University of Cambridge

Mathematical Tripos

Combinatorics

Lecturer:
Prof. A.G. Thomason

Editor:
H-J. Wagenaar

Contents

Chapter 1

Antichains

Definition. $\mathcal{A} \subseteq \mathcal{P}X$ is a **chain** if for all A, $B \in \mathcal{A}$ we have $A \subseteq B$ or $B \subseteq A$.

$\mathcal{A} \subseteq \mathcal{P}X$ is an **antichain** if for all A, $B \in \mathcal{A}$ we have $A \neq B$ implies $A \not\subset B$.

Consider the n-cube, given by considering the elements $0, \ldots, n$ as axis and the indicator functions as coordinates, connecting exactly those vertices which differ by one element. This can be thought of as a layered diamond shape of increasing size of subsets. As n increases it becomes a bell-curve and limits to two bounded axes.

Theorem 1.1 (Sperner, 1928). Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be an antichain. Then

$$|\mathcal{A}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. It is enough to decompose $\mathcal{P}[n]$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains.

To do this, it is enough to inject $[n]^{(r)} \to [n]^{(r+1)}$ for each $r < n/2$ so that A maps to B implies $A \subset B$ and to inject $[n]^{(r+1)} \to [n]^{(r)}$ for $r \geqslant n/2$ so A maps to B implies $A \supset B$.

This is the same as finding matchings in the bipartite graph of the n-cube between layer r and $r+1$.

Without loss of generality $r < n/2$, $S \subseteq [n]^{(r)}$ and let T be the neighbours of S in $[n]^{(r+1)}$.

Each member of S has $n-r$ neighbours in T and each member of T has at most $r+1$ neighbours in S. $e(S,T)$ is the number of edges between S and T, then $|S|(n-r) = e(S,T) \leqslant |T|(r+1)$, so

$$|T| \geqslant \frac{n-r}{r+1}|S| \geqslant |S|,$$

and by Hall's Theorem the matching exists. $\square$

Definition. For $\mathcal{A} \subseteq [n]^{(r)}$ the **lower shadow** of $\mathcal{A}$ is

$$\partial\mathcal{A} = \partial^-\mathcal{A} = \{B \in [n]^{(r-1)} \colon B \subset A \text{ some } A \in \mathcal{A}\}.$$

Lemma 1.2 (Local LYM). For all $\mathcal{A} \subseteq [n]^{(r)}$ we have

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geqslant \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

Shadows have greater density.

Proof. By same argument as in proof of Theorem 1.1:

$$r|\mathcal{A}| = e(\mathcal{A}, \partial\mathcal{A}) \leqslant |\partial\mathcal{A}|(n-r+1).$$ $\square$

Get equality if and only if $\mathcal{A} = \emptyset$ and $\mathcal{A} = [n]^{(r)}$ by the connectivity of the bipartite graph on $[n]^{(r)} \cup [n]^{(r-1)}$, hence showing uniqueness.

Theorem 1.3 (LYM inequality, Yamamoto, 1954). Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{\left|\mathcal{A} \cap [n]^{(r)}\right|}{\binom{n}{r}} \leqslant 1.$$

Proof 1. Let $\mathcal{A}_r = [n]^{(r)} \cap \mathcal{A}$ and

$$\mathcal{B}_r = \partial^{n-r}\mathcal{A}_n \cup \partial^{n-r-1}\mathcal{A}_{n-1} \cup \cdots \cup \mathcal{A}_r.$$

Then $\mathcal{B}_r = \partial\mathcal{B}_{r+1} \cup \mathcal{A}_r$ and the union is disjoint because $\mathcal{A}$ is an antichain.

Then

$$\begin{aligned} 1 \geqslant \frac{|\mathcal{B}_0|}{\binom{n}{0}} &= \frac{|\partial\mathcal{B}_1|}{\binom{n}{0}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}} \geqslant \frac{|\mathcal{B}_1|}{\binom{n}{1}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}} = \frac{|\partial\mathcal{B}_2|}{\binom{n}{1}} + \frac{|\mathcal{A}_1|}{\binom{n}{1}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}} \\ &\geqslant \frac{|\mathcal{B}_2|}{\binom{n}{1}} + \frac{|\mathcal{A}_1|}{\binom{n}{1}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}} \geqslant \cdots \geqslant \sum_r \frac{|\mathcal{A}_r|}{\binom{n}{r}} \end{aligned}$$

using the local LYM inequality. □

Remark. Equality holds if and only it holds always in local LYM, that is, if and only if $\mathcal{A} = [n]^{(r)}$ for some r.

Proof 2. Consider a random maximal chain $C = A_0 \subset A_1 \subset \cdots \subset A_n$ where $|A_i| = i$. The probability that A_r is some given r-set is $1/\binom{n}{r}$. Hence

$$\mathbb{P}(A \in \mathcal{A} \cap [n]^{(r)}) = \frac{\left|\mathcal{A} \cap [n]^{(r)}\right|}{\binom{n}{r}}.$$

$\mathcal{A}$ is an antichain, so these events are mutually exclusive, hence probabilities sum to $\leqslant 1$. □

Definition. A chain is **SYMMETRIC** if it is of the form $A_k \subset A_{k+1} \subset \cdots \subset A_{n-k}$ where $A_r \in [n]^{(r)}$.

Theorem 1.4. $\mathcal{P}[n]$ has a decomposition into symmetric chains.

Proof. Apply induction on n. For each chain $\mathcal{C} = A_k, A_{k+1}, \ldots, A_{n-1-k}$ is a symmetric chain decomposition of $\mathcal{P}[n-1]$, let

$$\begin{aligned} \mathcal{C}' &= A_k,\ A_{k+1},\ \ldots,\ A_{n-1-k},\ A_{n-1-k} \cup \{n\} \\ \mathcal{C}'' &= A_k \cup \{n\},\ A_{k+1} \cup \{n\}, \ldots, A_{n-2-k} \cup \{n\}. \end{aligned}$$

Notice that every subset of $[n]$ lies in exactly one of these chains, and the chains are symmetric. □

Does this mean we double the number of chains each time? But $2\binom{n-1}{\lfloor (n-1)/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor}$ only if n is even. But not if n is odd, then sometimes $\mathcal{C}$ has length 1, so $\mathcal{C}'' = \emptyset$. Check that we get $l_k(n) = \binom{n}{k} - \binom{n}{k-1}$ chains of length $n - 2k + 1$, where $0 \leqslant k \leqslant \lfloor n/2 \rfloor$, because $l_k(n) = l_k(n-1) + l_{k-1}(n-1)$.

Theorem 1.5 (Kleitman, 1965). Let $x_1, \ldots, x_n$ be points in $\mathbb{R}^m$ (or normed space) with $\|x_i\| \geqslant 1$. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ such that $\|x_A - x_B\| < 1$ for all $A, B \in \mathcal{A}$, where $x_A = \sum_{i \in A} x_i$. Then $|\mathcal{A}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$.

Erdős (1945) observed that if $x_n \in \mathbb{R}$, then the result follows because the sets $\{i \in A \colon x_i > 0\} \cup \{i \notin A \colon x_i < 0\}$ form an antichain.

Proof. We call a partition of $\mathcal{P}[n]$ into classes **quasi-symmetric** if there are $l_k(n) = \binom{n}{k} - \binom{n}{k-1}$ classes with $n - 2k + 1$ elements.

Such a partition has $\sum_k l_k(n) = \binom{n}{\lfloor n/2 \rfloor}$ classes in total.

Notice that the following procedure generates a quasi-symmetric partition of $\mathcal{P}[n]$: take a quasi-symmetric partition of $\mathcal{P}[n-1]$. Let $\mathcal{C}$ be a class, let $A^+ \in \mathcal{C}$, and let

$$\begin{aligned} \mathcal{C}' &= \mathcal{C} \cup \{A^+ \cup \{n\}\} \\ \mathcal{C}'' &= \{A \cup \{n\} \colon A \in \mathcal{C},\ A \neq A^+\}. \end{aligned}$$

Because $l_k(n) = l_k(n-1) + l_k(n-1)$ this works. We call a class $\mathcal{C}$ **dispersed** if $\|x_A - x_B\| \geqslant 1$ for all $A, B \in \mathcal{C}$ where $A \neq B$. If we can find a quasi-symmetric partition into dispersed classes we are done.

So: if $\mathcal{C}$ is dispersed, can we pick A^+ so that $\mathcal{C}'$ and $\mathcal{C}''$ are dispersed. Notice $\mathcal{C}''$ is always dispersed because

$$\left\| x_{A \cup \{n\}} - x_{B \cup \{n\}} \right\| = \|x_A - x_B\| \geqslant 1.$$

Let $e_n = x_n / \|x_n\|$. Pick A^+ so that $\langle x_{A^+}, e_n \rangle$ is maximal ranging over $A \in \mathcal{C}$. Then for all $A \in \mathcal{C}$:

$$\begin{aligned} \left\| x_{A^+ \cup \{n\}} - x_A \right\| &\geqslant \langle x_{A^+ \cup \{n\}} - x_A, e_n \rangle \\ &= \langle x_{A^+ \cup \{n\}}, e_n \rangle - \langle x_A, e_n \rangle \\ &= x_n - \langle x_{A^+}, e_n \rangle - \langle x_A, e_n \rangle \geqslant x_n \geqslant 1. \end{aligned}$$

□

Chapter 2

Saturation

Definition. An r-UNIFORM HYPERGRAPH is a pair $H = (V, E)$ where $E \subseteq V^{(r)}$.

2-uniform hypergraphs are graphs.

The COMPLETE HYPERGRAPH of order k is $K_k^{(r)} = ([k], [k]^{(r)})$.

We say such a hypergraph H is (STRONGLY) k-SATURATED if the addition of any missing edges creates an extra copy of $K_k^{(r)}$.

Consider $r = 2$. The Turán graph $T_{k-1}(n)$ is k-saturated, but there exists k-saturated graphs with many fewer edges. For example, for $k = 3$ a star is saturated, and in general $K_{k-2} + \overline{K}_{n-k+2}$ is k-saturated. Erdős-Hajnal-Moon (1964) showed that no k-saturated graph has fewer edges.

Theorem 2.1 (Bollobás, 1965). Let $\{(R_i, S_i) \colon i \in I\}$ be a collection of pairs of subsets of $[n]$ with $R_i \cap S_i = \emptyset$ but $R_i \cap S_j \neq \emptyset$ if $i \neq j$ for all i, $j \in I$. Then

$$\sum_{i \in I} \binom{r_i + s_i}{r_i}^{-1} \leqslant 1$$

where $r_i = |R_i|$ and $s_i = |S_i|$.

Remark. Putting $S_i = [n] - R_i$ gives global LYM.

Proof 1. By induction on n; $n = 1$ is trivial. For each $x \in [n]$, let

$$I_x = \{i \in I \colon R_i \subseteq [n] \setminus x\}.$$

For each $i \in I_x$, let $S_i^x = S_i \setminus x$ and $s_i^x = |S_i^x|$. By the induction hypothesis applied to (R_i, S_i^x) with $i \in I_x$ we have $\sum_{i \in I_x} \binom{r_i + s_i^x}{r_i}^{-1} \leqslant 1$. $i \in I_x$ for $n - r_i$ values of x of which s_i of the xs have $s_i^x = s_i - 1$ and $n - r_i - s_i$ have $s_i^x = s_i$. Hence

$$\begin{aligned} n &\geqslant \sum_x \sum_{i \in I_x} \binom{r_i + s_i^x}{r_i}^{-1} \\ &= \sum_{i \in I} (n - r_i - s_i) \binom{r_i + s_i}{r_i}^{-1} + s_i \binom{r_i + s_i - 1}{r_i}^{-1} \\ &= n \sum_{i \in I} \binom{r_i + s_i}{r_i}^{-1}. \end{aligned}$$

□

Proof 2 (Lubell). Take a random ordering of $[n]$ then $\binom{r_i + s_i}{s_i}^{-1}$ is the probability that R_i precedes S_i in the ordering, and these are mutually exclusive. □

Theorem 2.2. Let H be a strongly-$(r+t)$-saturated r-uniform hypergraph of order n. Then H has as least $\binom{n}{r} - \binom{n-t}{r}$ edges.

Remark. This is tight because it is the size of $[n]^{(r)} - [n-t]^{(r)}$.

Proof. For each edge $S_i \subseteq [n]^{(r)}$ not in H, set $R_i = [n] - V(K)$ where K is some $K_{r+t}(r)$ formed by adding i. Then (R_i, S_i) satisfy the condition of Theorem 2.1, so there are at most $\binom{n-t}{r}$ *is*. □

Definition. We call H **weakly k-saturated** if the edges not in H can be added one by one, in some order, each addition edge creating a new copy of $K_k^{(r)}$.

For example, for $r = 2$ and $k = 3$ any tree will do—and is the best possible as the graph needs to be connected. As a general principle, since the minimal example is not unique, the result is likely harder.

Theorem 2.3 (Alon, 1985). Let $R_i \in [n]^{(r)}$ and $S_i \in [n]^{(s)}$ satisfying $R_i \cap S_i = \emptyset$, $R_i \cap S_j \neq \emptyset$ for $1 \leqslant i < j \leqslant h$. Then $h \leqslant \binom{r+s}{s}$.

Remark. We use the exterior algebra

$$\bigwedge V = \bigoplus_{i=0}^{r+s} \bigwedge^{i} V$$

of a real vector space V. The simplest way to think of $\bigwedge^k V$ is to take a basis $e_1, \ldots, e_d$ of V; then a basis for $\bigwedge^k V$ is $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $\{i_1, \ldots, i_n\} \in [d]^{(k)}$ on the understanding that if two indices are interchanged, the product is multiplied by -1. If two indices are equal the product is zero. Extend to every product by linearity. Then $v_1 \wedge \cdots \wedge v_k \neq 0$ if and only if $\{v_1, \ldots, v_k\}$ is linearly independent.

Proof. Let $V = \mathbb{R}^{r+s}$. Let $\{v_i \colon i \in [n]\}$ be n vectors in general position, that is, any $r+s$ are linearly independent.

For $A \in \mathcal{P}[n]$, let $v_A = \bigwedge_{x \in A} v_x$. Then $v_A \wedge v_B \neq 0$ if and only if $A \cap B = \emptyset$ and $\{v_x \colon x \in A \cup B\}$ is linearly independent.

Now $v_{R_i} \wedge v_{S_i} \neq 0$, and $v_{R_i} \wedge v_{S_j} = 0$ if $1 \leqslant i < j \leqslant h$. But then $\{v_{R_i} \colon 1 \leqslant i \leqslant h\}$ is a linearly independent subset of $\bigwedge^r V$, for if $\sum_i c_i v_{R_i} = 0$, $c_i \in \mathbb{R}$, let $j = \max\{i \colon c_i \neq 0\}$ then

$$0 = \left(\sum_i c_i v_{R_i} \right) \wedge v_{S_j} = c_j v_{R_j} \wedge v_{S_j} \neq 0,$$

contradiction. Thus $h \leqslant \dim \bigwedge^r V = \binom{r+s}{r}$. □

Theorem 2.4. Let H be weakly $(r+t)$-saturated of order n. Then H has at least $\binom{n}{r} - \binom{n-t}{r}$ edges.

Proof. Add missing edges $S_1, S_2, \ldots$ in the apposite order and we let $R_i = [n] - V(K)$ where K is created by adding S_i. Then $R_i \cap S_i = \emptyset$ and $R_i \cap S_j \neq \emptyset$ if $i < j$. The result then follows from Theorem 2.3. □

Chapter 3

Shadows

Definition. There are two important orderings on $[n]^{(r)}$:

lexicographic (lex): $A < B$ if $\min A \triangle B \in A$;

colexicographic (colex): $A < B$ if $\max A \triangle B \in B$.

Colex is "lex reversed on $[n]$ reversed".

Clearly in colex all r-sets in $[k]^{(r)}$ precede any r-sets with $k+1$. n-sets with maximal element $k+1$ are ordered by colex on $[k]^{(r-1)} + \{k+1\}$.

