INTRODUCTION

THE AXIOM OF DETERMINACY concerns games of infinite length between two players, and asserts that in each such game where there are countably many moves available to each player at each turn, one of the players has a winning strategy.

The construction of strategies for such games has been the object of study among set theorists for thirty years; AD, the full Axiom of Determinacy implies weak forms of the Axiom of Choice — sufficient, say, to develop the theory of Lebesgue measure — but contradicts the full Axiom of Choice in many ways: it implies, for instance, that all sets of reals are Lebesgue measurable. It is natural, therefore, to consider restricted forms of the axiom, such as BD, the *Axiom of Borel Determinacy*, which asserts the existence of strategies for those games of which the pay-off set (the set of outcomes which are wins for the first player) is a Borel set in the space of all outcomes, or PD, the *Axiom of Projective Determinacy*, which does the same for games of which the pay-off set is projective in the sense of Kuratowski. These restricted forms, being compatible with the Axiom of Choice, are perhaps more palatable but still have striking consequences.

The axiom AD was proposed some thirty years ago by Mycielski and Steinhaus and has led to considerable new insights into the area of mathematics known as descriptive set theory. It is therefore of continuing interest. The question of its consistency was open for a very long time, but has now been solved, and this book aims to present a full account of the answer.

It has been known since the 1960's from work of Solovay that a proof of the consistency of AD would require some large cardinal axiom; from work of Martin that, conversely, weak forms of PD could be proved from large cardinal axioms; and from work of Friedman that this phenomenon would be finely tuned to the definitional complexity of any particular game. But it is only in the last few years that an exact calibration of the determinacy of various classes of games in terms of large cardinals has been achieved, and the work of many set theorists has been involved. What has emerged is the concept of a *Woodin cardinal*. Being a Woodin cardinal is a property stronger than being measurable and weaker than being supercompact. That this property is the correct large cardinal property for treating determinacy can hardly be doubted in view of the following theorems:

(i) AD is equiconsistent with the existence of $\aleph_0$ Woodin cardinals;
(ii) Δ^1_2 determinacy is equiconsistent with the existence of one Woodin cardinal;
(iii) $\boldsymbol{\Delta^1_2}$ determinacy is equivalent to the existence for each real x of an inner model M_x, containing x and all ordinals, and a countable ordinal which in M_x is a Woodin cardinal;
(iv) for each $k = 1, 2, \ldots$ determinacy for games in the first k levels of the projective hierarchy can be proved from the existence of a measurable cardinal greater than $k-1$ Woodin cardinals but not just from the existence of $k-1$ Woodin cardinals.

This book, to be published in two parts, gives complete proofs of the above equiconsistency results. This first Part is divided into seven chapters, and aims to explain how Woodin cardinals can be extracted from principles of determinacy. The arguments that we shall use are drawn from various areas of logic, and in the opening chapters we shall review the material that will form our starting point. In Chapter I, we summarise concepts from definability theory and establish some results for $L(\mathbb{R})$ that do not need determinacy in their proof; in chapter II we summarise Moschovakis' general recursion-theoretic framework for descriptive set theory, and in Chapter III we review some well-established consequences of the Axiom of Determinacy. In Chapter IV we discuss the notion of a Woodin cardinal, and give precise statements of the results to be proved in the two parts of the book.

The next three chapters contain the arguments for retrieving Woodin cardinals from various forms of determinacy. Chapter V is the heart of the argument: it comprises the proof that if AD is true in $L(\mathbb{R})$ then Θ is a Woodin cardinal in the HOD of that model.

In Chapter VI, the argument of Chapter V is refined in two related ways, to show that the statement that Θ is Woodin in HOD is true if V is the constructible closure of a single real and that all light-face Δ^1_2 games are determined; and to show that for any class S of ordinals it is also true in inner models of the form

$L[S; x]$ for almost all reals x in the sense of the Martin measure, provided that $V = L(\boldsymbol{R})$ and AD holds.

In Chapter VII, models of ZFC plus the statement that there are infinitely many Woodin cardinals are constructed in three ways, the first an inner model construction, the second a generic extension, the third an inner model construction analysed by means of generic elementary embeddings. All start from the assumption that AD holds in $L(\boldsymbol{R})$, using the results of Chapter VI.

Part II will begin with Chapter VIII, in which we review the underlying concepts used in the proof of the Martin-Steel theorem, and apply them in the following three chapters to the converse problem of building models of determinacy starting from Woodin cardinals. The final chapter is of a more speculative character: it aims to discuss various open problems and to suggest directions for future research.

