Calculus I

Notes from TAU Course with Additional Information

Gabriel Domingues

2020-02-14

Contents

1 Real Analysis

1.1 Sequences

Definition 1.1.1 (Common Sets)**.** *We denote:*

Natural Numbers: $\mathbb{N} = \{1, 2, \cdots\}$

Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

Rationals: $\mathbb{Q} = \left\{ \dfrac{m}{n} \;\middle|\; m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$

Reals: $\mathbb{R}$

Definition 1.1.2 (Sequence)**.** *A sequence of real numbers is a set of numbers which are written in some particular order. There are infinitely many terms to the sequence:*

$$a_1, a_2, \cdots, a_n, \cdots$$

Denote the entire sequence by: $\{a_n\}_{n=1}^{\infty}$ *or just* $\{a_n\}$*. The set of all sequences with in a set* $\mathcal{U}$ *is denoted* $\mathcal{U}^{\mathbb{N}}$*.*

Definition 1.1.3 (Bounded Sequences)**.** *A sequence* $\{a_n\}$ *is:*

Bounded from ***above*** *if:* $\exists\, M \in \mathbb{R} : \forall\, n \in \mathbb{N}\,,\; a_n \leq M$

Bounded from ***below*** *if:* $\exists\, m \in \mathbb{R} : \forall\, n \in \mathbb{N}\,,\; a_n \geq m$

Bounded *if it is bounded from both above and below.*

We call every such M *and* m *upper and lower bounds, respectively.*

Lemma 1.1.1. *A sequence* $\{a_n\}$ *is bounded iff:*

$$\exists\, M > 0 : \forall\, n \in \mathbb{N}\,,\; |a_n| \leq M$$

Proof. Let A and a be upper and lower bounds of $\{a_n\}$, respectively. Choose $M = \max\{|A|, |a|\}$. $\square$

Definition 1.1.4 (Monotonic Sequences)**.** *A sequence* $\{a_n\}$ *is:*

> *Monotonic* ***increasing*** *if:* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n \leq a_{n+1}$
>
> *Monotonic* ***decreasing*** *if:* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n \geq a_{n+1}$
>
> ***Stricly*** *monotonic* ***increasing*** *if:* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n < a_{n+1}$
>
> ***Stricly*** *monotonic* ***decreasing*** *if:* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n > a_{n+1}$

Usually, we omit N *and shift the sequence so that it is monotonic* $\forall\, n \in \mathbb{N}$*.*

Example 1.1.1. $\left\{ \dfrac{n^2}{2^n} \right\}_{n=1}^{\infty}$ *is stricly monotonically decreasing:*

$$a_{n+1} < a_n \Leftrightarrow \frac{(n+1)^2}{2^{n+1}} < \frac{n^2}{2^n} \Leftrightarrow (n+1)^2 < 2n^2 \Leftrightarrow n > \sqrt{2}+1 > 2.4 \Leftrightarrow n \geq 3$$

Therefore, if we pick $N = 3$*, the sequence is stricly monotonically decreasing.*

Lemma 1.1.2 (Triangle Inequality)**.** $\forall\, x, y \in \mathbb{R} \,,\, |x+y| \leq |x| + |y|$

Definition 1.1.5 (Limit of a Sequence)**.** *We say* L *is the limit of* $\{a_n\}$ *if:*

$$\forall\, \epsilon > 0 \,,\, \exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, |a_n - L| < \epsilon$$

We say: $\lim\limits_{n \to \infty} a_n = L$ *or* $\lim a_n = L$ *or even* $a_n \to L$*. If the limit exists, the sequences* ***converges*** *(to L), otherwise, it* ***diverges****. Further, we write the choice of* N *as* $N(\epsilon)$*.*

Remark 1.1.1. *If we take a sequence* $\{a_n\}$*, the idea of the limit* L *is such that, for any interval* $(L-\epsilon, L+\epsilon)$*, we have a finite number of terms outside the interval and an infinite number of terms inside.*

Theorem 1.1.1 (Uniqueness of Limit)**.** *If a sequence* $\{a_n\}$ *has a limit* L*, then it is unique.*

Proof. Suppose that there are two limits $L_1 \neq L_2$. By definition:

$$\forall\, \epsilon > 0 \,,\, \exists\, N_1 \in \mathbb{N} : \forall\, n \geq N_1 \,,\, |a_n - L_1| < \epsilon$$
$$\forall\, \epsilon > 0 \,,\, \exists\, N_2 \in \mathbb{N} : \forall\, n \geq N_2 \,,\, |a_n - L_2| < \epsilon$$

Then, by the triangle inequality:

$$\forall\, n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}\,,\ |L_1 - L_2| \leq |L_1 - a_n| + |a_n - L_2| < 2\epsilon$$

Which is a contradiction for $\epsilon = \frac{1}{2}|L_1 - L_2| > 0$. □

Theorem 1.1.2 (Limit ⇒ Bounded)**.** *Every convergent sequence is bounded.*

Proof. Let $N = N(1)$ and $M = \max\left\{|a_1|, \cdots, |a_{N-1}|, 1 + |L|\right\}$. Then, by definition of $a_n \to L$:

$$\forall\, n \geq N\,,\ |a_n| \leq |a_n - L| + |L| < 1 + |L|$$

Hence, $\forall\, n \in \mathbb{N}\,,\ |a_n| \leq M$. □

Lemma 1.1.3 (Linearity of Limits)**.** *Suppose* $\lim a_n = A$*,* $\lim b_n = B$ *and* C *is a constant. Then, the following are true:*

1. $\lim C = C$
2. $\lim(C \cdot a_n) = C \cdot A$
3. $\lim(a_n \pm b_n) = A \pm B$

Proof. By definition,

1. Pick $N = 1$, we have: $\forall\, \epsilon > 0\,,\ \forall\, n \geq 1\,,\ |c - c| = 0 < \epsilon$
2. Pick $N = N_A\left(\dfrac{\epsilon}{|c|}\right)$, we have:

$$\forall\, \epsilon > 0\,,\ \forall\, n \geq N\,,\ |c \cdot a_n - c \cdot A| = |c| \cdot |a_n - A| < \epsilon$$

3. Let $N = \max\left\{N_A\left(\dfrac{\epsilon}{2}\right), N_B\left(\dfrac{\epsilon}{2}\right)\right\}$ Then, by the triangle inequality: $\forall\, n \geq N\,,$

$$|(a_n \pm b_n) - (A \pm B)| = |(a_n - A) \pm (b_n - B)| \leq |a_n - A| + |b_n - B| < \epsilon$$

Hence, it is a linear operation on the convergent sequences. □

Remark 1.1.2. *If* $\exists \lim a_n$ *and* $\nexists \lim b_n$, *then* $\nexists \lim(a_n + b_n)$. *If* $\nexists \lim a_n$ *and* $\nexists \lim b_n$, *then there is no rule for* $\lim(a_n + b_n)$.

Corollary 1.1.1. $\lim a_n = A \Leftrightarrow \lim(a_n - A) = 0$

Theorem 1.1.3. *Let* $\{b_n\}$ *be bounded and* $\lim a_n = 0$. *Then,* $\lim(a_n \cdot b_n) = 0$.

Proof. Let M be a two-sided bound of b_n. if $\lim a_n = 0$, take $N = N\left(\frac{\epsilon}{M}\right)$:

$$\forall \epsilon > 0\,,\ \exists N \in \mathbb{N} : \forall n \geq N\,,\ |a_n| < \frac{\epsilon}{M}$$
$$\forall \epsilon > 0\,,\ \exists N \in \mathbb{N} : \forall n \geq N\,,\ |a_n \cdot b_n - 0| = |a_n| \cdot |b_n| \leq |a_n| \cdot M < \epsilon$$

that is the definition of the limit being 0. $\square$

Lemma 1.1.4 (Arithmetic of Limits)**.** *Suppose* $\lim a_n = A$, $\lim b_n = B$. *Then, the following are true:*

1. $\lim(a_n \cdot b_n) = A \cdot B$
2. *If* $B \neq 0$ *and* $b_n \neq 0$, $\lim \dfrac{a_n}{b_n} = \dfrac{A}{B}$

Proof. We prove each one:

1. $\lim(a_n \cdot b_n) = \lim \left[\overbrace{(a_n - A)}^{\to 0} \cdot \overbrace{b_n}^{\text{bounded}} + A \cdot b_n \right] = 0 + \lim(A \cdot b_n) = A \cdot B$
2. By the previous result, $\underbrace{\lim b_n}_{B} \cdot \lim \dfrac{a_n}{b_n} = \underbrace{\lim a_n}_{A}$

$\square$

Lemma 1.1.5 (Positivity)**.** *Let* $\{a_n\}$ *be a positive sequence (bounded from below by* 0*) then* $L = \lim a_n$, *if is exists, is also positive.*

Proof. By contrary, suppose $L < 0$, then: $\forall n \in \mathbb{N}\,, |a_n - L| \geq |L|$. The statement for $a_n \to L$ is a contradiction for $\epsilon = |L|$. $\square$

Corollary 1.1.2 (Monotinicity)**.** *If* $M \geq a_n \geq m \Rightarrow M \geq \lim a_n \geq m$

Theorem 1.1.4 (Sandwich/Squeeze Theorem)**.** *Given three sequences* $\{a_n\}$*,* $\{b_n\}$ *and* $\{c_n\}$*. If* $\lim a_n = \lim c_n = L$ *and*

$$\exists\, N_0 \in \mathbb{N} : \forall\, n \geq N_0\,,\ a_n \leq b_n \leq c_n$$

then, $\lim b_n = L$*.*

Proof. Take $\epsilon > 0$:

$$\begin{aligned}
&\exists\, N_1 \in \mathbb{N} : \forall\, n \geq N_1\,,\ L - \epsilon < a_n < L + \epsilon \\
&\exists\, N_2 \in \mathbb{N} : \forall\, n \geq N_2\,,\ L - \epsilon < c_n < L + \epsilon \\
&\exists\, N_0 \in \mathbb{N} : \forall\, n \geq N_0\,,\ a_n \leq b_n \leq c_n
\end{aligned}$$

Pick $N_3 = \max\{N_0, N_1, N_2\}$. Then, $\forall\, n \geq N_3\,,$

$$L - \epsilon < c_n \leq b_n \leq a_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon$$

□

Definition 1.1.6 (Wide Sense)**.** *We say that the sequence* $\{a_n\}$ *converges to* $+\infty$ *(eq.* $-\infty$*) if:*

$$\forall\, M \in \mathbb{R}\,,\ \exists\, N \in \mathbb{N} : \forall\, n \geq N\,,\ a_n > M \quad (\textit{eq. } a_n < M)$$

This a wide sense of the limit.

1.2 Cauchy Sequences

Definition 1.2.1 (Supremum and Infimum)**.** *For a set $A \subset \mathbb{R}$, we define:*

> *If A is bounded from above, the supremum of A is the lowest upper bound (denoted* $\sup A$*). Otherwise* $\sup A = \infty$*.*
>
> *If A is bounded from below, the infimum of A is the greatest lower bound (denoted* $\inf A$*). Otherwise* $\inf A = -\infty$*.*

Theorem 1.2.1 (Monotone Convergence)**.** *A bounded monotonic sequence converges. Furthermore:*

> *If $\{a_n\}$ is increasing,* $\lim a_n = \sup\{a_n\}$
>
> *If $\{a_n\}$ is decreasing,* $\lim a_n = \inf\{a_n\}$

Proof. We prove for increasing, since decreasing is analogous with $b_n = -a_n$. Let $M = \sup\{a_n\}$. For every $\epsilon > 0$, there exists N such that $a_N > M - \epsilon$, since otherwise $M - \epsilon$ is an upper bound of $\{a_n\}$, which contradicts to the definition of M. Then, since $\{a_n\}$ is increasing, and M is its upper bound, for every $n \geq N$, we have $|a_n - M| = M - a_n \leq M - a_N < \epsilon$. Hence, by definition, the limit of $\{a_n\}$ is $\sup\{a_n\}$. $\square$

Definition 1.2.2 (Subsequence)**.** *Given a sequence $\{a_n\}$, a subsequence $\{b_k\}$ is such that:*

$$b_k = a_{n_k}$$

where $\{n_k\}$ is a stricly increasing sequence of natural numbers.

