A question on the last example sheet of DM I.

RTP

$$x^{m+n} = x^n \cdot x^m$$

That's what it sez on the example sheet, and of course it's true for any kind of number—but the example sheet is on induction!

Observe that all these three variables could be variables ranging over $\mathbb{N}$. So any of these variables could become embroiled in an induction. It looks fairly clear that induction on 'x' is not going to be much use to us. Pretty obviously we want to do a UG on 'x'. But what do we do with the other two variables? I can't see an easy way to find the correct approach, and this is typical of this kind of problem. I just wrestled with it until it came out.

We fix x and prove by induction on 'm' that, for each m,

$$(\forall n)(x^{n+m} = x^n \cdot x^m)$$

Start with $m = 0$.

$$(\forall n)(x^{n+0} = x^n \cdot x^0)$$

That was easy. Now for the induction step.

For the induction assume

$$(\forall n)(x^{n+m} = x^n \cdot x^m)$$

So certainly

$$(\forall n)(x^{(n+1)+m} = x^{n+1} \cdot x^m)$$

whence

$$(\forall n)(x^{(n+1)+m} = x^n \cdot x \cdot x^m)$$

Then we do lots of rearrangement using associativity and commutativity of addition.

$$(\forall n)(x^{(n+1)+m} = x^n \cdot x^{1+m})$$

$$(\forall n)(x^{(n+1)+m} = x^n \cdot x^{m+1})$$

$$(\forall n)(x^{n+(1+m)} = x^n \cdot x^{m+1})$$

$$(\forall n)(x^{n+(m+1)} = x^n \cdot x^{m+1})$$

which is the same statement for $m + 1$.

You have to chose your induction carefully in order to not get into a tangle.

Observe that these three variables are treated in three different ways! UG on 'z', induction on 'm', and the 'n' is carried around and not inducted on.