Mechanical Resonance

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OUTLINE

Distributed Parameter Oscillator

- **Definition**
- **Rayleigh's Principle**
- **Rayleigh's Energy Method**
- **Rayleigh Example**
- **Other Methods**
- **Beam Fixity Study**
- **Beam Frequencies**
- **Plate Frequencies**
- **Equivalent Systems**

Definition of Distributed Parameter Oscillator

- Mass evenly distributed
- Compliance evenly distributed
- Theoretically infinite degrees of freedom
- Theoretically infinite number of mode shapes
- Mass and compliance inseparable
- Governed by partial differential equations
- Density and elastic modulii supplant mass and spring rate
- Analysis methods
    * large body of classical problems with special conditions and methods
    * approximate, energy methods adequate for simple problems
    * numerical approach frequently essential

Rayleigh's Principle

- Textbook Definition
  * The frequency of vibration of a conservative system vibrating about an equilibrium position has a stationary value in the neighborhood of a natural mode
- Deciphered
  * undamped system
  * vibrating at a frequency near or at resonance
  * frequency is constant with respect to parametric changes at resonance

Rayleigh's Energy Method

- Utilizes Rayleigh's Principle
- Approximate, Energy Method
- Premise: Kinetic Energy equals Potential Energy

$$T_{max} = V_{max}$$

- Reference Kinetic Energy T^*

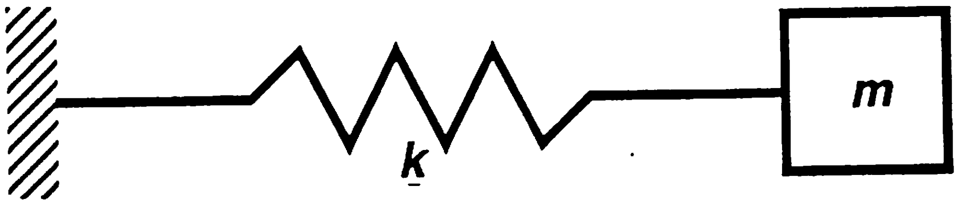
$$T_{max} = \omega^2 T^*$$

- The Rayleigh's Energy Method

$$\omega^2 = \frac{T^*}{V_{max}}$$

Rayleigh's Method Example

- **Apply to simple oscillator**



- **Assume single mode**

$x(t) = x_0 \cos \omega t$

$$x = x_0 \qquad \dot{x}(t) = x_0 \omega \sin \omega t$$

- **Potential energy, max**

$$V_{max} = 1/2 \; k x_0^2$$

k : spring constant

- **Reference kinetic energy**

$$T^* = 1/2 \; m x_0^2 \qquad T_{max} = \frac{1}{2} m (x_0 \omega)^2$$

- **Rayleigh's Quotient**

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{1/2 \; k x_0^2}{1/2 \; m x_0^2}$$

- **Frequency**

$$\omega = \sqrt{\frac{k}{m}}$$

Rayleigh's Method for the Continuum

- **Continuum Vibration**

$$u(x,t) = U(x)f(t)$$

- **Separable functions for mode shape and temporal term**
- **Continuum velocity**

$$\dot{u}(x,t) = U(x)\dot{f}(t)$$

$\dot{u} = \frac{\partial u}{\partial t}$ = Velocity

- **Continuum potential**

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial U(x)}{\partial x}f(t)$$

$\frac{\partial u}{\partial x}$ = strain

Rayleigh's Method for the Continuum

- **Potential energy**

$$V = 1/2 \int_0^L EI\left[\frac{\partial U(x)}{\partial x} f(t)\right]^2 dx$$

- **Kinetic engery**

$$T = 1/2 \int_0^L m\left[U(x)\dot{f}(t)\right]^2 dx$$

- **Equate T and V**

$$\left[\frac{\dot{f}(t)}{f(t)}\right]^2 = \frac{1/2 \int_0^L EI\left[\frac{\partial U(x)}{\partial x}\right]^2 dx}{1/2 \int_0^L m\left[U(x)\right]^2 dx}$$

- **Form Rayleigh's Quotient**

$$\omega^2 = \frac{\int_0^L EI\left[\frac{\partial U(x)}{\partial x}\right]^2 dx}{\int_0^L m(x)\left[U(x)\right]^2 dx}$$

