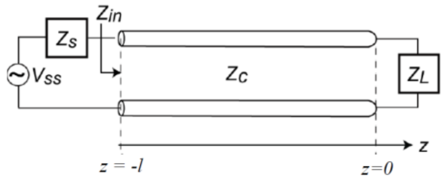
HARMONIC SIGNAL



$$\beta = \omega\sqrt{L'C'} = \omega\sqrt{\varepsilon\mu} = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}\left[\frac{1}{m}\right]$$

$$V^+(z) = {V_0}^+ e^{-j\beta z}$$
$$V^-(z) = {V_0}^- e^{+j\beta z}$$

Reflection coefficient and entrance impedance:

$$\Gamma(0) \equiv \frac{{V_0}^-}{{V_0}^+} = \Gamma_L = \frac{Z_L - Z_C}{Z_L + Z_C} \qquad Z_{in}(z) = Z_C\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

$$\Gamma(z) \equiv \frac{V^-(z)}{V^+(z)} = \Gamma_L \exp(2j\beta z) \qquad \Gamma(z) = \frac{Z_{in}(z) - Z_C}{Z_{in}(z) + Z_C}$$

$$Z_{in}(z=-L) = Z_C\frac{Z_L + jZ_C\tan(\beta L)}{Z_C + jZ_L\tan(\beta L)}$$

$$Z_{in}\left(z = -\frac{\lambda}{4}\right) = \frac{{Z_C}^2}{Z_L}$$

$$\begin{cases} V(z) = {V_0}^+ e^{-j\beta z}\left[1 + \Gamma(z=0)e^{j2\beta z}\right] \\ I(z) = \dfrac{{V_0}^+}{Z_C} e^{-j\beta z}\left[1 - \Gamma(z=0)e^{j2\beta z}\right] \end{cases}$$

POWER AND ENERGY CALCULATIONS

$$P = P_{av} = \langle \boldsymbol{\mathcal{V}}(z,t)\boldsymbol{\mathcal{I}}(z,t)\rangle = \frac{1}{2}\mathrm{Re}[V(z)I^*(z)]$$

$$= \frac{1}{2}|V(z)|^2\mathrm{Re}\left[\frac{1}{Z(z)}\right] = \frac{1}{2}|I(z)|^2\mathrm{Re}[Z(z)]$$

Power of the wave:

$$P^\pm = \frac{1}{2}\mathrm{Re}\{V^\pm I^{\pm *}\} = \pm\frac{1}{2}|V^\pm|^2\,\mathrm{Re}\left[\frac{1}{Z_C}\right]$$

$$P = P^+ + P^-$$

Lossless circuit: entrance power equals power on the load

$$P_L = P_{in} = \frac{1}{2}\mathrm{Re}[V_{in}{I^*}_{in}] = \frac{1}{2}|V_{in}|^2\,\mathrm{Re}\left[\frac{1}{Z_{in}}\right]$$

From the Telegraph Eq. we get the wave Eq.:

$$\frac{d^2V}{dz^2} = L'C'\frac{d^2V}{dt^2}$$
$$\frac{d^2I}{dz^2} = L'C'\frac{d^2I}{dt^2}$$

The general solution:

$$\begin{cases} V(z,t) = V^+\left(t - \dfrac{z}{v_p}\right) + V^-\left(t + \dfrac{z}{v_p}\right) \\ I(z,t) = I^+\left(t - \dfrac{z}{v_p}\right) + I^-\left(t + \dfrac{z}{v_p}\right) \end{cases}$$

The propagation velocity:

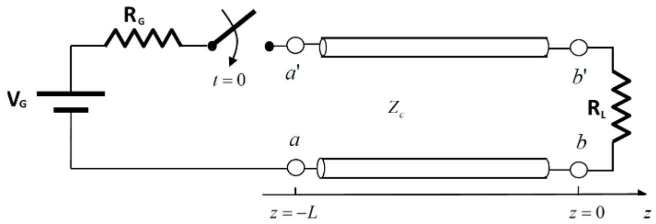
$$v_p = \frac{1}{\sqrt{L'C'}}$$

The characteristic Impedance:

$$Z_c = \sqrt{\frac{L'}{C'}} = \frac{V^+}{I^+} = -\frac{V^-}{I^-}$$

Time Domain solution:

$$\begin{cases} V(z,t) = V_0 + \displaystyle\sum_{n=1}^{\infty} {V_n}^+\left(t - \frac{z}{v_p}\right) + \sum_{n=1}^{\infty} {V_n}^-\left(t + \frac{z}{v_p}\right) \\ I(z,t) = I_0 + \displaystyle\sum_{n=1}^{\infty} {I_n}^+\left(t - \frac{z}{v_p}\right) + \sum_{n=1}^{\infty} {I_n}^-\left(t + \frac{z}{v_p}\right) \end{cases}$$