Example. Find the first 41 elements in $\mathbb{N}^{(4)}$.

We have $\binom{7}{4} = 35$ elements in $[7]^{(4)}$.

We have $\binom{4}{3} = 4$ elements in $[4]^{(3)} + \{8\}$.

We have $\binom{2}{2} = 1$ element in $[2]^{(2)} + \{5, 8\}$.

We have $\binom{1}{1} = 1$ element in $[1]^{(1)} + \{3, 5, 8\}$.

Observation. Each $m \in \mathbb{N}$ has a unique expression as

$$m = \binom{m_r}{r} + \binom{m_{r-1}}{r-1} + \cdots + \binom{m_s}{s}$$

where $m_r > m_{r-1} > \cdots > m_s \geqslant s \geqslant 1$.

If $\mathcal{I}$ is the initial segment of length m of colex on $[n]^{(r)}$, then

$$|\partial \mathcal{I}| = \binom{m_r}{r-1} + \binom{m_{r-1}}{r-2} + \cdots + \binom{m_s}{s-1}.$$

Observation. Let $1 \in A \in [n]^{(r)}$ and let B be the first r-set in colex after A with $1 \notin B$. Then $A - \{1\} \subseteq B$.

Proof. Consider the maximal string of successive integers beginning with 1 that lies in A, say $A = [l] \cup C$ with $l+1 \notin C$. Then the sets following A are:

$$\begin{aligned} \{1, 2, \ldots, l-1, l+1\} &\cup C, \\ \{1, 2, \ldots, l-2, l+1\} &\cup C, \\ \{1, 3, 4 \ldots, l+1\} &\cup C, \\ &\vdots \\ \{2, 3, \ldots, l+1\} &\cup C = B. \end{aligned}$$

□

Definition. Let $1 \leqslant i < j \leqslant n$. For $A \in [n]^{(r)}$ define

$$C_{ij}(A) = \begin{cases} A - \{j\} \cup \{i\} & \text{if } j \in A,\ i \notin A \\ A & \text{otherwise} \end{cases}$$

For $\mathcal{A} \subseteq [n]^{(r)}$ define

$$C_{ij}\mathcal{A} = \{C_{ij}(A) \colon A \in \mathcal{A}\} \cup \{A \colon C_{ij}(A) \in \mathcal{A}\}.$$

This is called the ij **compression** of $\mathcal{A}$, it replaces A by $C_{ij}(A)$ if it is not already in the set.

Clearly $|C_{ij}\mathcal{A}| = |\mathcal{A}|$.

Lemma 3.1. $|\partial C_{ij}\mathcal{A}| \leqslant |\partial \mathcal{A}|$.

Proof. We will show the injection $B \mapsto B \triangle \{i,j\}$ maps $\partial C_{ij}\mathcal{A} \backslash \partial \mathcal{A}$ to $\partial \mathcal{A} \backslash \partial C_{ij}\mathcal{A}$, thus proving the lemma.

Let $B \in \partial C_{ij} \backslash \partial \mathcal{A}$. Then, in particular there exists $A \in \mathcal{A}$ with $C_{ij}A \notin \mathcal{A}$ and $B \subset C_{ij}A$. Let $A = Z \cup j$, $j \notin Z$. We have $C_{ij}A = Z \cup \{i\}$.

$Z \in ([n] - i, j)^{(r-1)}$. Now $B \neq Z$ else $B \subset A$, so $B \in \partial \mathcal{A}$. Thus $B = Z - \{k\} \cup \{i\}$ for some $k \neq i$, j. Hence

$$B \mapsto B \triangle \{i,j\} = Z - \{k\} \cup \{j\} = B',$$

say. Then $B' \subset A$, so $B' \subset \partial \mathcal{A}$ as claimed.

To complete the proof, we show $B' \notin \partial C_{ij}\mathcal{A}$. Suppose to the contrary that $B' \subset A' \in C_{ij}\mathcal{A}$. Then $A' = Z - k \cup \{j, l\}$ for some l. If $l = i$, then $A' \in C_{ij}\mathcal{A}$, so $A' \in \mathcal{A}$, whence $B = A' - j \in \partial \mathcal{A}$, a contradiction.

If $l \neq i$, then $A' = z - k \cup \{j, l\} \in C_{ij}\mathcal{A}$ meaning $C_{ij}A' = Z - k \cup \{i, l\} \in \mathcal{A}$, so once again $B \subset C_{ij}A'$, hence $B \in \partial \mathcal{A}$, the final contradiction. $\square$

Same kind of argument could show $\partial C_{ij}\mathcal{A} \subseteq C_{ij}\partial \mathcal{A}$, giving another proof.

Definition. $\mathcal{A} \subseteq [n]^{(r)}$ is ij-COMPRESSED if $C_{ij}\mathcal{A} = \mathcal{A}$.

$\mathcal{A}$ is LEFT-COMPRESSED if it is ij-compressed for all $1 \leqslant i < j \leqslant n$.

Corollary 3.2. Given $\mathcal{A} \subseteq [n]^{(r)}$ there exists $\mathcal{B} \subseteq [n]^{(r)}$ with

(i) $|\mathcal{B}| = |\mathcal{A}|$.

(ii) $|\partial \mathcal{B}| \leqslant |\partial \mathcal{A}|$.

(iii) $\mathcal{B}$ is left-compressed.

Proof. If $\mathcal{A}$ is not ij-compressed, replace $\mathcal{A}$ by $\mathcal{A}' = C_{ij}\mathcal{A}$. By Lemma 3.1

$$|\partial \mathcal{A}'| \leqslant |\partial \mathcal{A}| .$$

Moreover

$$\sum_{A \in \mathcal{A}'} \sum_{k \in A} k < \sum_{A \in \mathcal{A}} \sum_{k \in A} k.$$

So after finite number of such moves, have a left-compressed family $\mathcal{B}$. $\square$

Initial segments of colex are left-compressed, but there are other left-compressed families. For example, initial segments of lex.

Theorem 3.3 (Kruskal-Katona, 1960, 1968). Let $\mathcal{A} \subseteq [n]^{(r)}$, $|\mathcal{A}| = m$ and let $\mathcal{I}$ be the initial segment of length m in colex order. Then $|\partial\mathcal{A}| \geqslant |\partial\mathcal{I}|$. Explicitly, if

$$|\mathcal{A}| = m = \binom{m_1}{r} + \binom{m_2}{r-1} + \cdots + \binom{m_s}{s}$$

where $m_r > m_{r-1} > \cdots > m_s \geqslant s \geqslant 1$. Then

$$|\partial\mathcal{A}| \geqslant \binom{m_r}{r-1} + \binom{m_{r-1}}{r-2} + \cdots + \binom{m_s}{s-1}.$$

Proof. Proceed by induction on $m+r$. By Corollary 3.2, we may assume $\mathcal{A}$ is $1j$-compressed for all $j \geqslant 2$. Let $\mathcal{I}$ be the initial segment of colex of length $|\mathcal{A}|$. Let

$$\begin{aligned}
\mathcal{A}_1 &= \{A - \{1\} \colon 1 \in A \in \mathcal{A}\} \\
\mathcal{A}_2 &= \{A \in \mathcal{A} \colon 1 \notin A\} \\
\mathcal{I}_1 &= \{A - \{1\} \colon 1 \in A \in \mathcal{I}\} \\
\mathcal{I}_2 &= \{A \in \mathcal{I} \colon 1 \notin A\}.
\end{aligned}$$

Note that $\mathcal{I}_1$ is an initial segment of colex of $\{2, 3, 4, \dots\}^{(r-1)}$ and also $\mathcal{I}_2$ is an initial segment of colex of $\{2, 3, 4, \dots\}^{(r)}$.

Also

$$\begin{aligned}
|\mathcal{A}| &= |\mathcal{A}_1| + |\mathcal{A}_2| \\
|\mathcal{I}| &= |\mathcal{I}_1| + |\mathcal{I}_2|
\end{aligned}$$

Observe that, since $\mathcal{A}$ is $1j$-compressed for all $j \geqslant 2$ we have $\partial\mathcal{A}_2 \subseteq \mathcal{A}_1$.

Hence $\partial\mathcal{A} = \mathcal{A}_1 \cup (\partial\mathcal{A}_1 + \{1\})$ where union is disjoint. Thus

$$\begin{aligned}
|\partial\mathcal{A}| &= |\mathcal{A}_1| + |\partial\mathcal{A}_1| \\
|\partial\mathcal{I}| &= |\mathcal{I}_1| + |\partial\mathcal{I}_1|\,.
\end{aligned}$$

Hence if $|\mathcal{A}_1| \geqslant |\mathcal{I}_1|$ then $|\partial\mathcal{A}_1| \geqslant |\mathcal{I}_1|$ by induction and $|\mathcal{A}| \geqslant |\partial\mathcal{I}|$ follows from the previous equation, and we are done.

But if $|\mathcal{A}_1| < |\mathcal{I}_1|$, then we have $|\mathcal{A}_2| > |\mathcal{I}_2|$, so $|\mathcal{A}_2| \geqslant \left|\mathcal{I}_2^+\right|$ where $\mathcal{I}_2^+$ is $\mathcal{I}_2$ is $\mathcal{I}_2$ together with the next element after $\mathcal{I}_2$ in colex on $\{2, 3, 4, \dots\}^{(r)}$.

By the observation, $\mathcal{I}_1 \subseteq \partial\mathcal{I}_2^+$. But recalling $\partial\mathcal{A}_2 \subseteq \partial\mathcal{A}_1$, we have by induction $|\mathcal{A}_1| \geqslant |\partial\mathcal{A}_2| \geqslant \left|\partial\mathcal{I}_2^+\right| \geqslant |\mathcal{I}_1|$. This is a contradiction. □

Theorem 3.4 (Weak but useful Kruskal-Katona). Let $\mathcal{A} \subseteq [n]^{(r)}$, $|\mathcal{A}| = \binom{x}{r}$ where $x \in \mathbb{R}$, $x > r-1$. Then $|\partial\mathcal{A}| \geqslant \binom{x}{r-1}$.

Proof 1. By Theorem 3.3. □

Proof 2. Since $\binom{x}{r}$ is increasing for $x > r-1$ and $\mathcal{A} \geqslant 1$. So $x \geqslant r$. Moreover if $x = r$ then the theorem is trivial so we may assume $x > r$.

Now proceed as in the previous proof, we may assume $\mathcal{A}$ is $1j$-compressed, so $\partial\mathcal{A}_2 \subseteq \mathcal{A}_1$ and $|\mathcal{A}| = |\mathcal{A}_1| + |\partial\mathcal{A}_1|$.

If $|\mathcal{A}_1| \geqslant \binom{x-1}{r-1}$ then $|\partial\mathcal{A}_1| \geqslant \binom{x-1}{r-2}$ by induction, so

$$|\partial\mathcal{A}| \geqslant \binom{x-1}{r-1} + \binom{x-1}{r-2} = \binom{x}{r-1}$$

as claimed. But if $|\mathcal{A}_2| = |\mathcal{A}| - |\mathcal{A}_1| > \binom{x-1}{r}$, so (by induction, since $x-1 > r-1$), $|\partial\mathcal{A}_2| \geqslant \binom{x-1}{r-1} > |\mathcal{A}_1|$. □

Note. For example, though lex is left-compressed, we could move 125 to 234 by $C_{34,15}$.

Definition. Let U, $V \in \mathcal{P}[n]$, $U \cap V = \emptyset$. For $A \in \mathcal{P}[n]$ let

$$C_{U,V}(A) = \begin{cases} A \cup U - V & \text{if } V \subseteq A,\ U \cap A = \emptyset \\ A & \text{otherwise} \end{cases}$$

For $\mathcal{A} \subseteq \mathcal{P}[n]$ let

$$C_{U,V}\mathcal{A} = \{C_{U,V}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon C_{U,V}(A) \in \mathcal{A}\}.$$

Notes. (i) $C_{ij}\mathcal{A} = C_{\{i\},\{j\}}\mathcal{A}$.

(ii) $|C_{U,V}\mathcal{A}| = |\mathcal{A}|$.

(iii) If $\mathcal{A} \subseteq [n]^{(r)}$ and $|U| = |V|$ then $C_{U,V}\mathcal{A} \subseteq [n]^{(r)}$.

The bad news is that $|\partial C_{U,V}\mathcal{A}| > |\partial\mathcal{A}|$ is possible for U, V and $\mathcal{A}$. There is some good news as well.

Lemma 3.5 (Bollobás, Leader (1987)). Let $\mathcal{A} \subseteq [n]^{(r)}$, $U \cap V = \emptyset$ and $|U| = |V|$. Suppose that for all $u \in U$ there exists $v \in V$ such that $C_{U-\{u\},v-\{v\}}\mathcal{A} = \mathcal{A}$. Then $|\partial C_{u,v}\mathcal{A}| \leqslant |\partial\mathcal{A}|$.

Proof. We show the bijection $\mathcal{P}[n] \to \mathcal{P}[n]$ given by $Y \mapsto Y \triangle (U \cup V)$ injects $\partial\mathcal{A}' \backslash \partial\mathcal{A} \to \partial\mathcal{A} \to \partial\mathcal{A}'$, where $'$ denotes $C_{U,V}$.

GIven $B \in \partial\mathcal{A}' \backslash \mathcal{A}$, we have $B \cup \{x\} \in \mathcal{A}' \backslash \mathcal{A}$ some x. Hence $U \subseteq B \cup \{x\}$ and $(B \cup \{x\}) \cap Y = \emptyset$ and $((B \cup \{x\}) \backslash u) \cup V \in \mathcal{A}$.

If $x \in U$ then $C_{U-\{x\},V-\{v\}}((B \cup \{x\}) \backslash U) \cup V = B \cup \{v\}\alpha$ for some $v \in V$ by the assumption.

This implies $B \in \partial\mathcal{A}$, a contradiction. Hence $x \notin U$, so $B \triangle (U \cup V) = (B \backslash U) \cup V \in \partial\mathcal{A}$.

Suppose now that $B \triangle (U \cup V) \in \partial\mathcal{A}'$. Hence for some y, $(B \backslash U) \cup V \cup \{y\} \in \mathcal{A}'$. If $y \in U$ then by the assumption, there is a $v \in V$ with $(B \backslash U) \cup V \cup \{y\} \in \mathcal{A}$ such that $G_{U-\{y\},V-\{v\}}(B \backslash U) \cup V \cup \{y\} = B \cup \{v\} \in \mathcal{A}$. So $B \in \partial\mathcal{A}$, a contradiction. Thus $y \notin U$. Now both $(B \backslash U) \cup V \cup \{y\}$ and $C_{U,V}B \cup \{y\} \in \mathcal{A}'$ and hence both in $\mathcal{A}$. But then $B \in \partial\mathcal{A}$, the final contradiction. $\square$

Definition. We say $\mathcal{A}$ is (U,V)-**compressed** if $C_{U,V}\mathcal{A} = \mathcal{A}$.

$$\Gamma = \{(U,V) \in \mathcal{P}[n] \times \mathcal{P}[n] \colon\ |U| = |V|\,,\ U \cap V = \emptyset, \\ \text{and } U = V = \emptyset \text{ or } \max U < \max V\}.$$

Lemma 3.6. $\mathcal{A}$ is an initial segment of colex on $[n]^r$ if and only if $C_{U,V}\mathcal{A} = \mathcal{A}$ for all $(U,V) \in \Gamma$.

Proof. If $\mathcal{A}$ is not an initial segment, pick $A' < A$ with $A' \notin \mathcal{A}$, $A \in \mathcal{A}$. Let $U = A' \backslash A$, $V = A \backslash A'$, then $(U, V) \in \Gamma$ and $C_{U,V} = A' = A$ we have that $C_{U,V} A' = A$, so $C_{U,V}\mathcal{A} \neq \mathcal{A}$: a contradiction.

