter on ω_1.

Proof of the Proposition: We use the Kleene boundedness theorem to seize control of the game.

This boundedness theorem is Corollary II·4·5. In terms of the following notation it says that if $A \subseteq \mathit{WORD}$ and A is $\Sigma^1_1(\beta)$, then $\sup\{|\alpha| \mid \alpha \in A\} < \lambda(\beta)$.

For $\alpha \in \mathit{WORD}$, we write $|\alpha|$ for the unique countable ordinal isomorphic to the relation coded by α; and we write $\lambda(\alpha) =_{\mathrm{df}} \sup\{|\beta| \mid \beta \leqslant_{\mathrm{Turing}} \alpha \;\&\; \beta \in \mathit{WORD}\}$.

3·12 Proposition $\lambda(\alpha) = \inf\{\xi \mid J_\xi\alpha \models \mathtt{KP}\}$.

We prove two lemmas without using AD from which the above remarks immediately follow.

3·13 Lemma *If X contains a closed unbounded set C then Adam has no winning strategy in $\mathcal{G}(\pi^{-1}``X)$.*

Proof : let σ be an Adam strategy. Then if σ observes Rule 0, $\forall e\, (\sigma\star[e])^0 \in \mathit{WORD}$, so $\{\beta \mid \exists e\; \beta = (\sigma\star[e])^0\}$ is a $\Sigma^1_1(\sigma)$ subset of WORD; hence the ordinals coded are bounded below $\lambda(\sigma)$. Pick $e_0 \in \mathit{WORD}$ with $|e_0| = \inf(C \setminus \lambda(\sigma))$.

If σ observes Rule 2, $\forall e\, (e^0 = e_0 \implies (\sigma\star[e])^1 \in \mathit{WORD}$; hence $\{\beta \mid \exists e\, (e^0 = e_0 \;\&\; \beta = (\sigma\star[e])^1\}$ is a $\Sigma^1_1(\sigma \oplus e_0)$ subset of WORD, and hence the ordinals coded are bounded below $\lambda(\sigma\oplus e_0)$, ($\oplus$ here denoting the recursive join of the two reals.)

Choose $e_1 \in \mathit{WORD}$ coding $\inf(C \setminus \lambda(\sigma \oplus e_0))$, and continue for ω steps. Let e be the real with $\forall k\, e^k = e_k$. When Eve plays e against σ, the decisive ordinal is $\sup\{|e^k| \mid k \in \omega\}$, which is in $C \subseteq X$; hence the strategy σ has failed to win this play for Adam. ⊣ (3·13)

We may here say that Eve has *seized control of the game*.

3·14 Lemma *Suppose that Eve has a winning strategy τ in $\mathcal{G}(X)$. Then X contains a club.*

Proof : Let $D = \{\delta < \Omega \mid J_\delta(\tau) \models \mathtt{KP}\}$, and let $C = D' =$ the derived set of D, $\{\theta < \Omega \mid 0 < \theta = \bigcup(D \cap \theta)\}$. C is club: we assert that $C \subseteq X$.

Let $\zeta \in C$: then there is a strictly increasing sequence ζ_n of elements of D with $\sup_n \zeta_n = \zeta$. This time it is Adam who seizes control of the game: we find a play a for him against τ for which ζ will be the critical ordinal.

For any countable ordinal η we write η^+ for $\inf D \setminus (\eta + 1)$, and in particular we write ξ for ζ^+.

Before play starts, Adam carries out the following calculations.

He picks a_0 generic over $J_\xi(\tau)$ with respect to the Lévy collapse of ζ_0, so that $|a_0| = \zeta_0$, and for each $\delta \in D \cap (\zeta_0, \xi]$, $J_\delta(\tau \oplus a_0) \models \mathtt{KP}$. In particular, $\lambda(\tau \oplus a_0) = \zeta_0^+ \leq \zeta_1$.

As in the previous lemma, he may use the boundedness theorem to argue that

$$\forall a\big[a^0 = a_0 \implies ([a]\star\tau)^0 \in \mathit{WORD} \;\&\; |([a]\star\tau)^0| < \lambda(\tau\oplus a_0)\big].$$

He now picks a_1 in WORD, Lévy generic over $J_\xi(\tau \oplus a_0)$, with $|a_1| = \zeta_1$: then $\lambda(\tau\oplus a_0 \oplus a_1) = \zeta_1^+ \leq \zeta_2$. As before, he uses the boundedness theorem to argue that

$$\forall a\big[a^0 = a_0 \;\&\; a^1 = a_1 \implies ([a]\star\tau)^0 \in \mathit{WORD} \;\&\; |([a]\star\tau)^0| < \lambda(\tau\oplus a_0 \oplus a_1)\big].$$

He continues for ω steps, and now plays the real a with $\forall k\, a^k = a_k$ against τ: the decisive ordinal will be ζ, which, as Eve has won, is therefore in X. ⊣

A third proof

A third proof of the measurability of ω_1 will follow, *modulo* a few facts about sharps, from the following classical result, which opened the doors of determinacy to set theorists.

3·15 Theorem [Solovay, 1967] *(AD) Every subset of ω_1 is constructible from a real.*

Proof : Let $A \subseteq \omega_1$: suppose that A is non-empty. Adam and Eve play a game.

Rule 1: Adam's play a must be in WORD: if it is not, he loses.

Rule 2: If Adam obeys Rule 1, Eve must play e such that for every n, $(e)_n \in \mathit{WORD}$; and then, writing η_n for $|(e)_n|$, she wins provided $|(a)_0| < \eta_0$ and $A \cap \eta = \{\eta_n \mid 1 \leq n < \omega\}$.

3·16 Remark It is safe to allow repetitions among Eve's ordinals.

Now Adam can have no winning strategy: for if σ were a strategy for him that never broke Rule 1, then for every e, $\sigma\star[e] \in \mathit{WORD}$; in other words, $\{c \mid \exists e\, c = \sigma\star[e]\}$ would be a $\Sigma^1_1(\sigma)$ subset of WORD and hence by Kleene's boundedness theorem,there is an ordinal ξ less than ω_1^σ such that $\forall e\ |\sigma \star [e]| < \xi$. Eve may now defeat σ by picking η_0 greater than $\xi \cup \bigcap A$, and η_n $(n > 0)$ so that $\{\eta_n \mid 0 < n < \omega\} = A \cap \eta_0$, and playing an e with $\forall n (e)_n \in WORD$ and $|(e)_n| = \eta_n$.

Thus by AD, Eve has a winning strategy, τ. We shall show that $A \in L[\tau]$, and shall use some elementary facts about Lévy forcing to do so.[C 1]

For κ an infinite countable ordinal, let $LG(\kappa)$ be the set of those $g \in WORD$ with $|g| = \kappa$ which are generic over $L[\tau]$ with respect to the Lévy collapse of κ, conditions for which, it will be remembered, are injections from finite subsets of ω into κ. We write $LC(\kappa)$ for the collapsing algebra in $L[\tau]$.

We already know that ω_1 is strongly inaccessible in $L[\tau]$, since AC is true there. It follows that for each κ, $LG(\kappa)$ is non-empty.

Fix now κ. We want to know whether κ is in A.

Let f and g be any two members of $LG(\kappa)$.

We consider the result of two plays of the game, in which Adam plays f and then g against τ.

We know that $\kappa \in A \iff \exists n{:}{\in}(\,0,\omega)\ |([f] \star \tau)_n| = \kappa$ and that $\kappa \in A \iff \exists n{:}{\in}(\,0,\omega)\ |([g] \star \tau)_n| = \kappa$; hence

$$\exists n{:}{\in}(\,0,\omega)\ |([f] \star \tau)_n| = \kappa \iff \exists n{:}{\in}(\,0,\omega)\ |([g] \star \tau)_n| = \kappa$$

and so if $\dot{f}$ is a name for the generic code of κ added by the Lévy forcing, the Boolean truth value $[\![\bigvee n{:}\ \epsilon\,(0,\omega)\ |([\dot{f}] \star \tau)_n| = \kappa]\!]$ of the corresponding statement of the forcing language is either 0 or 1, since no two generic objects evaluate it to different truth values. Hence

$$A = \{\kappa < \omega_1 \mid \text{In } L[\tau],\ [\![\bigvee n{:}\ \epsilon\,(0,\omega)\ |([\dot{f}] \star \tau)_n| = \kappa]\!]^{LC(\kappa)} = 1\}$$

and thus $A \in L[\tau]$, as required.

We may cover the case of finite κ by looking at $LC(\kappa \cup \omega)$ instead of $LC(\kappa)$. $\dashv$

The above result, combined with the rather more advanced result that AD, or indeed $\mathbf{\Pi^1_1}$-determinacy, entails the existence of $\alpha^\sharp$ for every real α, yields again the result that every subset of ω_1 contains or is disjoint from a club: if A is constructible from α, then $\alpha^\sharp$ yields a club of indiscernibles (greater than the parameters used to define A) which will either all be in A or all avoid it.

3·17 Remark The exact strength of the theory "Every subset of ω_1 is constructible from a real" plus "$\forall \alpha\ \alpha^\sharp$ exists" has yet to be determined. The theory of sharps will be reviewed in Chapter X.

[C 1] These facts may be found in ???

4: The Coding Lemma

We turn now to the highly important Coding Lemma. This is the place where ideas from recursion theory are injected into set theory. We begin with a simple form, and our proof is in essence that given in Moschovakis 7D.5 but with some additional terminology to emphasize the structure of the argument. In the next section we shall prove an extension, the Uniform Coding Lemma, by essentially the same method.

4·0 THE CODING LEMMA (AD) *Let $\preceq$ be a prewellordering of a set $X \subseteq \mathcal{N}$ and let $\Gamma = pos\ \Sigma^1_1(\prec)$. Let G be universal for $\mathbf{\Gamma} \restriction \mathcal{N} \times \mathcal{Y}$. Let $Z \subseteq Fld \preceq \times \mathcal{Y}$. Then there is an a in $\mathcal{N}$ such that*

(i) $G_a \subseteq Z$ and

(ii) for each $(\alpha, y) \in Z$ there is $(\alpha', y') \in G_a$ such that $\alpha \simeq \alpha'$.

Proof: In this proof we shall use the notation $\{\varepsilon\}(z)$ to denote a computation of Σ^0_1-recursion: that is, we write $\{\varepsilon\}(z)$ for $\{\varepsilon\}^{\mathcal{N},\mathcal{N}}_{\Sigma^0_1}(z)$ as in definition (II·2·1).

Call a *acceptable* if $G_a \subseteq Z$. For example, if $G_a = \emptyset$, a is acceptable; if $(\alpha, y) \in Z$ and $G_a = \{(\alpha, y)\}$ then a is acceptable; and if $G_a = \bigcup\{G_b \mid b \in S\}$ and each $b \in S$ is acceptable, so is a.

For acceptable a we define the *crash point* of a, in symbols, $crp(a)$, to be the least ζ such that (ii) of the Theorem fails for an α with $|\alpha|_{\preceq} = \zeta$. In one possible notation, this is saying that $(G_a)_1 \cap ((Z)_1)_\zeta = \emptyset$.

