Filling the gaps in geometry and relativity

Thomas F. C. Bastos

May 4, 2024

Abstract

The interplay between geometry and physics has been known for many decades, but it's only in fundamental physics where we can fully appreciate this relationship. In these notes, we will uncover one of these theories, General Relativity, and see how differential geometry plays an essential role. In the first part, we will strive to understand what curvature is, using ideas from Gauss, Cartan and Riemann. The second part is concerned with gravity: how it is an effect of the curvature of spacetime, and how such ideas came into being. It is meant to be an introduction to the introduction of general relativity, and for those who are already acquainted with the theory, it should fill some gaps that may have been left when you studied GR for the first time.

1 What is curvature?

In Book XI of the Confessions (397) Saint Augustine was trying to understand time. There he said something that struck me for a while:

> "What then is time? If no one asks me, I know; if I want to explain it to a questioner, I do not know." (Augustine, p. 242)

I believe the same is true for many concepts, but it is specially true for curvature: you know exactly what it is, except when you have to compute it. Instead of looking at intuition as being a bug in our brains, I like to think it's actually a feature. It's time to roll up our sleeves and try to build curvature and geometry from our intuition. We will thank Augustine for this insight latter.

We start with one dimensional geometries, i.e. curves, and quickly exhaust everything there is to know about them using frame fields. It turns out that this method is extremely powerful and, provided some suitable modifications, it will carry over to any number of dimensions.[1] In our path we will naturally encounter concepts like manifolds, tangent spaces, metrics, connections and covariant derivatives. Hopefully this will make you see the whole picture and don't be bothered asking yourself questions like: why in the world is curvature a (1,3) tensor?

[1] The method of frames is so awesome that it can be used to understand curvature in fiber bundles!

1.1 Curves and the power of Frenet Frames

We start with the simplest possible geometry: a parametrized curve $\gamma(t) : I \to \mathbb{R}^3$ in three dimensions. In figures 1 and 2, we can clearly see that 1 has no curvature while 2 has some curved sections. Curvature is a local property. Another basic fact is that a circle with a small radius is more curved than a circle with a large radius. This is easy to see: if zoom in on any circle, it starts to look like a straight line. Therefore the curvature k of a circle should be inversely proportional to its radius R:

$$k \sim \frac{1}{R} \tag{1}$$

Now consider the helix in figure. It is intuitive that the helix not only have curvature but also twist as it moves, a feature that is not shared with the circle. Notice that all of these features are defined by how the curve changes it's directions thought space. Let's formalize all of these statements.

Since we are only interested in the directions, we may only consider paths with unit speed $\|\gamma'\| = 1$ without loss of generality.

Definition 1. *A regular curve is a differentiable map $\gamma : I \to \mathbb{R}^3$ such that $\|\gamma'(s)\| = 1$*

There is a standard procedure to take any path $\gamma(t)$ without cusps and make it a regular curve. First calculate the integral

$$s = \int_0^t \|\gamma'\| \, dt$$

and then take the inverse of the function $s = s(t)$ so that $t = t(s)$ and $\gamma(s) = \gamma(t(s))$ where $\|\gamma'(s)\| = 1$. This is sometimes called arc lenght parametrization.[2] Making the rather confusing relabel $T \doteq \gamma'$ we are ready to define curvature:

Definition 2. *The curvature $k(s)$ of a regular curve γ is given by:*

$$T' = kN \tag{2}$$

where $N(s)$ is a unit vector field $\|N\| = 1$.

At first this may look abstract but it's precisely what we wanted: k measures the amount of change of direction of the unit tangent T. We can use the inner product $\langle T, T \rangle = 1$ to show that the vector fields are orthogonal:

$$\langle T, T \rangle' = 2k\langle T, N \rangle = 0$$

Now define another vector field $B \doteq T \times N$ which is again unitary and orthogonal to both T and N. You should picture these vector fields T, N, B as moving frames attached to the curve.

Definition 3. *Given a regular curve γ its Frenet frame is the set of vector fields $\{T, N, B\}$.*

[2] Forget about the parameters t, s as being time, it's just a real variable.

