Some work for you to do over the holidays

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The first question is bookwork, but two later questions have a rather open-ended character, in some ways more like an essay or a minor research project than like routine coursework. If there is any unclarity in either question please draw it to my attention so that I can clear it up: others may be experiencing difficulty too. Although there is no provision for students to be offered supervision in respect of this work, I am in principle happy to meet students to offer them hints if they need them.

1 Sequents

Find a proof of the sequent $\vdash (\exists x)(\forall y)(F(y) \to F(x))$

2 A puzzle about blood groups

First some preliminaries:

You will need this exercise [hint, hint!!!]

Prove that $\mathcal{P}(\{0, 1, 2\})$ and $\{\perp, \top\}^3$ are isomorphic.

$\mathcal{P}(\{0, 1, 2\})$ is the power set of $\{0, 1, 2\}$ ordered by $\subseteq$ (set inclusion). Clearly we are ordering $\{\perp, \top\}$ by $\perp < \top$.

Equivalence relations

(An equivalence relation $\sim$ is a **congruence relation for** an n-ary function f if, whenever $x_i \sim y_i$ for $i \leq n$, then $f(x_1 \ldots x_n) \sim f(y_1 \ldots y_n)$.

An equivalence relation $\sim$ is a **congruence relation for** an n-ary **relation** R if, whenever $x_i \sim y_i$ for $i \leq n$, then $R(x_1 \ldots x_n) \longleftrightarrow R(y_1 \ldots y_n)$.)

The Exercise

Consider the relation between humans "It is safe for x to receive a transfusion of blood from y." Ignoring the blood-borne diseases like HIV, CJD, Hep C and so on, we find that if x can safely receive a transfusion of blood from y, and y'

belongs to the same blood group as y, then x can safely receive a transfusion of blood from y'. That is to say, the equivalence relation of having-the-same-blood-group is a congruence relation for the binary relation "x can safely receive a transfusion of blood from y".

That way we can think of the relation "x can safely receive a transfusion of blood from y" as really a relation between the blood groups, and summarise it in the following matrix.

Columns are donors, rows are recipients.

	$O-$	$O+$	$B-$	$B+$	$A-$	$A+$	$AB-$	$AB+$
$O-$	$\times$							
$O+$	$\times$	$\times$						
$B-$	$\times$		$\times$					
$B+$	$\times$	$\times$	$\times$	$\times$				
$A-$	$\times$				$\times$			
$A+$	$\times$	$\times$			$\times$	$\times$		
$AB-$	$\times$		$\times$		$\times$		$\times$	
$AB+$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

So the table is telling you that O- people can donate blood to everyone, and so on.

The blood groups themselves (O-, O+, B-, B+, A-, A+, AB- and AB+) are the equivalence classes under this equivalence relation.

Look at the compatibility table for blood groups above. You may be struck by the pleasing pattern made by the $\times$s: I certainly was. In every column the number of '$\times$'s is a power of two! The same goes for rows. What I want you to do is follow the same path I trod when I spotted this pattern, namely:

1. Does this pattern tell you anything significant about the properties of this compatibility relation? I'm thinking of reflexivity, transitivity, symmetry, antisymmetry etc. etc. When you've got that sorted out, procede to part 2.

2. There are other ways of representing binary relations. Try some of them on this data and see if you get any pretty pictures. I tried it and I got a three-dimensional shape. I want you to find that three-dimensional shape. The fact that it is a *three*-dimensional shape is important.

3. One moral I want to draw from this coursework is that Discrete Mathematics and mere thought can be useful in developing profitable hypotheses in the sciences. You probably know enough biology to know that characters are inherited by genes, and that we each have two copies of each gene. Each gene can come in several forms called *alleles*. Some alleles are *dominant* in that you express them even if you have only one copy (brown eyes); some are *recessive* in that you do not express them unless you have two copies (blue eyes, green eyes). The picture you have developed will suggest to you a hypothesis about how many genes there are that control

your blood group, and how many alleles there are at each gene, and which are dominant. Formulate this hypothesis.

Do not forget that this is a logic/discrete-mathematics question, not a biology question. There is nothing to be gained—and time to be lost—in surfing the web for information about blood groups.

This exercise was originally designed as a group coursework, and will certainly benefit from brainstorming. It is inevitable that you will discuss this with your fellow students; do at least indicate how much of your answer is your own work, and whom you discussed it with. That way I will be able to form a picture of the progress you are making.

3 Duality in Propositional Logic

The **dual** $\widehat{A}$ of a propositional formula A is the result of replacing every propositional letter in A by its negation.

(Thus if p and q are letters then $\widehat{p \vee q}$ is $\neg p \vee \neg q$; $\widehat{\neg p \wedge q}$ is $p \wedge \neg q$ and so on. Notice that $\widehat{A}$ is typically *not* logically equivalent to $\neg A$! A formula A is **self-dual** if it is logically equivalent to its own dual: that is to say that $A \longleftrightarrow \widehat{A}$ is a tautology. For example: p `XOR` q is self-dual—'p' and 'q' being literals—even tho' A `XOR` B is not self-dual in general.)

A `XOR` $(B$ `XOR` $C)$ is not self-dual: it is in fact dual to its negation. But $(A$ `XOR` $D)$ `XOR` $(B$ `XOR` $C)$ *is* self-dual.

1. Show that the hat commutes with all connectives:
   $\widehat{A \vee B} \longleftrightarrow (\widehat{A} \vee \widehat{B})$
   $\widehat{A \wedge B} \longleftrightarrow (\widehat{A} \wedge \widehat{B})$
   $\widehat{A \to B} \longleftrightarrow (\widehat{A} \to \widehat{B})$
   $\widehat{\neg A} \longleftrightarrow \neg \widehat{A}$

2. Show that any propositional formula of the form $A \longleftrightarrow \widehat{A}$ is self-dual;

3. Show that if A is a self-dual formula so is $\neg A$;

4. Show that whenever A is a self-dual formula there is a formula B such that A is logically equivalent to $B \longleftrightarrow \widehat{B}$; In how many ways can this be done?

5. Is there a similar result for Predicate Calculus?