Radio Frequency Circuits & Antenna

Thomas Glezer
Tel Aviv University

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Homework: 3

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Question 1

A horizontal infinitesimal eletric dipole of constant current I_0 and length l is placed symmetrically about the origin and directed along the x-axis. Determine:

a) Eletromagnetic field in all space:

$$\vec{\boldsymbol{A}} = \mu I_0 l \frac{e^{-jkr}}{4\pi r}\hat{x}$$

$$\hat{x} = \sin\theta\cos\varphi \cdot \hat{r} + \cos\theta\cos\varphi \cdot \hat{\theta} - \sin\varphi \cdot \hat{\varphi}$$

$$\therefore \vec{\boldsymbol{A}} = \mu I_0 l \frac{e^{-jkr}}{4\pi r}[\sin\theta\cos\varphi \cdot \hat{r} + \cos\theta\cos\varphi \cdot \hat{\theta} - \sin\varphi \cdot \hat{\varphi}]$$

$$\begin{aligned}
\vec{\boldsymbol{H}} &= \frac{1}{\mu}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} \\
&= \frac{\mu I_0 l}{4\pi} \cdot \Big[\\
&\frac{1}{r\sin\theta}\Big(\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\Big) \cdot \hat{r} \\
&+ \frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\Big) \cdot \hat{\theta} \\
&+ \frac{1}{r}\Big(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\Big) \cdot \hat{\varphi}\Big]
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{H}}_r &= \frac{I_0 l e^{-jkr}}{4\pi r^2 \sin\theta}\Big(\frac{\partial(-\sin\varphi\sin\theta)}{\partial\theta} - \frac{\partial(\cos\theta\cos\varphi)}{\partial\varphi}\Big) \\
&= \cdots \Big(-\sin\varphi\cos\theta - \cos\theta \cdot (-\sin\varphi)\Big) \\
&= \cdots \Big(\cancelto{0}{-\sin\varphi\cos\theta + \cos\theta\sin\varphi}\Big)
\end{aligned}$$

$$\therefore \vec{\boldsymbol{H}}_r = 0, \quad \text{As expceted, as derived in class:} \quad \vec{\boldsymbol{H}} = \frac{1}{\mu}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}_T$$

$$\vec{\boldsymbol{H}}_\theta = \frac{I_0 l}{4\pi}\frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\theta\cos\varphi\cdot\frac{e^{-jkr}}{r}\right) - \frac{\partial}{\partial r}\left(r\cdot -\sin\varphi\frac{e^{-jkr}}{r}\right)\Big)$$

$$= \cdots\Big(\frac{e^{-jkr}}{r}\frac{\partial}{\partial\varphi}(\cos\varphi) + \sin\varphi\frac{\partial}{\partial r}\big(e^{-jkr}\big)\Big)$$

$$= \cdots\left(-\frac{e^{-jkr}}{r}\sin\varphi - jk\sin\varphi\cdot e^{-jkr}\right)$$

$$\vec{\boldsymbol{H}}_\theta = -\frac{I_0 l\sin\varphi\cdot e^{-jkr}}{4\pi}\Big(\frac{1}{r^2} + \frac{jk}{r}\Big)$$

$$\vec{\boldsymbol{H}}_\varphi = \frac{I_0 l}{4\pi}\frac{1}{r}\Big(\frac{\partial}{\partial r}\left(r\cos\theta\cos\varphi\frac{e^{-jkr}}{r}\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-jkr}}{r}\sin\theta\cos\varphi\right)\Big)$$

$$= \cdots\Big(-jk\cos\theta\cos\varphi\cdot e^{-jkr} - \frac{e^{-jkr}}{r}\cos\theta\cos\varphi)\Big)$$

$$\vec{\boldsymbol{H}}_\varphi = -\frac{I_0 l\cos\theta\cos\varphi}{4\pi}\cdot e^{-jkr}\Big(\frac{jk}{r} + \frac{1}{r^2}\Big)$$