Reflection coefficient at the load:

$$\Gamma_L = \frac{V_n^-(t, z=0)}{V_n^+(t, z=0)} = \frac{R_L - Z_c}{R_L + Z_c}$$

Reflection coefficient at the source:

$$\Gamma_G = \frac{V_{n+1}^+(t, z=-l)}{V_n^-(t, z=-l)} = \frac{R_G - Z_c}{R_G + Z_c}$$

GENERAL

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

$$\mu_0 = 4\pi\cdot 10^{-7}\frac{\mathrm{Henry}}{\mathrm{m}}, \quad \varepsilon_0 = 8.85\cdot 10^{-12}\frac{\mathrm{Farad}}{\mathrm{m}}$$

$$c = \frac{1}{\sqrt{\mu_o\varepsilon_0}} = 2.9979\cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$S_n \equiv \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}; S_\infty = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$\sin(jx) = j\sinh(x); \quad \cos(jx) = \cosh(x)$$
$$e^{jx} = \cos(x) + j\sin(x)$$

$$z = x + jy = |z|e^{j\theta}$$
$$|z| = \sqrt{x^2 + y^2} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

PHASORS

$$V(t) = V_0\cos(\omega t + \theta) = \mathrm{Re}\{V_0 e^{j\omega t}e^{j\theta}\} = \mathrm{Re}\{\tilde{V}e^{j\omega t}\}$$
$$.\tilde{V} \equiv V_0 e^{j\theta}$$

$$\frac{d}{dt}V(t) \Leftrightarrow j\omega\tilde{V}$$

DISTRIBUTED LINE IN LC MODEL (WITHOUT LOSSES)

Inductance and Capacitance per unit length

$$L'[\mathrm{H/m}] \text{ ו } C'[\mathrm{F/m}]$$

$$L = L'\Delta z$$
$$C = C'\Delta z$$

The Telegraph Eq. in the limit of $\Delta z \to 0$

$$\frac{dV}{dz} = -L'\frac{dI}{dt}$$
$$\frac{dI}{dz} = -C'\frac{dV}{dt}$$

For this solution we get:

$$\vec{E} \perp \vec{H} \perp \vec{k}$$

$$\frac{|\vec{E}|}{|\vec{H}|} = \eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}\frac{\mu_r}{\varepsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{\eta_0}{n}$$

$$\vec{H} = \frac{\hat{k} \times \vec{E}}{\eta} = \frac{1}{\eta}(\hat{k} \times \vec{E}_0)e^{-j\vec{k}\cdot\vec{r}}$$

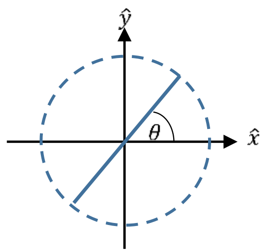
POLARIZATION OF a PLANE WAVE

In frequency:

$$\vec{E}(z_0) = \hat{x}|E_x| + \hat{y}|E_y|e^{j\phi_y - \phi_x}$$

In time:

$$\vec{E}(z_0, t) = \hat{x}|E_x|\cos(\omega t) + \hat{y}|E_y|\cos(\omega t + \phi_y - \phi_x)$$



LINEAR POLARIZATION

$$\phi_y - \phi_x = m \cdot \pi \qquad ; \; m \in \mathbb{Z}$$

$$\vec{E}(z_0, t) = \hat{x}|E_x|\cos(\omega t) \pm \hat{y}|E_y|\cos(\omega t)$$

$$\vec{E}(z_0) = |E_x|\hat{x} \pm |E_y|\hat{y}$$

Notating:

$$\theta = \tan^{-1}\left(\pm\frac{|E_y|}{|E_x|}\right); \quad 0 < \theta < \pi$$

$$|E_x| = E_0\cos(\theta), \quad \pm|E_y| = E_0\sin(\theta)$$

REFLECTION WITH LOSSES

$$\Gamma(0) = \frac{Z_L - Z_C}{Z_L + Z_C}, \qquad \Gamma(z) = \Gamma(0)e^{+2\gamma z}$$

$$\Gamma(-L) = \Gamma(0)e^{-2\gamma L} = \Gamma(0)e^{-2j\beta L - 2\alpha L}$$

$$Z_{in}(-L) = Z_C\frac{1 + \Gamma(-L)}{1 - \Gamma(-L)} = Z_C\frac{Z_L + Z_C\tanh(j\beta L + \alpha L)}{Z_C + Z_L\tanh(j\beta L + \alpha L)}$$