Rayleigh Example

(Axial deflection of beam)

Example 6.1

As an illustration of Rayleigh's energy method let us estimate the fundamental frequency of a nonuniform clamped-free bar vibrating longitudinally (Figure 6.1). The mass per unit length is given by

$$m(x) = 2m\left(1 - \frac{x}{L}\right) \tag{a}$$

$2m > m(x) \geq 0$

average = m

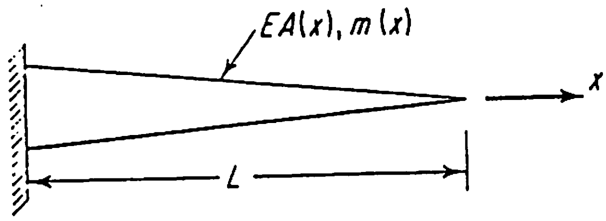


FIGURE 6.1

and the stiffness has the expression

$2EA > EA(x) \geq 0$

average = EA

$$EA(x) = 2EA\left(1 - \frac{x}{L}\right). \tag{b}$$

Although the eigenvalue problem can be solved in closed form, we shall use Rayleigh's energy method for comparison purposes. Let us assume as the fundamental mode the first eigenfunction of a uniform clamped-free bar

$$U(x) = \sin\frac{\pi x}{2L}, \tag{c}$$

$\therefore \; u(0) = 0$
$u(L) = 1$

Sinusoidal mode shape

Rayleigh Example

because it obviously satisfies the boundary conditions of the problem. Next form the integrals

$$\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx = 2EA\left(\frac{\pi}{2L}\right)^2 \int_0^L \left(1-\frac{x}{L}\right)\cos^2\frac{\pi x}{2L}\,dx$$

$$= \frac{EA}{2L}\left(1+\frac{\pi^2}{4}\right), \tag{d}$$

$$\int_0^L m(x)U^2(x)\,dx = 2m\int_0^L\left(1-\frac{x}{L}\right)\sin^2\frac{\pi x}{2L}\,dx = \frac{mL}{2}\left(1-\frac{4}{\pi^2}\right). \tag{e}$$

Introducing (d) and (e) in (6.14) we obtain

$$\omega^2 = R(U) = \frac{\int_0^L EA(x)[dU(x)/dx]^2\,dx}{\int_0^L m(x)U^2(x)\,dx} = \frac{1+(\pi^2/4)}{1-(4/\pi^2)}\frac{EA}{mL^2} = 5.8305\frac{EA}{mL^2},$$

from which we obtain the estimated fundamental frequency

$$\omega = 2.4146\sqrt{\frac{EA}{mL^2}}. \tag{f}$$

Other Approximate Methods

RAYLEIGH'S ENERGY METHOD
RAYLEIGH-RITZ METHOD
RAYLEIGH-RITZ METHOD. RE-EXAMINATION OF THE BOUNDARY CONDITION REQUIREMENTS
ASSUMED-MODES METHOD. LAGRANGE'S EQUATIONS
GALERKIN'S METHOD
COLLOCATION METHOD
ASSUMED-MODES METHOD. INTEGRAL FORMULATION
GALERKIN'S METHOD. INTEGRAL FORMULATION
COLLOCATION METHOD. INTEGRAL FORMULATION
HOLZER'S METHOD FOR TORSIONAL VIBRATION
MYKLESTAD'S METHOD FOR BENDING VIBRATION
LUMPED-PARAMETER METHOD EMPLOYING INFLUENCE COEFFICIENTS
LUMPED-PARAMETER METHOD. SEMIDEFINITE SYSTEMS

