Ordinary Differential Equations

Notes from TAU Course with Additional Information

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Contents

*Note: Chapter 2 can be skipped on a first reading.

1 First Order ODEs

1.1 General Analysis

Definition 1.1.1 (Implicit ODE)**.** *Given a function $\mathcal{F} : U \subset \mathbb{R}^{1+n} \to \mathbb{R}$, an implicit first-order ODE is an equation of the following form:*

$$\mathcal{F}(x, y, y', \cdots, y^{(n)}) \equiv 0$$

for a function $y : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable. Further, we call n the order of the ODE. If, possible, we way write it in explicit form (y).

Definition 1.1.2 (First Order ODE)**.** *Given a function $F : U \subset \mathbb{R}^2 \to \mathbb{R}$, an (explicit) first-order ODE is an equation of the following form:*

$$y' = F(x, y)$$

for a function $y : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ differentiable.

Lemma 1.1.3 (Constant Function Solutions)**.** *For $F : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$, if $\exists\, \lambda \in \mathbb{R} : \forall\, x \in \mathcal{A}\,,\ F(x, \lambda) = 0$, then $y(x) \equiv \lambda$ is a solution to the differential equation $y' = F(x, y)$.*

Proof. $y(x) \equiv \lambda \Rightarrow y'(x) \equiv 0 \Rightarrow y'(x) \equiv 0 \equiv F(x, \lambda) = F(x, y(x))$. □

Remark 1.1.4 (Integration)**.** *If F is independent of y, that is, $F(x, y) = G(x)$, then the ODE $y' = F(x, y)$ can be resolved by simple integration.*

Definition 1.1.5 (Autonomous)**.** *If F is independent of x, that is, $F(x, y) = G(y)$, then the ODE $y' = F(x, y)$ is called autonomous.*

Lemma 1.1.6. *If $y(x) = \gamma(x)$ is a solution to an autonomous ODE, then $y_a(x) = \gamma(x + a)$ is also a solution, for any $a \in \mathbb{R}$.*

Proof. Simply, notice $y_a'(x) = \gamma'(x+a) = F(x+a, \gamma(x+a)) = G(\gamma(x+a)) = G(y_a(x)) = F(x, y_a(x))$, since $y = \gamma(x)$ is a solution. □

Lemma 1.1.7 (Substitution / Change of Variables)**.** *For the equation $y' = F(x, y)$, let $y = G(x, z)$ for some $G : U \subset \mathbb{R}^2 \to \mathbb{R}$ differentiable. We get:*

$$F(x, G(x, z)) = y' = \partial_x G(x, z) + z' \cdot \partial_y G(x, z)$$

which is now an ODE in z, which we may solve $z(x)$ and substitute back $y(x) = G(x, z(x))$.

Definition 1.1.8 (Higher Order ODE)**.** *Given a function $F : U \subset \mathbb{R}^{1+n} \to \mathbb{R}$, an (explicit) n-th order ODE is an equation of the following form:*

$$y^{(n)} = F(x, y, \cdots, y^{(n-1)})$$

for a function $y : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ twice differentiable.

Lemma 1.1.9 (Integration)**.** *The solution for $y^{(n)} = f$ is:*

$$y(x) = \sum_{k=0}^{n-1} \frac{C_k}{k!} (x-a)^k + \frac{1}{n!} \int_a^x (x-t)^n \cdot f(t)\, dt$$

Proof. $y'(x) = \displaystyle\sum_{k=1}^{n-1} \frac{C_k}{(k-1)!} (x-a)^{k-1} + \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \cdot f(t)\, dt$ and, by induction, $y^{(n-1)}(x) = C_{n-1} + \displaystyle\int_a^x f(t)\, dt$, then $y^{(n)}(x) = f(x)$. □

Lemma 1.1.10 (Reduction of Order)**.** *If $F(x, y, y')$ is independent of:*

1. *y, that is, $F(x, y, y') = G(x, y')$, then: let $p(x) = y'(x)$. We get the first-order equation: $p'(x) = G(x, p(x))$*

2. *x, that is, $F(x, y, y') = G(y, y')$, then: let p be such that $y' = p(y)$. We get the first-order equation: $p'(y) \cdot p(y) = G(y, p(y))$*

Proof. We only prove the last relation, the rest is immediately clear. By the chain rule, $y''(x) = \dfrac{d(p(y))}{dx} = \dfrac{dp}{dy} \cdot \dfrac{dy}{dx} = p'(y) \cdot y'(x) = p'(y) \cdot p(y)$ □

Remark 1.1.11. *In the first case, we retrieve y from p by direct integration (cf. 1.1.4). In the second, we use separation of variables (cf. 1.3.4).*

1.2 Linearizing ODEs

Theorem 1.2.1 (General First Order Linear ODE)**.** *For the equation:*

$$y' + P(x) \cdot y = Q(x)$$

let $\mu(x) = \exp\left[\int P(x)\,dx\right] = \exp\left[\int_a^x P(t)\,dt\right]$*, called the integrating factor. The solution is:*

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) \cdot Q(x)\,dx = \frac{1}{\mu(x)} \left[y(a) + \int_a^x \mu(t) \cdot Q(t)\,dt\right]$$

where $y(a)$ *is arbitrary.*

Proof. By definition, we get: $\mu'(x) = P(x) \cdot \mu(x)$ and $\mu(a) = 1$. By multiplying both sides of the integrating factor:

$$\begin{aligned} y'(x) \cdot \mu(x) + y(x) \cdot \overbrace{P(x) \cdot \mu(x)}^{\mu'(x)} &= \mu(x) \cdot Q(x) \\ (y \cdot \mu)'(x) &= \mu(x) \cdot Q(x) \text{ (Integrating both sides)} \\ y(x) \cdot \mu(x) - y(a) &= \int_a^x \mu(t) \cdot Q(t)\,dt \end{aligned}$$

We have a solution. ∎

Remark 1.2.2. *We wrote the indefinite integrals where we can pick any antiderivative.*

Our goal now is to find substitutions to transform into a general linear ODE.

