John Rickard's Answer to the Impossible Question

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Prove that every theory has an independent axiomatisation. Many years ago PTJ put this in a question sheet. What he *meant* to ask was "prove that every theory *in a countable language* has an independent axiomatisation. There ensued a debate about whether or not the qualification '*countable*' was needed. Imre Leader had an example that he thought ought to be a counterexample, but he couldn't prove that it was a counterexample. In fact, it isn't! Indeed, it was Rickard's analysis of the candidate counterexample that led to a proof of the result. So it's true, so it's *still* the case that no-one has ever caught PTJ making a mistake!

This document is now decades old, and may need a bit of tweaking. Feedback welcome: contributions will be acknowledged in later versions.

1 The countable case

Suppose $\{A_i : i \in \mathbb{N}\}$ is an axiomatisation of T. For $i \in \mathbb{N}$, let B_i be $(\bigwedge_{j<i} A_j) \to A_i$, and $B_0 = A_0$.

Evidently the $\{B_i : i \in \mathbb{N}\}$ axiomatise T. If we discard any B_i that follow from earlier B_i then the remainder form an axiomatisation that is independent. For suppose $i < j$ and B_i is false. This can only happen if A_i is false. But if A_i is false, then the big conjunction that is the antecedent of B_j is false too, since $i < j$, and so B_j is true. Thus $B_i \vee B_j$. Now suppose *per impossibile* that

$\{B_i : i \in I\}, B_j \vdash B_k$ for some B_j, B_k and subset I (not necessarily nonempty!) of the axiomatisation. Then

$$\{B_i : i \in I\} \vdash B_j \to B_k$$

by the deduction theorem. But in any case

$$\{B_i : i \in I\} \vdash B_j \vee B_k$$

so

$$\{B_i : i \in I\} \vdash B_k$$

Now B_j was arbitrary, so we can delete any one of the premisses. Since we have not assumed that I was nonempty we can delete them all, so B_k is derivable only if it is valid.

2 The uncountable case

2.1 Imre's candidate counerexample

We start with Imre's candidate counterexample, and show how it goes wrong.

Imre says: for each $\alpha < \omega_2$ let p_α be a propositional letter and let S be the set of formulæ $p_\alpha \to p_\beta$ for $\alpha > \beta$. This theory is of size $\aleph_2$.

Consider the subset $T \subseteq S$ consisting of those axioms

$$p_{\alpha+1} \to p_\alpha$$

This set is also of size $\aleph_2$. It is also a **strongly independent subset of** S in the sense that no member ($p_{\alpha_0+1} \to p_{\alpha_0}$ say) of T follows from all the other members of S. This is because we can make every p_α false for $\alpha \geq \alpha_0$ except p_{α_0+1} and true otherwise.

Next we notice that both T and $(S \setminus T)$ are of size $\aleph_2$ so there is a bijection between them: call it π. We now claim that the scheme

$$\{t \wedge \pi(t) : t \in T\}$$

is independent and has the same deductive consequences as S does. It certainly has the same deductive closure. Why is it independent? Well, suppose some formula $t \wedge \pi(t)$ were deducible from finitely many others of that form. Then t would be derivable from a hatful of members of S and we have just seen that it isn't.

This trick (of finding a large strongly independent set and using a bijection between it and the whole of S) will be useful in what follows.

2.2 John Rickard's clever generalisation

Imre's report of this result of Rickard's works for propositional languages but there does not seem to be any insuperable difficulty preventing us from making it work for predicate calculi too. Accordingly I am going to write out the predicate version. Let us consider a set $S \subseteq \mathcal{LPC}$ where $\mathcal{LPC}$ is some first-order language with a large number of predicate letters, constants etc. We will assume that the language is wellorderable, so that the set of variables and the language itself are of the same size, and that they are wellordered in some fixed way to length κ, which is an initial ordinal.

Since we are primarily interested in the deductive closure of S we can trim it as follows: For each s in S, pick the first s' containing a minimal number of predicate letters and with the property that $S \vdash s'$ and $\mathcal{LPC} \vdash s' \to s$. Then take the collection of all the s'.

