Church's Generalised Cardinals

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SYNOPSIS

Concretising mathematical objects as equivalence classes. Church's motivation for CUS: k-equivalence relations intended to capture isomorphism of structures. If every bijection can be extended to a permutation of V we get a nice metatheorem about stratified formulæ. Equivalence classes as orbits. (Is there a Cantor-Bernstein theorem?) However *pace* Church, this doesn't quite capture isomorphism of mathematical structures. We need the relation of isomorphism between cumulative hierarchies over sets of atoms.

This note is a result of Tim Button asking me in February 2024 about the relation of "k-equinumerosity" in Church [2]. It's something that over the years I had taken some care to not think about, and I am grateful to Tim for prodding me. *Very* grateful, actually, since as a result of his prodding I located some embarrassing errors in my mental picture, and I now think I know what is going on − which is a feeling I never really had before − and in consequence feel emboldened to write out this treatment. Thank you, Tim.

Much if not all that follows below is presumably known to the *cognoscenti* and I make no claims of originality. I suspect however that that material has not been assembled in one place before, and I offer this thought as an excuse for the present document.

1 Background and Definitions

There is an idea in Foundations of Mathematics – going back as least as far as Frege, and enthusiastically embraced by the Neo-Logicists – that (some) mathematical objects can be conceptualised as equivalence classes wrt suitable equivalence relations: the stock example is the natural number (not the *real* number!) 1, which we would like to think of as the set of all singletons. There are obstacles to this as we all know, but the idea has its merits and is worth salvaging.

In [2] Church expounds a set theory with a universal set, nowadays known as 'CUS'. CUS is a spiced-up version of ZF in which one can actually pull off this stunt of thinking of things like the number 1 as equivalence classes. To this end he defines a suite of equivalence relations whose kth member he calls *k-equivalence*; he calls their equivalence classes *Generalised Cardinals*. CUS has axioms to say that if x is wellfounded then its k-equivalence classes are sets. You can have the equivalence classes as *proper classes* (in GB) but that's nowhere near as much fun. (For example you can't talk about sets of cardinals in GB). It is these equivalence relations of Church's that I intend to clarify in what follows below.

One piece of notation which will be very helpful is the following: 'j' connotes 'jump'.

$$j(f) = \lambda x. f``x.$$

So $j(f)$ maps sets-of-arguments-for-f to sets-of-values-of-f; if $f : X \to Y$, then $j(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$.

This notation has become standard in some circles (chiefly among NFistes) and we can use it to expeditiously and readably define two families of equivalence relations.

We start with a definition from Church [2] of one of these families.

Definition 1
*A 1-**correspondence** between X and Y is a bijection between X and Y.*
*f is an $(n+1)$-**correspondence** between X and Y if f is $j(g)$ for some n-correspondence g between $\bigcup X$ and $\bigcup Y$.*
*Two sets are n-**equinumerous** iff there is an n-correspondence between them.*

We will continue to use the word *equipollent* to relate two things that have the same cardinal, are in 1-1 correspondence, or are 1-equinumerous in terms of definition 1. We will not use the notation 'n-equipollence'. Sufficient unto the day is the evil thereof.

One (possibly helpful?) way of characterising Church's k-equinumerosity is to say that A and B are k-equinumerous iff there is a bijection f between $\bigcup^k A$ and $\bigcup^k B$ which "lifts" to a bijection between A and B, so that $j^k f$ is a bijection (in fact a k-correspondence) between A and B.

The reader should perhaps be warned against making the mistake (that I made!) of misreading Church's definition as saying that 0-equinumerosity is the universal relation and that A and B are $(j+1)$-equinumerous iff there is a bijection $f : A \longleftrightarrow B$ such that $(\forall x \in A)(x$ and $f(x)$ are j-equinumerous). This is a truly horrible notion, so it is no surprise that is not what Church had in mind. I owe Tim a grovelling apology for having wasted his time by saying that that was what Church meant.

Had I infinite time at my disposal I would perhaps explain in detail how these relations do not do what Church wants from his equivalence relations. However I do not have infinite time, and in any case the explanation would be useful only to people who had misread Church the way I had. With any luck I am the only such person.

