Electromagnetism

Notes from TAU Course with Additional Information

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Contents

1 Mathematical Basics

1.1 Vector Calculus

Definition 1.1.1 (Scalar and Vector functions)**.** *We call a function:*

- $\vec{\gamma} : [a,b] \to \mathbb{R}^n$ *(a parametrization of) a curve.*
- $\phi : \mathbb{R}^n \to \mathbb{R}$ *a scalar function.*
- $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ *a vector (-valued) function or vector field.*

Definition 1.1.2 (Differentials)**.** *We define:*

- *Derivative:* $\vec{\gamma}'(\lambda) = \dfrac{d\vec{\gamma}}{d\lambda} = \lim_{\delta\to 0} \dfrac{\vec{\gamma}(\lambda+\delta) - \vec{\gamma}(\lambda)}{\delta}$
- *Partial Derivative:* $\dfrac{\partial \phi}{\partial x_i} = \lim_{\delta\to 0} \dfrac{\phi(\vec{r} + \delta\, \hat{x}_i) - \phi(\vec{r})}{\delta}$
- *Gradient:* $\vec{\nabla}\phi = \sum_{i=1}^{n} \hat{x}_i \, \dfrac{\partial \phi}{\partial x_i}$
- *Divergence:* $\vec{\nabla} \cdot \vec{F} = \sum_{i=1}^{n} \dfrac{\partial F_i}{\partial x_i}$
- *Curl:* $\vec{\nabla} \times \vec{F} = \hat{x}\left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right) + \hat{y}\left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right) + \hat{z}\left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)$
- *(Directional Derivative)* $\dfrac{\partial \phi}{\partial \vec{v}} = \lim_{\delta\to 0} \dfrac{\phi(\vec{r} + \delta \cdot \vec{v}) - \phi(\vec{r})}{\delta} = \vec{v} \cdot \vec{\nabla}\phi(\vec{r})$

Lemma 1.1.3. *The following relations hold:*

- $\vec{\nabla} \cdot (\phi \cdot \vec{F}) = (\vec{\nabla}\phi) \cdot \vec{F} + \phi \cdot (\vec{\nabla} \cdot \vec{F})$
- $\vec{\nabla} \times (\phi \cdot \vec{F}) = (\vec{\nabla}\phi) \times \vec{F} + \phi \cdot (\vec{\nabla} \times \vec{F})$
- $\vec{\nabla} \times (\vec{\nabla}\phi) \equiv \vec{0}$
- $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

Remark 1.1.4. $\dfrac{d(\phi \circ \vec{\gamma})}{d\lambda} = \vec{\nabla}\phi\Big(\vec{\gamma}(\lambda)\Big) \cdot \vec{\gamma}'(\lambda)$

Definition 1.1.5 (Line Integral)**.** *Let* Γ *be a piecewise differentiable curve. Given a vector field* $\vec{F}$*, we define the line integral (circulation) along* Γ*:*

$$\int_\Gamma \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \int_a^b \vec{F}\Big(\vec{\gamma}(\lambda)\Big) \cdot \vec{\gamma}'(\lambda)\, d\lambda$$

for any parametrization $\vec{\gamma} : [a, b] \to \Gamma \subset \mathbb{R}^3$*, where* γ *is piecewise* $\mathcal{C}^1$*. That is,* $d\vec{\mathbb{r}}$ *is tangent to the curve.*

Remark 1.1.6. *We denote using* $\vec{\mathbb{r}}$ *to make explicit the dummy variable in the integration.*

Theorem 1.1.7 (Gradient)**.** $\displaystyle\int_{A\to B} \vec{\nabla}\phi(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \phi(B) - \phi(A)$

Proof. Let Γ be a curve from A to B and $\vec{\gamma} : [a, b] \to \Gamma \subset \mathbb{R}^n$ with $\vec{\gamma}(a) = A$ and $\vec{\gamma}(b) = B$ By the chain rule: $\displaystyle\int_\Gamma \vec{\nabla}\phi(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \int_a^b \vec{\nabla}\phi\Big(\vec{\gamma}(\lambda)\Big) \cdot \vec{\gamma}'(\lambda)\, d\lambda =$ $\displaystyle\phi\big(\vec{\gamma}(\lambda)\big)\Big|_a^b = \phi(B) - \phi(A)$ □

Definition 1.1.8 (Path Independence)**.** *A vector field* $\vec{F}$ *is path-independent if* $\displaystyle\int_{A\to B} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}}$ *only depends on* A *and* B*, that is, it is the same for any path* Γ *from* A *to* B*. Equivalently, for any closed curve* Γ*:* $\displaystyle\oint_\Gamma \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = 0$

Theorem 1.1.9 (Converse of Gradient)**.** *A vector field* $\vec{F}$ *is path-independent iff* $\exists\, \phi : \mathbb{R}^n \to \mathbb{R} : \vec{F} = \vec{\nabla}\phi$

Proof. Take a fixed $\vec{r}_0$, let: $\displaystyle\phi(\vec{r}) = \int_{\vec{r}_0\to\vec{r}} \vec{F}(\vec{\mathbb{r}})\, d\vec{\mathbb{r}}$. Then, for any $v \in \mathbb{R}^n$:

$$\begin{aligned} \vec{v} \cdot \vec{\nabla}\phi(\vec{r}) = \frac{\partial\phi}{\partial\vec{v}} &= \lim_{\delta\to 0} \frac{1}{\delta} \left[\int\limits_{\vec{r}_0\to\vec{r}+\delta\cdot\vec{v}} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} - \int\limits_{\vec{r}_0\to\vec{r}} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} \right] \\ &= \lim_{\delta\to 0} \frac{1}{\delta} \int\limits_{\vec{r}\to\vec{r}+\delta\cdot\vec{v}} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \lim_{\delta\to 0} \frac{1}{\delta} \int_0^\delta \vec{F}(\vec{r} + \lambda \cdot \vec{v}) \cdot \vec{v}\, d\lambda = \vec{F}(\vec{r}) \cdot \vec{v} \end{aligned}$$

We chose the linear parametrization of $\vec{r} \to \vec{r} + \delta \cdot \vec{v}$ since $\vec{F}$ is path-independent. The last step is due to the Fundamental Theorem of Calculus. Hence, $\forall\, v \in \mathbb{R}^n$, $\vec{v} \cdot \vec{\nabla}\phi = \vec{v} \cdot \vec{F}$, so $\vec{F} = \vec{\nabla}\phi$. □

Remark 1.1.10. *The previous function ϕ is called the potential of $\vec{F}$.*

Definition 1.1.11 (Boundary)**.** *We denote $\partial\Sigma$ the boundary (curve) of the (open) surface Σ. For a volume Ω, $\partial\Omega$ is a (closed) surface.*

Theorem 1.1.12 (Green)**.** *Let Γ be a positevely oriented (counterclockwise) curve in $\mathbb{R}^2$ and Σ a bounded surface s.t. $\partial\Sigma = \Gamma$. Then, for any differentiable $\vec{F}$:*

$$\oint\limits_{\Gamma} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \iint\limits_{\Sigma} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) d^2\mathbb{r}$$

Proof. We'll prove only for domains of the form (type III):

$$\begin{aligned} \Sigma &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x) \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; f_1(y) \leq x \leq f_2(y) \text{ and } \alpha \leq y \leq \beta \right\} \end{aligned}$$

We calculate $\displaystyle\oint_{\Gamma} F_x(\vec{\mathbb{r}})\, \hat{x} \cdot d\vec{\mathbb{r}}$. We split Γ into four curves:

$$\begin{aligned} \Gamma_1 &= \{(x, g_1(x)) \mid a \leq x \leq b\} \Rightarrow \oint_{\Gamma_1} F_x(\vec{\mathbb{r}})\, \hat{x} \cdot d\vec{\mathbb{r}} = \int_a^b F_x(x, g_1(x)) \cdot dx \\ \Gamma_2 &= \{(a, y) \mid g_1(a) \leq y \leq g_2(a)\} \Rightarrow \oint_{\Gamma_2} F_x(\vec{\mathbb{r}})\, \hat{x} \cdot d\vec{\mathbb{r}} = 0 \\ \Gamma_3 &= \{(x, g_2(x)) \mid a \leq x \leq b\} \Rightarrow \oint_{\Gamma_3} F_x(\vec{\mathbb{r}})\, \hat{x} \cdot d\vec{\mathbb{r}} = -\int_a^b F_x(x, g_2(x)) \cdot dx \\ \Gamma_4 &= \{(b, y) \mid g_1(b) \leq y \leq g_2(b)\} \Rightarrow \oint_{\Gamma_4} F_x(\vec{\mathbb{r}})\, \hat{x} \cdot d\vec{\mathbb{r}} = 0 \end{aligned}$$

since the curves Γ_2 and Γ_4 are perpendicular to the x-axis. Hence:

$$\oint\limits_{\Gamma} F_x(\vec{\mathbb{r}})\, \hat{x} \cdot d\vec{\mathbb{r}} = \int\limits_a^b \Big[F_x(x, g_1(x)) - F_x(x, g_2(x)) \Big]\, dx = -\int\limits_{x=a}^{b} \int\limits_{y=g_1(x)}^{g_2(x)} \frac{\partial F_x}{\partial y}\, dy\, dx$$

A similar calculation holds for $\displaystyle\oint\limits_{\Gamma} F_y(\vec{\mathbb{r}})\, \hat{y} \cdot d\vec{\mathbb{r}} = \int\limits_{y=\alpha}^{\beta} \int\limits_{x=f_1(y)}^{f_2(y)} \frac{\partial F_y}{\partial x}\, dx\, dy$. The result follows from linearity. $\square$

Definition 1.1.13 (Flux Integral)**.** *Let Σ be a surface. Given a vector field $\vec{F}$, we define the flux/surface integral of $\vec{F}$ over Σ:*

$$\Phi[\Sigma] = \iint\limits_{\Sigma} \vec{F}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \int\limits_{\lambda=a}^{b} \int\limits_{\mu=\alpha}^{\beta} \vec{F}\Big(\vec{\sigma}(\lambda,\mu)\Big) \cdot \left[\frac{\partial\vec{\sigma}}{\partial\lambda} \times \frac{\partial\vec{\sigma}}{\partial\mu}\right] d\lambda\, d\mu$$

for any piecewise C^1 parametrization $\sigma : [a,b] \times [\alpha,\beta] \to \Sigma \subset \mathbb{R}^3$. Further, the orientation (i.e. the choice of the order of λ, μ) is important. That is, $d^2\vec{\mathbb{r}}$ is normal to the surface.

Theorem 1.1.14 (Stokes')**.** *Let Γ be a positively-oriented (counterclockwise) closed curve and Σ a surface such that $\Gamma = \partial\Sigma$. Then, for any continuously differentiable $\vec{F}$:*

$$\oint\limits_{\Gamma} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \iint\limits_{\Sigma} (\vec{\nabla} \times \vec{F}) \cdot d^2\vec{\mathbb{r}}$$

Proof. Let $\sigma : [a,b] \times [\alpha,\beta] \to \Sigma$. Take $\vec{G} = \left(\vec{F} \cdot \frac{\partial\vec{\sigma}}{\partial\lambda}, \vec{F} \cdot \frac{\partial\vec{\sigma}}{\partial\mu}\right)$. Take the curve $\Delta = \partial([a,b] \times [\alpha,\beta])$, so that $\Gamma = \vec{\sigma}(\Delta)$, we get:

$$\oint\limits_{\Gamma} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \oint\limits_{\Delta} \vec{G}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \iint\limits_{[a,b]\times[\alpha,\beta]} \left[\frac{\partial G_\mu}{\partial\lambda} - \frac{\partial G_\lambda}{\partial\mu}\right] d\lambda\, d\mu$$

by Green's Theorem. By direct calculation, we have:

$$\frac{\partial G_\mu}{\partial\lambda} - \frac{\partial G_\lambda}{\partial\mu} = (\vec{\nabla} \times \vec{F}) \cdot \left[\frac{\partial\vec{\sigma}}{\partial\lambda} \times \frac{\partial\vec{\sigma}}{\partial\mu}\right]$$

The result follows by the definition of the flux integral. □

Corollary 1.1.15. *$\vec{F}$ is path-independent iff $\vec{\nabla} \times \vec{F} \equiv \vec{0}$.*

Theorem 1.1.16 (Gauß/Ostrogradsky)**.** *Let Σ be a positively-oriented (outwards) closed surface and Ω a solid such that $\Sigma = \partial\Omega$. Then, for any continuously differentiable $\vec{F}$:*

$$\oiint\limits_{\Sigma} \vec{F}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \iiint\limits_{\Omega} (\vec{\nabla} \cdot \vec{F})\, d^3\mathbb{r}$$

Proof. Analogous to Green's Theorem. $\square$

Theorem 1.1.17 (Green's Identities)**.** *For φ, ψ twice continuously differentiable.*

