CHAPTER X: Getting AD from ω Woodin cardinals

OUR GOAL IS IN SIGHT: we shall in this chapter piece together the component arguments to obtain a consistency proof for AD. We begin with a review, from forcing, of the concept of the $< \lambda$ symmetric collapse; we then use this concept, in the special case that λ is the supremum of an ω-sequence of Woodin cardinals, to study the reals of a certain ill-founded ω-model obtained as a direct limit of the well-founded generic ultralimits produced by the various Woodin cardinals, and to find a simple method for proving that certain formulæ satisfy the FC axioms. This leads to a consistency proof for AD starting from infinitely many Woodin cardinals. We then review the concept of $\mathbf{R}^\sharp$ and show that if we have enough large cardinals on top of the ω Woodins to build the sharp of the set of reals of the $< \lambda$ collapse, then we can prove that AD actually holds in an inner model, namely $L[\Re]$. In a final section we briefly consider extensions of these results to building models of stronger forms of AD.

1: the symmetric collapse

1·0 DEFINITION For an ordinal $\kappa \geq \omega$, $\mathrm{Coll}(\omega, \kappa)$ is the partial ordering that makes κ countable: for definiteness, conditions are injections $f : n \xrightarrow{1-1} \kappa$ for some $n \in \omega$, and $f \leq g$ iff $g = f \restriction dom(g)$.

$\mathrm{Coll}(\omega, A)$ where A is a set of infinite ordinals, is the product partial ordering, with finite supports, of the partial orderings $\mathrm{Coll}(\omega, \zeta)$ for $\zeta \in A$.

1·1 REMARK There is a slight ambiguity here, since an ordinal κ is also the set of ordinals less than κ; to maintain a distinction we write $Coll(\omega, < \kappa)$ in the second case. For κ singular the effects are the same, as we shall shortly see; not so for κ regular and uncountable, since $\mathrm{Coll}(\omega, \kappa)$ makes κ countable, whereas $\mathrm{Coll}(\omega, < \kappa)$ has, by a Δ-system argument, the κ-chain condition, and hence makes κ the first uncountable cardinal.

We shall need the following result of Krivine:

1·2 PROPOSITION *Let B be a separative notion of forcing of cardinality an uncountable cardinal κ, and suppose that $\Vdash_B \hat{\kappa}$ `is countable`. Then B has a dense subset isomorphic to $Coll(\omega, \kappa)$.*

Proof : Let C be the completion of B and $\dot{G}$ the canonical C-name for the (V, C) generic filter: so that for $c \in C$, $[\![\hat{c} \in \dot{G}]\!]^C = c$.

Since in V^C, B and therefore $G \cap B$ are countable, there is some name $\dot{f}$ such that

$$[\![f : \omega \longleftrightarrow \dot{G} \cap B]\!]^C = 1.$$

Note that if $c \Vdash \dot{f}(\hat{n}) = \hat{b}$ then $c \Vdash b \in \dot{G}$ and so $c \leq b$: for if $c \not\leq b$ then $\exists d :\leq c\ d \perp b$; but then $d \Vdash b \in \dot{G}$ (as c does) and $d \Vdash d \in \dot{G}$, contradicting the fact that the generic cannot contain contradictory things. Note also that each non-zero element of B splits into κ disjoint non-zero elements, otherwise B would somewhere have the κ chain condition and therefore $[\![\hat{\kappa}$ `is countable`$]\!] < 1$.

We now define a sequence of maximal anti-chains A_n in B.

Begin by setting $A_{-1} = \{1_C\}$. Now given A_{n-1}, form A_n as follows:

given $p \in A_{n-1}$, let $A'_n(p)$ be a maximal antichain in $\{q \in B \mid q \leq p\}$ such that to each $p' \in A'_n(p)$ there is a $b = b(p')$ in B with $p' \Vdash \dot{f}(\hat{n}) = \hat{b}$. $A_n(p)$ is now formed by splitting each $p' \in A'_n(p)$ into κ many disjoint non-zero pieces — possible as the κ-c.c. fails everywhere — and A_n will be $\bigcup\{A_n(p) \mid p \in A_{n-1}\}$.

Put $A = \bigcup\{A_n \mid -1 \leq n < \omega\}$.

We assert that A is the desired subset of B. To see that A is isomorphic to $\mathrm{Coll}(\omega, \kappa)$, it is enough (and easy) to check that

i for any n, any two distinct elements of A_n are incompatible;
ii for $n < m$, if $p \in A_m$ and $q \in A_n$ then either $p \leq q$ or $p \perp q$;
iii each p in A_m has exactly κ many r in A_{m+1} with $r \leq p$.

Finally we check that A is dense in C:

iv $\forall b{:}\in B\ \exists p{:}\in A\ \ p \le b$.

Proof of iv: given b there is a $c \le b$ and an n such that $c \Vdash \dot f(\hat n) = b$, as $\dot f$ is forced to be onto B. As A_n is a maximal antichain, c must be compatible with some $p \in A(n)$. By construction, there is a b' such that $p \Vdash \dot f(\hat n) = b'$; but then $b = b'$, otherwise p would be incompatible with c; so $p \Vdash \dot f(\hat n) = \hat b$, and so $p \le b$. Since $p \in A$ the proof is complete. $\dashv$

1·3 Corollary *For λ a singular strong limit cardinal, r.o.($Coll(\omega, <\lambda)$) $\cong$ r.o.($Coll(\omega, \lambda)$)*

Here r.o. stands for "the regular open algebra over", and denotes the canonical complete Boolean algebra given by a separative partial ordering.

1·4 Corollary *If B is a separative notion of forcing of cardinality at most κ, then $B \times Coll(\omega, \kappa) \cong Coll(\omega, \kappa)$.*

1·5 Remark The case $\kappa = \omega$ also works here and in the theorem, since all countable separative partial orderings are isomorphic to the Cohen ordering.

1·6 Definition If G is $(V, \mathrm{Coll}(\omega, <\lambda))$-generic, we write G_ν for $G \restriction \mathrm{Coll}(\omega, <\nu)$ and S_G for $\bigcup_{\nu<\lambda} \Re \cap V[G_\nu]$.

Now we want to prove the following:

1·7 Proposition *Let λ be a [singular] strong limit cardinal, and suppose that in some (not necessarily Boolean) extension, W, of V there is a set S of reals such that*

(i) $\forall x{:}\in S\ \ \exists P\ [P$ is in V a forcing of size less than λ and x is (V, P)-generic$]$

(ii) given any two members x, y of S, every real in $V[x, y]$ is in S.

(iii) each ordinal less than λ is recursive in at least one member of S.

Then in some Boolean extension of W there is a filter G that is $(V, Coll(\omega, <\lambda))$-generic and for which

$$S = S_G.$$

Proof : Work in W, and consider the following notion of forcing, call it $\mathcal{A}$: the conditions are objects g where $\exists x{:}\in S\ \ g \in V[x]$ and for some $\zeta = \zeta(g) < \lambda$, g is $(V, Coll(\omega, <\zeta))$ generic.

Note that (iii) implies that $\mathcal{A}$ is non-empty. We give it a natural partial ordering:

$$g_1 \le_{\mathcal{A}} g_2 \iff \zeta(g_1) \ge \zeta(g_2)\ \&\ g_2 = g_1 \restriction \zeta(g_2)$$

Let G be $(W, \mathcal{A})$-generic.

We may construe G as potentially a generic for $Coll(\omega, <\lambda)$, it being a collection of maps of ω onto the ordinals less than λ.

(i) G is indeed V-generic for that forcing.