We have become used to thinking of mathematical structures as tuples of carrier-sets plus assorted gadgets. Thus the reals-as-an-ordered-field is a set of (things called) *reals* decorated with various sets of tuples encoding the order, the multiplication and addition *etc. etc.* We have to decide how to represent the reals as sets, and there are various standard ways of doing that, as we all know. However the point is not just that every real is a pure set, and that the set of reals is a pure set, the point is that *the whole caboodle* – the bundle containing the carrier set, the relations, the constants, the tuples-within-a-tuple etc etc – *is a pure set*. So the property of two (groups, rings . . .) of being isomorphic must be captured by a set-theoretic relation, namely the relation of being connected by a particularly nice bijection. Church's idea is that those nice bijections will turn out to be the things he calls k-**correspondences** *between the caboodles*, as above.

While we are about it, let us elevate the word 'caboodle' from a piece of slang to a term of art:

DEFINITION 2 *The* **caboodle** *of a mathematical object is the set manifesting that object in set theory*[1]*.*

Incidentally, '*caboodle*' is not a typo for '*co-boodle*'; boodles haven't been invented yet . . . tho' perhaps the boodle of a set will be the mathematical object whose caboodle it is: the boodle of the $\subseteq$-minimal infinite hereditarily transitive set will be the ordinal ω.

2 What are Generalised Cardinals for?

The idea behind generalised cardinals is that isomorphism between two mathematical structures is to be explained as Church n-equinumerosity of two caboodles. This feels like an idea that ought to work, and in fact something very like it actually *does* work. Indeed this thought is the driving force behind the invention of CUS, which is a context within which these equivalence classes can be concretised as sets. It also motivates the definition of k-equinumerosity: two structures are k-equinumerous in virtue of (= their k-equinumerosity is witnessed by) a bijection satisfying some special properties. After all, an isomorphism isn't just

[1] It's odd that we do not already have a word for this!

a bijection; it's a bijection that does something extra, and the idea is that the extra is part constributed by the bijection being a k-equivalence.

It might be an idea to illustrate with a few examples the way in which isomorphism between mathematical structures is represented in set-theoretic foundationalist treatments by k-equinumerosity.

- A topological space $\mathcal{A}$ is a set A with a collection $\mathcal{O}_A$ of its subsets. So an isomorphism between two topological spaces $\mathcal{A}$ and $\mathcal{B}$ is a 2-isomorphism, namely a bijection $f : A \longleftrightarrow B$ with the property that $j(f)$ is a bijection between $\mathcal{O}_A$ and $\mathcal{O}_B$, which means that $j^2(f)$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$: isomorphic topological spaces are 3-equinumerous.

- What is an isomorphism between two orderings? If we think of an ordering as a subset of $X \times X$ where X is the carrier set of the ordering, and we use Wiener-Kuratowski ordered pairs, then an isomorphism between a ordering whose carrier set is A and one whose carrier set is B is going to be a bijection between the appropriate subset of $A \times A$ and the appropriate subset of $B \times B$ that arises from a bijection f between A and B. Because Wiener-Kuratowski pairs raise types by 2, $j^2(f)$ will send ordered pairs in $A \times A$ to ordered pairs in $B \times B$, so $j^3(f)$ will send the ordering of A to an ordering of B, meaning that the bijection f is witnessing the fact that the two orderings are 3-equinumerous.

  We *could* have chosen to think of orderings as sets of initial segments ("ordernestings") – and down that path an isomorphism is still a bijection between A and B – but the two orderings are now 2-equinumerous as sets rather than 3-equinumerous as sets.

  Reflect that – either way – we can now think of ordinals as k-equinumerosity classes of wellorderings (for suitable small k), as desired.

Thus we can think of isomorphism between two topological spaces (for example) as two k-correspondences, the first (a 1-correpondence) between the two carrier sets, and the second (a 2-correpondence) between the two collections of open sets. If we think of the space as a tuple of a set-and-a-family-of-open-sets then the isomorphism is a family of coordinatewise k-correspondences; we aren't at this stage thinking of it as a single k-correspondence between two ordered pairs (of a set and a family of open sets); that is for later.

Thus the idea is that isomorphism between two objects of an arbitrary flavour (topological space, wellordering . . .) of mathematical objects should turn out to be k-equinumerosity between two sets, sets obtained somehow from the mathematical objects. Naturally we are prepared for the precise value of k to be different for different flavours – as it might be $k = 2$ for topological spaces or $k = 3$ for wellorderings – but *not* to depend on the tokens between which the isomorphism holds. That's true, but there is a wrinkle. The value of k depends not only on the flavour but also on implementation decisions we have made about mathematical objects of that flavour . . . whether we use ordernestings or ordered pairs, and – if we use ordered pairs – what kind of ordered pairs we use.