Lemma 1.2.1. *A sequence $\{a_n\}$ converges to L in the wide sense if and only if any subsequence of $\{a_n\}$ converges to the same limit L.*

Definition 1.2.3 (Partial Limit)**.** *A real number a is called a partial limit of the sequence $\{a_n\}$ if there exists a subsequence of $\{a_n\}$ which converges to a in the wide sense.*

Remark 1.2.1. *For a partial limit a of $\{a_n\}$ for any interval $(a - \epsilon, a + \epsilon)$, we are only required to have an infinite number of terms inside. If there are infinitely many outside, there is no one limit L and if there finitely many outside, it is the one limit $L = a$.*

Theorem 1.2.2 (Bolzano-Weierstrass)**.** *To any bounded sequence, that exists a convergent subsequence, i.e., there exists at least one partial limit.*

Proof. By Monotone Convergence, we only need to prove that there exists a monotone subsequence of $\{a_n\}$. Call a_m a "peak" if $\forall\, n \geq m\,,\ a_m \geq a_n$.

> Case 1: $\{a_n\}$ has infinitely many peaks. Form a subsequence with these peaks denoted a_{n_k}. Since each of these terms is a peak and $n_1 < n_2 < ... < n_k < ...$ we have that $\{a_{n_k}\}_{k=1}^{\infty}$ is a monotonic decreasing subsequence of $\{a_n\}$.
>
> Case 2: $\{a_n\}$ has only a finite number of peaks denoted $a_{n_1}, a_{n_2}, ..., a_{n_k}$. Let $s_1 = n_k + 1$. Then, a_{s_1} is not a peak, and so then there exists an s_2 such that $s_1 < s_2$ and $a_{s_1} \leq a_{s_2}$. Also, a_{s_2} is not a peak and so there exists an s_3 such that $s_2 < s_3$ and $a_{s_2} \leq a_{s_3}$. By induction, we have a subsequence $a_{s_1} \leq a_{s_2} \leq ... \leq a_{s_m} \leq ...$, and so $\{a_{s_m}\}_{m=1}^{\infty}$ is a monotonic increasing subsequence of $\{a_n\}$.

□

Definition 1.2.4 (Limit Sup and Inf)**.** *The supremum of the set of all partial limits of a sequence is called* ***upper partial limit*** *of the sequence $\{a_n\}$ and is denoted as:*

$$\overline{\lim}\, a_n \quad or \quad \limsup a_n$$

The infimum of the set of all partial limits of a sequence is called ***lower partial limit*** *of the sequence $\{a_n\}$ and is denoted as:*

$$\underline{\lim}\, a_n \quad or \quad \liminf a_n$$

Theorem 1.2.3 (N&SC for the existence of limit)**.** *If a sequence $\{a_n\}$ is bounded and $\liminf a_n = \limsup a_n = a$, then $\lim a_n$ exists and is equal to a.*

Proof. Say $\{a_n\}$ is bounded by the interval $[m, M]$. Take an $\epsilon > 0$. Looking at the set $[m, M] \setminus (a - \epsilon, a + \epsilon) = [m, a - \epsilon] \cup [a + \epsilon, M]$, if there are infinetly many elements of the sequence, by Bolzano-Weierstrass there is a convergent subsequence, which contradicts either $\limsup a_n = a$ or $\liminf a_n = a$. Therefore, there are finetely many elements of the sequence in $[m, M] \setminus (a - \epsilon, a + \epsilon)$, hence in $\mathbb{R} \setminus (a - \epsilon, a + \epsilon)$, which is the definition of the limit. □

Definition 1.2.5 (Cauchy)**.** *A sequence* $\{a_n\}$ *is called a Cauchy sequence if:*

$$\forall \epsilon > 0 \,,\, \exists N \in \mathbb{N} : \forall m, n \geq N \,,\, |a_m - a_n| < \epsilon$$

equivalently:

$$\forall \epsilon > 0 \,,\, \exists N \in \mathbb{N} : \forall n \geq N \,,\, \forall p \in \mathbb{N} \,,\, |a_{n+p} - a_n| < \epsilon$$

Lemma 1.2.2 (Cauchy is Bdd)**.** *Every Cauchy sequence is bounded.*

Proof. Let $M = \max\left\{|a_1|, \cdots, |a_{N-1}|, \epsilon + |a_N|\right\}$. Then:

$$\forall n \geq N \,,\, |a_n| \leq |a_n - a_N| + |a_N| < \epsilon + |a_N|$$

Hence, $\forall n \in \mathbb{N} \,,\, |a_n| \leq M$. □

Lemma 1.2.3 (Convergent is Cauchy)**.** *Let* $\lim a_n = L$*, then* $\{a_n\}$ *is Cauchy.*

Proof. Then, by definition,

$$\forall \epsilon > 0 \,,\, \exists N \in \mathbb{N} : \forall n \geq N \,,\, |L - a_n| < \tfrac{\epsilon}{2}$$
$$\forall \epsilon > 0 \,,\, \exists N \in \mathbb{N} : \forall m \geq N \,,\, |a_m - L| < \tfrac{\epsilon}{2}$$

Since $|a_m - a_n| \leq |L - a_n| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, it is Cauchy. □

Theorem 1.2.4 (Cauchy Criterion)**.** $\{a_n\}$ *is convergent* $\Leftrightarrow$ *it is Cauchy.*

Proof. ($\Rightarrow$) is the previous lemma. ($\Leftarrow$): Suppose $\{a_n\}$ is Cauchy.

$$\forall \epsilon > 0 \,,\, \exists N \in \mathbb{N} : \forall n \geq N \,,\, |a_n - a_N| < \tfrac{\epsilon}{2}$$

Then:

$$|a_n - a_N| < \tfrac{\epsilon}{2} \Leftrightarrow a_N - \tfrac{\epsilon}{2} < a_n < a_N + \tfrac{\epsilon}{2}$$
$$a_N - \tfrac{\epsilon}{2} \leq \liminf a_n \leq \limsup a_n \leq a_N + \tfrac{\epsilon}{2}$$
$$\Rightarrow 0 \leq \limsup a_n - \liminf a_n \leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

Therefore, $\limsup a_n = \liminf a_n$. By the previous theorems, the sequence is bounded and $\limsup a_n = \liminf a_n$, therefore, it converges. □

1.3 Series

Definition 1.3.1 (Infinite Sum)**.** *Given a sequence $\{a_n\}$, the sum $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$ (also denoted $\sum a_n$) is called an (infinite) series. Consider partial sums:*

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

*If $\{S_n\}_{n=1}^{\infty}$ converges ($S_n \to S$), then the series is called **convergent**. Otherwise, the series is **divergent**. The number S is called the sum of the series, that is, $S = \sum a_k$.*

Example 1.3.1 (Basel)**.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent. Applying Cauchy to the partial sums, we get:*

$$\begin{aligned} |S_{n+p} - S_n| &= \sum_{k=n+1}^{n+p} \frac{1}{k^2} < \sum_{k=n+1}^{n+p} \frac{1}{k(k+1)} \\ &= \sum_{k=n+1}^{n+p} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n} < \epsilon \end{aligned}$$

So, choose $N(\epsilon) = \lceil \frac{1}{\epsilon} \rceil$.

Theorem 1.3.1 (Last Element)**.** *If $\sum a_n$ converges, then $\lim a_n = 0$.*

Proof. $\lim a_n = \lim \big(S_n - S_{n-1}\big) = \lim S_n - \lim S_{n-1} = S - S = 0$, where $S = \lim S_n$ and S_n are the partial sums. $\square$

Corollary 1.3.1. *If $\nexists \lim a_n$ or $\lim a_n = a \neq 0$, then $\sum a_n$ diverges.*

Remark 1.3.1. *The converse of the Last Element Theorem is not true.*

Example 1.3.2 (Harmonic)**.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent. Applying Cauchy to the partial sums, we get:*

$$|S_{n+p} - S_n| = \sum_{k=n+1}^{n+p} \frac{1}{k} \geq \frac{p}{n+p} \overset{p=n}{=} \frac{1}{2}$$

So, the sequence is not Cauchy.

Lemma 1.3.1 (Geometric Series)**.** $\sum_{n=0}^{\infty} a \cdot q^n$ *converges iff* $|q| < 1$.

Proof. Take the partial sums $S_n = \sum_{k=0}^{n} a \cdot q^k = a \cdot \dfrac{1 - q^{n+1}}{1 - q}$, which clearly converges iff $|q| < 1$. □

Lemma 1.3.2 (Linearity of Infinite Sums)**.** *If* $\sum a_n$ *and* $\sum b_n$ *converge, then* $\sum(a_n \pm b_n)$ *converges (to* $\sum a_n \pm \sum b_n$*) and* $\sum c \cdot a_n$ *converges (to* $c \cdot \sum a_n$*).*

Proof. Take $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. We supposed $\exists \lim A_n$ and $\exists \lim B_n$. Notice $\sum_{k=1}^{n}(a_k + b_k) = A_n + B_n$ and $\sum_{k=1}^{n} c \cdot a_k = c \cdot A_n$. By the linearity of the limits, we have the result. □

Lemma 1.3.3 (Bounded Series)**.** *If* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n \geq 0$*, then* $\sum a_n$ *converges iff* $\{S_n\}$ *is bounded.*

Proof. $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n = S_n - S_{n-1} \geq 0 \Leftrightarrow \{S_n\}$ is monotonically increasing. As a corollary of a previous theorem, a monotonically sequence has a limit iff it is bounded. □

Theorem 1.3.2 (Comparision)**.** *If* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, 0 \leq a_n \leq b_n$ *then,* $\sum b_n$ *converges* $\Rightarrow \sum a_n$ *converges.*

Proof. $\forall\, n \geq N \,,\, \sum_{k=N}^{n} a_k \leq \sum_{k=N}^{n} b_k$. By previous lemma, $\sum b_n$ converges $\Rightarrow \sum_{k=N}^{n} b_k$ is bounded from above. Hence, $\sum_{k=1}^{n} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} a_k$ is bounded from above (and below by 0), so it converges. □

Corollary 1.3.2. *If* $\exists\, N \in \mathbb{N} : \forall\, n \geq N \,,\, a_n \geq b_n \geq 0$ *then,* $\sum b_n$ *diverges* $\Rightarrow \sum a_n$ *diverges.*

Remark 1.3.2. *These follow from the comparison test and the definition of limit.*

1. *If* $\dfrac{a_n}{b_n} \to 0$*, then,* $\sum b_n$ *converges* $\Rightarrow \sum a_n$ *converges.*
2. *If* $\dfrac{a_n}{b_n} \to \infty$*, then,* $\sum b_n$ *diverges* $\Rightarrow \sum a_n$ *diverges.*

Theorem 1.3.3 (Limit Comparision)**.** *If* $\exists\, N \in \mathbb{N} : \forall\, n \geq N\,,\ a_n > 0$ *and* $b_n > 0$ *and* $\lim \dfrac{a_n}{b_n} = L > 0$ *then,* $\sum b_n$ *converges* $\Leftrightarrow \sum a_n$ *converges.*

Proof. $\lim \dfrac{a_n}{b_n} = L > 0 \Rightarrow \exists\, M > 0 : \forall\, n \in \mathbb{N}\,,\ \dfrac{a_n}{b_n} \leq M$. Hence, by linearity $\sum b_n$ converges $\Rightarrow \sum M \cdot b_n$ converges, by comparision, $\Rightarrow \sum a_n$ converges. To prove the converse, take $\lim \dfrac{b_n}{a_n} = \dfrac{1}{L} > 0$. □

Lemma 1.3.4 (Cauchy Condesation)**.** *Let* a_n *be a decreasing non-negative sequence. Then,* $\sum_{n=1}^{\infty} a_n$ *converges* $\Leftrightarrow \sum_{n=0}^{\infty} 2^n \cdot a_{2^n}$ *converges.*

Proof. Since a_n is decreasing, we have the estimate:

$$\forall\, n \in \mathbb{N}\,,\ \sum_{k=1}^{2^n-1} a_k \leq \sum_{k=0}^{n-1} 2^k \cdot a_{2^k} \leq 2 \cdot \sum_{k=1}^{2^n-1} a_k$$

Hence, by the Sandwich Theorem, we are done. □

Lemma 1.3.5 (P-Series)**.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges iff* $p > 1$.