The Coding Lemma asserts that there is an acceptable a with no crash point; and we shall prove it by supposing the contrary and then arranging for Adam and Eve to play the following game, CG(Z), which, be it noted, is definable from Z and $\prec$.

Adam is to play a real a; Eve a real e; then

Rule 1: If a is not acceptable, Adam loses.
Rule 2: If a is acceptable and e is not, Eve loses.
Rule 3: If both a and e are acceptable, then Eve wins iff $crp(a) < crp(e)$.

Suppose that the game is determined, and that σ is a winning strategy for Adam.

Since he always has to play acceptably, the set $\bigcup\{G_{\sigma\star[e]} \mid e \in \mathcal{N}\}$ is a union of acceptably coded sets. Moreover by the uniform closure theorem it is of the form $G_{\bar{a}}$ for some $\bar{a}$, which $\bar{a}$ is therefore acceptable. Let $\zeta = crp(\bar{a})$. We show that Eve can defeat Adam's play as follows. She picks $\bar{\alpha}$ of $\prec$-rank ζ and $\bar{y}$ with $\langle\bar{\alpha}, \bar{y}\rangle \in Z$. Now let $\tilde{e}$ be such that $G_{\tilde{e}} = G_{\bar{a}} \cup \{\langle\bar{\alpha}, \bar{y}\rangle\}$. Then $\tilde{e}$ is acceptable and has crash point greater than ζ, and thus defeats the strategy σ.

Suppose now that the game is determined, and that τ is a winning strategy for Eve.

This time, we have to be more circumspect, as Adam is constrained to play acceptably.

We make the following definition:

$$A_{\varepsilon,z} =_{\text{df}} \big\{x \bigm| \exists w\big(w \prec z \ \&\ \{\varepsilon\}(w)\downarrow \ \&\ x \in G_{\{\varepsilon\}(w)}\big)\big\}$$

Note that $A_{\varepsilon,z}$ is in $\mathbf{\Gamma}$: by the Uniform Closure Theorem, it equals $G_{\pi(\varepsilon,z)}$ where π is a total recursive function. Define now

$$f(\varepsilon, z) =_{\text{df}} [\pi(\varepsilon, z)] \star \tau.$$

f is a total function. By the recursion theorem there is a real ε^* such that for all z,

$$\{\varepsilon^*\}(z)\downarrow = f(\varepsilon^*, z)$$

We now prove, by recursion on the rank function ρ induced by the pre-well-ordering, a lemma:

4·1 LEMMA *For all z, $\{\varepsilon^*\}(z)$ is acceptable and, if $z \in Field(\prec)$, its crash point is greater than $\rho(z)$.*

Proof: Note first that if z is not in Field$(\prec)$, or if $\rho(z) = 0$, then $A_{\varepsilon,z}$ is empty; so $\pi(\varepsilon, z)$ is acceptable, and so therefore would be Eve's reply to it as a play by Adam, $[\pi(\varepsilon, z)] \star \tau$. That is true for every ε. For the case $\varepsilon = \varepsilon^*$, this reply is indeed $\{\varepsilon^*\}(z)$.

Suppose therefore that $z \in$ Field$(\prec)$ is a counterexample with $\rho(z)$ minimal. Then for all $w < z$, $\{\varepsilon^*\}(w)$ is acceptable with crash point greater than $\rho(w)$; hence $\pi(\varepsilon^*, z)$ is acceptable with crash point greater than or equal to $\rho(z)$. Hence $f(\varepsilon^*, z)$, being the response by Eve's winning strategy to Adam's play of $\pi(\varepsilon^*, z)$, is acceptable with crashpoint yet greater; thus strictly greater than $\rho(z)$. But $f(\varepsilon^*, z) = \{\varepsilon^*\}(z)$, and so our induction is complete. $\dashv$

Consider now the set $\bigcup\{G_{\{\varepsilon^*\}(a)} \mid a \in \mathcal{N}\}$: this, by the Uniform Closure Theorem, is $G_{\bar{e}}$ for some $\bar{e}$. By the lemma, each $\{\varepsilon^*\}(a)$ for $a \in \mathcal{N}$ is acceptable, and so $\bar{e}$ is acceptable; but with arbitrarily high crash point, and thus with no crash point at all. Contradiction ! $\dashv$

4·2 In the above version of the Coding Lemma two ideas are intertwined. One of them is that if P is as above, then any subset of $X_P \times \mathcal{Y}$ is pre-well-ordered by its first co-ordinate. To express it more formally:

4·3 PROPOSITION *Suppose that $P = (X_P, \prec_P)$ is a pre-well-ordering where $X_P \subseteq \mathcal{X}$ and that $A \subseteq X_P \times \mathcal{Z}$. Then P naturally induces a pre-well-ordering Q of A by setting*

$$(a, z) \preceq_Q (a', z') \iff_{\mathrm{df}} a \preceq_P a'.$$

The set of components of Q is in natural bijection with the set of those components of P touched by A; Q is induced by the norm $(a, z) \mapsto a \mapsto \phi_P(a)$.

The second idea is that given $P = (X_P, \prec_P)$ and any $Y \subseteq X_P$, there is a $Z \subseteq Y$ with $Z \in pos\Sigma^1_1(\prec_P)$ and Z meets every component that Y meets: what we might call coding-by-touch. More formally:

4·4 THEOREM (AD) *Let $\preceq$ be a prewellordering of a set $X \subseteq \mathcal{X}$ and let $\Gamma = pos\ \Sigma^1_1(\prec)$. Let $G \subseteq \mathcal{N} \times \mathcal{X}$ be universal for $\mathbf{\Gamma} \restriction \mathcal{X}$. Let $Z \subseteq X$ Then there is an a in $\mathcal{N}$ such that*

(i) $G_a \subseteq Z$ and

(ii) for each $\alpha \in Z$ there is $\alpha' \in G_a$ such that $\alpha \simeq_P \alpha'$.

Contrast that with another version of the Coding Lemma, what we might call *coding-by-fill*:

4·5 THEOREM (AD) *Let $\preceq_P$ be a prewellordering of a set $X \subseteq \mathcal{X}$ of length λ and let $\Gamma = pos\ \Sigma^1_1(\prec_P, =_P)$. Let $F \subseteq \mathcal{N} \times \mathcal{X}$ be universal for $\mathbf{\Gamma} \restriction \mathcal{X}$. Let $A \subseteq \lambda$. Then there is an b in $\mathcal{N}$ such that*

$$F_b = \bigcup\{(X)_\nu \mid \nu \in H\}.$$

Proof: take $Z = \bigcup\{(X)_\nu \mid \nu \in H\}$ in the coding-by-touch theorem, and let G, a be as given in that theorem. Note that if Q and R are two binary relations Q, R, and G is the canonical good universal set for $pos\Sigma^1_1(Q)$, then $\{x \mid \exists y\, (y \in G^Q_a \ \&\ R(x, y)\}$ is in $pos\Sigma^1_1(Q, R)$ and so there is an index b, independent of Q and R, such that

$$F^{Q,R}_b = \{x \mid \exists y\, (y \in G^Q_a \ \&\ R(x, y)\}$$

where $F^{Q,R}$ is the canonical good universal set for $pos\Sigma^1_1(Q, R)$

Then taking $Q = \prec_P$ and $R = =_P$, we have our result. $\dashv$ (4·5)

4·6 EXERCISE Why is the Coding Lemma as we have stated it not a corollary of the two ideas we have extracted from it ?

As further exercises, we give some consequences of the Coding Lemma:

4·7 EXERCISE For each ordinal λ, there is a map of $\mathcal{P}(\lambda)$ onto λ^+.

4·8 EXERCISE Under *AD* Θ is a limit cardinal.

Let $H \subseteq \Theta$, and write δ_H for τ_H and M_H for $J_{\delta_H}(\mathbb{R}; H)$.

4·9 EXERCISE *AD* implies that every subset of an ordinal less than δ_H is in M_H.

4·10 EXERCISE *AD* implies that if $\lambda < \delta_H$ then $\mathcal{P}(\lambda) \in M_H$.

4·11 EXERCISE Under *AD*, $M_H \models \bigwedge x$ `if` x `is well-orderable then` $\mathcal{P}(x) \in \dot{V}$.

4·12 EXERCISE *AD* implies that M_H is semi-strongly admissible in the sense that whenever u is an ordinal or more generally a well-orderable set in M_H, and Φ is Δ_0,

$$M_H \models \bigvee \mathfrak{v} \bigwedge \mathfrak{x} \colon \epsilon\, u [\bigvee \mathfrak{y}\, \Phi(\mathfrak{x}, \mathfrak{y}) \longrightarrow \bigvee \mathfrak{y} \colon \epsilon\, \mathfrak{v}\, \Phi(\mathfrak{x}, \mathfrak{y})].$$

4·13 EXERCISE *AD* implies that δ_H is a limit cardinal.

A theorem of Kunen

4·14 PROPOSITION (Kunen) $(AD + DC)$ *(i) Every ultrafilter on an ordinal less than Θ is a Rudin–Keisler image of the Martin measure on the set $\mathcal{D}$ of Turing degrees;*

(ii) the set of such ultrafilters is well-ordered.

Proof : let $\lambda < \Theta$. By the coding lemma, there is a function $f : \mathcal{N} \xrightarrow{\text{onto}} \mathcal{P}(\lambda)$. For each ultrafilter $\mathcal{U}$ on λ, define $f_{\mathcal{U}} : \mathcal{N} \to \mathcal{P}(\lambda)$ by

$$f_{\mathcal{U}}(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathcal{U} \\ \lambda \setminus f(x) & \text{otherwise} \end{cases}$$

Then set

$$g_{\mathcal{U}}(\mathfrak{d}) \ =_{\text{df}} \ \inf \bigcap \{ f_{\mathcal{U}}(x) \mid x \leq_T \mathfrak{d} \}$$

$g_{\mathcal{U}}(\mathfrak{d})$ is well-defined by 1·16, and is some ordinal less than λ. Let $X \subseteq \lambda$. If $X \in \mathcal{U}$, then $X = f(a)$ for some real a; so whenever $\mathfrak{d} \geq_T a$, $g_{\mathcal{U}}(\mathfrak{d}) \in X$: hence $g_{\mathcal{U}}^{-1}$“X contains a cone and is thus in the Martin ultrafilter. Similarly if $X \notin \mathcal{U}$, then $g_{\mathcal{U}}^{-1}$“X is disjoint from a cone. So $\mathcal{U} = g_{\mathcal{U}}^{*}\mu_{\text{Martin}}$

For each $\mathcal{U}$, the function $g_{\mathcal{U}}$ is ordinal-valued, and therefore represents some member of the ultrapower of λ by the Martin measure, $\Pi\lambda^{\mathcal{D}}/\mu_{\text{Martin}}$, which by DC will be well-founded; $g_{\mathcal{U}}$ therefore may be taken to be a von Neumann ordinal, which we call $\xi(\mathcal{U})$.