$$V(z) = V_0{}^{+}\left[e^{-\alpha z}e^{-j\beta z} + \Gamma_L e^{+\alpha z}e^{+j\beta z}\right]$$

$$= V_0{}^{+}e^{-\alpha z}e^{-j\beta z}\left[1 + \Gamma_L e^{+2\alpha z}e^{+2j\beta z}\right]$$

$$I(z) = \frac{V_0{}^{+}}{Z_0}\left[e^{-\alpha z}e^{-j\beta z} - \Gamma_L e^{+\alpha z}e^{+j\beta z}\right]$$

$$= \frac{V_0{}^{+}}{Z_0}e^{-\alpha z}e^{-j\beta z}\left[1 - \Gamma_L e^{+2\alpha z}e^{+2j\beta z}\right]$$

POWERS

$$P(z) = \frac{1}{2}\mathrm{Re}\{VI^*\} =$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left|V_0{}^{+}\right|^2 Y_C{}^{*}e^{-2\alpha z}\left(1 - |\Gamma_L|^2 e^{+4\alpha z}\right)\right\}$$

$$P_{in}(z = -l) = \frac{1}{2}\mathrm{Re}\left\{\left|V_0{}^{+}\right|^2 Y_C{}^{*}e^{2\alpha l}\left(1 - |\Gamma_L|^2 e^{-4\alpha l}\right)\right\}$$

$$P_{load}(z = 0) = \frac{1}{2}\mathrm{Re}\left\{\left|V_0{}^{+}\right|^2 Y_C{}^{*}\left(1 - |\Gamma_L|^2\right)\right\}$$

PLANE WAVES

$$\begin{cases} \nabla \times \vec{E} = -j\omega\mu\vec{H} \\ \nabla \times \vec{H} = j\omega\varepsilon\vec{E} \\ \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{H} = 0 \end{cases} \underbrace{\Rightarrow}_{\nabla\times\nabla\times\vec{A} = -\nabla^2 A + \nabla(\nabla\cdot A)} \begin{cases} \nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0 \\ \nabla^2\vec{H} + \omega^2\mu\varepsilon\vec{H} = 0 \end{cases}$$

The solution is:

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}$$

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

$$k = |\vec{k}| = 2\pi n/\lambda_0 = \omega\sqrt{\mu\varepsilon} = \omega n/c = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

STANDING WAVE RATIO (SWR)

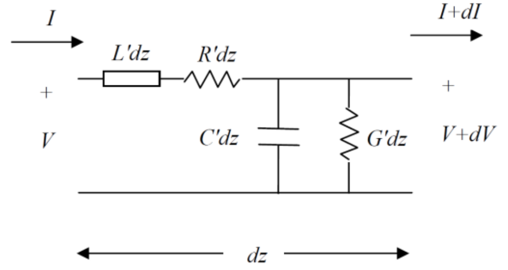
$$S = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \geq 1$$

$$|V_{\max}| = |V_0^+|(1 + |\Gamma|)$$

$$|V_{\min}| = |V_0^+|(1 - |\Gamma|)$$

$$|\Gamma(z)| = \left|\Gamma(0)e^{2j\beta z}\right| = |\Gamma(0)|$$

TRANSMISSION LINE WITH LOSSES



Conductivity (heat) losses – R'

Dielectric losses – G'

WAVE EQUATION WITH LOSSES

$$\begin{cases} \dfrac{dV}{dz} = -IR' - L'\dfrac{dI}{dt} \\ \dfrac{dI}{dz} = -VG' - C'\dfrac{dV}{dt} \end{cases}$$

$$\frac{d^2V}{dz^2} = L'C'\frac{d^2V}{dt^2} + (C'R' + L'G')\frac{dV}{dt} + R'G'V$$

The solution to the equation:

$$V = \mathrm{Re}\{V_0 e^{-\gamma z}e^{j\omega t}\}$$

$$\gamma = \alpha + j\beta = \pm\sqrt{(R' + j\omega L')(G' + j\omega C')}\ [1/\mathrm{m}]$$

$$Z_C(\omega) = \frac{V^+}{I^+} = R_0 + jX_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

Small losses: $R' << \omega L', G' << \omega C'$

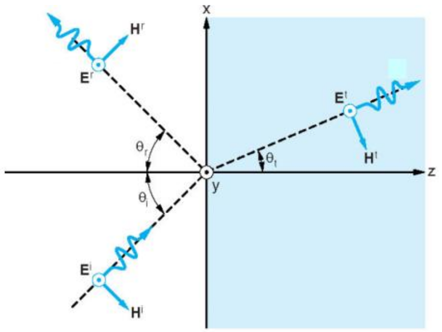
In first order approximation:

$$\beta \approx \omega\sqrt{L'C'} \qquad \alpha \approx \frac{1}{2}\sqrt{L'C'}\left(\frac{R'}{L'} + \frac{G'}{C'}\right)$$

Under this approximation, the decay coefficient is not frequency dependent and the propagation coefficient has a linear dependency (no dispersion).