We shall work mainly within ZF set theory with some form of the Axiom of Choice; but we shall also wish to use various strictly weaker subsystems. Some of these will be something like Kripke–Platek set theory, possibly with a few extra axioms; others will be a theory satisfied in some rank, and sufficient to show that the ultralimit of it by an extender will be a rank-initial segment of the ultralimit of the entire universe.

We shall be particularly interested in the inner model $L(\mathbb{R})$, where $\mathbb{R}$ is the set of all real numbers of some set-theoretical universe. As we shall, of course, wish to consider many different universes, we will find it convenient to make a notational distinction between a name for the set of reals of the universe we happen to be in, and a symbol denoting the term "the set of real numbers". For the latter we use the letter $\boldsymbol{R}$; so that a theorem stated in the form "If $V = L(\boldsymbol{R})$, then Θ is regular" will be true in many universes. When we have particular sets of reals in mind, we shall use the letter $\mathbb{R}$, possibly with subscripts. For the most part this distinction is not necessary, but in certain model-building it will be useful.

We shall wish to consider real numbers in various guises: almost never as Euclidean reals, but often as subsets of ω, and most often as functions from $\omega \to \omega$. We reserve the letter $\mathcal{N}$ for the set of such functions.

Among set theorists there are those who prefer the Gödel hierarchy $L_\xi(\boldsymbol{R})$ and those who prefer the Jensen hierarchy $J_\xi(\boldsymbol{R})$. Our arguments will usually be equally valid whether one considers the ξ^{th} stage of a hierarchy building $L(\boldsymbol{R})$ to be $J_\xi(\boldsymbol{R})$ or $L_{\omega\xi}(\boldsymbol{R})$. In those few places — perhaps only one, when the equivalence between two possible definitions of ∞-Borel is being proved — where we must scrutinise every level, we shall give two proofs, one for the L faction and one for the J. [C 1] **We emphasize that in the formal language used in building these hierarchies we always have a constant $\dot{\Re}$ denoting the set of reals.** Our notation for formal languages is this: The formulæ of these languages are built from symbols indicated by $\dot\epsilon$, $\dot=$, $\vee$, $\wedge$, $\bigvee$, $\bigwedge$, $\neg$; in addition we employ two devices to indicate that a concept or statement is to be formalised in one of these languages: a dot placed over a familiar symbol, for example $\dot\omega$, to denote the corresponding formal term, in our example the formal term for the first infinite ordinal; and a phrase in type-writer font, such as `every well-ordering is isomorphic to an ordinal` stands for the corresponding formal sentence.

We shall be interested in the *Axiom of Choice* which we take in the form that every set has a well-ordering. Among weaker forms of AC, we shall be interested in the axiom of dependent choice, DC and its restriction DCR to relations on the reals.

0·0 Exercise Show that DCR implies that each $J_\zeta(\boldsymbol{R}) \models DC$.

Before ending this introductory section, let us draw attention to some conventions of our notation.

Briefly expressible restrictions on the range of a quantified variable are signalled by a colon: for example

$$\forall x :\in y \; \exists z :\subseteq w \; \Phi(x, z)$$

says that to every x in the set y there is a subset z of w for which $\Phi(x, z)$ holds. We admit to the critical reader that the scope of the colon in this convention is left unclear.

We reserve the letter ϖ for *collapsing maps*: for X a set, we define $\varpi_X : X \to V$ by $\in$-recursion:

$$\varpi_X(x) \;=_{\text{df}}\; \{\varpi_X(y) \mid y \in x \cap X\}.$$

[C 1] The reader is warned that Steel's $J_1(\boldsymbol{R})$ is $V_{\omega+1}$, so not of height ω.

If, and only if, X is extensional, ϖ_X will be 1-1.

We reserve the letters φ and ϑ for formulæ of formalised languages: for example, we might write $J_\xi(\mathbb{R}) \models \varphi(a) \wedge \neg\vartheta(b)$.

We reserve the letter ϱ for rank functions associated with well-founded relations. If R is such a relation, we define

$$\varrho_R(x) \ =_{\rm df} \bigcup\{\varrho_R(y) + 1 \mid R(y, x)\}.$$

For example, if R is the membership relation, ϱ_R will be the set-theoretic rank function, which we shall denote by $\varrho_\in$.

We have two notations for pairing functions: we use $(x,\, y)$ for the usual Wiener–Kuratowski notion, and $\langle x,\, y\rangle$ when some coding is involved: for example when x and y are both reals and we wish to code the pair as a single real. When the need arises, we shall specify such coding with greater exactness.

We occasionally use the iota symbol $\iota y\, \Phi$ to mean the one and only y such that Φ.

[for the text of the latest draft, of some 150 pages, consult `ardm@univ-reunion.fr`]