If $\mathcal{U} \neq \mathcal{V}$, then $\xi(\mathcal{U}) \neq \xi(\mathcal{V})$: for let $X \subseteq \lambda$ be in $\mathcal{U}$ but not in $\mathcal{V}$, and suppose that $X = f(a)$. Then for $\mathfrak{d} \geq_T a$, $g_{\mathcal{U}}(\mathfrak{d}) \in X$ whereas $g_{\mathcal{V}}(\mathfrak{d}) \notin X$, and so $g_{\mathcal{U}}(\mathfrak{d}) \neq g_{\mathcal{V}}(\mathfrak{d})$ on a set of Martin measure 1, and so $\xi(\mathcal{U}) \neq \xi(\mathcal{V})$, and so the map $\mathcal{U} \mapsto \xi(\mathcal{U})$ supplies a well-ordering of the set of ultrafilters on λ. $\dashv$ (4·14)

4·15 REMARK The above argument will work more widely: one can show that every ultrafilter on an ordinal less than Θ is a Rudin-Keisler image of the Martin measure $\mathfrak{m}_B$ on the set of $=_B$-degrees for many other reducibilities $\leqslant_B$ and also of the Solovay ultrafilter $\mathfrak{W}(A)$ for many subsets A of $\mathcal{N}$.

5: The uniform coding lemma

For $P = (X, \prec_P)$ be a pre-well-ordering of a subset X of $\mathcal{N}$ of length λ, we write $(P)_\xi$ for $\{x \in \text{Field}(P) \mid \rho_P(x) = \xi$, and $P \restriction \zeta$ to be the restriction of $\prec_P$ to the set $\bigcup_{\xi<\zeta}(P)_\xi$. Thus $P \restriction 0$ is empty, $P \restriction \lambda = P$, for $\xi < \zeta$, $(P \restriction \zeta)_\xi = (P)_\xi$, and $X = \bigcup_{\xi<\lambda}(P)_\xi$.

Let G_ε be a good parametrisation of $pos\Sigma^1_1(\prec_P)$, as in §II.3. We write G^ζ_a for $G^{P\restriction\zeta}_a$.

The monotonicity remarked in (II·3·1) tells us that for $0 \leq \xi \leq \zeta \leq \lambda$ and any index a, $G^\xi_a \subseteq G^\zeta_a$.

5·0 THE UNIFORM CODING LEMMA (AD) *Let P be a pre-well-ordering of length λ, and $K \subseteq \lambda$. There is an index a such that*

$$\forall\zeta \leq \lambda \forall\xi < \zeta \big[G^\zeta_a \cap (P)_\xi \neq \emptyset \iff \xi \in K \big]$$

Proof : Monotonicity tells us that it is enough to find a such that

$$\begin{array}{cl} (i) & \xi \in K \implies G^{\xi+1}_a \cap (P)_\xi \neq \emptyset \\ (ii) & \xi \notin K \implies G^\lambda_a \cap (P)_\xi = \emptyset \end{array}$$

Call a *acceptable* if (ii) holds. The previous illustrative remarks about acceptable indices apply here: e.g. that an index (uniformly) coding the empty set is acceptable, and that an index (uniformly) coding the union of sets with acceptable indices is also acceptable. For acceptable a, define the *crash point* of a by

$$crp(a) = \begin{cases} \text{the first } \xi \in K \text{ such that } G^{\xi+1}_a \cap (P)_\xi = \emptyset & \text{if such } \xi \text{ exist} \\ \infty & \text{otherwise} \end{cases}$$

We play a game: Adam plays a real a, Eve a real e and Eve wins if a is not acceptable or if both a and e are acceptable and $crp(a) < crp(e)$.

The theorem in effect states that there is an acceptable index with crash point ∞; so if there is such an index, and Adam plays it, he automatically wins.

By AD the game is determined; so let σ be a winning strategy for Adam. Then for every e, $\sigma \star [e]$ is acceptable. Let $\bar{a}$ be an index such that for every T,

$$G^T_{\bar a} = \bigcup_{e\in\mathcal{N}} G^T_{\sigma\star[e]}.$$

Then $\bar a$ is acceptable: suppose that its crash point is $\bar\zeta < \infty$. Pick $\bar x \in (P)_{\bar\zeta}$. Let $\bar e$ be an index such that for each T,

$$G^T_{\bar e} = G^T_{\bar a} \cup \{x\}.$$

$\bar e$ is acceptable, with crash point greater than $\bar\zeta$, and so, if played by Eve, defeats σ. Thus the crash point of $\bar a$ is ∞ as desired.

Suppose now that τ is a winning strategy for Eve.

Define

$$A^T_{\varepsilon,w} = \big\{x \mid \exists z <_T w \;\; \{\varepsilon\}(z)\downarrow \text{ and } x \in G^T_{\{\varepsilon\}(z)}\big\}.$$

There is a recursive π such that $A^T_{\varepsilon,w} = G^T_{\pi(\varepsilon,w)}$. By the recursion theorem there is an ε^* such that for all w $\{\varepsilon^*\}(w)\downarrow = [\pi(\varepsilon^*,w)]\star\tau$.

5·1 Lemma *For any w, $\{\varepsilon^*\}(w)$ is acceptable and, if $w \in Field(P)$, its crash point is greater than $|w|_P$.*

Proof : As before, we handle first those $w \notin$ Field(P), for all which $\pi(\varepsilon,w)$ will code the empty set and therefore be acceptable. We then proceed by induction along P. Suppose true for all $z <_P w$. Then $\pi(\varepsilon^*,w)$ is acceptable, since it codes the union of sets with acceptable indices. Further,

$$crp(\pi(\varepsilon^*,w)) \geq \bigcup_{z<_P w} crp(z) \geq |w|_P,$$

by the induction hypothesis, since given $\zeta \in K\cap |w|_P$, let $z <_P w$ with $|z|_P = \zeta$. Then by induction, $crp(\{\varepsilon^*\}(z)) > \zeta$, and so $G^{\zeta+1}_{\{\varepsilon^*\}(z)}$ meets $(P)_\zeta$; and hence $G^{\zeta+1}_{\pi(\varepsilon^*,w)}$ does too.

But $\{\varepsilon^*\}(w)$, being Eve's response by her winning strategy τ to $\pi(\varepsilon^*,w)$, must therefore be acceptable with a yet larger crashpoint. $\dashv$

Now let $\tilde e$ be an index such that

$$G^T_{\tilde e} = \{x \mid \exists w\; x \in G^T_{\{\varepsilon^*\}(w)}\}$$

Then $\tilde e$ is acceptable and its crash point is ∞. $\dashv$

At the end of the last section we discussed the distinction between coding-by-touch and coding-by-fill. Having found our index a such that G_a is as prophesied, we may find an index b such that for any two relations Q,R,

$$G^{Q,R}_b = \{x \mid \exists y G^Q_a(y) \;\&\; R(x,y)\}$$

Then, taking Q to be $\prec_P$ and R to be $=_P$, we shall have this

5·2 Proposition *Let P be a pre-well-ordering of length λ. Let $K \subseteq \lambda$. Then there is an index b such that for all $\zeta < \lambda$,*

$$G^\zeta_b = \bigcup_{\nu\in\zeta\cap K} (P)_\nu.$$

Hence $\zeta \in K \iff G^{\zeta+1}_a \cap (P)_\zeta \neq \emptyset$.

6: Admissible coding and a digression on the strong partition property

The Uniform Coding Lemma allows us to code functions of ordinals by reals. We make this more precise and as an illustration establish certain combinatorial properties.

6·0 DEFINITION κ has the *strong partition property* if $\kappa > \omega$ and

$$\forall\lambda < \kappa\, \forall\pi : [\kappa]^\kappa \to \lambda\, \exists X :\in [\,\kappa]^\kappa\, \forall Y :\in [X]^\kappa \;\; \pi(Y) = \pi(X).$$

6·1 REMARK Such κ have been investigated by Kleinberg[R7]. An "automatic continuity" property of such cardinals is established in Henle and Mathias.[R8]

6·2 PROPOSITION (Kleinberg) *If κ has the strong partition property, then it is measurable.*

6·3 THEOREM *(AD) There are arbitrarily large cardinals below Θ with the strong partition property.*

6·4 REMARK Kechris and Woodin [R9] have shown that AD follows from the above statement if one assumes $V = L(\mathbb{R})$. Henle, Mathias and Woodin [R10] show that if $V = L(\mathbb{R})$ and AD holds, then in the Hausdorff extension, obtained by forcing with infinite subsets of ω *modulo* finite differences, there are no new sets of ordinals, all strong partition properties are preserved, but there is a Ramsey ultrafilter on ω and so AD has failed.

Coding functions

Let $\chi(\cdot,\cdot)$ be Gödel's primitive recursive pairing function: it is defined by well-ordering pairs of ordinals thus:

$$\begin{aligned}\langle\alpha,\,\beta\rangle < \langle\gamma,\,\delta\rangle \iff_{\text{df}} & \;\alpha\cup\beta < \gamma\cup\delta \;\text{ or}\\ & \;\alpha\cup\beta = \gamma\cup\delta \,\&\, \alpha<\gamma \;\text{ or}\\ & \;\alpha\cup\beta = \gamma\cup\delta \,\&\, \alpha=\gamma \,\&\, \beta<\delta\end{aligned}$$

and then $\chi(\alpha,\beta)$ is the place of $\langle\alpha,\,\beta\rangle$ in this well-ordering. So for cardinals λ, and many other ordinals in between,

$$\chi : \lambda\times\lambda \xrightarrow{1-1} \lambda.$$

Let $<_\Lambda$ be a prewellordering of $\Lambda \subseteq \mathcal{N}$ of length λ.

We are going to interpret a real a as a partial function from λ to λ, as follows:

$$a(\eta)\downarrow= \beta \iff_{\text{df}} G_a^{\chi(\eta,\beta)+1} \cap (\Lambda)_{\chi(\eta,\beta)} \neq \emptyset \;\&\; \forall\beta' :< \beta\; G_a^{\chi(\eta,\beta')+1} \cap (\Lambda)_{\chi(\eta,\beta')} = \emptyset$$

6·5 PROPOSITION (AD) *Every partial function has a code.*

Proof : Suppose that B is a partial function from λ to λ

Let $E_B = \{\chi(\alpha,\beta) \mid B(\beta)\downarrow= \alpha\}$.

By the Uniform Coding Lemma, there is an index a such that

$$\forall\xi :< \lambda \;\; G_a^{\xi+1} = \bigcup\{(\Lambda)_\nu \mid \nu \in E_B \cap (\xi+1)\}$$

Then
$$\forall\eta :< \lambda\; B(\eta) \simeq a(\eta)$$

where $\simeq$ is used in Kleene's sense that if either side is defined so is the other and they are equal.

Admissible codings

[R7] Kleinberg, book

[R8] Henle and Mathias, Supercontinuity, Math. Proc. Cam. Phil. Soc.

