A worked example using the curly brackets of set abstraction

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Consider the following term: what set of numbers does it denote?

$$\{2z \in \mathbb{N} \mid z \leq 5 \wedge 20/z \in \{w \in \mathbb{N} \mid w \leq z\}\} \tag{1}$$

The person who put this question together was using it to try to make a number of points. In this commentary I'll be trying to make those points for him/her.

The first point is that the curly bracket symbol is used in various distinct but related ways. We write things like '$\{a, b\}$' to denote the set that has 1 and 2 in it but nothing else. Let us call this the **primary** use of curly bracklets. But we also use them in expressions like

$$\{x \mid \Phi\} \tag{2}$$

In this usage the 'x' is a *variable*, and the expression denotes the set containing precisely those things that satisfy Φ, that being the thing after the vertical bar. Just a brief check, before we go on: make sure you are happy that $\{1, 2\}$ is the same set as $\{x \mid x = 1 \vee x = 2\}$.

(Notice *en passant* that some people—such as your humble correspondent—will use a colon instead of a vertical line after the variable in expressions like 2 above. Sadly there are lots of overlapping notations. I prefer the colon to the vertical bar beco's we also use the vertical bar to mean "cardinal of". Thus $|\{1, 2\}| = 2$. I think my way causes less confusion. You should be prepared to see both notations used.)

Notice that in formula 1 the thing between the left '$\{$' and the '$|$' isn't just a naked variable as it was in formula 2. This exploits two conventions which we must explain. We often write

$$\{x \mid x \in A \wedge \Psi\} \tag{3}$$

as

$$\{x \in A \mid \Psi\} \tag{4}$$

This looks perverse, but there is reason to it. We use it in situations where one feels that the set A provides a *context*. For example the expression $\{x \mid x^2 < 26\}$ denotes the set of all numbers whose square is less than 26. We might be interested in the set of all *real* numbers whose square is less than 26, in which case one would write

$$\{x \mid x^2 < 26 \wedge x \in \Re\} \tag{5}$$

or one might be interested in the set of all natural numbers whose square is less than 26, in which case one would write

$$\{x \mid x^2 < 26 \wedge x \in \mathbb{N}\} \tag{6}$$

but—beco's in circumstances like this one usually is thinking of $\Re$ or $\mathbb{N}$ as a *universe of discourse*, the place where it is happening—one would write these as

$$\{x \in \Re \mid x^2 < 26\} \tag{7}$$

and

$$\{x \in \mathbb{N} \mid x^2 < 26\} \tag{8}$$

Before you go any further check your understanding by writing out—in primary notation—what the set in formula 8 is. (I won't ask you to write out what the set from formula 7 is because it's infinite!)

But what we have before us in formula 1 is an example of s further liberalisation of this last step. This liberalisation will allow us to write things like

$$\{x^3 \in \mathbb{N} \mid x^2 < 26\} \tag{9}$$

You cannot deduce from what has gone before what this expression is used to mean, but you might be able to guess. What it denotes is the set that you get from the set denoted by formula 8, and replacing every number in that set by its cube.

We are now in a position to attack formula 1

The last point the questionmaster was trying to get across is that $w \in \{x : \phi(x)\}$ is the same as $\phi(w)$. Being a member of the set of all green things is just the same as being a green thing. Or rather, since the example is $20/z \in \{w \in \mathbb{N} \mid w \leq z\}$ the point is that being a member of the set of green slimy things is the same as being green and slimy. This formula is just the same as

$$20/z \in \mathbb{N} \wedge 20/z \leq z \tag{10}$$

This will simplify formula 1 to

$$\{2z \in \mathbb{N} \mid z \leq 5 \wedge 20/z \in \mathbb{N} \wedge 20/z \leq z\} \tag{11}$$

So we ascertain what

$$\{z \in \mathbb{N} \mid z \leq 5 \wedge 20/z \in \mathbb{N} \wedge 20/z \leq z\} \tag{12}$$

is and then multiply everything in it by 2. A simple case analysis shows that the only natural number z below 5 such that $20/z \leq z$ is 5. So the set of 12 is $\{5\}$. (Not 5 itself! That's another point the questionmaster was trying to make!)

Finally multiply everything in $\{5\}$ by 2 to obtain $\{10\}$.