You can use the inner product to show (exercise) that $B' = -\tau N$ where $\tau(s) \in \mathbb{R}$ is called the torsion. The most important thing about the Frenet frames is that their derivatives T', N', B' are expressed in terms of themselves T, N, B. In this way we can keep track of all the changes in all directions of the curve.

Theorem 1.1. *If γ is a regular curve and T, N, B are its Frenet frame fields, then*

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{3}$$

Proof. The first and last equations of 3 are just definitions. Since the Frenet frame fields are orthonomal:

$$N' = \langle N', T\rangle T + \langle N', N\rangle N + \langle N', B\rangle B$$

From the identity $\langle N, T\rangle = 0$ it follows that $\langle N', T\rangle = -\langle N, T'\rangle = -k$ and by similar arguments $\langle N', B\rangle = \tau$ □

At this point we should test if this definition agrees with our expectations. A straight line $\gamma(s) = a + bs$ where $\|b\| = 1$ has curvature $k = \|b'\| = 0$. A circle can be parametrized by $\gamma(t) = (R\cos t, R\sin t, 0)$ but its arc lenght parametrization is different:

$$s = \int_0^t \|\gamma'(t)\|\, dt = Rt$$

so that $\gamma(s) = (R\cos\frac{s}{R}, R\sin\frac{s}{R}, 0)$. The curvature is then

$$k = \|T'\| = \|\gamma''(s)\| = \left\| -\frac{1}{R}(\cos\frac{s}{R}, \sin\frac{s}{R}, 0) \right\| = \frac{1}{R}$$

and the torsion is $\tau = 0$ since $B' = 0$. This is exactly what we expected from equation 1 and also from the lack of twisting of a circle. An arc lenght parametrization of the helix is (exercise):

$$\gamma = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{b}{c}s)$$

where $c^2 = a^2 + b^2$. Then $k = \frac{a}{c^2}$ and $\tau = \frac{b}{c^2}$. Notice that when $b = 0$ the torsion goes to zero and the helix colapse in a circle.

These are some nice results, but there is a strong statement about curvature and torsion when you integrate the Frenet formulas 3:

Theorem 1.2. *Given two scalar fields $k, \tau : I \to \mathbb{R}$ there exists one regular curve (up to isometries) $\gamma : I \to \mathbb{R}^3$ such that its curvature and torsion are given by k, τ.*

In other words, k and τ completely determines the geometry of a curve. Well, that was kinda easy. The take away should be that if we want to understand curvature, we better have a way to construct frame fields attached to every point of our geometric object and express their derivatives in terms of themselves like in 3.

The thing is that there is nothing special about curves. We could choose a frame field that is defined on every point of $\mathbb{R}^3$. We would expect them to describe the curvature and torsion of the whole space $\mathbb{R}^3$, so it's interesting to generalize the formalism. But before we jump into that, we need to define extra objects that will greatly help us. These are covariant derivatives and forms. Let's start with some terminology:

Definition 4. *For each point $p \in \mathbb{R}^3$ the set of all vectors with base at p is denoted by $T_p(\mathbb{R}^3)$ and is called the tangent space at p. The set of all tangent spaces is called the tangent bundle $T\mathbb{R}^3$.*

Definition 5. *A smooth vector field is a smooth map $V : \mathbb{R}^3 \ni p \mapsto V(p) \in T_p(\mathbb{R}^3)$.*

Since we want to see how vector fields change in any direction, it's useful to define derivatives of these objects.

Definition 6. *Let W and V be smooth vector fields. The covariant derivative of W in the direction of V is another vector field $\nabla_V W$ such that:*

$$\nabla_V W(p) = W(p + V(p)t)'|_{t=0}$$

The definition of a covariant derivative looks like something you have never seen before, but you actually did. Take a point p, a vector $V(p)$ pointing somewhere with base on p and compute the directional derivative $W(p+V(p)t)'|_{t=0}$ of W in the direction $p+V(p)t$. For example, let $V = -y\hat{x} + x\hat{y}$ and $W = \cos(x)\hat{x} + \sin(x)\hat{y}$ be vector fields and $p = x\hat{x} + y\hat{y} + z\hat{z}$ be any point in $\mathbb{R}^3$. Then

$$p + V(p)t = (x - yt)\hat{x} + (y + xt)\hat{y} + z\hat{z}$$

$$\nabla_V W(p) = W(p + V(p)t)'|_{t=0} = y\sin(x)\hat{x} + x\cos y\hat{y}$$

Remember when your calculus teacher said that the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

is not well defined? They lied to you (for a good reason). We can define them using forms, which turns out to be *extremely* useful in physics and mathematics[3]