This collection has the same deductive closure as S (obviously), so without loss of generality we can take our S to be a set obtained in this way. Thus we may assume the following:

Proposition 1 *If $S' \vdash s$ where $S' \subseteq S$ then the set of predicate letters appearing in s is a subset of the set of predicate letters appearing in S'.*

Proof: This will be a corollary of the interpolation lemma. Suppose s contains some predicate letter F that does not appear in any formula in S'. Then, by the interpolation lemma, there is some interpolant s' containing only vocabulary common to S' and s s.t. $S' \vdash s'$ and $s' \vdash s$. This means that we would have put s' (instead of s) into S in the first place. (How do i know that we hadn't put in s for some other reason? Because any reason for putting in s is an even better reason for putting in s' instead!) ■

Now construct a sequence $\langle F_i : i < \kappa \rangle$ of predicate letters and a sequence $\langle \phi_i : i < \kappa \rangle$ of members of S as follows.

F_0 is the first predicate letter mentioned and ϕ_0 is the first member of S in which F_0 appears. Thereafter F_α is the first predicate letter not mentioned in any ϕ_β with $\beta < \alpha$ and ϕ_α is the first formula not already used in which F_α appears. Use "$\in$" for "is mentioned in".

We note that since κ is an initial ordinal this sequence is well defined. Let $\langle \psi_\alpha : \alpha < \kappa \rangle$ be a wellordering of the whole of S.

Now amend ϕ_α as follows. Set ϕ'_α to be

$$[\bigwedge_{\{\beta<\alpha : F_\beta \in \phi_\alpha\}} \phi_\beta] \to \phi_\alpha$$

and do the same for the ψ_αs getting ψ'_αs:

$$[\bigwedge_{\{\beta : F_\beta \in \psi_\alpha\}} \phi_\beta] \to \psi_\alpha$$

We note that the ϕ's have the same deductive closure as the ϕs. Once we have proved all the ϕs the ψ_αs all follow from the ψ'_αs. Therefore the ϕ's and the ψ's together have the same deductive closure as S. Now we want to show

Lemma 1 *The ϕ's are a strongly independent subset of the ϕ's $\cup$ the ψ's*

Proof: Suppose that ϕ'_{α_0} follows from some family $\{\phi'_\beta : \beta \in B\}$.

We must first establish a **sublemma** that without loss of generality ϕ_{α_0} never appears in the antecedent of any of these ϕ'_β. If it does we can argue as follows: we would have

$$\Gamma, \phi_{\alpha_0} \to Q \vdash P \to \phi_{\alpha_0}$$

and

$$\Gamma, \vdash (\phi_{\alpha_0} \to Q) \to (P \to \phi_{\alpha_0})$$

and by a long classical proof involving Peirce's law we could infer that

$$\Gamma \vdash P \to \phi_{\alpha_0}$$

and ϕ_{α_0} does not appear.[1] This proves the sublemma.

Notice that Γ is a set of conditionals. ϕ_{α_0} never appears in the antecedent of anything in Γ, so F_{α_0} never appears in the consequent of anything in Γ. (Remember that ϕ'_{α_0} is not in Γ.) So we have

$$\Gamma \vdash \phi'_{\alpha_0}$$

and ϕ'_{α_0} must be deducible from the consequents of things in Γ (after all, anything that follows from $p \to q$ must follow from q!). Therefore we have deduced ϕ'_{α_0} from things not mentioning F_{α_0}. But ϕ'_{α_0} is junk $\to \phi_{\alpha_0}$ where junk doesn't contain references to F_{α_0}. Since F_{α_0} appears in ϕ_{α_0} this contradicts proposition 1, and we conclude that the ϕ's are a strongly independent subset of the ϕ's $\cup$ the ψ's as desired. ■

Now we can prove the

THEOREM 1 *The set of formulæ of the form $\phi'_\alpha \wedge \psi'_\alpha$ is an independent axiomatisation of S*

Proof: The proof is as in the demonstration that Imre's counterexample is not a counterexample.

[1] A word is in order on this proof. $(p \to q) \to (r \to p)$ is intuitionistically the same as $r \to ((p \to q) \to p)$. The consequent is the antecedent of $(((p \to q) \to p) \to p)$ which is an instance of Peirce's law, whence $r \to p$. Notice that $((p \to q) \to (r \to p)) \to (r \to p)$ (which is after all what we are trying to prove) implies Peirce's law (just substitute $\perp$ for r) so we know this proof is not intuitionistically correct!