Beam Fixity Problem

- **Fixity is zero ----> simply supported**

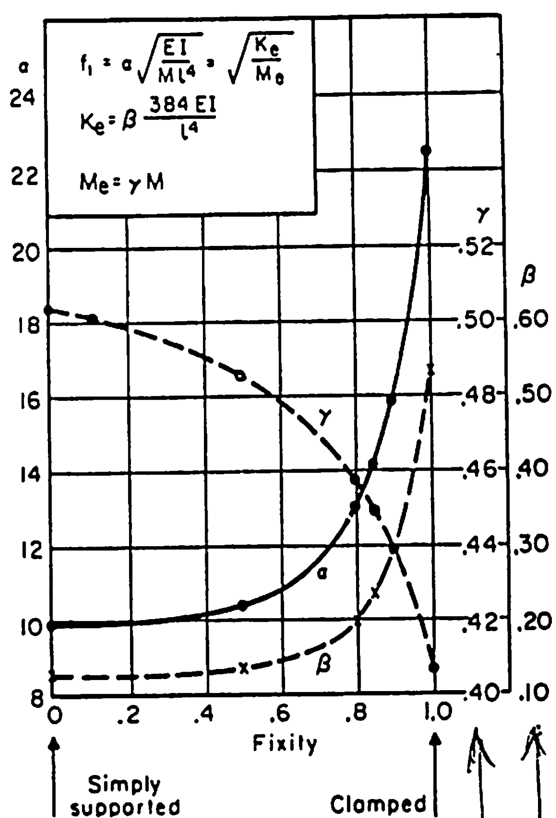
- **Fixity is one ----> built in**

As a second example, the Rayleigh procedure is applied to a single-span beam of uniform mass and stiffness. In this case, the end fixity F is defined as the moment developed at the ends, divided by the end moment for a perfectly rigid support. For instance, $F = 0$ for pinned supports, while $F = 1$ for rigidly clamped supports. Using statical deflection for any value of F, the stiffness and mass of an equivalent spring-mass system can be determined, from which the fundamental frequency can be expressed in the form

$$f_1 = \frac{\alpha}{2\pi}\sqrt{\frac{EIg}{Wl^3}} \qquad (17)$$

where E = modulus of elasticity
I = moment of inertia of cross-sectional area about neutral axis
W = total weight of beam
l = length of beam
g = 386 in. per sec²
α = coefficient depending on F

The coefficient α and the equivalent stiffness and mass are plotted in Fig. 4 as functions of F.



spring

frequency

equivalent mass

Mechanical Resonance

Beam Frequencies

Calculate factor, k_1 from column (E)
Beam length L

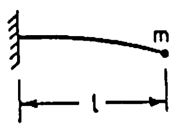
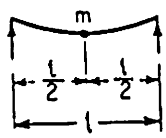
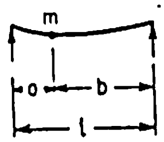
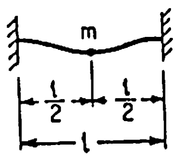
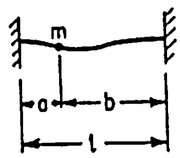
SUPPORTS	MODE n	(A) SHAPE AND NODES (NUMBERS GIVE LOCATION OF NODES IN FRACTION OF LENGTH FROM LEFT END)	(B) BOUNDARY CONDITIONS EQ (7.16)	(C) FREQUENCY EQUATION	(D) CONSTANTS EQ (7.16)	(E) kl EQ (7.14) $\omega_n = k^2\sqrt{\frac{EIg}{A\gamma}}$	(F) R RATIO OF NON-ZERO CONSTANTS COLUMN (D)
HINGED-HINGED	1		$x=0\begin{cases}X=0\\X''=0\end{cases}$	SIN $kl=0$	$A=0$	3.1416	1.0000
	2	0.50			$B=0$	6.283	1.0000
	3	0.333 0.667			$\frac{C}{D}=1$	9.425	1.0000
	4	0.25 0.50 0.75	$x=l\begin{cases}X=0\\X''=0\end{cases}$			12.566	1.0000
	n>4					$\approx n\pi$	1.0000
CLAMPED-CLAMPED	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	(COS kl)(COSH kl) $=1$	$A=0$	4.730	-0.9825
	2	0.50			$C=0$	7.853	-1.0008
	3	0.359 0.641			$\frac{D}{B}=R$	10.996	-1.0000-
	4	0.278 0.50 0.722	$x=l\begin{cases}X=0\\X'=0\end{cases}$			14.137	-1.0000+
	n>4					$\approx\frac{(2n+1)\pi}{2}$	-1.0000-
CLAMPED-HINGED	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	TAN $kl=$ TANH kl	$A=0$	3.927	-1.0008
	2	0.558			$C=0$	7.069	-1.0000+
	3	0.386 0.692			$\frac{D}{B}=R$	10.210	-1.0000
	4	0.294 0.529 0.765	$x=l\begin{cases}X=0\\X''=0\end{cases}$			13.352	-1.0000
	n>4					$\approx\frac{(4n+1)\pi}{4}$	-1.0000
CLAMPED-FREE	1		$x=0\begin{cases}X=0\\X''=0\end{cases}$	(COS kl)(COSH kl) $=-1$	$A=0$	1.875	-0.7341
	2	0.783			$C=0$	4.694	-1.0185
	3	0.504 0.868			$\frac{D}{B}=R$	7.855	-0.9992
	4	0.358 0.644 0.906	$x=l\begin{cases}X''=0\\X'''=0\end{cases}$			10.996	-1.0000+
	n>4					$\approx\frac{(2n-1)\pi}{2}$	-1.0000-
FREE-FREE	1		$x=0\begin{cases}X''=0\\X'''=0\end{cases}$	(COS kl)(COSH kl) $=1$	$B=0$	0 (REPRESENTS TRANSLATION)	
	2	0.224 0.776			$D=0$	4.730	-0.9825
	3	0.132 0.50 0.868			$\frac{C}{A}=R$	7.853	-1.0008
	4	0.094 0.356 0.644 0.906	$x=l\begin{cases}X''=0\\X'''=0\end{cases}$			10.996	-1.0000-
	5	0.0734 0.277 0.50 0.723 0.927				14.137	-1.0000+
	n>5					$\approx\frac{(2n-1)\pi}{2}$	-1.0000-