Proof. If $p \leq 0$, it diverges by last element. For $p > 0$, $a_n = \dfrac{1}{n^p}$ is a decreasing non-negative sequence. By condesation: $2^n \cdot a_{2^n} = \dfrac{2^n}{(2^n)^p} = 2^{n(1-p)} = \left(2^{1-p}\right)^n$, a geometric series. Hence it converges iff $2^{1-p} < 1 \Leftrightarrow p > 1$. □

Theorem 1.3.4 (Leibnitz Criteria)**.** *If* $\sum (-1)^{n-1} \cdot a_n$ *with* $(a_n > 0)$*, an alternating series, satisfies:*

1. $a_{n+1} \leq a_n$ *(i.e. monotonically decreasing)*
2. $\lim a_n = 0$

Then, the series converges.

Proof. Consider the parity n for the partial sums S_n.

- Even: $S_{2(m+1)} = S_{2m} + (a_{2m+1} - a_{2m+2}) \geq S_{2m} \Rightarrow \{S_{2m}\}$ is monotonically increasing. On the other hand,

$$S_{2m} = a_1 - a_{2m} - \sum_{k=1}^{m-1} \left(a_{2k} - a_{2k+1}\right) \leq a_1 - a_{2m} < a_1$$

  hence S_{2m} is bounded form above. Therefore, $\lim_{m\to\infty} S_{2m} = S$.

- Odd: $S_{2m+1} = S_{2m} + a_{2m+1} \Rightarrow \lim\limits_{m\to\infty} S_{2m+1} = \lim\limits_{m\to\infty} S_{2m} + \lim\limits_{m\to\infty} a_{2m+1} = \lim\limits_{m\to\infty} S_{2m} = S$.

Therefore, $\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} S_{2m} = S \Rightarrow \lim_{n\to\infty} S_n = S$. $\square$

1.4 Absolute Convergence

Definition 1.4.1 (Abs Convergence)**.** *A series* $\sum a_n$ *converges absolutely if the series* $\sum |a_n|$ *converges.*

Theorem 1.4.1 (Abs Convergence is Stronger)**.** *If a series converges absolutely, then it converges.*

Proof. $-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$. By comparision test, let $b_n = a_n + |a_n|$, then $\sum a_n = \sum \big(b_n - |a_n|\big)$ converges. □

Theorem 1.4.2 (Riemann Rearrangement)**.** *If the series* $\sum a_n$ *absolutely converges and the sequence* b_n *is obtained by a permutation of* a_n*, then* $\sum b_n$ *also absolutely converges and* $\sum b_n = \sum a_n$*.*

Theorem 1.4.3 (d'Alambert Criteria)**.** *Given a sequence* $\{a_n\}$*:*

1. *If* $\lim \left|\dfrac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum a_n$ *converges absolutely.*
2. *If* $\lim \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ *(including wide sense) then* $\sum a_n$ *diverges.*

Proof. By definition of limit,

1. $\exists\, N \in \mathbb{N} \,:\, \forall\, n \geq N\,,\ |a_{n+1}| \leq r|a_n|$, where $r < 1$. By induction, $\forall\, n \geq N\,,\ |a_n| \leq r^{n-N}|a_N|$. By comparision test with $r^{n-N}|a_N|$, $\sum |a_n|$ converges.
2. $\exists\, N \in \mathbb{N} : \forall\, n \geq N\,,\ |a_{n+1}| \geq |a_n| > 0 \Rightarrow \lim a_n \neq 0 \Rightarrow \sum a_n$ diverges.

□

Lemma 1.4.1. *Given a sequence* $\{a_n\}$*:*

1. *If* $\limsup \left|\dfrac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum a_n$ *converges absolutely.*

2. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$ *(including wide sense) then* $\sum a_n$ *diverges.*

Theorem 1.4.4 (Cauchy Criteria)**.** *Given a sequence* $\{a_n\}$*:*

1. *If* $\limsup \sqrt[n]{|a_n|} < 1$ *then* $\sum a_n$ *converges absolutely.*
2. *If* $\limsup \sqrt[n]{|a_n|} > 1$ *(including wide sense) then* $\sum a_n$ *diverges.*

Proof. By definition of limit superior, for infinetly many n :

1. $\sqrt[n]{|a_n|} \leq r$, that is, $|a_n| \leq r^n$, where $r < 1$. By comparision with r^n, $\sum |a_n|$ converges.
2. $|a_n| \geq 1 \Rightarrow \lim a_n \neq 0 \Rightarrow \sum a_n$ diverges.

□

Definition 1.4.2 (Power Series)**.** *A series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *in the variable* x *is called a power series.*

Lemma 1.4.2 (Radius of Convergence)**.** *Define*

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

if the RHS is finite and 0 *otherwise. Then,*

$$\forall\, x \in (a-R, a+R)\,,\ \sum_{n=0}^{\infty} a_n(x-a)^n \ \textit{converges}$$

$$\sum_{n=0}^{\infty} a_n(x-a)^n \ \textit{converges} \ \Rightarrow x \in [a-R, a+R]$$

Proof. It is a direct result of Cauchy's Criteria. The second proposition takes into account when the test is inconclusive. □

Remark 1.4.1. *We have to check the endpoints* $x = a \pm R$ *separately.*

2 Functions

2.1 Basic Definitions

Definition 2.1.1 (Functions)**.** *Let $\mathcal{D}$ and $\mathcal{R}$ be two subsets of $\mathbb{R}$. A function from $\mathcal{D}$ to $\mathcal{R}$ is a weell-defined law which, to each $x \in \mathcal{D}$, there is a unique number $y \in \mathcal{R}$. The set $\mathcal{D}$ is called the domain of f and the set $\mathcal{R}$ is caleld the range of f. Denote $f : \mathcal{D} \to \mathcal{R}$ or $y = f(x)$. The variable x is called the* ***independent variable*** *and y is a* ***dependent variable****. The variable x is also called the* ***origin*** *of y and y is the* ***image*** *of x.*

A set of points $\big\{\big(\, x, f(x)\,\big) \,\big|\, x \in \mathcal{D}\big\}$ in the plane $\mathbb{R}^2$ is called a graph of a function $y = f(x)$ (denoted $G(f)$ or only f).

Definition 2.1.2 (Image and Domain)**.** *For $f : \mathcal{D} \to \mathcal{R}$, the image is:*

$$\operatorname{Im}(f) = \{y \in \mathcal{R} \mid \exists\, x \in \mathcal{D} : y = f(x)\} = \{f(x) \mid x \in \mathcal{D}\}$$

i.e. the smallest possible range. We always have: $\operatorname{Im}(f) \subseteq \mathcal{R}$*.*

The biggest possible domain of f is called the ***existence domain*** *of f.*

Definition 2.1.3 (Parity)**.** *Given $f : \mathcal{D} \to \mathcal{R}$ such that $\mathcal{D}$ is symmetric, that is, $\forall x \in \mathcal{D},\ -x \in \mathcal{D}$, the function is called:*

- *Even:* $\forall\, x \in \mathcal{D}\,,\ f(-x) = f(x)$
- *Odd:* $\forall\, x \in \mathcal{D}\,,\ f(-x) = -f(x)$

Definition 2.1.4 (Periodicity)**.** *Given $f : \mathcal{D} \to \mathcal{R}$ is called periodical if:*

$$\exists\, T \neq 0 : \forall\, x \in \mathcal{D}\,,\ x + T \in \mathcal{D} \quad \textit{and} \quad f(x+T) = f(x)$$

Furthermore, T is called a period of f. The smallest such $T > 0$ (if it exists) is called ***the*** *period of f.*

Definition 2.1.5 (Monotonicity)**.** *Given $f : \mathcal{D} \to \mathcal{R}$ is called (eq. stricly) monotonic increasing if:*

$$\forall\, x, y \in \mathcal{D}\,,\ x < y \Rightarrow f(x) \leq f(y)\ \big(\textit{eq. } f(x) < f(y)\big)$$

And is called (eq. stricly) monotonic decreasing if:

$$\forall\, x, y \in \mathcal{D}\,,\, x < y \Rightarrow f(x) \geq f(y)\ \big(eq.\ f(x) > f(y)\big)$$

Definition 2.1.6 (Injectivity)**.** *Given $f : \mathcal{D} \to \mathcal{R}$ is called injective or one-to-one if:*

$$\forall\, x, y \in \mathcal{D}\,,\, x \neq y \Rightarrow f(x) \neq f(y)$$

equivalently, if:

$$\forall\, y \in \operatorname{Im}(f)\,,\, \exists!\, x \in \mathcal{D} : y = f(x)$$

Lemma 2.1.1 (Str Monotone is Inj)**.** *$f : \mathcal{D} \to \mathcal{R}$ is strictly monotonic, then it is injective.*

Proof. WLOG, f is stricly monotonic increasing. If $f(x) = f(y)$, then, we have three cases. If $x > y$, $f(x) > f(y)$. If $y > x$, $f(y) > f(x)$. None of these are true, so were left with $x = y$. $\square$

Definition 2.1.7 (Surjectivity)**.** *Given $f : \mathcal{D} \to \mathcal{R}$ is called surjective or onto if: $\mathcal{R} = \operatorname{Im}(f)$*

Definition 2.1.8 (Inverse Function)**.** *Given $f : \mathcal{D}(f) \to \operatorname{Im}(f)$ is one-to-one, one can define a function $g : \operatorname{Im}(f) \to \mathcal{D}(f)$ by $g(y) = x$, where x is the unique value such that $y = f(x)$. Therefore, $g(f(x)) = x$. The function g is called **inverse function** of f. Notation: $g = f^{-1}$. Notice that: $\mathcal{D}(f^{-1}) = \operatorname{Im}(f)$ and $\operatorname{Im}(f^{-1}) = \mathcal{D}(f)$.*

The graph of f and f^{-1} are symmetric with respect to the line $y = x$.

Definition 2.1.9 (Composition)**.** *Let $f : \mathcal{D}(f) \to \mathcal{R}$ and $g : \mathcal{D}(g) \to \mathcal{S}$, the composition $h = g \circ f : \mathcal{D}(h) \to \mathcal{S}$ is defined as follows:*

$$\mathcal{D}(h) = \{x \in \mathcal{D}(f) \mid f(x) \in \mathcal{D}(g)\}$$
$$\forall\, x \in \mathcal{D}(f)\,,\ h(x) = g(f(x))$$

where $\mathcal{D}(h) \neq \varnothing$.

2.2 Limits and Continuity

Definition 2.2.1 (Limit of a Function)**.** *Let f be defined in an open interval $\mathcal{U}$ about a, except possibly at a itself ("except"). A number L is called the limit of f at point a if:*

$$\text{(Heine) } \forall \{x_n\} \in \mathcal{U}^{\mathbb{N}}\,,\; x_n \to a \Rightarrow f(x_n) \to L$$

$$\text{(Cauchy) } \forall \epsilon > 0\,,\, \exists \delta > 0 : \forall x \in \mathcal{U}\,,\; 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

We denote $L = \lim\limits_{x\to a} f(x)$. Notice, by Heine, the limit is unique, if it exists.

Lemma 2.2.1 (H&C)**.** *The Heine definition and the Cauchy definition of the limit are equivalent.*

Proof. We prove each direction:

($\Rightarrow$) By contrary, suppose $\exists \epsilon > 0 : \forall \delta > 0\,,\, \exists x \in \mathcal{U} : 0 < |x-a| < \delta \Rightarrow |f(x) - L| \geq \epsilon$. Define the sequence $\{x_n\} \in \mathcal{U}$ by picking x_n such that $|x_n - a| < \dfrac{1}{n} \Rightarrow |f(x) - L| \geq \epsilon$. Hence $x_n \to a$ since $a - \dfrac{1}{n} < x_n < a + \dfrac{1}{n}$, by sandwich theorem. Therefore $f(x_n) \not\to L$, by definition, so Heine does not hold.

