Ultralimits

Thomas Forster

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Frayne's Lemma

(A reminder of two bits of jargon: an **expansion** of a structure $\mathcal{B}$ is a structure with the same carrier set and more gadgets. e.g. the rationals as a field are an expansion of the rationals as an additive group. The converse relation is a **reduction**: the rationals as an additive group are a reduction of the rationals as a field.)

LEMMA 1 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent. Then there is an ultrapower $\mathcal{A}^I/\mathcal{U}$ of A and an elementary embedding from $\mathcal{B}$ into it.*

Proof:

Supply names $\mathtt{b}$ for every member b of B. Let $\mathcal{L}$ be the language with the new constants. There is an obvious way of expanding $\mathcal{B}$ to a structure for this new language, namely to let each constant $\mathtt{b}$ denote that element b of B which gave rise to it. (Of course this is not the only way of doing it: *any* map $B \to B$ will give rise to an expansion of $\mathcal{B}$ of this kind—and later we will have to consider some of those ways). Let us write '$\mathcal{B}'$' to denote this obvious expansion of $\mathcal{B}$, and let I be the set of sentences of $\mathcal{L}$ true in $\mathcal{B}'$. (Use of the letter 'I' for this is a bit of a give-away!)

Consider ϕ a formula in I. It will mention finitely many constants—let us say two, for the sake of argument. Replace these two constants by new variables 'v_1' and 'v_2' (*not* mentioned in ϕ!) to obtain ϕ'' and bind them both with '$\exists$' to obtain $(\exists v_1)(\exists v_2)\phi''$ which we will call 'ϕ'' for short. This new formula is a formula of the original language which is true in $\mathcal{B}$ and is therefore also true in $\mathcal{A}$ (since $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent).

The next step is to expand $\mathcal{A}$ to a structure for the language $\mathcal{L}$ by decorating it with the with the extra constants $\mathtt{b}$ etc that we used to denote members of B. Of course any function $B \to A$ gives us a way of decorating $\mathcal{A}$ but with ϕ in mind we are interested only in those decorations which give us a structure that satisfies ϕ. If ϕ contained the constants $\mathtt{b}$ and $\mathtt{b'''}$ for example then the obvious way to expand A involves using those two constants to denote the witnesses in $\mathcal{A}$ for the two existential quantifiers in ϕ'. Since ϕ contains only finitely many constants this nails down denotations for only finitely many of the constant-names-for-members-of-B. However any finite map from B to A can be extended to a total function $B \to A$ so we can extend this to a way of labelling members of A with these constants in such a way that the decorated version of $\mathcal{A}$ satisfies the original formula ϕ.

Pick one such labelling and call it $a(\phi)$. (Thus $a(\phi)$ is merely an element of $B \to A$ satisfying an extra condition parametrised by ϕ. We can think of a as a function $\mathcal{L} \to (B \to A)$ or as a function $(\mathcal{L} \times B) \to A$ *ad libitum*). $\mathcal{A}$ expanded with this decoration we call $\langle \mathcal{A}, a(\phi) \rangle$. Now consider the set

$J(\phi) =: \{\psi \in I : \langle \mathcal{A}, a(\psi) \rangle \models \phi\}$

It is easy to check that the family $\{J(\phi) : \phi \in I\}$ of subsets of I has the finite intersection property and so gives rise to a ultrafilter[1] $\mathcal{U}$ on I and thence to an ultrapower $\mathcal{A}^I/\mathcal{U}$. Evidently if $\phi \in I$ then $J(\phi) \in \mathcal{U}$ and the ultrapower will believe ϕ.

We have to find an elementary embedding from $\mathcal{B}$ into this ultrapower. Given $b \in B$ whither do we send it? The obvious destination for b is the equivalence class of the function $\lambda\phi.a(\phi, b)$ that sends ϕ to $a(\phi, b)$. The function that sends b to $[\lambda\phi.a(\phi, b)]$ is $\lambda b.[\lambda\phi.a(\phi, b)]$—which we will write '$h$' for short. We must show that h is elementary.

The best way to understand what h does and why it is elementary is to think of the ultrapower as a reduction of the ultraproduct

$$\prod_{\psi \in I} \langle \mathcal{A}, a(\psi) \rangle.$$

("**expand** the factors; take an ultraproduct; **reduce** the ultraproduct—to obtain a ultrapower of the factors ...")

