Wave on a disk

Abstract

This project aims to simulate the propagation of a wave on a disk by solving the d'Alembert equation on MATLAB. The periodicity of the disk leads to a particular equation : the Bessel differential equation. Its solutions are the Bessel functions, which show the fundamental vibration modes of the disk.

Introduction

As many problems in physics are related to wave onto a disk (Chladni patterns, heat conduction in circular plate, quantum particle in a 2D circular well, etc). Therefore, the project purpose is to digitally simulate the fundamental aspect of these various physical phenomena: the propagation of a wave on a disk. The disk, made of a flexible material, is fixed along its rigid boundary of radius $R = 1$. An impulse at a point on the disk generates a mechanical wave, and the vertical displacement of each point on the disk is modeled using the wave equation in polar coordinates.

Theoretical Framework

Governing Equation

The wave propagation is governed by the two-dimensional wave equation in polar coordinates $(r,\ \theta)$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

where $u(r, \theta, t)$ represents the vertical displacement, and the Laplacian Δ in polar coordinates is:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

As r, θ and t are independent, the solution form is a product of functions of r, θ and t : $u(r,\theta,t) = R(r)\cdot\Theta(\theta)\cdot T(t)$

The temporal component $T(t)$ satisfies:

$$T''(t) + \lambda^2 T(t) = 0$$

with the physically meaningful solution:

$$T(t) = T_0\cos(\lambda t)$$

Thanks to the 2π-periodicity of the disk, the angular component $\Theta(\theta)$ satisfies:

$$\Theta''(\theta) + n^2\Theta(\theta) = 0$$

with n an integer. Therefore the solutions are:

$$\Theta(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$$

Finally, the radial component $R(r)$ satisfies Bessel's differential equation — which is obtained by replacing the derivatives of T and Θ with their constants λ and n :

$$r^2R''(r) + rR'(r) + (\lambda^2r^2 - n^2)R(r) = 0$$

with solutions given by Bessel functions of the first kind: $R(r) = J_n(\lambda r)$. Those solutions are chosen for their physical reality. Indeed, the first kind Bessel function do not diverge at $r = 0$.

Finally, the general solution is a superposition of the separable solutions:

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} J_n(\lambda_{mn}r)\Big(A_{mn}\cos(n\theta) + B_{mn}\sin(n\theta)\Big)\cos(\lambda_{mn}t)$$

Numerical Implementation

Diskretization

The disk is diskretized into a grid with radial spacing dr, angular spacing $d\theta$, and time step $d\theta$. The wave equation is approximated using finite differences:

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{dt^2}$$

and :

$$\Delta u \approx \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Stability Analysis

The numerical scheme is stable if the Courant-Friedrichs-Lewy (CFL) condition is satisfied:

$$\frac{dt}{dr, d\theta} \leq 1$$

The mean squared error between the numerical solution $u(r,\theta,t)$ and the analytical solution $v(r,\theta,t)$ is computed as:

$$\text{Error} = \frac{1}{N_r \times N_\theta} \sum_{i=1}^{N_r} \sum_{j=1}^{N_\theta} | v_{i,j} - u_{i,j} |^2$$

Results

The error remains bounded if the CFL condition is satisfied. The error between the numerical and real impulses arises because the discrete grid cannot perfectly capture the wave's continuous propagation, especially for large time or coarse spatial steps, leading to phase and amplitude discrepancies. The maximum observed error is approximately 35% when the CFL condition is met.

Conclusion

This study successfully modeled wave propagation on a circular disk using both analytical and numerical methods. The numerical scheme demonstrated stability under the CFL condition, providing a reliable simulation of the physical phenomenon. The analysis highlighted the importance of parameter selection in maintaining numerical accuracy.