Definition 7. *A one-form is a linear map $\omega_p : T_p(\mathbb{R}^3) \to \mathbb{R}$. A one-form field is a linear map $\omega : T\mathbb{R}^3 \to \mathbb{R}$.*

Definition 8. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. The differential df is a one-form such that:*

$$df(v) = v(f) = f(p + vt)'|_{t=0}$$

where $v \in T_p(\mathbb{R}^3)$.

Lemma 1.3. *The differentials dx, dy, dz form a basis of $T^*\mathbb{R}^3$*

Corollary 1.3.1. *The differential can be written as $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$*

Definition 9. *Wedge product*

Needs a lot of attention. Mention the electromagnetic field as a one-form, and faraday tensor as a two-form $F = dA$.

[3] That is an understatement.

Definition 10. *A frame field is a set of vector fields E_1, E_2, E_3 in $\mathbb{R}^3$ such that:*

$$\langle E_i, E_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.

We know that something funny happens when we express derivatives of these frames in terms of themselves. Let V be any vector field and

$$\begin{aligned}
\nabla_V E_1(p) &= \omega_{11} E_1(p) + \omega_{12} E_2(p) + \omega_{13} E_3(p) \\
\nabla_V E_2(p) &= \omega_{21} E_1(p) + \omega_{22} E_2(p) + \omega_{23} E_3(p) \\
\nabla_V E_3(p) &= \omega_{31} E_1(p) + \omega_{32} E_2(p) + \omega_{33} E_3(p)
\end{aligned}$$

In a shorter notation using Einstein's convention of repeated indices $\nabla_V E_i(p) = \omega_{ij} E_j(p)$, the coefficients ω_{ij} clearly depends on the vector V and looks like a one-form field:

Lemma 1.4. *Let E_i be frames fields in $\mathbb{R}^3$, $v \in T_p(\mathbb{R}^3)$ any vector and*

$$\omega_{ij}(v) = \langle \nabla_v E_i(p), E_j(p) \rangle$$

Then ω_{ij} is a matrix valued one-form and $\omega_{ij} = -\omega_{ji}$. They are called connection coefficients.

Proof. You can use definition 6 and the chain rule to prove that $\nabla_{av+bu} E = a\nabla_v E + b\nabla_v E$ for any vectors $u, v \in T_p(\mathbb{R}^3)$ and numbers $a, b \in \mathbb{R}$. Therefore

$$\omega_{ij}(av + bu) = \langle \nabla_{av+bu} E_i(p), E_j \rangle = a\omega_{ij}(v) + b\omega_{ij}(u)$$

To prove the antisymmetry of ω notice that

$$\nabla_v \langle E_i, E_j \rangle = \langle \nabla_v E_i, E_j \rangle + \langle E_i, \nabla_v E_j \rangle = 0$$

□

Now we can use the dual 1-forms θ_i of the frame fields:

$$\theta_i(E_j) = \delta_{ij}$$

to show a very interesting result that will clarify the geometry of $\mathbb{R}^3$

Theorem 1.5 (Cartan Structural Equations)**.** *Let E_i be frame fields, θ_i its duals and ω the connection coefficients. Then*

$$\begin{aligned}
d\theta_i - \omega_{ij} \wedge \theta_j &= 0 \\
d\omega_{ij} - \omega_{ij} \wedge \omega_{kj} &= 0
\end{aligned}$$

1.2 Surfaces and Gauss legacy

The next logical step is to go up a dimension. But what exactly is a 2 dimensional surface?