Now to find the eletric field:

$$E = \frac{1}{j\omega\varepsilon}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{H}}$$

$$E = \frac{1}{j\omega\varepsilon}\vec{\boldsymbol{\nabla}}\times -I_0 l\frac{e^{-jkr}}{4\pi}\Big(\frac{jk}{r} + \frac{1}{r^2}\Big)\Big(\sin\varphi\cdot\hat{\theta} + \cos\theta\cos\varphi\cdot\hat{\varphi}\Big)$$

$$E = \frac{I_0 lj}{4\pi\omega\varepsilon}\vec{\boldsymbol{\nabla}}\times e^{-jkr}\Big(\frac{jk}{r} + \frac{1}{r^2}\Big)\Big(\sin\varphi\cdot\hat{\theta} + \cos\theta\cos\varphi\cdot\hat{\varphi}\Big)$$

$$\vec{\boldsymbol{E}}_r = \frac{I_0 lj}{4\pi\omega\varepsilon\sin\theta}\cdot e^{-jkr}\Big(\frac{jk}{r^2} + \frac{1}{r^3}\Big)\Big(\frac{\partial}{\partial\theta}(H_\varphi\sin\theta) - \frac{\partial H_\theta}{\partial\varphi}\Big)\cdot\hat{r}$$

$$= \cdots\Big(\frac{\partial}{\partial\theta}(\cos\varphi\cos\theta\sin\theta) - \frac{\partial\sin\varphi}{\partial\varphi}\Big)\cdot\hat{r}$$

$$= \cdots\cos\varphi\Big(\cos^2(\theta) - \sin^2(\theta) - 1\Big)\cdot\hat{r}$$

$$= \frac{I_0 lj}{4\pi\omega\varepsilon\sin\theta}\cdot e^{-jkr}\Big(\frac{jk}{r^2} + \frac{1}{r^3}\Big)\cos\varphi(-2\sin^2(\theta))\hat{r}$$

$$\vec{\boldsymbol{E}}_r = -\frac{2I_0 lj}{4\pi\omega\varepsilon}\cdot e^{-jkr}\Big(\frac{jk}{r^2} + \frac{1}{r^3}\Big)\cos\varphi\sin\theta\cdot\hat{r}$$

$$
\begin{aligned}
\vec{\boldsymbol{E}}_\theta &= \frac{I_0 l j}{4\pi\omega\varepsilon}\frac{1}{r}\left(\cancelto{0}{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi}} - \frac{\partial}{\partial r}(rH_\varphi)\right)\hat{\theta} \\
&= \cdots \frac{\partial}{\partial r}\left(e^{-jkr}\left(jk + \frac{1}{r}\right)\right) \\
&= \cdots (e^{-jkr}(k^2 - \frac{jk}{r} - \frac{1}{r^2}))
\end{aligned}
$$

$$
\vec{\boldsymbol{E}}_\theta = -\frac{I_0 l j}{4\pi\omega\varepsilon}\frac{1}{r}(e^{-jkr}(k^2 - \frac{jk}{r} - \frac{1}{r^2}))\cdot\cos\theta\cos\varphi\cdot\hat{\theta}
$$

$$
\begin{aligned}
\vec{\boldsymbol{E}}_\varphi &= \frac{I_0 l j}{4\pi\omega\varepsilon}\frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\theta) - \cancelto{0}{\frac{\partial H_r}{\partial\theta}}\right)\hat{\varphi} \\
&= -\frac{I_0 l j}{4\pi\omega\varepsilon}\frac{1}{r}(e^{-jkr}(k^2 - \frac{jk}{r} - \frac{1}{r^2}))\cdot\sin\varphi\cdot\hat{\varphi}
\end{aligned}
$$

b) Far-zone fields radiated by the dipole:

$$
\vec{\boldsymbol{H}}_r = 0
$$

$$
\vec{\boldsymbol{H}}_\theta = -\frac{I_0 l j k\cdot\sin\varphi\cdot e^{-jkr}}{4\pi r}
$$

$$
\vec{\boldsymbol{H}}_\varphi = -\frac{I_0 l j k\cdot\cos\theta\cos\varphi}{4\pi r}\cdot e^{-jkr}
$$