So *that's* why Church does k-equinumerosity. However for many readers (certainly for an *NFiste* such as your humble correspondent) the most eye-catching feature of CUS is the axiom of complementation. Why does he do complementation? It doesn't help a whit with the concretisation project. I think he does it partly because he can – it's a striking *coup*; he does it also partly because he had an interest in set theories with a universal set (tho' this fact is not generally known) and of course complementation gives one a universal set. Finally it's a proof-of-concept – a *dry run* – for the *real* business of concretisation. The coding tricks used to furnish the k-equinumerosity classes can be displayed in a simple version that furnishes complements. A discussion of complementation is where the *aperçu* about all paradoxical sets being *intermediate* comes in. It's an astute observation, but it's not his focus. I shall say no more about it.

3 There is a second kind of equivalence relation

The definition of k-equinumerosity used by Church starts with a bijection between $\bigcup^k X$ and $\bigcup^k Y$. We then ask that j^k of this bijection map X onto Y.

We can modify this definition by asking additionally that this bijection between $\bigcup^k X$ and $\bigcup^k Y$ be a restriction of a permutation of V. Church wrote of the equivalence classes of his k-equivalence relations as *k-cardinals* and, in a kind of back-formation from this, the word *k-equinumerosity* is sometimes used to describe those equivalence relations; the modified relations to which we now turn tend to be known (in the NF literature at least) as **k-equivalence**. Let us cement this divergence of notation in place to record the difference between these two interesting families of relation. We are interested in both and we need to distinguish between them.

This modified definition of k-equivalence is important because of the following lemma – central to NF studies – due to Scott [6]. I have lifted this statement from [3], my NF monograph.

Lemma 1 *If Φ is stratifiable then*

$$\Phi(x_1, \ldots, x_k) \longleftrightarrow \Phi((j^{n_1}\sigma)(x_1), \ldots, (j^{n_k}\sigma)(x_k))$$

for any setlike permutation σ, where n_k is the integer assigned to the variable 'x_k' in some fixed stratification.

We explain 'setlike' on p. **??** below *Proof:*

By definition of j we have $x \in y$ iff $\tau(x) \in (j\tau)(y)$ for any permutation τ. In particular if 'x' has been assigned type n and 'y' the type $n+1$, we invoke the case where τ is $j^n(\sigma)$ to get $x \in y \longleftrightarrow j^n\sigma(x) \in j^{n+1}\sigma(y)$. (Remember that σ is a permutation of V iff $j\sigma$ is – at least if we have sufficient Replacement). By substitutivity of the biconditional we can do this simultaneously for all atomic subformulæ in $\Phi(x_1, \ldots, x_k)$. Variables 'y' in '$\Phi(x_1 \ldots x_k)$' now have prefixes like '$j^n\sigma$' in front of them but, since '$\Phi(x_1, \ldots, x_k)$' was stratifiable, they will be the same prefix for every occurrence of each such variable 'y'. We then use the fact that $j^n\sigma$ is a permutation of V so that any formula $Qy(\ldots j^n\sigma(y)\ldots)$ (Q a quantifier) is equivalent to $Qy(\ldots y\ldots)$. ■

We say an equivalence relation $\sim$ is a **congruence relation for** an n-ary function f if, whenever $x_i \sim y_i$ for $i \leq n$, then $f(x_1 \ldots x_n) \sim f(y_1 \ldots y_n)$.

We record here the important fact that Scott-style k-equivalence is a a congruence relation for (unary) functions captured by k-stratifiable expressions. Something similar holds for functions of higher degree but one has to specify that the same permutation witnesses $\sim_k$ at each coordinate.

The point for us is that there is no analogue of Lemma 1 – or the congruence property – for Church's k-equinumerosity – or at any rate none that anyone has ever thought worth stating. To illustrate: if there is a universal set then there is the possibility of a bijection between V and $V \setminus \{\emptyset\}$, but clearly no such bijection can be extended to a permutation of V. And of course this gives us an instance of how Lemma 1 does not apply to Church's k-equinumerosity. If $f : V \longleftrightarrow (V \setminus \{\emptyset\})$ then V and $V \setminus \{\emptyset\}$ are 1-equinumerous in virtue of f but if we take $\phi(\ )$ to be "has empty complement" we have $\phi(V)$ but not $\phi(V \setminus \{\emptyset\})$.