[R9] Kechris and Woodin, PNAS

[R10] Henle, Mathias and Woodin, Caracas 1983 meeting

Let $\mathcal{B}$ be a subset of $\mathcal{N} \times \mathcal{N}$ that forms a pre-well-ordering of $\mathcal{N}$. Form the inner model $L(\mathbb{R}, \mathcal{B})$, find the least ordinal $\delta_{\mathcal{B}}$ such that $J_{\delta_{\mathcal{B}}}(\mathbb{R}, \mathcal{B}) \preceq^{\mathcal{B}}_{\Sigma_1} L(\mathbb{R}, \mathcal{B})$, and let $M_{\mathcal{B}} = J_{\delta_{\mathcal{B}}}(\mathbb{R}, \mathcal{B})$. We have seen in earlier chapters that $\delta_{\mathcal{B}}$ is less than the Θ of $L(\mathbb{R}, \mathcal{B})$, and therefore less than Θ, and that $M_{\mathcal{B}}$ is admissible, $\mathbb{R} \in M_{\mathcal{B}}$, $\mathcal{B} \in M_{\mathcal{B}}$, and, using DC , $M_{\mathcal{B}}$ thinks that each pre-well-ordering is isomorphic to an ordinal, so that $|\mathcal{B}| < \delta_{\mathcal{B}}$.

6·6 Remark $\delta_{\mathcal{B}}$ will be the least ordinal not $\mathbf{\Delta^2_1}$ in $\mathcal{B}$, as discussed in (II·4·27).

Moreover, there is a natural pre-well-ordering $\Upsilon_{\mathcal{B}}$ of length $\delta_{\mathcal{B}}$ which is $\Sigma_1(M_{\mathcal{B}}, \mathcal{B})$ and of which every initial segment is $\Delta_1(M_{\mathcal{B}}, \mathcal{B})$ and therefore a member of $\mathcal{B}$. If we use that pre-well-ordering to code functions from $\delta_{\mathcal{B}}$ to $\delta_{\mathcal{B}}$ as described above, our coding will have this further property:

6·7 Proposition (AD) *For each η and β less than $\delta_{\mathcal{B}}$, $\{c \in \mathcal{N} \mid \forall \alpha < \eta\ c(\alpha) \downarrow < \beta\} \in M_{\mathcal{B}}$.*

6·8 Definition We shall refer to codings of that kind as *admissible*.

6·9 Corollary (AD) *Every function from $\delta_{\mathcal{B}}$ to $\delta_{\mathcal{B}}$ is $\Sigma_1(M_{\mathcal{B}}, \mathcal{B}, a)$ for some real a.*

Strong partition cardinals

The theorem stated at the start of the section will follow from the following together with the above remarks.

6·10 Theorem (AD) *Let M be an admissible set of height $\delta_{\mathcal{B}} < \Theta$, which contains all reals, with $\mathbb{R} \in M$. Suppose that there is a pre-well-ordering $<_\Upsilon$ with field $\Upsilon \subseteq \mathcal{N}$ which is $\Sigma_1(M, p)$ for some parameter $p \in M$, and with each initial segment a member of M. Then $\delta_{\mathcal{B}}$ is a strong partition cardinal.*

Proof: all codings of functions in this proof refer to the admissible coding defined from the pre-well-ordering $<_\Upsilon$.

We first show that $\delta_{\mathcal{B}}$ is a regular cardinal: suppose $\kappa < \delta_{\mathcal{B}}$ and $f : \kappa \longrightarrow \delta_{\mathcal{B}}$. By Corollary 6·9, we know that f is Σ_1 over $\delta_{\mathcal{B}}$ in a real a that codes f. $\forall \eta < \kappa \exists \beta\ a(\eta) \downarrow = \beta$; so by admissibility there is a $\gamma < \delta_{\mathcal{B}}$ which bounds the values of f. [Hence f is a $\Delta_1(M)$ subset of $\kappa \times \gamma$ and is thus a member of M.]

Now we treat the case with only two colours. Let $\pi : [\delta_{\mathcal{B}}]^{\delta_{\mathcal{B}}} \to 2$. Then we shall play a game $\mathcal{G}(\pi)$, in which Adam plays a and Eve plays e;

Rule 1: if $\exists \eta\, (a(\eta) \uparrow$ or $e(\eta) \uparrow)$ then for the least such η, if $a(\eta) \uparrow$, Adam loses, otherwise, if $a(\eta) \downarrow$, he wins.

If there is no decision under Rule 1, then $a(\eta)$ and $e(\eta)$ converge for all η, and we now define

$$\tilde{C}_{a,e}(\eta) = \bigcup \{a(\xi) \cup e(\xi) \mid \xi < \omega(\eta + 1)\}$$

This function is weakly increasing; we would like it to be strictly increasing, so we now ensure tidy play by picking a player we hate and declaring him to lose if it is not. For definiteness, we have

Rule 2: If $\tilde{C}_{a,e}$ is not strictly increasing, Adam loses.

If both Rules 1 and 2 have led to no decision, write $C_{a,e}$ for the image of $\tilde{C}_{a,e}$: This is consistent with the convenient notation, when discussing strong partition properties, of writing $\tilde{C}$ for the monotonic enumeration of a subset C of $\delta_{\mathcal{B}}$.

Rule 3 If $\pi(C_{a,e}) = 0$, Adam wins; otherwise he loses.

Now we shall show that if ρ is a winning strategy for this game, for either player, there is a set P in $[\delta_{\mathcal{B}}]^{\delta_{\mathcal{B}}}$ such that for each $R \in [P]^{\delta_{\mathcal{B}}}$ there is some play (a, e) following ρ such that $R = C_{a,e}$: it follows that P is homogeneous for the partition π.

First let σ be a winning strategy for Adam. We shall, for given η and β less than $\delta_{\mathcal{B}}$, prove a bound $F_\sigma(\eta, \beta)$ less than $\delta_{\mathcal{B}}$ for certain plays following σ.

The set $D =_{\text{df}} \{e \mid \forall \alpha < \eta\ e(\alpha) \downarrow < \beta\}$ is in M by Proposition 6·7. For each $e \in D$, $\sigma \star [e](\eta) \downarrow$: for otherwise e would defeat σ under rule 1.

Hence $\{(\sigma \star [e](\eta) \mid e \in D\}$ is bounded below $\delta_{\mathcal{B}}$, being the image under a $\Sigma_1(M)$ function of a subset of its domain which is a member of M. Let $F_\sigma(\eta, \beta)$ be the least strict upper bound less than $\delta_{\mathcal{B}}$ for this set.

Thus we have proved that for all η and β less than $\delta_{\mathcal{B}}$ there is an ordinal $F_\sigma(\eta, \beta)$ less than $\delta_{\mathcal{B}}$ such that

$$\forall e\big[(\forall\alpha<\eta\; e(\alpha)\downarrow<\beta) \implies \sigma\star[e](\eta)\downarrow< F_\sigma(\eta,\beta)\big].$$

Armed with the function F_σ, we define

$$Q \;=_{\rm df}\; \{\lambda<\delta_{\mathcal{B}} \mid \forall\eta<\lambda\forall\beta<\lambda F_\sigma(\eta,\beta)<\lambda\}.$$

By the regularity of $\delta_{\mathcal{B}}$ we know that Q is non-empty and indeed closed unbounded in $\delta_{\mathcal{B}}$.

For $S\in[\delta_{\mathcal{B}}]^{\delta_{\mathcal{B}}}$, we set

$${}_\omega S \;=_{\rm df}\; \{\bigcup_{n<\omega}\{\tilde S(\omega\eta+n)\mid \eta<\delta_{\mathcal{B}}\}$$

Set $P={}_\omega Q$. I assert that P has the desired property.

For let $R\subseteq P$: then there is an $S\subseteq Q$ with ${}_\omega S=R$. [Each element ζ of R is of the form $\tilde Q(\omega(\eta+1))$ for a uniquely determined $\eta=\eta_\zeta$: take $S=\{\tilde Q(\omega\eta_\zeta+n)\mid \zeta\in R\;\&\; n\in\omega\}$.]

Let Eve play an e such that $\forall\eta\; e(\eta)\downarrow=\tilde S(\eta)$, and let $a=\sigma[e]$.

Let $\eta<\delta_{\mathcal{B}}$ and let $\xi<\omega(\eta+1)$.

$\forall\nu<\xi\; e(\nu)\downarrow<\tilde S(\xi)$, so $a(\xi)<F_\sigma(\xi,\tilde S(\xi))<\tilde S(\xi+1)=e(\xi+1)$.

Tracing through the definitions, we see that $C_{a,e}={}_\omega S=R$, as required.

If Eve wins, with strategy τ, a similar argument shows that there is a homogeneous set with Eve's colour 1 throughout.

Now we turn to the case of λ colours.

Let $\pi:[\delta_{\mathcal{B}}]^{\delta_{\mathcal{B}}}\to\lambda$ where $\lambda<\delta_{\mathcal{B}}$. Fix a prewellordering $<_\Lambda$ in M of a set $\Lambda\subseteq\mathcal{N}$ of length λ; for x in Λ write $|x|_\Lambda$ for its rank in this pre-well-ordering.

For $\nu<\lambda$ let

$$\pi_\nu(P)=\begin{cases}0 & \text{if } \pi(P)=\nu;\\ 1 & \text{otherwise.}\end{cases}$$

and consider the games $\mathcal{G}(\pi_\nu)$. If Adam wins one of those then there will be a homogeneous set for π. We assume therefore that Eve wins them all, and derive a contradiction.

Let

$$Z=\{(x,\tau)\mid x\in\Lambda\;\&\;\tau \text{ is a winning strategy for Eve in } \mathcal{G}(\pi_{|x|_\Lambda}\}.$$

By the Coding Lemma, there is a $Z^*\subset Z$ with $Z^*\in pos\Sigma^1_1(<_\Lambda)$ such that, setting $\mathcal{S}^*=\{\tau\mid\exists x(x,\tau)\in Z^*\}$, $\mathcal{S}^*$ contains at least one strategy for Eve in each game $\mathcal{G}(\nu)$, $(\nu<\lambda)$. Since $<_\Lambda$ is in M, Z^* and $\mathcal{S}^*$ will be in M.

Now we establish that for each η and β less than $\delta_{\mathcal{B}}$, there is a uniform bound $F(\eta,\beta)$:

$$\forall\tau\in\mathcal{S}^*\forall a\big[\big(\forall\alpha\le\eta\; a(\alpha)\downarrow<\beta\big) \implies ([a]\star\tau)(\eta)\downarrow<F(\eta,\beta)\big].$$

That follows as before, by using the admissibility of M to show that the set $D' \;=_{\rm df}\; \{a\in\mathcal{N}\mid\forall\alpha\le\eta\, a(\alpha)\downarrow<\beta\}$ is in M, and then coupling that with the fact that $\mathcal{S}^*$ is in M to show that the set

$$\{\xi\mid\exists\tau{:\in}\mathcal{S}^*\;\exists a{:\in}D'\;\xi=([a]\star\tau)(\eta)\}$$

is in M. Now let

$$Q \;=_{\rm df}\; \{\lambda<\delta_{\mathcal{B}}\mid\forall\eta<\lambda\forall\beta<\lambda F(\eta,\beta)<\lambda\}.$$

and let $P={}_\omega Q$. Let $\nu=\pi(P)$. Let $\tau\in\mathcal{S}^*$ be a winning strategy for Eve in $\mathcal{G}(\pi_\nu)$, and let Adam play a so that $\forall\eta\; a(\eta)\simeq\tilde Q(\eta)$ in this game; let $e=[\alpha]\star\tau$. Then as before we find that $P=C_{a,e}$ and so (as Eve has won this play of $\mathcal{G}(\pi_\nu)$), $\pi(P)\ne\nu$, a contradiction. $\dashv$

6·11 Historical Note The theorem is due to Kechris, Kleinberg, Moschovakis and Woodin,[R11] who indeed prove a more refined form couched in terms of Spector classes satisfying certain conditions.

