A note on KF + ¬AxInf

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Remark 1 *$KF \vdash$ there is no $\subseteq$-maximal finite set.*

Proof:

Say a set x is finite iff every set that contains x and is closed under subcision ($x, y \mapsto x \setminus \{y\}$) contains $\emptyset$. We record for future use that the (class of) finite sets thus defined is closed under adjunction ($x, y \mapsto x \cup \{y\}$), as follows. Suppose x is finite, and $x \cup \{y\} \in A$, with A closed under subcision. We desire $\emptyset \in A$. If A is closed under subcision and contains $x \cup \{y\}$, then $x \in A$, whence $\emptyset inA$ (since x is finite).

Suppose x is a $\subseteq$-maximal finite set; we will obtain a contradiction. Let y be arbitrary. The (class of) finite sets is closed under adjunction, so $x \cup \{y\}$ is also finite, and therefore – by $\subseteq$-maximality of x – must be equal to x. So $y \in x$. But y was arbitrary, so $x = V$. But KF + $\exists$ universal set $=$ NF, so we are in NF. But NF $\vdash V$ is not finite. So our hypothesised $\subseteq$-maximal finite set wasn't a $\subseteq$-maximal finite set after all.

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Am i bovvered? Probably not; I think this signifies no more than that the arithmetic of KF is somewhat richer than i had hitherto naïvely supposed. Mac, too, of course proves that there is no $\subseteq$-maximal finite set. It is *perhaps* worthy of note that this fact can be proved even in KF.

I think it also means that KF + ¬AxInf proves that in some sense $\mathbb{N}$ is not a set, as follows.

Work in KF+ ¬AxInf. Suppose there is a set N s.t. every finite set is equinumerous with a unique member of N. We ask: "Is N finite?" If it's finite then it has a largest member. Long story short, this largest member must be a $\subseteq$-maximal finite set, and – by the above – there ain't no such animal. So N is infinite. But this contradicts our assumption of ¬AxInf. So there is no such N.

One would like to be able to say that this implies that $\mathbb{N}$ isn't a set, but that is something that is very hard to express in the language of pure set theory. One would have to say that there is no classifier for equinumerosity whose codomain is a set. Annoyingly this involves a quantifier over function classes. However one can do the following.

Remark 2 *If f is a classifier for equinumerosity of finite sets, then the range (codomain) of f is not a set.*

Since there is nothing more to being $\mathbb{N}$ than being-the-codomain-of-a-classifier-for-equinumerosity-of-finite-sets ("There is no saying what the numbers are; there is only Arithmetic" as Quine says somewhere) this is as close as we can get to saying that $\mathbb{N}$ is not a set.

Proof:

Reason in the second-order theory, and suppose f is a classifier for equinumerosity whose codomain is a set:

$$(\exists N)((\forall x)(f(x) \in N) \wedge (\forall n \in N)(\exists x)(n = f(x))).$$

We will obtain a contradiction.

We want to pick, for each $n \in N$, an x s.t. $f(x) = n$. The obvious move is to use AC on the set $\{f^{-1}``\{n\} : n \in N\}$, which is a finite family of pairwise disjoint sets. There is no problem using AC on this disjoint family since it is finite, but there *is* a problem with replacement since we need replacement to show that it exists. (KF does not have replacement). However the problem can be circumvented. We are assuming the negation of the Axiom of Infinity, so N must be finite. That means it must have a top element (it's a finite total order, after all, and finite total orders have maximum elements). So let X be a set s.t. $f(X)$ is this maximum element. Think about $\mathcal{P}(X)$ and its partition into equinumerosity classes. This partition is of course finite, and we have AC for finite sets. We use AC to pick one representative from each piece; the set of representatives is now an example of a set N s.t. every finite set is equinumerous with a unique member of N – wot we showed above was not to be had.

I think we can express this by saying that

KF + ¬AxInf proves that the codomain of any classifier for equinumerosity is not a set.

or perhaps even:

If we expand $\mathcal{L}(\in, =)$ by adding a function symbol 'f', and extend KF + ¬ AxInf to a theory T by adding axioms to say that f is a classifier for equinumerosity, and that allow 'f' to appear in set existence axioms, then T proves that the range of f is not a set.