Conversely, if $C_{U,V}\mathcal{A} \neq \mathcal{A}$, some $(U, V) \in \Gamma$, pick $A \in \mathcal{A}$ with $C_{U,V} A \notin \mathcal{A}$. Then $\max(A \triangle C_{U,V} A) = \max(U \cup V) \in V$, so $C_{U,V} A < A$ in colex, and $\mathcal{A}$ is not an initial segment. $\square$

Proof of Kruskal-Katona Theorem. If $\mathcal{A} \neq \mathcal{I}$, by Lemma 3.6 there exists $(U, V) \in \Gamma$ with $C_{U,V}\mathcal{A} \neq \mathcal{A}$. Pick such (U, V) with $|U| = |V|$ minimal. Thus the condition of Lemma 3.5 holds; for let $v \in \min V$, then for all $u \in U$, $(U - \{u\}, V - \{v\}) \in \Gamma$. Replace $\mathcal{A}$ by $\mathcal{A}' = C_{U,V}\mathcal{A}$; by the lemma $|\partial \mathcal{A}'| \leqslant |\partial \mathcal{A}|$. Since the members of $\mathcal{A}'$ are to the left of $\mathcal{A}$ in colex, with a finite number of steps we reach $\mathcal{I}$. $\square$

Definition. The UPPER SHADOW of $\mathcal{A}$ is

$$\partial^+ \mathcal{A} = \{B \in [n]^{(r+1)} \colon B \supset A \text{ some } A \in \mathcal{A}\}.$$

Corollary 3.7. Let $\mathcal{A} \subseteq [n]^{(r)}$. Let $\mathcal{J}$ be an initial segment of lex of length $|\mathcal{A}|$. Then $|\partial^+ \mathcal{A}| \geqslant |\partial^+ \mathcal{J}|$.

Proof. This follows from the relationship between colex and lex, being "colex reverses on $[n]$ reversed". Now observe if we write $\overline{\mathcal{A}} = \{[n] - A \colon A \in \mathcal{A}\} \subseteq [n]^{(n-r)}$, then $\partial^+ \mathcal{A} = \overline{\partial \overline{\mathcal{A}}}$. $\square$

Chapter 4

Intersecting Systems

Definition. $\mathcal{A} \subseteq \mathcal{P}[n]$ is **INTERSECTING** if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$.

Proposition 4.1. If $\mathcal{A} \subseteq \mathcal{P}[n]$ is intersecting, then $|\mathcal{A}| \leqslant 2^{n-1}$.

Proof. $\mathcal{A}$ can contain at most one of each pair A, $[n] - A$. $\square$

Theorem 4.2 (Erdős-Ko-Rado, 1938, 1961). Let $\mathcal{A} \subseteq [n]^{(r)}$ be intersecting, $r \leqslant n/2$, then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}$.

Proof (Daykin, 70s, Katona, 1964). Let $\overline{\mathcal{A}} = \{[n] - A \colon A \in \mathcal{A}\} \subseteq [n]^{(n-r)}$. Then $\mathcal{A}$ and $\partial^{n-2r}\overline{\mathcal{A}}$ are disjoint. Now $\left|\overline{\mathcal{A}}\right| = |\mathcal{A}|$. If $|A| > \binom{n-1}{r-1}$ then $\left|\overline{\mathcal{A}}\right| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$.

So by Kruskal-Katona, $\left|\partial^{n-2r}\overline{\mathcal{A}}\right| \geqslant \binom{n-1}{r}$ and then

$$\left|\overline{\mathcal{A}}\right| + \left|\partial^{n-2r}\overline{\mathcal{A}}\right| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

which is a contradiction. $\square$

Proof (Katona, 1968). Consider all $n!$ cyclic orderings of $[n]$. Given $A \in \mathcal{A}$, it appears as an arc in $n \cdot r! \cdot (n-r)!$ orderings (there are n possible arcs). On the other hand, given a cyclic ordering at most r arcs

can correspond to elements of $\mathcal{A}$ (if $c_1c_2\dots c_r \in \mathcal{A}$, then at most one of $\dots c_i$ and $c_{i+1}\dots$ arcs can be in $\mathcal{A}$).
Hence $n \cdot r! \cdot (n-r)!\,|\mathcal{A}| \leqslant r \cdot n!$. □

Remark. Both proofs can be inspected to show equality holds for $r < n/2$ only if there exists $x \in [n]$ such that $x \in A$ for all $A \in \mathcal{A}$.

Definition. $\mathcal{A}$ is t-**INTERSECTING** if $|A \cap B| \geqslant t$ for all A, $B \in \mathcal{A}$.

Lemma 4.3. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be t-intersecting. Then $C_{U,V}\mathcal{A}$ is t-intersecting provided $|U| \geqslant |V|$ and

(i) $C_{U,V-\{v\}}\mathcal{A} = \mathcal{A}$ for all $v \in V$,

(ii) For all $u \in U$, there exists $v \in V$ such that $C_{U-\{u\},V-\{v\}}\mathcal{A} = \mathcal{A}$.

Proof. Suppose not. Then there exists A, $B \in C_{U,V}\mathcal{A}$ with $|A \cap B| < t$. Clearly not both A, $B \in \mathcal{A}$, so we suppose $A = C_{U,V}A'$ where $A' \in \mathcal{A}$ and $A \notin \mathcal{A}$. Hence $A = (A' - V) \cup U$. If $B \notin \mathcal{A}$, then $B = C_{U,V}B'$, $B' \in \mathcal{A}$, $B' \neq B$. So $B = (B' - V) \cup U$. Thus

$$\begin{aligned} |A \cap B| &= |[(A' - V) \cup U] \cap [(B' - V) \cup U]| \\ &= |A' \cap B'| - |V| + |U| \geqslant |A' \cap B'| \\ &\geqslant t, \end{aligned}$$

a contradiction. Thus $B \in \mathcal{A}$. Now if $C_{U,V}B \neq B$, then $C_{U,V}B \in \mathcal{A}$ since $B \in \mathcal{A}$. Then

$$\begin{aligned} |A \cap B| &= |[(A' - V) \cup U] \cap B| \\ &= |A' \cap (B - V \cup U)| \\ &= |A' \cap C_{U,V}B| \geqslant t, \end{aligned}$$

a contradiction. Hence $C_{U,V}B = B$, meaning either $V \not\subseteq B$ or $V \subseteq B$ but $U \cap B \neq \emptyset$. If $V \not\subseteq B$, let $v \in V \backslash B$. Now by the conditions, $C_{U,V-\{v\}}\mathcal{A} = \mathcal{A}$, but then

$$\begin{aligned} |A \cap B| &= |[(A' - (V - \{u\})) \cup U] \cap B| \\ &= \left|C_{U,V-\{v\}}A' \cap B\right| \geqslant t \end{aligned}$$

because $C_{U,V-\{v\}}A'$, $B \in \mathcal{A}$. So $v \subseteq B$, so $U \cap B \neq \emptyset$. Pick $u \in U \cap B$. By the second condition, there is a $v \in V$ such that $C_{U-\{u\},V-\{v\}}\mathcal{A} = \mathcal{A}$, then

$$|A \cap B| = |(A' - (V - \{v\}) \cup (U - \{u\})) \cap B| \geqslant t$$

because u, $v \in B$, because $C_{U-\{u\},V-\{v\}}A'$, $B \in \mathcal{A}$, a contradiction. $\square$

Theorem 4.4 (Katona). Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be t-intersecting. Then $|\mathcal{A}| \leqslant \left|[n]^{(\geqslant k)}\right| = \sum_{i=k}^{n} \binom{n}{i}$ if $n + t = 2k$, and if $n + t = 2k + 1$ have $|\mathcal{A}| \leqslant \left|[n]^{(>k)} + [n-1]^{(k)}\right| = \binom{n-1}{k} + \sum_{i=k+1}^{n} \binom{n}{i}$.

Proof. Order all pairs (U, V) with U, $V \subseteq [n]$, $U \cap V = \emptyset$, $|U| > |V|$, in order of increasing V. Repeatedly apply $C_{U,V}$ with the least (U, V) in this order such that $C_{U,V}\mathcal{A} = \mathcal{A}$, as big as possible.

If $V = \emptyset$, then $C_{U,V}\mathcal{A}$ is certainly t-intersecting, and if $V \neq \emptyset$, then $C_{U,V}\mathcal{A}$ is t-intersecting by Lemma 4.3. Since $\sum_{A \in \mathcal{A}} |A|$ increases each time, the process stops when $C_{U,V}\mathcal{A} = \mathcal{A}$ for all (U, V).

Let $r = \min\{j \colon \mathcal{A} \cap [n]^{(j)} \neq \emptyset\}$. Then $[n]^{(j)} \subseteq \mathcal{A}$ for all $j \geqslant r$ else pick $A \in [n]^{(r)}$, $B \in [n]^{(j)}$ with $A \in \mathcal{A}$ and $B \notin \mathcal{A}$. Put $U = B \backslash A$, $V = A \backslash B$. $C_{U,V}A = B$, $C_{U,V}\mathcal{A} \neq \mathcal{A}$.

Pick $A \in [n]^{(r)} \cap \mathcal{A}$ pick $B \in [n]^{(r+1)} \subseteq \mathcal{A}$ such that $A \cap B = r + (r+1) - n$. Thus $r + (r+1) - n \geqslant t$. Hence $r \geqslant k$. If $n + 2t = 2k$, then $\mathcal{A} \subseteq [n]^{(\geqslant k)}$ which is t-intersecting so done.

If $n+2t = 2k+1$, then put $\mathcal{A}_k = [n]^{(k)} \cap \mathcal{A}$ then $\mathcal{A} = [n]^{(>k)} \cup \mathcal{A}_k$ which is t intersecting if and only if $\mathcal{A}_k$ is t-intersecting which is true if and only if $|A \cup B| \neq [n]$ for A, $B \in \mathcal{O}_k$, this holds if and only if $\{[n] - A \colon A \in \mathcal{A}_k\}$ is intersecting. By Erdős-Ko-Rado, $|\mathcal{A}_k| \leqslant [n-1]^{(k)}$. $\square$

Theorem 4.5. Let $1 \leqslant t \leqslant r$. Let $\mathcal{A} \subseteq [n]^{(r)}$ be t-intersecting. If n is large ($n \geqslant (16r)^r$ will do) then $|\mathcal{A}| \leqslant \binom{n-t}{r-t}$

Proof. Without loss of generality $t < r$ and $\mathcal{A}$ is maximal t-intersecting. So we can find $|A \cap B| = t$. If $\mathcal{A} = \{Y \in [n]^{(r)} \colon A \cap B \subseteq Y\}$ then we are done. Hence there exists $C \in \mathcal{A}$ such that $A \cap B \nsubseteq C$.

Then for each $D \in \mathcal{A}$, we must have $|D \cap (A \cup B \cup C)| \geqslant t+1$. So

$$|\mathcal{A}| \leqslant 2^{|A \cup B \cup C|} \left\{ \binom{n-t}{r-t-1} + \binom{n-t}{r-t-2} + \cdots + \binom{n-t}{0} \right\}$$

which is is $\mathcal{O}(n^{r-t-1})$ if n large, that is, $< \binom{n-t}{r-t}$. □

Theorem 4.5 can fail if n is not big. Define

$$\mathcal{F}_i = \{A \in [n]^{(r)} \colon\ |A \cap [t+2i]| \geqslant t+i\}$$

for $0 \leqslant i \leqslant r-t$. Then $\mathcal{F}_i$ is t-intersecting. Theorem 4.5 says $\mathcal{F}_0$ is best for large n. Let $M(n,r,t) = \max\{|\mathcal{A}| : \mathcal{A} \subseteq [n]^{(r)},\ \mathcal{A}\ t\text{-intersecting}\}$. So we aim to prove $M(n,r,t) = \max_i |\mathcal{F}_i|$ combinatorially.

By comparing the sums of $\mathcal{F}_{i+1} \backslash \mathcal{F}_i$ and $\mathcal{F}_i \backslash \mathcal{F}_{i+1}$ we see $\max |F_i| = |F_j|$ where

$$(r-t+1)\left(2 + \frac{t-1}{j+1}\right) < n < (r-t+1)\left(2 + \frac{t-1}{j-1}\right).$$

Where the upper bound is infinity when $j = 0$. If $n = (r-t+1)(2 + (t-1)/(j+1))$ for some j then $|\mathcal{F}_j| = |\mathcal{F}_{j+1}|$. However the $\mathcal{F}_i$ are never isomorphic if $n \geqslant 2r-t$, since $t+2i \leqslant n$ (we can identify the interval $t+2i$ elements in the set uniquely).

Theorem 4.6 (Wilson's Theorem). *$M(n,r,t) = |\mathcal{F}_0|$ if $n > (r-t+1)(t+1)$.*

Theorem 4.7. *$M(n,r,t) = \max |\mathcal{F}_i|$. Moreover the only cases of equality are isomorphic to some $\mathcal{F}_i$.*

Remark. We may assume $n \geqslant 2r-t+1$ since for $n \leqslant 2r-t$, $[n]^{(r)} = \mathcal{F}_t$ is t-intersecting.

Ingeniously use two ideas: generating sets and complementary families.

Definition. The set $\mathcal{G} \subseteq \mathcal{P}[n]$ **generates** $\mathcal{A} \subset [n]^{(r)}$ if $\mathcal{A}$ comprises all r-sets that contain some member of $\mathcal{G}$.

Note. $\mathcal{A}$ generates $\mathcal{A}$.

Notice crucially that, since $n \geqslant 2r - t + 1$, then $\mathcal{A}$ is t-intersecting if and only if $\mathcal{G}$ is, for if G_1, $G_2 \in \mathcal{G}$ with $|G_1 \cap G_2| \leqslant t - 1$ then we can find A_1, $A_2 \in [n]^{(r)}$, $A_i \supseteq G_i$ and $|A_1 \cap A_2| \leqslant t - 1$.

Assume $\mathcal{A}$ is t-intersecting and $|\mathcal{A}| = M(n, r, t)$.

We write $A \leqslant B$ if $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ with $a_1 < a_2 < \cdots < a_r$, $b_1 < b_2 < \cdots < b_r$, $a_i \leqslant b_i$, $1 \leqslant i \leqslant r$. Notice A can be obtained from B by a sequence of compressions C_{ij}, $i < j$.

By Lemma 4.3, we may assume $C_{ij}\mathcal{A} = \mathcal{A}$ for all $i < j$. In particular if $B \in \mathcal{A}$ and $A \leqslant B$ then $A \in \mathcal{A}$. The bulk of the proof lies in the next lemma.

Lemma 4.8. Let $\mathcal{A} \subseteq [n]^{(r)}$ be as described above. Let $n > (r - t + 1)(2 + (t - 1)/(k + 1))$ for some $k \in \mathbb{N} \cup \{0\}$. Then $\mathcal{A}$ is generated by some $\mathcal{G} \subseteq \mathcal{P}[t + 2k]$.

Proof. Note $n > 2r - 2t + 2 + (r - t + 1)(t - 1)/(k + 1)$. If $n \leqslant 2r - t + 1$, then $k + 1 > r - t + 1$, then since integers $k \geqslant r - t + 1$, so $t + 2k \geqslant 2r - t + 2 > n$. So result is trivial. So we may assume $n \geqslant 2r - t + 2$.

Given generators $\mathcal{G}$ of $\mathcal{A}$, let $m(\mathcal{G}) = \min\{m \colon \mathcal{G} \subseteq \mathcal{P}[m]\}$. Choose $\mathcal{G}$ with $m(\mathcal{G})$ minimal. Let $\mathcal{H} = \{H \colon H \leqslant G \text{ for some } G \in \mathcal{G}\}$. Then $\mathcal{H}$ also generates $\mathcal{A}$. Now let

$$\mathcal{J} = \{H \in \mathcal{H} \colon H' \in \mathcal{H} \text{ and } H' \subseteq H \text{ then } H' = H\},$$

that is $\mathcal{J}$ is subfamily of $\mathcal{H}$ minimal with respect to inclusion. Then $\mathcal{J}$ generates $\mathcal{A}$ and $m\mathcal{J} \leqslant m\mathcal{G}$. Since $n \geqslant 2r - t + 1$, $\mathcal{J}$ is t-intersecting. Moreover if $J \in \mathcal{J}$ and $K \leqslant J$ then K contains K' for some $K' \in \mathcal{J}$.