6·12 Remark In the form stated, our theorem does not include that of Martin, who showed that under AD, ω_1 is a strong partition cardinal. A proof of that will be found in Kechris' exposé.[R12]

[R11] KKMH, Cabal

[R12] Kechris, first Cabal.

7: The existence of definable strategies

[*This section requires some further work*]

We use ideas purloined from the pages of Moschovakis to prove a weak form of his Third Periodicity Theorem, which yields a proof that if the first player has a winning strategy for a Σ^1_2 game, he has one which is ordinal definable. That result will be sufficient for our purposes; Moschovakis proves by more refined calculations that in those circumstances the player has a winning strategy which is Σ^1_3, and proves similar results for a general appropriate pointclass Γ.

The result we actually prove is this:

7·0 THEOREM (DC) *Let λ be an infinite ordinal and let $A \subseteq \mathcal{N}$ be the projection of a tree $T \subseteq {}^{<\omega}(\omega \times \lambda)$. If each of countably many games closedly related to $\mathcal{G}(A)$ is determined, and if the first player has a winning strategy in $\mathcal{G}(A)$, then he has one that is ordinal definable from T.*

7·1 PROBLEM Can we do better ? Can we get one in $L[T]$? Presumably not, for Σ^1_2 games are projections of constructible trees, and there are such games that yield the existence of $0^\sharp$.

Perhaps we can get one in $L(R, T)$? That is worse than useless, since we are assuming $V = L(\mathbb{R})$ anyway.

Third try: perhaps the strategy is ∞-Borel in T, whatever that means.

Finally: is *DC* necessary, or will countable choice for families of reals do. ?

Our starting point is the concept of a semi-scale: so some definitions:

7·2 DEFINITION A *putative scale* on a pointset A is a sequence of norms $\langle \varphi_i \mid i \in \omega \rangle$ on A.

7·3 DEFINITION A putative scale on A is a *semiscale* if for each convergent sequence α_n of points in A with $\lim_{n\to\infty} \alpha_n = \alpha$ such that for each i, $\varphi_i(\alpha_n)$ is eventually constant as $n \to \infty$, with eventual value λ_i say, we have $\alpha \in A$.

The said semiscale is a *scale* if in the above circumstances we have additionally that for each i $\varphi_i(\alpha) \le \lambda_i$: we may say that a scale is a semiscale that is *lower semi-continuous*.

We show first that the possession of a semi-scale is the analytic equivalent of being representable as the projection of a tree.

Left-most branches

If $\alpha \in \mathfrak{p}[T]$, we define the (progressively) left-most branch f^T_α, associated to α, recursively, thus:

7·4 DEFINITION $f^T_\alpha(k)$ is the least ν such that for some g with $\langle \alpha, g\rangle \in [T]$ and $g \restriction k = f^T_\alpha \restriction k$, $g(k) = \nu$.

For $\alpha \in \mathfrak{p}[T]$, set $\phi_k(\alpha) = f^T_\alpha(k)$.

7·5 PROPOSITION *$\langle \phi_k \mid k \in \omega \rangle$ is a semi-scale on $\mathfrak{p}[T]$.*

Proof : Suppose that $\alpha_n \to \alpha$ and that there are ordinals ζ_i such that for each i, $\phi_i(\alpha_n) = \zeta_n$ for all large n: by passing to a subsequence of the α_i's if necessary we may assume that $\phi_i(\alpha_n) = \zeta_i$ whenever $i < n$.

Let $h(i) = \zeta_i$. We show that $\langle \alpha, h\rangle \in [T]$. Take any k, then for n sufficiently large, $\alpha \restriction k = \alpha_n \restriction k$ [COULD TAKE $n = k$????] and $h \restriction k = f_{\alpha_n} \restriction k$, and so $\langle \alpha \restriction k, h \restriction k\rangle \in T$; whence $\langle \alpha, h\rangle \in [T]$ and $\alpha \in \mathfrak{p}[T]$, as required. $\dashv$ (7·5)

On the other hand, suppose that a subset A of $\mathcal{N}$ carries a semi-scale $\langle \phi_i \rangle$. Form

$$T =_{\rm df} \{\langle \alpha(0), \phi_0(\alpha), \alpha(k), \phi_k(\alpha)\rangle \mid \alpha \in A, k \in \omega\}.$$

T is evidently closed under shortening, and A is visibly a subset of $\mathfrak{p}[T]$. Suppose that $\beta \in \mathfrak{p}[T]$; so let $\langle \beta, g\rangle \in [T]$. Then for each k there is an $\alpha_k \in \mathfrak{p}[T]$ such that $\beta \restriction k = \alpha_k \restriction k$ and $g \restriction k = \langle \phi_i(\alpha_k) \mid i < k\rangle$; but then $\lim_k \alpha_k = \beta$ and for each i, $\phi_i(\alpha_k)$ is eventually constant (with eventual value $g(i)$; so the hypotheses for the semi-scale property are satisfied and therefore $\beta \in A$.

Thus $A = \mathfrak{p}[T]$.

Thus we have proved the

7·6 PROPOSITION *A subset A of $\mathcal{N}$ admits a semi-scale iff it is the projection of a tree.*

7·7 PROBLEM Comparison of tree and length of semiscales ?

7·8 PROBLEM What about games on other sets besides ω ?

We shall use the following simple lemma:

7·9 LEMMA *Suppose that we have a putative scale ψ_i. Let*

$$\varphi_i(\alpha) = \langle \psi_0(\alpha), \psi_1(\alpha), \dots \psi_i(\alpha) \rangle$$

ordered lexically. Suppose that α_n is a sequence such that

$$\varphi_n(\alpha_{n+1}) \leq \varphi_n(\alpha_n)$$

for each n. Then each $\psi_i(\alpha_n)$ is eventually constant.

Proof : for each $n \geq 0$, $\psi_0(\alpha_{n+1}) \leq \psi_0(\alpha_n)$; since there is no infinite descent in the ordinals, the sequence $\psi_0(\alpha_n)$ must be eventually constant, for $n \geq n_0$, say. Now for $n \geq n_0$, $\psi_1(\alpha_{n+1}) \leq \psi_1(\alpha_n)$, and hence for n greater than or equal to some n_1, $\psi_1(\alpha_n)$ is constant. In general, once we know that for each $j \leq i$ and for $n \geq n_i$, $\psi_j(\alpha_n) = \psi_j(\alpha_{n_i})$, we know that thereafter $\psi_{i+1}(\alpha_{n+1}) \leq \psi_{i+1}(\alpha_n)$ and hence after some finite time ψ_{i+1} will become constant. $\dashv$

Now besides the norms ψ_i in our putative scale we may also consider the maps $\alpha \mapsto \alpha(i)$, giving us further norms [but unsquashed]. So if we incorporate these, we get excellent semi-scales.

Suppose therefore that we define

$$\chi_i(\alpha) = \langle \psi_0(\alpha), \alpha(0), \psi_1(\alpha), \alpha(1), \dots, \psi_i(\alpha), \alpha(i) \rangle$$

Then if the ψ's are a semiscale on A, each $\alpha_n \in A$ and for each n,

$$\chi_n(\alpha_{n+1}) \leq \chi_n(\alpha_n)$$

we shall know that for each i, each $\alpha_n(i)$ is eventually constant, with value $\alpha(i)$, say, and hence that LIMIT α_n exists $= \alpha$; further that for each i, $\psi_i(\alpha_n)$ is eventually constant, and thus that $\alpha \in A$.

We shall use this observation in proving that our definable strategy works.

The Moschovakis comparison game

Suppose now that ϕ is a norm on $A = \mathfrak{p}[T]$ and that u and v are two positions of THE SAME odd length starting from which the first player has a winning strategy in $\mathcal{G}(\mathfrak{p}[T])$.

We shall introduce a game of comparing u and v as winning positions.

The game $H_k(u, v)$

We probably want H_ϕ.

I don't see why we shouldn't compare u's and v's of different odd lengths.

But what would happen if we did ?

The following diagram sets out the moves.

$$\begin{array}{ccccccccccc} u & \alpha^*(0) & & & \alpha^*(1) & \alpha^*(2) & & & \alpha^*(3) & \dots & \alpha^* \\ v & & \gamma^*(0) & \gamma^*(1) & & & \gamma^*(2) & \gamma^*(3) & & \dots & \gamma^* \end{array}$$

In the upper line we see a play of $\mathcal{G}(A)$ starting from u. Players F and S choose α^* alternately: which one is the bad guy ?

In the lower line we see a play of $\mathcal{G}(A)$ starting from v. Players F and S choose γ^* alternately: which one is the bad guy ?

We set $\alpha = u ^\frown \alpha^*$ and $\gamma = v ^\frown \gamma^*$.

Rule 1: if $\alpha \notin A$, S loses.

Rule 2: if $\alpha \in A$ and $\gamma \notin A$, F loses.

Rule 3: if $\alpha \in A$ and $\gamma \in A$, S wins if and only if $\phi(\alpha) \leqslant \phi(\gamma)$.

7·10 DEFINITION We shall write $u \leqslant_\psi v$ iff S wins $H_\phi(u, v)$.
We shall show that $\leqslant_\psi$ is indeed a norm on the set of u's and v's.

Diagram for proving the transitivity of $\leq_\phi$

$$
\begin{array}{llccccccccccc}
 & u & \alpha^*(0) & & & & & & & & & \alpha^*(1) & \alpha^*(2)\ \ldots \\
H_k(u,v) & & \uparrow & \searrow^{\tau_0} & & & & & & & {}^{\tau_0}\nearrow & \downarrow & \uparrow \\
 & v & \uparrow & & \gamma^*(0) & & & & & \gamma^*(1) & & \downarrow & \uparrow \\
 & & \uparrow & & \downarrow & & & & & \uparrow & & \downarrow & \uparrow \\
 & v & \uparrow & & \gamma^*(0) & & & & & \gamma^*(1) & & \downarrow & \uparrow \\
H_k(v,w) & & \uparrow & & & \searrow^{\tau_1} & & & {}^{\tau_1}\nearrow & & & \downarrow & \uparrow \\
 & w & \uparrow & & & & \varepsilon^*(0) & \varepsilon^*(1) & & & & \downarrow & \uparrow \\
 & & \uparrow & & & & \downarrow & \uparrow & & & & \downarrow & \uparrow \\
 & u & \boxed{\alpha^*(0)} & & & & \downarrow & \uparrow & & & & \alpha^*(1) & \boxed{\alpha^*(2)}\ \ldots \\
H_k(u,w) & & & & & & \downarrow & \uparrow & & & & & \\
 & w & & & & & \varepsilon^*(0) & \boxed{\varepsilon^*(1)} & & & & &
\end{array}
$$

Here u and v are in W_k. The vertical arrows are just copying. The boxed moves are the free moves of the opponent. Eve uses her strategies τ_0 in $H_k(u, v)$ and τ_1 in $H_k(v, w)$ to build a strategy for herself in $H_k(u, w)$.