It might be worth thinking how lemma 1 comes about. A key move in the proof is *relettering*. When σ is a permutation, any expression $(\exists x)(\phi(\sigma(x)))$ is equivalent to $(\exists x)(\phi(x))$, because σ is surjective. That means we can replace '$(\exists x)(\phi(\sigma(x)))$' by '$(\exists x)(\phi(x))$', and thereby progressively get rid of occurrences of 'σ', as lemma 1 promises. If σ is not a surjection then we cannot reliably infer $(\exists x)(\phi(\sigma(x)))$ from $(\exists x)(\phi(x))$, because the bare existence of witnesses doesn't guarantee the existence of witnesses in the range of σ.

A word is perhaps in order on the significance of the qualification 'setlike'. A function f is *setlike* iff, for all x and all n, $j^n f(x)$ is defined, and it is this that makes relettering possible. If replacement holds then every function is setlike. Setlikeness becomes an issue when replacement is not to be relied on. Replacement holds of course in ZF; low replacement holds in CUS.

Lemma 1 does *not* say "when ϕ is stratifiable and, for each i, x_i is k-equivalent to y_i for some suitable k depending on x_i, then $\phi(x_1, \ldots x_n)$ iff $\phi(y_1, \ldots y_n)$"; we do need the extra condition that all the k-equivalences have to be witnessed by a single permutation.

The lemma does say *something* about formulæ with a single free variable: if ϕ is stratifiable and has a single free variable then $\phi(x) \longleftrightarrow \phi(j^n\sigma(x))$ holds for any permutation σ and a suitable n depending on ϕ. We can in fact infer the general version from the single-free-variable case by considering the special case where x is a tuple – and this fact will loom large later on – but for the moment we will not pursue it.

It's probably worth saying a bit about when the two kinds of equivalence relation are the same, since this topic has not been covered in the literature – despite it being presumably well-understood by the *cognoscenti*.

The Scott-style equivalence relations allude to permutations of the whole universe; the Church equivalence relations allude to bijections between sets. When is a bijection between two sets a restriction of a permutation of the universe? It is clear that in ZF (for example) any bijection between two sets can be so extended. Whenever we have a bijection $f : A \longleftrightarrow B$ we make countably many copies of $A \setminus B$ and $B \setminus A$ and extend f to a permutation ϕ of the union of all these copies. We obtain a permutation of the whole universe by ruling that the permutation fixes everything outside that union. This is something we can arrange by a judicious choice of pairing function . . . as long as we are able to make pairwise disjoint copies of arbitrary objects *ad libitum*. This is a principle of some foundational interest that I have discussed elsewhere. (See [4] where a discussion is provided of the situation we consider here).

Very well, we can extend a bijection $f : A \longleftrightarrow B$ to a permutation ϕ of a superset of $A \cup B$. Any permutation of a set x can be extended to a permutation of the whole of V by fixing everything in $V \setminus x$. Any permutation of V with set support (moves only a set of things) can be coded by a set — simply by saying nothing about the things that it fixes. ϕ is clearly such a permutation. However lemma 1 works in ZF even for permutations that do not have set support (move only a set of things) because ZF has replacement. It works in Zermelo too, but only for permutations that are sets, which is to say, for permutations that move only a set of things.

That was in ZF. What did we really need? We needed the principle that every bijection can be extended to a permutation of the universe. To prove Lemma 1 we need a certain amount of replacement: either postulated as such or by restricting ourselves to setlike maps in the first place.

How does this play out in CUS? If we want to assert Lemma 1 for low sets and permutations that move only a set of things then we are OK. CUS proves that any bijection between two low sets can be extended to a permutation of V, as long as we think of that permutation as a low set of pairs (by not mentioning the things that it fixes). In fact it sort-of works for arbitrary sets, but only for the silly reason that in basic CUS there are no bijections between non-low sets (not even identity maps – the sets according to CUS do not constitute a category!)

The Scott version in terms of permutations rather than bijections enables us to think of an equivalence class of a set x as an orbit of x under an action of the full symmetric group on V. This can be a helpful perspective, allowing us – as it does – to deploy a raft of group-theoretic ideas.