1. $$\iiint_\Omega \left(\psi \cdot \nabla^2 \varphi + \vec{\nabla}\psi \cdot \vec{\nabla}\varphi\right) d^3\mathbb{r} = \oiint_{\partial\Omega} \psi \cdot \vec{\nabla}\varphi \cdot d^2\vec{\mathbb{r}} = \oiint_{\partial\Omega} \psi \cdot \frac{\partial \varphi}{\partial \hat{\mathbb{n}}}\, d^2\mathbb{r}$$
2. $$\iiint_\Omega \left(\psi \cdot \nabla^2 \varphi - \varphi \cdot \nabla^2 \psi\right) d^3\mathbb{r} = \oiint_{\partial\Omega} \left(\psi \cdot \vec{\nabla}\varphi - \varphi \cdot \vec{\nabla}\psi\right) \cdot d^2\vec{\mathbb{r}}$$
   $$= \oiint_{\partial\Omega} \left(\psi \cdot \frac{\partial \varphi}{\partial \hat{\mathbb{n}}} - \varphi \cdot \frac{\partial \psi}{\partial \hat{\mathbb{n}}}\right) d^2\mathbb{r}$$

Proof. Follows directy from 1.1.16 and 1.1.3. $\square$

Lemma 1.1.18 (Kelvin-Helmholtz)**.**

$$\frac{d}{dt}\oint_{\Gamma(t)} \vec{F}(\vec{\mathbb{r}}, t) \cdot d\vec{\mathbb{r}} = \oint_{\Gamma(t)} \left[\frac{\partial \vec{F}}{\partial t} - \dot{\vec{\mathbb{r}}}(t) \times (\vec{\nabla} \times \vec{F}(\vec{\mathbb{r}}, t))\right] \cdot d\vec{\mathbb{r}}$$

1.2 Distributions and Integration

Definition 1.2.1 (Heaviside)**.** *Let $H : \mathbb{R} \to \mathbb{R}$ s.t.* $H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

Definition 1.2.2 (Test Functions)**.** *We define $\varphi \in C_0^\infty(\mathbb{R}^n)$ if $\varphi \in C^\infty(\mathbb{R}^n)$ and $\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}$ is bounded.*

Definition 1.2.3 (Dirac Delta)**.** *We define $\delta = H'$. This is made rigourous by integration by parts:*

$$\forall \varphi \in C_0^\infty(\mathbb{R}) \,,\, \forall R \in \mathbb{R}^+ \,,\, \int_{-R}^{R} \varphi(x) \cdot \delta(x)\, dx = \varphi(0)$$

which, if made use of the definition $\delta = H'$ and appyling integration by parts, is a valid result. This concept is reffered to as a ***weak derivative****. Further, we extend: $\delta^n(\vec{r} - \vec{a}) = \prod_{i=1}^{n} \delta(x_i - a_i)$*

Lemma 1.2.4. $\forall \varphi \in C_0^\infty(\mathbb{R}) \,,\, \forall R \in \mathbb{R}^+ \,,$

$$\varphi(0) = H(x) \cdot \varphi(x)\Big|_{-R}^{R} - \int_{-R}^{R} \varphi'(x) \cdot H(x)\, dx$$

Proof. $\displaystyle\int_{-R}^{R} \varphi'(x) \cdot H(x)\, dx = \int_{0}^{R} \varphi'(x)\, dx = \varphi(x)\Big|_{0}^{R} = \varphi(R) - \varphi(0)$ $\square$

Problem 1.2.5.

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = \frac{\delta(r)}{r^2} = 4\pi\, \delta^3(\vec{r})$$

Solution: *The first relation is obtained by taking:*

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = \vec{\nabla} \cdot \left(\frac{H(r)\,\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{H(r)}{r^2}\right) = \frac{H'(r)}{r^2} = \frac{\delta(r)}{r^2}$$

The second relation is taken by switching coordinates on a sphere of radius R: $\forall \varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\iint\limits_{\Omega} \int\limits_{r=0}^{R} \varphi(\vec{r}) \cdot \frac{\delta(r)}{r^2} \cdot \overbrace{r^2\, dr\, d\Omega}^{d^3\mathbb{r}} = 4\pi \cdot \varphi(\vec{0}) = \iiint\limits_{S^1(R)} \varphi(\vec{r}) \cdot 4\pi \cdot \delta^3(\vec{r})\, d^3r$$

where Ω is the solid angle.

Definition 1.2.6 (Distributions). *The set of all bounded linear functions of $C_0^\infty(\mathbb{R}^n)$ is denoted $D(\mathbb{R}^n)$. We identify every element with an improper function f so that if T is a bounded linear function corresponding to f, we get:*

$$\forall\, \varphi \in C_0^\infty(\mathbb{R}^n)\,,\; T(\varphi) = \int_{\mathbb{R}^n} \varphi(\vec{\mathbb{r}}) \cdot f(\vec{\mathbb{r}})\, d^n\mathbb{r}$$

Remark 1.2.7. *The delta function $\delta^n(\vec{r})$ is the unique function so that*

$$\forall\, \varphi \in C_0^\infty(\mathbb{R}^n)\,,\; \int_{\mathbb{R}^n} \varphi(\vec{\mathbb{r}}) \cdot \delta^n(\vec{\mathbb{r}})\, d^n\mathbb{r} = \varphi(\vec{0})$$

Definition 1.2.8 (Indicator Function). *For a set $A \subseteq \mathbb{R}^n$, we define the function $\mathbb{1}_A : \mathbb{R}^n \to \mathbb{R}$ s.t.:*

$$\mathbb{1}_A(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \in A \\ 0 & \text{if } \vec{r} \notin A \end{cases}$$

Example 1.2.9 (Heaviside as Indicator). $H = \mathbb{1}_{(0,\infty)}$

2 Electrostatics

2.1 Electric Field

Definition 2.1.1 (Electric Force). *The force acting on a particle with charge q due to an electric field $\vec{E}$ is $\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r})$. In that case, q is called a test charge for the field $\vec{E}$.*

Lemma 2.1.2 (Superposition Principle). *If there are two distinct fields $\vec{E}_1$ and $\vec{E}_2$ for two distinct sources, the total electrical field is $\vec{E}_1 + \vec{E}_2$.*

Lemma 2.1.3 (Electric Potential). *For a static electric field (charges that induce the field are static),*

$$\vec{\nabla} \times \vec{E} \equiv \vec{0}$$

hence $\exists\,\phi : \mathbb{R}^3 \to \mathbb{R} : \vec{E} = -\vec{\nabla}\phi$. Further, $\mathcal{E}[\partial\Sigma] = \oint_{\partial\Sigma} \vec{E}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = 0$ for any closed curve $\partial\Sigma$ (cf. 1.1.15, 1.1.9).

Theorem 2.1.4 (Coulomb's Law). *The electric field due to a point charge Q at the origin is:*

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0\, r^2}\,\hat{r}$$

more generally, for a charge at $\vec{r}_0$, we get: $\vec{E}(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0\,\|\vec{r}-\vec{r}_0\|^3}\,(\vec{r}-\vec{r}_0)$

Corollary 2.1.5. *The Coulomb electric potential is: $\phi(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0\,\|\vec{r}-\vec{r}_0\|}$*

Definition 2.1.6 (Charge Distribution). *We define the following quantity: $\rho \in D(\mathbb{R}^3)$ is the distribution of charge in a system. That is:*

$$Q[\Omega] = \iiint_\Omega \rho(\vec{r})\, d^3r$$

Example 2.1.7. *We have the following charge densities:*

- *A point charge: $\rho(\vec{r}) = Q \cdot \delta^3(\vec{r}-\vec{r}_0)$*
- *A system of charges: $\rho(\vec{r}) = \sum_{i=1}^{N} Q_i \cdot \delta^3(\vec{r}-\vec{r}_i)$*
- *A uniformly charged spherical shell: $\rho(\vec{r}) = \sigma \cdot \delta(r-R) = \dfrac{Q}{4\pi R^2} \cdot \delta(r-R)$*
- *A uniformly charged sphere: $\rho(\vec{r}) = \rho_0 \cdot \mathbb{1}_{[0,R]} = \dfrac{3Q}{4\pi R^3} \cdot \mathbb{1}_{[0,R]}$*

Remark 2.1.8. *We may have surface or linear charge density, denoted σ or λ respectively, where the charge is found only on a surface or a line. Moreover, ρ would have δ functions to restric the integral to that surface or curve.*

Theorem 2.1.9 (Extended Coulomb's Law)**.** *The electric field due to a charge distribution ρ on a volume $\mathcal{V}$ is:*

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\vec{r} - \vec{\mathbb{r}}}{\|\vec{r} - \vec{\mathbb{r}}\|^3} \rho(\vec{\mathbb{r}})\, d^3\mathbb{r}$$

Proof. This follows directly for the superposition of the infinitesimal charge $dQ = \rho(\vec{\mathbb{r}})\, d^3\mathbb{r}$ in Coulomb field. □

Corollary 2.1.10. *The Coulomb electric potential due to a charge distribution ρ on a volume $\mathcal{V}$ is:*

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\vec{\mathbb{r}})}{\|\vec{r} - \vec{\mathbb{r}}\|}\, d^3\mathbb{r}$$

Usually, it is much simpler to calculate the potential an then get the electric field by $\vec{E} = -\vec{\nabla}\phi$. Further the boundary conditions here are neglected (cf. 3.2.1).

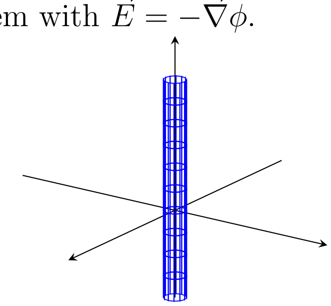
Remark 2.1.11. *If the charge is 2-dimensional or 1-dimensional, we can use λ or σ, respectively, directly into a simple or double integral.*

Remark 2.1.12. *The Coulomb potential given by the previous corollary was defined such that $\lim_{\vec{r}\to\infty} \phi(\vec{r}) = 0$*

Lemma 2.1.13. *We have: $\phi(\vec{r}) = \phi(\vec{r}_0) - \int_{\vec{r}_0 \to \vec{r}} \vec{E}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}}$*

Proof. Follows directly from the gradient theorem with $\vec{E} = -\vec{\nabla}\phi$. □

Problem 2.1.14 (Uniform Rod)**.**
Calculate the electric field and electric potential due to a rod of length $2c$ (endpoints at $(0,0,c)$ and $(0,0,-c)$) and uniform charge density λ.



Solution: *By definition, since it is symmetric around z, we integrate using cylindrical coordinates:*

$$\begin{aligned}\phi(\rho,z) &= \frac{1}{4\pi\epsilon_0}\int\limits_{z'=-c}^{c} \frac{\lambda}{\sqrt{\rho^2+(z-z')^2}}\,dz' = \frac{\lambda}{4\pi\epsilon_0}\,\mathrm{arcsinh}\left(\frac{z'-z}{\rho}\right)\Bigg|_{z'=-c}^{c} \\ &= \frac{\lambda}{4\pi\epsilon_0}\left[\mathrm{arcsinh}\left(\frac{z+c}{\rho}\right)-\mathrm{arcsinh}\left(\frac{z-c}{\rho}\right)\right] \\ &= \frac{\lambda}{4\pi\epsilon_0}\,\ln\left[\frac{z+c+\sqrt{(z+c)^2+\rho^2}}{z-c+\sqrt{(z-c)^2+\rho^2}}\right]\end{aligned}$$

Notice, we require $(\rho,z)\notin\{0\}\times[-c,c]=$ rod. For the electric field, we calculate:

$$\vec{E} = -\vec{\nabla}\phi = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{(z+c)\,\hat{\rho}-\rho\,\hat{z}}{\rho\sqrt{(z+c)^2+\rho^2}}+\frac{-(z-c)\,\hat{\rho}+\rho\,\hat{z}}{\rho\sqrt{(z-c)^2+\rho^2}}\right]$$

Observe: We could've solved it geometrically by defining $\alpha=\angle OF_1P$ and $\beta=\angle OF_2P$, where $P=(x,y,z), O=(0,0,0), F_1=(0,0,c), F_2=(0,0,-c)$. We get: $\vec{E}=\dfrac{\lambda}{4\pi\epsilon_0\,\rho}\Big[(\cos\alpha+\cos\beta)\,\hat{\rho}+(\sin\beta-\sin\alpha)\,\hat{z}\Big]$

Problem 2.1.15 (Infinite Line)**.** *Calculate the electric field and electric potential due to an infinite line of charge with uniform charge density λ.*

Solution: *We can calculate $\vec{E}$ directly:*

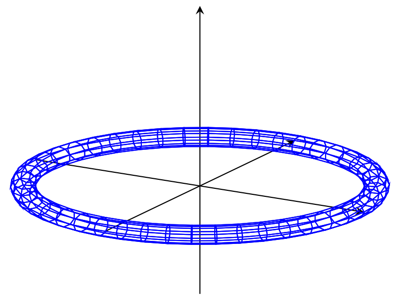
$$\begin{aligned}\vec{E}(\rho) &= \frac{1}{4\pi\epsilon_0}\int\limits_{z'=-\infty}^{\infty}\frac{\lambda\cdot\rho\,\hat{\rho}}{\left(\rho^2+(z')^2\right)^{\frac{3}{2}}}\,dz' = \frac{\lambda\cdot\rho\,\hat{\rho}}{4\pi\epsilon_0}\,\frac{z'}{\rho^2\sqrt{\rho^2+(z')^2}}\Bigg|_{z'=-\infty}^{\infty} \\ &= \frac{\lambda\,\hat{\rho}}{4\pi\epsilon_0\,\rho}\lim_{R\to\infty}\frac{\mathrm{sgn}\,z'}{\sqrt{1+\left(\frac{\rho}{z'}\right)^2}}\Bigg|_{z'=-R}^{R} = \frac{\lambda}{2\pi\epsilon_0\,\rho}\,\hat{\rho}\end{aligned}$$