$$
\vec{\boldsymbol{E}}_r = \frac{2I_0 l k}{\omega\varepsilon}\cdot\frac{e^{-jkr}}{4\pi r^2}\cdot\cos\varphi\sin\theta\cdot\hat{r}
$$

$$
\vec{\boldsymbol{E}}_\theta = -\frac{k^2 I_0 l j}{\omega\varepsilon}\frac{e^{-jkr}}{4\pi r}\cdot\cos\theta\cos\varphi\cdot\hat{\theta}
$$

$$
\vec{\boldsymbol{E}}_\varphi = -\frac{k^2 I_0 l j}{\omega\varepsilon}\frac{e^{-jkr}}{4\pi r}\cdot\sin\varphi\cdot\hat{\varphi}
$$

c) Directivity of the antenna:

Ignoring $\vec{\boldsymbol{E}}_r$ as it is $\propto 1/r^2$

$$\begin{aligned} U(\theta,\varphi) &\propto |\vec{\boldsymbol{E}}|^2 \\ &\propto \cos^2(\theta)\cos^2(\varphi) + \sin^2(\varphi) \end{aligned}$$

$$\begin{aligned} D_{max} &= \frac{4\pi U_{max}}{U(\theta,\varphi)_{avg}} \\ &= \frac{4\pi}{\int\limits_{\theta=0}^{\pi}\int\limits_{\varphi=0}^{2\pi} \cos^2(\varphi)\cos^2(\theta)\cdot\sin\theta\,d\theta\,d\varphi + \int\limits_{\theta=0}^{\pi}\int\limits_{\varphi=0}^{2\pi} \sin^2(\varphi)\cdot\sin\theta\,d\theta\,d\varphi} \\ &= \frac{4\pi}{\int\limits_{\theta=0}^{\pi} \cos^2(\theta)\cdot\sin\theta\,d\theta \int\limits_{\varphi=0}^{2\pi} \cos^2(\varphi)\,d\varphi + \int\limits_{\theta=0}^{\pi} \sin\theta\,d\theta \int\limits_{\varphi=0}^{2\pi} \sin^2(\varphi)\,d\varphi} \\ &= \frac{4\pi}{\frac{-\cos^3(\theta)}{3}\Big|_0^{\pi}\ \frac{\cos\varphi\sin\varphi+\varphi}{2}\Big|_0^{2\pi} + \ -\cos\theta\Big|_0^{\pi}\ \frac{2\varphi-\sin\left(\frac{2\varphi}{2}\right)}{4}\Big|_0^{2\pi}} \\ &= \frac{4\pi}{\frac{2}{3}\cdot\pi + 2\cdot\pi} \\ &= \frac{4\pi}{\frac{2\pi+6\pi}{3}} \\ &= \frac{4\pi}{8\pi}3 \\ &= 3/2 = 1.5 \end{aligned}$$

Question 2

A horizontal infinitesimal eletric dipole of constant current I_0 and length l is placed symmetrically about the origin and directed along the y-axis. Determine:

a) Eletromagnetic field in all space:

$$\vec{\boldsymbol{A}} = \mu I_0 l \frac{e^{-jkr}}{4\pi r}\hat{y}$$

$$\hat{y} = \sin\theta\sin\varphi\cdot\hat{r} + \cos\theta\sin\varphi\cdot\hat{\theta} + \cos\varphi\cdot\hat{\varphi}$$

$$\therefore \vec{\boldsymbol{A}} = \mu I_0 l \frac{e^{-jkr}}{4\pi r}[\sin\theta\sin\varphi\cdot\hat{r} + \cos\theta\sin\varphi\cdot\hat{\theta} + \cos\varphi\cdot\hat{\varphi}]$$

$$
\begin{aligned}
\vec{\boldsymbol{H}} &= \frac{1}{\mu}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A_T}} \\
&= \frac{\mu I_0 l}{4\pi} \cdot \Big[\\
&\frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\Big) \cdot \hat{\theta} \\
&+ \frac{1}{r}\Big(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\Big) \cdot \hat{\varphi}\Big]
\end{aligned}
$$