4 Actually there is yet another kind of equivalence relation

The idea with which we started is that, in a context of set-theoretic foundationalism, isomorphism of mathematical structures is simply the existence of a suitable bijection witnessing the fact that two sets (the caboodles of the two mathematical structures in play) are k-equivalent for some suitable k. For example a ring is a tuple $\langle X, +, \cdot, 0, 1\rangle$, of: a carrier set, two (graphs of) two-place functions and a couple of constants. Thanks to Wiener and Kuratowski such a tuple is in fact a set. We want to say that for another ring $\langle Y, +, \cdot, 0, 1\rangle$ to be isomorphic to $\langle X, +, \cdot, 0, 1\rangle$ it is necessary and sufficient for there to be a bijection f between X and Y in virtue of which the **set** $\langle X, +, \cdot, 0, 1\rangle$ is k-equinumerous to the **set** $\langle Y, +, \cdot, 0, 1\rangle$. Now, for this to be the case, it is necessary that the set – the *caboodle* – $\langle X, +, \cdot, 0, 1\rangle$ be a subset of $\mathcal{P}^k(X)$ for some suitable small k. The trouble is that this is not *literally* true – and for annoying silly reasons. The problem is that, in the tuple $\langle X, +, \cdot, 0, 1\rangle$, there are things – specifically the 0 and 1 – which appear at two different levels, once as components of the tuple on their own account, and once as members of X. This means that the tuple $\langle X, +, \cdot, 0, 1\rangle$ is not a subset of $\mathcal{P}^k(X)$ for any suitable small k but is rather a subset of the "cumulative hierarchy over the carrier set" (more precisely a finite initial segment thereof) ... by which we mean

$$V_0(X) = X; \quad \text{thereafter, for } \alpha > 0, \quad V_\alpha(X) = X \cup \mathcal{P}(\bigcup_{\beta<\alpha} V_\beta(X))$$

where X is the carrier set, thought of as a set of empty atoms. (That is to say, the internal structure of the members of X is not visible to the power set constructor.) $V(X)$ is then the union of all the V_α.

It's not clear to me whether Church did not spot this difference, or instead merely didn't think it was worth remarking on. Given Church's famous eye for detail it's more likely to be the second. That would suggest that Church did not know Lemma 1.

There are two ways to react to this news.

- The first is to say that one has to modify the definition of n-tuples ($n > 2$) to ensure that each object appears at only one level. That way we can continue to explain isomorphism in terms of k-equivalence or k-equinumerosity.

- The other is to give up on k-equivalence/equinumerosity and instead say that a caboodle is a member of $V(\bigcup^k A)$.

  Any bijection between sets A and B lifts by recursion on $\in$ to an $\in$-isomorphism between $V(A)$ and $V(B)$. Then we say that two mathematical structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic iff the corresponding $\mathcal{L}(\in, =)$-structures A and B are identified by the isomorphism between $V(\bigcup^n A)$ and $V(\bigcup^n B)$.

Which way one jumps will depend to a certain extent on how seriously one takes stratification. The $\in$-recursion that defines bijections between the $V(A)$ is not stratified. Holmes prefers the first bullet; I think most mathematicians – certainly most set-theorists – will prefer the second. You pays yer money and you takes yer choice.

If the moral is to be that Church should have used this notion of isomorphism rather than the k-equinumerosity that he did in fact use, then the obvious question is: were we to modify the axioms L_k accordingly, would we still be able to prove the consistency of the revamped CUS? And the answer is a clear 'yes'.

A word may be in order here on the situation in NF. NF supports Scott-style equivalence relations for all sets: in NF abstract mathematical objects (cardinals, ordinals ...) are implemented as equivalence classes for *arbitrary* sets, as Church intended; not merely equivalence classes (cardinals, ordinal ...) for *wellfounded* sets. And it uses k-equivalence relations. However NF does not support isomorphism between caboodles in the third style, using isomorphisms between the various "local" cumulative hierarchies $V(A)$, since these isomorphisms do not have stratified descriptions.

In CUS Church has axioms – he calls them L_k – to say that the k-equivalence classes of wellfounded sets are sets.

Why does he use Church-style not Scott-style equivalence relations, despite the Scott-style equivalence relations being better behaved? Part of the answer is that basic CUS doesn't actually prove the existence of any permutations of V at all, at least not if we want to think of them as sets of pairs (which is a reasonable things to want – we are doing Set Theory after all). Permutations *of low support* (permutations that move only a low set of things) are provided by low replacement. And, since the only k-equivalence or k-equinumerosity classes we are interested in are k-equivalence/equinumerosity classes of wellfounded (or low) sets, the only permutations we need to consider – even if we want Scott-style equivalence – are permutations of low support. And Scott-style equivalence relations using only permutations of low support are the same as Church-style equivalence relations. So *as long as you are interested in k-equivalence classes for low sets only* it makes sense to use Church-style equivalence (equinumerosity) relations. As it happens, had Church wished to use Scott-style k-equivalence he would have been able to do so, since he was interested in executing this programme for low sets only.

