# Associativity of Relational Composition

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The exercise is to show that relational composition is associative.

We write '$xTy$ for "$x$ is related to $y$ by $T$". We define relational composition in the obvious way:

$x(T \circ S)y$ iff $(\exists z)(xTz \wedge zSy)$

We will show that $R \circ (S \circ T) = (R \circ S) \circ T$

That is to say, for all $x$ and $y$, $xR \circ (S \circ T)y$ iff $x(R \circ S) \circ Ty$

(I am using capital Roman letters both as relation symbols and as variables in an algebra.)

Now, by definition of relational composition,

$$xR \circ (S \circ T)y$$

is

$$(\exists z)(xRz \wedge z(S \circ T)y)$$

and expand the second '$\circ$' to get

$$(\exists z)(xRz \wedge (\exists w)(zSw \wedge wTy))$$

We can pull the quantifiers to the front because[1] '$(\exists u)(A \wedge \phi(u))$' is the same as '$A \wedge (\exists u)\phi(u)$' getting

$$(\exists z)((\exists w)(xRz \wedge (zSw \wedge wTy)))$$

and

$$(\exists z)(\exists w)(xRz \wedge (zSw \wedge wTy))$$

and we can certainly permute the quantifiers getting

$$(\exists w)(\exists z)(xRz \wedge (zSw \wedge wTy))$$

we can permute the brackets in the matrix of the formula because '$\wedge$' is associative getting

$$(\exists w)(\exists z)((xRz \wedge zSw) \wedge wTy)$$

[1] At least as long as '$u$' is not free in $A$.

import the existential quantifier again getting

$$(\exists w)((\exists z)(xRz \wedge zSw) \wedge wTy)$$

and reverse the first few steps by using the definition of $\circ$ to get

$$(\exists w)(x(R \circ S)w \wedge wTy)$$

and

$$x(R \circ S) \circ Ty$$

as desired.

The more I think about this the odder it seems. Is this a proof? Is there really no more to the associativity of $\circ$ than the fact that $\wedge$ is associative? The proof above is certainly correct, but does it enlighten? What did the person who set this question expect by way of an answer? Did they think about it at all? I bet they didn't ... I think what happened is that the author thought it was a trivial fact which should accordingly be given to beginners to chew over—perhaps as part of a rite of passage. The pictures one draws of circles with dots inside them that are joined to dots in adjacent circles by lines corresponding to ordered pairs in $S$ and $T$ looks like the kind of notation that conceals a logical truth. Or is the associativity of relational composition something even more banal than a logical truth? Is it actually a *good thing* to have a notation that conceals it??

This example serves to remind me of how important it is for researchers to do some teaching. There are some things about your subject that you won't really understand properly until you have to think about how to explain them. The above is a case in point.