Diagram for proving that there is no descending sequence in $<_\phi$

$$
\begin{array}{llcccccccc}
 & u_0 & \overset{\sigma_0}{\longmapsto} \alpha_0^*(0) & & & & \boxed{\alpha_0^*(1)} \xrightarrow{\sigma_0} \alpha_0^*(2) & & & & \boxed{\alpha_0^*(3)}\ \ldots \\
H_{01} & & & & & & & & & & \\
 & u_1 & & & \alpha_1^*(0) \xrightarrow{\sigma_0} \alpha_1^*(1) & & & & \alpha_1^*(2) \xrightarrow{\sigma_0} \alpha_1^*(3) & & \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & u_1 & \overset{\sigma_1}{\longmapsto} \alpha_1^*(0) & & & & \alpha_1^*(1) \xrightarrow{\sigma_1} \alpha_1^*(2) & & & & \alpha_1^*(3) \ldots \\
H_{12} & & & & & & & & & & \\
 & u_2 & & & \alpha_2^*(0) \xrightarrow{\sigma_1} \alpha_2^*(1) & & & & \alpha_2^*(2) \xrightarrow{\sigma_1} \alpha_2^*(3) & & \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & u_2 & \overset{\sigma_2}{\longmapsto} \alpha_2^*(0) & & & & \alpha_2^*(1) \xrightarrow{\sigma_2} \alpha_2^*(2) & & & & \alpha_2^*(3) \ldots \\
H_{23} & & & & & & & & & & \\
 & u_3 & & & \alpha_3^*(0) \xrightarrow{\sigma_2} \alpha_3^*(1) & & & & \alpha_3^*(2) \xrightarrow{\sigma_2} \alpha_3^*(3) & & \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & u_3 & \overset{\sigma_3}{\longmapsto} \alpha_3^*(0) & & & & \alpha_3^*(1) \xrightarrow{\sigma_3} \alpha_3^*(2) & & & & \alpha_3^*(3) \ldots \\
H_{34} & & & & & & & & & & \\
 & u_4 & & & \alpha_4^*(0) \xrightarrow{\sigma_3} \alpha_4^*(1) & & & & \alpha_4^*(2) \xrightarrow{\sigma_3} \alpha_4^*(3) & & \\
 & & \ldots & \nearrow & & \searrow & \ldots & \nearrow & & \searrow & \ldots
\end{array}
$$

Here the sloping arrows are copying. Others follow the indicated strategies. The boxed moves are the Devil's free moves which will ensure that $\alpha_0 = u_0 {}^\frown \alpha_0^*$ is in the Σ_2^1 set A.

Therefore it is impossible to have neither $u \leqslant v$ nor $v \leqslant u$: otherwise we would have $u > v > u > v$..... Similarly we must have $u \leqslant u$.

FEELS AS THOUGH WE HAVE A WQO !

The minimal policy

Now to business. We have a set $A = \mathfrak{p}[T]$, on which we have the natural norms ϕ_k given by left-most paths. We put all those together as indicated above to get our immproved norms χ_k.

[We have ALREADY defined for each even k the set W_k of positions of length $k+1$ from which Adam still has a winning strategy in the game A and we have seen how to define a norm χ_k on W_k which derives ultimately from ϕ_k.]

WE ASSUME that the first player has a winning strategy for A, and that for all u, v OF EQUAL ODD LENGTH, and k the game $H_{\chi_k}(u,v)$ is determined. Henceforth $H_k(u,v)$.

That gives us norms ψ_k on the set of such u and v.

We define the *minimal policy* for Adam in $\mathcal{G}(A)$: at each position w of **even** length, when he is a winning position, he considers all plays a which leave him in a winning position, and admits into the policy only those a with $\psi_k(u^\frown\langle a\rangle)$ minimal, that is, for all other b, $u^\frown a \leqslant u^\frown b$.

HOW FAST DOES k go up ? Is it defined only for even k ?

The *best strategy* will be that of choosing the numerically least integer offered by the minimal policy at each stage.

Suppose now that the sequence $\alpha = a_0, a_1, a_2, \ldots$ is in accordance with this policy. That is, for each positive even k the sequence $a_0, a_1, ..., a_{k-2}$ was in the policy, Eve has just played a_{k-1}, the position $a_0, \ldots a_k$ is in W_k, and for all moves b for Adam at $[a_{k-1}]$, if $b \in W_k$ then $\chi_k([a_k]) \leq \chi_k((a_0, \ldots a_{k-1}, b))$: that is,

$$[a_k] \leq^*_{\chi_k} (a_0, \ldots a_{k-1}, b).$$

We must prove that $\alpha \in A$. To do so, we shall define for each even k a real α_k which will be a member of A: the sequence of the α_k's will converge to α: indeed we shall have for each even k and each $j < k$,

$$\alpha(j) = \alpha_k(j).$$

The definition of the α_k's will gradually evolve. As soon as we know the value $\alpha_k(k)$ we shall set b_k equal to it, we shall write

$$\begin{aligned} \bar{a}_k &=_{\rm df} (a_0, a_1, \ldots a_{k-1}, a_k) \\ \bar{b}_k &=_{\rm df} (a_0, a_1, \ldots a_{k-1}, b_k) \end{aligned}$$

By the minimality of a_k at $[a_{k-1}]$, we know that the player S has a winning strategy in the game $H_k =_{\rm df} H_k(\bar{a}_k, \bar{b}_k)$, and (using DC) shall pick a winning strategy τ_k. We shall then start a play of H_k in which S uses τ_k. In due course this will lead to b_{k+2} being defined and hence to the next game starting up. Moves will be copied between the games.

The first few rounds are shown in Diagram ?? . The story begins by our setting $\alpha_0(0)$ to be the first move for Adam given by the strategy σ; we immediately set b_0 to be this integer, define $\bar{a}_0$, $\bar{b}_0$ and pick a winning strategy for S in the game H_0. This game is played between S and F: F's first move will be a_1: which appears as a box in the diagram. All of F's other moves in the right-hand column of H_0 will be copied from moves by S in H_2; all of F's moves in the left-hand column will be dictated by σ; all of S's moves in this game will be dictated by τ_0.

After four moves have been played, giving $\alpha_2(1) = a_1$, $\alpha_0(1)$, $\alpha_0(2)$ and $\alpha_2(2)$ we may define b_2 as the last-named number, and proceed to set up the game H_2, in which F will play a_3 as the first move . All F's subsequent right-hand moves will be copied from H_4, all F's left-hand moves will be copied from H_0, all S's moves will be according to τ_2. After eight moves (four in H_2 and four in H_0) we shall have computed $\alpha_4(3) = a_3$, $\alpha_2(3)$, $\alpha_0(3)$, $\alpha_0(4)$, $\alpha_2(4)$ and finally $\alpha_4(4)$, which we define b_4 to be: we may now set up the next game and pick a strategy τ_4 for S in it.

F's first move will be a_5; all other moves will copied from H_6 or H_2.

At the end of time we shall have each α_k totally defined with $\alpha_k \longrightarrow= \alpha$. As α_0 obeys σ, it is a member of A. S has won the play of H_0 and so $\alpha_2 \leq^*_0 \alpha_0$: hence $\alpha_2 \in A$ and $\phi_0(\alpha_2) \leq \phi_0(\alpha_0)$. S has won the play of H_2; hence $\alpha_4 \in A$ and $\phi_2(\alpha_4) \leq \phi_2(\alpha_2)$. Inductively each $\alpha_k \in A$ and $\phi_k(\alpha_{k+2}) \leq \phi_k(\alpha_k)$.

The reasoning of Lemma 7·9 now applies to tell us that for each i, $\psi_i(\alpha_k)$ is eventually constant; hence the semi-scale property applies to tell us that $\alpha \in A$, as required.

Thus we have shown that the minimal policy, and hence the best strategy, indeed win for Adam in the game A.

It is evident that this strategy is definable: Moschovakis in his book shows that in the case that A is Σ^1_2, the best strategy that we have just defined is actually Δ^1_3.

We refer the reader to that book for those details; WE PAUSE FOR A DIAGRAM and then we essay a more set-theoretical calculation.

Diagram for demonstrating that the minimal policy succeeds

$$
\begin{array}{cccccccccccccc}
 & & H_0 & & & & & & & & & & & \\
\xmapsto{\sigma} \alpha_0(0) \longrightarrow & b_0 & & a_0 & & & & & & & & & & \\
 & {\scriptstyle H(\bar a_0,\bar b_0)} & \mathit{now} & \mathit{starts}: & & & & & & & & & & \\
 & & & \boxed{a_1} & & & & & & & & & & \\
 & & {\scriptstyle\tau_0}\swarrow & & & & & & & & & & & \\
 & \alpha_0(1) & & & & & H_2 & & & & & & & \\
 & \sigma\downarrow & & & & & & & & & & & & \\
 & \alpha_0(2) & & & & a_0 & & a_0 & & & & & & \\
 & & \searrow{\scriptstyle\tau_0} & & & a_1 & & a_1 & & & & & & \\
 & & & \alpha_2(2) & \longrightarrow & b_2 & & a_2 & & & & & & \\
 & & & & & {\scriptstyle H(\bar a_2,\bar b_2)} & \mathit{now} & \mathit{starts}: & & & & & & \\
 & & & & & & & \boxed{a_3} & & & & & & \\
 & & & & & & {\scriptstyle\tau_2}\swarrow & & & & & & & \\
 & & & \alpha_2(3) & \longleftarrow & \alpha_2(3) & & & & & & & & \\
 & & {\scriptstyle\tau_0}\swarrow & & & & & & & & & & & \\
 & \alpha_0(3) & & & & & & & & & H_4 & & & \\
 & \sigma\downarrow & & & & & & & & & & & & \\
 & \alpha_0(4) & & & & & & & & a_0 & & a_0 & & \\
 & & \searrow{\scriptstyle\tau_0} & & & & & & & a_1 & & a_1 & & \\
 & & & \alpha_2(4) & \longrightarrow & \alpha_2(4) & & & & a_2 & & a_2 & & \\
 & & & & & & \searrow{\scriptstyle\tau_2} & & & a_3 & & a_3 & & \\
 & & & & & & & \alpha_4(4) & \longrightarrow & b_4 & & a_4 & & \\
 & & & & & & & & & {\scriptstyle H(\bar a_4,\bar b_4)} & \mathit{now} & \mathit{starts}: & & \\
 & & & & & & & & & & & \boxed{a_5} & & \\
 & & & & & & & & & & {\scriptstyle\tau_4}\swarrow & & & \\
 & & & & & & & \alpha_4(5) & \longleftarrow & \alpha_4(5) & & & & \\
 & & & & & & {\scriptstyle\tau_2}\swarrow & & & & & & & \\
 & & & \alpha_2(5) & \longleftarrow & \alpha_2(5) & & & & & & & & \\
 & & {\scriptstyle\tau_0}\swarrow & & & & & & & & & & & \\
 & \alpha_0(5) & & & & & & & & & & & & \\
 & \sigma\downarrow & & & & & & & & & & & & \\
 & \alpha_0(6) & & & & & & & & & & & & \\
 & & \searrow{\scriptstyle\tau_0} & & & & & & & & & & & \\
 & & & \alpha_2(6) & \longrightarrow & \alpha_2(6) & & & & & & & & \\
 & & & & & & \searrow{\scriptstyle\tau_2} & & & & & & & \\
 & & & & & & & \alpha_4(6) & \longrightarrow & \alpha_4(6) & & & & \\
 & & & & & & & & & & \searrow{\scriptstyle\tau_4} & & & \\
 & & & & & & & & & & & \alpha_6(6) & \longrightarrow & \ldots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & \alpha_0 & & \alpha_2 & & \alpha_2 & & \alpha_4 & & \alpha_4 & & \alpha_6 & & \ldots
\end{array}
$$

Here the horizontal arrows are copying. The boxed moves are those of Eve against Adam's minimal strategy.

