Paris-Harrington: a talk for the Logic Seminar VUW July 20th 2021

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ABSTRACT

The Paris-Harrington theorem [1] is famous as the first (1977) example of a mathematically natural assertion in the language of arithmetic that is demonstrably not provable in Peano Arithmetic; a concrete example of Gödel-style incompleteness. In this tutorial-style exposition (designed to be accessible to advanced students) I shall prove the theorem and tell a fairly detailed story about why it is stronger than the Infinite-Ramsey theorem that it so closely resembles. However, a full proof that PH implies con(PA) would be a bridge too far.

I take it that my readers are familiar with the finite and infinite versions of Ramsey's theorem, and with proofs of them. Our concern today is with the finite theorem, and with a spiced-up version of the finite theorem due to Paris and Harrington. The finite version of Ramsey's theorem [aka FR] can be proved by an analogue of the usual proof of the infinite theorem, but it can also be obtained by a compactness argument from the statement of the infinite version. That second proof is not very parsimonious and people don't draw attention to it. The spiced up version of the finite theorem (due to Paris and Harrington, and hereafter 'PH') cannot be proved by any proof analogous to the usual proof of Ramsey's original finite version; it can be proved only by an argument analogous to the compactness proof of the finite version from the infinite version.

Some notation

- $[X]^j$ is the set of unordered j-tuples from X.
- $[n, m]$ is $\{x \in \mathbb{N} : n \leq x \leq m\}$. Sorry about the overloading of square brackets!
- I take it my readers are familiar with the notation $a \to (b)^c_d$, which means that if A is a set with $|A| = a$ then for every partition of $[A]^c$ into d pieces there is $B \subseteq A$ with $|B| = b$ and $[B]^c$ included in one of the d pieces.
- `butlast`(x) is the list x shorn of its last element. (Don't ask what `butlast` of the empty list is!)
- Finally: $\rotatebox[origin=c]{180}{$\mathcal{P}$}(x)$ is $\{y : y \cap x \neq \emptyset\}$. (It's an upside-down '$\mathcal{P}$' reflecting[1] the fact that the operation is dual to power set.) By abuse of notation we will write

[1] "reflecting" (joke!—geddit??)

'$Ƅ(x)$' when $x \subseteq X$ (X clear from context) to mean the "localised" version $\{y \subseteq X : y \cap x \neq \emptyset\}$.

Let us start with a statement of the usual finite version of Ramsey's theorem, FR:

Theorem 1. *FR*

$$(\forall mnk)(\exists j)(j \to (m)^n_k).$$

There is a standard proof of this result by a method precisely analogous to the proof of the infinite version, usually written[2] as $\omega \to (\omega)^n_m$. This proof proceeds by induction on the exponent n and can be carried out in PA. However the finite version can also be deduced from the infinite version by a kind of reverse compactness argument. Normally no one would try to prove the finite version in this way, because the reverse compactness proof is less informative (less effective) than the usual inductive proof. However we need to know about it here, since it is a modification of this second proof that gives us the spiced up version of Finite Ramsey that we know as *Paris-Harrington*, [1], aka PH.

Let us start with the reverse compactness proof of FR. We will then modify it to obtain a proof of PH.

Proof:

Suppose for a contradiction that FR is false, and that there are n, m, k in $\mathbb{N}$ such that, for all $p \in \mathbb{N}$, there is a set P with $|P| = p$ and a colouring $f : [P]^m \to [1, k]$ such that there is no set $X \subseteq P$ with $|X| = n$ and $|f``[X]^m| = 1$. Fix n, m and k and, for each p, let Y_p be the set

$$\{f : f : [\{1, \cdots p\}]^m \to [1, k] \wedge \neg(\exists X) \bigwedge \left\{ \begin{array}{l} X \subseteq [1, p] \\ |X| \geq n \\ |f``[X]^m| = 1 \end{array} \right\}\}.$$

of bad k-colourings of the m-tuples of the naturals below p. (k and m are fixed.)

For any k, the set F_k of all k-colourings of m-tuples of initial segments of $\mathbb{N}$ is countable. (Each initial segment $[1, p]$ has only a finite set of m-membered subsets and there are only finitely many ways of colouring the set of those subsets). So we can uniformly wellorder F_k. Suppose this to be done, somehow. Then, for each p, we set f_p to be the first element of Y_p in the sense of that ordering.

