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H. WOODIN

Theorem: λ is a limit of Woodin cards. Suppose $V(\mathbb{R})$ is a symmetric extension of V for $Coll(\omega, <\lambda)$. In $V(\mathbb{R})$, define

$$\Gamma = \{ p[T] : \text{for some } x \in \mathbb{R},\ T \in V[x], \text{ and } T \text{ is } <\lambda \text{ weakly-homo. in } V[x] \}$$

Then, ① $L(\mathbb{R}, \Gamma) \models AD$

② (Reflection) For each $\alpha \in OR$, there is $\bar{\alpha}, \bar{\Gamma}$ such that $L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \equiv L_\alpha(\mathbb{R}, \Gamma)$ and $\mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \subseteq \Gamma$.

Pf: Let α be given. Clearly, $Th(L_\alpha(\mathbb{R}, \Gamma)) \in V$ (since the collapse is homogeneous, etc...).

Γ is determined in $V(\mathbb{R})$ and Γ is closed under complement.

So, all Wadge games from Γ are determined.

Lemma: The Wadge degrees are "well-founded" in $V(\mathbb{R})$

(The problem is that we don't have DC in $V(\mathbb{R})$)

(So, "well-founded" means "ordinal ranking")

Pf: By passing to $V[x]$ if necessary, we may assume that, for some tree T on $(\omega \times \lambda)$ in V which is $<\lambda$ weakly-homo. in V, in $V(\mathbb{R})$ the lemma fails below the Wadge degree of $p[T]$. (i.e., we cannot rank the Wadge degrees means: as much ZFC as we want

below $p[T]$).

Choose $\delta >> \lambda$, $V_\delta \models \underbrace{ZFC^{-\epsilon}}$, $cf(\delta) > \lambda$.

Let $S = \{ X \prec V_{\lambda+1} : X$ is countable and for all $\kappa \in X$, if κ is Woodin and not a limit of Woodin cards., then for all $D \subseteq \mathbb{Q}_{<\kappa}$, D dense, $D \in X$, $D \cap (\cup b) \in b$ for some $b \in X \cap D \}$.

As before, S is stationary in $\mathcal{P}_{\omega_1}(V_{\lambda+1})$.

Let $G \subseteq \mathbb{Q}_{<\delta}$ be generic with $S \in G$.

We get:

$j: V \longrightarrow (M, E)$ in $V[G]$, $j''a$ '$\in$' M, $a \in V_\delta$.

As before, if $\kappa < \lambda$ is Woodin and not a limit of Woodin cards., then $G \cap \mathbb{Q}_{<\kappa}$ is V-generic.

Let $I = \{\kappa < \lambda : \kappa$ is Woodin and not a limit of Woodin cards.$\}$

For each $\kappa \in I$, get

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "j_\kappa"'] & & M \\ & M_\kappa \arrow[ur, "k_\kappa"'] & \end{tikzcd}$$

The diagram restricts to V_δ.

$$\begin{tikzcd} V_\delta \arrow[rr, "j \restriction V_\delta"] \arrow[dr, "j_\kappa \restriction V_\delta"'] & & (j(V_\delta), E) \\ & j_\kappa(V_\delta) \arrow[ur, "k_\kappa \restriction"'] & \end{tikzcd}$$

and this is in M. Let $M^* = \lim j_\kappa(V_\delta)$

Now, $V(\mathbb{R}\cap M^*)$ is a symmetric extension of V for $\mathrm{Coll}(\omega,<\lambda)$.

$j^*: V \longrightarrow M^*$

$j^*(V_\delta) = \lim j_\kappa(V_\delta)$

Now, M^* contains δ. (Since $j^*(V_\delta)\in M$, $k^*\in M$,

where
$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "j^*"'] & M \\ & M^* \arrow[u, "k^*"'] \end{tikzcd}$$
)

<u>Claim</u>: There exists a ranking of the Wadge degrees of Γ in M. (let $\mathbb{R} =_{df} \mathbb{R}^{M^*}$)

<u>Pf</u>: <u>Note</u>: $V_\delta(\mathbb{R})\in M$.

Let $\Gamma_{<\lambda} =_{df} \{p[T] : T$ is $<\lambda$ weakly-homo$\}$, in V

So, the Wadge degrees of $\Gamma_{<\lambda}$ are well-founded.

