Filling the gaps in geometry and relativity

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Abstract

Your abstract.

1 What is curvature

1.1 Curves, surfaces and the Gaussian curvature

1.2 Higher dimensions, Riemann curvature and the modern stuff

1.3 Repère Mobile, Frame bundle and principal fiber bundles,

2 Gravity enters the scene...

Now that we've delved deeply into the intricacies of geometry and curvature, we are in a position to confront the second big question: How exactly does gravity manifest as the curvature of spacetime? How did Einstein come up with this idea in the first place?

lots to be revised. change the order and add spacetime diagrams

2.1 It's simple once you know geometry

The idea that gravity is not a force is needed for Newtonian physics to be consistent! Consider what Newton's first law have to say:

(Law of Inertia): A body follows uniform straight motion unless acted on by a force.

Now imagine a universe with only a single particle. How can this particle tell that it is moving at all? Well, it can't tell. We need at least two particles, one the observer and other the observed. The observer will check with its coordinates and clocks if Newton's law holds for the other particle. But wait a second: both particles have mass, so there is a gravitational force that deviates the uniform straight motion. And if one includes more particles it only gets worse! It is clear that the force of gravity and Newton's first law cannot be both true since this leads to a contradiction. How can we resolve this problem? some spacetime diagrams here

It turns out that gravity is not a force, so it does not deviate particles from straight motion in spacetime. needs further explanationBut for this to be true we must loose a bit our notion of *straight*. That is no problem for us! The straightest possible paths are geodesics in a more general geometry. If we can show that the effect of gravity is the same as a geodesic path in some sort of curved spacetime, there is no more contradiction with Newton's first law once we consider a larger class of straight motions, that of geodesics.

Let us look at the simplest physical system where gravity is present: a free falling body in a gravitational potential Φ. The equation of motion is:

$$m\ddot{x} = -m\nabla\Phi$$

Notice that the mass appears on both sides. This is the difference of gravity to other forces like the electromagnetic one: all particles follows the same path regardless of the masses. A positively charged particle will follow a trajectory that is very different than a negative charged one when put on the same electromagnetic field. We want the system of equations

$$\ddot{x}^i + (\nabla\Phi)^i = 0$$

to look like the geodesic equation, but first derivatives are lacking on the second term. If we include an extra coordinate $x^0 = t$ of time (and it's essential to do it), it obviously obeys $\ddot{x}^0 = 0$ and we get a system of equations:

$$\begin{aligned} \ddot{x}^0 &= 0 \\ \ddot{x}^i + (\nabla\Phi)^i\dot{x}^0\dot{x}^0 &= 0 \end{aligned}$$

which is exactly a geodesic of spacetime, where we make the identifications $\Gamma^i_{00} = (\nabla\Phi)^i$, $\Gamma i_{jk} = 0$ otherwise. With a connection at hand we find the Riemman curvature $R^i_{0j0} = -\partial_j\partial^i\Phi$. Therefore we conclude that gravity is truly an effect of the curvature of spacetime. A refinement of Newton's first law is:

(Enhanced Law of Inertia): In the absence of forces all particles follows geodesics in spacetime.

I should insist that in the new definition gravity is not a force. Its effect is encoded in the geodesic motion.

2.2 Generalize Special relativity

In the section above we gave some physical arguments to conclude that gravity must be the curvature of some geometry. But it may seem unnatural to mix space and time together, specially because we were dealing with a galilean structure of flat space and absolute time. A more natural arena is, of course, Minkowski spacetime of special relativity where the geometry is explicitly given to us by a metric $\eta_{\mu\nu}$. Our goal is to introduce gravity in special relativity. This will inevitably lead us to general relativity.

We know that in special relativity it is best to view space and time together, i.e. spacetime, as a set of points $X^\mu = (ct, x, y, z)$ that resembles the vector space $\mathbb{R}^4$ but has a very different notion of distance between points:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}dX^\mu dX^\nu \tag{1}$$

where the metric is defined as $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. When we look for coordinate transformations [1] $X'^\mu = \Lambda^\mu_\nu X^\nu$ that preserves (1) we get to the Lorentz transformations (given that it is proper orthocronus etc etc). They form a group $\Lambda^\mu_\nu \in SO(1, 3)$ where composition is given by the usual matrix multiplication.

The point is that all invariant quantities in SR, i.e. quantities that does not depend on a particular observer, should be made with the aid of the invariant metric (1). For example, the proper time:

$$d\tau = \frac{\sqrt{-ds^2}}{c}$$

is the time measured by a particle's clock in its rest frame $X^\mu = (ct', 0, 0, 0)$. Different observers will experience different times t but they all agree on the value of τ. With these tools we can easily reconstruct the dynamics of particles in a relativistic invariant way. We define the 4-velocity and 4-momentum as:

$$U^\mu = \frac{dX^\mu}{d\tau}$$
$$P^\mu = mU^\mu$$

Let X^μ be a 4-vector in spacetime. We define $X_\mu X^\mu = X^2 \doteq \eta_{\mu\nu}X^\mu X^\nu$ as the contraction of X with itself. Then you can show that $U^2 = -c^2$ and $P^2 = -(mc)^2$. Newton's second law should be something like:

$$\frac{dP^\mu}{d\tau} = F^\mu$$

That is a nice way of getting to the dynamics of the theory, but it is not clear at all how to include some force or potential $V(x)$ in the equation above. We have look at the Lagrangian formulation of SR: What is the action of a free relativistic particle? Since Lorentz transformations are a symmetry of the system, we should look for a relativistic invariant action. The simplest thing to come up with is the following:

$$S = -m\int d\tau \tag{2}$$

From here on I'll use natural units where $c = 1$. To understand equation (2) first we notice that a particle describes a path in spacetime $X^\mu(\sigma) : \mathbf{R} \to \mathbf{R}^{1,3}$ parametrized by σ. The parameter does not have to be the proper time, we can always reparametrize

[1]The set of coordinate transformations $X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu$ that preserves the metric (1) is called the Poincaré group $\mathcal{P} \simeq O(1, 3) \oplus R^{1,3}$. The reader should notice that this is the group of isometries of spacetime. The Lorentz transformations are encoded in the subgroup $SO(1, 3)$ of the Poincaré group connected to the identity.

the path just like we did when analysing curves in the Geometry part of the notes. Thus we can rewrite equation (2):

$$S = -m \int d\sigma \frac{d\tau}{d\sigma} = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} \tag{3}$$

The action is fully relativistic, and more than that: it has reparametrization invariance. This is a form of gauge invariance: changing the parameter should not affect the underlying physics, i.e. the worldline of the particle. The Euler Lagrange equations give us(exercise):

$$\frac{d}{d\sigma}\left(m \frac{\partial \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}}{\partial \dot{X}^\rho}\right) = m \frac{d^2 X_\rho}{d\tau^2} = 0$$

Which is exactly what we were expecting from our previous equation $\frac{dP^\mu}{d\tau} = F^\mu = 0$. In the Lagrangian formalism it is straightforward to insert a potential in the theory:

$$S = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} - \int d\sigma \Phi$$

But this does not keep the reparametrization invariance! Any change in the parameter $\sigma \to \sigma'$ will be felt by the jacobian $\frac{\partial \sigma}{\partial \sigma'}$ in the second term. We can get around this by considering a four-potential A_μ instead of the scalar Φ, and contract it with the four-velocity:

$$S = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} - \int d\sigma q A_\mu \dot{X}^\mu$$

where q is just a constant measuring the coupling with the potential. Notice that a reparametrization don't change the action because of the 4-velocity term! And everything keeps lorentz invariance as well. The suggestive notation is going to make more sense once we derive the Euler Lagrange equations for the system:

$$m \frac{d^2 X_\mu}{d\tau^2} = q\left(\frac{\partial A_\mu}{\partial X^\nu} - \frac{\partial A_\nu}{\partial X^\mu}\right)\dot{X}^\nu = q F_{\mu\nu} \dot{X}^\nu$$

This looks exactly like the (covariant) equation of a particle in an electromagnetic field, where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the electromagnetic tensor! You can easily show that it satisfies $\partial_{[\rho} F_{\mu\nu]} = 0$ which is equivalent to the two homogeneous Maxwell's equations. Of course, it will only be Maxwell's EM when the other two inhomogeneous equations are provided $\partial^\mu F_{\mu\nu} = J_\nu$ where $J_\nu = (\rho, \mathbf{J})$ is the 4-current.

Does that mean that relativistic gravity is somehow a kind of gravitomagnetism? You could go in this route but eventually you would find serious inconsistencies in the solutions of your equations. Stuff like infinite energy, instability of simple closed orbits and worse.(reference here)

What now? We could go on with the previous idea and insert a relativistic tensor field of rank 2 $h_{\mu\nu}$ instead of the 4-potential A_μ and see where that leads us. We would actually get linearized general relativity in vacuum! But the procedure is much more involved and subtle. Since this is a field theory we would like to have positive kinetic terms (free of ghosts, if these words even make sense to you) and kill some degrees of freedom. This have to do with the latter quantization of the theory. In the appendix we show how you can achieve this, but for know we go through a much more simple route.

Instead of subtracting a potential, we change the metric:

$$S = -m \int d\sigma \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} \tag{4}$$

where now $g_{\mu\nu} = \eta_{\mu\nu}$ except at the 00 component $g_{00} = \eta_{00} - 2\Phi$. This choice of lagrangian is inspired by what we did in the section above since this leads to the same connection $\Gamma^i_{00} = (\nabla\Phi)^i$ and we already have a clue of the geometric nature of gravity.