We are now going to define a (bad) partition π of $[\mathbb{N}]^{m+1}$ into k pieces. You are given a set $x \subseteq \mathbb{N}$ of size $m+1$ and have to decide which piece to put it into. Its last member is $p+1$ for some natural number p. $x \setminus \{p+1\}$ is now a subset of $[1, p]$ and is therefore a suitable input for f_p. $f_p(x \setminus \{p+1\})$ is now a number $< k$, and that tells you which piece to put x into. (Slightly more formally, put x

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An aside: another motivation for choosing this symbol is that it looks like a 'b'; $Ƅ(\{x\})$ is $\{y : x \in y\}$, and (among NFistes at least) the operation $x \mapsto \{y : x \in y\}$ is notated '$B(x)$' in honour of Maurice **B**offa, who noticed that it is an injective $\in$-homomorphism.

[2] It should be written as $\aleph_0 \to (\aleph_0)^n_m$, since we are dealing with cardinals not ordinals, but this malpractice is common.

into the $f_{(sup(x)-1)}(x \setminus \{sup(x)\})$th piece.) So π partitions $[\mathbb{N}]^{m+1}$ into k pieces. We will show that π is bad.

With a view to obtaining a contradiction suppose X to be an infinite set monochromatic for π. Let $p+1$ be a member of X (and we will want to be able to find arbitrarily large such $p+1$). Consider those $(m+1)$-tuples from $X \cap [1, p+1]$ whose last element is $p+1$. What does π do to them? It sends every such $(m+1)$-tuple x to $f_p(\texttt{butlast}(x))$, and—because X is monochromatic—all these $f_p(\texttt{butlast}(x))$ are the same, whatever x we pick up. Now *every* m-sized subset of $X \cap [1,p]$ can be turned into such an $(m+1)$-tuple by the simple expedient of sticking $p+1$ on the end, so f_p sends every m-tuple from $X \cap [1,p]$ to the same number $< k$. But that is simply to say that $X \cap [1,p]$ is a subset of $[1,p]$ that is monochromatic for f_p. Now f_p was chosen so that any set monochromatic for it was of size less than n. So $X \cap [1,p]$ is of size less than n. So—no matter how large we pick $(p+1) \in X$—we find that $X \cap [1,p]$ has at most n members. So $|X| \leq n+1$ and X was not infinite, contradicting the Infinite Ramsey theorem. ∎

The sole purpose of this proof is to prepare the reader for slightly more complicated proof of PH given by Paris and Harrington.

Paris-Harrington

A key notion for Paris-Harrington (it is not present in Finite Ramsey) is the idea of a relatively large set of naturals:

Definition 1. *A finite subset x of $\mathbb{N}$ is* **relatively large** *if $|x| > \texttt{min}(x)$.*

... and it is an *aperçu* of (among others) Harvey Friedman[3], that this notion is *prima facie* unstratified. One might suspect that this unstratification of PH (unlike FR, which is stratified) is the key to the greater strength of PH over FR. Interestingly this turns out not be so, but the investigation this question launches is informative in other—unexpected—ways.

Let us have a statement and a proof of the theorem.

Theorem 2. *(Paris-Harrington)*
For every n, m, $k \in \mathbb{N}$, there is $p \in \mathbb{N}$ so large that whenever

$$f : [\{1, 2, \ldots p\}]^m \to [1, k]$$

there is a relatively large $X \subseteq [1,p]$ such that $|X| \geq n$ and $|f``[X]^m| = 1$.

[3]Subject: [FOM] PA Incompleteness; Sun, 14 Oct 2007 10:02:58 -0400):

> "In "relatively large", an integer is used both as an element of a finite set and as a cardinality (of that same set). This is sufficiently unlike standard mathematics, that an effort began, at least implicitly, to find PA incompleteness that did not employ this feature, or this kind of feature."

Proof:

We argue by "reverse compactness", as before.

