Radio Frequency Circuits & Antenna

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Homework: 2

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Quesiton 1

The power radiated by a lossless antenna is 15 W. The directional characteristics of the antenna are represented by the radiation intensity of

$$U = B_0 \cos^3(\theta[W]), \quad 0 \le \theta \le \pi/2, \quad 0 \le \varphi \le 2\pi, W \to \text{Unit solid angle}$$

Assume that there is no radiation in the angular sector: $\pi/2 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$

(a) Compute the maximum power density (in watts per square meter) at a distance of 1000 m (assume far field distance). Specify the angle where this occurs:

$$P_{rad} = 15[W]$$

$$\begin{aligned}
P_{rad} &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/2} U(\theta,\varphi)(\sin\theta)\, d\theta\, d\varphi \\
&= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/2} B_0 \cos^3(\theta) \sin\theta\, d\theta\, d\varphi \\
&= 2\pi B_0 \int_{\theta=0}^{\pi/2} \cos^3(\theta) \sin\theta\, d\theta \\
&= 2\pi B_0 \Big[\frac{-\cos^4(\theta)}{4}\Big]_0^{\pi/2} \\
&= -\frac{2\pi}{4} B_0[\cos^4(\pi/2) - \cos^4(0)] \\
P_{rad} &= \frac{2\pi}{4} B_0 = 15 \\
\therefore \quad B_0 &= \frac{30}{\pi} \\
\therefore \quad U(\theta,\varphi) &= \frac{30}{\pi}\cos^3(\theta)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S} &= \frac{U(\theta,\varphi)}{r^2} \\
\mathcal{S}_{max} &= \frac{U(0)}{1000^2} = \frac{30}{\pi}\frac{1}{10^6}
\end{aligned}$$

(b) Compute the directivity of the antenna (dimensionless and in dB):

$$
\begin{aligned}
D &= \frac{U_{max}}{P_{rad}/4\pi} \\
D &= \frac{4\pi \cdot U_{max}}{P_{rad}} \\
&= \frac{4\pi \cdot 30/\pi}{15} \\
&= 8[dimensionless]
\end{aligned}
$$

$$D_{dBi} = 10\log_{10} D = 9.03[dBi]$$

(c) Compute the gain of the antenna (dimensionless and in dB): As the antenna is lossless, gain is the same as D.

$$\therefore \quad G = D = 8, \quad G_{dBi} = D_{dBi} = 9.03[dBi]$$

Quesiton 2

The normalized radiation intensity of a given antenna is given by

(a) $U(\theta, \varphi) = \sin^2(\theta)\sin(\varphi)$,

(b) $U(\theta, \varphi) = \sin^2(\theta)\sin^2(\varphi)$,

(c) $U(\theta, \varphi) = \sin^2(\theta)\cos^3(\varphi)$

The radiation intensity is non-zero only in the $0 \leq \theta \leq \pi, 0 \leq \varphi \leq \pi$ region, and it is zero elsewhere. Find:

(a) The exact directivity (dimensionless and in dB):

[a]

$$\begin{aligned}
P_{rad} &= \int\limits_{\varphi=0}^{\pi} \int\limits_{\theta=0}^{\pi} U(\theta,\varphi)\sin\theta\, d\theta\, d\varphi \\
&= \int\limits_{\varphi=0}^{\pi} \sin\varphi\, d\varphi \int\limits_{\theta=0}^{\pi} \sin^2(\theta)\sin\theta\, d\theta \\
&= \Big[-\cos\varphi\Big]_0^{\pi} \cdot \Big[\frac{\cos^3(\theta)}{3} - \cos\theta\Big]_0^{\pi} \\
&= 2\cdot 2[1-\frac{1}{3}] = 8/3
\end{aligned}$$

$$\begin{aligned}
D &= \frac{4\pi\cdot U_{max}}{P_{rad}} = \frac{4\pi\cdot 1}{8/3} = \frac{12\pi}{8} = \frac{3\pi}{2} = 4.712 \\
D_{dBi} &= 10\log_{10} D = 6.732[dBi]
\end{aligned}$$

[b]

$$\begin{aligned}
P_{rad} &= \int\limits_{\varphi=0}^{\pi} \int\limits_{\theta=0}^{\pi} U(\theta,\varphi)\sin\theta\, d\theta\, d\varphi \\
&= \int\limits_{\varphi=0}^{\pi} \sin^2(\varphi)\, d\varphi \int\limits_{\theta=0}^{\pi} \sin^2(\theta)\sin\theta\, d\theta \\
&= \left[\frac{2x-\sin 2x}{4}\right]_0^{\pi} \cdot \Big[\frac{\cos^3(\theta)}{3} - \cos\theta\Big]_0^{\pi} \\
&= [\frac{\pi}{2}]\cdot 2[1-\frac{1}{3}] = 2\pi/3
\end{aligned}$$

$$\begin{aligned}
D &= \frac{4\pi\cdot U_{max}}{P_{rad}} = \frac{4\pi\cdot 1}{2\pi/3} = 12 \\
D_{dBi} &= 10\log_{10} D = 10.8[dBi]
\end{aligned}$$