(i) For each $A \in \mathcal{A}$ there is a unique inital segment J of A with $J \in \mathcal{J}$ (meaning $J = a_1 < a_2 < \cdots < a_j$, where $A = a_1 < a_2 < \cdots < a_r$) just take left-most generator of A in $\mathcal{J}$.

(ii) If $m = m(\mathcal{J})$ and $m \in J$, then A contains no other member of $\mathcal{J}$ (else this would be a subset of J).

(iii) If P, $Q \in \mathcal{J}$, $m \in P \cap Q$ and $|P \cap Q| = t$, then $P \cup Q = [m]$, that is, $|P| + |Q| = m + t$ (else $\mathcal{J}$ would contain a subset $P' = C_{im}P$, say, and $|P' \cap Q| < T$).

For $0 \leqslant i \leqslant m$, let $\mathcal{J}_i = \{J \in \mathcal{J}\colon\ |J| - i, m \in J\}$. By the minimality of m, $\mathcal{J}_p \neq \emptyset$ for some p. Let $\mathcal{J}'_p = \{J - m\colon J \in \mathcal{J}_p\}$. Let $P' \in \mathcal{J}'$. By the third condition, $|P' \cap J| \geqslant t$ for $J \in \mathcal{J}$ except perhaps for $J \in \mathcal{J}_q$ where $p + q = m + t$.

Suppose first $p \neq q$, that is $p \neq (m+t)/2$. Then $(\mathcal{J} - (\mathcal{J}_p \cup \mathcal{J}_q)) \cup \mathcal{J}'_p$ is t-intersecting so generates a t-intersecting family $\mathcal{B} \subseteq [n]^{(r)}$.

If $A \in \mathcal{A} \backslash \mathcal{B}$ then A is generated by $Q \in \mathcal{J}_q$ so by the second note A is generated uniquely by Q, so

$$|\mathcal{A}\backslash\mathcal{B}| = |\mathcal{J}_q| \binom{n-m}{r-q}.$$

On the other hand, if $P' \in \mathcal{J}'_p$ then $\mathcal{B}\backslash\mathcal{A}$ contains $P' \cup [m+1, n]^{(r-p+1)}$ since an element of $\mathcal{A}$ in this family would have to be generated by a subset of P', contradicting $P \in \mathcal{J}$ and minimality of $\mathcal{J}$. So

$$|\mathcal{B}\backslash\mathcal{A}| \geqslant \left|\mathcal{J}'_p\right| \binom{n-m}{r-p+1}$$

Now $|\mathcal{B}| \leqslant |\mathcal{A}| = M(n, r, t)$ and $\left|\mathcal{J}'_p\right| = |\mathcal{J}_p|$. So

$$|\mathcal{J}_p| \binom{n-m}{r-p+1} \leqslant |f_q| \binom{n-m}{r-q}.$$

Now $|\mathcal{J}_p| \neq 0$ and, since $p \leqslant r$, $n - m \geqslant 2r - t + 2 - m \geqslant 2r - q - p + 2 \geqslant r - p + 2$. So the left-hand side is not zero, so $|\mathcal{J}_q| \neq 0$, so we can repeat above argument with p replaced by q, and q replaced by $m + t - q = p$, to get

$$|\mathcal{J}_q| \binom{n-m}{r-q+1} \leqslant |\mathcal{J}_p| \binom{n-m}{r-p}.$$

Therefore

$$\frac{n-m-r+p}{r-p+1} \cdot \frac{n-m-r+q}{r-q+1} \leqslant 1$$

and $m = p + q$, so

$$\frac{n-q-r+t}{r-p+1} \cdot \frac{n-p-r+1}{r-q+1} \leqslant 1.$$

But $n \geqslant 2r - t + 2$, so $n - p - r + t \geqslant r - p + 2$, $n - q - r + t \geqslant r - q + 2$, a contradiction.

Therefore $p = q = (m+t)/2$. Now there must be $x \in [m]$ contained in at most $\frac{p-1}{m-1}|\mathcal{J}_p|$ member of $\mathcal{J}_p$. Let $\mathcal{K} = \{P \in \mathcal{J}_p \colon x \notin P\}$. $\mathcal{K}' = \{P - \{m\} \colon P \in \mathcal{K}\}$. Then the third condition gives $(\mathcal{J} - \mathcal{J}_p) \cup \mathcal{K}'$ is t-intersecting.

Let this family generate $\mathcal{C} \subseteq [n]^{(r)}$. If $A \in \mathcal{A} \backslash \mathcal{C}$, then A is generated uniquely by some $J \in \mathcal{J}_p - \mathcal{K}$. So $|\mathcal{A} \backslash \mathcal{C}| = |\mathcal{J}_p - \mathcal{K}| \binom{n-m}{r-p}$. On the other hand, each $P' \in \mathcal{K}'$ generates $P' \cup [m+1, n]^{(r-p+1)}$ in $\mathcal{C} \backslash \mathcal{A}$, so $|\mathcal{C} \backslash \mathcal{A}| \geqslant |\mathcal{K}'| \binom{n-m}{r-p+1}$. Since $\mathcal{C} \leqslant \mathcal{A}$ we have

$$\begin{aligned}\left(1 - \frac{p-1}{m+1}\right)|\mathcal{J}_p|\binom{n-m}{r-p+1} &\leqslant |\mathcal{K}'|\binom{n-m}{r-p+1} \\ &\leqslant |\mathcal{J}_p - \mathcal{K}|\binom{n-m}{r-p} \\ &\leqslant |\mathcal{J}_p|\frac{p-1}{m-1}\binom{n-m}{r-p}.\end{aligned}$$

So $\frac{m-p}{m-1}\frac{n-m-r+p}{r-p+1} \leqslant \frac{p-1}{m-1}$ so $(m-p)(n-m-r+p) \leqslant (p-1)(r-p+1)$. Writing $p = t+u$, so $m = t+2u$, $u \in \mathbb{Z}$, rearrange $n \leqslant (r-t+1)(2+(t-1)/u)$. Hence $u \leqslant k$. $\square$

Note this lemma already gives Theorem 4.6

The completement of $\mathcal{A} \subseteq [n]^{(r)}$ is $\overline{\mathcal{A}} = \{[n] - A \colon A \in \mathcal{A}\} \subseteq [n]^{(n-r)}$. Observe that $\mathcal{A}$ is t-intersecting if and only if $\overline{\mathcal{A}}$ is $(n-2r+t)$-intersecting. So $|\mathcal{A}| = M(n,r,t)$ if and only if $\left|\overline{\mathcal{A}}\right| = M(n, n-r, n-2r+t)$. Moreover $C_{ij}\mathcal{A} = \mathcal{A}$ for all $i < j$ if and only if $C_{ji}\overline{\mathcal{A}} = \overline{\mathcal{A}}$ for all $j > i$.

Lemma 4.9. Let $\mathcal{A} \subseteq [n]^{(r)}$ be t-intersecting. Let $\mathcal{G}$ generate $\mathcal{A}$ and $\mathcal{H}$ generates $\overline{\mathcal{A}}$. If $n > 2r - t$ then $|G \cup H| \geqslant n - r + t$ for all $G \in \mathcal{G}$ and $H \in \mathcal{H}$.

Proof. For any A, $B \in [n]^{(r)}$ with $G \subseteq A$ and $B \cap H = \emptyset$, then A, $B \in \mathcal{A}$. If $|G \cup H| \leqslant n - r + t - 1$, pick $r - t + 1$ elements of B in $\overline{G \cup H}$, and the rest from $\overline{H}$. Pick A containing G but none of the $r - t + 1$. Then $|A \cap B| \leqslant t - 1$. $\square$

Proof of Theorem 4.7. Pick $k \in \mathbb{N} \cup \{0\}$ so that

$$(r-t+1)\left(2+\frac{t-1}{k+1}\right) \leqslant n < \left(r-t+1)(2+\frac{t-1}{k}\right).$$

Let $r' = n-r$, $t' = n-2r+t$. Note $r'-t' = r-t$. Let $k' = r-t-k$. Since $0 \leqslant k \leqslant r-t$ we have $0 \leqslant k' \leqslant r'-t'$. Moreover, rearranging gives

$$(r'-t'+1)\left(2+\frac{t'-1}{k'+1}\right) < n \leqslant \left(r'-t'+1)(2+\frac{t'-1}{k'}\right).$$

Suppose first the inequality is strict. By Lemma 4.8, $\mathcal{A}$ is generated by $\mathcal{G} \subseteq \mathcal{P}[t+2k]$. Likewise (remember $\overline{\mathcal{A}}$ is right-compressed), $\overline{\mathcal{A}}$ is generated by $\mathcal{H} \subseteq \mathcal{P}[n-t'-2k'+1, n]$ (the right hand $t'+2k'$ elements).

Note $n-t'-2k' = t+2k$. By Lemma 4.9, $|G \cup H| \geqslant n-r+t = (t+k)+(t'+k')$ for all $G \in \mathcal{G}$, $H \in \mathcal{H}$. Thus either $|G| \geqslant t+k$ for all $G \in \mathcal{G}$ or $|H| \geqslant t'+k'$ for all $H \in \mathcal{H}$. In the first case $\mathcal{A} \subseteq \mathcal{F}_k$ so we are done, and in second case $\overline{\mathcal{A}} \subseteq \mathcal{F}_k$ so $\mathcal{A} \subseteq \mathcal{F}_k$.

If equality holds, we can, by Lemma 4.8, say $\mathcal{G} \subseteq \mathcal{P}[t=2k+2]$. $\mathcal{H} \subseteq$ as before. If $|G| \geqslant t+k+2$ for all $G \in \mathcal{G}$ then $\mathcal{A} = \mathcal{F}_{k+1}$. If not, that is, $|G| \leqslant t+k$ for some G, then $|H| \geqslant t'+k'$ for all $H \in \mathcal{H}$, so as before $\overline{\mathcal{A}} = \overline{\mathcal{F}}_k$ so $\mathcal{A} = \mathcal{F}_k$. □

Chapter 5

Exact Intersections

Aside. a $(v,k)-\lambda$ **DESIGN** is a system $\mathcal{A} \subseteq [v]^{(r)}$ (some r), members of $\mathcal{A}$ called **BLOCKS**, such that every element of $[v]$ lies in exactly k blocks, every pair of elements lies in exactly λ blocks.

Clearly the parameters are constrained: for example $br = vk$ where $b = |\mathcal{A}|$ and $\lambda\binom{v}{2} = b\binom{r}{2}$. A less obvious constraint is $b \geqslant v$. It turns out this needs only the pair constraint: works for non-uniform $\mathcal{A}$.

The **DUAL SYSTEM** to $\mathcal{A} \subseteq \mathcal{P}[v]$ is the system $\mathcal{A}^* \subseteq \mathcal{P}\mathcal{A}$ where $\mathcal{A}^* = \{A_x^* \colon x \in [v]\}$ and $A_x^* = \{A \in \mathcal{A} \colon x \in A\}$. Think of bipartite graph, vertex classes $[v]$, $\mathcal{A}$ edges denoting containment. Then $\left|A_x^* \cap A_y^*\right| = \lambda$.

$x \in \mathcal{L} \bmod p$ means there exists $l \in \mathcal{L}$ with $x \equiv l \bmod p$.

Theorem 5.1 (Fisher's Inequality). Let $\mathcal{A} \subseteq \mathcal{P}[n]$ and let $\lambda \in \mathbb{N} \cup \{0\}$ be such that $|A \cap B| = \lambda$ for all distinct A, $B \in \mathcal{A}$. Then $|\mathcal{A}| \leqslant n$, unless $\lambda = 0$ when $|\mathcal{A}| \leqslant n+1$.

Proof. If $|A| = \lambda$ for some $A \in \mathcal{A}$ then $A \subseteq B$ for all $B \in \mathcal{A}$ and the sets $B \backslash A$ are disjoint, so $|\mathcal{A}| \leqslant 1+n-|A|$. So we may assume $|A| > \lambda$ for all $A \in \mathcal{A}$. Let $x_A \in \mathbb{R}^n$ the characteristic vector of A. $x_A = (\delta_1, \delta_2, \ldots, \delta_n)$. Then $x_A \cdot x_A = |A| > \lambda$, and $x_A \cdot x_B = \lambda$ for $A \neq B$.

Suppose $\sum_{A \in \mathcal{A}} c_A x_A = 0$ where $c_A \in \mathbb{R}$. Then by dotting with x_B we have $c_B(x_B \cdot x_B - \lambda) = -\lambda C$ where $C = \sum_{A \in \mathcal{A}} c_A$. If $\lambda = 0$ this means $C_B = 0$ for all B. If $\lambda > 0$ then all c_B have opposite sign to C: only if $c_B = 0$ for all B. Either way, the x_A are independent, so $|\mathcal{A}| \leqslant n$. $\square$

Theorem 5.2. Let $\mathcal{L} \subseteq \{0\} \cup \mathbb{N}$ and let $\mathcal{A} \subseteq [n]^{(r)}$ be such that $|A \cap B| \in \mathcal{L}$ for $A \neq B \in \mathcal{A}$. Suppose $\operatorname{hcf}(\mathcal{L}) \nmid r$. Then $|\mathcal{A}| \leqslant n$.

Proof. For $A \in \mathcal{A}$ let $x_A \in \mathbb{Q}^n$ be its characteristic vector. Suppose $\sum_{A \in \mathcal{A}} j_A x_A = 0$, where we may suppose $j_A \in \mathbb{Z}$ and $\operatorname{hcf}\{j_A\} = 1$. Let p^k be some prime power with $p^k \mid l$ for all $l \in \mathcal{L}$ but $p^k \nmid r$. For all $B \in \mathcal{A}$, $0 = \langle x_B, \sum j_A x_A \rangle = \sum_A j_A |A \cap B|$. Then p^k divides every summand with $A \neq B$ so $p^k \mid j_B r$ so $p \mid j_B$. True for all $B \in \mathcal{A}$, contradiction. □

Theorem 5.3. Let $\mathcal{L} \subseteq \mathbb{N} \cup \{0\}$, $|\mathcal{L}| = s$. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that $|A \cap B| \in \mathcal{L}$ if A, $B \in \mathcal{A}$ distinct. Then $|\mathcal{A}| \leqslant \sum_{i=0}^{s} \binom{n}{i}$.

Proof. For $A \in \mathcal{A}$ define the polynomial $f_A \colon \mathbb{R}^n \to \mathbb{R}$ by

$$f_A(x) = \prod_{\substack{l \in \mathcal{L} \\ l < |A|}} (\langle x_A, x \rangle - l)$$

in the variables $x_1, \ldots, x_n$ where $x = (x_1, \ldots, x_n)$ and $\langle x_A, x \rangle = \sum_{i=1}^{n} \delta_i x_i$ here $x_A = (\delta_1, \delta_2, \ldots, \delta_n)$. Then $f_A(x_B) = 0$ if $A \not\subseteq B$. (In fact, true for $A \neq B$ if $|A| \notin \mathcal{L}$) and $f_A(x_A) \neq 0$. Let $\tilde{f}_A(x)$ be the polynomial obtained from f_A by replacing all powers of x_i greater than 1 by just x_i. Then $\tilde{f}_A(x_B) = f_A(x_B)$ for all B since x_B is a 0-1 vector. That is, $\tilde{f}_A(x_B) = 0$ if $A \not\subseteq B$. $\tilde{f}_A(x_A) \neq 0$.