For ϕ, we apply:

$$\phi(\rho)=\phi(\rho_0)-\int_{\rho_0}^{\rho}\frac{\lambda}{2\pi\epsilon_0\,\rho'}\,\hat{\rho}'\cdot\hat{\rho}'\,d\rho' = \phi(\rho_0)+\frac{\lambda}{2\pi\epsilon_0}\Big[\ln\rho_0-\ln\rho\Big]$$

Set $\rho_0=1$ and $\phi(\rho_0)=0$, we get: $\phi(\rho)=-\dfrac{\lambda}{2\pi\epsilon_0}\,\ln\rho$

Problem 2.1.16 (Uniform Ring)**.** *Calculate the electric potential due to a ring of charge (in the xy-plane) of radius R with uniform charge density λ.*



Solution: *We first calculate the potential, since it is symmetric about rotations around z, we integrate using cylindrical coordinates: We have $\|\vec{r}-\vec{\mathbb{r}}\| = \sqrt{(\rho - R\cos\varphi)^2 + (R\sin\varphi)^2 + z^2}$*

$$\phi(\rho, z) = \frac{\lambda R}{4\pi\epsilon_0} \int\limits_{\varphi=0}^{2\pi} \frac{d\varphi}{\sqrt{R^2 - 2\rho R\cos\varphi + \rho^2 + z^2}} = \{\textit{By parity}\}$$

$$= \frac{\lambda R}{2\pi\epsilon_0} \int\limits_{\varphi=0}^{\pi} \frac{d\varphi}{\sqrt{R^2 - 2\rho R\cos\varphi + \rho^2 + z^2}} = \left\{ \begin{array}{l} \theta = \dfrac{\pi - \varphi}{2} \\ -2\ d\theta = d\varphi \end{array} \right\}$$

$$= \frac{\lambda R}{\pi\epsilon_0} \int\limits_{\varphi=0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{R^2 - 2\rho R\big(2\sin^2\theta - 1\big) + \rho^2 + z^2}}$$

$$= \left\{ \begin{array}{l} \ell = \sqrt{(\rho + R)^2 + z^2} \\ k = \dfrac{\sqrt{4\rho R}}{\ell} \end{array} \right\} = \frac{\lambda R}{\pi\epsilon_0\,\ell} \int\limits_{\varphi=0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

$$\Rightarrow \phi(\rho, z) = \frac{\lambda R}{\pi\epsilon_0\sqrt{(\rho + R)^2 + z^2}} \cdot K\left(\sqrt{\frac{4\rho R}{(\rho + R)^2 + z^2}}\right)$$

where K is the complete elliptic integral of first kind. We see, it is barely possible to find a closed formula for ϕ, and even more so for $\vec{E}$. We can, however, calculate the value for $\rho = 0$, quite simply: $\phi(\rho = 0, z) = \dfrac{\lambda R}{2\epsilon_0\sqrt{R^2 + z^2}}$

2.2 Gauß's Law

Theorem 2.2.1 (Differential Form of Gauß's Law)**.** *The electric field due to a charge distribution ρ obeys:*

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

Proof. A first observation is to notice the Coulomb electric field obeys the relation: $\vec{\nabla} \cdot \vec{E}_{\text{Coulomb}}(\vec{r}) = \frac{Q}{\epsilon_0}\,\delta^3(\vec{r} - \vec{r}_0)$. For the general charge distribution, we calculate:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{\mathbb{r}}}{\|\vec{r} - \vec{\mathbb{r}}\|^3} \right) \rho(\vec{\mathbb{r}})\, d^3\mathbb{r} \\ &= \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} 4\pi\, \delta^3(\vec{r} - \vec{\mathbb{r}}) \rho(\vec{\mathbb{r}})\, d^3\mathbb{r} = \frac{1}{\epsilon_0}\rho(\vec{r})\end{aligned}$$

by definition of delta. □

Theorem 2.2.2 (Integral Form of Gauß's Law)**.** *For any solid Ω, the electric field due to a charge distribution ρ obeys:*

$$\Phi_E[\partial\Omega] = \oiint_{\partial\Omega} \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \frac{Q[\Omega]}{\epsilon_0}$$

Proof. By the differential form of Gauß's Law and Divergence Theorem:

$$\oiint_{\partial\Omega} \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \iiint_{\Omega} \vec{\nabla} \cdot \vec{E}(\vec{\mathbb{r}})\, d^3\mathbb{r} = \iiint_{\Omega} \frac{1}{\epsilon_0}\rho(\vec{\mathbb{r}})\, d^3\mathbb{r} = \frac{Q[\Omega]}{\epsilon_0}$$

□

Definition 2.2.3 (Equipotential Surface)**.** *$\mathcal{L}$ is an level curve of ϕ if*

$$\exists\, \phi_0 \in \mathbb{R} : \mathcal{L} = \{\vec{r} \in \mathbb{R}^n \mid \phi(\vec{r}) = \phi_0\}$$

Theorem 2.2.4 (Gradient Orthogonality)**.** *$\forall\, \vec{r} \in \mathcal{L}$, $\vec{E}(\vec{r})$ is normal to $\mathcal{L}$.*

Proof. Let Γ be a curve in $\mathcal{L}$ and γ a parametrization. Then, by definition,

$$\vec{\nabla}\phi\Big(\vec{\gamma}(\lambda)\Big) \cdot \vec{\gamma}'(\lambda) = \frac{d(\phi \circ \vec{\gamma})}{d\lambda} = 0$$

Therefore, $\vec{E} = -\vec{\nabla}\phi$ is perpendicular to any tangent vector, and hence the tangent plane, so, it is perpendicular to the surface. □

Problem 2.2.5 (Spherical Shell)**.** *Calculate the electric field of a spherical shell of radius R with uniform surface charge density σ.*

Solution: *By symmetry and Gauß's with Ω a (solid) sphere of radius r:*

1. $(r > R) : E(r) \cdot 4\pi\, r^2 = \dfrac{Q[\Omega]}{\epsilon_0} = \dfrac{\sigma \cdot \pi R^2}{\epsilon_0} \Rightarrow E(r) = \dfrac{Q}{4\pi\epsilon_0\, r^2} = \dfrac{\sigma}{\epsilon_0} \cdot \dfrac{R^2}{r^2}$
2. $(r < R) : E(r) \cdot 4\pi\, r^2 = 0 \Rightarrow E(r) = 0$

Hence: $\vec{E}(\vec{r}) = \begin{cases} 0 & \textit{if } \|\vec{r}\| < R \\ \dfrac{\sigma}{\epsilon_0} \cdot \dfrac{R^2}{\|\vec{r}\|^2}\,\hat{r} & \textit{otherwise} \end{cases}$

Since we picked an equipotential surface, the surface integral became a double integral. Further, due to symmetry, the electric field was constant in the equipotential surface.

Problem 2.2.6 (Solid Sphere)**.** *Calculate the electric field of a solid sphere of radius R with uniform charge density ρ.*

Solution: *By symmetry and Gauß's with Ω a (solid) sphere of radius r:*

1. $(r > R) : E(r) \cdot 4\pi\, r^2 = \dfrac{Q[\Omega]}{\epsilon_0} = \dfrac{\rho \cdot 4\pi R^3/3}{\epsilon_0} \Rightarrow E(r) = \dfrac{\rho}{3\epsilon_0} \cdot \dfrac{R^3}{r^2}$
2. $(r < R) : E(r) \cdot 4\pi\, r^2 = \dfrac{4\pi r^3}{3}\,\rho \Rightarrow E(r) = \dfrac{\rho}{3\,\epsilon_0}\, r$

Hence: $\vec{E}(\vec{r}) = \begin{cases} \dfrac{\rho}{3\,\epsilon_0}\,\vec{r} & \textit{if } \|\vec{r}\| < R \\ \dfrac{\rho}{3\epsilon_0} \cdot \dfrac{R^3}{\|\vec{r}\|^3}\,\vec{r} & \textit{otherwise} \end{cases}$

Problem 2.2.7 (Infinite Cylindrical Shell)**.** *Calculate the electric field of an infinite cylindrical shell with uniform surface charge density σ.*

Solution: *By symmetry and Gauß's with Ω a (solid) cylinder of radius ρ and height H:*

1. $(\rho > R) : E(\rho) \cdot 2\pi\, \rho \cdot H = \dfrac{Q[\Omega]}{\epsilon_0} = \dfrac{\sigma \cdot 2\pi R \cdot H}{\epsilon_0} \Rightarrow E(\rho) = \dfrac{\sigma}{\epsilon_0} \cdot \dfrac{R}{\rho}$
2. $(\rho < R) : E(\rho) \cdot 2\pi\rho \cdot H = 0 \Rightarrow E(\rho) = 0$

Hence: $\vec{E}(\vec{r}) = \begin{cases} 0 & \text{if } \rho < R \\ \dfrac{\sigma}{\epsilon_0} \cdot \dfrac{R}{\rho}\,\hat{\rho} & \text{otherwise} \end{cases}$ *where* $\rho = \|\vec{r} - (\vec{r} \cdot \hat{z})\,\hat{z}\| = \|\vec{r} \times \hat{z}\|$

Notice, by symmetry, the field is in the $\hat{\rho}$ direction. Therefore, the top and bottom circle of our cylinder $\partial\Omega$ don't contribute to the flux integral, since $\vec{E}$ is parallel to those two surfaces.

Problem 2.2.8 (Infinite Plane)**.** *Calculate the electric field of an infinite plane with uniform surface charge density σ.*

Solution: *By symmetry in the xy-plane and Gauß's with Ω a (solid) cylinder of radius R and height z centered at the plane:* $E(z) \cdot 2\pi\, R^2 = \dfrac{\sigma \cdot \pi R^2}{\epsilon_0} \Rightarrow$
$E(z) = \dfrac{\sigma}{2\epsilon_0} \Rightarrow \vec{E}(\vec{r}) = \dfrac{\sigma}{2\epsilon_0} \cdot \operatorname{sgn} z\,\hat{z}$

Notice, by symmetry, the field is in the $\hat{z}$ direction. Therefore, only the top and bottom circle of our cylinder $\partial\Omega$ contribute to the flux integral, since $\vec{E}$ is parallel to the lateral surface. Further, by changing $z \mapsto -z$, we should get $\vec{E} \mapsto -\vec{E}$, hence the $\operatorname{sgn} z$.

2.3 Dielectric Materials and Dipoles

Definition 2.3.1 (Dipole Moment)**.** *The dipole moment $\vec{p}$ due to a charge distribution ρ on a volume $\mathcal{V}$ is defined as:*

$$\vec{p} = \iiint_{\mathcal{V}} \vec{\mathbb{r}} \cdot \rho(\vec{\mathbb{r}})\, d^3\mathbb{r}$$

Theorem 2.3.2 (Multipole Expansion)**.** *Let $P_n(x) = \dfrac{1}{2^n\, n!}\dfrac{d^n}{dx^n}(x^2-1)^n$, the Legendre Polynomials, and $\cos\theta = \dfrac{\vec{r} \bullet \vec{\mathbb{r}}}{\|\vec{r}\|\, \|\vec{\mathbb{r}}\|}$. Then,*

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{\|\vec{r}\|^{n+1}}\iiint_{\mathcal{V}} \|\vec{\mathbb{r}}\|^n \cdot P_n(\cos\theta)\, \rho(\vec{\mathbb{r}})\, d^3\mathbb{r}$$

Proof. The Legendre Polynomials satisfy: $\dfrac{1}{\sqrt{1-2xt+t^2}} = \displaystyle\sum_{n=0}^{\infty} P_n(x)\, t^n$, where the right hand side converges for $x, t \in [-1, 1]$. Using Coulomb's Potential (cf. 2.1.10) and supposing $\dfrac{\|\vec{\mathbb{r}}\|}{\|\vec{r}\|} < 1$

$$\begin{aligned}
\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{V}} \frac{\rho(\vec{\mathbb{r}})}{\|\vec{r}-\vec{\mathbb{r}}\|}\, d^3\mathbb{r} = \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{V}} \frac{\rho(\vec{\mathbb{r}})}{\sqrt{\|\vec{r}\|^2 - 2\|\vec{r}\|\|\vec{\mathbb{r}}\|\,\cos\theta + \|\vec{\mathbb{r}}\|^2}}\, d^3\mathbb{r} \\
&= \frac{1}{4\pi\epsilon_0\, \|\vec{r}\|}\iiint_{\mathcal{V}} \frac{\rho(\vec{\mathbb{r}})}{\sqrt{1 - 2\cos\theta\, \frac{\|\vec{\mathbb{r}}\|}{\|\vec{r}\|} + \left(\frac{\|\vec{\mathbb{r}}\|}{\|\vec{r}\|}\right)^2}}\, d^3\mathbb{r} \\
&= \frac{1}{4\pi\epsilon_0\, \|\vec{r}\|}\iiint_{\mathcal{V}} \sum_{n=0}^{\infty}\left(\frac{\|\vec{\mathbb{r}}\|}{\|\vec{r}\|}\right)^n \cdot P_n(\cos\theta)\cdot\rho(\vec{\mathbb{r}})\, d^3\mathbb{r}
\end{aligned}$$