So, in M, the Wadge degrees of $j(\Gamma_{<\lambda})$ are well-founded.

We need, in M, an embedding of the Wadge degrees of Γ into the Wadge degrees of $j(\Gamma_{<\lambda})$.

We need a good (i.e., sufficiently elementary) map $\pi : \Gamma \longrightarrow j(\Gamma_{<\lambda})$.

For $A\in\Gamma$, choose $x\in\mathbb{R}$, $T\in V[x]$ s.t. T is $<\lambda$ weakly-homo. in $V[x]$ and $p[T]^{V(\mathbb{R})}=A$

Let $\kappa\in I$ be large enough with $x\in V[G\restriction\kappa]$.

Choose $\overline{T}\in M_\kappa$ s.t. $\overline{T}$ is $<\lambda$ weakly-homo in M_κ and $p[\overline{T}]=p[T]$ in $V(\mathbb{R})$.

Need to find $\overline{T}$:

Work in $V[G_\kappa]$.

We have: $j_\kappa : V \to M_\kappa \subseteq V[G_\kappa]$.

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H. WOODIN

<u>Theorem</u>: Suppose λ is a limit of Woodin cards. Suppose $V(\mathbb{R})$ is a symmetric extension for $\mathrm{Coll}(\omega, <\lambda)$. In $V(\mathbb{R})$, define

$$\Gamma = \{ p[T] : \text{for some } x \in \mathbb{R},\ T \in V[x] \text{ and } T \text{ is } <\lambda \text{ weakly homo. in } V[x] \}$$

Then, ① $L(\mathbb{R}, \Gamma) \models AD$

② for any α, there exists $\bar{\alpha}$, $\bar{\Gamma} \subseteq \Gamma$ such that $L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \equiv L_{\alpha}(\mathbb{R}, \Gamma)$ and $\mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \subseteq \Gamma$

More generally, if $\varphi(x)$ is a Σ_1 formula, then if $L(\mathbb{R}, \Gamma) \models \varphi[a]$, then $\exists \bar{\alpha}$, $\bar{\Gamma} \subseteq \Gamma$ with $L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \models \varphi[a]$ and $\mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \subseteq \Gamma$.

(This implies: (assume AD) if a $\underset{\sim}{\Sigma}^2_1(a)$ statement is true in $L(\mathbb{R}, \Gamma)$, then it has a witness in Γ.)

<u>Example</u>: V is "the" inner model of $ZFC + \exists\, \omega$-many Woodin cards. (or V is the inner model $HOD^{L(\mathbb{R})}$ constructed last spring, with ω-many Woodin cards.)

Under this assumption, $L(\mathbb{R}, \Gamma) = L(\mathbb{R})$.

[Why?: Γ is $\underset{\sim}{\Delta}^{2\,L(\mathbb{R})}_1$

Given this, the reflection follows:

(<u>Solovay</u>): (ZF) For all α, $\exists \bar{\alpha}$ with $L_{\bar{\alpha}}(\mathbb{R}) \equiv L_{\alpha}(\mathbb{R})$ and $\mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}) \subseteq \underset{\sim}{\Delta}^{2\,L(\mathbb{R})}_1$.

Also, if $\varphi(x)$ is a Σ_1 formula and $L(\mathbb{R}) \models \varphi[a]$, then $\exists \bar{\alpha}\ L_{\bar{\alpha}}(\mathbb{R}) \models \varphi[a]$ and $\mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}) \subseteq \Delta^{2\ L(\mathbb{R})}_1$.]

From last time:

① Every set of reals in Γ is determined.
② Γ is closed under complements and $\forall, \exists$, and more.

So, $G(A,B)$ are determined for all $A, B \in \Gamma$.

③ In $V(\mathbb{R})$, the Wadge degrees of the sets in Γ are well-ordered (modulo complements).

