Allen Hazen's notes on Modal Logic transcribed and copy-edited by

Thomas Forster

March 24, 2025

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`From: Allen Hazen <aphazen@ualberta.ca>`
`Sent: Saturday, March 22, 2025 7:54 PM`
`Subject: LRG: references on axiomatics of QML`

Dear Logic Group–

(Somewhere between a somewhat supplemented summary of and a bibliography for my presentation on 18 March 2025: I hope the bibliography is findable without having to read all the accompanying prose!) (Contents: introductory paragraphs about what was supposed to happen and what did happen at the 18 March meeting, with links to Greg Restall's papers and talk, and then. . . . In sorta random order:

(A) Systems of modal logic

(B) Adding quantifiers thereto

(C) Kripke's Helsinki paper

(D) Quine's axiomatisation of quantifier logic (this section has more stuff that I didn't talk about))

Greg Restall (St Andrews now, but from Queensland originally) was giving two talks in Calgary this past weekend, and the plan was that Bernie Linsky, Jeff Pelletier and I would go down to hear him, and then report at this week's (18.iii.2025) LRG meeting. The best of plans.... The weather forecast suggested dangerous driving, and I came down with a cold (sniff...), so...

Plan B was that, since, in his talk on Quantified Modal Logic, Greg was going to talk about a joint paper of Bernie Linsky and Ed Zalta's, Bernie would introduce the main points of that paper, and I would then say something about Greg's response....

So, Bernie did his part, and I went off on a ramble about the history of the axiomatics of Quantified Modal Logic and what they had to do Contingent Existence. Relevant references below. Of Greg's Calgary talks, the one on the philosophy of mathematics was based on a paper that has come out in the Proceedings of the Aristotelian Society and is available here: `http://dx.doi.org/10.1093/arisoc/aoaf002` The one on QML and Contingent Existence draws on stuff in a not-yet-published paper, `https://consequently.org/writing/mlce-ge2/` (consequently.org is Greg's personal website; a number of his papers are available there.)

> I don't know if I'll have anything more to say about this next week, but if you are looking at this paper, look at the example of (semi-) informal modal reasoning in the indented paragraph on page 11, deriving (Possibly(P) or Possibly(Q)) from Possibly(P or Q). The inference is valid in (most) modal logics, but the reasoning here is not reproducible in (most) conventional axiomatic or natural deduction formulations of (most) modal logics: the "where we started" bits aren't formalizable in the notation of modal propositional logic. The inference is typically recovered by a use of Indirect Proof: Fitch, who gives an Intuitionistic variant of modal logic as well as a classical one, postulates it as an extra rule in his Intuitionistic systems. Gregf's formalization of this reasoning on page 15, therefore, suggests that his "hypersequent" formalism for modal logic is somehow more "manœuverable" than conventional axiomatic or natural deduction formulations ...

And the version of his QML talk – including discussion with Ed Zalta – at Chapman University in California was recorded and is available here: `https://youtu.be/4A8pV5fT6lo` (No, I didn't know anything about Chapman University either, but gleaned one thing from its WWWeb site: its Philosophy department is NOT in the Humanities faculty but in the Science faculty!) —-

1 (A) Modal Logics

Back at the beginning of the (modern) study of QML, Ruth Barcan [1] considers first-order extensions of two of C.I. Lewis's modal logics: $S2$ and $S4$. $S4$ we all know and love, but $S2$ – though it seems to have been C.I. Lewis's favourite, and was anyway familiar to logicians of the 1940s – is weird and strange and (technical term) non-normal, and I thought it would take us too far afield to discuss in detail, so, in discussing this, I pretended she had worked with the (now!) better-known system T.

History of T. T was proposed by Feys (under the name "t") in 1937, and by Von Wright (under the name "M") in 1951. (References in [11] Hughes and

Cresswell *vide infra*.) (So older literature often calls it "the system of Feys and Von Wright.") It was also presented, in a natural deduction formulation, by Fitch in [5]. Fitch didn't give it a name, describing it only as "a system almost the same as the system Lewis calls $S2$," but since Feys's publication was over a decade before Von Wright's and Fitch's only one year after, it seems a bit unfair to say "system of Feys and Von Wright" and not "system of Feys, Von Wright, and Fitch..."

