Week 2

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Abstract

These are notes on topics I am currently interested in. This is also an attempt to be more organized, as was pointed out by my best friend.

1 Singularity Theorems

Requirements: basic notions of topology, differential geometry, black holes, and the cosmological FLRW model.

We don't know what a singularity is. What we do know is that:

- A singularity is not[1] a place in spacetime, i.e it is not a point on the manifold (M, g).
- A singularity is not necessarily a region where the curvature tensor R^d_{abc} blows up.
- A singularity is $\sim$ geodesic incompleteness!

The first two items should not be surprising to you. A singularity cannot have a place and time, since the metric is *not* defined there, hence the manifold is not defined at that point. Moreover, if you remove a cone of Minkowski spacetime, the curvature is everywhere zero except at the conical singularity where it is not defined. Hence we have a singularity, but the curvature does not blow up.

The best definition we have is that a spacetime has a singularity if there exists at least one inextendible geodesic that terminates at some finite affine parameter. A manifold with this property is called geodesically incomplete. We know that every timelike geodesic represents the wordline of a free particle, so an incomplete geodesic suggests that time and space cease to exist for such particle at a finite proper time.

This is a good definition because it does not rely on any choice of coordinate system, nor does it use any type of symmetry argument such as those used to derive the Schwarzschild solution (spherical symmetry) or the cosmological FLRW metric (isotropy). At the same time, proving that such geodesics exist inevitably involves sophisticated mathematical analysis of the global topological properties of manifolds.

[1] There were some attempts to 'locate' the singularity by adding them as a boundary of the spacetime manifold, but it didn't quite worked.

The singularity theorems give you the conditions under which spacetime has incomplete geodesics and really show that singularities are ubiquitous in general relativity.

Even more suspicious is the fact the all [2] singularities are hidden behind horizons, which means that they are inacessible to us. Where gravity start to behave badly, and quantum effects begin to dominate, it hides from us as if they don't want us to see quantum gravity effects! And, as you learned in kindergarten, something funny happens when you try to quantize gravity. All these facts add up to show that quantum gravity is, in fact, a real problem that should be addressed.

1.1 The intuition

The proof of the theorem will follow by contradiction. based on three key facts:

- Any globally hyperbolic spacetime has a path that maximizes proper time between any two points.
- Geodesics fail to be the path of maximal proper time after extended past a conjugate point.
- Any spacetime that satisfies the strong energy conditions and has a congruence of negative expansion $\theta < 0$ will contain conjugated points reachable after a finite affine parameter.

The first item would be in contradiction with the others, unless geodesics are incomplete. My goal in this section is to provide an intuitive understanding of each of these facts. The actual proof of the theorem and the lemmas will be presented in the next section. Let us first define some basic causal sets:

Definition 1. [3]*A path in spacetime $\lambda : I \to M$ is said to be:*

- *timelike, if its tangent vector field V obeys $g(V, V) < 0$*
- *causal, if its tangent vector field V obeys $g(V, V) \leq 0$*

Definition 2. *The set of all points connected to $p \in M$ by a future-directed timelike path*

$$I^+(p) = \{q \in M | \exists \lambda : I \to M,\ \textit{where } g(V, V) < 0 \textit{ and } \lambda(t) = p, \lambda(t') = q \textit{ given that } t < t'\}$$

is called the chronological future of p. The causal future $J^+(p)$ is defined similarly but with causal paths instead of timelike paths connecting p to q.

Definition 3. *Let $p, q \in M$ where $q \in J^+(p)$. The set of all continuous causal paths from p to q is defined by*

$$C(p, q) = \{\lambda : I \to M | g(V, V) \leq 0 \textit{ and } \lambda(t) = p, \lambda(t') = q \textit{ given that } t < t'\}$$

[2]...assuming Cosmic Censorship

[3]From here on $sgn(g) \sim (- + ++)$

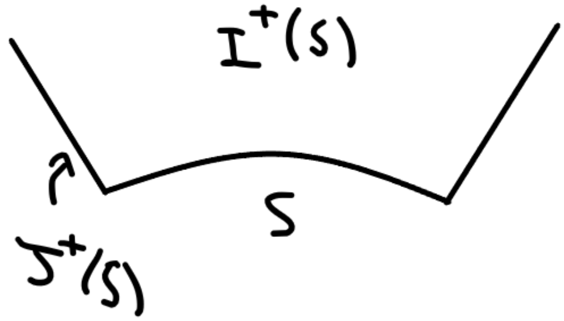

Figure 1: Causal structure of the subset $S \subset M$

The elapsed proper time (in loretzian signature) between two points $p, q \in M$ where $q \in J^+(p)$ is given by:

$$\tau(\lambda) = \int_p^q \sqrt{-g(V,V)}dt \tag{1}$$

where $V : M \to T(M)$ is the tangent vector field associated to the path $\lambda(t)$. It is clear that the proper time is a functional $\tau : C(p,q) \to \mathbb{R}$ defined on the space of all causal paths between two points. Therefore the existence of geodesics depends on the topological properties of the space $C(p,q)$, particularly the property of compactness and the continuity of τ. If both these conditions are met, then it is a basic result of topology that τ attains a maximum value for some $\lambda \in C(p,q)$. Topology plays an essential role! We will see latter that the compactness and (semi upper) continuity will be true if the spacetime under consideration is *globally hyperbolic*, a condition that is related to the predictability of general relativity.

Now, we are safe that any topologically well-behaved spacetime will always have a path that maximizes proper time between any two points $p, q \in M$. So how can a geodesic stop? That is a complicated question. So let us first look at how some geodesics stop being the path that maximizes proper time.