Definability of the best strategy

We have used AC$\aleph_0$R for picking strategies: WE SHOULD DO ALL THAT AT THE BEGINNING; that would be another application of $Det(OD_T)$.

The point of doing so is to show that the minimal strategy succeeds. NOT YET CERTAIN WHY IT DOES.

It is clear that if A is a projective set then the minimal strategy is projective.

$u \leqslant_k v$ iff there is a winning strategy in the game $H_\phi(u, k)$.

If $\alpha \in \mathfrak{p}[T]$, where is the left-most path for α ?

Now we need not have the left-most path. All we need is a definable way of choosing a path.

If $\alpha \in \mathfrak{p}[T]$, then $T(\alpha)$ is ill-founded, so it is ill-founded in $L[T, \alpha]$, so we could define the left-most path in that model. Will it be the left-most path ? Yes, I think it will.

We could say simply: take the first path constructible from α and T. How does that compare in complexity ?

7·11 CONJECTURE I believe that the natural hypothesis is "all games light-face projective in $\mathfrak{p}[T]$ are determined": no, we need its semi-scale. Cut "projective" down to Σ^1_2 or better.

To choose the strategies: play this game: first player says u, v; the second must produce σ or τ.

The definition of the best strategy requires the definition of the norms ψ_k.

I still feel ∞-Borel is in the air. The norms ϕ_k on A are ∞-Borel. Don't depend on the existence of strategies. So the games $H_k(u, v)$ are ∞-Borel.

Every Souslin set is ∞-Borel? Given T, $\alpha \in \mathfrak{p}[T]$ iff $T(\alpha)$ well-founded iff $T(\alpha)$ is well-founded in $L[T, \alpha]$.

Hugh has some theorem that given a set of reals A, if it is ∞-Borel then you can find an ∞-Borel code for it somewhere close at hand; but if it is Souslin, finding the Souslin tree is harder. But this sounds like junk; once you have found the Souslin tree, can you improve to get an ∞-Borel tree ?

If ∞-Borel is closed under projections then something happens; and then, also, the winning strategy is ∞-Borel.

Extent of scales

7·12 REMARK On the subject of scales in $L[\mathbb{R}]$, note the following.

Consider the sets

$$A =_{\text{df}} \{\langle x,\, y\rangle \mid y \text{ is OD from } x\}$$
$$B =_{\text{df}} \{\langle x,\, y\rangle \mid y \text{ is not OD from } x\}$$

A is Σ^2_1, since $\langle x,\, y\rangle \in A \iff \exists\zeta J_\zeta(\mathbb{R}) \models y$ `is ordinal definable from` x, and so A is $\Sigma_1(L[\mathbb{R}], \emptyset)$, and hence [**by Steel 1.12**] Σ^2_1. $\forall x \exists y\ \langle x,\, y\rangle \in B$ but B has no uniformisation in $L[\mathbb{R}]$, since any such would be ordinal-definable from some real, a say, and then the value given at a would indeed be ordinal definable from a, despite expectation. Hence B has no scale in $L[\mathbb{R}]$.

So, assuming AD plus $V = L(\mathbb{R})$, the sets with scales are precisely those in $\mathbf{\Sigma^2_1}$, by Wadge: let B be the above set, and C a scaled set. Then B is not (Lipschitz or Wadge) reducible to C, hence C is to $\neg B$, and so C is $\mathbf{\Sigma^2_1}$. On the other hand it is known that $\mathbf{\Sigma^2_1}$ has the scale property in $L[\mathbb{R}]$[R 13]

The main theorem of this section has the corollary

7·13 COROLLARY *Assume Det(Σ^1_2). Then if Adam has a winning strategy for a Σ^1_2 game, he has an* OD *one.*

which we shall apply in later sections.

Proof of the Corollary: immediate from Shoenfield's representation of Σ^1_2 sets as projections of constructible (class ?) trees. $\dashv$ (7·13)

[R 13] one of the Cabal volumes.

8: **Proof of the principle** *Slide*

Our goal in the present section is to establish the following

8·0 THEOREM $(ZF + V = L(\mathbb{R}) + AD + DC)$ *For any real* x, $HOD[x] = HOD_{\{x\}}$

which will be of pivotal importance in Chapter VII. The proof relies substantially on $V = L(\mathbb{R})$; AD is needed merely to supply a fine measure of the kind we have already seen to exist, and the rest of the argument is largely definability theory.

We shall refer to the conclusion of the theorem as the principle *Slide*.

Let $FM(\mu, A)$ be the statement that μ is a fine measure on $[A]^{\aleph_0}$. We shall use the existence of a fine measure μ, which we have established using AD, but which is rather weaker than AD so rather than assume $AD + V = L(\mathbb{R})$ in this section, we shall argue where possible just from the existence of μ and present our results in terms of $HOD_{\{\mu\}}$. Of course under AD such a μ is definable, so we would have $HOD = HOD_{\{\mu\}}$.

⊛ ⊛ ⊛ 8·1 The main steps in our argument are these:

8·2 PROPOSITION $(ZF + AD)$ *There is a definable* μ *with* $FM(\mu, \mathcal{N})$.

That is a special case of Theorem (2·5·3).

At the end of Chapter II we defined this set:

$$\mathcal{L} =_{\mathrm{df}} \{(x, \zeta) \mid x \in \mathcal{N} \ \&\ \zeta < \Theta \ \&\ \text{ the section at } x \text{ of the } \zeta^{\mathrm{th}} \text{ HB set is non-empty}\}$$

We shall prove, as promised, the following

8·3 THEOREM $(ZF + DC + FM(\mu, \mathcal{N}))$ *There is a formula* $\Phi(x, \zeta, S)$ *and an ordinal* $\kappa > \Theta$ *such that assuming* $AD{+}V = L(\mathbb{R})$*and that* S *is a subset of* Θ *such that* $HOD = L(S)$, *then for all reals* x *and ordinals* $\zeta < \Theta$,

$$(x,\, \zeta) \in \mathcal{L} \iff L_\kappa(S, x) \models \Phi(x, \zeta, S).$$

The proof uses general properties of forcing going back to Solovay's treatment of the measure problem.

8·4 COROLLARY $(ZF + DC + FM(\mu, \mathcal{N}))$ *If* X *is an* ∞*-Borel set of reals with code* S *then so is* $\exists^{\mathcal{N}} X$, *with code uniformly definable from* S *and* μ.

8·5 COROLLARY $(ZF + AD + DC)$ *If* A *is* HB, *so is* $\exists^{\mathcal{N}} A$.

The principle that if A is HB, so is $\exists^{\mathcal{N}} A$ enables us to complete our discussion of reals as generic over HOD, but using the *Borel algebras* $\mathbb{L}^k$ built from codes of HB sets rather than the *Vopěnka algebras* $\mathbb{K}^k$ built from codes of OD sets: we are able to show that all k-sequences of reals are generic over HOD w.r.t. the algebra $\mathbb{L}^k$, as before.

Our discussion of Borel algebras leads to a proof of the next point:

8·6 PROPOSITION $(V = L(\mathbb{R})\ )$ *If the class* HB *is closed under* $\exists^{\mathcal{N}}$, *then every* OD *set is* HB.

We prove that by repeating our previous argument that $L(\mathbb{R})$ is a co-generic extension of HOD, but now using the algebra $\mathbb{L}^\omega$ rather than $\mathbb{K}^\omega$, though in our context the two algebras will eventually prove to be isomorphic.

8·7 COROLLARY $(ZF + V = L(\mathbb{R}) + AD + DC)$ *Every* OD *set of reals is* HB.

8·8 PROPOSITION *If every* OD *set of reals is* HB, *then for each real* x, $HOD_{\{x\}} = HOD[x]$

Putting all the above points together we find that we have proved *Slide* to hold in the context in which we are most interested, and thereby have established Theorem 8·0.

8·9 REMARK It was proved by Kechris that DC is a consequence of $ZF + V = L(\mathbb{R}) + AD$: in this book we simply assume that DC holds, and neither use nor prove Kechris' result. In consequence there is a certain redundancy in the hypotheses of some of the above points.

8·10 REMARK The argument will relativise to show that if a set of reals is OD in the real α then it is ∞-Borel with code in HOD_α, which by *Slide* we shall know to be equal to HOD_α.

Actually there is an easy proof of that, as follows. Let $A = \{\beta \mid \phi(\alpha, \beta)\}$. Put $B = \{(\gamma, \beta) \mid \phi(\gamma, \beta)\}$. Then B is OD, and so HB. Any section of it, such as A, has a derived code in $HOD[\alpha]$: we need to know that taking a section of an ∞-Borel set yields an ∞-Borel set.

Action

We turn to the proof of 8·3. Weaken the definition of $\mathcal{L}$ for the moment: suppose merely that S is a set of ordinals which lists in a definable way tree-codes of HB sets. Each such has an OD ∞-Borel -code, and therefore a first such code, and we use $<_{OD}$ to list all first codes. Let θ be the length of this listing; for the moment we make no assumption that $\theta = \Theta$.

So we define

$$\mathcal{L}_S =_{\mathrm{df}} \{(x, \zeta) \mid x \in \mathcal{N} \ \&\ \zeta < \Theta \ \&\ \text{the section at } x \text{ of the } \zeta^{\text{th}} \text{ HB set in the list } S \text{ is non-empty}\}$$

8·11 THEOREM $(DC + FM(\mu, \mathcal{N}))$ *Let $S \subseteq \theta$ list, in length θ, one tree-code each of every* HB *set of reals. There is a formula $\Phi(x, \zeta, S)$ and an ordinal $\kappa_\infty > \theta$ such that for all reals a and ordinals $\zeta < \theta$,*

$$(a,\, \zeta) \in \mathcal{L}_S \iff J_{\kappa_\infty}(S, x) \models \Phi(\delta_\infty, a, \zeta_\infty, S).$$

Proof : by 3·0, we know that under the given hypotheses, $\aleph_1$ is measurable, incomparable with $2^{\aleph_0}$ and is strongly inaccessible in every inner model of AC. We shall use that fact to ensure that certain generic objects actually exist.