Fitch's natural deduction systems are very easy to use, and have been followed by many textbooks. Fitch himself surveyed natural deduction formulations of a variety of modal (and deontic) logics in [6].

If you really want to find out about S2, I recommend chapter 12, 13 and 15 of [11].

(NB. This is the original, 1968, Hughes and Cresswell, one of the best logic TEXTbooks ever written. Their later "New Introduction to Modal Logic" is a completely new – and rather less introductory – book, with only limited overlap with the earlier book, and doesn't cover the Lewis systems. ...If you really like going to original sources, you could go to Lewis and Langford's (1932) "Symbolic Logic," but I recommend Hughes and Cresswell, and, anyway, chapter 13 contains relevant later material not covered in L and L.)

Lewis's original formulation (see chapter 12 for details) is very strange to a modern eye. It has modus ponens for strict implication as a rule, but not modus ponens for the usual (material) implication, and it contains a primitive rule of substitution of proven equivalents. (My personal view is that any philosophical motivation for some of the system's other features would be likely to go against this rule, and that its adoption was a philosophical mistake. But Lewis was schooled in the older, algebraic, tradition of Peirce, Schröder and Royce, and so was accustomed to this sort of rule.)

$S2$ does have a Kripke-style model theory (presented in chapter 13 and originally set forth in [13].

Start with a Kripke model for T: possible worlds with a reflexive but otherwise unrestricted accessibility relation. Now classify the worlds into two sets, the normal and non-normal, with the distinguished (actual) world of the model being one of the normal ones. In calculating the truth value of a formula at a world

- do atomic formulas and truth functional compounds the right way,
- do modal formulas the right way at normal worlds, and
- at non-normal worlds, give every formula starting with a necessity operator the value `false` (and every formula starting with a possibility operator the value `true`).

This gives us a useable, non-weird, proof procedure. Add a new propositional constant, n, to the language (to be thought of, when you think of the models, as true at all and only normal worlds). Allow n as a new "axiom," without allowing necessitation to be applied to formulas derived with its aid. (Easier statement in Fitch-style natural deduction: n can be inserted freely in the main proof, but not "reiterated" into the "strict" subproofs of the Necessity Introduction and Possibility Elimination rules.) Now interpret the language of $S2$ in the (augmented) language of T:

> Atomic fmlas and truth functional compounds are translated straight,
> $S2$-neccessarily(P) as ($n \wedge T$-necessarily(P)); and
> $S2$-possibly(P) as ($n \rightarrow$T-possibly(P)).

...Marcus despaired of finding a "deduction theorem" for $S2$. For typical axiomatic formulations of logics, the deduction theorem amounts to the admissibility (theorem-hood preservingness) of an $\rightarrow$ Introduction rule. So this at least gives us what to aim for in revisiting her early work.

2 (B) Quantifiers in Modal Logic

In standard (and what were the most familiar from Whitehead-and-Russell and Hilbert and Bernays on down) axiomatic formulations of quantification theory, a proof is a column of formulas (which don't have to be closed formulas: they can fail of being sentences by containing. free variables). In particular, axioms may be open formulas. There is then a rule of Universal Generalization. allowing the "inference" of $(\forall x)F(x))$ from $F(y)$.