Lemma 1.2.3 (Bernoulli ODE)**.** *For* $\alpha \in \mathbb{R} \setminus \{0, 1\}$*:*

$$y' + P(x) \cdot y = Q(x) \cdot y^\alpha$$

let $z = y^{1-\alpha}$*. We get:*

$$z' + (1-\alpha)\,P(x) \cdot z = (1-\alpha)\,Q(x)$$

Proof. Calculate: $z' = (1-\alpha)\,y^{-\alpha} \cdot y' = (1-\alpha)\,y^{-\alpha} \cdot (-P(x)\,y + Q(x) \cdot y^n) = -(1-\alpha)\,P(x) \cdot y^{1-\alpha} + (1-\alpha)\,Q(x) = -(1-\alpha)\,P(x) \cdot z + (1-\alpha)\,Q(x)$ ∎

Definition 1.2.4 (Homogeneous function). *A function $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is homogeneous of degree k if: $\forall \lambda \in \mathbb{R}\,, \forall x, y \in U\,,\ F(\lambda \cdot x, \lambda \cdot y) = \lambda^k \cdot F(x, y)$. If F is homogeneous, then the ODE $y' = F(x, y)$ is called homogeneous.*

Lemma 1.2.5. *If $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is homogeneous of degree k, then exists $G : V \subseteq \mathbb{R} \to \mathbb{R}$ such that $F(x, y) = x^k \cdot G\left(\dfrac{y}{x}\right)$.*

Proof. Let $G(z) = F(1, z)$. We get: $F(x, y) = F\left(x \cdot 1, x \cdot \dfrac{y}{x}\right) = x^k \cdot F\left(1, \dfrac{y}{x}\right) = x^k \cdot G\left(\dfrac{y}{x}\right)$. $\square$

Lemma 1.2.6 (Homogeneous ODE). *Let $z = \dfrac{y}{x}$, then $y' = F(x, y)$ homogeneous of degree k becomes: $x \cdot z' + z = x^k \cdot G(z)$ where $G(z) = F(1, z)$.*

Proof. $y = x \cdot z$ and $F(x, y) = x^k \cdot G(z)$ (cf. 1.2.5). Apply 1.1.7. $\square$

1.3 Exact ODEs

Lemma 1.3.1 (Implicit Solution). *If $\phi(x, y) = const.$ is such that any curve that satifies it a solution to the ODE $y' = F(x, y)$, then:*

$$F(x, y) = -\frac{\partial_x \phi(x, y)}{\partial_y \phi(x, y)}$$

where ∂_x is the partial derivative.

Proof. We differentiate both sides of $\phi(x, y(x)) = \text{const.}$ wrt x, we get: $\partial_x \phi(x, y(x)) + y'(x) \cdot \partial_y \phi(x, y(x)) = 0$ $\square$

Definition 1.3.2 (Exact). *An ODE of the form $y' = F(x, y) = -\dfrac{M(x, y)}{N(x, y)}$, which is written:*

$$M\,dx + N\,dy = 0$$

such that $\partial_y M = \partial_x N$ is called exact.

Theorem 1.3.3 (N&SC Exact). *An ODE $M\,dx + N\,dy = 0$ is exact iff $\exists\, \phi : \mathbb{R}^2 \to \mathbb{R} : \phi(x, y) = const.$ is an implicit solution. Moreover $M = \partial_x \phi$ and $N = \partial_y \phi$.*

Proof. The converse was given in 1.3.1 with the observation that $\partial_y \partial_x \phi = \partial_x \partial_y \phi$. By Green, if $\partial_y M = \partial_x N$, then $\vec{L} = (M, N)$ is path-independent. By the converse of the gradient theorem, $\exists\, \phi : \mathbb{R}^2 \to \mathbb{R} : M = \partial_x \phi$ and $N = \partial_y \phi$. Then, $d\phi = M\,dx + N\,dy = 0 \Rightarrow \phi(x, y) = \text{const.}$, where y can be some (differentiable) function of x on the curve. $\square$

Theorem 1.3.4 (Separable First Order). *If there are functions ϕ, ψ , such that $F(x, y) = \dfrac{\varphi(x)}{\psi(y)}$, then there is a solution $y(x)$ given implicitly:*

$$\int_{y(a)}^{y} \psi(s)\,ds = \int_a^x \varphi(t)\,dt$$

where $y(a)$ is arbitrary. Or, equivaletly, $\Psi(y) - \Phi(x) = const.$ is an implicit solution, where Ψ and Φ are antiderivatives of ψ and φ, respectively.

Proof. Follows directly from 1.3.3 with $M(x, y) = -\varphi(x)$ and $N = \psi(y)$, we get $\phi(x, y) = \Psi(y) - \Phi(x)$ is an implicit solution. $\square$

Lemma 1.3.5. *For an exact ODE $M\,dx + N\,dy = 0$, the implicit solution is given by the following integral:*

$$\phi(x,y) = \int_{x_0}^{x} M(t,y)\,dt + \int_{y_0}^{y} N(x_0,s)\,ds = \int_{x_0}^{x} M(t,y_0)\,dt + \int_{y_0}^{y} N(x,s)\,ds$$

Proof. We'll only prove the first one. $\partial_x\phi(x,y) = M(x,y)$ and $\partial_y\phi(x,y) = N(x_0,y) + \displaystyle\int_{x_0}^{x} M_y(t,y)\,dt = N(x_0,y) + \int_{x_0}^{x} N_x(t,y)\,dt = N(x,y)$ □

Lemma 1.3.6 (Inexact)**.** *Let $M\,dx + N\,dy = 0$, where $\partial_y M \neq \partial_x N$. Consider an integrating factor $\mu(x,y) = \exp\Big[\beta(x,y)\Big]$ such that $\partial_y(\mu \cdot M) = \partial_x(\mu \cdot N)$, that is:*

$$\partial_y M - \partial_x N = N \cdot \partial_x \beta - M \cdot \partial_y \beta$$

We consider these inexact cases where we can express the following as functions of:

1. $\xi(x) = \dfrac{\partial_y M - \partial_x N}{N}$*:* $\mu(x) = \exp\left[\displaystyle\int \xi(x)\,dx\right] = \exp\left[\displaystyle\int_a^x \xi(t)\,dt\right]$
2. $\zeta(y) = \dfrac{\partial_y M - \partial_x N}{-M}$*:* $\mu(y) = \exp\left[\displaystyle\int \zeta(y)\,dy\right] = \exp\left[\displaystyle\int_b^y \zeta(s)\,ds\right]$
3. $\eta(z) = \dfrac{\partial_y M - \partial_x N}{N \cdot \partial_x z - M \cdot \partial_y z}$*:* $\mu(z) = \exp\left[\displaystyle\int \eta(z)\,dz\right] = \exp\left[\displaystyle\int_c^z \eta(r)\,dr\right]$ *where $z = f(x,y)$.*