Suppose $\sum_A c_A \tilde{f}_A = 0$. If not all c_A are zero, pick B with $c_B \neq 0$ and $|B|$ minimal. Then $0 = \sum_A c_A \tilde{f}_A(x_B) = c_B \tilde{f}_B(x_B) \neq 0$, contradiction. Hence $\tilde{f}_A$ are linearly independent and are in the space spanned by the monomials $\{x_T \colon T \subseteq [n] \wedge |T| \leqslant s\}$ with dimension $\sum_{i=0}^{s} \binom{n}{i}$. □

Theorem 5.4. Let p be prime. Let $\mathcal{L} \subseteq \mathbb{N} \cup \{0\}$, $|\mathcal{L}| = s$. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that $|A| \notin \mathcal{L} \bmod p$ for all $A \in \mathcal{A}$ and $|A \cap B| \in \mathcal{L} \bmod p$ for all A, $B \in \mathcal{A}$ distinct. Then $|\mathcal{A}| \leqslant \sum_{i=0}^{s} \binom{n}{i}$.

Proof. Repeat the proof of Theorem 5.3 with $\mathrm{GF}(p)$ in place of $\mathbb{R}$, and $f_A(x) = \prod_{l \in \mathcal{L}} (\langle x_A, x \rangle - l)$. Then $\tilde{f}_A(x_A) \neq 0$, $\tilde{f}_A(x_B) = 0$ if $B \neq A$. □

Theorem 5.5 (Frankel-Wilson, 1981). Let p be prime. Let $\mathcal{L} \subseteq \mathbb{N} \cup \{0\}$, $|\mathcal{L}| = s$. Let $r \in \mathbb{N}$ be such that $r \notin \mathcal{L} \bmod p$. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that $|A| \equiv r \bmod p$ for all $A \in \mathcal{A}$ and $|A \cap B| \in \mathcal{L} \bmod p$ for A, $B \in \mathcal{A}$ distinct. Suppose moreover that $r \notin \{0, 1, \dots, s-1\}$. Then $|\mathcal{A}| \leqslant \binom{n}{s}$.

Proof. Let $\mathcal{A}_i$ and M_i, $0 \leqslant i \leqslant s$ be the $|\mathcal{A}| \times \binom{n}{i}$ and $\binom{n}{s} \times \binom{n}{i}$ matrices, whose rows are indexed by $\mathcal{A}$, and $\binom{n}{s}$ respectively, columns indexed by by $[n]^{(i)}$. The entry in row A column B is 1 if $B \subseteq A$, 0 if not.

Let V be the vector space spanned by the columns of $\mathcal{A}_s$ over $\mathrm{GF}(p)$. Clearly $\dim V \leqslant \binom{n}{s}$. Note that for any $\binom{n}{s} \times t$ matrix M, the columns of $\mathcal{A}_s M$ lie in V.

Let $A \in \mathcal{A}$, $I \in [n]^{(i)}$. Then $(\mathcal{A}_s M_i)_{AI} = \big|\{S \subseteq [n]^{(s)} \colon A \supseteq S \supseteq I\}\big| = 0$ if $A \not\supset I$ and $\binom{|A|-i}{s-i}$ if $A \supset I$. Thus $\mathcal{A}_s M_i \equiv \binom{r-i}{s-i} \mathcal{A}_i \bmod p$.

The conditions on r imply $\binom{r-i}{s-i} \not\equiv 0 \bmod p$. So the columns of $\mathcal{A}_i$ lie in V. Therefore the columns of $\mathcal{B}_i = \mathcal{A}_i \mathcal{A}_i^\mathsf{T}$ lie in V. This is an $|\mathcal{A}| \times |\mathcal{A}|$ matrix, whose AB entry if A, $B \in \mathcal{A}$ is

$$(\mathcal{B}_i)_{AB} = \big|\{I \in [n]^{(i)} \colon I \subseteq A \wedge I \subseteq B\}\big| = \binom{|A \cap B|}{i}.$$

There exists $|\mathcal{A}| \times |\mathcal{A}|$ matrices $\mathcal{B}_i$, $(\mathcal{B}_i)_{AB} = \binom{|A \cap B|}{i}$ columns of $\mathcal{B}_i$ lie in a space V, $\dim V \leqslant \binom{n}{s}$. Consider the polynomial $\phi(x) = \prod_{l \in L}(x - l)$ over $\mathrm{GF}(p)$. Then $\phi(r) \not\equiv 0 \bmod p$, $\phi(l) = 0$ for all $l \in \mathcal{L}$. Since ϕ has degree s so there exist constants c_0, c_1, $\dots$, c_s such that $\phi(x) = \sum_{i=0}^{s} c_i \binom{x}{i}$. Now let $\mathcal{B} = c_0 \mathcal{B}_0 + c_1 \mathcal{B}_1 + \dots + c_s \mathcal{B}_s$. The diagonal entries of $\mathcal{B}$ are $\phi(|A \cap A|) = \phi(r) \neq 0$, and off-diagonal entries are $\phi(|A \cap B|)$. Thus B is non-singular, so $|\mathcal{A}| = \operatorname{rank} \mathcal{B} \leqslant \dim V \leqslant \binom{n}{s}$. □

Corollary 5.6 (Ray-Chaudhari, Wilson). Let $\mathcal{A} \subseteq [n]^{(r)}$, and let $\mathcal{L} = \{|A \cap B| : A, B \in \mathcal{A}, \text{ distinct}\}$. Then $|\mathcal{A}| \leqslant \binom{n}{|\mathcal{L}|}$.

Proof. Let $s = |\mathcal{L}|$. Clearly $s < r$ (else $s = r$ and result trivial). Pick $p > r$ and apply Theorem 5.5. □

Theorem 5.7. Let $q < r$ be a prime power. Let $\mathcal{A} \subseteq [n]^{(r)}$ be such that $|A \cap B| \not\equiv r \bmod q$ for distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leqslant \binom{n}{q-1}$.

Proof. Copy the proof of Theorem 5.5 but work over $\mathbb{Q}$ not $\mathrm{GF}(p)$. Take $s = q - 1$. $\mathcal{A}_s M_i = \binom{r-i}{s-i}\mathcal{A}_i$. So columns of $\mathcal{A}_i$ lie in $V/\mathbb{Q}$. Thus so do columns of $\mathcal{B}_i$. Let $\phi(x) = \binom{r-1-x}{q-1}$. Again choose c_i so $\phi(x) = \sum_{i=0}^{s} c_i \binom{x}{i}$.
Now $\phi(r) = \binom{-1}{q-1} = (-1)^{q-1} \neq 0$. On the other hand if $l < q$, $r \not\equiv l \bmod q$, $(r-l)\phi(l) = q\binom{r-l}{q}$ is an identity where each side is a product of two integers. Since $r - l \not\equiv 0 \bmod q$ then $\phi(l) \equiv 0 \bmod p$ where q is a power of p. Thus, as before, $\mathcal{B}$ is non-singular. □

Chapter 6

Amazing Consequences

Form a graph on $\mathbb{R}^n$ by joining vertices at distance 1. $4 \leqslant \chi(\mathbb{R}^2) \leqslant 7$ and easily shown $\chi(\mathbb{R}^n) \leqslant 3^n$.

Corollary 6.1. $\chi(\mathbb{R}^n) \geqslant (1.2 + o(1))^n$.

Proof. Let G be the subgraph of $\mathbb{R}^n$ spanned by $V = \{x_A/\sqrt{2q} \colon A \subseteq [n]^{(2q-1)}\}$. Then $|G| = \binom{n}{2q-1}$. An edge occurs if $|A \cap B| = q-1$. So if G is coloured each colour class by Theorem 5.7, has at most $\binom{n}{q-1}$ vertices. Thus $\chi(G) \geqslant \binom{n}{2q-1}/\binom{n}{q-1}$. Take $q = (2 - \sqrt{2} + o(1))n/4$. □

Corollary 6.2 (Kahn, Kalai, 1993). There is a set of diameter 1 in $\mathbb{R}^n$ that is not the union of $(1.2)^{\sqrt{n}}$ sets diameter < 1.

Proof. Choose $m = 4q$ with $\binom{m}{2} \approx n$. Let the coordinates of $\mathbb{R}^n$ be index by the edges of the complete graph on $[m]$. For each $A \subseteq [m]^{(m/2)}$ let v_A be the characteristic vector of the edges of the complete bipartite graph with vertex class A and $[m] - A$. Then $v_A = v_{[m]-A}$ and this vector has $(M/2)^2 = 4q^2$ ones. If A, B are two subsets then $d^2(v_A, v_B) = 4i(2q-i)$ where $i = |A \cap B|$. Thus $d^2(v_A, v_B) \leqslant 4q^2$ with equality if and only if $|A \cap B| = q$.

Let $S = \{v_A/2q \colon 1 \in A \in [m]^{(2q)}\}$. Then S has diameter 1 and $|S| = \frac{1}{2}\binom{M}{M/2}$. But if $T \subset S$ has diameter < 1, then $|A \cap B| \neq q$ if $v_A/2q$, $v_B/2q \in T$. Let $\mathcal{A} = \{A - 1 \colon v_A/2q\}$. Then $\mathcal{A} \subseteq [m]^{(2q-1)}$ and $|A \cap B| \not\equiv$

$q-1 \bmod q$ if A, B distinct members of $\mathcal{A}$. By Corollary 5.6 $|\mathcal{A}| \leqslant \binom{m-1}{q-1} = \frac{1}{4}\binom{M}{M/4}$. Thus $|S|\,/\,|T| \geqslant 2\binom{M}{M/2}\binom{M}{M/4} \geqslant (1.14+o(1))^M \geqslant (1.2+o(1))^{\sqrt{n}}$. □

Erdős showed there exist random colourings of K_n, $n \geqslant \sqrt{2}^t$ with no monochromatic K_t, that is, $R(t) > \sqrt{2}^t$. $(t-1)^2$ is easy as a lower bound, t^3 tricky.

Corollary 6.3. This construction shows $R(t) \geqslant \exp\{(1/4+o(1))\frac{\log^2 t}{\log\log t}\}$.

Proof. Let the vertices of the complete graph be $[h]^{(r)}$, so $n = \binom{h}{r}$. Here $r \equiv -1 \bmod q$, q prime power. Colour AB red if $|A \cap B| \not\equiv -1 \bmod q$. Theorem 5.7 shows there is no red K_t if $t > \binom{h}{q-1}$. Taking $r = q^2-1$. Corollary 5.6 shows no blue K_t if $t > \binom{h}{q-1}$.

Take $h = q^3$. □

We could have used Theorem 5.3 to get the same result. Constructions to give lower bounds for Ramsey numbers fail completely to give a lower bounded for the bipartite Ramsey—colour $K_{n,n}$ with no mono $K_{t,t}$. If you can bipartite you can do the ordinary (projection). No constructions were known beyond trivial t^2 until recently.

Chapter 7

Examples

7.1 Example Sheet 1

Exercise 7.1.1. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that whenever $A \in \mathcal{A}$ and $B \subseteq A$ then also $B \in \mathcal{A}$. Show that the average size of the members of $\mathcal{A}$ is at most n.

Solution. Set $\mathcal{A}_r = \mathcal{A} \cap [n]^{(r)}$. The condition for $\mathcal{A}$ is equivalent to $\partial\mathcal{A} \subseteq \mathcal{A}$. For all $r \geqslant n/2$ we have by the local LYM inequality applied $n - 2r$ times that

$$\frac{|\mathcal{A}_r|}{\binom{n}{r}} \leqslant \frac{|\partial^{n-2r}\mathcal{A}_r|}{\binom{n}{n-r}} = \frac{|\partial^{n-2r}\mathcal{A}|}{\binom{n}{r}} \leqslant \frac{|\mathcal{A}_{n-r}|}{\binom{n}{r}}.$$

Thus $|\mathcal{A}_{n-r}| \geqslant |\mathcal{A}_r|$ and for odd n:

$$\begin{aligned}
\mathbb{E}_{A\in\mathcal{A}} |A| &\leqslant \frac{1}{|\mathcal{A}|} \sum_{\lceil n/2\rceil \leqslant r \leqslant n} (|\mathcal{A}_r|\, r + |\mathcal{A}_{n-r}|\,(n-r)) \\
&\leqslant \frac{1}{|\mathcal{A}|} \sum_{\lceil n/2\rceil \leqslant r \leqslant n} (|\mathcal{A}_r|\, r + |\mathcal{A}_r|\,(n-r)) \\
&= \frac{n}{|\mathcal{A}|} \sum_{\lceil n/2\rceil \leqslant r \leqslant n} |\mathcal{A}_r| \leqslant \frac{n\,|\mathcal{A}|}{2\,|\mathcal{A}|} = \frac{n}{2}.
\end{aligned}$$

A similar argument shows the result for even n.

Exercise 7.1.2. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be an antichain not of the form $[n]^{(r)}$, $0 \leqslant r \leqslant n$. Must there exist a maximal chain that is disjoint from $\mathcal{A}$?

Solution. Yes. Let r be maximal such that $\mathcal{A}_r = \mathcal{A} \cap [n]^{(r)}$ is non-empty. This exists unless $n = 0$ which is a trivial case (take the empty chain). Observe that $\partial^+ \mathcal{A}_r = \overline{\partial \overline{\mathcal{A}_r}}$ and that equality in local LYM holds if and only if $\mathcal{A}_r$ is a full layer.

$\mathcal{A}_r = [n]^{(r)}$, so pick a $A_r \in \partial^+ \partial \mathcal{A}_r$ and a $A_{r-1} \in \partial \mathcal{A}_r$. By maximality of r, there is a chain down from $[n]$ to A_r, and any chain down from A_{r-1} is not in $\mathcal{A}$ since it is an antichain. Chaining these we get a maximal chain disjoint from $\mathcal{A}$.

Exercise 7.1.3. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ contains no chain of $s+1$ sets. Prove

$$\sum_{r=0}^{n} \left|\mathcal{A} \cap [n]^{(r)}\right| \binom{n}{r}^{-1} \leqslant s.$$

Deduce that, if n is even and $s = 2$, then $|\mathcal{A}| \leqslant \binom{n}{n/2} + \binom{n}{n/2-1}$.

Solution. Take Proof 2. in the notes, mutatis mutandis and replace "so these events are mutually exclusive" to "these events occur at most s times" and thus replace 1 by s in the following inequality, and we obtain the result.

In particular this means that $\mathcal{A}$ has size at most the sum of the s largest binomial coefficients, which is the corollary needed for $s = 2$, n even.

Exercise 7.1.4. Show that for every $k \geqslant 1$ there is a constant c_k such that, if $\mathcal{A} \subset \mathcal{P}[n]$ is an antichain with $\mathcal{A} \cap [n]^{(k)} \neq \emptyset$ and $\mathcal{A} \cap [n]^{(r)} = \emptyset$ for $k < r < n-k$, then $|\mathcal{A}| \leqslant c_k n^{k-1}$. What is the maximum of $|A|$ for $k = 1$? And for $k = 2$? Give bounds for c_k.

Solution. Without loss of generality, by the local LYM inequality, $\mathcal{A} \subseteq [n]^{(k)} \cup [n]^{(n-k)}$. Furthermore, by the local LYM inequality repeated $n - 2r$ times, we can also assume that $\mathcal{A} \cap [n]^{(n-k)}$ has only one element.

So by the Inclusion-Exclusion Principle:

$$\begin{aligned}
\frac{|\mathcal{A}|}{n^{k-1}} &= \frac{\binom{n}{k} - \binom{n-k}{k} + 1}{n^{k-1}} \\
&= \frac{n(n-1)\cdots(n-k) - (n-k)(n-k-1)\cdots(n-2k)}{n^{k-1}k!} + \frac{1}{n^{k-1}}. \\
&= \frac{n}{k!}((1-1/n)\cdots(1-k/n) - (1-k/n)\cdots(1-2k/n)) + \frac{1}{n^{k-1}} \\
&\leqslant \frac{n}{k!}(1-(1-3k^2/(2n))) + \frac{1}{n^{k-1}} = \frac{3k^2}{k!} + \frac{1}{n^{k-1}}.
\end{aligned}$$

Thus $c_k \leqslant 3k^2/k! + 1$. In particular case we get:

$$\begin{aligned}
k = 1\colon \binom{n}{1} - \binom{n-1}{1} + 1 = 2, \\
k = 2\colon \binom{n}{2} - \binom{n-2}{2} + 1 = 2n-2.
\end{aligned}$$

Exercise 7.1.5. The system $\mathcal{A} \subseteq \mathcal{P}[n]$ is said to *split* $[n]$ if, for all distinct i, $j \in [n]$, there exists $A \in \mathcal{A}$ with $i \in A$ but $j \notin \mathcal{A}$. Prove that if $\mathcal{A}$ splits $[n]$ then $n \leqslant \binom{|\mathcal{A}|}{\lfloor|\mathcal{A}|/2\rfloor}$.