($\Leftarrow$) By contrary, suppose $\exists \{x_n\} \in \mathcal{U}^{\mathbb{N}} : x_n \to a$, but $f(x_n) \not\to L$. By definition of the limits:

$$\forall \delta > 0\,,\, \exists N \in \mathbb{N} : \forall n \geq N\,, |x_n - a| < \delta$$
$$\exists \epsilon > 0 : \forall N \in \mathbb{N}\,,\, \exists n_0 \geq N : |f(x_{n_0}) - L| \geq \epsilon$$

Hence $|x_{n_0} - a| < \delta \not\Rightarrow |f(x_{n_0}) - L| < \epsilon$, so Cauchy does not hold.

$\square$

Definition 2.2.2 (Infinite Limit of Functions)**.** *Let f be defined in an open interval $\mathcal{U}$ about a, "except". We say $f(x) \to \infty$ (eq. $-\infty$) at $x = a$ if: (Cauchy)*

$$\forall M > 0\,,\, \exists \delta > 0 : \forall x \in \mathcal{U}\,,\; 0 < |x-a| < \delta \Rightarrow f(x) > M \big(\textit{eq. } f(x) < M\big)$$

$$\text{(Heine) } \forall \{x_n\} \in \mathcal{U}^{\mathbb{N}}\,,\; x_n \to a \Rightarrow f(x_n) \to \infty \;\big(\textit{eq. } -\infty\big)$$

Definition 2.2.3 (Limit at Infinity)**.** *Let f be defined in an unbounded open interval (a, ∞) or $(-\infty, a)$. A number L is called the limit of f at ∞ (eq. $-\infty$) if: (Cauchy)*

$$\forall \epsilon > 0 \,,\, \exists M > 0 : \forall x \in \mathcal{U} \,,\, x > M \, \big(eq. \; x < M\big) \Rightarrow |f(x) - L| < \epsilon$$

(Heine) $\forall \{x_n\} \in \mathcal{U}^{\mathbb{N}} \,,\, x_n \to \infty \, \big(eq. \; -\infty\big) \Rightarrow f(x_n) \to L$

Lemma 2.2.2. *A periodic function $f : \mathbb{R} \to \mathbb{R}$ has no limit at $\pm\infty$.*

Proof. Let T be the period of f. Since f is non-constant, $\exists\, x, y \in \mathbb{R} : f(x) \neq f(y)$. Take $x_n = x + nT$ and $y_n = y + nT$, we get $\forall\, n \in \mathbb{N} \,,\, f(x_n) = f(x) \neq f(y) = f(y_n)$. Hence $x_n, y_n \to \infty$, but $f(x_n) \to f(x) \neq f(y) \leftarrow f(y_n)$ ☐

Theorem 2.2.1 (Sandwich/Squeeze Theorem)**.** *Given three functions defined on an open interval $\mathcal{U}$ about a, "except". If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$ and $\forall\, x \in \mathcal{U} \setminus \{a\} \,,\, f(x) \leq g(x) \leq h(x)$, then, $\lim\limits_{x \to a} g(x) = L$*

Proof. Heine definition and the Sandwich Theorem for sequences. ☐

Definition 2.2.4 (Continuity)**.** *Let f be defined on an open interval $\mathcal{U}$ around a. We say that f is continuous at point a if:*

$$\lim_{x \to a} f(x) = f(a)$$

If $\forall\, a \in \mathcal{U} \,,\, f$ is continuous at a, then f is continuous at $\mathcal{U}$.

Lemma 2.2.3 (Arithmetic of Limits of functions)**.** *Suppose $\lim\limits_{x \to a} f(x) = F$, $\lim\limits_{x \to a} g(x) = G$ and C is a constant:*

1. $\lim\limits_{x \to a} c = c, \;\; \lim\limits_{x \to a} (c \cdot f(x)) = c \cdot F$
2. $\lim\limits_{x \to a} (f(x) \pm g(x)) = F \pm G$
3. $\lim\limits_{x \to a} (f(x) \cdot g(x)) = F \cdot G$
4. *If $G \neq 0$ and $g(x) \neq 0$,* $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{F}{G}$

Proof. Heine definition and the previous results for sequences. $\square$

Corollary 2.2.1 (Arithmetic of Continuity)**.** *If f and g are continuous, then: $c \cdot f(x)$, $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\dfrac{f(x)}{g(x)}$ are continuous. Further $f \circ g$ is continuous.*

Definition 2.2.5 (Limit from the Side)**.** *We say $x_n \to a^+$ iff $x_n \to a$ and $\forall\, n \in \mathbb{N}\,,\, x_n > a$. Analogously, $x_n \to a^-$ iff $x_n \to a$ and $\forall\, n \in \mathbb{N}\,,\, x_n < a$.*

Definition 2.2.6 (One-Sided Limits)**.** *We define:*

(From the right) $\lim\limits_{x \to a^+} f(x) = L$ *iff: (Cauchy)*

$$\forall\, \epsilon > 0\,,\, \exists\, \delta > 0 : \forall\, x \in \mathcal{U}\,,\; a < x < a + \delta \Rightarrow |f(x) - L| < \epsilon$$

(Heine) $\forall\, \{x_n\} \in \mathcal{U}^{\mathbb{N}}\,,\; x_n \to a^+ \Rightarrow f(x_n) \to L$

(From the left) $\lim\limits_{x \to a^-} f(x) = L$ *iff: (Cauchy)*

$$\forall\, \epsilon > 0\,,\, \exists\, \delta > 0 : \forall\, x \in \mathcal{U}\,,\; a - \delta < x < a \Rightarrow |f(x) - L| < \epsilon$$

(Heine) $\forall\, \{x_n\} \in \mathcal{U}^{\mathbb{N}}\,,\; x_n \to a^- \Rightarrow f(x_n) \to L$

Remark 2.2.1. *The proof of the equivalency of the definitions is the same.*

Definition 2.2.7 (Continuity on Closed Sets)**.** *We say f is continuous on $[a, b]$ if it is continuous on $\mathcal{U} = (a, b)$ and both $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$.*

Theorem 2.2.2 (Two-Sided Limit Criteria)**.** *Let f be defined on an open interval $\mathcal{U}$ around a, "except". Then:*

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Proof. The ($\Rightarrow$) direction is trivial. For the ($\Leftarrow$) let $x_n \to a$, define $\{x_n^+\}$ as the (largest) subsequence of x_n such that $\forall\, n \in \mathbb{N}\,, , x_n^+ > a$. Define x_n^- analogously. Then, $x_n^+ \to a^+$ and $x_n^- \to a^-$. By Heine, $f(x_n^+) \to L$ and $f(x_n^-) \to L$. Since $\{x_n\} = \{x_n^+\} \cup \{x_n^-\}$, we get: $f(x_n) \to L$. $\square$

2.3 Classifing Continuity

Definition 2.3.1 (Uniform Continuity)**.** *A function $f : \mathcal{U} \to \mathbb{R}$ is uniformly continuous iff: (Cauchy)*

$$\forall \epsilon > 0\,,\, \exists \delta > 0 : \forall x, y \in \mathcal{U}\,,\, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Remark 2.3.1. *The main difference between uniform continuity and regular continuity is that the δ is chosen independent of x and y, whereas the δ in regular continuity is, in general, dependent on the choice of a.*

Theorem 2.3.1 (UC $\Rightarrow C^0$)**.** *If f is uniformly continuous on $\mathcal{U}$, then it is continuous on $\mathcal{U}$.*

Proof. $a \in \mathcal{U} \Rightarrow \forall \epsilon > 0\,,\, \exists \delta > 0 : \forall x \in \mathcal{U}\,,\, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$, hence, by definition $\lim\limits_{x \to a} f(x) = f(a)$. □

Theorem 2.3.2 (Heine-Cantor)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.*

Proof. By contrary,

$$\exists \epsilon > 0 : \forall \delta > 0\,,\, \exists x, y \in [a, b] : |x - y| < \delta \Rightarrow |f(x) - f(y)| \geq \epsilon$$

Define $\{x_n\}$ and $\{y_n\}$ such that $\forall n \in \mathbb{N}\,,\,, |x_n - y_n| < \dfrac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$. Since x_n is bounded, there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit we denote $x_0 \in [a, b]$. By Triangle Inequality,

$$|y_{n_k} - x_0| \leq |x_{n_k} - x_0| + |x_{n_k} - y_{n_k}| \leq |x_{n_k} - x_0| + \frac{1}{n_k}$$

hence $y_{n_k} \to x_0$. Since f is continuous, $\lim\limits_{k \to \infty} f(x_{n_k}) = f(x_0) = \lim\limits_{k \to \infty} f(y_{n_k})$. But $\forall k \in \mathbb{N}\,,\, |f(x_{n_k}) - f(y_{n_k})| \geq \epsilon > 0$. By taking $k \to \infty$, there is a contradiction. □

Lemma 2.3.1. *If $f : [a, \infty) \to \mathbb{R}$ is continuous and $\lim\limits_{x \to \infty} f(x)$ exists and is finite, then, f is uniformly continuous.*

Proof. Let $\epsilon > 0$. By definition,

$$\exists\, M \in \mathbb{R} : \forall\, x \in \mathcal{U}\,,\; x > M \Rightarrow |f(x) - L| < \frac{\epsilon}{3}$$

Then, $\forall\, x, y > M\,,\; |f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \dfrac{2\epsilon}{3} < \epsilon$. Since $[a, M]$ is compact, f is UC there, so,

$$\exists\, \delta > 0 : \forall\, x, y \in [a, M]\,,\; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$$

$\forall\, x \in [a, M]\,,\; \forall\, y > M\,,\; |x - y| < \delta \Rightarrow |x - M| < \delta$ and $|y - M| < \delta \Rightarrow$ $|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| < \epsilon$. $\square$

Definition 2.3.2 (Lipschitz Continuity)**.** *Let $f : \mathcal{U} \to \mathbb{R}$ is Lipschitz continuous (on $\mathcal{U}$) if:*

$$\exists\, K > 0 : \forall\, x, y \in \mathcal{U}\,,\; |f(x) - f(y)| < K|x - y|$$

Theorem 2.3.3 (LpC $\Rightarrow$ UC)**.** *Let $f : \mathcal{U} \to \mathbb{R}$ is Lipschitz continuous (on $\mathcal{U}$), then f is uniformly continuous on $\mathcal{U}$.*

Proof. Choose $\delta(\epsilon) = \dfrac{\epsilon}{K}$ on the definition. $\square$

Remark 2.3.2. *The converse is not true. Counterexample: $f(x) = \sqrt{|x|}$ is uniformly continuous on $\mathbb{R}$, but not Lipschitz continuous (cusp at $x = 0$).*

Definition 2.3.3 (Types of Discontinuity)**.** *Let f be defined on an open interval about a, "except".*

The point a is a ***removable*** *discontinuity point of f if:*

(a) $\exists \lim_{x \to a} f(x)$

(b) $\lim_{x \to a} f(x) \neq f(a)$ *or $f(a)$ is not defined.*

The point a is a ***jump/first kind*** *discontinuity point of f if:*

(a) $\exists \lim_{x \to a^-} f(x)$ *and* $\exists \lim_{x \to a^+} f(x)$

(b) $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

The point a is a ***essential/second kind*** *discontinuity point of f if: At least one of $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ does not exist.*

2.4 Continuity Theorems

Definition 2.4.1 (Boundedness)**.** *A function f is bounded on $\mathcal{D}$ if*

$$\exists\, M > 0 : \forall\, x \in \mathcal{D}\,,\, |f(x)| \leq M$$

or equivalently, $\exists\, m, M \in \mathbb{R} : \forall\, x \in \mathcal{D}\,,\, m \leq f(x) \leq M$

Theorem 2.4.1. *Suppose that $g(x)$ is bounded and $\lim\limits_{x\to a} f(x) = 0$. Then, $\lim\limits_{x\to a} \big[f(x) \cdot g(x)\big] = 0$.*

Proof. Heine definition and the previous lemma for sequences. □

Lemma 2.4.1 (Boundedness Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous, then f is bounded.*

Proof. By contrary, suppose f is not bounded from above on $[a, b]$. Then, $\forall\, n \in \mathbb{N}\,,\,, \exists\, x_n \in [a, b] : f(x_n) > n$. This defines a sequence $\{x\}_{n=1}^{\infty}$. Because $[a, b]$ is bounded,by Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$. But $\forall\, k \in \mathbb{N}\,,\, f(x_{n_k}) > n_k \geq k$ which implies that $f(x_{n_k}) \to \infty$, contradiction. Therefore, f is bounded from above on $[a, b]$. Analogously, to prove it is bounded from below. □

Theorem 2.4.2 (Weierstrass/Extreme Value Theorem)**.** *Let f be continuous on the closed interval $[a, b]$. Then f gets its maximal and minimal values on $[a, b]$.*

Proof. We prove for the maximal, since the minimal is analogous. Since f is bounded, take $M = \sup\{f(x) \mid x \in [a, b]\}$. We just need to show $\exists\, x_0 \in [a, b] : f(x_0) = M$.