Proof. We check:

1. Dividing by N: $\xi(x) = \partial_x\beta - \frac{M}{N} \cdot \partial_y\beta$. We get: $\partial_y\beta \equiv 0$ and $\partial_x\beta = \xi$, it satifies the equation.
2. Dividing by $-M$: $\zeta(y) = \partial_y\beta - \frac{N}{M} \cdot \partial_x\beta$. We get: $\partial_x\beta \equiv 0$ and $\partial_y\beta = \zeta$, it satifies the equation.
3. Dividing by $N \cdot \partial_x z - M \cdot \partial_y z$: $\eta(z) = \dfrac{N \cdot \partial_x\beta - M \cdot \partial_y\beta}{N \cdot \partial_x z - M \cdot \partial_y z}$. We get: $\partial_x\beta = (\partial_x z) \cdot \eta(z)$ and $\partial_y\beta = (\partial_y z) \cdot \eta(z)$, it satifies the equation.

□

2 Theory of Analysis $^{(*)}$

2.1 Limit of Functions

Definition 2.1.1 (Banach Space). *A Banach space E is a normed vector space (norm $\|\cdot\|$) such that every Cauchy sequence converges.*

Definition 2.1.2 (Lipschitz). *A function $f : E \to F$ between two normed vector spaces, is Lipschitz continuous if, and only if:*

$$\exists\, K > 0 : \forall\, x, y \in X\,,\ \|f(x) - f(y)\|_F \leq K \cdot \|x - y\|_E$$

Then, K is called the Lipschitz constant of f. If $K < 1$,f is called a contraction.

Theorem 2.1.3 (Banach Fixed Point Theorem). *Given a Banach space $(E, \|\cdot\|)$, a contraction $F : E \to E$ has a unique fixed point x^*. In particular, $\forall\, x \in E$,*

$$x^* = \lim_{n\to\infty} F^n(x)$$

where $F^n = (F \circ \cdots \circ F)(x)$, n times.

Proof. First, if F has a fixed point, it is unique. If x and y are fixed points, $\|x-y\| = \|F(x)-F(y)\| \leq K\|x-y\| \Leftrightarrow 0 \leq (K-1)\|x-y\| \Rightarrow \|x-y\| = 0 \Leftrightarrow x = y$, since $K < 1$. Now, we show that F has a fixed point. By induction, $\forall\, x \in E\,,\ \|F^{n+1}(x) - F^n(x)\| \leq K^n\|F(x) - x\|$. Therefore, $\forall\, m, n \in \mathbb{N}$,

$$\|F^m(x) - F^n(x)\| \leq \left|\frac{K^n - K^m}{1 - K}\right| \cdot \|F(x) - x\|$$

Hence, the sequence $\{F^n(x)\}_{n\in\mathbb{N}}$ is Cauchy in E, so it converges to some element x^*. But, $\|F^{n+1}(x) - F^n(x)\| \to 0$. Therefore, $\|F(x^*) - x^*\| = 0 \Leftrightarrow F(x^*) = x^*$. □

Lemma 2.1.4. *Given a Banach space $(E, \|\cdot\|)$, a function $F : E \to E$, if F^k is a contraction, for some $k \in \mathbb{N}$, then F has a unique fixed point x^*, given the same as before, that is, $\forall\, x \in E\,,\ x^* = \lim\limits_{n\to\infty} F^n(x)$.*

$^{(*)}$This chapter can be skipped on a first reading.

Proof. By 2.1.3, F^k has a unique fixed point x^*. So, $F^k(F(x^*)) = F(F^k(x^*)) = F(x^*)$, since the fixed point is unique, we must have $F(x^*) = x^*$. To prove uniqueness, observe any fixed point of F is a fixed point of F^k, which is unique. □

Definition 2.1.5 (Continuous Function)**.** *The space of continuous functions $f : [a,b] \to \mathbb{R}$ is defined as:*

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ continuous}\}$$

Furthermore, it is a vector space $(C([a,b]), \mathbb{R})$ with pointwise addition and scalar multiplication: $(f+g)(x) = f(x) + g(x)$ and $(\alpha \cdot f) = \alpha \cdot f(x)$

Definition 2.1.6 (Uniform Norm)**.** *We define the following norm in $C([a,b])$:*

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$

which, we can check obeys every axiom of a norm. We say $f_n \to f$ uniformly iff $\|f - f_n\|_\infty \to 0$

Theorem 2.1.7. *$C([a,b])$ is a Banach space with $\|\cdot\|_\infty$. That is, if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy then, $\exists\, f \in C([a,b]) : f_n \to f$ uniformly.*

Proof. Given in Calculus II. □

Corollary 2.1.8. *$C([a,b], \mathbb{R}^n) = \{f : [a,b] \to \mathbb{R}^n \mid f_i \in C([a,b])\}$ is a Banach Space with norm $\|f\|_\infty = \sup_{x\in[a,b]} \|f(x)\|$.*

Lemma 2.1.9. *If $f_n \to f$ uniformly in $\mathcal{I}$, then:*

1. *$\forall\, x \in \mathcal{I}$, $f_n(x) \to f(x)$*
2. *if f_n are continuous, and $x_n \to x$, then $f_n(x_n) \to f(x)$.*

Proof. Exercise in Calculus II. □

Definition 2.1.10 (Equicontinuity)**.** *A family $\mathcal{F} \subset C([a,b])$ is equicontinuous at $t \in [a,b]$ if:*

$$\forall\, \epsilon > 0\,,\ \exists\, \delta > 0 : \forall\, x \in [a,b]\,,\ |x - t| < \delta \Rightarrow \forall\, f \in \mathcal{F}\,,\ |f(x) - f(t)| < \epsilon$$

A family with common Lipschitz constant K is immediately equicontinuous.