Proof : let $D \in V$ be dense in $Coll(\omega, <\lambda)$; we shall show that $\forall \bar g{:}\in \mathcal{A}\ \exists g{:}\in \mathcal{A}\ \ g \le_{\mathcal{A}} \bar g\ \&\ g$ meets D, where by "meets" we mean that $\exists q{:}\in D\ \ g \supseteq q$.

So let $\bar g \in \mathcal{A}$ be given, and set $\bar\nu = \nu(\bar g)$.

Any condition q in $Coll(\omega, <\lambda)$ has domain a finite subset of $\omega \times \lambda$ and for any ν splits naturally into $(q)_\nu = q \restriction \omega \times \nu$ and $(q)^\nu =\restriction q \restriction \omega \times [\nu, \lambda)$; so that the map $q \mapsto ((q)_\nu, (q)^\nu)$ defines an isomorphism between $\mathrm{Coll}(\omega, <\lambda)$ and $\mathrm{Coll}(\omega, <\nu) \times Coll(\omega, [\nu, \lambda))$.

Write $D/\bar g = \{(q)^{\bar\nu} \mid q \in D\ \&\ (q)_{\bar\nu} \subseteq \bar g\}$.

Then by the standard theory of two-stage iterations, $D/\bar g$ is in $V[\bar g]$ and is dense in $Coll(\omega, [\bar\nu, \lambda))$. Let $r \in D/\bar g$: pick γ with $\bar\nu < \gamma < \lambda$ and $r \in Coll(\omega, [\bar\nu, \gamma))$: possible as all conditions are finite sets.

Let $\Delta = \big(Power(Coll(\omega, [\bar\nu, \gamma)))\big)^{V[\bar g]}$. Pick $y \in S$ such that $\bar g \in V[y]$ and in $V[y]$, Δ is countable. This is possible since λ is a strong limit cardinal; so the members of Δ are described in V by a set of names of size less than λ, and so in $V[\bar g]$, $\overline{\overline{\Delta}} = \kappa$, say, $< \lambda$. Any y in S in which κ is recursive will do.

Inside $V[y]$ we may build a $(V[\bar g], Coll(\omega, [\bar\nu, \gamma))$-generic, h, with $r \in h$. Now put $g = \bar g \cup h$. Then g is $(V, Coll(\omega, <\gamma)$-generic, as its tail is generic over its head, and its head is generic over V; $g \le \bar g$; $r \in D_g$ so for some q in D $(q)^{\bar\nu} = r\ \&\ (q)_{\bar\nu} \subseteq \bar g$; hence $q \subseteq g$ and so g meets D.

(ii) If $x \in S_G$, there is by definition of S_G a $\nu < \lambda$ with $x \in V[G \restriction \nu]$; but $G \restriction \nu = g$, say, is in $\mathcal{A}$; hence $\exists y{:}\in S\ \ g \in V[y]$; so $x \in V[y]$, and so $x \in S$.

(iii) Suppose $a \in S$: we show that $\forall \bar{g} :\in \mathcal{A}\ \exists \} :\in \mathcal{A}\ \} \leq \breve{\}}\ \&\ \dashv \in \mathcal{V}[\}]$. Density will then imply that $\Vdash_{\mathcal{A}} \bigwedge a :\epsilon S \bigvee \nu :< \hat{\lambda}\ a :\epsilon \hat{V}[\dot{G} \restriction \nu]$.

To get our bearings, suppose first that $\bar{g}$ is the empty condition.

Let a be (V, P_a)-generic, where $\overline{\overline{P_a}} = \kappa$, say, $< \lambda$ and we may suppose that P_a is a complete Boolean algebra generated by the truth values $[\![\dot{a}(\hat{n}) = \hat{m}]\!]^{P_a}$. Let h be $(V[a], Coll(\omega, \kappa))$-generic. Then (a, h) is $\big(V, P_a \times Coll(\omega, \kappa)\big)$-generic. Since, by Krivine, $r.o.(P_a \times Coll(\omega, \kappa)) \cong r.o.(Coll(\omega, \kappa))$, there is a g which is $Coll(\omega, \kappa)$-generic such that $V[a][h] = V[g]$.

Now since we may predict in advance how much less than λ needs to be countable to build h and find g, we may pick $y \in S$ so that the argument of the last paragraph may be carried out in $V[y]$: hence our $g \in \mathcal{A}$; since $a \in V[g]$, we are done.

Now we modify this: given $\bar{g}$ (non-empty) in $\mathcal{A}$, let $b \in S$ be such that $\bar{g} \in V[b]$, and let $c = \langle a, b \rangle$. Then $c \in S$, so c is generic over V by an algebra P_c of size less than λ; since $\bar{g} \in V[c]$, c is also generic over $V[\bar{g}]$, by an algebra Q_c, say, of cardinality less than λ in $V[\bar{g}]$: call this cardinal κ. Actually Q_c will be a quotient algebra of P_c, by the filter generated by truth values associated with the P_c-name of $\bar{g}$.

Let $\bar{\nu} = \nu(\bar{g})$. For some $z \in S$ with $c \in V[z]$, we can build in $V[z]$ an object h which is $(V[c], Coll(\omega, [\bar{\nu}, \bar{\nu}+\kappa))$-generic. Since in $V[\bar{g}]$, $Q_c \times Coll\big(\omega, [\bar{\nu}, \bar{\nu}+\kappa)\big) \cong Coll(\omega, [\bar{\nu}, \bar{\nu}+\kappa))$, there is a $(V[\bar{g}], Coll[\omega, [\bar{\nu}, \bar{\nu}+\kappa))$-generic g' such that $V[\bar{g}][g'] = V[\bar{g}][c][h]$. Put $g = \bar{g} \cup g'$: then g is $(V, Coll(\omega, < \bar{\nu}+\kappa)$-generic; $g \in \mathcal{A}$ as $g \in V[z]$ and $z \in S$; and $a \in V[c] \subseteq V[g]$. $\dashv$

1·8 Proposition *Suppose that x, a real in $V[G]$, is in $V[G]$ definable from members of V and an initial segment G_α of G. Then $x \in V[G_\alpha]$ and so $x \in S$.*

Proof : Following an argument of Solovay and Lévy,[R1] note that $Coll(\omega, < \lambda)$ factors as $Coll(\omega, < \alpha) \times Coll(\omega, [\alpha, \lambda))$; move to $V[G_\alpha]$: and let $C = Coll(\omega, [\alpha, \lambda))$, (or its completion, in the model $V[G_\alpha]$.) Let G^α be the rest of G. Then

$$\forall n, m :\in \omega\ \ x(n) = m \Longleftrightarrow [\![\dot{x}(\hat{n}) = \hat{m}]\!]^C \in G^\alpha$$

but by the definability of x from parameters in $V[G_\alpha]$, and the homogeneity of C, the Boolean truth value in each case is 0 or 1, and hence $x \in V[G_\alpha]$. $\dashv$

Let N be the submodel of $V[G]$ consisting of those sets hereditarily definable from members of V and from members of $\{G_\alpha \mid \alpha < \lambda\}$. The above proposition implies that the reals of N are precisely those of S. Hence if we form $L[S]$, we see that $L[S] \subseteq N$, and $\Re \cap L[S] = S$.

N models ZF; it might possibly not satisfy DC as not all the reals of $V[G]$ are in it; and indeed a celebrated result of Lévy, one of the first applications of forcing, shows that if we start from L and form the $< \lambda$ symmetric collapse, where $\lambda = (\aleph_\omega)_L$, then in $L[S]$,[N1] $\lambda = \omega_1$, and is of cofinality ω in these models, being accessible by the sequence $\langle \omega_n^L \mid n < \omega \rangle$.

However we are far from L, and we shall see that in our case DC does in fact hold in $L[S]$.