1.3 Higher dimensions, Riemann curvature and the modern stuff

Even though we had much success you can already tell something fishy is going on. It's odd that we are studying a two (one) dimensional geometry by embedding it in a 3d space, and this 3d space has a lot of structure which we take as given: a vector space (really a manifold) equipped with a very specific inner product $\langle .,. \rangle_E$ where E stands for Euclidean. In fact, $\langle .,. \rangle$ is the very definition for something to have a geometry! But we will get there latter.

1.4 Repère Mobile, Frame bundle and principal fiber bundles

---

2 Gravity enters the scene...

Now that we've delved deeply into the intricacies of geometry and curvature, we are in a position to confront the second big question: How exactly does gravity manifest as the curvature of spacetime? How did Einstein come up with this idea in the first place?

lots to be revised. change the order and add spacetime diagrams

2.1 It's simple once you know geometry

The idea that gravity is not a force is needed for Newtonian physics to be consistent! Consider what Newton's first law have to say:

(Law of Inertia): A body follows uniform straight motion unless acted on by a force.

Now imagine a universe with only a single particle. How can this particle tell that it is moving at all? Well, it can't tell. We need at least two particles, one the observer and other the observed. The observer will check with its coordinates and clocks if Newton's law holds for the other particle. But wait a second: both particles have mass, so there is a gravitational force that deviates the uniform straight motion. And if one includes more particles it only gets worse! It is clear that the force of gravity and Newton's first law cannot be both true since this leads to a contradiction. How can we resolve this problem?

It turns out that gravity is not a force, so it does not deviate particles from straight motion in spacetime. But for this to be true we must loose a bit our notion of *straight*, which is equivalent to consider a curved geometry. That is no problem for us! The straightest possible paths are geodesics in a more general geometry. If we can show that the effect of gravity is the same as a geodesic path in some sort of curved space, there are no more contradictions with Newton's first law since gravity is no longer a force.

Let us look at the simplest physical system where gravity is present: a free falling body in a gravitational potential Φ. The equation of motion is:

$$m\ddot{x} = -m\nabla\Phi$$

Notice that the mass appears on both sides. This is the difference of gravity to other forces like the electromagnetic one: all particles follows the same path regardless of the masses. A positively charged particle will follow a trajectory that is very different than a negative charged one when put on the same electromagnetic field. Rewriting the equation of motion

$$\ddot{x}^i + (\nabla\Phi)^i = 0 \tag{4}$$

We want this to look like a geodesic equation, but first derivatives are lacking on the second term. We are stuck: there is only three dimensions of space and none of them can be put together to build a geodesic out of equation (4). Turns out it's 21 century and there is another dimension we can play with: time. If we include the extra coordinate $x^0 = t$ of time on our recipe, it obviously obeys $\ddot{x}^0 = 0$ and we get a system of equations:

$$\begin{cases} \ddot{x}^0 = 0 \\ \ddot{x}^i + (\nabla\Phi)^i \dot{x}^0 \dot{x}^0 = 0 \end{cases}$$

That is exactly a geodesic equation on *spacetime*, not just space! Notice that the law of inertia mentions space *and* time: a uniform straight motion. These are necessarily straight lines on spacetime diagrams, not just straight lines on space[4]. So it should not come as a surprise that we had to include time in the last step of the recipe.

With the geodesic equation we can obtain the connection $\Gamma^i_{00} = (\nabla\Phi)^i$, and $\Gamma i_{jk} = 0$ otherwise. With a connection at hand we find the Riemann curvature tensor $R^i_{0j0} = -\partial_j\partial^i\Phi$. Therefore we conclude that gravity is truly an effect of the curvature of spacetime.

A refinement of Newton's first law is:

(Enhanced Law of Inertia): In the absence of forces all particles follows geodesics in spacetime.

I should insist that in the new definition gravity is not a force. Its effect is encoded in the geodesic motion.

2.2 Generalize Special relativity

In the section above we gave some physical arguments to conclude that gravity must be the curvature of some geometry. But it may seem unnatural to mix space and time together, specially because we were dealing with a galilean structure of flat space and absolute time. A more natural arena is, of course, Minkowski spacetime of special relativity where the geometry is explicitly given to us by a metric $\eta_{\mu\nu}$. Our goal is to introduce gravity in special relativity. This will inevitably lead us to general relativity.