Exercise: Show that in the non relativistic limit $v << c$ the lagrangian 4 simplifies to $L \approx mv^2/2 - m\Phi$

The lagrangian 4 not only reproduces all of the effects of gravity, but also introduces some new phenomena. Take a photon with a certain frequency and shoot it from the bottom of a high building. At the top, someone measures the same photon and realizes the frequency has changed! The gravitational redshift is predicted by the new metric $g_{\mu\nu}$. The photon 4-momentum is:

$$k^\mu = (E, 0, 0, E)$$

and the 4-velocities of the observers are $U^\mu = (1, 0, 0, 0)$. The ratio of the measured frequencies is:

$$\frac{\omega_2}{\omega_1} = \frac{k^\mu U_\mu(2)}{k^\mu U_\mu(1)} = \frac{g_{00} k^0 U^0(2)}{g_{00} k^0 U^0(1)} = \frac{1 - \frac{2GM}{r_2}}{1 - \frac{2GM}{r_1}}$$

At this point there is nothing stopping us from thinking that the metric $g_{\mu\nu}$ may be anything, or at least anything that is consistent with the matter content of the physical system. Each $g_{\mu\nu}$ is interpreted as the geometry of spacetime or of just a portion of spacetime. The euler-lagrange equations of 4 gives us geodesics just like we did in the Geometry part. We thus arrive at the same conclusions drawn in the section above: gravity is the geometry of spacetime, and free particles move on geodesics. Furthermore, we have the necessary tools of differential geometry to make precise statements of these ideas in a much more general setting.

Definition: Spacetime is a four-dimensional smooth manifold equipped with a Lorentzian signature metric.

The four dimensions should be obvious. A smooth manifold structure is the least we can impose for something to look like spacetime without having any kind of weird topological phenomena.[2]. It also assures that everything is made without any reference

[2]By topological I mean continuity, paracompactness etc. The paracompactness of a manifold, for example, is a necessary and sufficient condition for the existence of a Riemmanian metric. For the existence of a Lorentzian metric we have to impose a further condition: it have to admit a non-vanishing vector field. For a non-compact manifold that is no problem, but for compact manifolds this is equivalent to have zero euler characteristic! Weird stuff.

to coordinate systems. A Lorentzian signature is necessary so that we maintain lorentz invariance locally (at a point), just like we showed that a riemmanian metric is trivial $g_{\mu\nu}(p) = \delta_{\mu\nu}$ at a point. It assures that when gravity (curvature) is not present, we go back to our beloved flat Minkowski spacetime full of twins and paradoxes.

How can we possible know the metric from the matter content in this general setting? Just like in electromagnetism where Maxwell's equations relate the electromagnetic field and the charge distribution, we need field equations relating $g_{\mu\nu}$ to the stress-energy-momentum tensor $T_{\mu\nu}$. You see, the energy momentum tensor is just a way of encoding the matter content in a tensorial, relativistic fashion. I will just say what it is: it's a matrix where the $T_{\mu\nu}$ component is the density of P^μ momentum in the ν direction. For example, the 00 component is just the energy density where the 11 component is the pressure along the x direction

$$T_{00} = \frac{P^0}{\Delta X \Delta Y \Delta Z} = \rho$$
$$T_{11} = \frac{P^1}{\Delta T \Delta Y \Delta Z} = p_x$$

In all of its glory

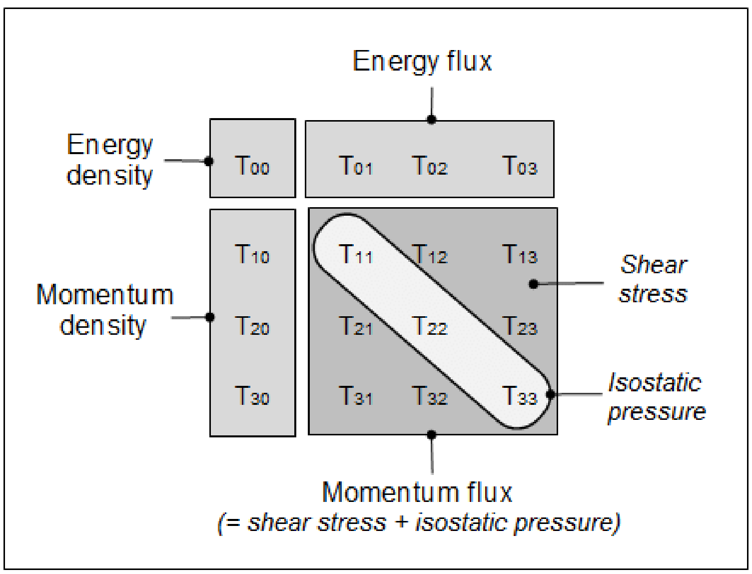

Figure 1: Energy momentum. baby

Recall that in classical mechanics Poisson's equation $\nabla^2\Phi = 4\pi G\rho$ determines the gravitation potential of a source. We know that Φ is just part of the 00 component of the metric tensor, so a natural generalization of Poisson's equation is

$$\nabla^2 g_{\mu\nu} = kT_{\mu\nu}$$

But the covariant derivative of the metric is zero by metric compatibility. Furthermore, we want the energy momentum tensor to be conserved $\nabla_\mu T^{\mu\nu} = 0$. On the left hand side we should have a symmetric tensor such that $\nabla_\mu G^{\mu\nu} = 0$ where G is made of second derivatives of the metric. We know just the guy to do it!

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = k T_{\mu\nu}$$

finish section, energy momentum tensor as a noether current of spacetime translations, black holes

2.3 It doesn't stop here: geometry in gauge theories

References