1·9 Proposition *Let $\dot{S}$ name the set S_G where G is $(V, Coll(\omega, \lambda))$-generic. Suppose that $\phi(a, \zeta, S)$ is a statement about a real a, an ordinal ζ and a set of reals S, and has no other free variables. Then for $a \in \Re$,*

$$[\![\phi(\hat{a}, \hat{\zeta}, \dot{S})]\!]^{\mathrm{Coll}(\omega,\lambda)} = 0 \text{ or } 1.$$

Proof : if not, there will be conditions p and q in $Coll(\omega, \lambda)$ such that $p \Vdash \phi(\hat{a}, \hat{\zeta}, \dot{S})$ and $q \Vdash \neg\phi(\hat{a}, \hat{\zeta}, \dot{S})$. We may extend p and q if necessary so that $dom(p) = dom(q)$. Then there will be a permutation π with finite support such that $\pi(p) = \pi(q)$; this lifts to an automorphism $\tilde{\pi}$ of the Boolean-valued model; $\tilde{\pi}(G)$ will differ very little from G and certainly will be inter-constructible with it, and will give rise to the same set of reals: $\Vdash S_G = S_{\tilde{\pi}(G)}$. This contradicts the fact that $q \Vdash \phi(\hat{a}, \zeta, \tilde{\pi}(\dot{S}))$ and $q \Vdash \neg\phi(\hat{a}, \zeta, \dot{S})$. $\dashv$

R1

N1 which equals N in this case

2: ω Woodins and the magic condition

Our aim in this section is to prepare the ground for our construction of models of AD. We fix an increasing sequence $\theta_0 < \theta_1 < \ldots$ of Woodin cardinals, and write $\lambda = \bigcup_n \theta_n$. To ease the world shortage of subscripts, we write Q_i for Q_{θ_i}. At intervals in the drama a character κ will appear: all that we know about κ for the moment is that $\kappa = \overline{\overline{V_\kappa}}$. At a later stage we shall assume κ to be the limit of Ramsey cardinals.

Our first object of discussion is the following *magic condition*:

2·0 Definition $MC = \{X \preceq V_{\lambda+1} \mid X$ is countable $\&$ $\langle \theta_i \mid i < \omega\rangle \in X$ $\&$
$\forall A{:}\in X$ (A predense in $Q_i \implies \exists a{:}\in X \cap A \bigcup a \cap X \in a)\}$

2·1 Proposition *MC is stationary in $V_{\lambda+1}$.*

Proof : Let F be a $V_{\lambda+1}$-function. Suppose that η is a limit ordinal greater than λ with $V_\eta \models KP$. Let $<_\eta$ be a well-ordering of V_η, and consider the structure

$$\mathcal{V} = \langle V_\eta, <_\eta, F, \langle \theta_i | i < \omega\rangle, \lambda\rangle.$$

Note that if Y is an elementary submodel of $\mathcal{V}$, then $Y \cap V_{\lambda+1}$ is an elementary submodel of $V_\lambda + 1$ and is F-closed.

From our alternative characterisation of "semi-proper" we know that if A is pre-dense in Q_θ, where θ is Woodin, then for $\delta < \theta$ with δ A-strong, every countable δ-small η-model X has a countable δ-neat extension Z which $(A \cap Q_\delta)$-improves it: or, in symbols, $\exists \zeta :< \delta\ X \cap V_\delta = X \cap V_\zeta = Z \cap V_\zeta$ and $\exists a{:}\in Z \cap A \cap Q_\delta \bigcup a \cap Z \in a$. Further, we know that if $\delta' \geq \delta$ and W is a δ'-neat extension of Z then $\bigcup a \cap Z = \bigcup a \cap W$.

So we start from some countable $Y \preceq \mathcal{V}$, and examine the countably many A's which are members of Y and predense in Q_0. We now set out to improve our starting model with respect to each such A, but of course after one step we shall have acquired new ones. We therefore must do some book-keeping to ensure that at stage $2i(2j+1)$ we improve the $j^{\rm th}$ predense set of the $i^{\rm th}$ model. We may help ourselves to a larger δ (but always less than θ_0) each time. After exactly ω steps we reach a model Y_0 which has the property that for each $A \in Y_0$ which is predense in Q_0 there is an $a \in Y_0 \cap A$ with $\bigcup a \cap Y_0 \in a$, because of our scrupulous use of neat extensions.

Now we start again working on the Q_1 pre-dense sets: the process is the same, we must only take care to use δ's in the interval (θ_0, θ_1) so as not to destroy the good work already done by creating new subsets of Q_0. After ω more steps we reach a model Y_1 which does what we want for both Q_0 and Q_1.

After a total of ω^2 steps, therefore, we reach a Y_ω which has the property that for all $i < \omega$ and every $A \in Y_\omega$ which is pre-dense in Q_i, there is an $a \in Y_\omega \cap A$ with $\bigcup a \cap Y \in a$.

Finally we note that $Y_\omega \cap V_{\lambda+1}$ is F-closed and in MC. $\dashv$

The magic condition is so called because of the following remarkable property:

2·2 Proposition *Suppose that $\kappa = \overline{\overline{V_\kappa}} > \lambda$, the supremum of the increasing sequence of Woodin cardinals θ_i. Consider the tower forcing P_κ: let $\dot G$ name the generic for this forcing. Then*

$$MC \Vdash_{P_\kappa} \bigwedge i\ \dot G \cap \hat Q_i \texttt{ is } (V, Q_i)\texttt{-generic}.$$

Proof : Let Δ be dense in Q_i. We show that $\forall b :\leq MC \exists c :\leq b\, \exists \bar a{:}\in \Delta\ \ c \leq \bar a$; whence $c \Vdash \neg[\dot G \cap Q_i \cap \Delta = \emptyset]$.

So let $b \leq MC$: then $\bigcup MC \subseteq \bigcup b$ and $b_{\bigcup MC} \subseteq MC$. $\bigcup MC = V_{\lambda+1}$, so by intersecting with the extension of a suitable club in $\bigcup MC$ to $\bigcup b$, we may assume that $\forall X{:}\in b\ ; \Delta \in X$.

Thus $\forall X :\in b\ X \cap \bigcup MC \in MC$, so $\forall X :\in b\ \exists a{:}\in X\ \cap \Delta \cap \bigcup MC\ \bigcup a \cap X \cap \bigcup MC \in a$; so by normality, $\exists c :\leq b\ \bigcup c = \bigcup b\ \&\ \exists \bar a \forall X :\in c\ \ \bar a \in X \cap \Delta \cap \bigcup MC\ \&\ \bigcup \bar a \cap X \cap \bigcup MC \in MC$.

Now $\bar a \in V_{\theta_i}$, so $\bigcup \bar a \subseteq V_{\lambda+1} = \bigcup MC$; so $\forall X :\in c\ \ X \cap \bigcup \bar a \in \bar a$; that is, $c_{\bigcup \bar a} \subseteq \bar a$; but $\bigcup \bar a \subseteq \bigcup MC \subseteq \bigcup b = \bigcup c$. Thus $c \leq \bar a$ and $\bar a \in \Delta$. $\dashv$

So now let G be P_κ generic with $MC \in G$. If it becomes necessary to emphasize the rôle of κ in the definition of G we shall call it $G^{(\kappa)}$. We now work inside $V[G]$, and let $n < \omega$. We write G_n for $G \cap Q_n$.

Since by the qualities of the magic condition, G_n is (V, Q_n)-generic, we may build a transitised inner model M_n of $V[G_n]$ and an elementary embedding $j_n : V \to M_n$.