[c]

$$P_{rad} = \int_{\varphi=0}^{\pi} \int_{\theta=0}^{\pi} U(\theta,\varphi)\sin\theta\, d\theta\, d\varphi$$

We must change the limits of integration as $U(\theta,\varphi) < 0$ is not possible

$$= \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{\pi} U(\theta,\varphi)\sin\theta\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \cos^3(\varphi)\, d\varphi \int_{\theta=0}^{\pi} \sin^2(\theta)\sin\theta\, d\theta$$

$$= \left[\sin\varphi - \frac{\sin^3(\varphi)}{3}\right]_0^{\pi/2} \cdot \left[\frac{\cos^3(\theta)}{3} - \cos\theta\right]_0^{\pi}$$

$$= [\frac{2}{3}] \cdot 2[1 - \frac{1}{3}] = 8/9$$

$$D = \frac{4\pi \cdot U_{max}}{P_{rad}} = \frac{4\pi \cdot 1}{8/9} = \frac{9\pi}{2} = 14.14$$

$$D_{dBi} = 10\log_{10} D = 11.5[dBi]$$

(b) The azimuthal and elevation half-power beamwidths (in degrees)

[a]

Azimuthal HPBW: move to φ plane, $\therefore \quad \theta = 90°$

$$HPBW_{AZ} = 2[90° - \sin^{-1}\left(\frac{1}{2}\right)] = 120°$$

Elevation HPBW: $\varphi = 90°$

$$HPBW_{EL} = 2[90° - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)] = 90°$$

[b]

$$HPBW_{AZ} = 2[90° - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)] = 90°$$

$$HPBW_{EL} = 2[90° - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)] = 90°$$

[c]

Azimuthal HPBW: move to φ plane, $\therefore \quad \theta = 90°$

$$HPBW_{AZ} = 2[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)] \approx 75°$$

Elevation HPBW: $\varphi = 0°$

$$HPBW_{EL} = 2[90° - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)] = 90°$$

Quesiton 3

The radiation intensity of an antenna is symmetric, and it can be approximated by:

$$U(\theta) = \begin{cases} \cos^{-1}(\theta) & 0° \leq \theta \leq 30° \\ 0.866 & 30° \leq \theta \leq 90° \\ 0 & 90° \leq \theta \leq 180° \end{cases} \tag{1}$$

and is independent of j . Find the directivity by integrating the function and elevation half-power beamwidths (in degrees):

Function is discontinuous, thus no possible solution.

If we were to assume thhat the intended function, thats actually solves the Antenna euqation without discontinous points:

$$U(\theta) = \begin{cases} \cos(\theta) & 0° \leq \theta \leq 30° \\ 0.866 & 30° \leq \theta \leq 90° \\ 0 & 90° \leq \theta \leq 180° \end{cases} \tag{2}$$

$$\begin{aligned} P_{rad} &= \int\limits_{\theta=0}^{\pi/2} U(\theta)\sin\theta \, d\theta \\ &= \int\limits_{\theta=0}^{\pi/6} \cos\theta \sin\theta \, d\theta + \int\limits_{\theta=\pi/6}^{\pi/2} 0.866 \sin\theta \, d\theta \\ &= \left[\frac{\sin^2(\theta)}{2}\right]_0^{\pi/6} + 0.866\Big[-\cos\theta\Big]_{\pi/6}^{\pi/2} \\ &= 1/8 + 3/4 = 7/8 = 0.875 \end{aligned}$$

$$D = \frac{4\pi \cdot U_{max}}{P_{rad}} = \frac{4\pi \cdot 1}{7/8} = \frac{32\pi}{7} = 14.36$$

Quesiton 4

The complex electric field of a uniform plane wave is given by:

$$\mathbf{E}(z) = E_0(\hat{x} + j\hat{y})e^{j\beta z}$$

Find the polarization of the wave (linear, circular, or other). State the direction of rotation (right-hand circular polarization (RHCP) or left-hand circular polarization (RHCP)):

Firstly we can see that the wave propagates in the $-\hat{z}$ direction, now let's observe the time-domain behavior:

$$\mathbf{E}(z,t) = E_0[\hat{x}\cos(\omega t + \beta z) - \hat{y}\sin(\omega t + \beta z)]$$

Analyzing it in the plane $z = 0$:

$$\mathbf{E}(0,t) = E_0[\hat{x}\cos(\omega t) - \hat{y}\sin(\omega t)]$$

Thus, it propagates in the RHCP.