Theorem 2.1.11 (Arzela-Ascoli)**.** *A sequence $\{f_n \in C([a,b])\}_{n\in\mathbb{N}}$ that is equicontinuous and bounded (that is, $\{\|f_n\|\}_{n\in\mathbb{N}}$ is bounded) has a converging subsequence.*

Proof. Given in Analysis. □

2.2 Existence and Uniqueness

Theorem 2.2.1 (Picard-Lindelöf/ E&U). *Let $F : \mathcal{R} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuous and Lipschitz on y (second variable) defined on the closed rectangle $\mathcal{R} = \mathcal{A} \times \mathcal{B}$. That is, $\forall\, x \in \mathcal{A}\,,\ \forall\, y_1, y_2 \in \mathcal{B}\,,\ |F(x, y_1) - F(x, y_2)| \leq K|y_1 - y_2|$ for some $K \in \mathbb{R}$. Then, there is a (closed) interval $\mathcal{I} \subseteq \mathcal{A}$ such that the solution to the differential equation:*

$$y'(x) = F(x, y(x)) \quad \textit{with} \quad y(x_0) = y_0$$

exists and is unique on $\mathcal{I}$, with $x_0 \in \mathcal{I}$.

Proof. First, let $\mathcal{R} = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ and $\mathcal{I} = [x_0 - \theta, x_0 + \theta]$ for some $a, b, \theta > 0$. Define $E = \{\gamma \in C(\mathcal{I}) \mid \gamma(x_0) = y_0 \text{ and } \|\gamma - y_0\|_\infty \leq b\}$ and require $\theta \leq a$. Then, E is a Banach space with uniform norm (cf. 2.1.7). We define a function $\Gamma : E \to E$ as follows:

$$\Gamma[\varphi](x) = y_0 + \int_{x_0}^{x} F(t, \varphi(t))\, dt$$

Let $M = \sup\limits_{(x,y)\in\mathcal{R}} |F(x, y)|$, which only depends on F. To guarantee that the range of Γ is correct, i.e. $\varphi \in E \Rightarrow \Gamma[\varphi] \in E$, we need:

$$|\Gamma[\varphi](x) - y_0| = \left| \int_{x_0}^{x} F(t, \varphi(t))\, dt \right| \leq \int_{x_0}^{x} \big|F(t, \varphi(t))\big|\, dt \leq M|x - x_0| \leq M \cdot \theta$$
$$\Rightarrow \forall\, x \in \mathcal{I}\,,\ \big|\Gamma[\varphi](x) - y_0\big| \leq M \cdot \theta \Rightarrow\ \big\|\Gamma[\varphi] - y_0\big\|_\infty \leq M \cdot \theta$$

If we require that $\theta \leq \dfrac{b}{M}$, we get: $\forall\, \varphi \in E\,,\ \Gamma[\varphi] \in E$, as needed. Now, let us prove Γ is a contraction. Since F is Lipschitz on y:

$$\begin{aligned}
\forall\, x \in \mathcal{I}\,, \big|\Gamma[\varphi](x) - \Gamma[\psi](x)\big| &= \left| \int_{x_0}^{x} \Big[F(t, \varphi(t)) - F(t, \psi(t))\Big]\, dt \right| \\
&\leq \int_{x_0}^{x} \Big|F(t, \varphi(t)) - F(t, \psi(t))\Big|\, dt \leq K \int_{x_0}^{x} \big|\varphi(t) - \psi(t)\big|\, dt \\
&\leq K \cdot |x - x_0| \cdot \big\|\varphi - \psi\big\|_\infty \leq K \cdot \theta \cdot \big\|\varphi - \psi\big\|_\infty
\end{aligned}$$

Hence $\big\|\Gamma[\varphi] - \Gamma[\psi]\big\|_\infty \leq K \cdot \theta \cdot \big\|\varphi - \psi\big\|_\infty$. If we choose $\theta < \dfrac{1}{K}$, Γ is a contraction. By 2.1.3, $\exists!\, \gamma \in E : \Gamma[\gamma] = \gamma$. By FTC, $\gamma'(x) = F(x, \gamma(x))$ and $\gamma(x_0) = y_0$, by construction of Γ. □

Now that we prove that the theorem is true, we shall try to improve the size/range of the interval $\mathcal{I}$ around x_0.

Lemma 2.2.2 (Interval of E&U)**.** *For F satisfing conditions of Picard-Lindelöf (cf. 2.2.1), the interval of solution $\mathcal{I} = [x_0 - \theta, x_0 + \theta]$, where $\theta = \min\left\{a, \dfrac{b}{M}\right\}$ and $M = \sup\limits_{(x,y)\in\mathcal{R}} |F(x,y)|$.*

Proof. The requirements $\theta \le a$ and $\theta \le \dfrac{b}{M}$ do not change. However, we may drop $\theta < \dfrac{1}{K}$ for the following reason: Instead of Γ being a contraction, we shall prove $\exists\, k \in \mathbb{N} : \Gamma^k$ is a contraction. As before,

$$\begin{aligned}\forall\, x \in \mathcal{I}\,, \left|\Gamma^k[\varphi](x) - \Gamma^k[\psi](x)\right| &\le K \int_{x_0}^{x} \left|\Gamma^{k-1}[\varphi](t) - \Gamma^{k-1}[\psi](t)\right| dt \\ &\le \cdots \le \frac{1}{k!}\, K^k \cdot |x - x_0|^k \cdot \left\|\varphi - \psi\right\|_\infty \le \frac{(K\cdot\theta)^k}{k!} \left\|\varphi - \psi\right\|_\infty\end{aligned}$$

Using the integral $\left|\displaystyle\int_{x_0}^{x}\int_{x_0}^{t_1}\int_{x_0}^{t_2}\cdots\int_{x_0}^{t_{k-1}} dt_1\, dt_2 \cdots dt_k\right| = \dfrac{|x-x_0|^k}{k!}$ (it's a simplex). For sufficiently large $k \in \mathbb{N}$, $\dfrac{(K\cdot\theta)^k}{k!} < 1$, so Γ^k is a contraction. By 2.1.4, this is sufficient for existence and uniqueness of a fixed point. $\square$