Suppose for a contradiction there are n, m, k in $\mathbb{N}$ such that, for all $p \in \mathbb{N}$, there is $f : [\{1, 2, \ldots p\}]^m \to [1, k]$ such that there is no relatively large $X \subseteq [1, p]$ such that $|X| = n$ and $|f``[X]^m| = 1$. Fix n, m, k and p and let Y_p be the set

$$\{f : f : [\{1, 2, \ldots p\}]^m \to [1, k] \wedge \neg(\exists X) \bigwedge \left\{ \begin{array}{l} X \subseteq [1, p] \\ |X| > \mathtt{min}(X) \\ |X| \geq n \\ |f``[X]^m| = 1 \end{array} \right\} \}.$$

That is to say: this time Y_p is—not the set of

> *colourings-that-are-bad-in-the-sense-of-lacking-a-monochromatic-set-of-size-n*

as before but, instead, the set of

> *colourings-that-are-bad-in-the-sense-of-not-having-any-monochromatic-sets-of-size-n-that-are-relatively-large.*

We define f_p as before, and—as before—initial segments of the monochromatic set X will be monochromatic for the colourings f_p. Now sets that are monochromatic for f_p are either smaller than n or are not relatively large. By considering initial segments of X that are long enough we can take care of the first condition, so the only way they can manage to be monochromatic for f_p will be by failing to be relatively large. So, for some large j, consider the initial segment consisting of the first j elements of X. We now know that this is not relatively large, so its first element must be bigger than j. So the first element of X is at least j. But j could have been taken to be arbitrarily large.

$\blacksquare$

Famously PH is stronger than FR: it actually proves Con(PA). One obvious difference between FR and PH is that PH doesn't talk about colourings of tuples from arbitrary finite *sets* but of colourings of tuples quite specifically from *initial segments of* $\mathbb{N}$. The key to understanding the relation between FR and PH is to rephrase them so they appear to be talking about the same things. To this end we now develop a notion of relative largeness for an arbitrary finite set. Then we will reformulate FR as an assertion about relative largeness.

A subset of $\mathbb{N}$ of size n is relatively large simply if it has a member smaller than n. We can generalise 'relatively large' to finite sets of things other than natural numbers if we acknowledge that 'relatively large subset of X' has a hidden parameter, namely an ordering $<$ of X, or equivalently an enumeration $e : X \hookrightarrow \mathbb{N}$. Accordingly we say that an n-sized subset $X' \subseteq X$ is relatively large *with respect to* $<$ iff one of its members is among the first n members of X *according to* $<$. That is to say $[X]^n \in \text{Ь}(N)$ where N is the set of the bottom n elements of X according to $<$. Let's set this definition up in lights:

Definition 2. *$Y' \subseteq Y$ is relatively large* **with respect to an enumeration** *$e : Y \to \mathbb{N}$ iff $e``Y'$ is relatively large as in definition 1.*

Given X it is clear that any $X' \in [X]^n$ is a relatively large subset of X with respect to *some* enumeration e of X, namely any e s.t. $e``X' = [0, n]$.

So here is FR reformulated as an assertion about relatively large monochromatic sets.

> For all n, m, j in $\mathbb{N}$
> There is k in $\mathbb{N}$ so large that
> For every set X of size k and
> For every m-colouring χ of $[X]^j$ (**FR1**)
> there is an enumeration e of X and
> there is $X' \subseteq X$ with $|X'| = n$
> with X' monochromatic wrt χ and relatively large wrt e.

Now that we have expressed FR in a syntax that is the same as that used to express PH we are in a better position to compare them. Here is PH.

> For all n, m, j in $\mathbb{N}$
> There is k in $\mathbb{N}$ so large that
> For every set X of size k and
> For every m-colouring χ of $[X]^j$ and (**PH1**)
> For every enumeration e of X
> there is $X' \subseteq X$ with $|X'| = n$ and X' monochromatic wrt χ and relatively large wrt e.

We have set out PH and FR above in something very like Prenex Normal form. The two of them have the same matrix (the stuff after the prefix) and the prefixes

$$(\forall mnj)(\exists k)(\forall X)(\forall \chi)\underline{(\exists e)}(\exists X') \quad \text{and} \quad (\forall mnj)(\exists k)(\forall X)(\forall \chi)\underline{(\forall e)}(\exists X')$$

are the same except for the underlined part, where FR has '$\exists e$' and PH has '$\forall e$'. Clearly we must expect PH to be stronger than FR.

These two novel formulations of FR and PH may—neatly—differ only in quantifier prefix, but they are both unstratified, so Friedman's concern remains to be addressed. I am now in a position to exhibit a pair of formulations of FR and PH, both stratified, which differ only in the quantifier prefix. This shows that—rather to my disappointment and surprise—stratification plays no role in the extra strength of PH.

