Another Logic Exercise

Thomas Forster

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First the patter ... Recall the principle of well-founded induction over a relation R;

$$(\forall x)((\forall y)(R(y,x) \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z))$$

REMARK 1 *Suppose $R' \subseteq R$ are wellfounded relations on a fixed domain. Then R'-induction is a weaker principle than R-induction.*

Proof:

If $R'(x,y) \to R(x,y)$ then $(\forall y)(R(y,x) \to \phi(y))$ implies $(\forall y)(R'(y,x) \to \phi(y))$ and
$(\forall x)((\forall y)(R'(y,x) \to \phi(y)) \to \phi(x))$ implies $(\forall x)((\forall y)(R(y,x) \to \phi(y)) \to \phi(x))$ and finally

$$(\forall x)((\forall y)(R(y,x) \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z))$$

implies

$$(\forall x)((\forall y)(R'(y,x) \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z))$$

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Reflect that if R' is the empty relation then R-induction is trivial. For consider: if R is the empty relation then

$$(\forall x)((\forall y)(R(y,x) \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z))$$

is

$$(\forall x)((\forall y)(\bot \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z))$$

which is

$$(\forall x)((\forall y)(\top) \to \phi(x)) \to (\forall z)(\phi(z))$$

which is

$$(\forall x)(\phi(x)) \to (\forall z)(\phi(z)).$$

EXERCISE 1 *Provide demonstrations, using sequent calculus, natural deduction or resolution that $(\forall x)((\forall y)(R(y,x) \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z))$ and $(\forall xy)(R'(x,y) \to R(x,y))$ together imply*

$$(\forall x)((\forall y)(R'(y,x) \to \phi(y)) \to \phi(x)) \to (\forall z)(\phi(z)).$$

I offer the usual inducements (bottle of port or claret) for a nicely LaTeX-ed answer to any of the above. (That is to say, a natural deduction proof, sequent proof or resolution proof. A JAPE box-proof would be acceptable too).