Take $f(x_n) = M - \dfrac{1}{n}$, so that $f(x_n) \to M$. $\{x_n\}$ is bounded, so by Bolzano-Weierstrass, there is $\{x_{n_k}\}_{k=1}^{\infty}$ with limit $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0) = M$. □

Theorem 2.4.3 (Cauchy/Intermediate Value Theorem)**.** *Let f be continuous on $[a, b]$ and let $y \in \mathbb{R}\ :\ f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then*

$$\exists\, c \in (a, b) : f(c) = y$$

Proof. We will prove for $f(a) < y < f(b)$. Let $S = \{x \in [a, b] \mid f(x) \leq y\}$. Then S is non-empty since $a \in S$, and S is bounded above by b. Let $c = \sup S$ so that $c \in (a, b)$. Let's prove that $f(c) = y$.

Let $\epsilon > 0$. Since f is continuous, $\exists\, \delta > 0\ :\ \forall\, x \in [a, b]\,,\ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. That is, $\forall\, x \in (c - \delta, c + \delta)\,,\ f(x) - \epsilon < f(c) < f(x) + \epsilon$. By the properties of the supremum, there exists some $\alpha \in (c - \delta, c]$ that is contained in S, and so $f(c) < f(\alpha) + \epsilon \leq y + \epsilon$. Picking $\beta \in (c, c + \delta)$, we know that $\beta \notin S$ because c is the supremum of S. This means that $f(c) > f(\beta) - \epsilon > y - \epsilon$ Since, $\forall\, \epsilon > 0\,,\ y - \epsilon < f(c) \leq y + \epsilon$, we get $f(c) = y$. ☐

Corollary 2.4.1. *Let f be continuous on $[a, b]$. If $f(a) \cdot f(b) < 0$, then $\exists\, c \in (a, b) : f(c) = 0$*

Theorem 2.4.4 (Brouwer's Fixed Point Theorem)**.** *$f : [a, b] \to [a, b]$ continuous, then $\exists\, x_0 \in [a, b] : f(x_0) = x_0$*

Proof. Let $g(x) = f(x) - x$. Then, we get: $g(b) = f(b) - b \leq b - b = 0$ and $g(a) = f(a) - a \geq a - a = 0$. Then, either $g(b) = 0$, or $g(a) = 0$ or $g(b) < 0 < g(a)$, so by IVT, $\exists\, c \in (a, b) : g(c) = 0$. Either way, $\exists\, c \in [a, b] : g(c) = 0$. ☐

Corollary 2.4.2. *Let f be continuous on $[a, b]$. Then $\mathrm{Im}(f)$ is a closed interval $[m, M]$, where M is the maximum of f and m is the minimum.*

Definition 2.4.2 (Intermediate Value Property)**.** *$f : [a, b] \to \mathbb{R}$ has intermediate value property if, for any $y \in \mathbb{R}\ :\ f(a) < y < f(b)$ or $f(a) > y > f(b)$, then*

$$\exists\, c \in (a, b) : f(c) = y$$

Theorem 2.4.5 (Darboux)**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b), then f' has the intermediate value property in $[\alpha, \beta] \subset (a, b)$.*

Proof. Supose $f'(\alpha) > y > f'(\beta)$. Let $g : [a,b] \to \mathbb{R}$ such that $g(x) = f(x) - yx$. Since g is continuous on $[\alpha, \beta]$, by Weierstrass (EVT), g attains its maximum in $[\alpha, \beta]$.

Because $g'(\alpha) = f'(\alpha) - y > 0$, we know g cannot attain its maximum value at α. Likewise, because $g'(b) = f'(b) - y < 0$, we know g cannot attain its maximum value at β.

Therefore, g must attain its maximum value at some point $x \in (a,b)$. Hence, by Fermat's theorem, $g'(x) = 0$, i.e. $f'(x) = y$ $\square$

Corollary 2.4.3. *If $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b), then there are no removable or jump discontinuity. Hence if $\exists \lim_{x\to c^-} f'(x)$ and $\exists \lim_{x \to c^+} f'(x)$ for $c \in (a,b)$, then $\lim_{x\to c^+} f'(x) = \lim_{x\to c^+} f'(x) = f'(c)$.*

3 Differential Calculus

3.1 Derivatives

Definition 3.1.1 (Derivative). *Given a function $f : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$, the derivative a point $a \in \mathcal{D}$, so that there is an open interval in $\mathcal{D}$ around a, is defined as:*

$$f'(a) = \lim_{x \to a} = \frac{f(x) - f(a)}{x - a}$$

If the limit exists, f is differentiable at a. We can also take the derivative at each point and get a function $f' : \mathcal{D}' \subseteq \mathcal{D} \to \mathbb{R}$ such that:

$$f'(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}$$

$\mathcal{D}'$ is the domain of differentiability.

Remark 3.1.1. *The line tangent to the curve of the graph of f has slope $f'(a)$ at point a. Actually the tangent line has formula $y = f'(a) \cdot \big(x - a\big) + f(a)$. Furthermore, the normal line to the curve (if $f'(a) \neq 0$) has equation $y = -\dfrac{1}{f'(a)} \cdot \big(x - a\big) + f(a)$*

Lemma 3.1.1 (Infinetesimal Function). *Let f be defined on an open interval $\mathcal{U}$ about a. Then, f is differentiable at a iff $\exists\, L \in \mathbb{R}$ and $\exists\, \varphi : \mathcal{U} \to \mathbb{R}$ with $\lim_{x \to a} \varphi(x) = 0$, such that: $\dfrac{f(x) - f(a)}{x - a} = L + \varphi(x)$. In that case, $L = f'(a)$.*

Theorem 3.1.1. *If f is differentiable at a, then f is continuous at a.*

Proof. By the previous lema,

$$f(x) = f(a) + \Big(f'(a) + \varphi(x)\Big) \cdot (x - a) \Rightarrow \lim_{x \to a} f(x) = f(a)$$

$\square$

Remark 3.1.2. *The derivative function might not be continuous. Example: $f(x) = x^2 sin(1/x)$ is differentiable everywhere, $f'(x)$ is not continuous at 0.*

Lemma 3.1.2 (Product Rule and Arithmetic of Limits). *If $\exists\, f'(x)$ and $\exists\, g'(x)$:*

1. $(c \cdot f(x))' = c \cdot f'(x)$
2. $(f(x) + g(x))' = f'(x) + g'(x)$
3. *(Product Rule)* $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. *(Quotient Rule)* $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$

Proof. The first two follow from linearity. We will only show the product rule.

$$\begin{aligned}
(f \cdot g)'(x) &= \lim_{\delta \to 0} \frac{f(x+\delta) \cdot g(x+\delta) - f(x) \cdot g(x)}{\delta} = \\
&= \lim_{\delta \to 0} \left[f(x+\delta) \cdot \frac{g(x+\delta) - g(x)}{\delta} + \frac{f(x+\delta) - f(x)}{\delta} \cdot g(x) \right] \\
&= f'(x) \cdot g(x) + f(x) \cdot g'(x)
\end{aligned}$$

□

Corollary 3.1.1 (Power Rule). $p(x) = \sum_{k=0}^{n} a_k \cdot x^k \Rightarrow p'(x) = \sum_{k=1}^{n} k \cdot a_k \cdot x^{k-1}$

Lemma 3.1.3. *Let f be an invertible and continuous in an open interval about a. If $\exists\, f'(a)$ and $f'(a) \neq 0$, then, the inverse function f^{-1} is differentiable at $b = f(a)$ and:*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = \frac{1}{f'\left(f^{-1}(b)\right)}$$

Proof. By definition of the derivative:

$$\begin{aligned}
\left(f^{-1}\right)'(b) &= \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{f\left(f^{-1}(y)\right) - f\left(f^{-1}(b)\right)} \\
&= \left(\lim_{y \to b} \frac{f\left(f^{-1}(y)\right) - f\left(f^{-1}(b)\right)}{f^{-1}(y) - f^{-1}(b)} \right)^{-1}
\end{aligned}$$

Since f is bijective and continuous, $f^{-1}(y) \to f^{-1}(b)$ as $y \to b$. Then, since $x = f^{-1}(y)$ and $a = f^{-1}(b)$:

$$\big(f^{-1}\big)'(b) = \left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right)^{-1} = \Big(f'(a)\Big)^{-1} = \Big(f'\big(f^{-1}(b)\big)\Big)^{-1}$$

□

Theorem 3.1.2 (Chain Rule)**.** *Let g be differentiable in an open interval about a and f be differentiable in an open interval about $b = g(a)$. Then, $f \circ g$ is differentiable at a and:*

$$\big(f \circ g\big)'(a) = f'\big(g(a)\big) \cdot g'(a) = \big(f' \circ g\big)(a) \cdot g'(a)$$

Proof. We can represent:

$$\begin{aligned} \frac{g(x) - g(a)}{x - a} &= g'(a) + \alpha(x) \\ \frac{f(y) - f(a)}{y - b} &= f'\big(b\big) + \beta(y) \end{aligned}$$

Putting it together, we get:

$$\begin{aligned} f\big(g(x)\big) - f\big(g(a)\big) &= \Big(f'\big(g(a)\big) + \beta\big(g(x)\big)\Big) \cdot \Big(g(x) - g(a)\Big) \\ &= \Big(f'\big(g(a)\big) + \beta\big(g(x)\big)\Big) \cdot \Big(g'(a) + \alpha(x)\Big) \cdot \big(x - a\big) \end{aligned}$$

By definition of the derivative: $(f \circ g)'(a) =$

$$= \lim_{x\to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{x - a} = \lim_{x\to a} \Big(f'\big(g(a)\big) + \beta\big(g(x)\big)\Big) \cdot \Big(g'(a) + \alpha(x)\Big)$$

Since $\lim\limits_{x\to a} \alpha(x) = 0$ and $\lim\limits_{x\to a} \beta\big(g(x)\big) = \lim\limits_{y\to b} \beta(y) = 0$, we get:

$$\left.\begin{cases} \lim\limits_{x\to a} \Big(f'\big(g(a)\big) + \beta\big(g(x)\big)\Big) = f'\big(g(a)\big) \\ \lim\limits_{x\to a} \Big(g'(a) + \alpha(x)\Big) = g'(a) \end{cases}\right\} \Rightarrow \big(f \circ g\big)'(a) = f'\big(g(a)\big) \cdot g'(a)$$

□

3.2 Mean Value Theorems and Taylor

Theorem 3.2.1 (Fermat)**.** *Let f be defined on an open interval $\mathcal{U}$ and let f be differentiable at $x_0 \in \mathcal{U}$. If f gets its maximal (or minimal) value at x_0, then: $f'(x_0) = 0$*

Proof. WLOG $f(x_0)$ is the maximal value of f at $\mathcal{U} = (a, b)$. Then:

$$\forall\, \delta \in (a - x_0, b + x_0)\,,\; f(x_0 + \delta) \leq f(x_0)$$

- $\delta > 0: \; \dfrac{f(x_0+\delta) - f(x_0)}{\delta} \leq 0 \Rightarrow \lim\limits_{\delta \to 0^+} \dfrac{f(x_0+\delta) - f(x_0)}{\delta} \leq 0$
- $\delta < 0: \; \dfrac{f(x_0+\delta) - f(x_0)}{\delta} \geq 0 \Rightarrow \lim\limits_{\delta \to 0^-} \dfrac{f(x_0+\delta) - f(x_0)}{\delta} \geq 0$

Hence, since the derivate exists, the one sided limits are equal:

$$0 \leq \lim_{\delta \to 0^-} \frac{f(x_0+\delta) - f(x_0)}{\delta} = f'(x_0) = \lim_{\delta \to 0^+} \frac{f(x_0+\delta) - f(x_0)}{\delta} \leq 0$$

$\Rightarrow f'(x_0) = 0.$ □

Theorem 3.2.2 (Rolle)**.** *Let f be defined on an closed interval $[a, b]$. If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then, $\exists\, c \in (a, b) : f'(c) = 0$.*

Proof. Since $[a, b]$ is compact (closed and bounded), f is bounded. Using Weierstrass Theorem, f gets its maximum M and minimum m on $[a, b]$. Consider the following possibilities:

- $m = M : \Rightarrow f(x) = \text{const.} \Rightarrow \forall\, c \in (a, b)\,,\; f'(x) = 0.$
- $m < M : \Rightarrow$ at least one of $\{m, M\}$ are obtained in (a, b), otherwise $f(a) \neq f(b)$. Therefore, by Fermat's Theorem, let c be the value for minimum or maximum, we have: $f'(c) = 0$.