Since the power series converges, we may exchange the series and integral. □

Corollary 2.3.3 (Dipole Approximation)**.**

$$\phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0\|\vec{r}\|} + \frac{\vec{p}\bullet\vec{r}}{4\pi\epsilon_0\|\vec{r}\|^3} + \mathcal{O}\left(\frac{1}{\|\vec{r}\|^3}\right)$$

Proof. Follows from 2.3.2, by using $P_0(x) \equiv 1$ and $P_1(x) = x$. Also, we calculate: $\vec{p}\bullet\vec{r} = \displaystyle\iiint_{\mathcal{V}} \vec{\mathbb{r}}\bullet\vec{r}\cdot\rho(\vec{\mathbb{r}})\, d^3\mathbb{r} = \|\vec{r}\|\iiint_{\mathcal{V}} \|\vec{\mathbb{r}}\|\,\cos\theta\cdot\rho(\vec{\mathbb{r}})\, d^3\mathbb{r}$. □

Lemma 2.3.4 (Electric Field of Dipole)**.**

$$\phi_{dipole}(\vec{r}) = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 \|\vec{r}\|^2} \qquad \vec{E}_{dipole}(\vec{r}) = \frac{3(\vec{p} \cdot \hat{r})\,\hat{r} - \vec{p}}{4\pi\epsilon_0 \|\vec{r}\|^3}$$

Proof.

$$\begin{aligned}
\vec{E}_{\text{dipole}}(\vec{r}) = -\vec{\nabla}\phi_{\text{dipole}}(\vec{r}) &= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{p_x x + p_y y + p_z z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) \\
&= -\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\left(\frac{p_{x_i}}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3}{2}\cdot\frac{2x_i\,(p_x x + p_y y + p_z z)}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right)\hat{x}_i \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3(p_x x + p_y y + p_z z)\,\vec{r}}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right]
\end{aligned}$$

□

Remark 2.3.5. *In a simplified case,* $\vec{p} = q \cdot \vec{d}$*.*

Lemma 2.3.6. *The force acting on a dipole* $\vec{p}$ *due to electric field* $\vec{E}$ *is:* $\vec{F} = (\vec{p} \cdot \nabla)\vec{E}$*.*

Proof. By direct calculation:

$$\vec{F}(\vec{r}) = \lim_{d\to 0} q\left[\vec{E}(\vec{r} + d\,\hat{n}) - \vec{E}(\vec{r})\right] = p\lim_{d\to 0}\frac{\vec{E}(\vec{r} + d\,\hat{n}) - \vec{E}(\vec{r})}{d} = p\hat{n}\cdot\nabla\vec{E}(\vec{r})$$

□

Definition 2.3.7 (Free and Bound Charges)**.** *The bound charges in a material* $\mathcal{M}$ *cannot be removed e.g. by grounding it. The free charges are the remaining ones, i.e.:* $\rho = \rho_f + \rho_b$*.*

Lemma 2.3.8. *Let* $\vec{P}$ *denote the polarization density of a material* $\mathcal{M}$*:*
(i) $\rho_b = -\vec{\nabla} \cdot \vec{P}$
(ii) $\sigma_b = \vec{P}\big|_{\partial\mathcal{M}} \cdot \hat{n}$

Proof. By 2.3.4, we integrate in $\mathcal{M}$: $\phi_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{M}}\frac{(\vec{r} - \vec{\mathbb{r}})\cdot\vec{P}(\vec{\mathbb{r}})}{\|\vec{r} - \vec{\mathbb{r}}\|^3}\,d^3\vec{\mathbb{r}} = \frac{1}{4\pi\epsilon_0}\oiint_{\partial\mathcal{M}}\frac{\vec{P}(\vec{\mathbb{r}})}{\|\vec{r} - \vec{\mathbb{r}}\|}\cdot d^2\vec{\mathbb{r}} - \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{M}}\frac{\vec{\nabla}\cdot\vec{P}(\vec{\mathbb{r}})}{\|\vec{r} - \vec{\mathbb{r}}\|}\,d^3\vec{\mathbb{r}}$, it follows by definition of volume and surface charge. □

Corollary 2.3.9. $\vec{D} = \epsilon_0 \cdot \vec{E} + \vec{P}$ *obeys the Gauß Law with respect to the free charges:*

$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad \oiint_{\partial\Omega} \vec{D}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = Q_f[\Omega]$$

Definition 2.3.10 (Relative Permeability)**.** *In a linear material $\mathcal{M}$, the polarization density is parallel to the electric field. We get $\vec{D}(\vec{r}) = \epsilon(\vec{r}) \cdot \vec{E}(\vec{r})$, where ϵ is called the permeability, sometimes we write $\epsilon(\vec{r}) = \kappa(\vec{r}) \cdot \epsilon_0$.*

Remark 2.3.11. $\vec{E}(\vec{r}) = \dfrac{1}{\kappa(\vec{r})} \cdot \vec{E}_{vacc}(\vec{r})$*, where $\vec{E}_{vacc}$ is the electric field calculated in vaccum. And $\rho = \epsilon_0 \, \vec{\nabla} \cdot \vec{E}$ and $\rho_f = \epsilon_0 \, \vec{\nabla} \cdot \vec{E}_{vacc}$.*

3 Electric Systems

3.1 Work and Energy

Definition 3.1.1 (Potential Energy)**.** *Define $U(\vec{r}) = q\,\phi(\vec{r})$ the potential energy of a test charge due to the charge configuration on ϕ. For $\phi(\vec{r}_0) = 0$:*

$$U(\vec{r}) = \int_{\vec{r}_0 \to \vec{r}} \vec{F}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = q \int_{\vec{r}_0 \to \vec{r}} \vec{E}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = q\,\phi(\vec{r})$$

Definition 3.1.2. *The energy stored in a system of particles is:*

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{Q_i\,Q_j}{\|\vec{r}_i - \vec{r}_j\|} = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{Q_i\,Q_j}{4\pi\epsilon_0 \|\vec{r}_i - \vec{r}_j\|} = \frac{1}{2} \sum_{i=1}^{N} Q_i\,\phi_i(\vec{r}_i)$$

where $\phi_i(\vec{r}_i) = \lim_{\vec{r} \to \vec{r}_i} \left[\phi(\vec{r}) - \frac{Q_i}{\|\vec{r} - \vec{r}_i\|} \right]$ is the potential for all charged appart from Q_i. For a continuous charge distribution ρ on a volume $\mathcal{V}$, we have:

$$U = \frac{1}{2} \iiint_{\mathcal{V}} \rho(\vec{\mathbb{r}})\,\phi(\vec{\mathbb{r}})\,d^3\mathbb{r}$$

Problem 3.1.3. *Calculate the energy stored in a spherical shell of radius R with charge distribution σ.*

Solution:

$$U = \frac{1}{2} \iint_{\Omega=0}^{4\pi} \sigma \frac{q}{4\pi\epsilon_0 R} R^2\,d\Omega = \frac{1}{2} \cdot \frac{q}{4\pi\epsilon_0 R}\,\sigma 4\pi R^2 = \frac{q^2}{8\pi\epsilon_0 R}$$

where Ω is the solid angle.

Lemma 3.1.4. *For a continuous charge distribution ρ on a volume $\mathcal{V}$ with $\phi\big|_{\partial\mathcal{V}} \equiv 0$ or $\left.\frac{\partial\phi}{\partial\hat{n}}\right|_{\partial\mathcal{V}} \equiv 0$:*

$$U = \frac{1}{2}\,\epsilon_0 \iiint_{\mathcal{V}} \|\vec{E}(\vec{\mathbb{r}})\|^2\,d^3\mathbb{r}$$

Proof. By Gauß's theorem (cf. 1.1.16), $\iiint_{\mathcal{V}} \vec{\nabla}\boldsymbol{\cdot}(\phi\cdot\vec{E})\, d^3\mathbb{r} = \oiint_{\partial\mathcal{V}} \phi\cdot\vec{E}\boldsymbol{\cdot}d^2\vec{\mathbb{r}} = 0$ since either $\phi\big|_{\partial\mathcal{V}} \equiv 0$ or $\left.\frac{\partial\phi}{\partial\hat{n}}\right|_{\partial\mathcal{V}} \equiv 0$. Hence, if we integrate $\vec{\nabla}\boldsymbol{\cdot}(\phi\cdot\vec{E}) = \phi\cdot(\vec{\nabla}\boldsymbol{\cdot}\vec{E}) + (\vec{\nabla}\phi)\boldsymbol{\cdot}\vec{E} = \frac{1}{\epsilon_0}\,\rho\,\phi - \|\vec{E}\|^2$, by Gauß's law (cf. 2.2.1), $\rho = \epsilon_0\,\vec{\nabla}\boldsymbol{\cdot}\vec{E}$, we get the result. □

Problem 3.1.5. *Calculate the energy stored in a solid sphere of radius R with charge distribution ρ.*

Solution:

$$U = \frac{1}{2}\,\epsilon_0 \iint_{\Omega=0}^{4\pi} \int_{r=0}^{R} \left(\frac{\rho\, r}{3\epsilon_0}\right)^2 r^2\, dr\, d\Omega = \frac{2\pi\,\rho^2}{9\epsilon_0}\cdot\frac{R^5}{5} = \frac{q^2}{40\pi\epsilon_0 R}$$

where Ω is the solid angle.

Remark 3.1.6 (Material Correction)**.** *For a continuous charge distribution ρ on a material $\mathcal{M}$ with $\phi\big|_{\partial\mathcal{M}} \equiv 0$ or $\left.\frac{\partial\phi}{\partial\hat{n}}\right|_{\partial\mathcal{M}} \equiv 0$:*

$$U = \frac{1}{2}\iiint_{\mathcal{M}} \epsilon(\vec{\mathbb{r}})\cdot\|\vec{E}(\vec{\mathbb{r}})\|^2\, d^3\mathbb{r}$$

3.2 Boundary Value Problems

Definition 3.2.1 (Poisson Equation). *Given $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ and $\vec{E} = -\vec{\nabla}\phi$, we get:*

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

The equation is given in a region $\mathcal{V}$. There are two opitions of what can be given in $\partial\mathcal{V}$:

Dirichlet Boundary Conditions: $\phi\big|_{\partial\mathcal{V}} = f$

Neumann Boundary Conditions: $\left.\dfrac{\partial\phi}{\partial\hat{n}}\right|_{\partial\mathcal{V}} = f$

Moreover, the equation $\nabla^2\phi = 0$ is called the Laplace Equation.

Theorem 3.2.2 (Uniqueness Theorem). *The solution to Poisson's equation is unique in $\mathcal{V}$ given either Dirichlet or the Neumann Boundary Conditions (up to a constant).*

Proof. Let ϕ_1 and ϕ_2 be two solutions and $\psi = \phi_1 - \phi_2$. By linearity, we have: $\nabla^2\psi = 0$ and either $\psi\big|_{\partial\mathcal{V}} \equiv 0$ or $\left.\dfrac{\partial\psi}{\partial\hat{n}}\right|_{\partial\mathcal{V}} \equiv 0$. By 1.1.17 $\varphi = \psi$:

$$\iiint_{\mathcal{V}} \|\vec{\nabla}\psi\|^2 \, d^3\mathbb{r} = \oiint_{\partial\mathcal{V}} \psi \cdot \frac{\partial\psi}{\partial\hat{\mathbb{n}}} \, d^2\mathbb{r} - \iiint_{\mathcal{V}} \psi \cdot \nabla^2\psi \, d^3\mathbb{r} = 0$$

Then, $\forall\, \vec{r} \in \mathcal{V}$, $\|\vec{\nabla}\psi\|^2 = 0$. Solving $\vec{\nabla}\psi \equiv \vec{0}$, then $\psi = \text{const.}$, then, $\forall\, \vec{r} \in \mathcal{V}$, $\phi_1(\vec{r}) = \phi_2(\vec{r}) + \text{const.}$ which is exactly what we seeked to prove. Also, if the Dirichlet conditions applies, the constant vanishes. □

Problem 3.2.3. *Calculate the potential in the region between two concentric sphere of radius a and $2a$ and potential 0 and V, respectively, and charge density $\rho = \rho_0$ inside.*

Solution: *By symmetry, the potential only depends on r. Hence, the laplacian becomes $\nabla^2\phi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\phi}{\partial r}\right) = -\dfrac{\rho_0}{\epsilon_0}$. By direct integration, we get: $r^2\dfrac{\partial\phi}{\partial r} = A - \dfrac{\rho_0}{\epsilon_0} \cdot \dfrac{r^3}{3} \Rightarrow \phi(r) = B - \dfrac{A}{r} - \dfrac{\rho_0}{\epsilon_0} \cdot \dfrac{r^2}{6}$. By substituting, we* *get:* $A = 2aV + \frac{\rho_0}{\epsilon_0}\, a^3,\ B = 2V + \frac{\rho_0}{\epsilon_0} \cdot \frac{7a^2}{6}$, *hence the potential is given by:* $\phi(r) = 2V\left(1 - \frac{a}{r}\right) + \frac{\rho_0\, a^2}{6\epsilon_0}\left(7 - \frac{6a}{r} - \frac{r^2}{a^2}\right)$ *for* $a \leq r \leq 2a$, *which solves the Poisson equation with Dirichlet conditions.*

Theorem 3.2.4 (Mean Value Property)**.** *Let* ψ *be a solution of Laplace equation on* $\mathcal{V} \subseteq \mathbb{R}^3$. *Then:* $\forall\, \vec{r} \in \mathcal{V}$,

$$\psi(\vec{r}) = \frac{1}{4\pi R^2} \oiint_{\partial B_R(\vec{r})} \psi(\vec{\mathbb{r}})\, d^2\mathbb{r} = \frac{3}{4\pi R^3} \iiint_{B_R(\vec{r})} \psi(\vec{\mathbb{r}})\, d^3\mathbb{r}$$

that is, the value at $\vec{r}$ *is the average value over any sphere or spherical surface centered at* $\vec{r}$.