For example, take the Riemaniann manifold given by the sphere $M = S^2$ and consider two neighbouring geodesics λ_1, λ_2 parting from the North pole (N) as in Figure 2. At finite affine length $t = \pi$ they will inevitably meet again at the South pole (S). We call the points N, S to be *conjugated points* or *focusing points*. Notice that if we were to extend any one the geodesics beyond their meeting point at S to some point p, the path would no longer represent the minimal distance between N and p. A similar situation occurs in Lorentzian manifolds that satisfy the strong energy conditions. To illustrate this, we need an equation that describes how nearby geodesics move relative to each other. The geodesic deviation equation provides exactly this information.

Definition 4. *Let $O \subset M$ be open. A congruence of paths in O is a set of paths such that each $p \in M$ is hit by a path in the congruence. This is equivalent to define a vector field in O.*

Definition 5. *Let λ be a timelike geodesic with tangent vector field V^a. A Jacobi field* *on λ is a solution η^a to the geodesic deviation equation*

$$V^a \nabla_a (V^b \nabla_b \eta^c) = -R_{abd}{}^c \eta^b V^a V^d \tag{2}$$

Two points $p, q \in \lambda$ are said to be conjugated if there exists a non-trivial Jacobi field such that $\eta^a(p) = 0$ and $\eta^a(q) = 0$

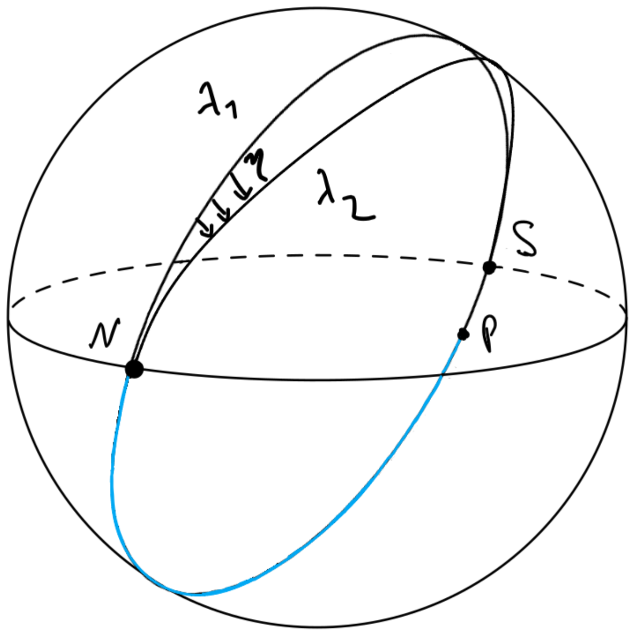

Figure 2: N and S are congujated points of the geodesic λ. After extending the geodesic to p, the path no longer represents the minimal distance which is the blue path. η represents the Jacobi field of λ_1.

This is very intuitive. The Jacobi field measures the infinitesimal displacement between nearby geodesic in a congruence. Hence it's zero at the conjugated points. In the next section we will show that the trace of equation 2 can be put into a simpler form know as the Raychaudhuri equation:

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}V^a V^b \tag{3}$$

where θ is the expansion of the congruence of geodesics. Intuition says that the geodesic congruence will converge provided that $\theta \to -\infty$. In fact we will show that a Jacobi field exists if $\theta \to -\infty$ at some point. Now if we assume the geodesics are hypersurface orthogonal $\omega_{ab} = 0$ and $R_{ab}V^a V^b \geq 0$ then equation 3 becomes:

$$\frac{d\theta}{d\tau} + \frac{\theta^2}{3} \leq 0 \implies \theta^{-1}(\tau) \geq \theta_0^{-1} + \tau/3$$

Here's the catch: if $\theta_0 \leq 0$ in the congruence, like the past direct geodesics in cosmology, then $\theta \to -\infty$ in a finite proper time $\tau \leq \frac{3}{|\theta_0|}$. We can use Einstein's field equations to show that these conditions are equivalent:

$$R_{ab}V^aV^b \geq 0 \implies T_{ab}V^aV^b \geq -T/2$$

This is called the strong energy condition: we have to impose a positive energy density $\rho = T_{00} \geq 0$ and low negative pressures $p = T_{ii}$ compared to ρ. This is satisfied by all know forms of matter. Under such conditions, geodesics will inevitably reach conjugated points after some finite affine parameter. As in the sphere case, if we try to extend any of the geodesics past the point q in figure 3, we see that the geodesic is no longer the path that maximizes proper time.

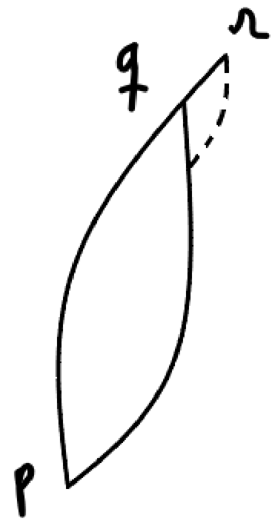

Figure 3: The path through the dashed line has proper time that is greater than the extended geodesic. You may think about the triangle inequality.

We can finally state the theorem: A globally hyperbolic spacetime which satisfy the strong energy conditions and $\theta_0 \leq 0$ at some point $r \in M$, will be geodesically incomplete. In particular, geodesics must stop at finite proper time $\tau \leq \frac{3}{|\theta_0|}$.

The proof follows by contradiction: supose that such spacetime does have a curve λ, $\lambda(0) = p$ with lenght $\tau > \frac{3}{|\theta_0|}$. Take any point in the curve beyond $\frac{3}{|\theta_0|}$. By global hyperbolicity, there exist a geodesic between such point and p. But this contradicts the fact that there is a conjugated point and hence it cannot be a geodesic. Therefore all paths must terminate at time $\tau \leq \frac{3}{|\theta_0|}$.

1.2 The proof

References