$$
\vec{\boldsymbol{H}}_r = 0
$$

$$
\begin{aligned}
\vec{\boldsymbol{H}}_\theta &= \frac{\mu I_0 l}{4\pi}\frac{1}{r}\Big(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\Big) \\
&= \cdots \Big(\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\left(\frac{e^{-jkr}}{r}\sin\theta\sin\varphi\right) - \frac{\partial}{\partial r}\left(r\frac{e^{-jkr}}{r}\cos\varphi\right)\Big) \\
&= \cdots \Big(\frac{e^{-jkr}}{r}\cos\varphi + jke^{-jkr}\cos\varphi)\Big) \\
\vec{\boldsymbol{H}}_\theta &= \frac{\mu I_0 l}{4\pi} \cdot e^{-jkr} \cdot \left(\frac{1}{r^2} + \frac{jk}{r}\right) \cdot \cos\varphi \cdot \hat{\theta}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\boldsymbol{H}}_\varphi &= \frac{\mu I_0 l}{4\pi}\frac{1}{r}\Big(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\Big) \\
&= \cdots \Big(\frac{\partial}{\partial r}\left(r\frac{e^{-jkr}}{r}\cos\theta\sin\varphi\right) - \frac{\partial}{\partial \theta}\left(\frac{e^{-jkr}}{r}\sin\theta\sin\varphi\right)\Big) \\
&= \cdots - \left(jke^{-jkr}\cos\theta\sin\varphi + \frac{e^{-jkr}}{r}\cos\theta\sin\varphi\right) \\
\vec{\boldsymbol{H}}_\varphi &= -\frac{\mu I_0 l}{4\pi} \cdot e^{-jkr}(\frac{jk}{r} + \frac{1}{r^2}) \cdot \cos\theta\sin\varphi \cdot \hat{\varphi}
\end{aligned}
$$

$$
\vec{\boldsymbol{H}} = \frac{\mu I_0 l}{4\pi} \cdot e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2}\right)[\cos\varphi \cdot \hat{\theta} - \cos\theta\sin\varphi \cdot \hat{\varphi}]
$$

$$
E = \frac{1}{j\omega\varepsilon}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}}
$$

$$
E = \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}\vec{\boldsymbol{\nabla}} \times e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2}\right)[\cos\varphi \cdot \hat{\theta} - \cos\theta\sin\varphi \cdot \hat{\varphi}]
$$

$$
\begin{aligned}
\vec{\boldsymbol{E}}_r &= \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}e^{-jkr}\Big(\frac{jk}{r^2}+\frac{1}{r^3}\Big)\frac{1}{\sin\theta}\Big(\frac{\partial}{\partial\theta}(H_\varphi\sin\theta)-\frac{\partial H_\theta}{\partial\varphi}\Big)\cdot\hat{r}\\
&= \cdots\Big(\frac{\partial}{\partial\theta}(-\cos\theta\sin\varphi\sin\theta)-\frac{\partial\cos\varphi}{\partial\varphi}\Big)\cdot\hat{r}\\
&= \cdots\Big(\sin\varphi(\sin^2(\theta)-\cos^2(\theta))+\sin\varphi\Big)\cdot\hat{r}\\
&= \cdots\sin\varphi\Big((\sin^2(\theta)-\cos^2(\theta))+1\Big)\cdot\hat{r}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\boldsymbol{E}}_r &= \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}e^{-jkr}\Big(\frac{jk}{r^2}+\frac{1}{r^3}\Big)\frac{1}{\sin\theta}\sin\varphi\cdot\sin^2(\theta)\\
&= \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}e^{-jkr}\Big(\frac{jk}{r^2}+\frac{1}{r^3}\Big)\sin\varphi\cdot\sin(\theta)\cdot\hat{r}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\boldsymbol{E}}_\theta &= \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}\frac{1}{r}(\cancelto{0}{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi}}-\frac{\partial}{\partial r}(rH_\varphi))\hat{\theta}\\
&= \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}\bigg(re^{-jkr}\Big(\frac{jk}{r}+\frac{1}{r^2}\Big)\bigg)\cos\theta\sin\varphi\\
&= \frac{1}{j\omega\varepsilon}\frac{\mu I_0 l}{4\pi}\frac{1}{r}e^{-jkr}\Big(-j^2k^2-\frac{jk}{r}-\frac{1}{r^2}\Big))\cos\theta\sin\varphi
\end{aligned}
$$

$$
\vec{\boldsymbol{E}}_\theta = \frac{\mu I_0 l}{j\omega\varepsilon}\frac{e^{-jkr}}{4\pi r}\Big(k^2-\frac{jk}{r}-\frac{1}{r^2}\Big))\cos\theta\sin\varphi\cdot\hat{\theta}
$$