(Note that if we converted open formulas to sentences by replacing free variables with constants, instances of this rule would not generally be truth-preserving. They are, however, *validity*-preserving, and this is what is required to justify their inclusion as "theoremhood preserving" rules in an axiomatic logic.) Standard natural deduction formulations of quantifier logic have things that, for our purposes, do the same work: the rule of Universal Quantifier Introducton replaces the rule of generalization (and also, in the context of the rest of the system, the axiom (-schema) of the distribution of the universal quantifier over the conditional: $\forall x(Fx \rightarrow Gx) \rightarrow (\forall x(Fx) \rightarrow \forall x(Gx))$. (Different versions of Natural Deduction have different formulations of the rule: it's as if coming up with a novel variant was a requirement on authors of elementary logic textbooks! My own personal favorite is Fitch's version, as in his book cited above and in Rich Thomason's 1970 textbook with the same title. ...For a bit of discussion on the WWWeb and illustrative derivations, and for the modifications needed for Free Logic, see the "Stanford Encyclopedia of Philosophy" article on Natural Deduction, by F.J. Pelletier and A.P. Hazen.)

Modern axiomatic formulations of (normal) modal logics have a rule of Necessitation (if A is a theorem so is $\Box A$) and the "K axiom" of distribution of necessity over the conditional,

$$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

in nice natural deduction systems for modal logic there is a rule of Necessity Introduction which (in the context of the rest of the system) does the same work. (There is a systematic analogy between the rules (and axioms) for the universal quantifier and those for the necessity operator:

Universal Quantifier	**Necessity Operator**
Rule of generalization	Necessitation rule
Distribution axiom	K axiom:
$\forall$-Introduction	Necessity Introduction

an analogy which is clearest in the modal logic $S5$ but gets attenuated in weaker modal logics. My own deep conviction is that this analogy is strong evidence for – or maybe is the ground for – the naturalness of construing necessity and possibility as a kind of quantification over (something like) possible worlds .... One of the things I like about Fitch's formulation of (and notation for) natural deduction is that it displays this analogy very clearly.)

The basic fact is that if you combine this sort of quantificational machinery with this sort of modal machinery you can derive the Converse Barcan Formula,

$$\Box\forall x(Fx) \rightarrow \forall x\Box Fx).$$

In Fitch style (for $S4$)

1	$\Box\forall x(Fx)$	Hypothesis
2	(a) $\mid$	Beginning of "general" subproof with proper vble a
3	$\mid\ \mid$	Beginning of "strict" subproof
4	$\mid\ \mid\ \Box\forall x(Fx)$	1, "reiterated"
5	$\mid\ \mid\ \forall x(Fx)$	4, Necessity Elimination
6	$\mid\ \mid\ Fa.$	5, Universal Quantifier Elimination
7	$\mid\ \ \Box(Fa)$	Necessity Introduction, from strict subproof 3–6
8	$\forall x\Box(Fx)$	$\forall$- Intro., from general subproof 2–7
9	$\Box\forall x(Fx) \rightarrow \forall x\Box Fx)$	$\rightarrow$-Intro, from (hypothetical) subproof 1–8

(Fitch's machinery uses general and strict subproofs as "premisses" for the U.Q. and Necessity Introduction rules: soundness of the rules is ensured by placing restrictions on what earlier formulas can be "reiterated" into such subproofs. (1), being closed and starting with a $\Box$ operator, qualifies for both kinds of subproof.)

Ruth Barcan, in her 1946 paper [1], showed that the converse Barcan Formula was provable in the straightforward combination of the usual axiomatic machinery of quantification with $S2$ or $S4$. (This was a lot harder for her than it is for us, as she was using C.I. Lewis's axiomatic formulations of the modal logics, which don't have Necessitation as a primitive rule!) She was unable to prove the Barcan Formula itself,

$$\forall x\Box(Fx) \rightarrow \Box\forall x(Fx)$$

(it can't be done in these logics), and so she proposed adding it as an additional axiom in the quantified modal systems.

> Historical accuracy time: Barcan actually stated the Barcan Formula and Converse Barcan Formula in their dual forms, with possibility and existential quantifier operators, and with strict implication rather than material as the main connective. Most later literature on the topic has used the version I use here.

Exploring the surrounding jungle a bit, it turned out that if you combine the stronger modal logic $S5$ with the same sort of quantificational machinery, the Converse Barcan Formula and the Barcan Formula itself are both provable; see Prior, [14].