Theorem 2.2.3 (E& U for Systems)**.** *Let $\underline{F} : \mathcal{R} \subseteq \mathbb{R}^{1+n} \to \mathbb{R}^n$ be continuous and Lipschitz on $\underline{y}$ (second variable) defined on the closed cylinder $\mathcal{R} = \mathcal{A} \times \mathcal{B}$ for $\mathcal{A} \subseteq \mathbb{R}$. That is,*

$$\forall\, x \in \mathcal{A}\,,\ \forall\, \underline{y}_1, \underline{y}_2 \in \mathcal{B}\,,\ \|\underline{F}(x,\underline{y}_1) - \underline{F}(x,\underline{y}_2)\| \le K\|\underline{y}_1 - \underline{y}_2\|$$

for some $K \in \mathbb{R}$. Then, there is a (closed) interval $\mathcal{I} \subseteq \mathcal{A}$ such that the solution to the differential system:

$$\underline{y}'(x) = \underline{F}(x, \underline{y}(x)) \quad \textit{with}\ \ \underline{y}(x_0) = \underline{y}_0$$

exists and is unique on $\mathcal{I}$, with $x_0 \in \mathcal{I}$.

Proof. Let $\mathcal{R} = [x_0 - a, x_0 + a] \times K_b(\underline{y}_0)$ (cf. Calculus II) and define the set $E = \left\{\underline{\gamma} \in C(\mathcal{I}, \mathbb{R}^n) \;\middle|\; \underline{\gamma}(x_0) = \underline{y}_0 \text{ and } \|\underline{\gamma} - \underline{y}_0\|_\infty \le b\right\}$ (cf. 2.1.8). The proof follows exactly as 2.2.1 and 2.2.2 with the same estimate $\theta = \min\left\{a, \dfrac{b}{M}\right\}$, where $M = \sup_{(x,\underline{y})\in\mathcal{R}} \|\underline{F}(x,\underline{y})\|$. $\square$

Lemma 2.2.4 (Picard Iteration)**.** *Let $\varphi_0(x) \equiv y_0$, define:*

$$\varphi_n(x) = y_0 + \int_{x_0}^{x} F(t, \varphi_{n-1}(t))\, dt$$

for $n \in \mathbb{N}$. Then, $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to the solution of $y' = F(x,y)$, with $y(x_0) = y_0$.

Proof. Follow directly from 2.1.3, using Γ defined in2.2.1. $\square$

Example 2.2.5. *If want to solve $y' = 2x \cdot y$ with $y(0) = 1$, we get, by induction:*

$$\varphi_0(x) = 1; \varphi_1(x) = 1 + \int_0^x 2t\, dt = 1 + x^2 \Rightarrow \varphi_n(x) = \sum_{k=0}^{n} \frac{x^{2k}}{k!}$$

Hence, taking the limit, $y(x) = e^{x^2}$.

Definition 2.2.6 (Uniqueness Property)**.** *An ODE said to have the uniqueness property if $\gamma_1 : \mathcal{I}_1 \to \mathbb{R}$ and $\gamma_2 : \mathcal{I}_2 \to \mathbb{R}$ are both solutions and $\exists\, x_0 \in \mathcal{I}_1 \cap \mathcal{I}_2 : \gamma_1(x_0) = \gamma_2(x_0)$, then:*

$$\forall\, x \in \mathcal{I}_1 \cap \mathcal{I}_2\,,\ \gamma_1(x) = \gamma_2(x)$$

Theorem 2.2.7 (Extension of Picard-Lindelöf)**.** *If $F : \mathcal{R} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous and Lipschitz on y (second variable), then $y' = F(x,y)$ has the uniqueness property (cf. 2.2.6).*

Proof. Let $\mathcal{J} = \{x \in \mathcal{I}_1 \cap \mathcal{I}_2 \mid \gamma_1(x) = \gamma_2(x)\}$. $x_0 \in \mathcal{J}$, so it is not empty. Since γ_1 and γ_2 are continuous, $\mathcal{J} = (\gamma_2 - \gamma_1)^{-1}(\{0\})$ is closed (on $\mathcal{I}_1 \cap \mathcal{I}_2$). Take $s_0 \in \mathcal{J}$. By Picard-Lindelöf, $\exists\, \theta > 0$: the solution to $y' = F(x,y)$ and $y(s_0) = \gamma_1(s_0) = \gamma_2(s_0)$ exists and is unique in $(s_0 - \theta, s_0 + \theta)$. Hence, $(s_0 - \theta, s_0 + \theta) \cap (\mathcal{I}_1 \cap \mathcal{I}_2) \subseteq \mathcal{J}$. Therefore, $\mathcal{J}$ is open (on $\mathcal{I}_1 \cap \mathcal{I}_2$). By connectedness (cf. Calculus II), a non-empty open and closed set must be $\mathcal{J} = \mathcal{I}_1 \cap \mathcal{I}_2$. $\square$

2.3 Existence and Approximate Solutions

Lemma 2.3.1 (Smooth Approximation)**.** *Let $\underline{F} : U \subseteq \mathbb{R}^k \to \mathbb{R}^n$ continuous, then there is a sequence $\{\underline{F}_n \in C^\infty(U, \mathbb{R}^n)\}_{n\in\mathbb{N}}$ the converge uniformly to $\underline{F}$ in compact subsets of U (cf. Calculus II).*

Proof. Define $\varphi_n \in C^\infty(\mathbb{R}^k, \mathbb{R})$ (called a mollifier) such that $\varphi_n(\underline{x}) = 0$ if $\|\underline{x}\| \geq \frac{1}{n}$, $\forall\, \underline{x} \in \mathbb{R}^k$, $\varphi_n(\underline{x}) \geq 0$ and $\displaystyle\int_{\mathbb{R}^k} \varphi_n(\underline{x})\, d^k\underline{x} = \int_{K_{\frac{1}{n}}(\underline{0})} \varphi_n(\underline{x})\, d^k\underline{x} = 1$ (there are many examples). Then, define

$$\underline{F}_n(\underline{x}) = \int_{\mathbb{R}^k} \varphi_n(\underline{x} - \underline{y}) \cdot \underline{F}(\underline{y})\, d^k\underline{y} = \int_{\mathbb{R}^k} \varphi_n(\underline{y}) \cdot \underline{F}(\underline{x} - \underline{y})\, d^k\underline{y}$$