TE SOLUTION (PERPENDICULAR)

$E_z = 0$



$\vec{k} = k_x\hat{x} + k_z\hat{z} = k(\hat{x}\sin\theta + \hat{z}\cos\theta)$

$\hat{e}_{\mathrm{TE}} = \hat{y}$

$\hat{h}_{\mathrm{TE}} = -\hat{x}\cos\theta + \hat{z}\sin\theta$

$\vec{E}_{\mathrm{TE}} = E_{\mathrm{TE}}\,\hat{y}\,e^{-jk\sin\theta x}e^{-jk\cos\theta z}$

$E_y = e^{-jk\sin\theta x}\left[V_0^+e^{-jk\cos\theta z} + V_0^-e^{jk\cos\theta z}\right], \qquad V_0^+ = E_{\mathrm{TE}}$

TL MODEL FOR TE POLARIZATION:

$E_y^+ \;\leftrightarrow\; V^+$

$-H_x^+ \;\leftrightarrow\; I^+$

$k_z = k\cos\theta \;\leftrightarrow\; \beta$

$\dfrac{\eta}{\cos\theta} \;\leftrightarrow\; Z_{C_{TE}}$

The longitudinal component:

$$H_z = \left[\frac{1}{Z_{C_{TE}}}\tan\theta\right]E_y$$

FRESNEL EQUATIONS FOR TE:

(For the transverse component Ey):

$$r_{21}^{\mathrm{TE}} = \frac{\dfrac{\eta_2}{\cos\theta_2} - \dfrac{\eta_1}{\cos\theta_1}}{\dfrac{\eta_2}{\cos\theta_2} + \dfrac{\eta_1}{\cos\theta_1}} = \frac{n_1\cos\theta_1 - \dfrac{\mu_1}{\mu_2}n_2\cos\theta_2}{n_1\cos\theta_1 + \dfrac{\mu_1}{\mu_2}n_2\cos\theta_2}$$

$$t_{21}^{\mathrm{TE}} = \frac{2\dfrac{\eta_2}{\cos\theta_2}}{\dfrac{\eta_2}{\cos\theta_2} + \dfrac{\eta_1}{\cos\theta_1}} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + \dfrac{\mu_1}{\mu_2}n_2\cos\theta_2}$$

$t_{21}^{\mathrm{TE}} = r_{21}^{\mathrm{TE}} + 1$

Skin depth:

$$\delta_s = \frac{1}{\alpha}$$

In a metal (good conductor):

$$\varepsilon_{\mathrm{eff}} = \varepsilon' - j\left[\varepsilon'' + \frac{\sigma}{\omega}\right] \xrightarrow[\frac{\sigma}{\omega} >> |\varepsilon'|,|\varepsilon''|]{} -j\frac{\sigma}{\omega}$$

$$k = \omega\sqrt{\mu\varepsilon_{\mathrm{eff}}} = \frac{1-j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}, \qquad \delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$\eta = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$$

Usually: $|Z_{metal}| << Z_{other}$

FRESNEL EQUATIONS FOR PERPENDICULAR INCIDENT:

Refractive index in a dielectric:

$$n = \sqrt{\varepsilon_r}$$

The incident wave is coming from medium 1 to medium 2:

$$\begin{aligned} n_1 &= \sqrt{\varepsilon_{r1}} \\ n_2 &= \sqrt{\varepsilon_{r2}} \end{aligned} \;\Rightarrow\; \begin{aligned} r_{21} &= \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{n_1-n_2}{n_1+n_2} \\ t_{21} &= 1 + r_{21} = \frac{2\eta_2}{\eta_2+\eta_1} = \frac{2n_1}{n_1+n_2} \end{aligned}$$

ANGLED INCIDENT:

$$\vec{E}(\vec{r}) = \vec{E}_0e^{-j\vec{k}\cdot\vec{r}}$$

$$= [E_{\mathrm{TE}}\hat{e}_{\mathrm{TE}} + E_{\mathrm{TM}}\hat{e}_{\mathrm{TM}}]e^{-j\vec{k}\cdot\vec{r}}$$

$$\vec{H}(\vec{r}) = \frac{1}{\eta}(\hat{k}\times\vec{E}_0)e^{-j\vec{k}\cdot\vec{r}}$$

$$= \left[\frac{E_{\mathrm{TE}}}{\eta}\hat{h}_{\mathrm{TE}} + \frac{E_{\mathrm{TM}}}{\eta}\hat{h}_{\mathrm{TM}}\right]e^{-j\vec{k}\cdot\vec{r}}$$