To summarise: if our reason for developing k-equinumerosity relations in CUS is in order to discuss isomorphism classes of structures over wellfounded (or even low) sets then to give even a Scott-style treatment it is sufficient to restrict attention to permutations of low support. [This is just as well, co's there are no (set) permutations of non-low support according to basic CUS]. And a Scott-style treatment using only permutations of low support is the same as a Church-style treatment. Had we wanted k-equinumerosity classes for co-low sets then we would need to be able to reason about set bijections between non-low sets, and this we cannot do in basic CUS. And if we could so reason (as we can in NF, for example) then the difference between a Scott treatment of k-equivalence and a Church treatment would matter because of pairs like V and $V \setminus \{V\}$.

5 Afterthoughts

There are other equivalence relations on sets that come to mind in this context, and a document such as this may be the right place to collect some elementary remarks about them – to tidy things up, as it were. We also record also some relevant basic observations about CUS which were surely known to Church but not spelled out in [2], and which might be helpful to people trying to find their bearings in CUS.

5.1 k-equinumerosity Classes for Low Sets not just Wellfounded Sets

On the face of it, Church's L_k axioms say merely that the collection of all things (wellfounded or otherwise) k-equinumerous to a given *wellfounded* set is a set. However we can prove in CUS that – for every k – every x such that $\bigcup^k x$ is low is k-equinumerous with a wellfounded set. After all, if $\bigcup^k x$ is low, it is in bijection with some wellfounded set (that's what *low* means, after all) and that makes x k-equinumerous with a wellfounded set. So, for every k, and for every x such that $\bigcup^j x$ is low, the k-equinumerosity class ("k-cardinal") of x is a set. In brief, we get k-equinumerosity classes (k-cardinals) for low sets as well as wellfounded sets.

However there is no suggestion that *non-low* sets should have k-equinumerosity classes that are sets. Such equivalence classes cannot be furnished by any straightforward modification of the ("CO") constructions that provide k-equinumerosity classes for low sets.

5.2 No Cantor-Bernstein–like theorems

Church's equinumerosity relations, with their talk of bijections, invite thoughts of Cantor-Bernstein–like theorems (substitute 'injection' for 'bijection' to get a '$\leq$' rather than a '$\sim$') in a way that the other suite (with their talk of permutations) does not. But actually this invitation is a mere tease, for there is no Cantor-Bernstein theorem to be had, as we shall now see.

Make two copies $\mathbb{N}_1$ and $\mathbb{N}_2$ of the naturals. Order one like $\mathbb{Q}$, and the other like $1+\mathbb{Q}+1$. Use Wiener-Kuratowski pairs, so we think of the two total orders as subsets of $\mathcal{P}^3(\mathbb{N}_1)$ and $\mathcal{P}^3(\mathbb{N}_2)$. Evidently there are injections $i_1 : \mathbb{N}_1 \hookrightarrow \mathbb{N}_2$ and $i_2 : \mathbb{N}_2 \hookrightarrow \mathbb{N}_1$ such that $j^3(i_1)$ is an order-preserving injection from the $\mathbb{Q}$-shaped order into the $1+\mathbb{Q}+1$-shaped order and $j^3(i_2)$ goes the other way. But there is no bijection i_3 between $\mathbb{N}_1$ and $\mathbb{N}_2$ that lifts to an isomorphism (a 3-correspondence) between the two orders ... co's they are not isomorphic! check the exponents

5.3 Equidecomposability

A good thing to read in this connection is [8]. A lovely book.

We start with some definitions.

We consider the following relations:

Definition 3

1. *$x \sim_1 y \longleftrightarrow_{\mathrm{df}} (\exists \pi)(\pi``x = y)$;*

2. *$x \approx y \longleftrightarrow_{\mathrm{df}}$ there is a partition $\mathbb{P}$ of the symmetric difference (x `XOR` y) into pairs such that each pair in $\mathbb{P}$ meets both x and y;*

3. *$|x| = |y|$;*

4. *x and y are* **equidecomposable (with n pieces)** *if*

   (a) *y can be partitioned into $y_1 \ldots y_n$, and n permutations $g_1 \ldots g_n$ can be found such that x is the union of the $g_i``y_i$; and*

   (b) *y can be built similarly from a partition of x;*

5. *Church's 1-equinumerosity.*

GC ("group choice") is the principle that says that every set of finite-or-countable sets has a selection function. It is precisely what is needed to establish that two permutations of the same cycle type are conjugate. (Two permutations σ and τ have the same cycle type if there is a bijection between the set of cycles of σ and the set of cycles of τ that preserves cardinality.) Such pairs of permutations are sometimes said to be *conformal*. See [5].

GC also enables us to prove the following.