Proof. Let Ω be the solid angle, define:

$$\gamma(R) = \frac{1}{4\pi R^2} \oiint_{\partial B_R(\vec{r})} \psi(\overbrace{\ \vec{\mathbb{r}}\ }^{\vec{r}+R\,\hat{n}})\, \overbrace{d^2\mathbb{r}}^{R^2 d\Omega} = \frac{1}{4\pi} \iint_{\Omega=0}^{4\pi} \psi(\vec{r} + R\,\hat{n})\, d\Omega$$

Deriving wrt R:

$$\gamma'(R) = \frac{1}{4\pi} \iint_{\Omega=0}^{4\pi} \frac{\partial \psi}{\partial \hat{n}}\, (\vec{r}+R\,\hat{n})\, d\Omega \underset{\text{Gauß}}{=} \frac{1}{4\pi R^2} \int_{r=0}^{R} \iint_{\Omega=0}^{4\pi} \nabla^2 \psi\, (\vec{r}+R\,\hat{n}) R^2\, dr\, d\Omega = 0$$

since ψ is a solution of Laplace equation. Hence, $\gamma(R) = \text{const.}$, therefore:

$$\gamma(R) = \lim_{R \to 0} \gamma(R) = \frac{1}{4\pi} \iint_{\Omega=0}^{4\pi} \lim_{R\to 0} \psi(\vec{r}+R\,\hat{n})\, d\Omega = \frac{1}{4\pi} \iint_{\Omega=0}^{4\pi} \psi(\vec{r}+\vec{0})\, d\Omega = \psi(\vec{r})$$

Now, for the volume result, we employ the following formula:

$$\iiint_{B_R(\vec{r})} \psi(\vec{\mathbb{r}})\, d^3\mathbb{r} = \int_{r=0}^{R} \left(\oiint_{\partial B_r(\vec{r})} \psi(\vec{\mathbb{r}})\, d^2\mathbb{r} \right) dr = \int_{r=0}^{R} 4\pi r^2 \cdot \psi(\vec{r})\, dr = \frac{4\pi R^3}{3} \cdot \psi(\vec{r})$$

□

Corollary 3.2.5 (Maximum Principle)**.** *Let* ψ *be a solution of Laplace equation on* $\mathcal{V} \subseteq \mathbb{R}^3$. *Then,* ψ *has no local maxima or minima on the interior of* $\mathcal{V}$. *Hence, the extreme values must occur at the boundary* $\partial\mathcal{V}$.

Proof. By definition, if there is a local extremum, we may enclose the point by a sufficiently small sphere such that the centre has a bigger value then any point on/inside the sphere. But, by 3.2.4 the value should be the average of the sphere. Contradiction. □

Remark 3.2.6. *Another proof for uniqueness (cf. 3.2.2) on Dirichlet condition is given by:*

Proof. For two solutions ϕ_1 and ϕ_2, let $\psi = \phi_1 - \phi_2$. By linearity, $\nabla^2 \psi = 0$ and $\psi\big|_{\partial \mathcal{V}} \equiv 0$, then $\psi \equiv 0$, since any non-zero value in $\mathcal{V}$ would contradict 3.2.5. Therefore, $\phi_1 = \phi_2$. □

3.3 Surface of Materials

Lemma 3.3.1 (Interface)**.** *In the boundary surface of a solid $\mathcal{V}$:*

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\,\hat{n}$$

Proof. 1. Take a small area A around $\vec{r}$. By 2.2.2 on a small box V around A: $\Phi_E[\partial V] = \vec{E}_{\text{above}} \cdot (A\,\hat{n}) + \vec{E}_{\text{below}} \cdot (-A\,\hat{n}) = \frac{Q}{\epsilon_0} = \frac{\sigma\,A}{\epsilon_0}$. Hence, $(\vec{E}_{\text{above}} - \vec{E}_{\text{below}}) \cdot \hat{n} = \dfrac{\sigma}{\epsilon_0}$.

2. Take a small curve Γ around $\vec{r}$. By 2.1.3 on a small area Σ around Γ: $\displaystyle\oint_\Gamma \vec{E}(\vec{\mathrm{r}}) \cdot d\vec{\mathrm{r}} = \vec{E}_{\text{above}} \cdot (\ell\,\hat{t}) + \vec{E}_{\text{below}} \cdot (-\ell\,\hat{t}) = 0$ for any tangent vector $\hat{t}$. Hence, $(\vec{E}_{\text{above}} - \vec{E}_{\text{below}}) \cdot \hat{t} = 0$.

Therefore, $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \dfrac{\sigma}{\epsilon_0}\,\hat{n}$. □

Corollary 3.3.2. $(\phi_{above} - \phi_{below})\big|_{\partial\mathcal{V}} = 0$ *and* $\sigma = -\epsilon_0 \left.\dfrac{\partial}{\partial \hat{n}}\right|_{\partial\mathcal{V}} (\phi_{above} - \phi_{below})$

Proof. By 2.1.3 and 3.3.1, there second equation follows. Further, by taking at straight line curve from below to above, $\phi_a - \phi_b = -\displaystyle\int_{b\to a} \vec{E}(\vec{\mathrm{r}}) \cdot d\vec{\mathrm{r}}$ which goes to 0 as the path tends toward the boundary. □

Lemma 3.3.3 (Interface of Materials)**.** *In the boundary surface of a solid $\mathcal{V}$: $\vec{D}_{above} - \vec{D}_{below} = \sigma_f\,\hat{n}$, in a linear medium, $\epsilon_{above} \cdot \vec{E}_{above} - \epsilon_{below} \cdot \vec{E}_{below} = \sigma_f\,\hat{n}$*

Definition 3.3.4 (Conductors)**.** *A conductor, heretofore denoted Π is an object which charges can move freely. Ideally, we would have an unlimited supply of free charges.*

Theorem 3.3.5. *In a conductor, Π:*

1. $\vec{E}\big|_\Pi \equiv \vec{0}$
2. $\rho\big|_\Pi = 0$ *and the charges are in $\partial\Pi$*
3. Π *is an equipotential (cf. 2.2.3), hence, $\vec{E} \perp \partial\Pi$*

Proof. 1. Consedering the conductor consists of coupled charges (i.e. atoms), if there is a non-zero $\vec{E}$ at a point, then the charges would move. Supposing electrostatics, the charges cannot move, hence the filed must vanish inside a conductor on the electrostatical regime.

2. By 2.2.1, $\rho\big|_{\Pi} = \epsilon_0 \, (\vec{\nabla} \cdot \vec{E})\big|_{\Pi} = 0$.
3. By 2.1.13, $\phi_a - \phi_b = -\displaystyle\int_{b \to a} \vec{E}(\mathbb{r}) \cdot d\vec{\mathbb{r}} = 0$, and the rest is 2.2.4.

$\square$

Corollary 3.3.6. $\sigma = \epsilon_0 \, \hat{n} \cdot \vec{E}\big|_{\partial\Pi} = -\epsilon_0 \, \dfrac{\partial \phi}{\partial \hat{n}}\Big|_{\partial\Pi}$

Lemma 3.3.7 (Cavity)**.** *If a conductor* Π *has a cavity inside (denoted* Π_c*), that is, it is not simply connected (cf. Calculus II), then, in* Π_c*:*

$$Q[\partial\Pi_c] = -Q[\Pi_c]$$

that is, the induced charged on the surface of the cavity is exactly opposite to the charge inside the cavity.

Proof. Direct application of 2.2.2 with Π_c and 3.3.5. $\square$

Theorem 3.3.8 (Faraday Cage)**.** *If* $Q[\Pi_c] = 0$*, then* $\vec{E}\big|_{\Pi_c} \equiv \vec{0}$

Proof. Observe $\vec{E}$ must be continuous inside Π_c since $\vec{\nabla} \cdot \vec{E}$ exists (cf. 2.2.1). Then, there is a loop Γ (part inside the conductor, part in the cavity) so that $\mathcal{E}[\partial\Sigma] > 0$. However, this contradicts 2.1.3. Therefore, $\vec{E}\big|_{\Pi_c} \equiv \vec{0}$. $\square$

3.4 Method of Images

Definition 3.4.1. *An image charge is a charged distribution on* $\mathbb{R}^3 \setminus \mathcal{V}$ *so that the Poisson Eq. (cf. 3.2.1) satisfies the boundary condition. By the uniqueness theorem (cf. 3.2.2), since we did not change* $\rho\big|_{\mathcal{V}}$*, and it satisfies Dirichlet boundary conditions, the solution is valid and unique.*

Problem 3.4.2. *A point charge of charge* q *is placed at* $(0, 0, a)$ *and in the plane* $z = 0$*, there is a grounded* ($\phi = 0$) *infinite conducting sheet. Find the potential everywhere above the sheet* ($z > 0$)*. What is the induced surface charge density on the sheet?*

Solution: *Place a charge* $-q$ *at* $(0, 0, -a)$*. By Coulomb:*

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z+a)^2}}$$

So, $\phi\big|_{z=0} = 0$*. Hence this expression gives us the potential for* $z > 0$*. Now, by 3.3.6,* $\sigma = -\epsilon_0 \left.\dfrac{\partial\phi}{\partial z}\right|_{z=0} = -\dfrac{qa}{2\pi\left(x^2 + y^2 + a^2\right)^{\frac{3}{2}}}$

Problem 3.4.3. *A point charge of charge* q *is placed at* $(0, 0, a)$ *(with* $a > R$*) and there is a grounded* ($\phi = 0$) *infinite conducting sphere* ($r = R$)*. Find the potential everywhere outside the sphere* ($r > R$)*. What is the induced surface charge density on the sphere?*

Solution: *Place a charge* q' *at* $(0, 0, a')$ *with* $a' < R$*. By Coulomb:*

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z-a)^2}} + \frac{q'}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z-a')^2}}$$

Setting $\phi(0, 0, R) = 0$ *and* $\phi(0, 0, -R) = 0$*, we get:* $a' = \dfrac{R^2}{a}$ *and* $q' = -\dfrac{R}{a}\,q$*.*

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{R\,q}{4\pi\epsilon_0\sqrt{a^2(x^2 + y^2) + (a\,z - R^2)^2}}$$

So, $\phi\big|_{r=R} = 0$*. Hence this expression gives us the potential for* $r > R$*. Now, by 3.3.6,* $\sigma = -\epsilon_0 \left.\dfrac{\partial\phi}{\partial r}\right|_{r=R} = \dfrac{q\,z}{4\pi\left(R^2 + a^2 - 2a\,z\right)^{\frac{3}{2}}}\left(1 - \dfrac{a^2}{R^2}\right)$ *or, in spherical coordinates,* $\sigma(\theta) = \dfrac{q\,R\cos\theta}{4\pi\left((a - R\cos\theta)^2 + R^2\sin^2\theta\right)^{\frac{3}{2}}}\left(1 - \dfrac{a^2}{R^2}\right)$

3.5 Capacitors

Definition 3.5.1. *A capacitor $\mathcal{C}$ is a system of two conductors Π_A and Π_B in a vacuum. When charged with $+Q$ and $-Q$, respectively, with potential difference V, we define the capacitance as:*

$$C = \frac{Q}{V} = \frac{Q}{\phi_A - \phi_B}$$

Remark 3.5.2. *A more correct definition is: $C = \dfrac{Q_f}{V}$.*

Theorem 3.5.3 (Second Uniqueness)**.** *In a volume $\mathcal{V}$, $\vec{E}(\vec{r})$ is unique in $\mathcal{V}$ given the charge density ρ between the conductors inside $\mathcal{V}$ and the total charge in each conductor. That is, let $\mathcal{V} = \mathcal{V}' \sqcup \left(\bigsqcup_i \Pi_i\right)$, the solution to this system is unique:*

$$\oiint_{\partial\Pi_i} \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \frac{1}{\epsilon_0} Q_i \qquad (\vec{\nabla} \cdot \vec{E})\big|_{\mathcal{V}'} = \frac{1}{\epsilon_0}\rho$$