[4]Think about some particle moving on a straight line with constant acceleration.

We know that in special relativity it is best to view space and time together, i.e. spacetime, as a set of points $X^\mu = (ct, x, y, z)$ that resembles the vector space $\mathbb{R}^4$ but has a very different notion of distance between points:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dX^\mu dX^\nu \tag{5}$$

where the metric is defined as $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. When we look for coordinate transformations [5] $X'^\mu = \Lambda^\mu_\nu X^\nu$ that preserves (5) we get to the Lorentz transformations (given that it is proper orthocronus etc etc). They form a group $\Lambda^\mu_\nu \in SO(1, 3)$ where composition is given by the usual matrix multiplication.

The point is that all invariant quantities in SR, i.e. quantities that does not depend on a particular observer, should be made with the aid of the invariant metric (5). For example, the proper time:

$$d\tau = \frac{\sqrt{-ds^2}}{c}$$

is the time measured by a particle's clock in its rest frame $X'^\mu = (ct', 0, 0, 0)$. Different observers will experience different times t but they all agree on the value of τ. With these tools we can easily reconstruct the dynamics of particles in a relativistic invariant way. We define the 4-velocity and 4-momentum as:

$$U^\mu = \frac{dX^\mu}{d\tau}$$
$$P^\mu = mU^\mu$$

Let X^μ be a 4-vector in spacetime. We define $X_\mu X^\mu = X^2 \doteq \eta_{\mu\nu} X^\mu X^\nu$ as the contraction of X with itself. Then you can show that $U^2 = -c^2$ and $P^2 = -(mc)^2$. Newton's second law should be something like:

$$\frac{dP^\mu}{d\tau} = F^\mu$$

That is a nice way of getting to the dynamics of the theory, but it is not clear at all how to include some force or potential $V(x)$ in the equation above. We have look at the Lagrangian formulation of SR: What is the action of a free relativistic particle? Since Lorentz transformations are a symmetry of the system, we should look for a relativistic invariant action. The simplest thing to come up with is the following:

$$S = -m \int d\tau \tag{6}$$

From here on I'll use natural units where $c = 1$. To understand equation (6) first we notice that a particle describes a path in spacetime $X^\mu(\sigma) : \mathbf{R} \to \mathbf{R}^{1,3}$ parametrized by σ. The parameter does not have to be the proper time, we can always reparametrize

[5] The set of coordinate transformations $X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu$ that preserves the metric (5) is called the Poincaré group $\mathcal{P} \simeq O(1, 3) \oplus R^{1,3}$. The reader should notice that this is the group of isometries of spacetime. The Lorentz transformations are the subgroup $SO(1, 3)$ of the Poincaré group connected to the identity.

the path just like we did when analysing curves in the Geometry part of the notes. Thus we can rewrite equation (6):

$$S = -m \int d\sigma \frac{d\tau}{d\sigma} = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} \tag{7}$$

The action is fully relativistic, and more than that: it has reparametrization invariance. This is a form of gauge invariance: changing the parameter should not affect the underlying physics, i.e. the worldline of the particle. The Euler-Lagrange equations gives us (exercise):

$$\frac{d}{d\sigma} \left(m \frac{\partial \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}}{\partial \dot{X}^\rho} \right) = m \frac{d^2 X_\rho}{d\tau^2} = 0$$

Which is exactly what we were expecting from our previous equation $\frac{dP^\mu}{d\tau} = F^\mu = 0$. In the Lagrangian formalism it is straightforward to insert a potential in the theory:

$$S = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} - \int d\sigma \Phi$$