Pf. of Reflection: Fix $\varphi(x)$, $a \in \mathbb{R}$. WMA, by passing to $V[a]$, that $a \in V$. (We allow Γ as a parameter in φ)
For some α, $L_\alpha(\mathbb{R}, \Gamma) \models \varphi(a, \Gamma)$ (Γ as a parameter)
We want to choose a "least" stage at which $\varphi(a, \Gamma)$ is true.
Let $\bar{\Gamma}$ be the least initial segment of Γ (in Wadge degree) such that $\exists \beta\ L_\beta(\mathbb{R}, \bar{\Gamma}) \models \varphi(a, \bar{\Gamma})$
Let $\bar{\alpha}$ be least with $L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \models \varphi(a, \bar{\Gamma})$.
We must show that $\mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma}) \in \Gamma$.
Note: Every element of $L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma})$ is definable with parameters from $\{\mathbb{R}\} \cup \mathbb{R} \cup \bar{\Gamma} \cup \{\bar{\Gamma}\}$.
Let $B \in \mathcal{P}(\mathbb{R}) \cap L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma})$. We want: $B \in \Gamma$.
Choose $b \in \mathbb{R}$, $A \in \bar{\Gamma}$ with B defined in $L_{\bar{\alpha}}(\mathbb{R}, \bar{\Gamma})$ with

parameter $\bar{\Gamma}$.

Choose $y \in \mathbb{R}$, $T \in V[y]$, T is $<\lambda$ weakly-homo. in $V[y]$, and $p[T] = A$.

We may assume b, y, and hence, T, are in V, by passing to $V[b, y]$.

Fix $\delta >> \lambda, \bar{\alpha}$, etc...

As before, let

$$S = \{ Z \prec V_{\lambda+1} : \lambda \in Z,\ Z \text{ is countable},\ Z \cap (\cup b) \in b \text{ for some } b \in Z \cap D, \text{ if } D \text{ is dense in } \mathbb{Q}_{<\kappa},\ \kappa \text{ is a Woodin card, not a limit of Woodin cards},\ \kappa < \lambda, \text{ and } \kappa \in Z \}.$$

S is stationary in $\mathcal{P}_{\omega_1}(V_{\lambda+1})$.

Let $G \subseteq \mathbb{Q}_{<\delta}$, $S \in G$.

We get: $j : V \to (M, E)$

if κ is Woodin, $\kappa < \lambda$, and κ not a limit of Woodin cards, then $G \cap V_\kappa$ is generic for $\mathbb{Q}_{<\kappa}$.

Have:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "j_\kappa"'] & (M, E) \\ & M_\kappa \arrow[u, "k_\kappa"'] \end{tikzcd}$$

$M_\kappa \subseteq V[G \cap V_\kappa]$.

Fix $\lambda < \bar{\delta} < \delta$, $cf(\bar{\delta}) > \lambda$.

Get:

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[d, "j_\kappa"'] \arrow[drr, "j^*"] & & (M, E) \\ M_\kappa \arrow[rr, "k^*_\kappa"'] & & (M^*, E^*) \arrow[u, "k^*"'] \end{tikzcd}$$

$M^* = \lim M_\kappa$

The restriction of this diagram to $\bar{\delta}$ lies in (M, E).
So, the well-founded part of $j^*(V_\delta) \supseteq \bar{\delta}$.
let $\mathbb{R} = \mathbb{R}^{M^*} = \bigcup_{\substack{\kappa<\lambda \\ \kappa \text{ Woodin}}} \mathbb{R}^{M_\kappa}$. So, $V(\mathbb{R})$ is a symmetric extension of V for $\mathrm{Coll}(\omega, <\lambda)$

In V, let $\Gamma_{<\lambda} = \{ p[T] : T \text{ is } <\lambda \text{ weakly-homo.} \}$

$j(\Gamma_{<\lambda})$ has similar def. in (M, E).
We want a "nice" map $\Gamma \longrightarrow j(\Gamma_{<\lambda})$.
means: $(\mathbb{R}, \vec{A}) \prec (j(\mathbb{R}), j(\vec{A}))$, for $\vec{A} \in \Gamma$
a finite seq. of elements of Γ

We have: $L_{\bar{\delta}}(\mathbb{R}, \bar{\Gamma})$, $b, y \in \mathbb{R}$, $A \in \bar{\Gamma}$, $A = p[T]$, $T \in V$, and
$B \in \mathcal{P}(\mathbb{R}) \cap L_{\bar{\delta}}(\mathbb{R}, \bar{\Gamma})$.