Proof. Let $\vec{E}_1$ and $\vec{E}_2$ be two electric fields that solve the system. Define $\vec{E} = \vec{E}_2 - \vec{E}_1$, by linearity, the system becomes: $\oiint_{\partial\Pi_i} \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = 0$ and $(\vec{\nabla} \cdot \vec{E})\big|_{\mathcal{V}'} = 0$. Moreover, $\partial\mathcal{V}' = \bigsqcup_i \partial\Pi_i$ and $\phi\big|_{\partial\mathcal{V}'} \equiv 0$. By 3.3.5, $\phi\big|_{\Pi_i} \equiv \phi_i$ (const.), then:

$$\iiint_{\mathcal{V}'} \vec{\nabla} \cdot (\phi \cdot \vec{E})(\vec{\mathbb{r}})\, d^3\mathbb{r} = \oiint_{\partial\mathcal{V}'} \phi(\vec{\mathbb{r}}) \cdot \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \sum_i \phi_i \cdot \oiint_{\partial\Pi_i} \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = 0$$
$$= \iiint_{\mathcal{V}'} \left[(\vec{\nabla} \cdot \vec{E})(\vec{\mathbb{r}}) \cdot \phi(\vec{\mathbb{r}}) - \|\vec{E}(\vec{\mathbb{r}})\|^2\right] d^3\mathbb{r} = -\iiint_{\mathcal{V}'} \|\vec{E}(\vec{\mathbb{r}})\|^2\, d^3\mathbb{r}$$

Hence, $\forall\, \vec{r} \in \mathcal{V}$, $\vec{E}(\vec{r}) = \vec{0}$, that is, $\vec{E}_1 \equiv \vec{E}_2$. □

Corollary 3.5.4. *The distribution of charge on the surface of the conductor does not matter for the electric field.*

Theorem 3.5.5. *The capacitance of a capacitor $\mathcal{C}$ (cf. 3.5.1) only depends on the geometry of the conductors. That is, there is a linear dependency between Q and V.*

Proof. Since they are in a vacuum, $\rho = 0$. Taking Q_i on the capacitor, we get the electrical field $\vec{E}_i$. Then, dividing by Q_i and using 3.5.3,

$$\frac{1}{Q_1}\,\vec{E}_1 = \frac{1}{Q_2}\,\vec{E}_2 \Rightarrow \frac{1}{Q_1}\,\phi_1 = \frac{1}{Q_2}\,\phi_2 \Rightarrow C_1 = C_2$$

so, the capacitance does not change by changing the charge. $\square$

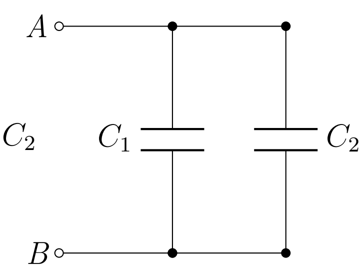
Problem 3.5.6. *Calculate the capacitance of two concentric spherical shells with radii $a < b$.*

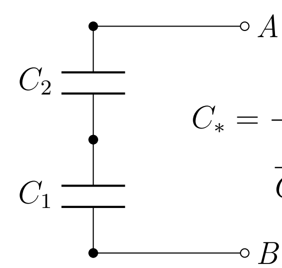
Solution: *We get a charge of Q (we're not assuming sign) in the inner shell. By spherical symmetry: $\vec{E} = \dfrac{Q}{4\pi\epsilon_0\, r^2}\,\hat{r} \Rightarrow$*

$$\phi(b) = \phi(a) - \int_{r=a}^{b} \frac{Q}{4\pi\epsilon_0\, r^2}\, dr = \phi(a) - \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \Rightarrow C = \frac{4\pi\epsilon_0}{\dfrac{1}{a} - \dfrac{1}{b}}$$

Lemma 3.5.7 (Associating Capacitors)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two capacitors, and combining them in parallel and series, respectively, will give the following equivalent capacitance:*

$$C_\parallel = C_1 + C_2$$





$$C_* = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

Proof. By definition:

- Parallel: $C_\parallel = \dfrac{Q}{\phi_A - \phi_B} = \dfrac{Q_1 + Q_2}{\phi_A - \phi_B} = C_1 + C_2$
- Series: $C_* = \dfrac{Q}{\phi_A - \phi_B} = \dfrac{Q}{\phi_A - \phi_C + \phi_C - \phi_B} = \dfrac{Q}{\dfrac{Q}{C_2} + \dfrac{Q}{C_1}} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$

Hence, we can associate capacitors by these formulas. $\square$

Lemma 3.5.8 (Capacitor Energy)**.** *For a capacitor, the energy stored inside is:* $U = \dfrac{Q^2}{2C}$

Proof. By 3.1.2, $U = \dfrac{1}{2} \iiint_{\mathcal{V}} \rho(\vec{\mathbb{r}}) \cdot \dfrac{Q}{C} \, d^3\mathbb{r} = \dfrac{Q^2}{2C}$. □

Corollary 3.5.9. *The capacitance is given by (cf. 3.1.4):*

$$\frac{1}{C} = \iiint_{\mathcal{V}} \epsilon(\vec{\mathbb{r}}) \left[\frac{\|\vec{E}(\vec{\mathbb{r}})\|}{Q} \right]^2 d^3\mathbb{r}$$

Problem 3.5.10. *Calculate the capacitance of two concentric spherical shells with radii $a < b$.*

Solution: *We get a charge of Q (we're not assuming sign) in the inner shell. By spherical symmetry:* $\vec{E} = \dfrac{Q}{4\pi\epsilon_0 \, r^2} \, \hat{r} \Rightarrow$

$$\frac{1}{C} = \iint_{\Omega=0}^{4\pi} \int_{r=a}^{b} \epsilon_0 \frac{1}{16\pi^2 \epsilon_0^2 \, r^4} \, r^2 \, dr \, d\Omega = \frac{1}{4\pi\epsilon_0} \int_{r=a}^{b} \frac{dr}{r^2} = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

where Ω is the solid angle.

4 Currents and Circuits

4.1 Current Density

Definition 4.1.1 (Currents). *Define $\vec{J}$ as the current density, is defined as:*

$$\vec{J}(\vec{r},t) = \rho(\vec{r},t) \cdot \vec{v}_{drift}(\vec{r},t)$$

where $\rho(\vec{r},t)$ is the charge density (which now depends on time) and $\vec{v}_{drift}(\vec{r},t)$ is the average drift velocity of the particles. Moreover, we can rewrite the density $\rho(\vec{r},t) = e\, n(\vec{r},t)$, where e is the electron's charge and n is the number of electrons per volume.
Define the current through a surface Σ as:

$$\mathcal{I}[\Sigma] = \iint_{\Sigma} \vec{J}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}}$$

Remark 4.1.2. *We may have surface or linear current density, denoted $\vec{K}$ or $\vec{I}$ respectively, where the charge is found only on a surface or a line.*

Theorem 4.1.3 (Continuity). *(Local) Conservation of Charge is equivalent to the following formula:*

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \equiv 0$$

Proof. Take a closed surface $\partial\Omega$, then by Local Conservation of Charge, the current though the surface is exactly minus the change in charge. That is,

$$\iiint_{\Omega} \vec{\nabla} \cdot \vec{J}(\mathbb{r},t)\, d^3\mathbb{r} = \oiint_{\partial\Omega} \vec{J}(\mathbb{r},t) \cdot d^2\vec{\mathbb{r}} = \mathcal{I}[\partial\Omega] = -\frac{\partial Q[\Omega]}{\partial t} = -\iiint_{\Omega} \frac{\partial \rho}{\partial t}\, d^3\mathbb{r}$$

since this is valid for all volumes Ω, the integrands should equal. □

Definition 4.1.4 (Steady Current). *A current is **steady** if:*

$$\frac{\partial \rho}{\partial t} \equiv 0 \quad \text{and} \quad \frac{\partial \vec{J}}{\partial t} \equiv \vec{0}$$

That is, the charges move individually and constant drift, but the charge density does not change. A direct consequence is $\vec{\nabla} \cdot \vec{J} \equiv 0$.

Lemma 4.1.5. *In the regime 4.1.4, the electric static equations (2.1.3, 2.2.1, 2.2.2) are still valid, hence so are every uniqueness theorem.*

Lemma 4.1.6. *The power dissipated by a current is:*

$$P = \iiint_{\mathcal{V}} \vec{E}(\vec{\mathbb{r}}) \cdot \vec{J}(\vec{\mathbb{r}})\, d^3\mathbb{r}$$

Proof. $dw = \vec{f} \cdot d\vec{r} = \rho(\vec{E} + \vec{r}' \times \vec{B}) \cdot \vec{r}'\, dt = \vec{E} \cdot (\rho \cdot \vec{v})\, dt \Rightarrow P = \iiint_{\mathcal{V}} \vec{E} \cdot \vec{J}\, d^3\vec{\mathbb{r}}$ $\square$

4.2 Ohm's Law

Theorem 4.2.1 (Ohm's Law)**.** *In a linear material, there is a scalar function* $\varrho : \mathcal{V} \subseteq \mathbb{R}^3 \to \mathbb{R}$*, called the resistivity, such that:*

$$\vec{E}(\vec{r}) = \varrho(\vec{r}) \cdot \vec{J}(\vec{r})$$

Definition 4.2.2. *A resistor* $\mathcal{R}$ *is a system consiting of a linear material between two conductors* A *and* B*. When passing with steady current* $\mathcal{I}$ *through each conductor, with potential difference* $V = \phi_A - \phi_B$*, we define resistance as:*

$$R = \frac{V}{\mathcal{I}}$$

Lemma 4.2.3. *The resistance of a resistor* $\mathcal{R}$ *(cf. 4.2.2) only depends on the resistivity of the material and geometry of both the conductors and material. That is, there is a linear dependency between* V *and* $\mathcal{I}$*.*

Proof. Since the currents are steady, $\vec{\nabla} \cdot \vec{J} \equiv 0$. Looking at the formula for resistence and capacitance:

$$\frac{1}{C} = \frac{\int \vec{E}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}}}{\oiint \epsilon(\vec{\mathbb{r}}) \cdot \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}}} \qquad R = \frac{\int \vec{E}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}}}{\oiint \frac{1}{\varrho(\vec{\mathbb{r}})} \cdot \vec{E}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}}}$$

Hence, all properties follow by analogy $R \leftrightarrow \dfrac{1}{C}$ by $\dfrac{1}{\varrho} \leftrightarrow \epsilon$. □

Problem 4.2.4. *Calculate the resistence of two concentric spherical shells with radii* $a < b$ *with uniform resistivity* ϱ *in between.*

Solution: *We get a current of* $\mathcal{I}$ *(we're not assuming sign) goint out the inner shell. By spherical symmetry:* $\vec{J} = \dfrac{\mathcal{I}}{4\pi\, r^2}\,\hat{r} \Rightarrow$

$$\phi(b) = \phi(a) - \int_{r=a}^{b} \frac{\varrho\,\mathcal{I}}{4\pi\, r^2}\, dr = \phi(a) - \frac{\varrho\mathcal{I}}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) \Rightarrow R = \frac{\varrho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$

This is exactly the same result we got for capacitors, by applying the analogy.