by Leibnitz Rule (cf. Calculus I), $\underline{F}_n^{(j)}(\underline{x}) = \displaystyle\int_{\mathbb{R}^k} \varphi_n^{(j)}(\underline{x} - \underline{y}) \cdot \underline{F}(\underline{y})\, d^k\underline{y}$. So, $\underline{F}_n$ is C^∞. Now, we prove that $\underline{F}_n \to \underline{F}$ uniformly on some compact set $\mathcal{R} \subseteq U$. Calculing: $\|\underline{F}_n(\underline{x}) - \underline{F}(\underline{x})\| = \left\| \displaystyle\int_{\mathbb{R}^k} \varphi_n(\underline{x} - \underline{y}) \cdot \left[\underline{F}(\underline{y}) - \underline{F}(\underline{x})\right] d^k\underline{y} \right\| \leq \displaystyle\int_{\mathbb{R}^k} \varphi_n(\underline{x} - \underline{y}) \cdot \left\|\underline{F}(\underline{y}) - \underline{F}(\underline{x})\right\| d^k\underline{y} = \int_{K_{\frac{1}{n}}(\underline{x})} \varphi_n(\underline{x} - \underline{y}) \cdot \left\|\underline{F}(\underline{y}) - \underline{F}(\underline{x})\right\| d^k\underline{y}$. Since $\underline{F}$ is continuous, it is uniformly continuous on $\mathcal{R}$ (cf. Calculus II). Hence $\forall\, \epsilon > 0\,, \exists\, \delta > 0 \,:\, \forall\, \underline{x}, \underline{y} \in \mathcal{R}\,,\ \|\underline{x} - \underline{y}\| < \delta \Rightarrow \|\underline{F}(\underline{x}) - \underline{F}(\underline{y})\| < \epsilon$. Choose $\dfrac{1}{n} \leq \delta$, then $\|\underline{F}_n(\underline{x}) - \underline{F}(\underline{x})\| < \epsilon$. Therefore, $\underline{F}_n \to \underline{F}$ uniformly. $\square$

Corollary 2.3.2. *The following are true:*

(i) $\forall\, n \in \mathbb{N}\,,\ \sup\limits_{\underline{x}\in U} \|\underline{F}_n(\underline{x})\| \leq \sup\limits_{\underline{x}\in U} \|\underline{F}(\underline{x})\|$

(ii) $\underline{F}_n$ *are Lipschitz with a common constant.*

Proof. Since φ_n is C^∞, it is Lipschitz continuous. $\|\underline{F}_n(\underline{x}_2) - \underline{F}_n(\underline{x}_1)\| \leq \displaystyle\int_{\mathcal{R}} \|\varphi_n(\underline{x}_2 - \underline{y}) - \varphi_n(\underline{x}_1 - \underline{y})\| \cdot \underline{F}(\underline{y})\, d^k\underline{y} \leq K \cdot \|\underline{x}_2 - \underline{x}_1\| \cdot \int_{\mathcal{R}} \underline{F}(\underline{y})\, d^k\underline{y}$. $\square$

Theorem 2.3.3 (Peano Existence)**.** *Let $\underline{F} : \mathcal{R} \subseteq \mathbb{R}^{1+n} \to \mathbb{R}^n$ be continuous, where $\mathcal{R}$ is a closed cylinder. Then, there is a (closed) interval $\mathcal{I}$ such that the solution to the differential equation/system:*

$$\underline{y}'(x) = \underline{F}(x, \underline{y}(x)) \quad \textit{with} \quad \underline{y}(x_0) = \underline{y}_0$$

exists on $\mathcal{I}$, with $x_0 \in \mathcal{I}$.

Proof. Let $\mathcal{R} = [x_0 - a, x_0 + a] \times K_b(\underline{y}_0)$. Take $\{\underline{F}_n \in C^\infty(,\mathbb{R}^n)\}_{n\in\mathbb{N}}$ that converge uniformly to F (cf. 2.3.1). By 2.2.1, 2.2.2 and 2.3.2, for each $n \in \mathbb{N}$ there is a unique solution $\underline{y}_n(x)$ to the equation $\underline{y}_n'(x) = F_n(x, \underline{y}_n(x))$ with $\underline{y}_n(x_0) = \underline{y}_0$ on the interval $[x_0 - \theta_n, x_0 + \theta_n]$ where $\theta_n = \min\left\{a, \dfrac{b}{M_n}\right\}$. Using 2.3.2, we define $\theta = \min\left\{a, \dfrac{b}{M}\right\} \le \theta_n$ and $\mathcal{I} = [x_0 - \theta, x_0 + \theta]$. Writing in integral form: $\underline{y}_n(x) = \underline{y}_0 + \displaystyle\int_{x_0}^{x} \underline{F}_n(t, \underline{y}_n(t))\,dt$. We now prove $\left\{\underline{y}_n\right\}_{n\in\mathbb{N}}$ is equicontinuous: (cf. 2.1.10)

$$\|\underline{y}_n(x_2) - \underline{y}_n(x_1)\| = \left\|\int_{x_1}^{x_2} \underline{F}_n(t, \underline{y}_n(t))\,dt\right\| \le M \cdot |x_2 - x_1|$$

By 2.1.11, there is a converging subsequence $\left\{\underline{y}_{n_k}\right\}_{k\in\mathbb{N}}$ let $\underline{y}_{n_k} \to \underline{y}$ uniformly in $\mathcal{I}$. Taking $n = n_k \to \infty$ on both sides of the integral equation, and using 2.1.9 $\underline{F}_{n_k}(t, \underline{y}_{n_k}(t)) \to \underline{F}(t, \underline{y}(t))$, then, $\underline{y}$ is a solution to $\underline{y}(x) = \underline{y}_0 + \displaystyle\int_{x_0}^{x} \underline{F}(t, \underline{y}(t))\,dt$, hence it is a solution to the initial value system in $\mathcal{I}$. □

Remark 2.3.4. *The proof of 2.3.3 is simply to take an approximation of $\underline{F}$, solve each ODE using 2.2.1 and show the solutions converge to a solution to the original ODE.*

Lemma 2.3.5 (Grönwall)**.** *Let $\forall\, x > a\,,\ u'(x) \le \beta(x) \cdot u(x)$, then*

$$\forall\, x \ge a\,,\ u(x) \le u(a) \cdot \exp\left[\int_a^x \beta(t)\,dt\right]$$