Solution. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$, then for all $x \in [n]$ we have $B_x = \{i\colon x \in A_i\} \subseteq [k]$, set $\mathcal{B} = \{B_x\colon x \in [n]\}$. $\mathcal{B}$ is an antichain, $|\mathcal{B}| = n \leqslant \binom{k}{\lfloor k/2\rfloor}$.

Exercise 7.1.6. A system $\mathcal{A} \subseteq \mathcal{P}[n]$ is called a *cross-cut* if for every $B \subseteq [n]$ there exists $A \in \mathcal{A}$ with $B \subseteq A$ or $A \subseteq B$. Prove that every cross-cut contains a cross-cut of size at most $\binom{n}{\lfloor n/2\rfloor}$. Does every cross-cut contain a cross-cut that is an antichain?

Solution. Without loss of generality $\mathcal{A}$ is a minimal cross-cut. Then there does not exists $A \subset B \subset C$ with A, B, $C \in \mathcal{A}$. Consider

$$\mathcal{A}' = \{A \in \mathcal{A}\colon \forall B \in \mathcal{A}\colon B \not\subset A\}.$$

For each $A \in \mathcal{A}\backslash\mathcal{A}'$, there exists a set $A'' \in \mathcal{P}[n]\backslash\mathcal{A}$ with $A'' \subset A$ such that if $B \in \mathcal{A}$ and $B \subseteq A''$ or $A'' \subseteq B$ then $B = A$, that is, a witness that A must be in $\mathcal{A}$ (remember $\mathcal{A}$ minimal), it needs one $\subset A$ for there is $B \in \mathcal{A}$ with $B \subset A$. If $\mathcal{A}\backslash\mathcal{A}'$ non-empty, consider

$$\mathcal{A}'' = \{A'' \colon A \in \mathcal{A}\backslash\mathcal{A}'\}.$$

Then $\mathcal{A}' \cup \mathcal{A}''$ is an antichain. So $|\mathcal{A}| = |\mathcal{A}' \cup \mathcal{A}''| \leqslant \binom{n}{\lfloor n/2 \rfloor}$, as if $A'' = B''$, then we have $A \supset A'' = B'' \subset B$ and thus $A = B$.

No. Take $n = 4$ and the cross-cut $\{\{1\}, \{3, 4\}, \{2, 4\}, \{1, 2, 3\}\}$.

Exercise 7.1.7. Let a, $x_1, \ldots, x_n \in \mathbb{R}^k$ satisfy $\|x_i\| \geqslant 1$ for $1 \leqslant i \leqslant n$. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the 2^n sums $\sum_1^n \epsilon_i x_i$, $\epsilon_i \in \{-1, 1\}$, lie in the open ball centre a and radius 1.

Solution. This is Theorem 1.5 (Kleitman) restated where we translate $\mathbb{R}^k$ by $-\sum_i^n x_i$ and transform $x_i \mapsto 2x_i$ and $\langle \varepsilon_i \colon 1 \leqslant i \leqslant k \rangle$ maps to $A = \{i \colon \varepsilon_i = 1\}$, finished off by shrinking the space by a factor 2.

Exercise 7.1.8. Let $x_1, \ldots, x_n \in \mathbb{R}$ satisfy $|x_i| \geqslant 1$ for $1 \leqslant i \leqslant n$, and let $T \subset \mathbb{R}$ be the union of s disjoint open intervals each of length 1. Show that the number of subsets $A \subseteq [n]$ with $\sum_{i \in A} x_i \in T$ is at most the sum of the s largest binomial coefficients $\binom{n}{j}$.

Solution. Using Erdős observation that if $s = 1$ then if I is an interval, $\mathcal{A}$ the subsets with $\sum_{i \in A} x_i \in I$, then

$$\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} \{i \in A \colon x_i > 0\} \cup \{i \notin A \colon x_i < 0\}$$

form an antichain of $[n]$. These sets for different intervals are disjoint, there cannot be a chain in $\mathcal{A}$ of size $s + 1$ and so by Exercise 7.1.3

$$\sum_{r=0}^{n} \left|\mathcal{A} \cap [n]^{(r)}\right| \binom{n}{r}^{-1} \leqslant s,$$

whence $|\mathcal{A}|$ is at most the sum of the s largest binomial coefficients $\binom{n}{j}$.

Exercise 7.1.9. Let G_1 and G_2 be bipartite graphs with vertex classes X_1, Y and X_2, Y respectively, where $|X_1| = |X_2| = m \leqslant |Y|$. Show that there exist matchings of X_1 and X_2 to the *same* subset of Y if and only if $|\Gamma(S) \cap \Gamma(T)| \geqslant |S| + |T| - m$ for all $S \subseteq X_1$ and $T \subseteq X_2$. (Here as usual $\Gamma(S) = \{y \in Y : sy \text{ is an edge, some } s \in S\}$.

Deduce that $\mathcal{P}[n]$ can be decomposed into symmetric chains.

Solution. $\Longrightarrow$ Without loss of generality $\Gamma(X_1) = Y = \Gamma(X_2)$.

Consider G bipartite graph with vertex class $X_1 \cup Y'$ and $X_2 \cup Y''$ where Y' and Y'' are copies of Y. An edge $x_1 \in X_1$ is connected to a $y' \in Y''$ if and only if $x_1 y \in G_1$ and similarly for X_2 and Y'. $y' \in Y'$ and $y'' \in Y''$ are connected if and only if they originate from the same $y \in Y$.

A matching in G exactly corresponds to a matching of X_1 and X_2 to the same subset of Y. Suppose $X \subseteq X_1 \cup Y'$, set $S = X \cap X_1$ and $T = X_2 \backslash \Gamma(X \cap Y')$. Now

$$\begin{aligned}
|\Gamma(X)| &= |\Gamma(S) \cup (Y \backslash \Gamma(T))| + m - |T| \\
&= |Y| - |\Gamma(T)| + |\Gamma(S) \cap \Gamma(T)| + m - |T| \\
&\geqslant |X \cap Y''| + |S| + |T| - m + m - |T| \\
&\geqslant |X \cap Y''| + |S| \\
&\geqslant |X| .
\end{aligned}$$

Now by Hall's Matching Theorem there is a matching as required.

$\Longleftarrow$ Clear.

There is now a natural way one will try to match $\mathcal{P}[n]$ into symmetric chains. Start with the middle (or empty chains for n even) and expand to the extremities one step at a time using this result and identifying a pre-existing chain to a point. By local LYM, the density increases and so to extend to $[n]^{(r)}$ and $[n]^{(n-r)}$ for $r < n/2$ we have

$$|\Gamma(S) \cap \Gamma(T)| \geqslant \frac{n-r}{r+1}(|S|+|T|) - \binom{n}{r+1} = \frac{n-r}{r+1}\left(|S| + |T| - \binom{n}{r}\right).$$

Since $\binom{n}{r} = m$ and $n - r \geqslant r + 1$ the result follows.

Exercise 7.1.10. What are the 99th, 100th and 101st elements in the colex order on $\mathbb{N}^{(4)}$? For which $A \in \mathbb{N}^{(4)}$ does A and its successor (in colex) have the same sum?

Solution. We have

$$\begin{aligned} 99 &= \binom{8}{4} + \binom{6}{3} + \binom{4}{2} + \binom{3}{1} \\ 100 &= \binom{8}{4} + \binom{6}{3} + \binom{5}{2} \\ 101 &= \binom{8}{4} + \binom{6}{3} + \binom{5}{2} + \binom{1}{1} \end{aligned}$$

and so the elements are $\{3,5,7,9\}$ and $\{4,5,7,9\}$ and $\{1,6,7,9\}$.

The sum of successors (in colex) have the same sum if and only if it contains 2 and 3 but not 4 and 1. So $\{2,3\}$ subset transforms to $\{1,4\}$.

Exercise 7.1.11. Extend the definition of colex to all finite subsets of $\mathbb{N}$. Verify that, if subsets are identified with integers via binary representation in the natural way, then colex is precisely the usual order on $\mathbb{N} \cup \{0\}$.

Solution. Suppose A, $B \in \mathbb{N}^{<\omega}$, then $A < B$ in colex iff $\max A \triangle B \in B$

We identify A with $\sum_{i \in A} 2^i$ to get a bijection between $\mathbb{N}^{<\omega}$ and $\mathbb{N} \cup \{0\}$ ($\{0\}$ corresponds to the empty set). The order carries over.

Exercise 7.1.12. Let $\mathcal{A} \subset [9]^{(3)}$ satisfy $|\mathcal{A}| = 28$. How small can $|\partial\mathcal{A}|$ be? And $|\partial^+\mathcal{A}|$?

Solution. We have $|\mathcal{A}| = \binom{6}{3}+\binom{4}{2}+\binom{2}{1}$ and so $|\partial\mathcal{A}| \leqslant \binom{6}{2}+\binom{4}{1}+\binom{2}{0} = 20$.

Similarly, $|\overline{\mathcal{A}}| = 56 = \binom{8}{6}28 + \binom{7}{5}21 + \binom{5}{4}5 + \binom{3}{3}$ and
$|\partial^+\mathcal{A}| \leqslant \binom{8}{5} + \binom{7}{4} + \binom{5}{3} + \binom{3}{2} = 104$.

Exercise 7.1.13. Let $m \in \mathbb{N}$. Show that there is a unique expression for m of the form

$$m = \binom{m_r}{r} + \binom{m_{r-1}}{r-1} + \cdots + \binom{m_s}{s}$$

where $m_r > m_{r-1} > \cdots > m_s \geqslant s$ and $r \geqslant s \geqslant 1$. Show that the size of the shadow of the first m elements in the colex order of $\mathbb{N}^{(r)}$ is

$$\binom{m_r}{r-1} + \binom{m_{r-1}}{r-2} + \cdots + \binom{m_s}{s-1}.$$

Solution. Such an expression exists by constructing it with a greedy algorithm and $\binom{n}{n} = 1$ and $\binom{n}{1} = n$ gives we do not run into trouble. Suppose we have two such expressions for some m. Consider least such m. By minimality we must have that $m_r \neq m_r'$. However, we have

$$m \leqslant \binom{m_r}{r} + \binom{m_r - 1}{r-1} + \binom{m_r - r + 1}{1} < \binom{m_r + 1}{r} \leqslant m,$$

as we have an injection: let the permutations in $\binom{m_r - i}{r-i}$ represent permutations $[m_r]^{(r-i)} + (m_r+1) + (m_r) + \cdots + (m_r - i + 1)$ which is missing the permutation $[m_r + 1] \backslash [m_r + 1 - r]$. This is a contradiction.

A similar argument shows that the first m members of colex are

$[m_r]^{(r)}$, $[m_{r-1}]^{(r-1)} + (m_r + 1)$, $[m_{r-2}]^{(r-2)} + (m_{r-1} + 1) + (m_{r-2} + 1)$, $\ldots$

Exercise 7.1.14. Find a set system $\mathcal{A}$ for which equality holds in the Kruskal-Katona theorem but which is not isomorphic to an initial segment of colex.

Solution. Take $r = 4$ and $|\mathcal{A}| = 4$, then the initial segment of colex is $\{12, 13, 23, 14\}$ whereas the non-isomorphic (consider the graph associated with both) $\{12, 23, 34, 14\}$ has the same shadow $\{1, 2, 3, 4\}$.

Exercise 7.1.15. Let $\mathcal{A} \subset \mathbb{N}^{(r)}$ and $\mathcal{B} \subset \mathbb{N}^{(r+1)}$ be initial segments of colex with $|\mathcal{A}| = |\mathcal{B}|$. Do we always have $|\partial\mathcal{A}| \leqslant |\partial\mathcal{B}|$? [Hint. Yes, use induction with (weak) Kruskal-Katona.]

Exercise 7.1.16. Fill in the details of this fast proof of the Kruskal-Katona Theorem, due to Bollobás and Leader.

Let $\mathcal{A}_{i^+} = \{A \in \mathcal{A} : i \in A\}$ and $\mathcal{A}_{i^-} = \mathcal{A} \backslash \mathcal{A}_{i^+}$. Let $\mathcal{A}''_{i+}$ be the first $|\mathcal{A}_{i^+}|$ elements of $([n]\backslash\{i\})^{(r-1)}$ in colex, and let $\mathcal{A}'_{i\_}$ be the first $|\mathcal{A}_{i\_}|$ elements of $([n]\backslash\{i\})^{(r)}$. The *squash* of $\mathcal{A}$ *in direction* i is $\mathcal{A}' = \mathcal{A}'_{i_+} \cup \mathcal{A}'_{i_-}$, where $\mathcal{A}'_{i\_}$ is $\mathcal{A}''_{i\_}$ with element i added to each set. Then (the crucial fact) $|\partial\mathcal{A}'| \leqslant |\partial\mathcal{A}|$. As long as you can find some direction in which the squash of $\mathcal{A}$ differs from $\mathcal{A}$, squash $\mathcal{A}$. You must stop. Then $\mathcal{A}$ is an initial segment of colex, or $\mathcal{A}$ is some particularly special case.

Solution. By induction on n and r. We have

$$\begin{aligned} |\partial\mathcal{A}\,| &= |\partial\mathcal{A}_{i^+}| + |\partial\mathcal{A}_{i^-}| - |\partial\mathcal{A}_{i^+}| \cap |\partial\mathcal{A}_{i^-}| \\ |\partial\mathcal{A}'| &= |\partial\mathcal{A}'_{i^+}| + |\partial\mathcal{A}'_{i^-}| - |\partial\mathcal{A}'_{i^+}| \cap |\partial\mathcal{A}'_{i^-}|\,. \end{aligned}$$

Anything in the intersection of the shadows, must not include i, and thus must be a member of $\mathcal{A}_{i^+}$ where i removed at each element or $\mathcal{A}''_{i+}$. Due to the previous argument and induction the last term is biggest for $\mathcal{A}'$. By induction we have the sizes of the first two sets in the two sums bound each other in the right direction. Thus we obtain $|\partial\mathcal{A}'| \leqslant |\partial\mathcal{A}|$.

If you have squashed in every direction, and $\mathcal{A}$ is not an initial segment, then there exists A, B where B comes before A in the lexicographic order and $A \in \mathcal{A}$ and $B \notin \mathcal{A}$. By squashing, for every $i \in [n]$, i must be in exactly one of A and B. Thus A must be the complement of B. For each element in $\mathcal{A}$ everything before is in $\mathcal{A}$ except at most thing, which is its complement. This only happens when $\mathcal{A}$ is an initial segment except for the second last element and this second last is the complement of the last. Replacing the last by the second last reduces the shadow.

Exercise 7.1.17. Let $\mathcal{A} \subset \mathcal{P}[n]$ be maximal intersecting. Show that $|\mathcal{A}| = 2^{n-1}$.

Solution. Suppose $\mathcal{A}$ is maximal intersecting. Given A, $\overline{A} \notin \mathcal{A}$. Then there exists B, $C \in \mathcal{A}$ such that $A \cap B$ and $\overline{A} \cap C$ are both empty, but then $B \cap C$ is empty, which is a contradiction. So we have a contradiction and so A or $\overline{A} \in \mathcal{A}$. So $|\mathcal{A}| = 2^n/2 = 2^{n-1}$.