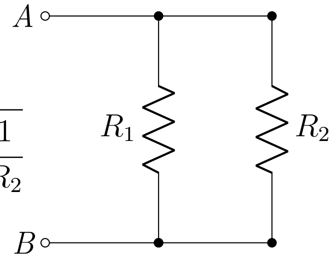
Remark 4.2.5. *The resistor has a (maybe non-trivial) capacitance, which makes it a RC circuit.*

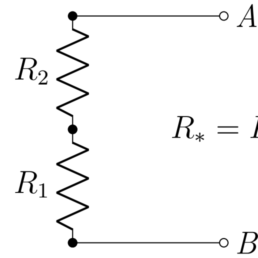
Theorem 4.2.6 (Kirchoff Laws)**.** *For any given circuit:*

- *Current: The sum of currents going into a node is equal to the sum going out. Equivalently, the algebraic sum of currents in a node is zero.*
- *Voltage: The directed sum of voltage differences in a loop is zero.*

Lemma 4.2.7 (Associating Resistors)**.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two resistors, and combining them in parallel and series, respectively, will give the following equivalent resistance:*

$$R_{\parallel} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$





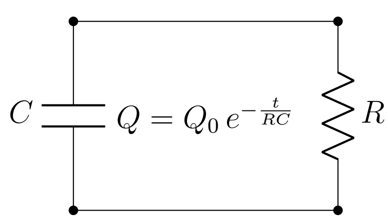
$$R_* = R_1 + R_2$$

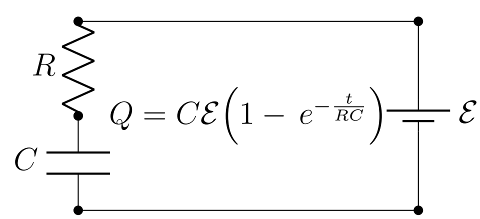
Proof. By definition:

- Parallel: $R_{\parallel} = \dfrac{\phi_A - \phi_B}{I} = \dfrac{\phi_A - \phi_B}{I_1 + I_2} = \dfrac{\phi_A - \phi_B}{\dfrac{\phi_A - \phi_B}{R_1} + \dfrac{\phi_A - \phi_B}{R_2}} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$
- Series: $R_* = \dfrac{\phi_A - \phi_B}{I} = \dfrac{\phi_A - \phi_C + \phi_C - \phi_B}{I} = \dfrac{R_2\, I + R_1\, I}{I} = R_1 + R_2$

Hence, we can associate resistors by these formulas. □

Lemma 4.2.8 (RC Circuit)**.** *We consider two cases:*





Proof. For each case:

1. $R(-Q') = V = \dfrac{Q}{C} \Rightarrow Q = Q_0\, e^{-\frac{t}{RC}}$
2. $\mathcal{E} = RQ' + \dfrac{Q}{C} \Rightarrow Q = C\mathcal{E}\left(1 - \, e^{-\frac{t}{RC}}\right)$

□

Lemma 4.2.9 (Resistance Power)**.** *For a resistor, the power dissipated is:* $P = R\,I^2$

Proof. By definition, $U = qV \Rightarrow P = \dfrac{dU}{dt} = V\dfrac{dq}{dt} = VI = RI^2$ by Ohm's Law. $\square$

Corollary 4.2.10. *The resistance is given by (cf. 4.1.6):*

$$R = \iiint_{\mathcal{V}} \varrho(\vec{\mathbb{r}}) \left[\frac{\|\vec{J}(\vec{\mathbb{r}})\|}{I} \right]^2 d^3\mathbb{r}$$

5 Magnetics

5.1 Magnetic Field

Definition 5.1.1 (Lorentz Force)**.** *The force acting on a test charge q in electric field $\vec{E}$ and magnetic field $\vec{B}$ is:*

$$\vec{F}(t,\vec{r},\vec{v}) = q\big[\vec{E}(t,\vec{r}) + \vec{v}\times\vec{B}(t,\vec{r})\big]$$

Corollary 5.1.2 (Cyclotronic Motion)**.** *Let $\vec{\omega}_B = -\dfrac{q}{m}\,\vec{B}$ and $\vec{a}_E = \dfrac{q}{m}\,\vec{E}$, then a particle with charge q moving with $\vec{r}(t)$ satifies the ODE:*

$$\ddot{\vec{r}}(t) = \vec{a}_E(t,\vec{r}) + \vec{\omega}_B(t,\vec{r})\times\dot{\vec{r}}(t)$$

where only the electromagnetic forces are present.

Lemma 5.1.3 (Superposition Principle)**.** *If there are two distinct fields $\vec{B}_1$ and $\vec{B}_2$ for two distinct sources, the total magnetic field is $\vec{B}_1 + \vec{B}_2$.*

Theorem 5.1.4 (Gauß's Law of Magnetism)**.** *For a static magnetic field (currents that induce the field are steady),*

$$\vec{\nabla}\cdot\vec{B} \equiv \vec{0}$$

hence $\exists\,\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3 : \vec{B} = \vec{\nabla}\times\vec{A}$. Further, $\Phi_B[\partial\Omega] = \displaystyle\oiint_{\partial\Omega} \vec{B}(\vec{\mathbb{r}})\cdot d^2\vec{\mathbb{r}} = 0$ for any closed surface $\partial\Omega$ (cf. 1.1.16).

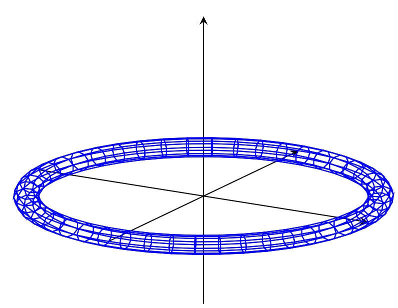
Theorem 5.1.5 (Biot-Savart Law)**.** *The magnetic field due to a steady current $\vec{J}$ on a volume $\mathcal{V}$ is:*

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_{\mathcal{V}} \vec{J}(\vec{\mathbb{r}})\times\frac{\vec{r}-\vec{\mathbb{r}}}{\|\vec{r}-\vec{\mathbb{r}}\|^3}\,d^3\mathbb{r}$$

where $\mu_0 = \dfrac{1}{c^2\,\epsilon_0}$ (c is the speed of light).

Corollary 5.1.6. *The vector potential $\vec{A}$ due to a steady current $\vec{J}$ on a volume $\mathcal{V}$ is: $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\iiint_{\mathcal{V}} \frac{\vec{J}(\vec{\mathbb{r}})}{\|\vec{r}-\vec{\mathbb{r}}\|}\,d^3\mathbb{r}$*

Problem 5.1.7 (Uniform Ring)**.**
Calculate the magnetic field due to a ring of charge (in the xy-plane) of radius R with steady current I going counterclockwise, at the z-axis.



Solution: *Since it is symmetric about rotations around z, we integrate using cylindrical coordinates: We have $\|\vec{r} - \vec{\mathbb{r}}\| = \sqrt{R^2 + z^2}$:*

$$\vec{B}(\rho) = \frac{\mu_0 I}{4\pi} \int_{\varphi=0}^{2\pi} \frac{\hat{\varphi} \times (z\,\hat{z} - R\,\hat{\rho}) R\,d\varphi}{(R^2 + z^2)^{\frac{3}{2}}} = \frac{\mu_0 I\,R}{4\pi} \int_{\varphi=0}^{2\pi} \frac{(z\,\hat{\rho} + R\,\hat{z})\,d\varphi}{(R^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{\mu_0 I\,R}{4\pi} \int_{\varphi=0}^{2\pi} \frac{R\,\hat{z}\,d\varphi}{(R^2 + z^2)^{\frac{3}{2}}} = \frac{\mu_0 I\,R^2}{2(R^2 + z^2)^{\frac{3}{2}}}\,\hat{z}$$

Lemma 5.1.8 (Magnetic Dipole Moment)**.** *Define $\vec{\mu} = \frac{1}{2} \iiint_{\mathcal{V}} \vec{\mathbb{r}} \times \vec{J}(\vec{\mathbb{r}})\,d^3\mathbb{r}$ we get the dipole approximation of the vector potential: $\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0\,\vec{\mu} \times \vec{r}}{4\pi\,\|\vec{r}\|^3}$*

Proof. Similar to the proof of electric dipole, with the added expression there are no magnetic monopole. □

Corollary 5.1.9. $\vec{B}_{dipole}(\vec{r}) = \dfrac{\mu_0\left[3(\vec{\mu} \cdot \hat{r})\,\hat{r} - \vec{\mu}\right]}{4\pi\,\|\vec{r}\|^3}$

Remark 5.1.10. *In a simplified case, $\vec{\mu} = \mathcal{I} \cdot \vec{S}$.*

Lemma 5.1.11. *The torque on a current loop due to a (locally constant) magnetic field $\vec{B}$ is: $\vec{\tau} = \vec{\mu} \times \vec{B}$.*

Proof. By direct calculation:

$$\vec{\tau} = \oint \vec{r} \times \left(\mathcal{I}\,d\vec{r} \times \vec{B}\right) = \mathcal{I} \oint \left[(\vec{r} \cdot \vec{B})\ d\vec{r} - \vec{B}(\vec{r} \cdot d\vec{r})\right]$$

$$= \mathcal{I} \oint (\vec{r} \cdot \vec{B})\ d\vec{r} - \vec{B}\mathcal{I} \oint \vec{r} \cdot d\vec{r} = \mathcal{I} \oint (\vec{r} \cdot \vec{B})\ d\vec{r}$$

$$\vec{\mu} \times \vec{B} = \frac{1}{2} \oint \left(\vec{r} \times \mathcal{I}\,d\vec{r}\right) \times \vec{B} = \frac{1}{2}\mathcal{I} \oint \left[(\vec{r} \cdot \vec{B})\ d\vec{r} - \vec{r}(\vec{B} \cdot d\vec{r})\right]$$

$$\Rightarrow \vec{\tau} - \vec{\mu} \times \vec{B} = \frac{1}{2}\mathcal{I} \oint \left[(\vec{r} \cdot \vec{B})\ d\vec{r} + \vec{r}(\vec{B} \cdot d\vec{r})\right]$$

$$= \frac{1}{2}\mathcal{I} \sum_{j,k} \hat{x}_k \oint \left[r_j\,B_j\ dx_k + r_k\,B_j\ dx_j\right] = \frac{1}{2}\mathcal{I} \sum_{j,k} B_j \hat{x}_k \oint r_j\ dx_k + r_k\ dx_j$$

which is zero taking Stokes. □

5.2 Ampère's Law

Theorem 5.2.1 (Differential Form of Ampère's Law)**.** *The magnetic field due to a steady current $\vec{J}$ obeys:*

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \, \vec{J}(\vec{r})$$

Proof. We calculate using 5.1.5 and 5.1.6:

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \\
\nabla^2 \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \vec{J}(\vec{\mathbb{r}}) \Big(- 4\pi \delta^3(\vec{r} - \vec{\mathbb{r}}) \Big) \, d^3\mathbb{r} = -\mu_0 \vec{J}(\vec{r}) \\
\vec{\nabla} \cdot \vec{A} &= \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \vec{J}(\vec{\mathbb{r}}) \cdot \frac{\vec{r} - \vec{\mathbb{r}}}{\|\vec{r} - \vec{\mathbb{r}}\|^3} \, d^3\mathbb{r} \\
&= -\frac{\mu_0}{4\pi} \oint\!\!\!\!\oint_{\partial\mathcal{V}} \frac{\vec{J}(\vec{\mathbb{r}})}{\|\vec{r} - \vec{\mathbb{r}}\|} \cdot d^2\vec{\mathbb{r}} \text{ due to } \vec{\nabla} \cdot \vec{J} \equiv 0 \\
&= 0 \text{ since } \mathcal{V} \text{ encloses all the current}
\end{aligned}$$

Hence $\vec{\nabla} \times \vec{B} = -\nabla^2 \vec{A} = \mu_0 \vec{J}$. □

Remark 5.2.2. *Taking the divergence of both sides,* $\vec{0} \equiv \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} \Rightarrow \vec{\nabla} \cdot \vec{J} \equiv \vec{0}$.

Corollary 5.2.3 (Coloumb Gauge)**.** *The vector potential is given by the PDE:* $\nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J}(\vec{r})$ *and* $\vec{\nabla} \cdot \vec{A}(\vec{r}) = 0$

Theorem 5.2.4 (Integral Form of Ampère's Law)**.** *For any surface Σ, the magnetic field due to a steady current $\vec{J}$ obeys:*

$$\oint_{\partial\Sigma} \vec{B}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \mu_0 \, \mathcal{I}[\Sigma]$$

Proof. By the differential form of Ampère's Law and Stokes' Theorem:

$$\oint_{\partial\Sigma} \vec{B}(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}} = \iint_{\Sigma} (\vec{\nabla} \times \vec{B}(\vec{\mathbb{r}})) \cdot d^2\vec{\mathbb{r}} = \iint_{\Sigma} \mu_0 \vec{J}(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \mu_0 \, \mathcal{I}[\Sigma]$$

The result follows. □

Problem 5.2.5 (Infinite Wire)**.** *Calculate the magnetic field due to an infinite wire with steady current I.*

Solution: *By symmetry and Ampère's with Σ a flat disk of radius ρ:*

$$2\pi\rho B_\varphi = \mu_0 I \Rightarrow \vec{B}(\rho) = \frac{\mu_0 I}{2\pi\rho}\,\hat{\varphi}$$

Problem 5.2.6 (Solenoid)**.** *Calculate the magnetic field due to an infinite solenoid (radius R) with steady current I and turn density n.*

Solution: *By symmetry and Ampère's with Σ a rectangle on the $\varphi = const.$ half-plane of sides ρ and L with one side in the z-axis:*

1. $\rho > R : L \cdot (B_z - B_{z0}) = -\mu_0 n \cdot L \cdot I \Rightarrow B_z = const.$, *for* $\lim\limits_{\rho\to\infty} B_z = 0$, *we need* $B_z = 0$.
2. $\rho < R : L \cdot (B_z - B_{z0}) = 0 \Rightarrow B_z = B_{z0} = \mu_0\, n\, I\, \hat{z}$.

Hence: $\vec{B}(\vec{r}) = \begin{cases} \mu_0\, n\, I\, \hat{z} & \textit{if } \rho < R \\ \vec{0} & \textit{otherwise} \end{cases}$

Lemma 5.2.7 (Interface)**.** *In the boundary surface of a solid $\mathcal{V}$:*

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{K} \times \hat{n}$$

Proof.

