Radio Frequency Circuits & Antenna

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Homework: 1

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Quesiton 1

Show that:

$$\left[\vec{\mathbf{\nabla}} \times \vec{\mathbf{F}}\right] \cdot \hat{y} = \frac{\partial (F_x[x,y])}{\partial z} - \frac{\partial (F_z[x,y])}{\partial x}$$

Firstly we can recall that the curl operator is given, in the cartesian form, as:

$$\vec{\mathbf{\nabla}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Then extending the determinant, we get the following expression:

$$\vec{\mathbf{\nabla}} \times \vec{\mathbf{F}} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \cdot \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \cdot \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \cdot \hat{z}$$

Now if we multiply the direction vector of y on both sides of the equation we would get:

$$\left[\vec{\mathbf{\nabla}} \times \vec{\mathbf{F}}\right] \cdot \hat{y} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \cdot \hat{x} \cdot \hat{y} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \cdot \hat{y} \cdot \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \cdot \hat{z} \cdot \hat{y}$$

But we recall that x, y, z are orthogonal to one another, and a direction vetor multiplied by itself yields 1:

$$\left[\vec{\mathbf{\nabla}} \times \vec{\mathbf{F}}\right] \cdot \hat{y} = \cancelto{0}{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \cdot \hat{x} \cdot \hat{y}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \cancelto{1}{\cdot \hat{y} \cdot \hat{y}} + \cancelto{0}{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \cdot \hat{z} \cdot \hat{y}}$$

Finally obtaning:

$$\left[\vec{\mathbf{\nabla}} \times \vec{\mathbf{F}}\right] \cdot \hat{y} = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$$

Quesiton 2

Show that:

$$\vec{\boldsymbol{\nabla}} \times \left[\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}}\right] = \vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}}\right) - (\vec{\boldsymbol{\nabla}} \cdot \vec{\nabla}) \cdot \vec{\mathbf{F}}$$

We can show it by a general vector cross product operation:

$$(u \times (v \times w))_x = u_y(v_x w_y - v_y w_x) - u_z(v_z w_x - v_z w_x)$$

$$= v_x(u_y w_y + u_z w_z) - w_x(u_y v_y + u_z v_z)$$

$$= v_x(u_y w_y + u_z w_z) - w_x(u_y v_y + u_z v_z) + (u_x v_x w_x - u_x v_x w_x)$$

$$v_x(u_x w_z + u_y w_y + u_z w_z) - w_x(u_x v_x + u_y v_y + u_z v_z)$$

$$(u \times (v \times w))_x = (u \cdot w)v_x - (u \cdot v)w_x$$

By symmetry on the cartesian system, we can also claim the following equalities:

$$(u \times (v \times w))_y = (u \cdot w)v_y - (u \cdot v)w_y$$

$$(u \times (v \times w))_z = (u \cdot w)v_z - (u \cdot v)w_z$$

Thus, we can combine them and obtain:

$$(u \times (v \times w)) = (u \cdot w)v - (u \cdot v)w$$

Finally we make the following substitutions:

$$u \quad \& \quad v = \vec{\nabla}, w = \vec{\mathbf{F}}$$

Thus we can't prove the claim but rather that:

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}}\right) = (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}})\vec{\nabla} - (\vec{\boldsymbol{\nabla}} \cdot \vec{\nabla})\vec{\mathbf{F}} = \vec{\nabla}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}}) - (\vec{\boldsymbol{\nabla}} \cdot \vec{\nabla})\vec{\mathbf{F}}$$

Quesiton 3

Show that if:

$$\vec{A} = A_r\hat{r} + A_\varphi\hat{\varphi} + A_\theta\hat{\theta}, \quad A_\varphi \wedge A_\theta \text{ are constant, then:}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} \cdot \vec{A} \approx \vec{k} \cdot \vec{k} \cdot \vec{A} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

The vector laplacian in spherical coordinates is given as:

$$\begin{aligned}\vec{\nabla}^2 A = &\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)\right)\hat{r} \\ &+\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)\right)\hat{\theta} \\ &+\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\varphi^2}\right)\hat{\varphi}\end{aligned}$$

Now using the fact that $A_\varphi \wedge A_\theta$ are constant, then:

$$\begin{aligned}\vec{\nabla}^2 A = &\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)\right)\hat{r} \\ &+\cancelto{0}{\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)\right)}\hat{\theta} \\ &+\cancelto{0}{\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\varphi^2}\right)}\hat{\varphi}\end{aligned}$$