Neither Marcus in her early papers nor Prior said anything about the intuitive meaning of these formulas: that seems to appear out of nowhere with Kripke's work in the 1960s... at least if you look only at the modal logic literature. I asked Ruth Barcan (Marcus) about this once, in a hotel lobby at an American Philosophical Association covention. She said the interpretation – under which both can fail if the "existent' objects the quantified variables range [over] is allowed to vary – was worked out in the 1950s in the context of temporal or tense logics: assume the quantified variables range over the objects in existence at a particular time, and replace the Necessarily operator with a "it's not *merely true now* but it will stay that way forever" operator. See Prior's papers and books from the 1950s for details.

When Kripke started thinking about the intuitive meaning of modal operators (and I think this was one of his most important and original contributions), the provability of the Converse Barcan Formula and the Barcan Formula in quantified versions of attractive modal logics... became an embarrassment. Because, if you think of the quantified variables as ranging over objects whose existence is contingent (objects, that is, like us rather than numbers), the Converse Barcan Formula and the Barcan Formula seem intuitively invalid!

3 (C) Kripke's Helsinki Paper

Kripke [12].

After arguing convincingly for the intuitive invalidity of the Converse Barcan Formula and the Barcan Formula, Kripke addresses the mismatch between the axiomatic systems familiar from the literature and the intuitive validity concept. The solution, of course, is to change the axiomatic system! After canvassing some more radical alternatives (Prior's system Q, for example (which can be seen as motivated by something akin to "serious actualism")), Kripke proposes a ... simple? ... alternative: replace the "standard" axiomatic treatment of quantification theory with one that does not have a rule of Universal Generalization: one in such no such rule is needed, since proofs are columns of sentences (= closed formulas). (In particular, all the axioms are sentences. Since no open formulas occur in a proof, there is no need to add quantifiers to them by means of a Universal Generalization rule.) This system was a slight modification of the axiomatic system of quantification theory of Quine's [15] [16] Mathematical Logic (1st edition 1940, revised edition 1951). (Further discussion in section 4, below.) This axiomatisation would have seemed familiar to many logicians in 1963, though I don't think it was ever as widely used as systems with Universal Generalization: the change from "industry standard" logical technology was minimal. (Cf. Quine's "Maxim of minimum mutilation" in theory change.) It doesn't involve adding new concepts – such as the Existence predicates often used in Free Logic – to the basic language of quantified modal logic. Rhetorically, it's brilliant! The "embarrassment" presented by Barcan's and Prior's results is dissolved by a simple change in the axiomatisation, with no need for anything new or strange.

How satisfactory is Kripke's system? Well. . . One thing to note is that it is a deductive system for a severely restricted modal language. Kripke's language has no individual constants in it. (So it is in one way a very Quinean formulation of QML: singular reference is completely handled by the quantified variables!) If it were extended to include individual names, the language would have to be enriched with further machinery – like the Existence predicate – from Free Logic. (Possibly by adding the identify predicate, $=$, and defining "a exists" as "$\exists x(x = a)$". I *think* that various features of Free Logic would then follow automatically from a Kripke-style axiomatisation.) . . . Even restricting ourselves to quantified variables, there are things we might think we should say, but can't. The Barcan Formula isn't valid, of course, but what about this weaker variant: $\forall x \Box(Fx) \rightarrow \Box \exists x(Fx)$? The Barcan Formula itself fails because, even if every actual object is necessarily F, there might be additional objects in other worlds which fail to be F. But at least, one is tempted to argue, the actual objects would exist there, and they would be F. No. What if the predicate F expresss a property – maybe that of "either existing or not existing" – which an object can have without existing? In that case the actual objects would still be F at other worlds, but there might not be – might not exist – anything F at some worlds. The problem is that the distinction between properties that hold only of existing objects and "funny" properties can't be expressed in Kripke's language (I think): the further conceptual resources of Free Logic really have to be added to QML to make it an adequate language for formulating metaphysical issues in.

I'd like to suggest that Kripke's axiomatisation of QML isn't really an alternative to versions using Free Logic, but rather amounts to a subsystem of them. (Which really isn't meant as a criticism of Kripke: see above about the rhetorical role played by his axiomatisation in the paper.)