Exercise 7.1.18. Prove the following extension of the Erdős-Ko-Rado theorem: if $\mathcal{A} \subseteq [n]^{(\leqslant r)}$ is an intersecting antichain and $r \leqslant n/2$, then $\mathcal{A} \leqslant \binom{n-1}{r-1}$.

Solution. We can push-up the lowest r' such that $A_{r'} = A \cap [n]^{(r')} \neq \emptyset$ and push it up (using ∂^+) until $r' = r$. Since $r \leqslant n/2$ and local LYM implies we have only increased the size of $\mathcal{A}$ as the density has gone up, and our set is still intersecting and an antichain. So $\mathcal{A} \leqslant \binom{n-1}{r-1}$ by the Erdős-Ko-Rado Theorem.

Exercise 7.1.19. Let $n = 2k$ be even, and let $\mathcal{A} \subseteq \mathcal{P}[n]$ be an intersecting antichain. Show that there is an intersecting antichain $\mathcal{B} \subseteq [n]^{(k)} \cup [n]^{(k+1)}$ with $|\mathcal{A}| \leqslant |\mathcal{B}|$.

For a given cyclic order σ of $[n]$ let $f(\sigma)$ be the number of member of $\mathcal{B}_k = \mathcal{B} \cap [n]^{(k)}$ that are intervals, and let $g(\sigma) \geqslant (k+1)f(\sigma)/k$; deduce that $|\mathcal{B}_k| \leqslant |\partial^+\mathcal{B}_k|$, and that $|\mathcal{A}| \leqslant \binom{n}{k+1}$.

Solution. Use ∂^+ for levels $r < k$ and ∂ for levels $r > k+1$ to get $|\mathcal{A}| \leqslant |\mathcal{B}|$ (by local LYM inequality) and $\mathcal{B} \subseteq [n]^{(k)} \cup [n]^{(k+1)}$. Doing ∂^+ it remains ∂^+, so do that first. If $r > k+1$ and we wish to take the shadow of the elements in $\mathcal{A} \cap [n]^{(r)}$, then these elements remain intersecting with all of this in levels $r' \geqslant k$ as $k + 2 - 1 + k > 2k$.

$f(\sigma) \leqslant k$ by the usual argument as the intervals must overlap. Suppose $f(\sigma)$ and let $A_1, \ldots, A_{\sigma(n)}$ be the intervals. Then first add the first element to the left of each interval (loop around if necessary) to each A_i, then these are all in $\partial^+\mathcal{B}_k$. If we take $A_{\sigma(n)}$ and add the first element to the right, then this is not one we have found before, so

$$g(\sigma) \geqslant f(\sigma) + 1 \geqslant f(\sigma) + f(\sigma)/k = (k+1)f(\sigma)/k.$$

We have that

$$\begin{aligned}\sum_\sigma f(\sigma) &= \frac{k!k!}{n!}\left|\mathcal{B}_k\right| \\ \sum_\sigma g(\sigma) &= \frac{(k+1)!(k-1)!}{n!}\left|\partial^+\mathcal{B}_k\right| \\ &\geqslant \frac{k+1}{k}\frac{k!k!}{n!}\left|\mathcal{B}_k\right|.\end{aligned}$$

Thus we find $|\partial^+\mathcal{B}_k| \geqslant |\mathcal{B}_k|$. As $\mathcal{B}$ is an antichain, we find $|\mathcal{A}| \leqslant |\mathcal{B}| \leqslant \binom{n}{k+1}$.

Exercise 7.1.20. Let $\mathcal{A} \subseteq [n]^{(r)}$. Show there is a t-intersecting $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| \geqslant \binom{r}{t}\binom{n}{t}^{-1}|\mathcal{A}|$.

Solution. For all $T \in [n]^{(t)}$ let $\mathcal{A}_T = \{A \in \mathcal{A} \colon T \subseteq A\}$. Then $\sum_T |\mathcal{A}_T| = |\mathcal{A}|\binom{r}{t}$, which is also the number of (A, T) such that $A \in \mathcal{A}$ and $T \in [n]^{(t)}$ and $T \subseteq A$. Consider $\mathcal{B}$ the largest $\mathcal{A}_T$.

Exercise 7.1.21. Let $\mathcal{A} \subset [n]^{(\leqslant r)}$ be an intersecting family such that $\bigcap_{A\in\mathcal{A}} A = \emptyset$. Prove there is a set I with $|I| \leqslant 3r-3$ and $|A \cap I| \geqslant 2$ for all $A \in \mathcal{A}$.

Solution. If every pair intersect in at least two points, pick any $A \in \mathcal{A}$. Or there exists A, B such that $|A \cap B| = 1$, $A \cap B = \{v\}$ and there exists C, $v \notin C$ such that $I = A \cup B \cup C$ works.

Exercise 7.1.22. Let s, $t \geqslant 1$ and suppose $\mathcal{A} \subseteq [n]^{(\leqslant s)}$ and $\mathcal{B} \subseteq [n]^{(\leqslant t)}$ are such that $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show there is a set I with $A \cap B \cap I \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $|I| \leqslant (s+t-1)\binom{s+t-2}{s-1}$.

Solution. Take I minimal with $A \cap B \cap I \neq I$ for all A, B. Take $\mathcal{A}$, $\mathcal{B}$ minimal so without loss of generality A, $B \subseteq I$ for all A, B (also replace A by $A \cap I$). For all $i \in I$ there exists $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$ such that $A_i \cap B_i = \{i\}$.

There exists $|I|!$ orderings of I. Order good for i if $A_i - \{i\} < i < B_i - \{i\}$.

An ordering not good for both i and j, as A_i and B_j would not be intersecting. The number of orderings which is good for i is ($a = |A_i|$ and $b = |B_i|$) $\binom{n}{a+b-1}(a-1)!(b-1)!(n-a-b-1)! \geqslant$ same with $a = s$, $b = t$. $n = |I|$ and so

$$n! > n\binom{n}{a+b-1}(a-1)!(b-1)!(n-a-b+1)!.$$

So

$$1 \geqslant n\frac{1}{(a+b-1)}(a-1)!(b-1)!.$$

Whence

$$n \leqslant \binom{a+b-2}{a-1}(a+b-1).$$

Exercise 7.1.23. Let $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$ be an intersecting family of finite sets. Must there exist a finite $F \subseteq \mathbb{N}$ such that the family $\{A \cap F : A \in \mathcal{A}\}$ is intersecting? What if $\mathcal{A} \subseteq \mathbb{N}^{(r)}$.

Solution. No. Consider $\mathcal{A} = \bigcup_{n\in\mathbb{N}}\{A_n, B_n\}$ where

$$A_n = \{1, 3, 5, 7, \ldots, 2n-1, 2n, 2n+2\}$$
$$B_n = \{2, 4, 6, 8, \ldots, 2n, 2n+1, 2n+2\}.$$

Yes. Suppose not, in particular for all n there exists A, $B \in \mathcal{A}$ such that $A \cap [n] \cap B = \emptyset$. Pick n_i and A_i, $B_i \in \mathcal{A}$ such that $A_i \cap [n_i] \cap B_i = \emptyset$ and $A_i \cup B_i \subseteq [n_{i+1}]$ which is possible since all sets in $\mathcal{A}$ are finite.

Consider A_{r+1}. Either it contains two elements in $[n_1]$ belonging to A_1 and B_1 or it contains an element in $[n_2]\backslash[n_1]$ belonging to A_1 or B_1 (or both). In either case, this element does not belong both to A_2 and B_2 and hence we pick up another element in $[n_2]$ et cetera. Hence A_r contains at least $r+1$ elements by considering up to A_r and B_r, but it also shares an element with B_{r+1}. Contradiction with $A_{r+1} \in \mathbb{N}^{(r)}$.

Or, consider $\mathcal{F} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$. $\mathcal{F}^* = \{F \in \mathcal{F} : \nexists F' \in \mathcal{F}, F \subseteq F'\}$. Pick A_F, B_F, $A_F \cap B_F = F$ for all $F \in F^*$. Show $|F^*| \leqslant \binom{s+t-2}{s-1}$.

Or, consider the stronger statement, dependent on r: given any finite graph G (with loops allowed) collection of families $\mathcal{A}_V \subseteq \mathbb{N}^{(\leqslant r)}$, $U \in V(G)$, if $UW \in E(G)$ then $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}_U$ and $B \in \mathcal{A}_W$. Then there exists finite F where everything same.

For all $\sigma \in A$ let $\mathcal{A}_\sigma = \{B \in \mathcal{A} \colon B \cap A = \sigma\}$. $\mathcal{A}_\sigma^a = \{B \backslash \sigma \colon B \in \mathcal{A}_\sigma\}$. $\sigma \cap \tau \neq \emptyset$. Done $\sigma \cap \tau = \emptyset$, $v(G)\mathcal{P}(H)$. Prove $p(R)$ true by induction:

Let $F' = \bigcup$ one set chosen from each family. every set in every family meets F' (unless some $v \in G$ has degree 0).

For all $\sigma \subseteq F'$ let $\mathcal{A}_{V,\sigma}$ be the members of $\mathcal{A}_V$ meeting F' exactly at σ. $\mathcal{A}_{V,\sigma^*} = \mathcal{A}_{V,\sigma}$ with σ removed. Let G^* be graph vertices one for each $\mathcal{A}^*_{V,\tau,\sigma}$ edges are constraints required by G but not satisfied in F' by $\sigma \cap \tau = \emptyset$.

7.2 Example Sheet 2

Exercise 7.2.1. Let $\mathcal{F}_i = \{A \in [n]^{(r)} \colon |A \cap [t+2i]| \geqslant t+i\}$ and $n_k = (r-t+l)(2+(t-l)/k)$. By comparing $\mathcal{F}_{i+1} \backslash \mathcal{F}_i$ with $\mathcal{F}_i \backslash \mathcal{F}_{i+1}$, or otherwise, show that if $n_{k+1} < n < n_k$ then $\max_i |\mathcal{F}_i| = |\mathcal{F}_k|$. Show also that no two $\mathcal{F}_i$s are isomorphic if $n \geqslant 2r - t$.

Solution. We have after rewriting:

$$\begin{aligned}
\mathcal{A}_{i+1} \backslash \mathcal{A}_i = \{A \in [n]^{(r)} \colon & |A \cap [t+2i]| = t+i-1 \wedge \\
& t+2i+2 \in A \wedge t+2i+1 \in A\} \\
\mathcal{A}_i \backslash \mathcal{A}_{i+1} = \{A \in [n]^{(r)} \colon & |A \cap [t+2i]| = t+i \wedge \\
& t+2i+2 \notin A \wedge t+2i+1 \notin A\}.
\end{aligned}$$

Thus they have sizes:

$$\begin{aligned}
|\mathcal{A}_{i+1} \backslash \mathcal{A}_i| &= \binom{t+2i}{t+i-1}\binom{n-t-2i-2}{r-t-i-1} \\
|\mathcal{A}_i \backslash \mathcal{A}_{i+1}| &= \binom{t+2i}{t+i}\binom{n-t-2i-2}{r-t-i}.
\end{aligned}$$

The ratio is:

$$\begin{aligned}
\frac{|\mathcal{A}_{i+1}\backslash\mathcal{A}_i|}{|\mathcal{A}_i\backslash\mathcal{A}_{i+1}|} &= \frac{\binom{t+2i}{t+i-1}\binom{n-t-2i-2}{r-t-i-1}}{\binom{t+2i}{t+i}\binom{n-t-2i-2}{r-t-i}} \\
&= \frac{\frac{(t+2i)!}{(t+i-1)!(i+1)!}\frac{(n-t-2i-2)!}{(r-t-i-1)!(n-r-i-1)!}}{\frac{(t+2i)!}{(t+i)!i!}\frac{(n-t-2i-2)!}{(r-t-i)!(n-r-i-2)!}} \\
&= \frac{(t+i)!i!(r-t-i)!(n-r-i-2)!}{(t+i-1)!(i+1)!(r-t-i-1)!(n-r-i-1)!} \\
&= \frac{(t+i)(r-t-i)}{(i+1)(n-r-i-1)}
\end{aligned}$$

We are interested when this ratio is $\geqslant 1$, this happens exactly when (setting $i+1=k$)

$$\begin{aligned}
n-r-k \geqslant \frac{(r-t-i)(t+i)}{k} \\
n &\geqslant (r-t+1)\frac{(t+k-1)}{k} - (t+k-1) + r + k \\
n &\geqslant (r-t+1)(1+(t-1)/k) + r - t + 1 \\
n &\geqslant (r-t+1)(2+(t-1)/k).
\end{aligned}$$

as required. This gives the result as the n_k are increasing. If $n \geqslant 2r-t$, then $t+2i \leqslant n$ and thus we can identify $[t+2i]$ uniquely in each $\mathcal{A}_i$, and thus they cannot be isomorphic.

Exercise 7.2.2. Given $\mathcal{Z} \subseteq \mathcal{P}[n]$, let $m(\mathcal{Z})$ be the maximum size of a family $\mathcal{A} \subseteq \mathcal{P}[n]$, such that for all A, $B \in \mathcal{A}$, $A \cap B \supseteq Z$ holds for some $Z \in \mathbb{Z}$. Define $\overline{m}(\mathcal{Z})$ similarly except only $(A \triangle B) \cap Z = \emptyset$ is required. Clearly $m(\mathcal{Z}) \leqslant \overline{m}(\mathcal{Z})$; show that equality holds.

[Hint. Take a family with $|\mathcal{A}| = \overline{m}(\mathcal{Z})$. Choose, if possible, $i \in [n]$ and $A \in \mathcal{A}$ with $A \cup \{i\} \notin \mathcal{A}$. Replace simultaneously all such A by $A \cup \{i\}$. Repeat.]

Solution. Take a family with $\overline{m}(\mathcal{Z})$. Choose, if such exists, $i \in [n]$ and $A \in \mathcal{A}$ with $A \cup \{i\} \notin \mathcal{A}$. Replace simultaneously all such A by $A \cup \{i\}$ to obtain $\mathcal{A}'$.

We show $\mathcal{A}'$ still has the required property. If A, $B \in \mathcal{A}'$ have both been upgraded or stayed the same, there is no problem, so without loss of generality we have $A \cup \{i\}$ with $A \in \mathcal{A}$, $A \cup \{i\} \notin \mathcal{A}$ and $B \in \mathcal{A}$. If $\{i\} \in B$, we are done as A, $B \in \mathcal{A}$ and we have only made $A \triangle B$ smaller by adding $\{i\}$ to A. So without loss of generality, $\{i\} \notin B$, and so $(A \cup \{i\}) \triangle B = (A \triangle B) \cup \{i\} = A \triangle (B \cup \{i\})$ and by assumption $B \cup \{i\} \in \mathcal{A}$, and thus we can find $Z \in \mathcal{Z}$ with required property.

Do this until it stops, as we are adding elements to the sets in $\mathcal{A}$ this must stop eventually. The newly constructed $\mathcal{A}'$ has the same number of elements as $\mathcal{A}$. Suppose A, $B \in \mathcal{A}$. We have that there is $Z \in \mathcal{Z}$ such that $(A \triangle B) \cap Z = \emptyset$. Suppose $i \in Z \backslash (A \cap B)$. We find that i cannot be in both (then in $A \cap B$) or in one of then (then in $A \triangle B$), so i is in neither. Thus we must have $B' = B \cup (Z \backslash (A \cap B)) \in \mathcal{A}$ by assumption and induction. Thus there is $Z' \in \mathcal{Z}$ such that $(A \triangle B') \cap Z' = \emptyset$ and thus $i \notin Z' \backslash (A \cap B')$. However, $A \cap B' = A \cap B$ by construction.

Exercise 7.2.3. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that, for all A, $B \in \mathcal{A}$, $A \cap B \supseteq \{i, i+1\}$ hold for some $i \in [n-1]$. By consider the projections of $\mathcal{A}$ onto the odd or even numbers, show that $|\mathcal{A}| \leqslant 2^{n-2}$.