The new thought is that we should think of PH as saying not so much that there is a monochromatic set with special properties, but rather that (in contrast to FR, which only promises *one* monochromatic set) the set of χ-monochromatic subsets of X is *large*—in the sense that, for every total order $<$ of X, it meets Ь of the initial segment containing the first n elements of $\langle X, < \rangle$. That sounds like a quantifier.

We can rephrase the last line of the above formulation (FR1) of FR to get

For all $n, m, j \in \mathbb{N}$
There is $k \in \mathbb{N}$ so large that
For every set X of size k and **(FR2)**
For every m-colouring χ of $[X]^j$
The set M_n^χ of n-sized subsets of X monochromatic for χ is nonempty.

Now for the new formulation of PH:

For all $n, m, j \in \mathbb{N}$
There is $k \in \mathbb{N}$ so large that
For every set X of size k and **(PH2)**
For every m-colouring χ of $[X]^j$
The set M_n^χ of n-sized subsets of X monochromatic for χ meets Ƅ(A) for every $A \in [X]^n$.

…and the difference between these two [which—lest we forget—resides wholly in the last line] is that (FR2) says merely that the set of monochromatic subsets of X of size n is nonempty, whereas (PH2) says that it meets every member of a fairly large set. (so there are so many monochromatic sets that every $A \in [X]^n$ meets one).

It's probably worth saying a few words about why this version PH2 of PH is equivalent to the usual version that asserts the relative largeness of the monochromatic set. Definition 2 says in effect that an n-sized subset $X' \subseteq X$ is *relatively large with respect to* $<$ iff one of its members is among the first n members of X *according to* $<$. ($<$ is of course the ordering of X induced by the enumeration e.) That is to say $X' \subseteq X$ is relatively large wrt $<$ iff $X' \in$ Ƅ(N) where N is e^{-1}“$[0, n-1]$, the set of the bottom n elements of X according to $<$.

For the other direction, if Y meets some element $X' \in [X]^n$ then it is relatively large wrt any enumeration of X that counts X' using only the naturals $< n$.

Thus, to infer the new formulation PH2 from the original version PH we argue as follows. PH2 is going to tell us that there are lots of monochromatic sets for the two-colouring we have in mind. Here's how. Fix an enumeration of the set X whose n-tuples you are going to colour. Then think of the members of X as natural numbers and apply the original form of PH. You get a monochromatic *large* set of numbers. But you can do this whichever enumeration of X you choose; for any enumeration of X that you might choose, PH will give you a monochromatic set that is relatively large wrt that enumeration. So you get lots of monochromatic sets—which is what PH2 is saying.

This formulation prompts some natural questions. Does this version have a slick compactness proof? Does it give a slick proof of Con(PA)? Does it suggest formulations of analogues of PH for uncountable cardinals?

The new formulation of PH asserts that M_χ^n meets every set in the family Ƅ“$[X]^n$. Now we know the size of this family (it's $\binom{k}{n}$) and we know the size of all the members of that family (and all these values of L are of size $(2^n-1)\cdot 2^{k-n}$) so we have a lower bound on the size of M^n_χ purely in terms of k and n. Specifically we can find k/n pairwise disjoint subsets of X of size n, and no monochromatic set of size n can meet more than n of them, so $|M^n_\chi|$ must be at least k/n^2.

I close with some hints about how to prove that PH is stronger than FR in the sense of actually implying Con(PA). There is a proof in [1] of course, but there are more recent proofs that are more informative and pleasing. Stan Wainer tells me there is a story along the following lines:

(1) PH implies H_{ϵ_0} is total;

(2) Totality of H_{ϵ_0} implies Σ_1 Reflection;.

(3) Σ_1 Reflection implies Con(PA).

Some items in the bibliography are present beco's they are useful, even tho' they aren't cited.

References

[1] Paris, J. and Harrington, L. A Mathematical Incompleteness in Peano Arithmetic. In Handbook of Mathematical Logic (Ed. J. Barwise). Amsterdam, Netherlands: North-Holland, 1977.

[2] J. Ketonen and R. Solovay Rapidly growing Ramsey functions, Annals of Mathematics **113** (1981) pp 267–314.

[3] Martin Loebl and Jaroslav Nesetril An unprovable Ramsey-type theorem *Proc AMS* **116** nov 1992 pp 819–824.