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Challenge: deduce the axiom of empty set from the axiom of infinity.

The morally correct way to do this is to observe that the axiom of infinity has the form "there is a set with a special property". If there is even one set, then—as long as we have separation—there will be an empty set, since the subsets consisting of all those elements of the set that are not equal to themselves will be a set by separation.

Now the axiom of infinity can also come in the form "There is a successor set", or

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))$$

In the presence of the axiom scheme of replacement this can be deduced from the bare assertion there is an infinite set (a set not the same size as any proper subset of itself) but the axiom is often taken in this more specialised form because it makes it easy to give immediately an implementation of arithmetic. Fair enough. However, this muddies the waters slightly, in that it enables us to give a different proof of the existence of the empty set. People sometimes say that the axiom of infinity *presupposes* the existence of the empty set, but that's not quite right. Let's get this 100% straight. The axiom in the form "there is a successor set" says

$$(\exists x)((\exists e \in x)(\forall w)(w \notin e) \wedge (\forall y)(y \in x \to y \cup \{y\} \in x)) \tag{1}$$

I have written out the '$\emptyset \in x$' bit in primitive notation so we know there are no tricks being played.

The expression 1 is of the form

$$(\exists x)(p \wedge F(x))$$

where p is $(\exists e \in x)(\forall w)(w \notin e)$ and $F(x)$ is $(\forall y)(y \in x \to y \cup \{y\} \in x)$. Anything of the form $(\exists x)(p \wedge F(x))$ is going to imply $(\exists x)p$, namely

$$(\exists x)((\exists e \in x)(\forall w)(w \notin e)$$

whence

$$(\exists e)(\forall w)(w \notin e)$$

which says that there is an empty set, which is what we wanted.