For comparison. . . There are many presentations of formulations of QML that use Free Logic to allow for contingent existents. My favorite – I'm not claiming this is the best, but it's one I know how to find – is in [10].

($S5$ with two kinds of quantifiers: the Barcan Formula and the converse Barcan Formula holding for one but not the other. Written to be expository, has displayed sample derivations which may be useful.)

The basic idea is to have quantifier rules that have extra premisses. In Universal Quantifier Elimination, Fa is inferred from $\forall x(Fx)$ along with "a exists: Existential Quantifier Introduction is similar. With Universal Quantifier Introduction and Existential Quantifier Elimination, the "general" subproof used is allowed an extra hypothesis, to the effect that the proper variable of the subproof designates an existent.

4 (D) Quine's Axiomatisation of Quantifier Logic

((D1) Basic description, followed by

(D2) The modification leading to Kripke's axiomatisation of QML, followed by

(D3) A discussion of why it works, followed by

(D4) ruminations about what the "simplest" version of QML might be, and ending with (D5) references about a further complication, not immediately relevant to QML)

4.1 (D.1)

Quine used an axiomatisation of quantification theory in which only sentences (= closed formulas) occur as lines in proofs in his Mathematical Logic [15], [16].

> Aside: I have argued – in unpublished talks – that there is a philosophical motivation for this. Giving an adequate semantic account of open formulas is difficult: see various comments in Bertrand Russell's writings in the first decade of the 20th century about the mysterious and problematic nature of "the variable." And Quine, in the period when he was writing [15] was supervising his first Ph.D. student, Leigh Steinhardt (later, as Leigh Cauman, the managing editor of the "Journal of Philosophy"). Her thesis? "The Variable in its Relation to Semantic Problems." Quine was perhaps unwilling to mar the purity of his logic book with metaphysical preaching, but I think he was consciously showing by example that the free (or "real") variable could be replaced in logic by the bound (or "apparent") one.

It doesn't look that way if you just glance at the book. Displayed formal proofs contain what look like (abbreviations for) open formulas, and even the axioms are specified (*100 to *104, on page 88 of the revised edition [16]) by what look like schemata for open formulas. Officially, however. . . by a carefully explained notational convention, all these open-looking formulas are to be thought of as abbreviations for (certain of) their universal closures: the sentences formed by binding all their free variables (in a specified order) by initial universal quantifiers.

But more is involved than simple notational convention. What looks, in displayed proofs, like an instance of modus ponens (inferring open formula B from open formulas $A \to B$ and A) has to be interpreted as an inference of the sentence ZB from the sentences $X(A \to B)$ and YA, where X, Y, and Z are appropriate strings of universal quantifiers. This is NOT a form of *modus ponens*: the main operator of $X(A \to B)$ is a universal quantifier, not the displayed $\to$. It is an instance of a new rule. . . or perhaps more accurately a new set of rules, distinguished by the number of invisible universal quantifiers involved. To show that this (these) rule(s) is sound, and admissible in Quine's official system requires a substantial metatheorem, *111 (proof on pp. 90-91).

The end result is an axiomatisation of quantification which – once the notational conventions and a few basic metatheorems have been internalized – seems about as usable as the ordinary ones, but without the (dangerous!) rule of universal generalization.

4.2 (D.2)

A modification in the direction of free logic. $\exists x(Fx \vee \neg Fx)$ is a theorem of standard quantification theory. Formally, this leads to the adjective in the usual definition of validity: a formula is valid if and only if it holds in all non-empty domains. Metaphysically it suggests that the existence of at least one thing is provable by logic alone: eat your heart out, St. Anselm! Russell had famously commented (in ?? Introduction to Mathematical Philosophy ??) that the provability of theorems like this was "a defect of logical purity." In the early 1950s a variety of thinkers thought about it.

For a reasonably short, but still depressing, display of ordinary language philosophy at its worst, see the articles – winning submissions in a contest! – in Analysis 14 (1953), pp. 1-5.