$\hat{e}_{\mathrm{TM}} \perp \hat{e}_{\mathrm{TE}} \perp \hat{k}$

CIRCULAR POLARIZATION

$\phi_y - \phi_x = \pm\dfrac{\pi}{2} + 2\pi m \;\;;\;\; m \in \mathbb{Z}$

And the modulus of the amplitudes are equal

$|E_x| = |E_y| = |E|/\sqrt{2}$

RIGHT CIRCULAR

$\boldsymbol{\phi_y - \phi_x = -\pi/2}$

$$\{\mathrm{RC}\}: \vec{E}(z_0,t) = \frac{|E|}{\sqrt{2}}(\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y})$$

$$\{\mathrm{RC}\}: \vec{E}(z_0) = \frac{|E|}{\sqrt{2}}\cdot(\hat{x} - j\hat{y})$$

LEFT CIRCULAR

$\boldsymbol{\phi_y - \phi_x = +\pi/2}$

$$\{\mathrm{LC}\}: \vec{E}(z_0,t) = \frac{|E|}{\sqrt{2}}(\cos(\omega t)\,\hat{x} - \sin(\omega t)\,\hat{y})$$

$$\{\mathrm{LC}\}: \vec{E}(z_0) = \frac{|E|}{\sqrt{2}}\cdot(\hat{x} + j\hat{y})$$

PERPENDICULAR INCIDENT

$$\vec{E}(z) = \vec{E}_0e^{-j\beta z} = [E_{0x}\hat{x} + E_{0y}\hat{y}]e^{-j\beta z}$$

$$\vec{H}(z) = \frac{1}{\eta}(\hat{z}\times\vec{E}_0)e^{-j\beta z} = \left[\frac{E_{0x}}{\eta}\hat{y} - \frac{E_{0y}}{\eta}\hat{x}\right]e^{-j\beta z}$$

TL MODEL FOR PERPENDICULAR INCIDENT

$E_x^+ \,/\, E_y^+ \;\leftrightarrow\; V^+$

$H_y^+ \,/\, -H_x^+ \;\leftrightarrow\; I^+$

$k = \omega\sqrt{\epsilon\mu} \;\leftrightarrow\; \beta$

$\eta = \sqrt{\dfrac{\mu}{\epsilon}} \;\leftrightarrow\; Z_C$

MEDIUM WITH LOSSES AND CONDUCTIVITY

$\gamma = jk = j\beta + \alpha$

Dielectric and magnetic losses:

$\varepsilon = \varepsilon' - j\varepsilon'', \qquad \mu = \mu' - j\mu''$

With a finite conductivity:

$$\eta = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}}, \qquad k_2 = \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)},$$

PLANE WAVE SPECTRUM

Spectrum of the field in the z plane:

$$E_T(x;z) = \int_{-\infty}^{\infty} \tilde{E}_T(k_x;z)\cdot e^{-jk_x x}dk_x$$

$$\tilde{E}_T(k_x;z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_T(x;z)\cdot e^{jk_x x}dx$$

Spectrum of the field at z=0:

$$E_T(x;0) = \int_{-\infty}^{\infty} \tilde{E}_T(k_x;0)\cdot e^{-jk_x x}dk_x$$

$$\tilde{E}_T(k_x;0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_T(x;0)\cdot e^{jk_x x}dx$$

The connection between the spectrum at z and the spectrum at z=0:

$$\tilde{E}(k_x;z) = \tilde{E}(k_x;0)\cdot e^{-jk_z z} = \tilde{E}_0(k_x)\cdot e^{-j\sqrt{k^2-k_x^2}\cdot z}$$

The field in z>0:

$$E(x;z) = \int_{-\infty}^{\infty} \tilde{E}_0(k_x)\cdot e^{-j\sqrt{k^2-k_x^2}\cdot z}\cdot e^{-jk_x x}dk_x$$

For far field we can neglect the decay waves with $: |k_x| > k$

$$E(x;z) = \int_{-k}^{k} \tilde{E}_0(k_x)\cdot e^{-jk_x x}\cdot e^{-j\sqrt{k^2-k_x^2}\cdot z}dk_x$$

Useful functions:

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

$$\operatorname{comb}\left(\frac{x}{a}\right) = |a|\cdot\sum_{-\infty}^{\infty}\delta(x-na)$$