All these M_n's are of course submodels of $V[G]$, and we may readily define maps $j^{(\kappa)}_{n,m} : M_n \to M_m$ for $n < m$: the typical element of M_n is represented by a pair $\langle a,\, f\rangle$ where $a \in G_n$ and $f : a \to V$ is in V. But such objects, *modulo* a different equivalence class, represent objects in M_m: so it is natural to define

$$j_{n,m}([\langle a,\, f\rangle]_{G_n}) = [\langle a,\, f\rangle]_{G_m}$$

Of course, we now want to form the direct limit $\mathcal{M}_\omega$ of this system: but we must beware, since it may not be well-founded. However, it will be an ω-model, since all the elements of $\omega + 1$ are left fixed by all the embeddings, so there is no room for non-standard integers to creep in. So we may assume that the well-founded initial part of $\mathcal{M}_\omega$ has been transitised, and that this extends past ω, indeed up to the original ω_1.

Now what can we say about the reals of $\mathcal{M}_\omega$? First, our remarks in the last paragraph show that $j^{(\kappa)}_{n,m}(x) = x$ for any real x in M_n, and so $\Re \cap M_n \subseteq \Re \cap M_{n+1}$. Hence

$$\Re \cap \mathcal{M}_\omega = \bigcup_n \Re \cap M_n$$

by general properties of direct limits. Let us write $S^{(\kappa)}$ for this set of reals in $V[G]$.

S meets the three requirements for being the set of reals of a symmetric $< \lambda$ collapse: each real in S is generic over V by an algebra of size less than λ, namely one of the Q_n's. Given two of them, x and y, say, let n be such that both x and y have been added by Q_n: then $\Re \cap V[x,y] \subseteq \Re \cap M_n \subseteq S$. Thirdly, every ordinal less than λ is recursive in some member of S, since Q_n renders countable all ordinals less than θ_n. Thus we have shown

2·3 PROPOSITION *S^κ is the set of reals of some $< \lambda$ symmetric collapse.*

Back to $\mathcal{M}$. We have, as before, an elementary embedding $j : V \to \mathcal{M} \subseteq V[G]$ and we know that the reals of $V[G]$ are the reals of $\mathcal{M}$; but we should notice that these need not be the same as the reals of $\mathcal{M}_\omega$: this will depend very much on κ, and what G has been up to since it left the magic condition behind. We do know, though, that we may assume $j(\omega) = \omega$, essentially because if a is stationary and $f : a \to \omega$, then for some n, $f^{-1}``\{n\}$ is stationary, and so $j``\omega = j(\omega)$.

The embedding j may be seen to factor through j_ω, as do each of the j_n; indeed if we define $k_n : M_n \to \mathcal{M}$ as the map $[\langle a,\, f\rangle]_{G\cap Q_n} \mapsto [\langle a,\, f\rangle]_G$ for $a \in G \cap Q_n$, these will plainly commute with our maps $j_{n,m}$, so that $k_n = k_m \circ j_{n,m}$ and will thus induce a map $k : \mathcal{M}_\omega \to \mathcal{M}$.

These maps are all elementary by Łoś. We shall return shortly to the question of computing these maps inside $\mathcal{M}$. For the moment, let us look ahead.

A suitable restriction of the map j_ω induces an elementary embedding from $L[\Re]$ to $(L[S^{(\kappa)}])_{\mathcal{M}^{(\kappa)}_\omega}$ This latter is a strange model: let us call it $\mathcal{L}[S^{(\kappa)}]$. It is the constructible closure of $S^{(\kappa)}$ computed inside a possibly ill-founded model; and is therefore not the same as $L[S^{(\kappa)}]$ constructed along the ordinary well-founded ordinals. Indeed, we shall examine later a case where the two are not even elementarily equivalent, AD holding in one and not in the other.

Note that the theory of $L[S^{(\kappa)}]$ does not depend on κ, even if we allow names for reals from the ground model V or for ordinals: we proved this in our section on symmetric collapses. .

However, $\mathcal{L}[S^{(\kappa)}]$ does depend on κ: we shall see that as we increase κ, so the well-founded part of of $\mathcal{M}^{(\kappa)}_\omega$ and therefore of $\mathcal{L}[S^{(\kappa)}]$ increases, and we shall exploit this fact in proving that DC and then AD hold in $L[S^{(\kappa)}]$.

Our plan for proving that $DC + AD$ holds in $L[S^{(\kappa)}]$ is broadly to say that if not, some counterexample to these principles occurs in some $J_\gamma[\Re]$, and that this γ will be independent of κ, by the invariance of the theory of the model, including names for ordinals. We may then choose κ larger than γ, so that γ is in the well-founded part of $\mathcal{M}^{(\kappa)}_\omega$, and thus we have manœuvred the counterexample into a model into which V is elementarily embedded. We delay the details and the rest of the argument as we must now establish the last technical fact we shall need:

2·4 PROPOSITION $\kappa \subseteq wfp(\mathcal{M}^{(\kappa)}_\omega)$.

2·5 REMARK So far, the Q version of tower forcing has been much more in play than the P version: we particularly needed Q to build our tree representing $\{x \mid \phi(x)\}$ where ϕ satisfies the three FC axioms, as we had to work with countable models. It is in the first line of this proof that the importance of using P_κ not Q_κ becomes clear.

Proof of 2.4: Let $\gamma < \kappa$: then $j``V_\gamma$ is in $\mathcal{M}$, since it is represented by $\langle Power(V_\gamma),\, id\rangle$:[C 1] note that conditions like $Power(V_\gamma)$ are as we have seen inseparable from the trivial condition 1, and so are in the generic. Although we do not know how well-founded $\mathcal{M}$ is, this object $j``V_\gamma$ is genuinely well-founded, and if within $\mathcal{M}$ we "collapse it" we get something well-founded, E-transitive, and isomorphic to V_γ. If we suppose that we have transitised as much of $\mathcal{M}$ as possible, we may conclude that $V_\gamma \in \mathcal{M}$; but also the inverse of the collapsing map is $j \restriction V_\gamma$, and so that is in $\mathcal{M}$. So in particular, $\kappa \subseteq wfp(\mathcal{M})$.

Now we know that for all $a \in V_\kappa$, $a \in G \iff j``\bigcup a \in j(a)$, and hence $G \restriction V_\gamma \in \mathcal{M}$, being computable from $j \restriction V_\gamma$.

So in particular, setting $G_i = G \cap V_{\theta_i}$, $\mathcal{M}$ has as a member the sequence $\langle G_i | i \in \omega\rangle$. From this sequence and V_γ it can build a structure isomorphic to $(\mathcal{M}_\omega)_{j_\omega(\gamma)}$.[C 2] It is tempting to argue that as j factors through j_ω, and $j(\gamma)$ is well-founded, so is $j_\omega(\gamma)$. But we don't know that $j(\gamma)$ is well-founded, only that $j``\gamma$ is there. Therefore further argument is needed.

So we now show that $\mathcal{M}$ also knows $k \restriction (\mathcal{M}_\omega)_{j_\omega(\gamma)}$. To see this, we prove a general

2·6 Lemma $[\langle a,\, f\rangle]_G = j(f)(j``\bigcup a)$.

Proof : Consider three maps from a to V, namely f, the constant function c_f with $c_f(x) = f$, and the identity i with $i(x) = x$. Now

	$c_f(x)$	applied to	$i(x)$	yields	$f(x)$;
so by Łoś,	$j(f)$	applied to	$j``\bigcup a$	yields	$[\langle a,\, f\rangle]_G$ $\dashv$

Hence k is computable inside $\mathcal{M}$: each element of $\mathcal{M}_\omega$ is $[\langle a,\, f\rangle]_G \cap V_{\theta_n}$ for some n, and is sent to $[\langle a,\, f\rangle]_G$, *i.e.* to $j(f)(j``\bigcup a)$, which $\mathcal{M}$ can compute if the element has in $\mathcal{M}_\omega$ rank less than $j_\omega(\gamma)$, for then it is represented by a function all of whose values have rank less than γ, so choosing a γ' greater than $\lambda \cup \gamma + 2$, $\mathcal{M}$ can compute k from a knowledge of $j \restriction V_{\gamma'}$.