We only change the sinosoidal relations and phase, so we can easily derive the next term for the eletric field:

$$
\begin{aligned}
\vec{\boldsymbol{E}}_\varphi &= \frac{1}{r}(\frac{\partial}{\partial r}(rH_\theta)-\cancelto{0}{\frac{\partial H_r}{\partial\theta}})\hat{\varphi}\\
&= -\frac{\mu I_0 l}{j\omega\varepsilon}\frac{e^{-jkr}}{4\pi r}\Big(k^2-\frac{jk}{r}-\frac{1}{r^2}\Big))\cos\varphi\cdot\hat{\varphi}
\end{aligned}
$$

b) Far-zone fields radiated by the dipole:

$$
\vec{\boldsymbol{H}}_r = 0
$$

$$
\vec{\boldsymbol{H}}_\theta = jk\mu I_0 l\cdot\frac{e^{-jkr}}{4\pi r}\cdot\cos\varphi\cdot\hat{\theta}
$$

$$\vec{\boldsymbol{H}}_\varphi = -jk\mu I_0 l \frac{e^{-jkr}}{4\pi r} \cdot \cos\theta \sin\varphi \cdot \hat{\varphi}$$

$$\vec{\boldsymbol{E}}_r = \frac{jk\mu I_0 l}{j\omega\varepsilon} \frac{e^{-jkr}}{4\pi r^2} \sin\varphi \cdot \sin\theta \cdot \hat{r}$$

$$\vec{\boldsymbol{E}}_\theta = \frac{k^2 \mu I_0 l}{j\omega\varepsilon} \frac{e^{-jkr}}{4\pi r} \cos\theta \sin\varphi \cdot \hat{\theta}$$

$$\vec{\boldsymbol{E}}_\varphi = -\frac{k^2 \mu I_0 l}{j\omega\varepsilon} \frac{e^{-jkr}}{4\pi r} \cos\varphi \cdot \hat{\varphi}$$

c) Directivity of the antenna:

We get a Hertzian dipole as in the first problem, and as the radiation directivity shouldn't reduce based on a preference of direction. Thus, we must get the same value as before.

$$D_{max} = 1.5$$

Question 3

A horizontal infinitesimal eletric dipole of constant current I_m and length l is placed symmetrically about the origin and directed along the z-axis. Determine:

a) Eletromagnetic field in all space:

$$\vec{\boldsymbol{A}} = \mu I_m l \frac{e^{-jkr}}{4\pi r} \hat{z}$$

$$\hat{z} = \cos\theta \cdot \hat{r} - \sin\theta \cdot \hat{\theta}$$

$$\therefore \vec{\boldsymbol{A}} = \mu I_m l \frac{e^{-jkr}}{4\pi r} [\cos\theta \cdot \hat{r} - \sin\theta \cdot \hat{\theta}]$$

$$\vec{\boldsymbol{H}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}_{\boldsymbol{T}}$$

$$\vec{\boldsymbol{H}}_r = 0$$

$$\vec{\boldsymbol{H}}_\theta = \frac{\mu I_m l}{4\pi}\frac{1}{r}\left(\cancelto{0}{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi}} - \cancelto{0}{\frac{\partial r H_\varphi}{\partial r}}\right) = 0$$

$$\begin{aligned}
\vec{\boldsymbol{H}}_\varphi &= \frac{\mu I_m l}{4\pi}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial H_r}{\partial\theta}\right) \\
&= \cdots\left(-\frac{\partial}{\partial r}\left(r\frac{e^{-jkr}}{r}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-jkr}}{r}\cos\theta\right)\right) \\
&= \cdots\left(jk(e^{-jkr}\sin\theta) + \frac{e^{-jkr}}{r}\sin\theta\right) \\
\vec{\boldsymbol{H}}_\varphi &= \mu I_m l \cdot \frac{e^{-jkr}}{4\pi r}(jk + \frac{1}{r}) \cdot \sin\theta \cdot \hat{\varphi}
\end{aligned}$$