TEM TRANSMISSION LINES

Structures supporting TEM mode:

$$E_z = H_z = 0$$

$$k \leftrightarrow \beta$$

$$\vec{E}(\vec{r}) = \vec{e}(x,y)[V_0{}^{+}e^{-jkz} + V_0{}^{-}e^{+jkz}]$$

$$\vec{H}(\vec{r}) = \vec{h}(x,y)[I_0{}^{+}e^{-jkz} + I_0{}^{-}e^{+jkz}]$$

$$\vec{h}(x,y) = \frac{Z_C}{\eta}\vec{z}\times\vec{e}(x,y)$$

BRUSTER ANGLE:

The angle for which there is no reflection for TM only:

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) \Rightarrow r_{21}^{\mathrm{TM}} = 0$$

In general:

$$\theta_B = \sin^{-1}\left[\sqrt{\frac{\mu_2\varepsilon_2(\eta_2{}^2-\eta_1{}^2)}{\eta_2{}^2\mu_1\varepsilon_1-\eta_1{}^2\mu_2\varepsilon_2}}\right]$$

SNELL'S LAW:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\sqrt{\mu_{r2}\varepsilon_{r2}}}{\sqrt{\mu_{r1}\varepsilon_{r1}}} = \frac{n_2}{n_1}$$

$$n_1 > n_2$$

The critic angle:

$$\sin\theta_{cr} = \frac{n_2}{n_1}$$

For incident angle greater than the critic: $: \theta_i > \theta_{cr} :$

$$|\Gamma_{\mathrm{TE}}| = |\Gamma_{\mathrm{TM}}| = 1 = e^{j\phi}$$

$$\cos\theta_2 = -j\sqrt{\frac{\sin^2\theta_1}{\sin^2\theta_{cr}}-1}$$

Plugging in: $\vec{E}_2(z) = \vec{E}_{0_2}e^{-jk_2(x\sin\theta_t+z\cos\theta_t)}$

leads to propagation in x and decay in z:

$$\vec{E}_2(z) = \vec{E}_{0_2}e^{-jk_2x\frac{\sin\theta_1}{\sin\theta_{cr}}}e^{-k_2z\sqrt{\frac{\sin^2\theta_1}{\sin^2\theta_{cr}}-1}}$$

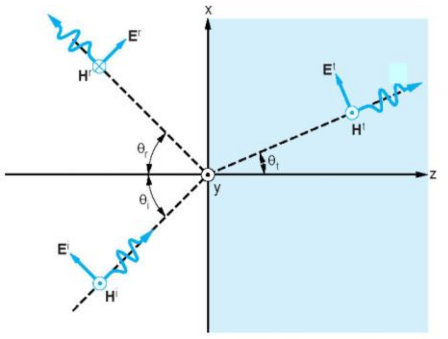
In a full reflection:

$$\Gamma_{\mathrm{TM}} = e^{j\phi_{\mathrm{TM}}} \quad \phi_{\mathrm{TM}} = 2\tan^{-1}\left(-\frac{\sqrt{\sin^2\theta_1-\sin^2\theta_{cr}}}{\cos\theta_1\sin^2\theta_{cr}}\right)$$

$$\Gamma_{\mathrm{TE}} = e^{j\phi_{\mathrm{TE}}} \quad \phi_{\mathrm{TE}} = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_1-\sin^2\theta_{cr}}}{\cos\theta_1}\right)$$

TM SOLUTION (PARALLEL)

$$H_z = 0$$



$$\vec{k} = k_x\hat{x} + k_z\hat{z} = k(\hat{x}\sin\theta + \hat{z}\cos\theta)$$

$$\hat{e}_{\mathrm{TM}} = \hat{x}\cos\theta - \hat{z}\sin\theta$$

$$\hat{h}_{\mathrm{TM}} = \hat{y}$$

$$\vec{E}_{\mathrm{TM}} = E_{\mathrm{TM}}\,(\hat{x}\cos\theta - \hat{z}\sin\theta)e^{-jk\sin\theta x}e^{-jk\cos\theta z} =$$

$$E_x = e^{-jk\sin\theta x}[V_0^{+}e^{-jk\cos\theta z} + V_0^{-}e^{jk\cos\theta z}],$$

$$V_0^{+} = E_{\mathrm{TM}}\cos\theta$$

TL MODEL FOR TM POLARIZATION:

$$E_x^{+} \leftrightarrow V^{+}$$

$$H_y^{+} \leftrightarrow I^{+}$$

$$k_z = k\cos\theta \leftrightarrow \beta$$

$$\eta\cos\theta \leftrightarrow Z_{C_{TM}}$$

The longitudinal component:

$$E_z = [-Z_{C_{TM}}\tan\theta]H_y$$

FRESNEL EQUATIONS FOR TM POLARIZATION:

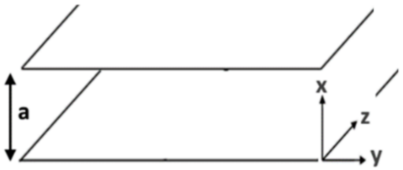
(For the transverse component Ex):

$$r_{21}^{\mathrm{TM}} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = \frac{n_1\cos\theta_2 - \frac{\mu_1}{\mu_2}n_2\cos\theta_1}{n_1\cos\theta_2 + \frac{\mu_1}{\mu_2}n_2\cos\theta_1}$$

$$t_{21}^{\mathrm{TM}} = \frac{2\eta_2\cos\theta_2}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = \frac{2n_1\cos\theta_2}{n_1\cos\theta_2 + \frac{\mu_1}{\mu_2}n_2\cos\theta_1}$$

$$t_{21}^{\mathrm{TM}} = r_{21}^{\mathrm{TM}} + 1$$

PLATES WAVEGUIDE:



Wave vector:

$$\beta_m = \sqrt{k^2 - k_{c,m}{}^2} = \omega\sqrt{\mu\varepsilon\left[1-\left(\frac{f_{c,m}}{f}\right)^2\right]}$$

$$k_{c,m} = \left(\frac{m\pi}{a}\right)$$

Cut-off frequency:

$$f_{c,m} = \frac{1}{2\sqrt{\mu\varepsilon}}\left(\frac{m}{a}\right)$$

TEM MODE:

$$E_x = Ae^{-jkz}$$

$$H_y = \frac{E_x}{\eta}e^{-jkz}$$

TE MODES:

$$E_y = C_m\, sin\left(\frac{m\pi}{a}x\right)e^{-j\beta_m z}$$

$$H_x = -\frac{\beta_m}{\omega\mu}C_m\, sin\left(\frac{m\pi}{a}x\right)e^{-j\beta_m z}$$

$$H_z = j\frac{m\pi}{\omega\mu a}C_m\, cos\left(\frac{m\pi}{a}x\right)e^{-j\beta_m z}$$

$$Z_m^{\mathrm{TE}} = \frac{\omega\mu}{\beta_m} = \frac{k\eta}{\beta_m}$$

TM MODES:

$$H_y = C_m\, cos\left(\frac{m\pi}{a}x\right)e^{-j\beta_m z}$$

$$E_x = \frac{\beta_m}{\omega\varepsilon}C_m\, cos\left(\frac{m\pi}{a}x\right)e^{-j\beta_m z}$$

$$E_z = \frac{jm\pi}{\omega\varepsilon a}C_m\, sin\left(\frac{m\pi}{a}x\right)e^{-j\beta_m z}$$

$$Z_m^{\mathrm{TM}} = \frac{\beta_m}{\omega\varepsilon} = \frac{\beta_m\eta}{k}$$

Wavelength:

$$\lambda_{nm} = \frac{2\pi}{\beta_{nm}}$$

TE MODES:

$$H_z = A_{nm}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$E_x = \frac{j\omega\mu}{k^2-\beta_{nm}^2}\frac{m\pi}{b}A_{nm}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$E_y = -\frac{j\omega\mu}{k^2-\beta_{nm}^2}\frac{n\pi}{a}A_{nm}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$H_x = \frac{j\beta_{nm}}{k^2-\beta_{nm}^2}\frac{n\pi}{a}A_{nm}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$H_y = \frac{j\beta_{nm}}{k^2-\beta_{nm}^2}\frac{m\pi}{b}A_{nm}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$Z_{nm}^{\mathrm{TE}} = -\frac{E_y}{H_x} = \frac{E_x}{H_y} = \frac{k\eta}{\beta_{nm}} = \frac{\eta}{\sqrt{1-\left(\frac{f_{c,nm}}{f}\right)^2}}$$

TM MODES:

$$E_z = B_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$H_x = \frac{j\omega\varepsilon}{k^2-\beta_{nm}^2}\frac{m\pi}{b}B_{nm}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

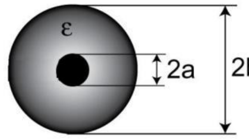
$$H_y = -\frac{j\omega\varepsilon}{k^2-\beta_{nm}^2}\frac{n\pi}{a}B_{nm}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$E_x = -\frac{j\beta_{nm}}{k^2-\beta_{nm}^2}\frac{n\pi}{a}B_{nm}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$E_y = -\frac{j\beta_{nm}}{k^2-\beta_{nm}^2}\frac{m\pi}{b}B_{nm}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-j\beta_{nm}z}$$

$$Z_{nm}^{\mathrm{TM}} = -\frac{E_y}{H_x} = \frac{E_x}{H_y} = \frac{\beta_{nm}\eta}{k} = \eta\sqrt{1-\left(\frac{f_{c,nm}}{f}\right)^2}$$