We also know that $\vec{\boldsymbol{A}}$ is the solution of the Antenna equation, thus its expressed as:

$$\vec{\boldsymbol{A}} = \frac{e^{-jkr}}{4\pi r}\hat{r}$$

Then we get:

$$
\begin{aligned}
\vec{\nabla}^2 A &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{4\pi r}\right)\right)\Bigg) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\left[(-jk)\cdot\frac{e^{-jkr}}{4\pi r}+(-1)\cdot\frac{e^{-jkr}}{4\pi r^2}\right]\right)\Bigg) \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\left[(jkr)\cdot\frac{e^{-jkr}}{4\pi}+\frac{e^{-jkr}}{4\pi}\right]\right) \\
&= -\frac{1}{r^2}\left[(jk)\cdot\frac{e^{-jkr}}{4\pi}+(-jk)(jkr)\cdot\frac{e^{-jkr}}{4\pi}+(-jk)\frac{e^{-jkr}}{4\pi}\right] \\
&= (-jk)\cdot\frac{e^{-jkr}}{4\pi r^2}+(j^2k^2r)\cdot\frac{e^{-jkr}}{4\pi r^2}+(jk)\frac{e^{-jkr}}{4\pi r^2} \\
&= -k^2\cdot\frac{e^{-jkr}}{4\pi r}
\end{aligned}
$$

Thus the laplacian wasn't the intendent operation on $\vec{\boldsymbol{A}}$, let us analyze the grad of the divergance:

$$
\begin{aligned}
\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}\right) &= \vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}}\right) \\
&= \vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{A_\theta}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\right) \\
&= \vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{e^{-jkr}}{4\pi r}\right) + \frac{A_\theta}{r\sin\theta}\cos\theta\right) \\
&= \vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{e^{-jkr}}{4\pi}\right) + \frac{A_\theta}{r\tan\theta}\right) \\
&= \vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\left[\frac{e^{-jkr}}{4\pi} + (-jkr)\frac{e^{-jkr}}{4\pi}\right] + \frac{A_\theta}{r\tan\theta}\right) \\
&= \vec{\boldsymbol{\nabla}}\left(\frac{e^{-jkr}}{4\pi r^2} + (-jk)\frac{e^{-jkr}}{4\pi r} + \frac{A_\theta}{r\tan\theta}\right) \\
&= \left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta} + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}}\right)\left(\frac{e^{-jkr}}{4\pi r^2} + (-jk)\frac{e^{-jkr}}{4\pi r} + \frac{A_\theta}{r\tan\theta}\right) \\
&= \left[(-jk)\frac{e^{-jkr}}{4\pi r^2} + (-2)\frac{e^{-jkr}}{4\pi r^3}\right] \\
&+ \left[(j^2k^2)\frac{e^{-jkr}}{4\pi r} + (jk)\frac{e^{-jkr}}{4\pi r^2}\right] \\
&+ \left[(-1)\frac{A_\theta}{r^2\tan\theta} + (-1)\frac{1}{r}\frac{A_\theta\sec^2(\theta)}{r\tan\theta}\right] \\
&= -k^2\cdot\frac{e^{-jkr}}{4\pi r} - \frac{A_\theta}{r^2\tan\theta}(1+\sec^2(\theta)) - \frac{e^{-jkr}}{2\pi r^3} \\
&\approx -k^2\vec{\boldsymbol{A}}_{\boldsymbol{r}} - \frac{A_\theta}{r^2\tan\theta}(1+\sec^2(\theta)) \\
&\approx -k^2\vec{\boldsymbol{A}}_{\boldsymbol{r}} + \mathcal{O}\left(\frac{1}{r^2}\right)
\end{aligned}
$$

Quesiton 4

Show that:

$$\vec{\boldsymbol{\nabla}} \cdot \left[\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}}\right] = 0$$

Using general vectors manipulations:

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)$$

For our case $u = v = \vec{\nabla}$, thus:

$$= u_1(u_2w_3 - u_3w_2) - u_2(u_1w_3 - u_3w_1) + u_3(u_1w_2 - u_2w_1)$$

$$= u_1(u_2w_3 - u_3w_2) - u_2(u_1w_3 - u_3w_1) + u_3(u_1w_2 - u_2w_1)$$

$$= u_1u_2w_3 - u_1u_3w_2 - u_2u_1w_3 + u_2u_3w_1 + u_3u_1w_2 - u_3u_2w_1$$

$$= w_1(u_2u_3 - u_3u_2) + w_2(u_3u_1 - u_1u_3) + w_3(u_1u_2 - u_2u_1)$$

As the multiplication order is not relevant we concluded our demonstration, as the equation above yields zero.