□

Theorem 3.2.3 (Lagrange/Mean Value Theorem)**.** *Let f be defined on an closed interval $[a, b]$. If f is continuous on $[a, b]$ and differentiable on (a, b), then,*

$$\exists\, c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let $g(x) = f(a) - f(x) + \dfrac{f(b) - f(a)}{b - a} \cdot (x - a)$, so that $g(a) = g(b) = 0$. Also, notice that $g'(x) = -f'(x) + \dfrac{f(b) - f(a)}{b - a}$. Then, by Rolle's Theorem, $\exists\, c \in (a, b)$:

$$0 = g'(c) = -f'(c) + \frac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\square$

Theorem 3.2.4 (Cauchy Mean Value Theorem)**.** *Let f and g be defined on an closed interval $[a, b]$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then,*

$$\exists\, c \in (a, b) : f'(c) \cdot \big(g(b) - g(a)\big) = g'(c) \cdot \big(f(b) - f(a)\big)$$

Proof. Let $h(x) = \big(f(x) - f(a)\big) \cdot \big(g(b) - g(a)\big) - \big(f(b) - f(a)\big) \cdot \big(g(x) - g(a)\big)$, so that $h(a) = h(b) = 0$. Also, notice that $h'(x) = f'(x) \cdot \big(g(b) - g(a)\big) - \big(f(b) - f(a)\big) \cdot g'(x)$. Then, by Rolle's Theorem, $\exists\, c \in (a, b)$:

$$0 = h'(c) = f'(c) \cdot \big(g(b) - g(a)\big) - \big(f(b) - f(a)\big) \cdot g'(c)$$

$\Rightarrow f'(c) \cdot \big(g(b) - g(a)\big) = g'(c) \cdot \big(f(b) - f(a)\big)$ $\square$

Definition 3.2.1 (Higher Order Derivatives)**.** *Suppose that f is differentiable. If f' is differentiable, we say that f is* ***twice*** *differentiable. Notation: $f''(x) = (f')'(x)$. It is called the second derivative of f. Similarly, we define the n-th derivative, denoted $f^{(n)}(x)$.*

Definition 3.2.2 (C^k classes)**.** *We define $f \in C^0$ if f is continuous. Further, we define $f \in C^k \Leftrightarrow f' \in C^{k-1}$. If f is infinetly differentiable, we write C^∞.*

Theorem 3.2.5 (Taylor's Formula)**.** *Let f be $n+1$ times differentiable (with $n \in \mathbb{N}_0$) on an open interval $\mathcal{U}$ about a and let $x \in \mathcal{U}$. Then, $\exists\, c$ between a and x, which might depend on x, such that:*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(x)$$

With $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$ *, called the Lagrange Remainder.*

Proof. Denote $T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$. Fixing x, let:

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} (x-t)^k \; - \Big[f(x) - T_n(x)\Big] \frac{(x-t)^{n+1}}{(x-a)^{n+1}}$$

Notice that g is continuous between a and x (on $[a,x]$ or $[x,a]$). Also, $g(a) = g(x) = 0$. Moreover, g is differentiable between a and x with:

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \; + (n+1) \Big[f(x) - T_n(x)\Big] \frac{(x-t)^n}{(x-a)^{n+1}}$$

Hence, by Rolle's Theorem, $\exists\, c$ between a and x such that:

$$0 = g'(c) = -\frac{f^{(n+1)}(t)}{n!} (x-c)^n \; + (n+1) \Big[f(x) - T_n(x)\Big] \frac{(x-c)^n}{(x-a)^{n+1}}$$

$$\Leftrightarrow f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} = R_n(x)$$

□

Remark 3.2.1. *In order of stronger to weaker: Taylor $>$ Lagrange $>$ Rolle.*

Definition 3.2.3 (Analytical Function)**.** *If $\forall\, x \in \mathcal{U}$, $\lim_{n\to\infty} R_n(x) = 0$ and $f \in C^\infty$, we have:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

the Taylor Series. If $a = 0$, the series is called MacLaurin Series. Then, we call f an analytical function (denoted $f \in C^\omega$), that is, it's Taylor Series converges pointwise to f in some open interval.

Remark 3.2.2 (Error)**.** *If we want to calculate with an error of less than* ε *we only need to find* n *such that:*

$$\Big|R_n(x)\Big| = \frac{|f^{(n+1)}(c)| \cdot |x|^{n+1}}{(n+1)!} < \varepsilon$$

Theorem 3.2.6 (L'Hôpital's Rule)**.** *Suppose that* f *and* g *are continuous on a closed interval* $[a, b]$*, and are differentiable on the open interval* (a, b)*. Suppose that* $g'(x)$ *is never zero on* (a, b)*, and that* $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)}$ *exists, and that* $f(a) = g(a) = 0$*. Then,*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

It is also true for $x \to a^-$*, so also true for* $x \to a$*.*

Proof. $\forall\, x \in (a, b)\,,\ f$ and g are continuous on $[a, x]$ and differentiable on (a, x). By Cauchy's MVT $\exists\, c \in (a, x) : f'(c) \cdot g(x) = f(x) \cdot g'(c)$. Then:

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(c)}{g'(c)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)}$$

Since $c \to a^+$ as $x \to a^+$. □

Remark 3.2.3. *We can continue the theorem to the* n*-th derivative:*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$$

It is also valid for $a = \pm\infty$ *and the limit in the wide sense.*

Lemma 3.2.1 (SC for LpC)**.** *If* $f : \mathcal{U} \to \mathbb{R}$ *is differentiable at* $\mathcal{U}$ *and the derivative is bounded, then* f *is Lipschitz continuous.*

Proof. Let $K = \sup|f'(x)|$ then, by Lagrange's $\forall\, x, y \in \mathcal{U}\,, \exists\, c \in (x, y)$: $\dfrac{f(x) - f(y)}{x - y} = f'(c) \Rightarrow |f(x) - f(y)| \leq K|x - y|$ □

3.3 Investigation of Functions

Definition 3.3.1 (Extremum Point)**.** *Let f be defined on an open interval $\mathcal{U}$ about a. We say that f has a **local minimum (eq. maximum)** at a if exists an open interval $\mathcal{I} \subseteq \mathcal{U}$ about a such that:*

$$\forall\, x \in \mathcal{I}\,,\; f(x) \geq f(a)\; \Big(eq.\;\; f(x) \leq f(a)\Big)$$

*The point a which is either a local minimum or local maximum point is called a **local extremum point**.*

Definition 3.3.2 (Critical Point)**.** *We say that a is a **critical point** of f if either $f'(a) = 0$ or $\nexists f'(a)$.*

Lemma 3.3.1 (NC for Local Extremum)**.** *a is a local extremum $\Rightarrow$ a is a critical point.*

Proof. If f is differentiable at a, we use Fermat's Theorem. Otherwise, $\nexists\, f'(a)$. Either way, a is a critical point. □

Lemma 3.3.2 (SC I for Local Extremum)**.** *If $\exists f'$ and $\exists f''$ which are on continuous on an open interval about a and $f'(a) = 0$ and $f''(a) \neq 0$. Then, a is local extremum and:*

- *If $f''(a) < 0$, a is local maximum.*
- *If $f''(a) > 0$, a is local minimum.*

Proof. If $f''(a) > 0$ and f'' is continuous, then $f''(a) > 0$ in some open interval around a. From Taylor's Formula we get:

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{1}{2}\,f''(c)\,(x-a)^2 \geq f(a) + f'(a)\,(x-a) = f(a)$$

since $\frac{1}{2}\,f''(c)\,(x-a)^2 \geq 0$. Analogously, for local maximum. □

Lemma 3.3.3 (SC II for Local Extremum)**.** *Let a is critical point of f. f is continuous at a and differntiable on an open interval about a, "except".*

- *If f' changes sign from negative to positive at a, then a is a local minimum.*
- *If f' changes sign from positive to negative at a, then a is a local maximum.*
- *Else, a is not a local extremum point.*

Definition 3.3.3. *Let f be defined on a domain $\mathcal{D}$. We say that f has a **global minimum (eq. maximum)** at a if*

$$\forall\, x \in \mathcal{D}\,,\; f(x) \geq f(a)\; \Big(\textit{eq. } f(x) \leq f(a)\Big)$$

Theorem 3.3.1. *Let f be continuous on $[a, b]$, then f has at least one global maximum and at least one global minimum. Further, if c is a global extremum, then it is either an endpoint or a critical point.*

Proof. The first part is exactly Weierstrass' Theorem. The second part follow from: if c is a global extremum, then it is either a local extremum on (a, b), in which case it is a critical point. Else, $c = a$ or $c = b$. □

Definition 3.3.4 (Convexity)**.** *Let f be differentiable at a. We say that f is **convex upward (eq. downward)** at a if exits an open interval about a in which the graph of the function is situated under (eq. above) the tangent line to f at $(a, f(a))$, i.e.*

$$\exists\, \delta > 0 : \forall\, x \in (a - \delta, a + \delta)\,, f(x) \leq f(a) + f'(a)(x - a)$$
$$\Big(\textit{eq. } f(x) \geq f(a) + f'(a)(x - a)\Big)$$

*In an interval, f is **convex upward (eq. downward)** if, for every point in the interval, it is convex upward (eq. downward).*

Theorem 3.3.2. *Let f be twice differentiable on the interval (a, b).*

1. *If $\forall\, x \in (a, b)\,,\; f''(x) > 0,\; f$ is convex downward on (a, b)*
2. *If $\forall\, x \in (a, b)\,,\; f''(x) < 0,\; f$ is convex upward on (a, b)*

Proof. $\forall\, x \in (a,b)\,,$ by Taylor's formula, for $n=1$, $\exists\, c \in (a,x):$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}\,f''(c)\,(x-a)^2$$

Then, we have either $f''(c) > 0 \Rightarrow f(x) \geq f(a) + f'(a)(x-a)$ or, on the other case, $f''(c) < 0 \Rightarrow f(x) \leq f(a) + f'(a)(x-a)$ □

Definition 3.3.5 (Inflection Point)**.** *Let f be continuous on an open interval about a and is differentiable at a in a wide sense. We say a is an **inflection point** of f if exists an interval $(a-\delta, a+\delta)$ such that the function changes its convexity passing through a.*

Remark 3.3.1. *It may happen that at an inflection point a the derivative equals 0 (ex. $f(x) = x^3$, $a=0$) or ∞ (ex. $f(x) = \sqrt[3]{x}$, $a=0$) or any value L (ex. $f(x) = L\,\sin x$, $a=0$)*