Quesiton 5

A uniform plane wave is given by:

$$\mathbf{E}(z) = \left(E_{0x}e^{j\varphi_x}\hat{x} + E_{0y}e^{j\varphi_y}\hat{y}\right)e^{-j\beta z}$$

Find the polarization of the wave (linear, circular, or elliptical), sense of rotation (RHCP or LHCP), axial ratio (AR), and tilt angle (in degrees) when

(a) $E_{0x} \neq E_{0y}, \Delta\varphi = \varphi_y - \varphi_x = 0$

$$E_0(0,t) = (E_{0x}\hat{x} + E_{0y}\hat{y}) \cdot \cos(\omega t + \varphi_x)$$

Linear Polarization,

$$AR = \frac{OA}{OB},$$

$$OA = \sqrt{\frac{1}{2}[E_{0x}^2 + E_{0y}^2 + \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\Delta\varphi)}]}$$

$$OB = \sqrt{\frac{1}{2}[E_{0x}^2 + E_{0y}^2 - \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\Delta\varphi)}]}$$

$$\therefore AR = \frac{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0y}^2 + \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2}]}}{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0y}^2 - \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2}]}}$$

$$\text{tilt angle: } = \tau = \frac{\pi}{2} - \frac{1}{2}\arctan[\frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos(\Delta\varphi)]$$

$$\therefore \tau = \frac{\pi}{2} - \frac{1}{2}\arctan[\frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}]$$

(b) $E_{0x} = E_{0y}, \Delta\varphi = \varphi_y - \varphi_x = -\pi/2$

$$\begin{aligned} E_0(0,t) &= E_{0x}(\cos(\omega t + \pi/2)\hat{x} + \cos(\omega t)\hat{y}) \\ &= E_{0x}\sin(\omega t + \pi/2)\hat{x} + E_{0x}\sin(\omega t)\hat{y} \end{aligned}$$

$$AR = \frac{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0x}^2 + \cancelto{0}{\sqrt{E_{0x}^4 + E_{0x}^4 - 2E_{0x}^2E_{0x}^2}}\;]}}{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0x}^2 - \cancelto{0}{\sqrt{E_{0x}^4 + E_{0x}^4 - 2E_{0x}^2E_{0x}^2}}\;]}}$$

$RHCP$

$$AR = \frac{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0x}^2]}}{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0x}^2]}} = 1$$

$$\tau = \frac{\pi}{2} - \frac{1}{2}\arctan[\frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos(\Delta\varphi)] \Rightarrow \tau \text{ is undefined}$$

If we were to estimate $E_{0x} = E_{0y} + \delta, \quad \delta > 0$

$$\begin{aligned} \tau &= \frac{\pi}{2} - \frac{1}{2}\arctan[\frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos(\Delta\varphi)] = \frac{\pi}{2} - \frac{1}{2}\arctan[-\infty] \\ &= \frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4} \end{aligned}$$

(c) $E_{0x} = E_{0y}, \Delta\varphi = \varphi_y - \varphi_x = -\pi/4$

$$
\begin{aligned}
E_0(0,t) &= E_{0x}(\cos(\omega t + \varphi)\hat{x} + \cos(\omega t + \varphi - \pi/4)\hat{y}) \\
&= E_{0x}\sin(\omega t + \pi/2)\hat{x} + E_{0x}\sin(\omega t)\hat{y}
\end{aligned}
$$

Elliptical polarization,

$$
\begin{aligned}
AR &= \frac{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0x}^2 + \sqrt{E_{0x}^4 + E_{0x}^4}]}}{\sqrt{\frac{1}{2}[E_{0x}^2 + E_{0x}^2 - \sqrt{E_{0x}^4 + E_{0x}^4}]}} \\
&= \frac{\sqrt{\frac{1}{2}[2E_{0x}^2 + \sqrt{2E_{0x}^4}]}}{\sqrt{\frac{1}{2}[2E_{0x}^2 - \sqrt{2E_{0x}^4}]}} \\
&= \frac{\sqrt{\frac{1}{2}[2E_{0x}^2 + E_{0x}^2\sqrt{2}]}}{\sqrt{\frac{1}{2}[2E_{0x}^2 - E_{0x}^2\sqrt{2}]}} \\
&= \frac{\sqrt{\frac{1}{2}[2 + \sqrt{2}]}}{\sqrt{\frac{1}{2}[2 - \sqrt{2}]}} \\
&= \frac{1.306}{0.5411} \\
AR &= 2.4142
\end{aligned}
$$

$$
\tau = \frac{\pi}{2} - \frac{1}{2}\arctan[\frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos(\Delta\varphi)] \Rightarrow \tau \text{ is undefined}
$$