Now we may show that $wfp(\mathcal{M}_\omega)$ contains κ. For let $\gamma < \kappa$: then $j_\omega(\gamma + 2)$ is isomorphic to an ordinal of $\mathcal{M}$, using portions of k; so $\mathcal{M}$ has a map $k \restriction j_\omega(\gamma + 2) \to j(\gamma + 2)$. We already know that $\gamma + 1$ is an ordinal of $\mathcal{M}$; these two ordinals can therefore be compared in $\mathcal{M}$; but it is absurd to suppose that $j_\omega(\gamma + 2)$ (which will contain, pointwise, a copy of $\gamma + 2$) can be mapped in an order-preserving way into $\gamma + 1$, and therefore $\gamma + 1$ is isomorphic to an initial segment of $j_\omega(\gamma + 2)$: thus $\mathcal{M}_\omega$ is equipped with well-founded ordinals reaching up to κ.

[C 1] we have shown this above, but it also follows from the more general Lemma 2.6 below

[C 2] In verifying this, we need to note that $j_{0n}(\gamma) < \kappa$.

3: Axiom FC2*

In sections 4 and 6 we shall complete the proofs of theorems IV.3.3 and IV.3.4: in doing so we shall rely on our three axioms that guarantee the existence of a forcible complement. In each of the two possible formulæ ϕ that we shall consider, the verification of the second Axiom presents the most problems. Now that we have the concept of a symmetric $< \lambda$-collapse to hand, we may present a variant of that second axiom, which we now show to imply the original version. We shall then verify directly that this variant holds of one of our two formulæ; for the other we must be slightly more circumspect.

Axiom FC2* *Whenever P is a forcing of cardinality less than λ and g is (V, P)-generic and M is a $< \lambda$ symmetric collapse over $V[g]$, then for any real x in $V[g]$,*

$$V[g] \models \phi[x] \Longleftrightarrow M \models \phi[x]$$

3·0 Proposition *Axiom FC2* implies Axiom FC2.*

Proof : in Axiom $FC2$, we are comparing the truth of $\phi(x)$ in $V[g]$ as against $V[g][h]$. Let M be a $< \lambda$-symmetric collapse of $V[g][h]$. Then M is also a $< \lambda$-symmetric collapse of $V[g]$. So by Axiom FC2*,

$$\begin{aligned} V[g] \models \phi[x] &\Longleftrightarrow M \models \phi[x] \\ &\Longleftrightarrow V[g][h] \models \phi[x]. \end{aligned}$$

4: AD in a symmetric extension, from ω Woodins

Consider an ω-model $\mathcal{K}$, possibly a proper class, such that $\mathcal{K} \models L[\Re] \models \neg DC$. In the theory $ZF + V = L[\Re]$ we may show formally that $DC_{\Re} \implies DC$; hence if $L[\Re] \models \neg DC$, then there is some $\mathcal{K}$-ordinal γ such that $\mathcal{K}$ believes that in $J_\gamma[\Re]$ there is a relation on the reals with no descending chain when there ought to be. Call the least such γ the *DC-ordinal of* $\mathcal{K}$.

If DCR fails it fails in some $J_\gamma[\Re]$; and if it fails in some $J_\gamma[\Re]$ that counterexample is a failure in $L[\Re]$ as all the descending chains of reals are codable as single reals and therefore exist in advance in $J_1[\Re]$

Again we may prove formally in $ZF + V = L[\Re]$ that if $\neg AD$ then there is a least ordinal δ such that $J_\delta[\Re]$ contains a game with no winning strategy; as all strategies are reals and all reals are in $J_1[\Re]$, this $J_\delta[\Re] \models \neg AD$; and if any $J_\delta[\Re] \models \neg AD$, they have it right.

Moreover some undetermined game will be definable solely from a real parameter, as any ordinal parameter can be minimised as "the least ordinal such that this wff defines an indeterminate game"; hence we may define the *test ordinal* as the least δ such that for some wff χ and some real parameter z, $\{y \mid J_\delta \models \chi[y, z]\}$ is an indeterminate game; the first such χ, in a Gödel numbering of wffs, we shall call the *test formula*; and we shall speak of any real z for which the above formula yields an indeterminate game as a *test parameter*. Note of course that we cannot speak of **the** test parameter.

So whenever we have an ω-model $\mathcal{K}$ which believes that in $L[\Re]$, AD fails we may speak of the test ordinal of $\mathcal{K}$ and the test formula of $\mathcal{K}$, and of certain reals of $\mathcal{K}$ as being test parameters in $\mathcal{K}$.

With these formal concepts in mind, we now define the formula $\phi_0(x)$ as follows:

$$\begin{aligned} \phi_0(x) \ =_{\mathrm{df}} \ & (\neg AD)_{L[\Re]} \ \& \ x(0) = \text{ the Gödel number} \ulcorner \chi \urcorner \text{ of the test formula } \chi \ \& \ (x^*)^2_1 \text{ is a test parameter} \\ & \& \ J_\gamma[\Re] \models \chi((x^*)^2_0, (x^*)^2_1) \quad \text{where } \gamma \text{ is the } AD\text{-ordinal.} \end{aligned}$$

In the definition of ϕ_0 we have used some notations introduced in Chapter III : x^*, $(\cdot)^2_0$, $(\cdot)^2_1$.

We shall prove below that this ϕ_0 satisfies the three Axioms. Granting that for the moment, we know that $\{x \mid \phi_0(x)\}$ is, in V, forcibly complemented, by VIII.; and so is $< \lambda$ weakly homogeneous.

Our aim is to prove that AD holds in $L[S]$ where S are the reals of a symmetric $< \lambda$ collapse.

If that is false, then let γ_1 be the test ordinal of this model: by the homogeneity of the algebra, the value of γ_1 is fixed; we may now choose κ larger than γ_1 and form our tower of models culminating in $\mathcal{M}^{(\kappa)}_\omega$, the reals of which we know to be those of a symmetric $< \lambda$ collapse. Thus we know that $\mathcal{M}_\omega \models \neg AD$. Hence using j_ω, we may pull the failure of AD back to V: so we may define γ_0 as the test ordinal of V, and let χ_0 be the test formula of V, and the real a a test parameter.

Thus $Z =_{\rm df} \{y \mid J_{\gamma_0} \models \chi_0(y, a\}$ is not determined in V.

But now define the function f by $f(y)$ =the real x with $x(0)$ = the Gödel number of χ_0, $(x^*)^2_0 = y$ and $(x^*)^2_1 = a$; then f is continuous and $\forall y\ y \in Z \iff \phi_0(f(y))$: thus Z, being continuously reducible to a $< \lambda$-weakly homogenous set, is $< \lambda$-weakly homogenous, and so, by Martin-Steel, determined.

It remains to show that ϕ_0 satisfies the three axioms. For FC1, it is enough to notice that the reals of $V[g]$ and of $V_\kappa[g]$ are the same and that any set of reals, in particular any undetermined game, constructible from that set of reals will be constructed before stage κ, since κ is strongly inaccessible. The same argument yields FC3 since the reals of $V[G]$ and of M are the same. For FC2 it is enough to check that FC2* holds.