$$\vec{\boldsymbol{H}} = \mu I_m l \cdot \frac{e^{-jkr}}{4\pi r}(jk + \frac{1}{r}) \cdot \sin\theta \cdot \hat{\varphi}$$

$$E = \frac{1}{j\omega\varepsilon}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}}$$

$$E = \frac{1}{j\omega\varepsilon}\mu I_m l \cdot \vec{\boldsymbol{\nabla}} \times \left(\frac{e^{-jkr}}{4\pi r}(jk + \frac{1}{r}) \cdot \sin\theta\right) \cdot \hat{\varphi}$$

$$\begin{aligned}
\vec{\boldsymbol{E}}_r &= \left(\frac{\partial}{\partial\theta}(H_\varphi \sin\theta) - \cancelto{0}{\frac{\partial H_\theta}{\partial\varphi}}\right) \cdot \hat{r} \\
&= \frac{\mu I_m l}{j\omega\varepsilon} \cdot \frac{\partial}{\partial\theta}\left(\frac{e^{-jkr}}{4\pi r}(jk + \frac{1}{r}) \cdot \sin\theta\sin\theta\right) \\
&= \frac{\mu I_m l}{j\omega\varepsilon} \cdot \frac{e^{-jkr}}{4\pi r}(jk + \frac{1}{r}) \cdot 2\cos\theta\sin\theta \cdot \hat{r}
\end{aligned}$$

$$\vec{\boldsymbol{E}}_\theta = \frac{\mu I_m l}{j\omega\varepsilon}\cdot\frac{1}{r}\cdot\Big(\cancelto{0}{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi}} - \frac{\partial}{\partial r}(rH_\varphi)\Big)\hat{\theta}$$

$$= \cdots - \frac{\partial}{\partial r}\left(r\frac{e^{-jkr}}{4\pi r}(jk+\frac{1}{r})\cdot\sin\theta\right)\hat{\theta}$$

$$= \cdots - \Big(\frac{e^{-jkr}}{4\pi}(k^2 - \frac{jk}{r} - \frac{1}{r^2})\Big)\cdot\sin\theta$$

$$= -\frac{\mu I_m l}{j\omega\varepsilon}\cdot\frac{e^{-jkr}}{4\pi r}(k^2 - \frac{jk}{r} - \frac{1}{r^2})\cdot\sin\theta\cdot\hat{\theta}$$

$$\vec{\boldsymbol{E}}_\varphi = \Big(\cancelto{0}{\frac{\partial}{\partial r}(rH_\theta)} - \cancelto{0}{\frac{\partial H_r}{\partial\theta}}\Big)\hat{\varphi} = 0$$

b) Far-zone fields radiated by the dipole:

$$\vec{\boldsymbol{H}}_r = \vec{\boldsymbol{H}}_\theta = 0$$

$$\vec{\boldsymbol{H}}_\varphi = jk\mu I_m l\cdot\frac{e^{-jkr}}{4\pi r}\cdot\sin\theta\cdot\hat{\varphi}$$

$$\vec{\boldsymbol{E}}_r = \frac{k\mu I_m l}{\omega\varepsilon}\cdot\frac{e^{-jkr}}{4\pi r}\cdot 2\cos\theta\sin\theta\cdot\hat{r}$$

$$\vec{\boldsymbol{E}}_\theta = \frac{k^2 j\mu I_m l}{\omega\varepsilon}\cdot\frac{e^{-jkr}}{4\pi r}\cdot\sin\theta\cdot\hat{\theta}$$

$$\vec{\boldsymbol{E}}_\varphi = 0$$

c) Directivity of the antenna:

We get a Hertzian dipole as in the first and second problem, and as the radiation directivity shouldn't reduce based on a preference of direction. Thus, we must get the same value as before.