<u>Claim</u>: $j^*(\Gamma_{<\lambda}) \supseteq \Gamma$

Pf: Special case: Suppose $C \in \Gamma$ and $C = p[T]$, and $T \in V$ and T is $<\lambda$ weakly-homo.
Choose S a tree for $\omega^\omega \setminus C$, $T, S \subseteq (\omega \times \lambda)^{<\omega}$
Need: $C = p[j^*(T)]^{M^*}$
We know:
$$p[T]^{V(\mathbb{R})} = \mathbb{R} \setminus p[S]^{V(\mathbb{R})}$$
So, if M^* were well-founded, we would be done.
We need: $p[T]^{V(\mathbb{R})} \subseteq p[j^*(T)]^{M^*}$
$p[S]^{V(\mathbb{R})} \subseteq p[j^*(S)]^{M^*}$

To see $p[T]^{V(\mathbb{R})} \subset p[j^*(T)]^{M^*}$, fix $x \in p[T]^{V(\mathbb{R})}$.
Choose $\kappa < \lambda$, Woodin and not a limit of Woodin cardinals, with $x \in V[G \cap \mathbb{Q}_{<\kappa}]$, $G_\kappa = G \cap \mathbb{Q}_{<\kappa}$.
We have:

$$\begin{tikzcd} V \arrow[r, "j_\kappa"] \arrow[rr, bend left, "j^*"] & M_\kappa \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & M^* \\ & V[G_\kappa] & \end{tikzcd}$$

So, $x \in M_\kappa$. Further, $p[T]^{V[G_\kappa]} = p[j_\kappa(T)]^{V[G_\kappa]} = p[j_\kappa(T)]^{M_\kappa}$

So, $x \in p[j_\kappa(T)]^{M_\kappa}$.
So, $x \in p[j^*(T)]^{M_\kappa} = p[j^*(T)]^{M^*}$.

This takes care of the special case.
The general case is similar: Suppose $C \in \Gamma$. Choose $x \in \mathbb{R}$, $T \in V[x]$, T $<\lambda$ weakly homo in $V[x]$ and $p[T]^{V(\mathbb{R})} = C$.
Choose κ as above large enough with $x \in V[G_\kappa]$.
Have: $j_\kappa : V \to M_\kappa \subset V[G_\kappa]$

$T \in V[G_\kappa]$, T is $<\lambda$ weakly-homo in $V[G_\kappa]$.

Subclaim: ① $p[T]^{V[G_\kappa]} \in M_\kappa$ (let's call $p[T]^{V[G_\kappa]}$, C_κ)

② $M_\kappa \models C_\kappa$ is $<\lambda$ weakly-homo Suslin

$j_\kappa(\lambda)$, provided λ is regular, or $cf(\lambda) < \kappa$

③ $k_\kappa(C_\kappa) = C$

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H. Woodin

Have: $V \xrightarrow{j^*} M^*$, $\mathbb{R}^{M^*} = \mathbb{R}^*$

$V(\mathbb{R}^*)$ a symmetric extension of V for $Coll(\omega, <\lambda)$

assume $cf(\lambda) = \omega$, or λ regular.

$\Gamma \subseteq j^*(\Gamma_{<\lambda})$, $\Gamma \in M^*$

But M^* is not well-founded.

Also, have: $V \xrightarrow{j_N} N \subseteq V[G_\gamma]$

$$j_N(\Gamma_{<\lambda}) \supseteq (\Gamma_{<\lambda})^{V[G_\gamma]}$$

Case 1: Γ contains $A \subseteq \mathbb{R}^*$ with $\Gamma \subseteq L(A, \mathbb{R}^*)$.

In this case, can assume that $\exists T \in V$, etc..., with $p[T] = A$ in $V(\mathbb{R}^*)$

(i.e., can do a little bit of forcing to catch all the reals we need to get the tree)

Choose $\bar\Gamma$, then $\bar\alpha$, least with $A \in \bar\Gamma$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p[T]$.

By the argument done before, relativized to A, we get $L(A, \mathbb{R}^*) \models AD$.

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Case 2: ~~Choose $\bar\Gamma$, then $\bar\alpha$.~~
~~If $\bar\Gamma$ happens to be Γ, then "easy".~~
~~Otherwise:~~ We have to see that $\bar\Gamma^{V(\mathbb{R}^*)} = \bar\Gamma^{M^*}$.

Case 2: ON.