But to see this we may argue has we have just done: whether something is true in the symmetric collapse will have truth value 0 or 1 and in the latter case will hold in $\mathcal{M}^{(\kappa)}_\omega$, independently of κ; and κ may now be chosen larger than the test ordinal γ_1, and under our elementary embedding j_ω, γ_0 will be sent to γ_1, χ_0 to χ_1, and the test parameter, being a real, will stay put (as indeed will the test formula).

So $\phi(x)$ if true in $V[g]$ will be true in the $< \lambda$-symmetric collapse over $V[g]$.

5: Review of the theory of sharps

The study of $0^\sharp$ begins with the investigation of a possible theory T of models J_ζ with names c_i $(i < \omega)$ for ordinal indiscernibles. T is consistent, complete, extends $KP + V = J$; contains all instances of axioms saying that the indiscernibles are indiscernible ordinals; and there is a *witness condition* saying that if $\bigvee x\Phi \in T$ then for some term $t(\vec{c})$ in the indiscernibles, $\Phi(t(\vec{c}))$ is in T.

Given such a theory with the indiscernibles indexed by a linearly ordered set I, so that $i < i' \iff T \vdash c_i \epsilon c_{i'}$, and where I is non-empty and has no last point, and if J is another topless linear ordering, we may produce a similar theory T_J in a language where we jettison the names c_i for $i \in I$ and have names d_j for $j \in J$: we take $T_J = \{\Phi(\vec{d}) \mid \Phi(\vec{c}) \in T\}$.

From such a T_J we may build a model M_J out of the terms of the theory, identifying two terms when the theory asserts that they are equal, and considering the equivalence class of the term t to stand in the relationship E_J to that of the term t' when $T \vdash t \epsilon t'$. The rôle of the witness condition is to ensure that a proof by induction on the length of formulæ will succeed in showing that

$$M_J \models \Phi[\vec{d}] \iff T_J \vdash \Phi(\vec{d})$$

where we have not distinguished notationally between the name d_j of an indiscernible and its denotation in the model M_J, which will of course be the equivalence class of the term d_j.

We write ON_J for the "ordinals" of the model M_J, and C_J for the indiscernibles: that is, the denotations of d_j for $j \in J$; so C_J is a subset of ON_J.

Four lemmata apply: we state them in rapid succession; a more careful statement would require the truth of the clauses of each to be included in the hypotheses of the next.

All four state the equivalence of three statements, and are roughly of this kind: the first says of a property that "it always happens"; the second says that "it happens in at least one non-trivial instance"; and the third that it will happen if certain trivial requirements are met. In the first three lemmata the trivial requirement is that certain sentences are in the theory T; in the fourth, that the property holds when J is countable. Here they are.

The first is concerned with the question whether the indiscernibles are cofinal in the ordinals. Do not forget our convention that the I's and J's are required to be non-empty topless linear orderings.

5·0 Lemma *The following are equivalent:*

(i) For every J, C_J is cofinal in ON_J
(ii) There is a J for which C_J is cofinal in ON_J
(iii) Each sentence "$\vec{s} < c \longrightarrow t(\vec{s}) < c$" is in T.

The definition of M_J from J is satisfyingly functorial: so that if I can be embedded in J, then the embedding lifts to an embedding of M_I in M_J, and, remembering that the map $i \mapsto (c_i)^{M_I}$ embeds I in M_I, where the superscript, which we shall in future omit, means "the meaning of this constant symbol in the model M_I", we may express this by a commutative diagram:

$$\begin{array}{ccc} I & \xrightarrow{\pi} & J \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ M_I & \xrightarrow{\xi} & M_J \end{array}$$

The case that π embeds I as an initial segment of J is of particular interest to us, and in this case it will be seen that ξ is then an elementary embedding.

First though, we should ask whether, when I is an initial segment of J, that means that ON_I will be an initial segment of ON_J ? The next Lemma addresses that.

5·1 LEMMA *The following are equivalent:*

(i) for all I, J, if I is an initial segment of J, then ON_I is an initial segment of ON_J under the natural embedding of M_I in M_J;

(ii) there are I and J with I a proper initial segment of J and ON_I an initial segment of ON_J;

(iii) each sentence "$t(\vec{r}, c, \vec{s}) < c < \vec{s} \wedge c < \vec{s'} :\longrightarrow t(\vec{r}, c, \vec{s}) = t(\vec{r}, c, \vec{s'})$" is in T.

The next lemma is concerned with whether C_I is a closed subset of ON_I.

5·2 LEMMA *The following are equivalent:*

(i) for every I, non-empty $A \subseteq I$ and $c \in I$, if $c = sup\ A$ in the ordering I then $c = sup\, A$ also in the ordering ON_I;

(ii) there is an I and an $A \subseteq I$ and a $c \in I$ such that A is non-empty but has no top point, and such that $c = supA$ both in relation to the ordering I and to the ordering ON_I;

(iii) each sentence "$t(\vec{r}, c, \vec{s}) < c < \vec{s} \wedge \vec{r} < c' < \vec{s'} :\longrightarrow t(\vec{r}, c, \vec{s}) = t(\vec{r}, c', \vec{s'})$" is in T.

The final lemma addresses the question of the well-foundedness of the models M_J for wellorderings J:

5·3 LEMMA *The following are equivalent:*

(i) for every well-ordering J, ON_J is a well-ordering;

(ii) there is an uncountable well-ordering J with ON_J a well-ordering;

(iii) for every countable well-ordering J, ON_J is a well-ordering.

We shall say that T is a possible $0^\sharp$ if it satisfies clause (iii) of each lemma above, and also the initial requirements about its being a consistent complete theory in a language with countable many indiscernibles, etc.

We can not of course prove that possible $0^\sharp$'s exist, and indeed if $V = L$ they do not. Nor do we yet know that at most one possible $0^\sharp$ can exist. We shall see that its uniqueness is provable in ZF; to derive its existence from, say, the existence of a Ramsey cardinal κ, we proceed as follows. First we introduce three formulæ:

$$\begin{aligned} \Phi_0(X,\kappa) &=_{\rm df} X \text{ is a set of indiscernibles for } J_\kappa \ \&\ \overline{\overline{X}} = \kappa \\ \Phi_1(Y,\kappa) &=_{\rm df} \Phi_0(Y,\kappa) \ \&\ \mathrm{Hull}(J_\kappa, Y) = J_\kappa \\ \Phi_2(Z,\kappa) &=_{\rm df} \Phi_1(Z,\kappa) \ \&\ Z \text{ is closed in } \kappa \end{aligned}$$

We then prove in quick succession four facts:

5·4 LEMMA *(i) If κ is a Ramsey cardinal then $\exists X\ \Phi_0(X,\kappa)$;*

(ii) if $\Phi_0(X,\kappa)$ then $\exists Y\ \Phi_1(Y,\kappa)\ \&\ \tilde{Y}(\omega) \le \tilde{X}(\omega)$;

(iii) if $\Phi_1(Z,\kappa)\ \&\ \forall Y \Phi_1(Y,\kappa) \implies \tilde{Z}(\omega) \le \tilde{Y}(\omega)$ then $\Phi_2(Z,\kappa)$;

(iv) if $\Phi_2(Z,\kappa)\ \&\ \Phi_2(W,\kappa)$ then $Z = W$.

We have thus reached a closed unbounded set C_κ of indiscernibles for J_κ.

From now on one works only from the existence of C_κ, and may forget that κ had the Ramsey property. One may now prove that for all other uncountable cardinals λ there is a set C_λ with $\Phi_2(C_\lambda, \lambda)$, and that if $\lambda < \kappa$ then $C_\lambda = \lambda \cap C_\kappa$; and that the existence of such C_κ is equivalent to the existence of an object satisfying the definition of being $0^\sharp$; and conversely we may use the uniqueness of the club set of indiscernibles to conclude that there is at most ONE $0^\sharp$.