Proof. Let $v(x) = \exp\left[\displaystyle\int_a^x \beta(t)\,dt\right]$, so $v'(x) = \beta(x) \cdot v(x)$ and $v(x) \ge 0$. By the quotient rule:

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2} = \frac{u'(x) - \beta(x) \cdot u(x)}{v(x)} \le 0$$

Then, $\dfrac{u(x)}{v(x)}$ is monotonically decreasing (technically, non-increasing). Then, $\forall\, x \le a\,,\ \dfrac{u(x)}{v(x)} \le \dfrac{u(a)}{v(a)} = u(a)$. □

Corollary 2.3.6. *Same statement as 2.2.7.*

Proof. Define: $u(x) = \displaystyle\int_{x_0}^{x} \big\|\gamma_1(t) - \gamma_2(t)\big\|\, dt$. Then:

$$u'(x) = \|\gamma_1(x) - \gamma_2(x)\| = \left\| \int_{x_0}^{x} \big[F(t, \gamma_1(t)) - F(t, \gamma_2(t))\big]\, dt \right\|$$

$$\leq \int_{x_0}^{x} \big\|F(t, \gamma_1(t)) - F(t, \gamma_2(t))\big\|\, dt \leq K \cdot \int_{x_0}^{x} \big\|\gamma_1(t) - \gamma_2(t)\big\|\, dt = K \cdot u(x)$$

Moreover, $u(x_0) = 0$ and $u(x) \geq 0$. By 2.3.5, $u(x) \leq u(x_0) \cdot e^{K(x-x_0)} = 0$. Hence, $u(x) \equiv 0$, then $u'(x) \equiv 0$, so $\gamma_1 \equiv \gamma_2$. $\square$

3 Linear ODEs

3.1 General Analysis

Definition 3.1.1 (Linear ODE). *For $\mathcal{F} : U \subset \mathbb{R}^{1+n} \to \mathbb{R}$, the implicit ODE $\mathcal{F}(x, y', \cdots, y^{(n)}) = 0$ is called linear iff F is linear on $y', \cdots, y^{(n)}$, equivaletly:*

$$\sum_{k=0}^{n} a_k(x) \cdot y^{(k)} = b(x)$$

For functions $a_k(x)$ and $b(x)$. Where $y^{(0)} = y$.

Definition 3.1.2 (Homogeneous Linear ODE). *If $b(x) \equiv 0$ in the following definition, the ODE is homogeneous.*

Definition 3.1.3 (Differential Operator). *Define the operator:*

$$\begin{aligned} L : C^n(\mathbb{R}) &\to C(\mathbb{R}) \\ \varphi &\mapsto L[\varphi](x) = \sum_{k=0}^{n} a_k(x) \cdot \varphi^{(k)}(x) \end{aligned}$$

Then, the ODE $\sum_{k=0}^{n} a_k(x) \cdot y^{(k)} = b(x)$ is expressed as $L[y](x) = b(x)$. We may rewrite it as:

$$L = \sum_{k=0}^{n} a_k(x) \cdot \mathfrak{D}^k$$

where $\mathfrak{D}$ is the differentiating operator.

Lemma 3.1.4. *The operator L is linear, where $C^n(\mathbb{R})$ is equipped with addition and scalar multiplication of functions.*

Proof. The differentiating operator is linear (cf. Calculus I) and multiplication operators $a_k(x) \cdot \mathrm{id}$ are also linear. By composition and sum (cf. Linear Algebra), L is linear. □

Corollary 3.1.5. *The set of solutions to a homogeneous ODE $L[y] \equiv 0$ is* $\ker(L)$*, hence it is a vector subpace of $C^n(\mathbb{R})$.*

3.2 Fundamental Set and Wronskian

Definition 3.2.1. *A set* $\{y_1, y_2, \cdots, y_n\} \subset C^n(\mathcal{I})$ *is linearly dependent if:*

$$\exists\, (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\} : \sum_{k=1}^{n} a_k \cdot y_k(x) \equiv 0$$

It is linear independent otherwise. For that set, $\mathcal{Y} = (y_1, \cdots, y_n)$ *ordered as a sequence, we define the fundamental matrix:*

$$M_{\mathcal{Y}}(x) = \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

and the Wronskian: $W_{\mathcal{Y}}(x) = \det(M_{\mathcal{Y}}(x))$

Lemma 3.2.2. *A set* $\mathcal{Y} = \{y_1, y_2, \cdots, y_n\} \subset C^n(\mathcal{I})$ *is linearly dependent iff* $W_{\mathcal{Y}} \equiv 0$.

Proof. By definition, it is linearly independent iff $\exists\, (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\} : \sum_{k=1}^{n} a_k \cdot y_k(x) \equiv 0$. Differentiating, we obtain the following system:

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which, for a given $x \in \mathcal{I}$, has a non-trivial solution iff the determinant is zero (cf. Linear Algebra), that is, $W_{\mathcal{Y}}(x) = 0$. $\square$

Corollary 3.2.3. $\exists\, x_0 \in \mathcal{I} : W_{\mathcal{Y}}(x_0) \neq 0$ *iff* $\mathcal{Y}$ *is linearly independent.*

Definition 3.2.4. *For a linear ODE* $L[y](x) = b(x)$*, a linear independent set of solutions for the homogeneous equation is called a fundamental set.*

Theorem 3.2.5 (Abel's Formula)**.** *Let* $y_1, y_2, \cdots y_n$ *be solutions to the homogeneous ODE* $\sum_{k=0}^{n} a_k(x) \cdot y^{(k)}(x) \equiv 0$*, then:*

$$W_{\mathcal{Y}}(x) = C \exp\left[-\int \frac{a_{n-1}(x)}{a_n(x)}\, dx\right] = W_{\mathcal{Y}}(a) \cdot \exp\left[-\int_a^x \frac{a_{n-1}(t)}{a_n(t)}\, dt\right]$$