But this does not keep the reparametrization invariance! Any change in the parameter $\sigma \to \sigma'$ will be felt by the jacobian $\frac{\partial \sigma}{\partial \sigma'}$ in the second term. We can get around this by considering a four-potential A_μ instead of the scalar Φ, and contract it with the four-velocity:

$$S = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} - \int d\sigma q A_\mu \dot{X}^\mu$$

where q is just a constant measuring the coupling with the potential. Notice that a reparametrization don't change the action because of the 4-velocity term! And everything keeps lorentz invariance as well. The suggestive notation is going to make more sense once we derive the Euler Lagrange equations for the system:

$$m \frac{d^2 X_\mu}{d\tau^2} = q \left(\frac{\partial A_\mu}{\partial X^\nu} - \frac{\partial A_\nu}{\partial X^\mu} \right) \dot{X}^\nu = q F_{\mu\nu} \dot{X}^\nu$$

This looks exactly like the (covariant) equation of a particle in an electromagnetic field, where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the electromagnetic tensor! You can easily show that it satisfies $\partial_{[\rho} F_{\mu\nu]} = 0$ which is equivalent to the two homogeneous Maxwell's equations. Of course, it will only be Maxwell's EM when the other two inhomogeneous equations are provided $\partial^\mu F_{\mu\nu} = J_\nu$ where $J_\nu = (\rho, \mathbf{J})$ is the 4-current.

Does that mean that relativistic gravity is somehow a kind of gravitomagnetism? You could go in this route but eventually you would find serious inconsistencies in the solutions of your equations. Stuff like infinite energy, instability of simple closed orbits and worse.(reference here)

What now? We could go on with the previous idea and insert a relativistic tensor field of rank 2 $h_{\mu\nu}$ instead of the 4-potential A_μ and see where that leads us. We would actually get linearized general relativity in vacuum! But the procedure is much more involved and subtle. Since this is a field theory we would like to have positive kinetic terms (free of ghosts, if these words even make sense to you) and kill some degrees of freedom. This have to do with the latter quantization of the theory. In the appendix we show how you can achieve this, but for know we go through a much more simple route.

Instead of subtracting a potential, we change the metric:

$$S = -m \int d\sigma \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} \tag{8}$$

where now $g_{\mu\nu} = \eta_{\mu\nu}$ except at the 00 component $g_{00} = \eta_{00} - 2\Phi$. This choice of lagrangian is inspired by what we did in the section above since this leads to the same connection $\Gamma^i_{00} = (\nabla\Phi)^i$ and we already have a clue of the geometric nature of gravity.

Exercise: Show that in the non relativistic limit $v << c$ the Lagrangian in (8) simplifies to $L \approx mv^2/2 - m\Phi$

The Lagrangian (8) not only reproduces all of the effects of gravity, but also introduces some new phenomena. Take a photon with a certain frequency and shoot it from the bottom of a high building. At the top, someone measures the same photon and realizes the frequency has changed! The gravitational redshift is predicted by the new metric $g_{\mu\nu}$. The photon 4-momentum is:

$$k^\mu = (E, 0, 0, E)$$

and the 4-velocities of the observers at the bottom and top of the building are $U^\mu_{\text{bottom}} = U^\mu_{\text{top}} = (1, 0, 0, 0)$. The ratio of the measured frequencies is:

$$\frac{\omega_2}{\omega_1} = \frac{k^\mu U_\mu(\text{top})}{k^\mu U_\mu(\text{bottom})} = \frac{g_{00} k^0 U^0(\text{top})}{g_{00} k^0 U^0(\text{bottom})} = \frac{1 - \frac{2GM}{r_2}}{1 - \frac{2GM}{r_1}}$$

Where it's clear that the frequency of the top ω_2 gets redshifted since $r_2 > r_1$. At this point there is nothing stopping us from thinking that the metric $g_{\mu\nu}$ may be anything, or at least anything that is consistent with the matter content of the physical system. Each $g_{\mu\nu}$ is interpreted as the geometry of spacetime or of just a portion of spacetime. The euler-lagrange equations of 8 gives us geodesics just like we did in the Geometry part. We thus arrive at the same conclusions drawn in the section above: gravity is the geometry of spacetime, and free particles move on geodesics. Furthermore, we have the necessary tools of differential geometry to make precise statements of these ideas in a much more general setting.

Definition: Spacetime is a four-dimensional smooth manifold equipped with a Lorentzian signature metric.

