Lots of funny symbols—a FAQ

Thomas Forster

November 17, 2007

The following symbols appear at various times, and people often wonder about the differences. $\to$, $\Rightarrow$, `->`, $\models$, $\vdash$.

$\to$ and `->` are the same symbol in the sense of being different manifestations of the same symbol in two different fonts.

They have several uses. $A \to B$ is a notation denoting the set of all functions from A into B where A and B are sets (or types). Also $\to$ is a constructor of the recursive datatype of (propositional) formulæ: sticking a '$\to$' between two formulæ results in another formula. Punning between these two uses of the one symbol plays an important part in understanding λ calculus.

When '$\to$' is used as a constructor of the recursive datatype of (propositional) formulæ it is often provided with semantics to make $p \to q$ mean the same as '$\neg p \vee q$'. It was not always thus. There is also the symbol '$\supset$' which is also a constructor and is only ever used for this purpose. '$\to$' was first used in propositional logic quite explicitly to *not* mean the same thing as the truth-functional connective '$\supset$' (since there was an ideological debate around whether or not the relation encapsulated by '$\neg p \vee q$' could properly be regarded as implication!). Sadly this symbol is hardly used any longer, and '$\to$' is now used for both.

The symbol '$\vdash$' is usually placed between a term denoting a theory or a set of formulæ (to its left) and a single formula (to its right). Occasionally even the thing on the left can be a single formula. In these circumstances '$\vdash$' belongs to a metalanguage and means that there is a deduction of the formula on the right from the theory or set of formulæ on the left. Sometimes it has a subscript ('T' for example) denoting a theory or Logic in which the deduction is to be performed. The relation it captures is sometimes called **syntactic**

entailment (to be contrasted with semantic entailment below).

There are people (like me!) who write sequents with '$\vdash$' in the middle instead of '$\Rightarrow$' (as Dr. Paulson does). In this usage '$\vdash$' is not a piece of metalanguage but is a constructor of a formal language. However the commonest use of the double-shafted arrow is probably its use as a piece of slang metalanguage, where it is put between two formulæ (or more typically between two lines of a proof) to mean that the second follows (in some sense to be divined from context) from the first.

Notice the difference between this use and the way we can put '$\to$' between two formulæ. Putting '$\to$' between two formulæ results in **another formula of the same language that the two given formulæ belonged to**. The slang use of '`=>`' really involves putting '$\Rightarrow$' between two *names* of formulæ so as to make an assertion about how those two formulæ are related, and this assertion is not an object of the same language that the two given formulæ belonged to, but is a piece of English (or rather: English with knobs on).

'$\mathcal{M} \models T$' and '$\mathcal{M} \models \phi$' are two correct uses of '$\models$'. The first means that the theory (set of formulæ) T is true in the structure $\mathcal{M}$, and the second means that the formula ϕ is true in the structure $\mathcal{M}$.

There is another use of '$\models$'. It can be used, like '$\vdash$' between a formula (or something denoting a set of formulæ) on the left, and a formula on the right. The resulting expressions mean that every interpretation making the stuff on the left true makes the stuff on the right true too, and this relation is usually called **semantic entailment**.