Mechanical Resonance

Beam Frequencies

MASSLESS BEAMS WITH CONCENTRATED MASS LOADS

m = MASS OF LOAD, LB-SEC^2/IN.
l = LENGTH OF BEAM, IN.
I = AREA MOMENT OF INERTIA OF BEAM CROSS SECTION, IN.^{4}
E = YOUNG'S MODULUS, LB/IN.^{2}
ω_n = ANGULAR NATURAL FREQUENCY, RAD/SEC

FIXED-FREE END LOAD	HINGED-HINGED CENTER LOAD	HINGED-HINGED OFF-CENTER LOAD	FIXED-FIXED CENTER LOAD	FIXED-FIXED OFF-CENTER LOAD
				
$\omega_n = \sqrt{\frac{3EI}{ml^3}}$	$\omega_n = 4\sqrt{\frac{3EI}{ml^3}}$	$\omega_n = \frac{1}{ab}\sqrt{\frac{3EIl}{m}}$	$\omega_n = 8\sqrt{\frac{3EI}{ml^3}}$	$\omega_n = \sqrt{\frac{3EIl^3}{ab}}$

MASSIVE SPRINGS (BEAMS) WITH CONCENTRATED MASS LOADS

m = MASS OF LOAD, LB-SEC^2/IN.
$m_s(m_b)$ = MASS OF SPRING (BEAM), LB-SEC^2/IN.
k = STIFFNESS OF SPRING LB/IN.
l = LENGTH OF BEAM, IN.
I = AREA MOMENT OF INERTIA OF BEAM CROSS SECTION, IN.^{4}
E = YOUNG'S MODULUS, LB/IN.^{2}
ω_n = ANGULAR NATURAL FREQUENCY, RAD/SEC

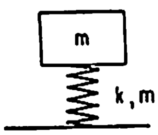
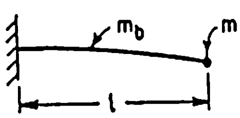
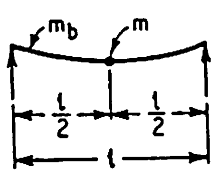
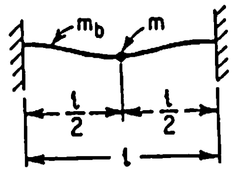
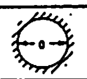
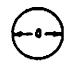
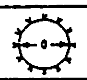
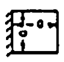
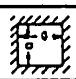
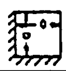
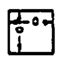
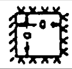
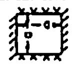
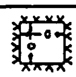
MASS - HELICAL SPRING	FIXED - FREE END LOAD	HINGED - HINGED CENTER LOAD	FIXED - FIXED CENTER LOAD
			
$\omega_n = \sqrt{\frac{k}{(m + \frac{m_s}{3})}}$	$\omega_n = \sqrt{\frac{3EI}{l^3(m+0.23m_b)}}$	$\omega_n = \sqrt{\frac{48EI}{l^3(m+0.5m_b)}}$	$\omega_n = 14\sqrt{\frac{EI}{l^3(m+0.375m_b)}}$