Each of the factors $\langle \mathcal{A}, a(\psi) \rangle$ is a structure for $\mathcal{L}$ and therefore the ultraproduct is too. By the same token, for each $b \in B$, each of the factors has an element which is pointed to by $\mathtt{b}$-the-constant-name-of-b, and therefore the ultraproduct will too. The key fact is that h is the function that sends each $b \in B$ to the thing in the ultraproduct that is pointed to by $\mathtt{b}$ the constant-name-of-b.

As for the elementarity of h, suppose $\mathcal{B} \models \phi(\vec{v})$. Then, for some choice of constants $\vec{\mathtt{b}}$, $\mathcal{B} \models \phi(\vec{\mathtt{b}})$, and $\mathcal{B}' \models \phi'$. But now $J(\phi)$ is $\mathcal{U}$-large, so the ultrapower believes ϕ. ■

But what we really need is Scott's lemma:

Lemma 2 *Suppose $g : \mathcal{A} \hookrightarrow \mathcal{B}$ is an elementary embedding. Then there is an ultrapower $\mathcal{A}^I/\mathcal{U}$ of A and an elementary embedding from $\mathcal{B}$ into it making the triangle commute.*

Proof:

The ideas are the same, but we need to be slightly more careful in the definition of $a(\phi)$. Fix once for all a member a of A. As before, we extend the language by adding names for every member of B, thus obtaining the language $\mathcal{L}$ as before. Now we expand $\mathcal{B}$ by decorating B with these names, but not in

[1] There doesn't seem to be any reason to conclude that this ultrafilter will be nonprincipal, but then nor does it seem to matter if it *isn't*. Bell and Slomson don't say that it will be nonprincipal. Thanks to Phil Ellison for drawing my attention to this point.

the obvious way. If b is in the range of g we allow $\mathtt{b}$ the constant-name-of-b to denote b; if b is not in the range of g, then $\mathtt{b}$ will denote $g(a)$. Let's call this expanded structure $\mathcal{B}'$.

If we are to expand $\mathcal{A}$ to obtain a structure for $\mathcal{L}$ then we must ensure that, for each $b \in B$, the constant-name-$\mathtt{b}$-of-b points to something in A. The obvious way to do this is to ordain that $\mathtt{b}$ point to g^{-1} of the thing that that $\mathtt{b}$ points to in the expansion $\mathcal{B}'$ of $\mathcal{B}$. This decorated version of $\mathcal{A}$ and the decorated version $\mathcal{B}'$ of $\mathcal{B}$ are elementarily equivalent (with respect to the extended language with the names) (*)

As before, let I be the set of sentences of $\mathcal{L}$ true in $\mathcal{B}'$. Consider a formula $\phi \in I$. Recall what we did at the same stage in the proof of Frayne's Lemma. This time we replace with existentially-quantified variables only those constants denoting elements of B not in the range of g. Let's call this formula ϕ' like last time. Evidently $\mathcal{B}' \models \phi'$ and so, by the remark (*) at the end of the last paragraph, the decorated version of $\mathcal{A}$ also satisfies ϕ'. So, as before, there is another decoration of $\mathcal{A}$ which actually satisfies the original ϕ. Pick one such decoration and call it $a(\phi)$, and call the structure thus decorated $\langle \mathcal{A}, a(\phi) \rangle$. We define

$J(\phi) =: \{\psi \in I : \langle \mathcal{A}, a(\phi) \rangle \models \phi\}$

as before, and it has the finite intersection property as before and gives us an ultrafilter $\mathcal{U}$ as before, and we have the same elementary embedding h from $\mathcal{B}$ into the ultrapower as before. It remains only to check that the diagram is commutative. I think this can safely be left as an exercise to the reader. ■

Now comes the fun part, and i wish i had the diagrams package working (hint! hint!!)

Coda

This document is the result of my attempting to digest the relevant section (around pp 150-160) of John Bell and Alan Slomson: *Models and Ultraproducts*: a lovely book. Thanks to Alan and John. Since I intend to leave this document on my web-page as a resource for my Cambridge students, I welcome comments from them that might help make the document more useful.