Then of course we define $0^\sharp$ as the unique theory $T = T_{(\omega,<)}$ in ω indiscernibles of which clause (iii) of each Lemma is true. The first three, plus the ground conditions, are arithmetical requirements; the final lemma is a Π^1_2 requirement.

The above treatment readily generalises to models of the axiom $V = J[A]$ where A is any set of ordinals, of which we would have to include the atomic diagram in the theory, and leads to the concept of $A^\sharp$.

6: AD in an inner model, from ω Woodins and a bit more

For "a bit more", a measurable cardinal is more than enough. The minimal requirement is believed to be the existence of ${V_\lambda}^\sharp$: we shall prove a less exhausting version:

6·0 THEOREM *Suppose there are ω Woodin cardinals and, greater than them, ω_1 Ramsey cardinals. Then AD holds in $L[\Re]$.*

Proof : Let λ be the supremum of the ω Woodins, and $\kappa > \lambda$ the supremum of Ramsey cardinals, with $cf(\kappa)$ uncountable.[N 2]

Let G be (V, P_κ)-generic: in $V[G]$ we build the models $\mathcal{M}^{(\kappa)}$ and $\mathcal{M}^{(\kappa)}_\omega$; and from our previous work we know that κ is a subset of the well founded part of both.

6·1 PROPOSITION *$\{\eta < \kappa \mid \eta$ is a Ramsey cardinal and in $V[G]$, $j(\eta) = \eta\}$ is cofinal in κ.*

Proof : let $\eta_0 < \kappa$ and let p be a condition in P_κ. Let $\eta \in (\eta_0, \kappa)$ be Ramsey with $p \in V_\eta$, and let

$$q \ =_{\text{df}} \ \{X \preceq V_\eta \mid X \cap \bigcup p \in p \ \&\ otp(X \cap \eta) = \eta\}.$$

We shall show that q is a condition stronger than p which forces the statement $j(\hat\eta) = \hat\eta$. Density arguments will do the rest.

Evidently $q_{\bigcup p} \subseteq p$; we must show that $\bigcup p \subseteq \bigcup q$ and that q is stationary in V_η. So let $t \in \bigcup p$ and let F be a V_η-function. We shall find an F-closed $Y \in q$ with $t \in Y$.

Consider the two structures

$$\mathcal{A} \ =_{\text{df}} \ \langle V_\eta, \varepsilon, <_\eta, p, F, t, \rangle$$

and

$$\mathcal{A}^+ \ =_{\text{df}} \ \langle V_\eta, \varepsilon, <_\eta, p, F, t, x\rangle_{x \in \bigcup p}$$

The language even of $\mathcal{A}^+$ is of size less than η, so as η is Ramsey, we may find an $S \subseteq \eta$ of order type η which forms a set of indiscernibles for $\mathcal{A}^+$. Let $\mathcal{B}$ be the set of all $\bigcup p$-functions that are definable in $\mathcal{A}$ from the set S_ω of the first ω elements of S. Then $\mathcal{B}$ is countable, so as p is stationary, we may find an $X \in p$ closed under all members of $\mathcal{B}$. In particular, as we have named t, the constant function c_t is in $\mathcal{B}$, and so $t \in X$.

Let Y be the elementary substructure of $\mathcal{A}$ generated by $X \cup S$. Then $Y \cap \bigcup p = X \cap \bigcup p$: for let $y \in Y \cap \bigcup p$; so y is the evaluation in $\mathcal{A}$ of a term, say $y = \tau^{\mathcal{A}}(\vec{x}, \vec{s})$ where $\vec{x}$ is a finite subset of X and $\vec{s}$ is a finite subset of S. $X \subseteq \bigcup p$, as $X \in p$; so as both y and all the x's are in $\bigcup p$, we may, using the indiscernibility of S with respect to $\mathcal{A}^+$, replace $\vec{s}$ by $\vec{s'} \subseteq S_\omega$ and still have $y = \tau^{\mathcal{A}}(\vec{x}, \vec{s'})$. But the map $\vec{z} \mapsto \tau^{\mathcal{A}}(\vec{z}, \vec{s'})$ is in $\mathcal{B}$, and so $y \in X$. Thus $Y \cap \bigcup p \subseteq X \cap \bigcup p$; the other direction is obvious.

Hence we have shown that $Y \cap \bigcup p \in p$; plainly $otp(Y \cap \eta) = \eta$; $t \in Y$ as $t \in X$; and Y is F-closed as we have named Y in the structure $\mathcal{A}$, and $Y \preceq \mathcal{A}$. Hence $Y \in q$.

Now we assert that

$$q \Vdash j(\hat\eta) = \hat\eta.$$

Certainly q forces $\langle q,\, h\rangle$ to represent $j``\eta$, where $\forall x :\in q \ \ h(x) = x \cap \eta$. But for $x \in q$, the order type of $h(x) = \eta$: so

$$q \Vdash j``\hat\eta \texttt{ has order type } j(\hat\eta).$$

[N 2] What we seem to need about κ is that it is the limit of Ramsey cardinals and that P_κ works sensibly. For the latter, $cf(\kappa) = \omega$ will not do; κ a strong limit of uncountable cofinality seems to be enough.

and, visibly, $j``\eta$ has order type η. $\dashv$

Now let

$$\phi_1(x) \ =_{\rm df}\ \mathbf{R}^\sharp \text{ exists and } x \in \mathbf{R}^\sharp.$$

where as before x is a variable ranging over reals.

When $\mathbf{R}^\sharp$ exists, its elements, though we may often want to code them as reals, are statements about reals and names for indiscernibles. So we think of a possible member x of $\mathbf{R}^\sharp$ as a wff $\chi(\underline{a}_0, \dots \underline{a}_{m-1}, c_0, \dots c_{n-1})$ where the $\underline{a}$'s name reals $a_0, \dots a_{m-1}$, and the c's name ordinal indiscernibles.

We wish to show that, in the presence of ω_1 Ramsey cardinals above λ, the three FC axioms hold of ϕ_1.

First a lemma familiar from the theory of forcing:

6·2 Lemma *If η is Ramsey and $\mathcal{B}$ is a complete Boolean algebra of cardinality less than η, then*

$$\Vdash_{\mathcal{B}} \hat{\eta} \texttt{ is Ramsey}.$$

Proof : Let $\Vdash \dot{\pi} : [\hat{\eta}]^{<\omega} \longrightarrow \hat{\lambda}$, where $\lambda < \kappa$. For each $s \in [\eta]^{<\omega}$ pick a maximal anti-chain A_s in $\mathcal{B}$ and a function $f_s : \mathcal{B} \to \lambda$ such that

$$q \in A_s \implies q \Vdash \dot{\pi}(\hat{s}) = \hat{f}_s(\hat{q})$$

Define a partition of $[\eta]^{<\omega}$ by $\rho(s) = \langle A_s,\ f_s\rangle$. The range of ρ is of cardinality less than η, and so there is an X of size η which is homogeneous of ρ. We assert that $\Vdash \hat{X} \texttt{ is homogenous for } \dot{\pi}$. For given $p,\, n \in \omega$ and $s,\, t$ in $[x]^n$, p will be compatible with some $r \in A_s = A_t$, so $p \wedge r \Vdash \dot{\pi}(\hat{s}) = \hat{f}_s(\hat{r}) = \hat{f}_t(\hat{r}) = \dot{\pi}(\hat{t})$. $\dashv$

Thus in every extension of V by an algebra of size less than η_0, the first Ramsey cardinal greater than λ, and any set of reals S in the extension with $S = \Re \cap L[S]$, $S^\sharp$ will exist in the extension.