Plate Frequencies

NATURAL FREQUENCIES OF THIN FLAT PLATES OF UNIFORM THICKNESS

$$\omega_n = B\sqrt{\frac{E t^2}{\rho a^4 (1-\nu^2)}} \text{ RAD/SEC}$$

E = YOUNG'S MODULUS, LB/IN^2
t = THICKNESS OF PLATE, IN.
ρ = MASS DENSITY, LB-SEC^2/IN^4
a = DIAMETER OF CIRCULAR PLATE OR SIDE OF SQUARE PLATE, IN.
ν = POISSON'S RATIO

SHAPE OF PLATE	DIAGRAM	EDGE CONDITIONS	VALUE OF B FOR MODE:							
			1	2	3	4	5	6	7	8
CIRCULAR		CLAMPED AT EDGE	11.84	24.61	40.41	46 14	103.12			
CIRCULAR		FREE	6 09	10.53	14.19	23.80	40.88	44.68	61.38	69.44
CIRCULAR		CLAMPED AT CENTER	4.35	24.26	70.39	138.85				
CIRCULAR		SIMPLY SUPPORTED AT EDGE	5.90							
SQUARE		ONE EDGE CLAMPED-THREE EDGES FREE	1.01	2.47	6.20	7.94	9.01			
SQUARE		ALL EDGES CLAMPED	10.40	21.21	31.29	38.04	38.22	47.73		
SQUARE		TWO EDGES CLAMPED-TWO EDGES FREE	2.01	6.96	7.74	13.89	18.25			
SQUARE		ALL EDGES FREE	4.07	5.94	6.91	10.39	17.80	18.85		
SQUARE		ONE EDGE CLAMPED-THREE EDGES SIMPLY SUPPORTED	6.83	14.94	16.95	24.89	28.99	32.71		
SQUARE		TWO EDGES CLAMPED-TWO EDGES SIMPLY SUPPORTED	8.37	15.82	20 03	27.34	29 54	37.31		
SQUARE		ALL EDGES SIMPLY SUPPORTED	5.70	14.26	22.82	28.52	37.08	48.49		

MASSLESS CIRCULAR PLATE WITH CONCENTRATED CENTER MASS

CLAMPED EDGES 

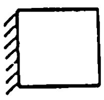
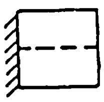
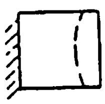
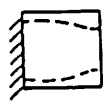
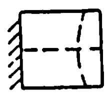
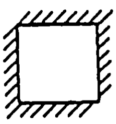
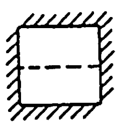
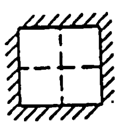
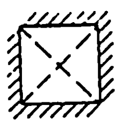
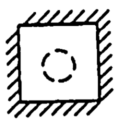
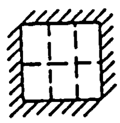
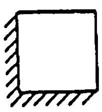
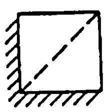
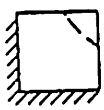
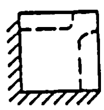
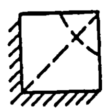
$$\omega_n = 4.09\sqrt{\frac{E h^3}{m a^2 (1-\nu^2)}}$$

SIMPLY SUPPORTED EDGES 

$$\omega_n = 4.09\sqrt{\frac{E h^3}{m a^2 (1-\nu)(3+\nu)}}$$

Plate Frequencies

Table 7.7. Natural Frequencies and Nodal Lines of Square Plates with Various Edge Conditions (*After D. Young.*[29])

	1ST MODE	2ND MODE	3RD MODE	4TH MODE	5TH MODE	6TH MODE
$\omega_n/\sqrt{Dg/\gamma ha^4}$	3.494	8.547	21.44	27.46	31.17	
NODAL LINES						
$\omega_n/\sqrt{Dg/\gamma ha^4}$	35.99	73.41	108.27	131.64	132.25	165.15
NODAL LINES						
$\omega_n/\sqrt{Dg/\gamma ha^4}$	6.958	24.08	26.80	48.05	63.14	
NODAL LINES						