Mostowski had a better idea: modify the axiomatisation of First Order Logic so as to have as theorems only formulas true in all domains, empty as well as non-empty. But he was working with axiomatisations having open formulas as theorems and ... guessed wrong? ... about the best way of handling the complexities this presented. At which point Theodore Hailperin, [7] avoided the trap by starting with Quine's system, in which all theorems are sentences: a fairly minor change yielded a system allowing for proof of all and only the sentences (of pure First Order Logic: no identity, no individual constants) true in all domains, empty and non-empty. Quine, in the next volume of the JSL, published a note [17]. In this, Quine makes useful (the note is still worth reading!) conceptual points, and makes (only partially successful: see section 4.5) for suggestions for simplifying Hailperin's axiomatisation.

The form of First Order Logic axiomatised in these papers is called inclusive f.o.l.: the empty domain is included in the definition of validity. Since its language doesn't have individual constants, it doesn't say anything about how to handle non-designating singular terms. The term "Free Logic" is sometimes used for a version of f.o.l. that requires a non-empty domain (so: has theorems like $\exists x(Fx \vee \neg Fx)$), but allows non designating terms (so: does not have theorems like $F(a) \rightarrow (\exists x)F(x)$. Versions of f.o.l. dropping both kinds of "existential presupposition" are pedantically called "Universally Free Logic"; I'm happy to use "Free Logic" in this more, um, inclusive sense.

Kripke's "Semantical considerations" paper [12] essentially proposes the adoption of the Hailperin/Quine modification of Quine's Mathematical Logic, adding modal operators to the language (and allowing necessity operators to be interspersed ad libitum in the initial sequences of universal quantifiers in the axioms).

(Hailperin went on to use inclusive quantification theory in defining a powerful and flexible extension of the idea of a many-sorted first order logic: see [8] and [9]).

4.3 (D.3) How it works

The axioms of Hailperin/Quine "inclusive" logic include (universal closures of) formulas of the form $\forall x(Fx) \rightarrow F(x/y)$, where $F(x/y)$ is the formula you get from Fx by replacing the (free) occurrences of x in Fx with (free: by section 18 and metatheorem *170 in section 21 of [15] and [16], I think, we can avoid cases where there would be a clash of bound variables) occurrences of y. For inclusive logic (and so for Kripke's QML), replace these with (universal closures of) formulas of the form $\forall y(\forall x(Fx) \rightarrow F(x/y))$. How is this supposed to help?

You might think it isn't any change at all! We're talking about universal closures, after all. So the variable y would be bound by one of the string of "invisible" universal quantifiers beginning the sentence, and permuting these quantifiers we could make $\forall y$ into the last of the string. But then the universal closures of $\forall x(Fx) \rightarrow F(x/y)$ and of $\forall y(\forall x(Fx) \rightarrow F(x/y))$ would turn out to be the same sentence!

True enough ... if y is there!

But now consider the (easily overlooked) degenerate case in which the quantifier $\forall x$ in $\forall x(Fx)$ is vacuous: the case in which Fx contains no (free) occurrence of x, and in which Fx and $F(x/y)$ are the same formula. In this case, the original axioms from Mathematical Logic let you erase the vacuous quantifier, whereas the revised axioms don't.

It turns out that vacuous quantifiers, which are an annoying but ultimately trivial notational feature of "exclusive" f.o.l., are non-trivial in inclusive f.o.l. (This point is nicely discussed in Kathleen Johnson Wu, [19].

On the sensible treatment (see Quine's note for a defence of this evaluation) of inclusive logic, any sentence beginning with a universal quantifier is true, and any sentence beginning with an existential quantifier false, in the empty domain. (As Quine points out, if you can't remember a proof procedure for inclusive logic, this gives you a substitute: prove your sentence in ordinary, non-inclusive, f.o.l., and then mechanically check that it is true in the empty domain!) So consider something like

$\forall x \exists y(Fy \vee \neg Fy)$, where x has no occurrence in $\exists y(Fy \vee \neg Fy)$. The quantifier $\forall x$ is vacuous, but dropping it would convert a sentence true in the empty domain into one false in the empty domain.