$$D_{max} = 1.5$$

Question 4

A circularly polarized wave, travelling in the $+\hat{z}$, is recieved by an elliptically polarized antenna, whose reception characteristics are given by:

$$E_a = (2\hat{x} - j\hat{y})f(r, \theta, \varphi)$$

Find the PLF (dimensionless and in dB) when the incident wave is

$$|2\hat{x} - j\hat{y}|^2 = (2-j)(2+j) = 2^2 - 2j + 2j - j^2 = 4 + 1 = 5$$

$$\hat{\rho}_a = \frac{2\hat{x} - j\hat{y}}{\sqrt{5}}$$

a) $E_{inc} = (\hat{x} - j\hat{y})e^{-jkz}$:

$$\begin{aligned} |\hat{x} - j\hat{y}|^2 &= (1-j)(1+j) = 2 \\ &= \hat{\rho}_w = \frac{\hat{x} - j\hat{y}}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} PLF &= |\hat{\rho}_a \cdot \hat{\rho}_w|^2 \\ &= \left| \frac{2\hat{x} - j\hat{y}}{\sqrt{5}} \cdot \frac{\hat{x} - j\hat{y}}{\sqrt{2}} \right|^2 \\ &= \left| \frac{(2 - \cancelto{0}{2j\hat{x}\hat{y}} - \cancelto{0}{j\hat{x}\hat{y}} + j^2)}{\sqrt{10}} \right|^2 \\ &= \left| \frac{(2-1)}{\sqrt{10}} \right|^2 \\ &= \frac{1}{10} \end{aligned}$$

$$PLF_{dB} = 10\log(1/10) = -10[dB]$$

b) $E_{inc} = (\hat{x} + j\hat{y})e^{-jkz}$:

$$\begin{aligned} |\hat{x} + j\hat{y}|^2 &= (1+j)(1-j) = 2 \\ &= \hat{\rho}_w = \frac{\hat{x} + j\hat{y}}{\sqrt{2}} \end{aligned}$$

$$
\begin{aligned}
PLF &= |\hat{\rho}_a \cdot \hat{\rho}_w|^2 \\
&= \Big|\frac{2\hat{x} - j\hat{y}}{\sqrt{5}} \cdot \frac{\hat{x} + j\hat{y}}{\sqrt{2}}\Big|^2 \\
&= \Big|\frac{(2 + \cancelto{0}{2j\hat{x}\hat{y}} - \cancelto{0}{j\hat{x}\hat{y}} - j^2)}{\sqrt{10}}\Big|^2 \\
&= \Big|\frac{(2+1)}{\sqrt{10}}\Big|^2 \\
&= \frac{9}{10}
\end{aligned}
$$

$$PLF_{dB} = 10\log(9/10) = -0.457[dB]$$

Question 5

A lineraly polarized wave travelling in the $+\hat{z}$-direction has a tilt angle of $45°$. It is incident upon antenna whose polarization characteristics are given by:

$$\rho_a = \frac{4x + jy}{\sqrt{17}}$$

Find the polarization loss factor, PLS (dimensionless and in dB).
Wave parameters:

$$\hat{\rho}_w = \frac{\hat{x} + \hat{y}}{\sqrt{2}}$$

$$
\begin{aligned}
PLF &= |\hat{\rho}_a \cdot \hat{\rho}_w|^2 \\
&= \left| \frac{4\hat{x} - j\hat{y}}{\sqrt{17}} \cdot \frac{\hat{x} + \hat{y}}{\sqrt{2}} \right|^2 \\
&= \left| \frac{(4 + \cancelto{0}{4\hat{x}\hat{y}} - \cancelto{0}{\hat{x}\hat{y}} - j)}{\sqrt{34}} \right|^2 \\
&= \left| \frac{(4 - j)}{\sqrt{34}} \right|^2 \\
&= \frac{(4 - j)(4 + j)}{34} \\
&= \frac{16 + 4j - 4j - j^2}{34} \\
&= \frac{17}{34} \\
&= 0.5
\end{aligned}
$$

$$
PLF_{dB} = 10\log(0.5) = -3[dB]
$$