Claim: If $A \in \Gamma$, then $L(A, \mathbb{R}) \models AD + DC$

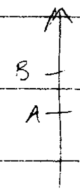
(Postpone the proof of Claim)

Fact: (AD + DC + ZF)

Suppose $A \subset \mathbb{R}$ and $\mathcal{P}(\mathbb{R}) \not\subseteq L(A,\mathbb{R})$. Then, $A^\#$ exists.

(Here, $A^\# = Th(\langle L(A,\mathbb{R}), A, \underbrace{c_1, \ldots, c_n, \ldots}_{\text{indiscernibles (finitely many)}}, \mathbb{R}\rangle)$)

By first choosing $\bar{\Gamma}$, there's no problem in $V \xrightarrow{j_N} N \subseteq V[G_\gamma]$.
i.e., N and $V[G_\gamma]$ compute the same thing.
The problem is in $V \xrightarrow{j^*} M^* \subseteq V(\mathbb{R}^*)$, since M^* is not well-founded, M^* and $V(\mathbb{R}^*)$ might compute different things.



Suppose $B \notin L(A,\mathbb{R})$.

Suppose $A \in \Gamma$.

Claim: $A^\# \in M^*$

Pf: Choose $B \in M$, $B \geq_W A$, $B \notin L(A,\mathbb{R})$.
Since $L(B,\mathbb{R}) \models AD$, by the Fact above, $B^\# \in L(B,\mathbb{R})$
$L_\delta(B,\mathbb{R}) \subseteq M^*$, some δ

Claim: $M^* \models (A^\#)^{V(\mathbb{R}^*)} = A^\#$

(Claim: $A^\# \in \Gamma$)

Need: $(\bar{\Gamma})^{M^*} = (\bar{\Gamma})^{V(\mathbb{R}^*)}$: $\subseteq$ is clear.

Suppose $(\bar{\Gamma})^{V(\mathbb{R}^*)}$ is strictly greater than $(\bar{\Gamma})^{M^*}$.
Choose $A \in (\bar{\Gamma})^{V(\mathbb{R}^*)} \setminus (\bar{\Gamma})^{M^*}$.
We have $L(A,\mathbb{R}) \models AD$.
So, $(\bar{\Gamma})^{M^*} \in L(A,\mathbb{R})$.
$A^\#$ exists in $V(\mathbb{R}^*)$
" " M^*

Thus, $M^* \models$ "$L(A,\mathbb{R}) \models AD$", and so $(\bar{\alpha})^{M^*}$ is $<$ least indiscernible of $L(A,\mathbb{R})$. So, $\bar{\alpha} \in$ well-founded part of M^*.

So, modulo the Fact above, we are done.

Application of the theorem, together with what we did last spring

Theorem: Suppose λ is a limit of Woodin cards. Define $\Gamma_{<\lambda}$ as above. Then,

1. Scale$(\Sigma^2_1(\Gamma_{<\lambda}))$
2. Every $\Sigma^2_1(\Gamma_{<\lambda})$ fact has a $\Delta^2_1(\Gamma_{<\lambda})$ witness in $\Gamma_{<\lambda}$.

(where $A \in \Sigma^2_1(\Gamma_{<\lambda})$ if the existential quantifier ranges over $\Gamma_{<\lambda}$)

This is the best candidate for a well-ordering of $\mathbb{R}$ consistent with large cardinals.

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H. Woodin

Lemma: (AD + ZF + DC)

Suppose $A \subseteq \mathbb{R}$ and $\mathcal{P}(\mathbb{R}) \not\subseteq L(A,\mathbb{R})$.

Then, $A^\#$ exists.

Pf:

Use the following:

Fact: (AD + DC + ZF) $\Theta = \sup\{\kappa : \kappa \longrightarrow (\kappa)^\kappa\}$ (actually, need only the Θ = sup of reg. cards.)

Corollary: If $S \subseteq \lambda < \Theta$, then $S^\#$ exists.

Suppose $B \notin L(A,\mathbb{R})$. $\Theta_A = \Theta^{L(A,\mathbb{R})}$, $\Theta_B = \Theta^{L(B,\mathbb{R})}$.

Let G be $L(B,\mathbb{R})$ generic for $Coll(\omega,\mathbb{R})$.

Then, $L(A,\mathbb{R}) \subseteq L[G,A] = L[x]$.

Note: (In ZFC) $HOD^{L[x]} = L[S]$ for some $S \subseteq \omega_2^{L[x]}$.