1. Take a small area A around $\vec{r}$. By 5.1.4 on a small box V around A: $\Phi_B[\partial V] = \vec{B}_{\text{above}} \cdot (A\,\hat{n}) + \vec{B}_{\text{below}} \cdot (-A\,\hat{n}) = 0$. Hence, $(\vec{B}_{\text{above}} - \vec{B}_{\text{below}}) \cdot \hat{n} = 0$.
2. Take a small curve Γ around $\vec{r}$. By 5.2.4 on a small area Σ around Γ: $\oint_\Gamma \vec{B}(\vec{\mathfrak{r}}) \cdot d\vec{\mathfrak{r}} = \vec{B}_{\text{above}} \cdot (\ell\,\hat{t}) + \vec{B}_{\text{below}} \cdot (-\ell\,\hat{t}) = \mu K \cdot \ell$ for the vector $\hat{t}$ tangent to the surface but perpendicular to $\vec{K}$, that is: $\hat{t} = \hat{n} \times \hat{K}$. Hence, $\hat{n} \times (\vec{B}_{\text{above}} - \vec{B}_{\text{below}}) = \mu_0 \vec{K}$.

Therefore, $\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 \vec{K} \times \hat{n}$. □

5.3 Faraday's Law

Theorem 5.3.1 (Faraday-Maxwell Differential Law)**.** *The electric and magnetic fields resultant of the same source obey:*

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}$$

Theorem 5.3.2 (Faraday-Maxwell Integral Law)**.** *For any surface Σ, the electric and magnetic fields resultant of the same source obey:*

$$\oint_{\partial\Sigma} \vec{E}(\vec{\mathbb{r}}, t) \cdot d\vec{\mathbb{r}} = -\iint_{\Sigma} \frac{\partial \vec{B}(\vec{\mathbb{r}}, t)}{\partial t} \cdot d^2\vec{\mathbb{r}}$$

Proof. By the differential form of Maxwell-Faraday's Law and Stokes' Theorem:

$$\oint_{\partial\Sigma} \vec{E}(\vec{\mathbb{r}}, t) \cdot d\vec{\mathbb{r}} = \iint_{\Sigma} \left[\vec{\nabla} \times \vec{E}(\vec{\mathbb{r}}, t)\right] \cdot d^2\vec{\mathbb{r}} = -\iint_{\Sigma} \frac{\partial \vec{B}(\vec{\mathbb{r}}, t)}{\partial t} \cdot d^2\vec{\mathbb{r}}$$

The result follows. □

Corollary 5.3.3. *Substituing the vector potential:*

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) \equiv \vec{0} \Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$

Moreover, ϕ is exactly the electric potential, as before. Hence, it obeys: $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ if we require $\vec{\nabla} \cdot \vec{A} \equiv 0$.

Definition 5.3.4 (EMF)**.** *The electromotive force (emf) around a curve $\Gamma(t)$ (which may depend on time) is defined as:*

$$\mathcal{E}[\Gamma(t)] = \oint_{\Gamma(t)} \vec{f}(\vec{\mathbb{r}}, t) \cdot d\vec{\mathbb{r}} = \oint_{\Gamma(t)} \left[\vec{E}(\vec{\mathbb{r}}, t) + \dot{\vec{\mathbb{r}}}(t) \times \vec{B}(\vec{\mathbb{r}}, t)\right] \cdot d\vec{\mathbb{r}}$$

where $\vec{F}(\vec{r}, t) = q\,\vec{f}(\vec{r}, t)$, that is, $\vec{f}$ is the force density.

Theorem 5.3.5 (Faraday's Flux Rule)**.** *For any surface $\Sigma(t)$ (that may change with time), the magnetic field due to an arbitrary current $\vec{J}$ obeys:*

$$\mathcal{E}[\partial\Sigma(t)] = -\frac{d\Phi_B[\Sigma(t)]}{dt} = -\frac{d}{dt} \iint_{\Sigma(t)} \vec{B}(\vec{\mathbb{r}}, t) \cdot d^2\vec{\mathbb{r}}$$

Proof. By 1.1.18 and the potential formulation of $\vec{E}$, then:

$$\begin{aligned}
-\frac{d\Phi_B[\Sigma(t)]}{dt} &= -\frac{d}{dt}\iint_{\Sigma(t)} \vec{B}(\vec{\mathbb{r}},t)\cdot d^2\vec{\mathbb{r}} = -\frac{d}{dt}\oint_{\partial\Sigma(t)} \vec{A}(\vec{\mathbb{r}},t)\cdot d\vec{\mathbb{r}} \\
&= -\oint_{\partial\Sigma(t)} \left[\frac{\partial\vec{A}(\vec{\mathbb{r}},t)}{\partial t} - \dot{\vec{\mathbb{r}}}\times(\vec{\nabla}\times\vec{A}(\vec{\mathbb{r}},t))\right]\cdot d\vec{\mathbb{r}} \\
&= \oint_{\partial\Sigma(t)} \left[\vec{E}(\vec{\mathbb{r}},t) + \vec{\nabla}\phi(\vec{\mathbb{r}},t) + \dot{\vec{\mathbb{r}}}\times(\vec{\nabla}\times\vec{A}(\vec{\mathbb{r}},t))\right]\cdot d\vec{\mathbb{r}} \\
&= \oint_{\partial\Sigma(t)} \left[\vec{E}(\vec{\mathbb{r}},t) + \dot{\vec{\mathbb{r}}}(t)\times\vec{B}(\vec{\mathbb{r}},t)\right]\cdot d\vec{\mathbb{r}} = \mathcal{E}[\partial\Sigma(t)]
\end{aligned}$$

As required. □

Corollary 5.3.6 (Lenz's Law)**.** *The induced current (Eddy current) on a resistive material will generate an opposing magnetic field, so as to reduce the change in flux.*

Definition 5.3.7 (Magnetic Energy)**.**

$$U = \frac{1}{2\mu_0}\iiint_{\mathcal{V}} \|\vec{B}(\vec{\mathbb{r}})\|^2 d^3\mathbb{r}$$

Proof. This energy comes exactly from induction:

$$\begin{aligned}
U &= \frac{1}{2}\iiint_{\mathcal{V}} \vec{A}(\vec{\mathbb{r}})\cdot\vec{J}(\vec{\mathbb{r}})\,d^3\mathbb{r} = \frac{1}{2\mu_0}\iiint_{\mathcal{V}} \vec{A}(\vec{\mathbb{r}})\cdot\vec{\nabla}\times\vec{B}(\vec{\mathbb{r}})\,d^3\mathbb{r} \\
&= \frac{1}{2\mu_0}\iiint_{\mathcal{V}} \|\vec{B}(\vec{\mathbb{r}})\|^2\,d^3\mathbb{r} - \frac{1}{2\mu_0}\oiint_{\partial\mathcal{V}} \vec{A}(\vec{\mathbb{r}})\times\vec{B}(\vec{\mathbb{r}})\cdot d^2\vec{\mathbb{r}}
\end{aligned}$$

And the surface term is zero by enforcing a boundary condition $\vec{A}\big|_{\partial\mathcal{V}} \equiv \vec{0}$. □

5.4 Inductance

Definition 5.4.1 (Mutual Inductance)**.** *Given n current loops $\Gamma_i = \partial \Sigma_i$ with current I_i passing through, we define:* $M_{i,j} = \dfrac{\Phi_{i,j}}{I_j}$, *where*

$$\Phi_{i,j} = \Phi_{B_j}[\Sigma_i] = \iint_{\Sigma_i} \vec{B}_j(\vec{\mathbb{r}}) \cdot d^2\vec{\mathbb{r}} = \oint_{\Gamma_i} \vec{A}_j(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}}$$

Lemma 5.4.2 (Neumann Formula)**.** *The mutual inductances depend only on the geometry of the two current loops and:*

$$M_{i,j} = \frac{\mu_0}{4\pi} \oint_{\Gamma_i} \oint_{\Gamma_j} \frac{d\vec{\mathbb{r}}_i \cdot d\vec{\mathbb{r}}_j}{\|\vec{\mathbb{r}}_i - \vec{\mathbb{r}}_j\|}$$

Proof. By 5.1.6 on a loop: $\vec{A}_j(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\oint_{\Gamma_j} \frac{I_j \; d\vec{\mathbb{r}}}{\|\vec{r} - \vec{\mathbb{r}}\|}$. Hence, it follows since I_j does not depend on position, we can plug it into $M_{i,j}\, I_j = \Phi_{i,j} = \displaystyle\oint_{\Gamma_i} \vec{A}_j(\vec{\mathbb{r}}) \cdot d\vec{\mathbb{r}}$ and divide through. □

Corollary 5.4.3. $M_{i,j} = M_{j,i}$

Definition 5.4.4 (Self Inductance)**.** *Define:* $L_i = \dfrac{\Phi_{i,i}}{I_i}$, *hence, we get:*

$$\mathcal{E}_i = -L_i \frac{dI_i}{dt}$$

Moreover, can take $L_i = \lim_{\Gamma_j \to \Gamma_i} M_{i,j}$

Lemma 5.4.5 (Inductor Energy)**.** *For a inductor, the energy stored inside is:*

$$U = \frac{L\, I^2}{2}$$

Proof. By 4.1.6, $U = \displaystyle\int V\, I\, dt = \int L \frac{dI}{dt} I\, dt = \frac{L\, I^2}{2}$. □

Corollary 5.4.6. *The Inductance is given by (cf. 5.3.7):*

$$L = \iiint_{\mathcal{V}} \frac{1}{\mu_0} \left[\frac{\|\vec{B}(\vec{\mathbb{r}})\|}{I} \right]^2 d^3\mathbb{r}$$

6 Maxwell's Equations

6.1 Maxwell's Correction and Waves

Remark 6.1.1. *So, far, our equations are:*

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

However, the last equation cannot be correct, in general, since taking divergence of both sides would give $\vec{\nabla} \cdot \vec{J} = \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$

Theorem 6.1.2 (Ampère Law with Maxwell Correction)**.**

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Proof. Say $\vec{\nabla} \times \vec{B} = \mu_0(\vec{J} + \vec{J}_D)$ for some $\vec{J}_d$ (called the displacement current). Taking the divergence and the curl:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 = \mu_0(\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{J}_d) \Rightarrow \vec{\nabla} \cdot \vec{J}_d = -\vec{\nabla} \cdot \vec{J} = \frac{\partial \rho}{\partial t}$$

and a solution to that, using Gauß's Law is: $J_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. ☐

Lemma 6.1.3 (Inhomogeneous Wave Equation)**.** *Let* $\Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ *and* $\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \equiv 0$ *(Lorentz Condition), then the Maxwell Equations become:*

$$\Box^2 \phi = -\frac{\rho}{\epsilon_0} \qquad \Box^2 \vec{A} = -\mu_0 \vec{J}$$

Proof. Direct application of Maxwell's Equations. ☐

Corollary 6.1.4 (E&M Waves)**.** *In a charge-free region ($\rho = 0$ and $\vec{J} = \vec{0}$), the electromagnetic fields obey: $\Box^2 \vec{E} = \Box^2 \vec{B} = \vec{0}$.*

Lemma 6.1.5. *In a charge-free region, let ψ be a solution to the wave equation. Then,*

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cdot \psi(\hat{k} \cdot \vec{r} - ct) \Rightarrow \vec{B}(\vec{r}, t) = \frac{\hat{k} \times \vec{E}_0}{c} \cdot \psi(\hat{k} \cdot \vec{r} - ct)$$

Proof. Follows from Faraday's Law. ☐

6.2 Special Relativity

Theorem 6.2.1 (Lorentz Transformation of Fields)**.** *The transformation of electromagnetic fields from a frame S to S' moving at velocity $\vec{v}$.*

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{v} \times \vec{B}\right) \qquad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}\right)$$

Proof. Let $A_0 = \phi/c$, then, we define: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the result follows by applying a Lorentz boost to this tensor. Another derivation would be using the Lorentz force (cf. 5.1.1) with the boost in velocities and forces. $\square$

Corollary 6.2.2. *Magnetic field is a Lorentz transformation of Electric Field. If $\vec{B} = \vec{0}$:*

$$\vec{B}' = -\frac{\gamma}{c^2}\vec{v} \times \vec{E} = -\frac{1}{c^2}\vec{v} \times \vec{E}'$$

Theorem 6.2.3 (Jeffimenko's Equation)**.** *A solution to Maxwell's Equations is:*

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\vec{\mathbb{r}}, t_r)}{\|\vec{r} - \vec{\mathbb{r}}\|} \, d^3\vec{\mathbb{r}} \qquad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \frac{\vec{J}(\vec{\mathbb{r}}, t_r)}{\|\vec{r} - \vec{\mathbb{r}}\|} \, d^3\vec{\mathbb{r}}$$

where $t_r = t - \dfrac{\|\vec{r} - \vec{\mathbb{r}}\|}{c}$ is the retarded time.

Proof. The derivation of this solution envolves taking Fourier transformation to find a Green's function. Outside the scope. $\square$