Remark 1 *(GC) Every permutation is a product of two involutions.*

Proof:

Given a permutation τ we construct two involutions σ and π such that $\tau = \sigma \cdot \tau$ cyclewise. We think of any infinite τ-cycle as a copy of $\mathbb{Z}$, by choosing an element w "to be 0" (We can do this simultaneously for all infinite τ-cycles by using *GC*). The restriction of π to this cycle is $\lambda x.(-x)$. To be explicit: send x to the unique z such that $(\exists n \in \mathbb{Z})(\tau^n(z) = w \wedge \tau^n(w) = x)$, and the restriction of σ is $\lambda x.(1-x)$ similarly. For an n-cycle we do the same *mod* n. Thus for each τ-cycle we pick a pair of finite permutations. We do not need any AC to do this, since once we have picked w we can tell the two permutations apart since precisely one of them fixes w. We then take σ to be the union of all the permutations that *fix* the chosen w and π to be the union of all their mates, those that *move* w.

■

The following basic observation might help the reader orient themselves.

Proposition 1 *If $|X| = |Y|$, and $|V \setminus X| = |V \setminus Y|$, then $X \sim_1 Y$.*

Proof: Simply take the union of the two bijections considered as sets of ordered pairs. It is a permutation of V witnessing $X \sim_1 Y$. ■

In fact there is a natural generalisation. For each n we can show that:
For all n-tuples $\vec{a}$ and $\vec{b}$

- there is a permutation π of V such that, for each $i \leq n$, $\pi``a_i = b_i$ iff
- each of the 2^n boolean combinations of the as is the same size as the corresponding boolean combination of the bs.

(Equally obvious! – probably by induction on 'n'.)

Equidecomposibility can mean various things depending on the condition we put on the partition, the set of pieces. A general definition would be

Definition 4 *A and B are equidecomposible iff there is a partition $\mathbb{P}$ of A and an injective function $f : \mathbb{P} \hookrightarrow \mathcal{P}(B)$ such that $f``\mathbb{P}$ is a partition of B and, for each $p \in \mathbb{P}$, $p \sim_1 f(p)$.*

This idea has its roots in 19th century dissection problems going back to the Wallace-Bolyai-Gerwien theorem (one such problem was in Hilbert's famous list) so normally one takes $\mathbb{P}$ to be finite. It is those roots in dissection problems that cause us to define the matching using $\sim_1$ rather than equipollence, which is why the final clause is "$p \sim_1 f(p)$." rather than "$|p| = |f(p)|$".

Equidecomposibility with one piece ($\mathbb{P}$ is a singleton) is of course just $\sim_1$.
If A and B are equidecomposible with $\mathbb{P}$ finite then clearly $|A| = |B|$.
Is there a converse? Yes: it even turns out that equidecomposibility with *two* pieces is already equipollence:

Remark 2 *$|x| = |y|$ iff x and y are equidecomposable using two pieces.*

Proof:

R → L

Suppose $x = x_1 \cup x_2$ with $x_1 \cap x_2 = \emptyset$, $y = y_1 \cup y_2$ with $y_1 \cap y_2 = \emptyset$ and σ and τ are two permutations such that $\tau``x_1 = y_1$ and $\sigma``x_2 = y_2$. Then f defined by

$$f(u) = \begin{cases} \tau(u) \text{ if } u \in x_1 \\ \sigma(u) \text{ if } u \in x_2 \\ u \text{ otherwise} \end{cases}$$

maps x 1-1 onto y.

L → R.

Let x and y be of size m and have complements of sizes p and q respectively. Proposition 1 deals with the case where $p = q$. To show x and y are equidecomposable using two pieces in the remaining case, we need to show that a set of size m can be split into two sets of size m_1 and m_2 such that $m_1 + p = m_1 + q$ and $m_2 + p = m_2 + q$.

If we can do this, then

$x = x_1 \cup x_2$ with $|x_1| = m_1$ and $|x_2| = m_2$ and $x_1 \cap x_2 = \emptyset$.

$y = y_1 \cup y_2$ with $|y_1| = m_1$ and $|y_2| = m_2$ and $y_1 \cap y_2 = \emptyset$.

and x_1 is mapped onto y_1 by a permutation that we construct by noting that $|x_1| = |y_1|$ and that $|V \setminus x_1| = |x_2| + |V \setminus x|$ so $|V \setminus x_1| = m_2 + p$. Also $|V \setminus y_1| = |y_2| + |V \setminus y|$ so $|V \setminus y_1| = m_2 + q$, which equals $m_2 + p$. x_2 will be mapped onto y_2 similarly.