$\omega_n = 2\pi f_n$

$D = Eh^3/12(1-\mu^2)$

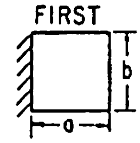
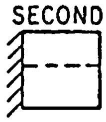
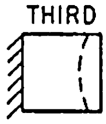
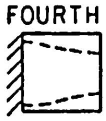
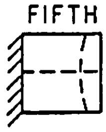
γ = WEIGHT DENSITY

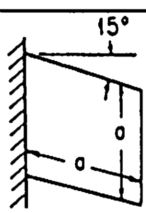
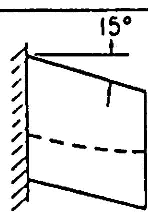
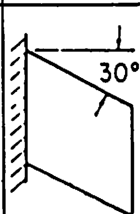
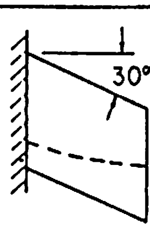
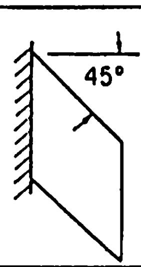
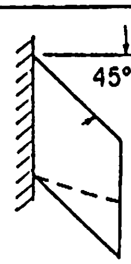
h = PLATE THICKNESS

a = PLATE LENGTH

Plate Frequencies

Table 7.8. Natural Frequencies and Nodal Lines of Cantilevered Rectangular and Skew Rectangular Plates ($\mu = 0.3$)* (*M. V. Barton.*[30])

MODE \ a/b	1/2	1	2	5
FIRST 	3.508	3.494	3.472	3.450
SECOND 	5.372	8.547	14.93	34.73
THIRD 	21.96	21.44	21.61	21.52
FOURTH 	10.26	27.46	94.49	563.9
FIFTH 	24.85	31.17	48.71	105.9

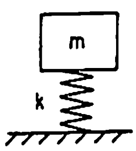
MODE	FIRST	SECOND	FIRST	SECOND	FIRST	SECOND
$\omega_n/\sqrt{Dg/\gamma h a^4}$	3.601	8.872	3.961	10.190	4.824	13.75
NODAL LINES						

* For terminology, see Table 7.7.

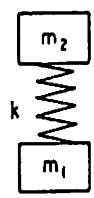
Equivalent Systems

MASS - SPRING SYSTEMS IN TRANSLATION
(RIGID MASS AND MASSLESS SPRING)

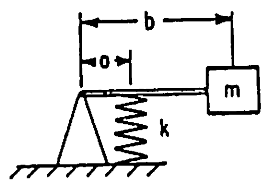
k = SPRING STIFFNESS, LB/IN.
m = MASS, LB-SEC^2/IN.
ω_n = ANGULAR NATURAL FREQUENCY, RAD/SEC



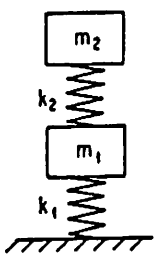
$$\omega_n = \sqrt{\frac{k}{m}}$$



$$\omega_n = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$$

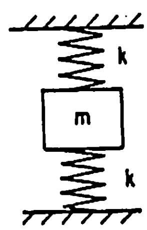


$$\omega_n = \frac{a}{b}\sqrt{\frac{k}{m}}$$



$$\omega_n = \frac{1}{2}\left[\frac{k_1}{m_1}+\frac{k_2}{m_2}\left(1+\frac{m_2}{m_1}\right)\pm\sqrt{\left[\frac{k_1}{m_1}+\frac{k_2}{m_2}\left(1+\frac{m_2}{m_1}\right)\right]^2-\frac{4k_1k_2}{m_1m_2}}\,\right]$$

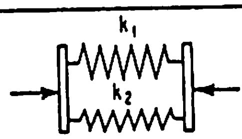
2 Frequenzen



$$\omega_n = \sqrt{\frac{2k}{m}}$$

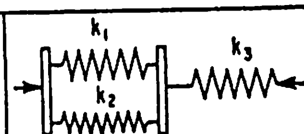
SPRINGS IN COMBINATION

k_r = RESULTANT STIFFNESS OF COMBINATION



$$k_r = k_1 + k_2$$

$$k_r = \frac{k_1 k_2}{k_1+k_2}$$



$$k_r = \frac{(k_1+k_2)\,k_3}{k_1+k_2+k_3}$$