Definition 3.3.6 (Asymptotes)**.** *Let f be defined (for some $\delta > 0$) on either $(a-\delta, a)$ or $(a, a+\delta)$ or $(a-\delta, a+\delta)$ "except". If at least one of the one-sided limits of f is equal to $\pm\infty$, the we say that the straight line $x=a$ is a **vertical asymptote** of f.*

*The straight line $y = ax+b$ is called an **oblique asymptote** of f at $+\infty$ (eq. $-\infty$) if:*

$$\lim_{x\to+\infty}\big[f(x) - (ax+b)\big] = 0 \;\left(\textit{eq.}\;\lim_{x\to-\infty}\big[f(x) - (ax+b)\big] = 0\right)$$

*In the case $a=0$, the asymptote is also called a **horizontal asymptote**.*

Lemma 3.3.4. *The asymptote at $\pm\infty$ (if it exists) is unique.*

Proof. Let a_1x+b_1 and a_2x+b_2 be asymptote of f at ∞.

$$\left.\begin{cases}\lim\limits_{x\to\infty}\big[f(x) - (a_1x+b_1)\big] = 0\\ \lim\limits_{x\to\infty}\big[(a_2x+b_2) - f(x)\big] = 0\end{cases}\right\} \Rightarrow \lim_{x\to\infty}\big[(a_2x+b_2) - (a_1x+b_1)\big] = 0$$

$\Rightarrow a_2 = a_1$ and $b_1 = b_2$. Analogously, for $-\infty$. □

Theorem 3.3.3 (Calculating Asymptotes)**.** *Let f be defined on (c,∞). If there exists $a = \lim\limits_{x\to\infty}\dfrac{f(x)}{x}$ and $b = \lim\limits_{x\to\infty}\big[f(x) - ax\big]$, then the straight line $y = ax+b$ is a unique oblique asymptote of f at $+\infty$. Analogously, for $-\infty$.*

4 Integral Calculus

4.1 Indefinite Integral

Definition 4.1.1 (Indefinite Integral)**.** *A function F is called an antiderivative of f in the domain $\mathcal{D}$ if $\forall\, x \in \mathcal{D}\,,\ \ F'(x) = f(x)$. If F is an antiderivative of f, then $F(x) + C$ also is. The set of all antiderivatives of f,i.e. $F(x) + C$ is called the **indefinite integral** of f and is denoted by:*

$$\int f(x)\,dx = F(x) + C$$

Remark 4.1.1. *We have the following properties:*

1. *If $\exists f'$, then* $\displaystyle\int f'(x)\,dx = f(x) + C$
2. *The integration is linear:* $\displaystyle\int \Big[a \cdot f(x) + b \cdot g(x)\Big]\,dx = a \cdot \int f(x)\,dx + b \cdot \int g(x)\,dx$

Example 4.1.1. *We have:*

1. $\displaystyle\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$, *if* $\alpha \neq -1$
2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$
3. $\displaystyle\int \Big[a \cdot \cos x + b \cdot \sin x\Big]\,dx = a \cdot \sin x - b \cdot \cos x + C$
4. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$
5. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

Lemma 4.1.1 (Substitution)**.** *The chain rule states:*

$$[f(u(x))]' = f'(u(x)) \cdot u'(x) \Rightarrow \int f(u) \cdot u'\, dx = \int f(u)\, du$$

We also write: $\displaystyle\int f(x)\, dx = \left\{ \begin{array}{l} x = g(t) \\ dx = g'(t)\, dt \end{array} \right\} = \int f(g(t))\, g'(t)\, dt$

Lemma 4.1.2 (Integration by Parts)**.** *The product rule states:*

$$(u \cdot v)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x) \Rightarrow u \cdot v' = (u \cdot v)' - u' \cdot v$$
$$\Rightarrow \int u \cdot v'\, dx = uv - \int u' \cdot v\, dx$$

Lemma 4.1.3 (Integration of Trigonometric Functions)**.** *We have:*

1. *Prosthaphaeresis:*

$$\begin{aligned} \sin(\alpha) \cdot \sin(\beta) &= \frac{1}{2}\Big(\cos(\alpha - \beta) - \cos(\alpha + \beta)\Big) \\ \cos(\alpha) \cdot \cos(\beta) &= \frac{1}{2}\Big(\cos(\alpha - \beta) + \cos(\alpha + \beta)\Big) \\ \sin(\alpha) \cdot \cos(\beta) &= \frac{1}{2}\Big(\sin(\alpha + \beta) + \sin(\alpha - \beta)\Big) \\ \cos(\alpha) \cdot \sin(\beta) &= \frac{1}{2}\Big(\sin(\alpha + \beta) - \sin(\alpha - \beta)\Big) \end{aligned}$$

2. $\displaystyle\int \sin^{2k+1}(x) \cdot \cos^m(x)\, dx = \int (1 - \cos^2(x))^k \cdot \sin(x) \cdot \cos^m(x)\, dx = \left\{ \begin{array}{l} u = \cos x \\ u' = -\sin x \end{array} \right\} = -\int (1 - u^2)^k \cdot u^m\, du$

3. *We use recursively:* $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$. $\displaystyle\int \sin^{2k}(x) \cdot \cos^{2m}(x)\, dx = \int \left(\frac{1 - \cos(2x)}{2}\right)^k \left(\frac{1 + \cos(2x)}{2}\right)^m dx$

Example 4.1.2. *We have:*

1. $\int \sin(3x)\,\cos(7x)\,dx = \frac{1}{2}\int \Big[\sin(10x) - \sin(4x)\Big]\,dx = \frac{1}{2}\Big[-\frac{\cos(10x)}{10} + \frac{\cos(4x)}{4}\Big] + C = -\frac{\cos(10x)}{20} + \frac{\cos(4x)}{8} + C$

2. $\int \sin^3(x)\,\cos^2(x)\,dx = \left\{\begin{array}{l} u = \cos x \\ u' = -\sin x \end{array}\right\} = -\int (1-u^2)\cdot u^2\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$

3. $\int \cos^4(x)\,dx = \frac{1}{4}\int \big[1+2\cos(2x)+\cos^2(2x)\big]\,dx = \frac{1}{8}\int \big[3+4\cos(2x)+\cos(4x)\big]\,dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$

Lemma 4.1.4 (Substitutions)**.** *We get:*

Expression	Substitution	Differential	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ $\theta \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$	$dx = a\cos\theta\,d\theta$	$\sqrt{1-\sin^2\theta} = \cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$ $\theta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$	$dx = a\sec^2\theta\,d\theta$	$\sqrt{1+\tan^2\theta} = \sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ $\theta \in \big[0, \frac{\pi}{2}\big)$	$dx = a\tan\theta\,\sec\theta\,d\theta$	$\sqrt{\sec^2\theta - 1} = \tan\theta$

Example 4.1.3. *We get:*

1. $\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \left\{\begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\,dx \end{array}\right\} = \int \cot^2\theta\,d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C$

2. $\int \frac{x}{\sqrt{x^2+4}}\,dx = \left\{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta\,dx \end{array}\right\} = 2\int \tan\theta\cdot\sec\theta\,d\theta = 2\sec\theta + C = \sqrt{x^2+4} + C$

Definition 4.1.2 (Simple Rational Function). *A rational function is a function* $f(x) = \dfrac{P(x)}{Q(x)}$*, where* P *and* Q *are polynomials. A rational function* $f(x) = \dfrac{P(x)}{Q(x)}$ *is called* ***simple*** *is* $\deg P < \deg Q$.

Lemma 4.1.5 (Simplifing RFs). *If a rational function is not simple, it can be written as the sum of a polynomial and a simple rational function.*

Proof. By polynomial long division, we have: $P(x) = Q(x) \cdot S(x) + R(x)$ with $\deg R < \deg Q$, then $f(x) = S(x) + \dfrac{R(x)}{Q(x)}$, so $\dfrac{R(x)}{Q(x)}$ is simple. □

Definition 4.1.3 (Basic RF). *We identify four basic simple rational functions:*

$f(x)$	$\int f(x)\,dx(+C)$
$\dfrac{A}{x-\alpha}$	$A\ln\lvert x-\alpha\rvert$
$\dfrac{A}{(x-\alpha)^n}$	$A\dfrac{(x-\alpha)^{1-n}}{1-n}$
$\dfrac{Ax+B}{x^2+2px+q}$	$\dfrac{A}{2}\ln\lvert x^2+2px+q\rvert + \dfrac{B-Ap}{\sqrt{q-p^2}}\arctan\left(\dfrac{x+p}{\sqrt{q-p^2}}\right)$
$\dfrac{Ax+B}{(x^2+2px+q)^n}$	$\dfrac{A}{2}\dfrac{(x^2+2px+q)^{1-n}}{1-n} + \dfrac{B-Ap}{(q-p^2)^{n-\frac{1}{2}}}\mathcal{I}_n\left(\dfrac{x+p}{\sqrt{q-p^2}}\right)$

so that $p^2 - q < 0$ *and* $n \in \mathbb{N}_{>1}$.

1. $$\int \frac{Ax+B}{x^2+2px+q}\,dx = \int \frac{Ax+B}{(x+p)^2+\sqrt{q-p^2}^2} = \left\{\begin{array}{l} a=\sqrt{q-p^2} \\ t = x+p \\ dt = dx \end{array}\right\} = \int \frac{At\,dt}{t^2+a^2} + (B-Ap)\int\frac{dt}{t^2+a^2} = \frac{A}{2}\ln\lvert t^2+a^2\rvert + \frac{B-Ap}{a}\arctan\left(\frac{t}{a}\right) + C = \frac{A}{2}\ln\lvert x^2+px+q\rvert + \frac{B-Ap}{\sqrt{q-p^2}}\arctan\left(\frac{x+p}{\sqrt{q-p^2}}\right) + C$$

2. $\displaystyle\int \frac{Ax+B}{(x^2+2px+q)^n}\,dx = \left\{\begin{array}{l} a=\sqrt{q-p^2} \\ t=x+p \\ dt=dx \end{array}\right\} = A\int \frac{t\,dt}{(t^2+a^2)^n} + (B - Ap)\int \frac{dt}{(t^2+a^2)^n} = \frac{A}{2}\frac{(t^2+a^2)^{1-n}}{1-n} + (B-Ap)\frac{1}{a^{2n-1}}\mathcal{I}_n\left(\frac{t}{a}\right) + C = \frac{A}{2}\frac{(x^2+2px+q)^{1-n}}{1-n} + \frac{\big(B-Ap\big)}{\sqrt{q-p^2}^{2n-1}}\mathcal{I}_n\left(\frac{x+p}{\sqrt{q-p^2}}\right) + C$

Where $\mathcal{I}_n(u) = \displaystyle\int \frac{du}{(u^2+1)^n}$ *is given by a recursive formula.*

Theorem 4.1.1 (Partial Fraction Decomposition)**.** *Every simple rational function can be written as a sum of basic rational functions.*

Proof. Let $f(x) = \dfrac{R(x)}{Q(x)}$

1. If $Q(x) = A\,(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_k}{x-\alpha_k} = \sum_{i=1}^{k}\frac{A_k}{x-\alpha_k}$$

where they are given explicitly by: $A_i = \lim\limits_{x\to\alpha_i}(x-\alpha_i)\,f(x)$.

2. If $Q(x) = A\,(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_k)^{m_k}$, $m_i \in \mathbb{N}_{>1}$:

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{A_{i,j}}{(x-\alpha_i)^j}$$

where they are given by: $A_{i,j} = \dfrac{1}{(m_i-j)!}\lim\limits_{x\to\alpha_i}\dfrac{d^{\,m_i-j}}{dx^{\,m_i-j}}(x-\alpha_i)^{m_i}\,f(x)$.