Solution. Project $\mathcal{A}$ onto the even numbers, by assumption it is still intersecting and thus has size at most $2^{n/2-1}$ (n even). The projection maps $2^{\lfloor n \rfloor/2}$ sets to the same set, at most $2^{\lfloor n/2 \rfloor - 1}$ of these are allowed, as these still have to be intersecting on the odd numbers (use projection again, this time injective). So in total there are at most 2^{n-2}.

Exercise 7.2.4. Prove that a decomposition of the edges of K_n into a non-trivial edge-disjoint complete subgraphs requires at least n subgraphs. Show that this bound can be achieved.

Solution. Suppose $\mathcal{A} \subseteq [n]^{(r)}$ is such a decomposition. Consider the dual system with $\mathcal{A}^* \subseteq \mathcal{P}\mathcal{A}$ where $\mathcal{A}^* = \{A_x^* \colon x \in [n]\}$ and $A_x^* = \{A \in \mathcal{A} \colon x \in A\}$. We then have $\left|\mathcal{A}_x^* \cap \mathcal{A}_y^*\right| = 1$ for all x, $y \in [n]$ distinct. Thus by Fisher's Inequality, $|\mathcal{A}^*| = n \leqslant |\mathcal{A}|$.

Remains to show that if $V_1, \ldots, V_{n+1}$ is a cover of K_{n+1} then it is of the required form.

Exercise 7.2.5. Let $\pi \subseteq X^{(r+1)}$, $|\pi| \geqslant 1$, be such that $|l \cap l'| = 1$ for all distinct l, $l' \in \pi$, and $|\{l \in \pi \colon x \in l, x' \in l\}| = 1$ for all distinct x, $x' \in X$. Show $|\pi| = |X| = r^2 + r + 1$. ($\pi$ is called a **PROJECTIVE PLANE OF ORDER** r.)

Solution. We can apply Fisher's equality to X and π and their dual system, and so $|\pi| = |X|$. There are $\binom{|X|}{2}$ pairs (x, x') with $x \neq x'$ and any line l contains exactly $\binom{r+1}{2}$ of these, so

$$\frac{|X|\,(|X|-1)}{2} = \binom{|X|}{2} = |\pi| \binom{r+1}{2} = \frac{|\pi|\,(r+1)r}{2} = \frac{|X|\,(r^2+r)}{2}.$$

Thus we have $|X| = r^2 + r + 1$.

Exercise 7.2.6. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that $|A \cap B|$ is even for all A, $B \in \mathcal{A}$, $A \neq B$. Prove that if $|A|$ is odd for all $A \in \mathcal{A}$ then $|\mathcal{A}| \leqslant n$, whilst if $|\mathcal{A}|$ is even for all $A \in \mathcal{A}$ then $|\mathcal{A}| \leqslant 2^{n/2}$.

Solution. Theorem 5.4 applies with $p = 2$ and $\mathcal{L} = \{0\}$. We have $s = 1$ and so $|\mathcal{A}| \leqslant \binom{n}{1} = n$.

In the even case, let $x_A \in \mathbb{Z}_2^n$ be the characteristic vector of $A \in \mathcal{A}$ and let U be the subspace of $\mathbb{Z}_2^n$ spanned by these. We have $\dim(U) + \dim(U^\perp) = n$ but U is self-orthogonal, thus $U \subseteq U^\perp$ and so $\dim(U) \leqslant n/2$ and $|\mathcal{A}| \leqslant |U| \leqslant 2^{n/2}$.

Exercise 7.2.7. Let $\mathcal{A} \subseteq [n]^{(k)}$ be such that $|A \cap B| \equiv l \bmod m$ whenever $A \neq B \in \mathcal{A}$, where $k \not\equiv l \bmod m$. Show that $|\mathcal{A}| \leqslant n$.

Solution. Let $\mathcal{A} \subset [n]^{(k)}$ be such that $|A \cap B| \equiv l \pmod m$ whenever $A \neq B \in A$, where $k \not\equiv l \pmod m$. Show that $|A| \leqslant n$.

Follow the proof of Frankel-Wilson. Work over the field $\mathbb{Q}$.

The matrix $\mathcal{A}_1$ is the $|\mathcal{A}| \times n$ matrix whose rows are the characteristic vectors of the sets in $\mathcal{A}$. The matrix $\mathcal{A}_0$ is the $|\mathcal{A}|\,| \times n$ all-one vector.

Since $\mathcal{A}_1$ has all row sums equal to k, the columns of $\mathcal{A}_0$ (there is only one) are in the space V spanned by the cols of $\mathcal{A}_1$. (Here we have used that the sets in $\mathcal{A}$ are all size k, not just congruent to k.)

Setting $\mathcal{B}_i = \mathcal{A}_i\mathcal{A}_i^\mathsf{T}$, the columns lie in V. Here, the (i,j) entry of $\mathcal{B}_1$ is the intersection size of the ith and jth members of A, and $\mathcal{B}_0$ is the all-one $|\mathcal{A}| \times |\mathcal{A}|$ matrix. Let $\mathcal{B} = \mathcal{B}_1 - l\mathcal{B}_0$. Then the entries of $\mathcal{B}$ are integers, its columns lie in V. The diagonal entries of B are equal to $k-l$, the off-diagonals are divisible by m. Since $k-l$ is not divisible by m, there is some prime power $q = p^t$ which divides m but which doesn't divide $k-l$.

If $\mathcal{B}$ is non-singular then as the columns $\mathcal{C}_j$ of $\mathcal{B}$ are in V, we find that $|\mathcal{A}| \leqslant n$ as required. If not, there is a linear combination $\sum c_j\mathcal{C}_j = 0$. We may suppose c_j are integers with no common factor. But, for the jth row, the only number in the sum not divisible by p^t is the diagonal entry, so c_j must be divisible by p. This contradicts the c_j having no common factor.

Exercise 7.2.8. Let p be a prime and let $\mathcal{L} \subseteq \mathbb{N}\cup\{0\}$. Let $B_i \subseteq A_i \subseteq [n]$ be such that $|B_i| \notin \mathcal{L} \bmod p$, $1 \leqslant i \leqslant m$, but $|A_j \cap B_i| \in \mathcal{L} \bmod p$ for $1 \leqslant i < j \leqslant m$. Show that $m \leqslant \sum_{j=0}^{|\mathcal{L}|} \binom{n}{j}$.

Solution. Repeat the proof of Theorem 5.3 with Theorem 5.3 with GF(p) in place of $\mathbb{R}$ and $f_i \prod_{l\in\mathcal{L}}(\langle x_A, x\rangle - l)$. Then $\tilde{f}_i(x_{A_i}) = \tilde{f}_i(x_{B_i}) \neq 0$ as $B_i \subseteq A_i$ and $|B_i| \notin \mathcal{L} \bmod p$ and we have $\tilde{f}_i(x_{A_j}) = 0$ for $i < j$.

To show the f_i are independent, let j be least such that $\{f_1, \ldots, f_j\}$ is dependent, so $0 = \sum_{i\leqslant j} c_i\tilde{f}_i = 0$ where without loss of generality $c_j \neq 0$ by minimality of j. Then $0 = \sum_{i\leqslant j} c_i\tilde{f}_i(x_{A_j}) = 0$. Contradiction.

Exercise 7.2.9. Let p be a prime and let $\mathcal{L} \subseteq \mathbb{N} \cup \{0\}$. Let $k \in \mathbb{N}$. Let $\mathcal{A} \subseteq \mathcal{P}[n]$ be such that $|A| \notin \mathcal{L} \bmod p$ for each $A \in \mathcal{A}$, but $|A_1 \cap A_2 \cap \cdots \cap A_k| \in \mathcal{L} \bmod p$ for any collection of k distinct sets from $\mathcal{A}$. Show that $|\mathcal{A}| \leqslant (k-1)\sum_{j=0}^{|\mathcal{L}|} \binom{n}{j}$.

[Hint. Partition $\mathcal{A}$ into subfamilies $\{C_1, \ldots, C_d\}$, put $A_i = C_1$, $B_i = \bigcap_{t=1}^{d} C_t$.]

Solution. Note $\mathcal{A}$ still has this property if we remove any number of sets. Fix $A \in \mathcal{A}$ consider maximal set $\{C_1, \ldots, C_d\}$ where $C_i \in \mathcal{A}$, $A = C_1$ and $C_1 \cap \cdots \cap C_l \notin \mathcal{L} \bmod p$ for all $l \leqslant d$. Such a sequence exists as $A \notin \mathcal{L} \bmod p$ and $d < k$ by assumption. Repeat with $\mathcal{A}\backslash\{C_1, \ldots, C_d\}$

to find a partition of $\mathcal{A}$. Set A_i the initial members of these maximal sets, B_i the intersection of the members.

For each $d < k$ ($k-1$ of these) apply Exercise 7.2.8 to the A_i and B_i belonging to maximal sets of size d. These satisfy the hypothesis as $|A_j \cap B_i| \in \mathcal{L} \bmod p$ for $i < j$ by maximality of the chain starting at A_i, which was created before A_j. Thus our partition of $\mathcal{A}$ and thus $\mathcal{A}$ can have size at most $(k-1)\sum_{j=0}^{\mathcal{L}} \binom{n}{j}$.

Exercise 7.2.10. Prove there is only one (up to isomorphism) extremal hypergraph for the saturation property, but in general many for the weak-saturation property.

Solution. Let us first show equality can only hold in Theorem 2.1 only if all the r_i are the same, say r, and all the s_i are the same, say s, and the system consists of all partitions of some $r+s$ set into an r-set and an s-set.

Note that if any R or S is empty then $|I| = 1$ and we are done. So all sets are non-empty. Proceed by induction on n. It is easy to check for small n. We may assume that every $x \in [n]$ is in at least one R or S set; otherwise throw this x away and finish immediately by induction.

Following the proof, we find that equality must hold for all I_x, in particular for each x, I_x is non-empty and the sets R_i with $x \notin R_i$ are all the same size, say r_x. Moreover, the pairs (R_i, S_i^x) are all complementary pairs from some set Z_x of size $r^x + s^x$.

Suppose $s^x = 0$ for every x. Then, for each x, I_x has one member, and the corresponding S is non-empty so $S = \{x\}$. Given any R_i, pick $x \notin R_i$, then $S_i = \{x\}$. So $R = [n] - \{x\}$. So the result follows for $r = n-1$ and $s = 1$.

Otherwise, $s^x > 0$ for some x, so $r^x < n-1$. Pick y different from x. Then there's an r^x-set in Z_x not containing y. Hence this R-set must have size r^y. So $r^x = r^y$. That is, all R_i have the same size, r say.

But note the result, unlike the proof, is symmetric in R and S; so could do a corresponding proof with $J_x = \{i \in I \colon S_i \subseteq [n] \backslash x\}$. So we may conclude that all S_i have the same size, s say.

Now consider again the complementary pairs (R_i, S_i^x) in Z_x. The sets S_i^x all have the same size; either this size is s, and $S_i^x = S_i$ for all i, or it is $s-1$, and $S_i^x = S_i - \{x\}$ for all i. But the first case is impossible, because x must be in some R_i (or some S_i), and we could then pick a pair (R_j, S_j) in Z_x with R_i disjoint from S_j (or S_i disjoint from R_j).

So, for all x, the second case applies, that is, if $x \notin R_i$ then $x \in S_i$. Thus $r+s=n$ and $R \cup S = [n]$ for every pair. But we must have all such pairs to achieve equality, and the claim is proved.

The result now follows by following the proof of Theorem 2.2: we have shown the (R_i, S_i) are pairs of complementary subsets from some $n-t$ set. In particular, all missing edges are from this set, and we are done.

Exercise 7.2.11. Let $X_1, \ldots, X_n$ be n disjoint sets. For each i, $1 \leqslant i \leqslant n$, let A_{ij} and B_{ij}, $1 \leqslant j \leqslant h$, be subsets of X_i satisfying $|A_{ij}| \leqslant r_i$ and $|B_{ij}| \leqslant s_i$, $(\bigcup_i A_{ij}) \cap (\bigcup_i B_{ij}) = \emptyset$ for $1 \leqslant j \leqslant h$ and $(\bigcup_i A_{ij}) \cap (\bigcup_i B_{il}) \neq \emptyset$ for $1 \leqslant j < l \leqslant h$. By considering the vector space $V = V_1 \oplus \cdots \oplus V_n$, where V_i is the space generated by X_i, prove that

$$h \leqslant \prod_{i=1}^{n} \binom{r_i + s_i}{s_i}.$$

Solution. This is a generalisation of Theorem 2.3.

Let $W_i = \mathbb{R}^{r_i+s_i}$ and $V_i = \bigwedge W_i$. Let $v_{A_{ij}}$, $v_{A_{ij}} \in V_i$ as in that proof. Note $v_{A_{ij}} \wedge v_{B_{il}} = 0$ if and only if $A_{ij} \cap B_{il} \neq \emptyset$. Consider

$$\begin{aligned} x_j &:= v_{A_{1j}} \oplus \cdots \oplus v_{A_{nj}} \\ y_j &:= v_{B_{1j}} \oplus \cdots \oplus v_{B_{nj}} \end{aligned}$$

$x_j \cap y_j \neq 0$ as $A_{ij} \cap B_{ij} = \emptyset$ for all i. Similarly, $x_j \wedge y_l = 0$ if $j < l$ for there is i such that $A_{ij} \cap B_{il} \neq \emptyset$. Suppose $\sum_j c_j x_j = 0$, let $j = \max\{i \colon c_i \neq 0\}$ then

$$0 = \left(\sum_j c_j x_j \right)_l \wedge y_l = c_j x_l \wedge y_l \neq 0$$

So $\{x_j \colon 1 \leqslant j \leqslant h\}$ is a linearly independent subset of $\bigwedge^{r_1} V_1 \oplus \cdots \oplus \bigwedge^{r_n} V_n$ and the result follows.

Exercise 7.2.12. An r-uniform r-partite hypergraph with n_i vertices ith class (each edge having one vertex per class) is denoted by $G(n_1, \ldots, n_r)$. The complete r-uniform r-partite hypergraph $G(s_1, \ldots, s_r)$ is denoted by $K^{(r)}(s_1, \ldots, s_r)$. The graph $G(n_1, \ldots, n_r)$ is said to be **WEAKLY** $K^{(r)}(s_1, \ldots, s_r)$**-SATURATED** if there is a sequence of graphs $G = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_t = K^{(r)}(n_1, \ldots, n_r)$ such that G_i is obtained from G_{i-1} by the addition of an edge, and G_i contains more copies of $K^{(r)}(s_1, \ldots, s_r)$ than does G_{i-1}. Show that such a $G(s_1, \ldots, s_r)$ has size at least $\prod_{i=1}^r n_i - \prod_{i=1}^r (n_i - s_i + 1)$.

Solution. Let $S_j = \{b_{ij} \colon 1 \leqslant i \leqslant r\}$ denote the added edge that created a copy of $K^{(r)}(s_1, \ldots, s_r)$ in G_i that was not present in G_{i-1}, say K. $B_{ij} = \{b_{ij}\}$. Set $A_{ij} = \{[n_i] - V_i(K_j)\}$. Then we have $|B_{ij}| = 1$ for all i and j. $A_{ij} \cap B_{ij} = \emptyset$ for all i as K was created at this stage, not earlier. Suppose $1 \leqslant j < l \leqslant t$, then there is an i such that $A_{ij} \cap B_{il} \neq \emptyset$, as this just means that the edge added is not completely in $V_i(K_j)$, which it cannot be, otherwise it would have already been in G_j, and thus cannot be S_j.

By Exercise 7.2.11, we have that the number of edges added is bounded, and thus get a lower bound on the size of $G(s_1, \ldots, s_r)$ by

$$\prod_{i=1}^r n_i - \prod_{i=1}^r \binom{n_i - s_i + 1}{1} = \prod_{i=1}^r n_i - \prod_{i=1}^r (n_i - s_i + 1).$$