The four dimensions should be obvious. A smooth manifold structure is the least we can impose for something to look like spacetime without having any kind of weird topological phenomena.[6]. It also assures that everything is made without any reference

[6] By topological I mean continuity, paracompactness etc. The paracompactness of a manifold is a sufficient condition for the existence of a Riemannian metric. For a Lorentzian metric we have to impose

to coordinate systems. A Lorentzian signature is necessary so that we maintain lorentz invariance locally (at a point), just like we showed that a Riemannian metric is trivial $g_{\mu\nu}(p) = \delta_{\mu\nu}$ at a point. It assures that when gravity (curvature) is not present, we go back to our beloved flat Minkowski spacetime full of twins and paradoxes.

How can we possible know the metric from the matter content in this general setting? Just like in electromagnetism where Maxwell's equations relate the electromagnetic field and the charge distribution, we need field equations relating $g_{\mu\nu}$ to the stress-energy-momentum tensor $T_{\mu\nu}$. You see, the energy momentum tensor is just a way of encoding the matter content in a tensorial, relativistic fashion. I will just say what it is: it's a matrix where the $T_{\mu\nu}$ component is the flux of P^μ momentum across the hypersurface perpendicular to the x^ν direction. For example, the 00 component is just the energy density where the 11 component is the pressure along the x direction

$$T_{00} = \frac{P^0}{\Delta X \Delta Y \Delta Z} = \rho$$
$$T_{11} = \frac{P^1}{\Delta T \Delta Y \Delta Z} = p_x$$

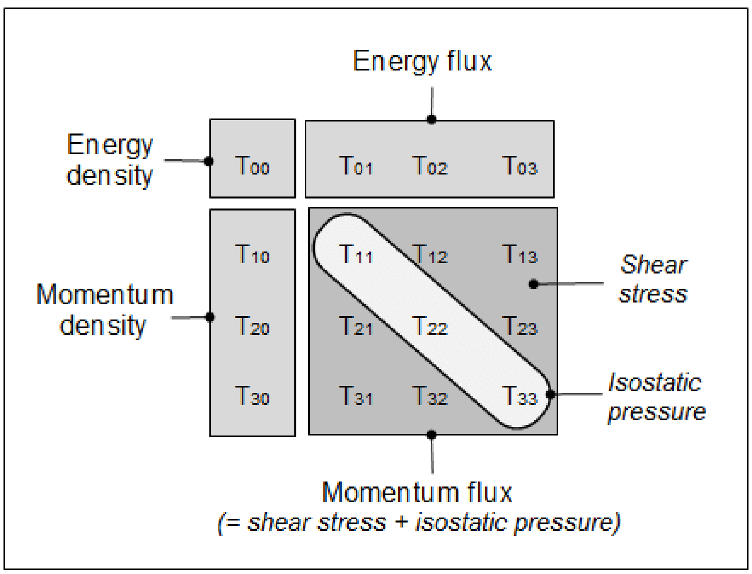

Figure 1: Energy momentum in all of its glory jumpscare.

If you know some field theory you may think that $T_{\mu\nu}$ is the Noether current associated to spacetime translations. Unfortunately this is not the same tensor we use in general relativity, since it's defined only in the context of special relativity. Hence it is not covariant under general coordinate transformations.

Recall that in classical mechanics Poisson's equation $\nabla^2\Phi = 4\pi G\rho$ determines the gravitation potential of a source. We know that Φ is just part of the 00 component of the metric tensor, so a natural generalization of Poisson's equation is

one further condition: it has to admit a non-vanishing vector field. For a non-compact manifold that is no problem, but for compact manifolds this is equivalent to have zero Euler characteristic. Compact spacetimes could be 4-torus!

$$\nabla^2 g_{\mu\nu} = kT_{\mu\nu}$$

But the covariant derivative of the metric is zero by metric compatibility. Furthermore, we want the energy momentum tensor to be conserved $\nabla_\mu T^{\mu\nu} = 0$. On the left hand side we should have a symmetric tensor such that $\nabla_\mu G^{\mu\nu} = 0$ where G is made of second derivatives of the metric. We know just the guy to do it!

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = kT_{\mu\nu}$$

finish section, black holes

2.3 It doesn't stop here: geometry in gauge theories

References