Turn now to the FC axioms, set out in Chapter VIII.

FC3 will be true of ϕ_1, for V, and therefore, by the Lemma, also $V[G]$, contains enough Ramsey cardinals to prove the existence of $\mathbf{R}^\sharp$; and M too will think that $\mathbf{R}^\sharp$ exists, since V does and j is elementary.

Since $V[G]$ and M have the same reals and therefore compute the same $L[\Re]$, and as they both think that $\mathbf{R}^\sharp$ exists, their two concepts of $\mathbf{R}^\sharp$ will coincide.

Axiom FC1 may be verified by a similar argument: $V[g]$ is equipped to prove the existence of $\mathbf{R}^\sharp$, which is a set of reals and therefore (as P is in V_κ),a member of $V_\kappa[g]$; $V_\kappa[g]$ has the same reals as $V[g]$, and hence correctly identifies the $\mathbf{R}^\sharp$ of $V[g]$ as being $\mathbf{R}^\sharp$.

Axiom FC2 remains to be checked. Broadly we wish to repeat our device of using Axiom FC2*, but we must be careful as $S^\sharp$ is not a member of $L[S]$.

6·3 Lemma *Suppose that x is $\chi(\underline{a}_1, \dots \underline{a}_{m-1}, c_0, \dots c_{n-1})$ and that κ is the supremum of ω_1 Ramsey cardinals greater than λ, the supremum of ω Woodin cardinals. Let $\eta_0 < \eta_1 < \eta_2 < \dots \eta_{n-1} < \eta_n$ be Ramsey cardinals greater than λ and less than κ. Then*

$$x \in \mathbf{R}^\sharp \iff [\![\, J_{\hat{\eta}_n}[\dot{S}] \models \chi[\hat{a}_0, \dots \hat{a}_{m-1}, \hat{\eta}_0, \dots \hat{\eta}_{n-1}]\,]\!]^{\mathrm{Coll}(\omega,\lambda)} = 1.$$

Proof : consider the elementary embeddings $j^{(\kappa)}$ and $j_\omega^{(\kappa)}$ as defined above. We have shown that $j(\eta) = \eta$ for many such η, so first suppose that $\forall i :\leq n\ j(\eta_i) = \eta_i$. Note that as j factors through j_ω, we shall then have that $j_\omega(\eta_i) = \eta_i$ for each $i \leq n$.

Then

$$\begin{aligned} x \in (\mathbf{R}^\sharp)_V &\implies J_{\eta_n}[\Re] \models \chi[a_0, \dots a_{m-1}, \eta_0, \dots, \eta_{n-1}] \\ &\implies J_{\eta_n}[S] \models \chi[a_0, \dots a_{m-1}, \eta_0, \dots, \eta_{n-1}] \\ &\implies [\![\, J_{\hat{\eta}_n}[\dot{S}] \models \chi[\hat{a}_0, \dots \hat{a}_{m-1}, \hat{\eta}_0, \dots, \hat{\eta}_{n-1}]\,]\!]^{\mathrm{Coll}(\omega,\lambda)} = 1 \end{aligned}$$

Here we have applied j_ω^κ to get from the first line to the second, and $\dot{S}$ is of course the Boolean name, definable from the name $\dot{G}$ of the generic collapsing map, for the set of reals of the symmetric collapse.

We have only proved the implication in one generation; replacing x by its formal negation $\neg\chi[\vec{a}, \vec{c}]$ will give us a strong form of the other direction, showing indeed that for $x \notin \mathbf{R}^\sharp$ the Boolean truth value is 0.

Finally we may drop the assumption that $j(\eta) = \eta$ for the η's under discussion, since in the extension of V by $Coll(\omega, \lambda)$, all the η's remain Ramsey, and in particular remain cardinals bigger than $2^{\aleph_0}$, and hence are all indiscernibles for $L[\dot{S}]$; thus η's which are fixed points of j may be replaced by those which are not without altering the Boolean truth values. $\dashv$

Now, turning to the verification of Axiom FC2 for ϕ_1, we wish to check that the truth or otherwise of $\phi_1(x)$ is absolute from $V[g]$ to $V[g][h]$; but this is plain from the Lemma, since the relevant Boolean truth values are 0 or 1 in each model, and they cannot differ, since a symmetric collapse over $V[g][h]$ is *â fortiori* a symmetric collapse over $V[g]$. $\dashv$

Hence our general structure theorem tells us that $\{x \mid \phi_1(x)\}$ is the projection of a $< \lambda$ weakly homogeneous tree. But every set in $L[\Re]$ is continuously reducible to this set, and therefore is also the projection of a weakly homogeneous tree; a final application of the Martin-Steel theorem implies that it is therefore determined; its strategy is in $L[\Re]$, and hence AD is true in $L[\Re]$. $\dashv$

6·4 Remark We really want to prove that under the present hypotheses there is an elementary embedding of $L[\Re]$ into $L[S]$ — the true $L[S]$ and not the bogus $\mathcal{L}[S]$! But that follows once we know that the sharps cohere. That would then yield another proof of AD in $L[\Re]$ since, in the order in which we have done things, we already know that AD holds in $L[S]$.

The following remarks of Solovay may assist the reader in understanding the difference between the two situations:

> "With ω Woodins with nothing on top, there is an elementary embedding from L[R] into an ill-founded version of L[S], what above is called $\mathcal{L}[S]$. Why it doesn't lead to immediate contradictions is that the ill-founded $\mathcal{L}[S]$, which is $(L[S])_{\mathcal{M}_\omega}$ is not elementarily equivalent to the well-founded $L[S]$
>
> "To have something definite to think about, assume that (1) there are ω Woodin cardinals; (2) There is no countable transitive model of ZFC + "There are ω Woodin cardinals"; (3) We are in Woodin's inner model for ω Woodins.
>
> "(So really, we are working in the model obtained as follows. Take the smallest ordinal such that there is a model of "ZFC + There are ω Woodins". Perform the appropriate inner model construction.)
>
> "First remark: the model $L[R]$ definitely has a sharp. The model $L[S]$ definitely does not. By the way, V is the constructible closure of a subset of λ (where λ is the sup of the Woodin cardinals), and this set also doesn't have a sharp, though every bounded subset of λ does.
>
> "Second remark: in $L[R]$, there is a definable well-ordering of the reals. In fact this well-ordering is Σ^2_1, and letting δ be the ordinal δ^2_1, then this well-ordering is Σ_1 over $L_\delta[R]$. However, the model has a lot of determinacy. In fact every set of reals in $L_\delta[R]$ is determined.
>
> "What happens in the fake version, $\mathcal{L}[S]$? It agrees with the true version as far as its ordinals are well-ordered. At some ill-ordered fake ordinal, a subset of S is constructed which is not determined (relative to the reals S.) This bad set does not appear in the true model $L[S]$. The bogus model has a sharp (a bogus sharp since it produces ill-founded models from well-ordered input) but the true $L[S]$ does not.
>
> "Let me emphasize that these remarks pertain to the precise situation I outlined above. In the real world, every set of reals has a sharp, of course, since there are arbitrarily large measurable cardinals."

7: More than ω Woodins

We wish to explore the structure of a world in which there are more than ω many Woodin cardinals. This exploration will eventually lead to the construction of models of $AD\Re$, the version of AD where the players choose reals rather than integers. This is known to be significantly stronger than AD: for example Solovay proved in his paper in the first Cabal volume that $AD\Re$ implies the existence of $\mathbf{R}^\sharp$, which of course cannot exist if $V = L[\Re]$, and thus is *not* a consequence of AD, (always provided that AD is consistent).