3. If $Q(x) = A\,(x^2+2p_1x+q_1)\cdot(x^2+2p_kx+q_k)\,(x-\alpha_1)\cdots(x-\alpha_l)$;, where $p_i^2 - q_i < 0$:

$$\frac{R(x)}{Q(x)} = \frac{P_1x+Q_1}{x^2+2p_1x+q_1} + \cdots + \frac{P_kx+Q_k}{x^2+2p_kx+q_k} + \frac{A_1}{x-\alpha_1} + \cdots + \frac{A_l}{x-\alpha_l}$$

4. If $Q(x) = A\,(x^2 + 2p_1 x + q_1)^{m_1}(x^2 + 2p_2 x + q_2)^{m_2} \cdots (x^2 + 2p_k x + q_k)^{m_k}$, where $m_i \in \mathbb{N}_{>1}$ and $p_i^2 - q_i < 0$:

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{P_{i,j}\,x + Q_{i,j}}{(x^2 + 2p_i x + q_i)^j}$$

□

Corollary 4.1.1. *Every rational function has an antiderivative in closed form, that is, there is an expression using only elementary functions for its antiderivative.*

4.2 Definite Integrals

Definition 4.2.1 (Riemann Sums and Integral)**.** *Let f be defined on the closed interval $[a, b]$. Let $n \in \mathbb{N}$ and T is a partition for $[a, b]$ on n sub-intervals.*

$$T : a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

Denote $\Delta x_i = x_i - x_{i-1}$ and we denote $\|T\| = \max_{i=1,2,\cdots,n} \Delta x_i$, the norm of the partition.

Choose a point $c_i \in [x_{i-1}, x_i]$. Then, the sum $\sum_{i=1}^{n} f(c_i)\,\Delta x_i$ is called the ***Riemann sum*** *of T.*

A function f(x) is called Riemann integrable over $[a, b]$ if there exists the limit $I = \lim_{\|T\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$ and the limit is independent on the choice of T and c_i. This limit is denoted by $I = \int_a^b f(x)\,dx$ and is called the ***definite integral*** *(or the Riemann integral).*

Remark 4.2.1. *The Riemann sum approximates the area under the curve $y = f(x)$ and the x-axis by rectangles of base Δx_i and height $f(c_i)$. Therefore, the Riemann integral is exactly that area.*

Lemma 4.2.1. *We have:*

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_b^a c\,dx = c\,(b - a)$
4. *If f is not bounded on $[a, b]$, then f is not Riemann integrable on $[a, b]$.*

Example 4.2.1. *We define the following bounded function:*

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

It is not Riemann integrable on any $[a, b]$*.*

Proof. Take any partition T on $[a, b]$ and choose:

- $c_i \in \mathbb{Q} \, : \, \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a$
- $c_i \notin \mathbb{Q} \, : \, \sum_{i=1}^{n} f(c_i)\, \Delta x_i = 0$

Since $b - a \neq 0$, $\nexists \lim_{\|T\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$, since it depends on the choice of c_i. $\square$

Definition 4.2.2 (Darboux Upper and Lower integrals)**.** *Let f be defined and bounded on the closed interval $[a, b]$. Let $n \in \mathbb{N}$ and T is a partition for $[a, b]$ on n sub-intervals.*

$$T : a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

Denote $\Delta x_i = x_i - x_{i-1}$ and we denote $\|T\| = \max_{i=1,\,2,\,\cdots\,,\,n} \Delta x_i$, the norm of the partition. Define $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$.

*Then, the sum $L(T) = \sum_{i=1}^{n} m_i\, \Delta x_i$ is called the **lower Darboux sum** of T and the sum $U(T) = \sum_{i=1}^{n} M_i\, \Delta x_i$ is called the **upper Darboux sum** of T.*

Denoting $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$, we notice:

$$m\,(b-a) = \sum_{i=1}^{n} m\,\Delta x_i \le \sum_{i=1}^{n} m_i\,\Delta x_i = L(T) \le A$$
$$A \le U(T) = \sum_{i=1}^{n} M_i\,\Delta x_i \le \sum_{i=1}^{n} M\,\Delta x_i = M(b-a)$$

*We define: $\sup_T L(T) = L$ the **lower Darboux integral** and $\inf_T U(T) = U$ the **upper Darboux integral** denoted*

$$\underline{\int_a^b} f(x)\,dx = L \quad and \quad U = \overline{\int_a^b} f(x)\,dx$$

*If $U = L$, the function is said to be **Darboux integrable**.*

Lemma 4.2.2 (N&SC for Darboux Integrability)**.** *f is Darboux integrable iff $\forall\,\epsilon > 0\,,\,\exists\,T : U(T) - L(T) < \epsilon$*

Proof. That condition is equivalent to $\lim_{\|T\|\to 0} \big(U(T) - L(T)\big) = 0$ □

Lemma 4.2.3 (DI ⇔ RI)**.** *Any Riemann sum $R(T,c) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$ is between the Darboux sums: $U(T) \ge R(T,c) \ge L(T)$. If $\lim_{\|T\|\to 0} U(T) = \lim_{\|T\|\to 0} L(T) = I$, by Sandwich, $\lim_{\|T\|\to 0} R(T) = I \Rightarrow f$ is Riemann integrable over $[a,b]$. That is, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$*

Theorem 4.2.1. *Every continuous function is Riemann Integrable.*

Proof. Since f is defined on a closed interval $[a,b]$, it is uniformly continuous. So, $\forall\,\epsilon > 0\,,\,\exists\,\delta > 0 : \forall\,x,y \in [a,b]\,,\,|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. Then, given $\epsilon > 0$, choose the constant partition $\|T_n\| = \frac{1}{n} < \min\{1,\delta\}$. Therefore, $M_i - m_i < \epsilon$, so $U(T_n) - L(T_n) < \sum_{i=1}^{n} \epsilon \cdot \frac{1}{n} = \epsilon$. □

Definition 4.2.3 (Piecewise Continuity)**.** *A function f which is defined and bounded on $[a, b]$ is called piecewise continuous if it has at most a finite number of discontinuity points and all of them are removable or of the first kind.*

Theorem 4.2.2. *Every piecewise continuous function if Riemann Integrable.*

Proof. By linearity of the integral, split the interval so that we are left with the integral of only continuous functions. □

Theorem 4.2.3. *Every bounded monotonic function is Riemann Integrable.*

Proof. WLOG, let f be increasing on $[a, b]$. Given $\epsilon > 0$, choose the constant partition $\|T_n\| = \dfrac{1}{n} < \dfrac{\epsilon}{f(b) - f(a)}$. Further $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore, $U(T) - L(T) = \displaystyle\sum_{i=1}^{n} \Big[f(x_i) - f(x_{i-1})\Big] \frac{1}{n} = \frac{f(b) - f(a)}{n} < \epsilon$ □

Theorem 4.2.4 (Intermediate Integral Theorem)**.** *Let f be continuous on $[a, b]$, then $\exists\, c \in [a, b]$:*

$$\int_a^b f(x)\, dx = f(c)\,(b - a)$$

Proof. Take $g(t) = f(t) \cdot (b - a) - \displaystyle\int_a^b f(x)\, dx$, which is continuous, since f is continuous. By Weierstrass' theorem, f achieves a maximum M and a minimum m on $[a, b]$. By Darboux Integration, $m \cdot (b - a) \leq \displaystyle\int_a^b f(x)\, dx \leq M \cdot (b - a)$, hence, by IVT, $\exists\, c \in [a, b] : g(c) = 0$. □

4.3 Fundamental Theorem of Calculus

Theorem 4.3.1 (FTC I)**.** *Let f be continuous on (a,b) and $c \in (a,b)$. Then, the function $F(x) = \displaystyle\int_c^x f(t)\,dt$ is an antiderivative function of f on (a,b), i.e. $\forall\, x \in (a,b)\,,\ F'(x) = f(x)$.*

Proof. Given $\delta > 0$, for any $x \in (a,b)$ such that $x+\delta \in (a,b)$, by Intermediate Integral Theorem, $\exists\, \tau \in [x, x+\delta]$:

$$
\begin{aligned}
F(x+\delta) - F(x) = \int_c^{x+\delta} f(t)\,dt - \int_c^x f(t)\,dt = \int_x^{x+\delta} f(t)\,dt = f(\tau)\,\delta \\
\Rightarrow F'(x) = \lim_{\delta\to 0} \frac{F(x+\delta) - F(x)}{\delta} = \lim_{\delta\to 0} f(\tau) = f(x)
\end{aligned}
$$

since $\tau \in [x, x+\delta]$. □

Remark 4.3.1. *The theorem is also true for a piecewise continuous function, with the caveat $F'(x) = f(x)$ for x a continuity point.*

Theorem 4.3.2 (Newton-Leibnitz / FTC II)**.** *Let f be continuous on $[a,b]$ and let F be an arbitrary antiderivative function of f on (a,b). Then:*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof. Let $c \in (a,b)$. By FTC, $G(x) = \displaystyle\int_c^x f(t)\,dt$ is an antiderivative of f. Taking any other antiderivative $F(x) = G(x) + C$. Calculate:

$$
\begin{aligned}
F(b) - F(a) &= \big(G(b) + C\big) - \big(G(a) + C\big) = G(b) - G(a) \\
&= \int_c^b f(t)\,dt - \int_c^a f(t)\,dt = \int_a^b f(t)\,dt
\end{aligned}
$$

□

Notation: $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

Remark 4.3.2. *If f is piecewise continuous then:*

$$\int_a^b f(x)\,dx = F(b^-) - F(a^+)$$

where $F(b^-) = \lim_{x\to b^-} F(x)$ and $F(a^+) = \lim_{x\to a^+} F(x)$.

Lemma 4.3.1 (Leibnitz Formula)**.**

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t,x)\,dt = f\big(b(x),x\big)\cdot b'(x) - f\big(a(x),x\big)\cdot a'(x) + \int_{a(x)}^{b(x)} \partial_x f(t,x)\,dt$$

where $\partial_x f(t,x) = \lim_{\delta\to 0}\dfrac{f(t,x+\delta)-f(t,x)}{\delta}$.

Definition 4.3.1 (Improper Integral)**.** *An integral is said to be improper if either (or both) bounds are infinite or the function is unbounded in the interval of integration. We calculate by the following:*

$$\begin{aligned}
\int_a^\infty f(x)\,dx &= \lim_{R\to\infty}\int_a^R f(x)\,dx \\
\int_{-\infty}^b f(x)\,dx &= \lim_{R\to\infty}\int_{-R}^b f(x)\,dx \\
\int_a^b f(x)\,dx &= \lim_{\epsilon\to 0}\left[\int_{c+\epsilon}^b f(x)\,dx + \int_a^{c-\epsilon} f(x)\,dx\right]
\end{aligned}$$

Example 4.3.1.

$$\begin{aligned}
\int_1^\infty x^{-\frac{4}{3}}\,dx &= \lim_{R\to\infty}\int_1^R x^{-\frac{4}{3}}\,dx = \lim_{R\to\infty} -3\,x^{-\frac{1}{3}}\Big|_1^R = 3 \\
\int_{-\infty}^{-1} x^{-\frac{4}{3}}\,dx &= \lim_{R\to\infty}\int_{-R}^{-1} x^{-\frac{4}{3}}\,dx = \lim_{R\to\infty} -3\,x^{-\frac{1}{3}}\Big|_{-R}^{-1} = 3 \\
\int_1^{-1} x^{-\frac{1}{3}}\,dx &= \lim_{\epsilon\to 0}\left[\int_\epsilon^1 x^{-\frac{1}{3}}\,dx + \int_{-1}^{-\epsilon} x^{-\frac{1}{3}}\,dx\right]
\end{aligned}$$

Theorem 4.3.3 (Integral Test)**.** *Let $f : [N,\infty)\to\mathbb{R}$ be an integrable function that is monotone decreasing.*

$$\sum_{n=N}^\infty f(n) \text{ converges} \Leftrightarrow \int_N^\infty f(x)\,dx \text{ converges}$$

Proof. We have the following estimate: $\forall\, M \in \mathbb{N}\,,$

$$\int_N^{M+1} f(x)\,dx \le \sum_{n=N}^{M} f(n) \le f(N) + \int_N^{M} f(x)\,dx$$

Taking limit of both sides, we get the result. $\square$