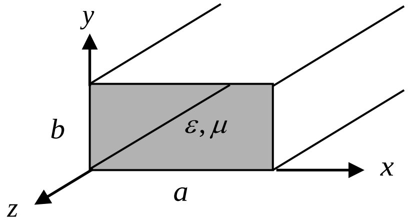
COAXIAL LINE:



$$\vec{e}(r,\phi) = \hat{r}\frac{1}{\ln(b/a)}\frac{1}{r}, \qquad \vec{h}(r,\phi) = \hat{\phi}\frac{1}{2\pi r}$$

$$Z_C = \eta\frac{\ln(b/a)}{2\pi}$$

RECTANGULAR WAVEGUIDE:



Wave vector:

$$\beta_{nm} = \sqrt{k^2 - k^2{}_{c,nm}}, \qquad k = \omega\sqrt{\mu\varepsilon}$$

$$k_{c,nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Cut-off frequency:

$$f_{c,nm} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

Phase velocity:

$$v_p = \frac{\omega}{\beta_{nm}} = \frac{\frac{1}{\sqrt{\mu\varepsilon}}}{\sqrt{1-\left(\frac{f_{c,nm}}{f}\right)^2}} \underset{\substack{\text{propagating}\\ f>f_{c,nm}}}{\geq} \frac{1}{\sqrt{\mu\varepsilon}}$$

Group velocity:

$$v_g = \frac{\beta_{nm}}{k}\frac{1}{\sqrt{\mu\varepsilon}} \leq \frac{1}{\sqrt{\mu\varepsilon}}$$

$\underline{E} \simeq \underline{\hat{\theta}} j\omega\mu g(r) I_0 d \sin\theta \, e^{jk\underline{\hat{r}}\cdot\underline{r}'}, \underline{H} \simeq \underline{\hat{\phi}} \eta^{-1} E_\theta$

Radiation Pattern:

$$F(\theta,\phi) = \frac{|E(\theta,\phi)|}{|E|_{max}}$$

POINTING VECTOR:

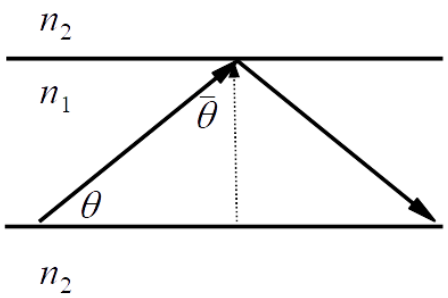
$$\underline{\boldsymbol{S}} = \frac{\mathbf{1}}{\mathbf{2}} \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^* = \underline{\hat{\boldsymbol{r}}} \frac{|\boldsymbol{E}|^2}{\mathbf{2}\boldsymbol{\eta}}$$

DISPERSION:

$$v_p = \frac{\omega}{\beta(\omega)}$$

$$v_g = \frac{1}{\partial\beta(\omega)/\partial\omega}$$

DIELECTRIC WAVEGUIDE:



$$\bar{\theta}_c = \cos^{-1}\left(\frac{n_2}{n_1}\right)$$

Solutions for TE modes:

$$\tan\left(\frac{\pi d}{\lambda} n_1 \sin\theta - m\frac{\pi}{2}\right) = \sqrt{\frac{\sin^2\bar{\theta}_c}{\sin^2\theta} - 1}$$

RADIATION:

$$\underline{J} = \underline{\hat{z}} I_0 d \delta(\underline{r} - \underline{r}') e^{j\omega t}$$

The dipole location:

$\underline{r}' = x'\underline{\hat{x}} + y'\underline{\hat{y}} + z'\underline{\hat{z}}$

The observation point:

$$\underline{r} = x\underline{\hat{x}} + y\underline{\hat{y}} + z\underline{\hat{z}}$$

$$= r(\sin\theta\cos\phi\,\underline{\hat{x}} + \sin\theta\sin\phi\,\underline{\hat{y}} + \cos\theta\,\underline{\hat{z}})$$

The magnetic vector potential of the dipole:

$$\underline{A} = \underline{\hat{z}} \mu I_0 d \frac{e^{-jkR}}{4\pi R}, R = |\underline{r} - \underline{r}'|$$

Far field approximation:

$$R = r - \underline{\hat{r}} \cdot \underline{r}'$$

$$\underline{A} = \underline{\hat{z}} \mu g(r) I_0 d e^{jk\underline{\hat{r}}\cdot\underline{r}'}, g(r) = \frac{e^{-jkr}}{4\pi r}$$