We write S_ζ for the ζ^{th} code listed by S and S^ζ for the HB set that it codes.

Fix a and ζ and suppose that b in S^ζ. Let σ be any countable set of reals with both a and b as members. By the discussion of §3 of Chapter I, any real in $L(S, \sigma)$, such as b, is generic over $M_\sigma =_{\mathrm{df}} HOD^{L(S,\sigma)}_{S,\sigma}$ with respect to a 1-dimensional Vopěnka algebra. Let us denote the corresponding ω-dimensional Vopěnka algebra, $(\mathbb{K}^\omega_S)^{L(S,\sigma)}$, more simply, by $\mathbb{K}(S, \sigma)$. Then that latter model is of the form $L(T_\sigma)$ where T_σ is a set of ordinals obtained uniformly from σ and coding $\mathbb{K}(S, \sigma)$, the canonical name, $\dot{x}(S, \sigma)$, in M_σ for the sequence being added, and the set S. S of course is likely to be an uncountable object.

We wish to show that $\mathbb{K}(S, \sigma)$ is countable in V, indeed that its power set in M_σ is countable in V. $L(S, \sigma)$ need not be a model of AC, but let $\gamma \in \mathcal{N}$ be any real such that $\sigma = \{(\gamma)_n \mid n \in \omega\}$. Then we know that there is an injection of $\mathbb{K}(S, \sigma)$ into $\big(\mathcal{P}(\mathcal{P}(\omega))\big)^{L(S,\sigma)}$, which is a subset of $\big(\mathcal{P}(\mathcal{P}(\omega))\big)^{L(S,\gamma)}$, which is in bijection with some ordinal η, since AC is true in $L(S, \gamma)$. Hence by 3·0 η is countable in V, and therefore by composition of injections, so is $\mathbb{K}(S, \sigma)$. Its power set, of cardinal $\delta(\sigma)$, say, in $HOD^{L[S,\sigma]}_{\{S\}}$, a model of AC, will also be countable in V, again by 3·0.

We have assumed that σ contains a b such that $\langle a,\, b\rangle \in S^\zeta$. This b will be the evaluation of $\dot{x}(S, \sigma)$ with respect to the $(M_\sigma, \mathbb{K}(S, \sigma)$ generic filter G^1_b. Some condition, p say, will force the statement

$$(\hat{a}, \dot{x}(S, \sigma)) \mathrel{\epsilon} \texttt{the set coded by } \widehat{S_\zeta}$$

and generics for $\mathbb{K}(S, \sigma)$ containing p are easily created from an enumeration in order type ω of the ground model's power set of the algebra in question. Thus if we treat $L(T_\sigma, a)$ as the ground model and make an extension by $Coll(\omega, \delta(\sigma))$, the usual Lévy collapse generated by finite conditions $f : n \xrightarrow{1-1} \lambda$ and making $\delta(\sigma)$ countable, we shall find in that extension a generic for $\mathbb{K}(S, \sigma)$ containing p and therefore a real y with $(a, y) \in S^\zeta$.

The statement "`the section at` $\hat{a}$ `of the set coded by` $\widehat{S_\zeta}$ `is non-empty`" contains only names of objects in the ground model, and therefore by the well-known homogeneity of collapsing algebras can only have truth value $\mathbb{0}$or $\mathbb{1}$. Therefore let $\Psi(\delta, a, \zeta, S)$ be the assertion that that statement has truth value $\mathbb{1}$: so

$$\Psi(\delta, a, \zeta, S) \iff [\![\texttt{the section at } \hat{a} \texttt{ of the set coded by } \widehat{S_\zeta} \texttt{ is non-empty}]\!]^{Coll(\omega,\delta)} = \mathbb{1}.$$

The measure μ is fine, and hence the set of σ containing a given pair a and b is of measure 1. Thus we have shown that if $(S^\zeta)_a$ is non-empty, then for μ-almost all σ, and an appropriate *kappa*,

$$J_\kappa(T_\sigma, a) \models \Psi(\delta(\sigma), a, \zeta, S).$$

Conversely, if for some σ, $J_\kappa(T_\sigma, a) \models \Psi(\delta(\sigma), a, \zeta, S)$, then since by 3·0 a $(L(T_\sigma, a), \mathrm{Coll}(\omega, \delta(\sigma))$- generic H exists, $(S^\zeta)_a$ will actually be non-empty, having an element in $J_\kappa(T_\sigma, a)[H]$.

Thus, remembering that T_σ codes S,

$$(a,\zeta)\in\mathcal{L}\iff \text{ for } \mu\text{-almost all } \sigma,\ J_\kappa(T_\sigma,a)\models\Psi(\delta(\sigma),a,\zeta,S).$$

We form therefore the ultrapower over all σ of the structures $P_\sigma =_{\mathrm{df}} \langle J_\kappa(S,T_\sigma)\rangle$, $Q_\sigma =_{\mathrm{df}} \langle J_\kappa(S,T_\sigma,a)\rangle$, and of the T_σ's, $\delta(\sigma)$'s, a, S, and ζ. Call the results P_∞, Q_∞, T_∞, δ_∞, a_∞, S_∞ and ζ_∞. These structures all have definable well orderings (uniformly in a where applicable) so model AC and so we shall be able to establish Łoś's theorem. Fortunately, the measure is countably complete, and so these structures are, by DC, well-founded sets and may be taken to be transitive. Further $a_\infty = a$, being essentially a countable sequence of integers. T_∞ and S_∞ will be sets of ordinals. Q_∞ will equal $J_{\kappa_\infty}(S_\infty,T_\infty,a)$. We shall have

$$(a,\zeta)\in\mathcal{L}\iff J_{\kappa_\infty}[S_\infty,T_\infty,a]\models\dot\Psi[\delta_\infty,a,\zeta_\infty,S_\infty],$$

and thus have found an HB description of $\mathcal{L}$. ⊣ (8·11)

8·12 Corollary $(ZF+DC+FM(\mu,\mathcal{N}))$ *If X is an ∞-Borel set of reals with code S then so is $\exists^{\mathcal{N}}X$, with code uniformly definable from S and μ.*

8·13 Corollary *Assume DC and that there is a countably complete fine measure μ on $[\mathcal{N}]^{\aleph_0}$. Then the class* HB *is closed under $\exists^{\mathcal{N}}$.*

Proof: any member of $\exists^{\mathcal{N}}$HB is of the form $\{a\mid (a,\zeta)\in\mathcal{L}\}$ for some ζ. ⊣ (8·13)

Higher dimensional Borel algebras

Now we assume that HB is closed under $\exists^{\mathcal{N}}$, which was the point at which our discussion in Chapter II halted. With that assumption, it may resume its progress. We may define in analogy to the work in Chapter I, §3, algebras $\mathbb{L}^k$ and an algebra $\mathbb{L}^\omega$, and establish harmonious relations between them.

We work from our HB description of the set $\mathcal{L}$. First note that each a is generic over $L(S)$. Note then that we can in any $L(S,a)$ recover $\mathbb{L}^{1,a}$: we have seen how to extract codes of a-sections uniformly, and we can read from $\mathcal{L}$ the partial order: to see if $C_a\subseteq D_a$, find an index ζ for $C\setminus D$ and ask if $(a,\zeta)\in\mathcal{L}$.

$\mathcal{L}$ can handle sections of any finite number of dimensions — that after all is merely coding. Thus in $L(S,a)$ we can build all the algebras $\mathbb{L}^{1,k}$ for any k, giving us the extension from where we are; and also the algebra $\mathbb{L}^{1,\omega}$, giving us the full extension. We write $\mathbb{L}_a$ for that algebra.

8·14 Proposition *$\mathbb{L}_x$ is weakly homogeneous and thus satisfies the 0-1 law.*

Proof: $\mathbb{L}_x$ is generated by non-empty sections at x of HB sets. Our argument in chapter 1 will go through: in effect, we consider permutations of finite support of $\omega\setminus\{0\}$. ⊣ (8·14)

The diagram returns

8·15 Proposition *If $V=L(\mathbb{R})$ and the class* HB *is closed under $\exists^{\mathcal{N}}$, then every* OD *set is* HB.

Proof: Let A be an OD set, D its ∞-Borel code, so $D\in HOD$.

The assumption is enough, as we have seen in our discussion of Vopěnka algebras, to produce the picture

$$\begin{array}{ccc} L[\mathbb{R}] & \xrightarrow{\mathbb{E}} & L[\vec\alpha] \\ & & \uparrow \mathbb{L} \\ & & HOD \end{array}$$

where, to recapitulate, $\mathbb{E}$ is the forcing that adds an enumeration of $\mathbb{R}$, and $\mathbb{L}=\mathbb{L}(\omega)$ is the forcing formed from the forcings $\mathbb{L}(k)$ (for $k\in\omega$) with respect to which each k-sequence of reals is generic over HOD.

For any real a, the extension from $HOD[a]$ up to $L[\vec\alpha]$ is given by the algebra $\mathbb{L}^{1,a}$. Hence we can say that for any cardinal η comfortably larger than $\bigcup S$ and ζ,

$$x\in A\iff J_\eta[S,x]\models [\![\texttt{In the submodel } L[\{\alpha_i\mid i\in\omega\}] \texttt{ of the extension } L[\vec\alpha],$$
$$\hat x\ \epsilon\ \texttt{the set coded by } \hat D]\!]^{\mathbb{L}_x}=1$$

That is plainly an HB definition of A. ⊣ (8·15)

8·16 PROPOSITION *if every* OD *set of reals is* HB, *then for each real* x, $HOD_{\{x\}} = HOD[x]$.

Proof : We already know that $HOD_{\{x\}} = HOD[G_x]$,and that $HOD[x] \subseteq HOD[G_x]$; we must show that $G_x \in HOD[x]$. Let $F(C^o)$ be the first ∞ Borel code for C. F is in HOD. Then

$$\begin{aligned} C^o \in G_x &\iff x \in C \\ &\iff x \in \text{the set } \infty\text{-Borel-coded by } F(C^o) \end{aligned}$$ ⊣ (8·16)

We have now reached the result that we shall use in chapter VII in building a model containing ω Woodin cardinals starting from a model of AD:

8·17 THEOREM *Assume that* $V = L(\mathbb{R})$ *and that* AD *holds. Let* S *be the canonical definable set of ordinals such that* $HOD = L[S]$. *Then for each real* x,

$$HOD_{\{S,x\}} = HOD[x] = L[S, x].$$

Proof: In the circumstances, $HOD_{\{S,x\}} = HOD_{\{x\}}$, since S is definable; $HOD_{\{x\}} = HOD[x]$ by the results of this chapter ; $HOD = L[S]$ and so $HOD[x] = L[S, x]$. ⊣ (8·17)