Set Theory and Logic Extra Example sheet on Propositional Logic

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February 10, 2022

You might need to look up conjunctive and disjunctive normal forms.

(i)

Show how $\wedge$, $\vee$ and $\neg$ can each be defined in terms of $\to$ and $\bot$. Why can you not define $\wedge$ in terms of $\vee$? Can you define $\vee$ in terms of $\to$? Can you define $\wedge$ in terms of $\to$ and $\vee$?

(ii)

Explain briefly the relation between truth-tables and Disjunctive Normal Form.

Explain briefly why every propositional formula is equivalent both to a formula in CNF and to a formula in DNF.

Establish that the class of all propositional tautologies is the maximal propositional logic in the sense that any superset of it that is a propositional logic (closed under $\models$ and substitution) is trivial (contains all well-formed formulæ).

(iii) 

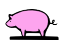
(a) Suppose A is a propositional formula and 'p' is a letter appearing in A. Explain how to find formulæ A_1 and A_2 not containing 'p' such that A is logically equivalent to $(A_1 \wedge p) \vee (A_2 \wedge \neg p)$.

(b) Hence or otherwise establish that, for any two propositional formulæ A and B with $A \models B$, there is a formula C, containing only those propositional letters common to both A and B, such that $A \models C$ and $C \models B$. (Hint: for the base case of the induction on the size of the common vocabulary you will need to think about expressions over the empty vocabulary).

(iv)

Why does T not follow from K and S?

Show that Peirce's Law: $((A \to B) \to A) \to A$ cannot be deduced from K and S.

(v) 

A *type* in a propositional language $\mathcal{L}$ is a countably infinite set of formulæ.

For T an $\mathcal{L}$-theory a T-*valuation* is an $\mathcal{L}$-valuation that satisfies T. A valuation v *realises* a type Σ if v satisfies every $\sigma \in \Sigma$. Otherwise v *omits* Σ. We say a theory T *locally omits* a type Σ if, whenever ϕ is a formula such that T proves $\phi \to \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg\phi$.

(a) Prove the following:

Let T be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type. If T locally omits Σ then there is a T-valuation omitting Σ.

(b) Prove the following

Let T be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq \mathcal{L}(T)$ be a type. If T locally omits every Σ_i then there is a T-valuation omitting all of the Σ_i.

(vi)

Prove that, for every formula ϕ in Conjunctive Normal Form, there is a formula ϕ' which

(i) is satisfiable iff ϕ is;

(ii) is in CNF where every conjunct contains at most three disjuncts.

(Hint: there is no assumption that $\mathcal{L}(\phi') = \mathcal{L}(\phi)$.)