So, $S^\#$ exists $\Longrightarrow$ $x^\#$ exists $\Longrightarrow$ $L[A,\mathbb{R}]^\#$ exists

Also, $\Theta^{L(A,\mathbb{R})} = \omega_1^{L[x]}$

Θ_B

κ

Θ_A

Suppose $\kappa > \Theta^{L(A,\mathbb{R})} = \Theta_A$ and κ is regular, $\kappa < \Theta_B$, $\kappa \longrightarrow (\kappa)^\kappa_{<\kappa}$

Then, κ is regular in $L[x]$.

Thus, κ is regular in $L[x]$.

$\kappa \geq \omega_2^{L[x]}$

So, $S \subseteq \kappa < \Theta_B$, $S \in L(A,\mathbb{R})$.

So, $S^\#$ exists in $L(B,\mathbb{R})$. $\square$

Theorem: If λ is a limit of Woodin cardinals and $V(\mathbb{R})$ is a symmetric extension of V for $\mathrm{Coll}(\omega, \mathbb{R})$.
Define Γ as before in $V(\mathbb{R})$.
Then, every set in Γ has a scale in Γ. So, by reflection, $L(\mathbb{R}, \Gamma) \models AD^+$.

Fact: (AD^+) Either $AD_{\mathbb{R}}$ holds or there is a largest Suslin cardinal

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Theorem: Suppose λ is: ① a limit of Woodin cardinals.
② a limit of card. which are $< \lambda$ strong
(Note: this is a weaker hypothesis than λ is Mahlo to Woodin's)
Suppose $V(\mathbb{R})$ is a symmetric extension of V for $\mathrm{Coll}(\omega, <\lambda)$. Then, with Γ defined in $V(\mathbb{R})$ as before, then $L(\mathbb{R}, \Gamma) \models AD_{\mathbb{R}}$.

Theorem: Suppose κ is a Woodin card. Suppose $g \subset \mathrm{Coll}(\omega, \kappa)$ is generic. Then, $V[g] \models \Delta^1_2$-det.

Corollary: TFAE: ① $\utilde{\Delta}^1_2$-det. $(= \utilde{\Sigma}^1_2\text{-det})$
② ~~$\forall x \in \mathbb{R}\ \exists$ transitive model $\supseteq \{x\} \cup OR$ of ZFC + $\exists$ a Woodin card~~
$\forall x \in \mathbb{R}\ \exists \alpha_x < \omega_1\ \exists$ transitive model $\supseteq OR \cup \{x$

of ZFC + α_x is a Woodin cardinal.

Pf: ① ⇒ ② Done

① ⇒ ③ as theorem.

② ⇒ $x^{\#}$ exists, all $x \in \mathbb{R}$. □

Pf of theorem (Sketch):

Basic tool #1 : lemma: Suppose $V = L[x]$ and $L[x] = L[x][g]$ if $g \subseteq \omega$ and g is ccc over $L[x]$.

Then, $L[x] \not\models \Delta^1_2$-det.

Fact: $\underset{\sim}{\Delta}^1_2$-det $\Rightarrow$ $Th(L[x])$ is constant on a cone.

Δ^1_2-det + #'s $\Rightarrow$ " "

Suppose $M \supseteq OR$, $M \models \kappa$ is Woodin, and M is iterable, in the sense of Martin-Steel (i.e., every iteration tree has a branch).

(*) Also, assume $\delta < \kappa \Rightarrow \delta$ is not Woodin in $L(M_\delta)$.

Assume $M = L[M_\kappa]$.

Fact: $\exists B \in M$, B is κ-cc of size κ, B is countably generated such that $\forall x\, \exists$ iteration $j^*: M \longrightarrow M^*$ such that x is M^*-generic for $B^* = j^*(B)$.

(only need M is iterable and $M \models \kappa$ is Woodin).

Assume $\kappa < \omega_1$.

Assuming (*) holds, then if $M_\kappa \in L[x]$, then $M^* \in L[x]$ and $j^*(\kappa) = \omega_1^{L[x]}$.

Then, $L[x] = M^*[x]$, x generic for B^*

So, $L[x]^{\mathrm{Coll}(\omega, \omega_1^{L[x]})} = M^*[x]^{\mathrm{Coll}(\omega, \kappa^*)} = M^{*\,\mathrm{Coll}(\omega, \kappa^*)} \equiv \Gamma$