(The same principle is at work in formulations of Free Logic with an existence predicate. To infer Fa from $\forall x(Fx)$, you need a second premiss: a exists. If the $\forall x$ in $\forall x(Fx)$ is vacuous, there is no a visible, but the rule still requires a second premiss. But what extra premiss? a exists? b exists? ... It doesn't matter: if any such second premiss is true, the domain has to be non-empty, and dropping the vacuous quantifier is harmless.)

4.4 D.4

The simplest QML? Linsky and Zalta have argued that QML formulated with a Universal Generalization rule is the "simplest" quantified modal logic. But there are alternative axiomatisations of quantification theory without this rule. So is (to give it a name) Barcan-style QML really simpler than Kripke-style? I think we can address this issue most simply by ignoring modality! Is "conventional" axiomatic quantification theory somehow simpler than its Quinean alternative? And this, I think, requires thinking harder about the relevant notion of simplicity.

Thesis. Quantification theory with Universal Generalization is simpler than Quine's in that it gives simpler proofs. Quine's system is usable, but as we have seen this involves a bit of a cheat: notational conventions and tacit appeal to metatheorems allow us to give Quine-proofs that look conventional. But if we look at the "official" proofs.... I haven't even tried to verify this (I think it might be an interesting project for a logic or computational complexity student), but the proof of metatheorem *111 in *Mathematical Logic* at least suggests an algorithm for converting "practical" proofs into "official" ones (with the initial universal quantifiers all written out, and inferences from $X(A \to B)$ and YA to ZB derived from first principles) which would lead to an exponential increase in the length of the proof.

Antithesis. Quine's version of quantification theory is conceptually simpler. A rule of inference allowing a sentence to be inferred from sentences corresponds to a possible real inference: one in which we draw a conclusion (a proposition) from accepted or assumed premisses – acceptance and assumption being propositional attitudes. An inference from one or two open formulas ... isn't an inference at all. An open formula is not a sentence: it doesn't express a proposition we could accept or assume. (Related to this conceptual point... In ML, Quine notes that his treatment of quantification is similar to that of Fitch [4]. In that paper Fitch proved the consistency of a system – a bit stronger than what you get by just leaving Reducibility and Choice out from the system of Principia Mathematica but leaving Infinity in – by semantic means, appealing to what was later dubbed the "substitution interpretation of the quantifiers." In this context, the conceptual issue I have alluded to is reflected in a real technical difficulty: how is one to give a semantic account of open formulas?)

So ...

4.5 (D.5)

A side issue. In [15] Mathematical Logic, (1940) first edition, Quine included "permutation" axioms: universal closures of formulas of the form $\forall x \forall y(P) \to \forall y \forall x(P)$. Some time later, his student George Berry showed him how these could be made redundant by a change in the specification of the order of the "invisible" quantifiers at the front of axioms, and he made this change in the (1951) revised edition [16] of Mathematical Logic. Thinking that this was finished business, he followed this formulation in his 1954 note on inclusive logic. It turned out, however, that Berry's derivation of permutation from the other axioms depends on having the original $\forall x(Fx) \to F(x/y)$ axioms and not just their inclusive replacements. Oops! Quine added a note about this at the end of the reprinting of "Quantification and the empty domain," in the enlarged (1995) edition of Selected Logic Papers, not in the first (1966) edition. There is a (slightly) longer discussion on pp. 31-32 of Quine's "intellectual autobiography" [18]

(Kripke avoided the whole issue by allowing as axioms closures in which the free variables of a formula are bound in arbitrary order by universal quantifiers.)

The non-redundancy of these permutation axioms in the first edition of Mathematical Logic was proven by Kit Fine in [3] The means used in his proof are related to the, umm. . . semantical analysis of free variables which he developed in [2]

Apologies for long-windedness.

Be well, Allen

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