To find m_1 and m_2, we need a theorem of Tarski's,

Remark 3 *(Tarski, [7])*

If $m + p = m + q$ then there are n, p_1 and q_1 such that
$p = n + p_1$,
$q = n + q_1$; *and*
$m = m + p_1 = m + q_1$.

Proof:

In the case we are considering, m, p, and q are as in the hypothesis of the statement of remark 2. The desired m_1 and m_2 can be found as follows:

$$m_1 = m$$

$$m_2 = \aleph_0 \cdot (p_1 + q_1).$$

We need to verify that $m_1 + p = m_1 + q$, $m_2 + p = m_2 + q$, and $m_1 + m_2 = m$. We know m absorbs p_1 and q_1 so $m_1 + p = m_1 + q$ since they are both equal to $m + n$. Also m absorbs $p_1 + q_1$, so it absorbs $\aleph_0 \cdot (p_1 + q_1)$. Thus $m_1 + m_2 = m$ as desired. To verify $m_2 + p = m_2 + q$ we expand and rearrange, noting that $(\forall x)(\aleph_0 \cdot x + x = \aleph_0 \cdot x)$. ■

If we allow the partition $\mathbb{P}$ to be infinite then equidecomposibility becomes a relation that – unless we have some AC – could in principle be weaker than equipollence. If this "infinite equidecomposibility" has to be the same as equipollence then, for example, any two sets that have countable partitions into pairs must be equipollent, and indeed both countable.

5.4 The relation $\approx$

(see definition 3.)

We will make use of the following

Lemma 2 *If x is a set and τ an involution, then $x \approx \tau``x$*

Proof:

$$\tau``(x \setminus \tau``x) \;=\; \tau``x \setminus \tau^2``x \;=\; \tau``x \setminus x$$

so τ bijects $x \setminus \tau``x$ with $\tau``x \setminus x$ and thereby witnesses $x \approx \tau``x$. ■

Clearly $x \sim_1 y$ is a weaker condition than $x \approx y$, since it is compatible with x being a proper subset of y whereas $x \approx y$ (which implies it) is not. Clearly $\approx^*$ (the transitive closure of $\approx$) is potentially a weaker relation than $\approx$, and it would tidy things up greatly if $\sim_1$ and $\approx^*$ were the same relation. It turns out that we can prove that they are if we have some weak choice assumptions.

Theorem 1 *(weak choice assumptions)*

$$(\forall xy)(x \sim_1 y \longleftrightarrow x \approx^* y)$$

Proof:

First we prove this on the assumption that $|V|^2 = |V|$.

L $\to$ R

Bowler-Forster [1] tells us that if $|V|^2 = |V|$ then every permutation of V is a product of involutions, so $x \sim_1 y$ implies that $x \approx^n y$, for some n – using n appplications of lemma 2 whence $x \approx^* y$ as desired.

R $\to$ L

Observe that $x \approx y \to x \sim_1 y$, whence $x \approx^n y \to x \sim_1 y$ for any n.

In fact [1] puts a bound on the number of involutions needed to capture a permutation, so we have actually proved $\approx^n = \approx^*$ for some small n ... a dozen or so, as I recall.

The other way we can prove it is if we are willing to use GC; using GC we can prove $\approx^2 = \approx^*$ – and we don't even need to assume that $|V|$ is idemmultiple. We claim

$$\forall xy(x \sim_1 y \longleftrightarrow \exists z(x \approx z \approx y)).$$

R $\to$ L

$x \approx y \to x \sim_1 y$, since any partition of x `XOR` y witnessing $x \approx y$ extends naturally to a permutation of V (an involution, in fact) mapping x onto y.

L $\to$ R:

This is the direction for which we need GC.

Assume the LHS, so $x \sim_1 y$ which is to say there is a permutation π of V such that $\pi``x = y$. Now remark 1 tells us there are two involutions τ and σ with $\sigma \cdot \tau = \pi$.

Now lemma 2 tells us that τ witnesses $x \approx \tau``x$, and analogously $\sigma$ witnesses $y \approx \sigma``y$.

But $\tau``x = \sigma``y$, and this thing is the desired witness to the '$\exists z$'.

■

Perhaps a word is in order at this point about the possibility of Cantor-Bernstein–like theorems. We observed earlier that there is no C-B–like theorem for Church's k-cardinals. However there are C-B–like theorems for equidecomposibility and $\approx$, but they are neither sufficiently difficult to prove nor sufficiently closely related to our current concerns for us to want to write out proofs.

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