Proof. By the product rule,

$$
\begin{aligned}
W'_{\mathcal{Y}}(x) &= \cancel{\begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}} + \cancel{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}} + \\
&\quad \cdots + \cancel{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \\
&= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -\sum_{k=0}^{n-1} \frac{a_k(x)}{a_n(x)} \cdot y_1^{(k)} & -\sum_{k=0}^{n-1} \frac{a_k(x)}{a_n(x)} \cdot y_2^{(k)} & \cdots & -\sum_{k=0}^{n-1} \frac{a_k(x)}{a_n(x)} \cdot y_n^{(k)} \end{vmatrix} \\
&= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -\frac{a_{n-1}(x)}{a_n(x)} \cdot y_1^{(n-1)} & -\frac{a_{n-1}(x)}{a_n(x)} \cdot y_2^{(n-1)} & \cdots & -\frac{a_{n-1}(x)}{a_n(x)} \cdot y_n^{(n-1)} \end{vmatrix} \\
&= -\frac{a_{n-1}(x)}{a_n(x)} \cdot W_{\mathcal{Y}}(x)
\end{aligned}
$$

The result follows by 1.2.1. □

Corollary 3.2.6. *For solutions of a homogeneous ODE, the Wronskian is either zero everywhere or non-zero everywhere. Hence, for a fundamental set* $\mathcal{Y}$*,* $\forall\, x \in \mathcal{I}\,,\ W_{\mathcal{Y}}(x) \neq 0$.

Theorem 3.2.7. *For a fundamental set* $\mathcal{Y}$ *of the homogeneous ODE* $L[y] \equiv 0$ *, then*

$$\ker(L) = \mathrm{Span}(\mathcal{Y})$$

Proof. By existence and uniqueness theorem, we need to prove that every initial condition can be achieved by a linear combination of $y_1, y_2, \cdots, y_n$. Let $\sum_{k=1}^{n} \alpha_k \cdot y_k(x) = y(x)$, by differentiating and checking the initial conditions:

$$
\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}
$$

the matrix is invertible since $W_{\mathcal{Y}}(x_0) \neq 0$ (due to 3.2.6) $\square$

Corollary 3.2.8. *The solution to the ODE $L[y] \equiv 0$ with an initial values at x_0 is given by:*

$$y_h(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}^t M_{\mathcal{Y}}^{-1}(x_0) \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}$$

where $\mathcal{Y} = (y_1, y_2, \cdots, y_n)$ is a fundamental set.

3.3 Variation of Parameters

Theorem 3.3.1. *A particular solution to* $L[y](x) = \sum_{k=0}^{n} a_k(x) \cdot y^{(k)} = b(x)$ *is given by:*

$$y_p(x) = \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot y_k(x) \right] b(t)\, dt = \sum_{k=1}^{n} \left[\int_a^x \omega_k(t) \cdot b(t)\, dt \right] \cdot y_k(x)$$

where $\mathcal{Y} = (y_1, \cdots, y_{n-1}, y_n)$ *is a fundamental set and* ω_k *are given by:*

$$\begin{bmatrix} \omega_1(x) \\ \vdots \\ \omega_{n-1}(x) \\ \omega_n(x) \end{bmatrix} = \frac{1}{a_n(x)}\; M_{\mathcal{Y}}^{-1}(x)\, \vec{e}_n$$

Proof. We differentiate the formula for $y_p(x)$:

$$\begin{aligned}
y_p'(x) &= b(x) \cdot \sum_{k=1}^{n} \omega_k(x) \cdot y_k(x) + \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot y_k'(x) \right] b(t)\, dt \\
&= \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot y_k'(x) \right] b(t)\, dt \\
y_p^{(j)}(x) &= \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot y_k^{(j)}(x) \right] b(t)\, dt \;\text{ for } 0 \le j \le n-1 \\
y_p^{(n)}(x) &= b(x) \cdot \sum_{k=1}^{n} \omega_k(x) \cdot y_k^{(n-1)}(x) + \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot y_k^{(n)}(x) \right] b(t)\, dt \\
&= \frac{b(x)}{a_n(x)} + \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot y_k^{(n)}(x) \right] b(t)\, dt
\end{aligned}$$

Finally, we have:

$$\sum_{j=0}^{n} a_j(x) \cdot y_p^{(j)}(x) = b(x) + \int_a^x \left[\sum_{k=1}^{n} \omega_k(t) \cdot \left(\sum_{j=0}^{n} a_j(x) \cdot y_k^{(j)}(x) \right) \right] b(t)\, dt = b(x)$$

hence, it is a solution. Further, it is the unique solution for the initial condition $y^{(j)}(a) = 0$ for $0 \le j \le n-1$. $\square$

Corollary 3.3.2. *The solution to the ODE $L[y](x) = b(x)$ with the initial values at x_0 is given by: $y(x) = y_p(x) + y_h(x)$ (cf. 4.1.9,3.3.1).*

Theorem 3.3.3. *Let $y_0(x)$ be a solution to $L[y] \equiv 0$. Then,*

$$y(x) = y_0(x) \cdot \int \gamma(x)\, dx = y_0(x) \cdot \left(\gamma_0 + \int_a^x \gamma(t)\, dt\right)$$

where $\gamma(x)$ is a solution a $(n-1)$-th degree linear ODE.

Proof. Let $y(x) = Y(x) \cdot y_0(x)$. Leibnitz: $y^{(k)} = \sum_{j=0}^{k} \binom{k}{j} Y^{(j)}(x) \cdot y_0^{(k-j)}(x)$

$$\begin{aligned} 0 \equiv \sum_{k=0}^{n} a_k(x) \cdot y^{(k)} &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} a_k(x) \cdot Y^{(j)}(x) \cdot y_0^{(k-j)}(x) \\ = Y(x) \cdot \sum_{k=0}^{n} a_k(x) \cdot y_0^{(k)}(x) &+ \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{k}{j} a_k(x) \cdot Y^{(j)}(x) \cdot y_0^{(k-j)}(x) \\ &= \sum_{j=1}^{n} \left[\sum_{k=j}^{n} \binom{k}{j} a_k(x) \cdot y_0^{(k-j)}(x) \right] Y^{(j)}(x) \equiv 0 \end{aligned}$$

Hence, let $Y(x) = \gamma_0 + \displaystyle\int_a^x \gamma(t)\, dt$. We get the ODE: $\displaystyle\sum_{k=0}^{n-1} b_k(x) \cdot \gamma^{(k)}(x) \equiv 0$ defined by $b_j(x) = \displaystyle\sum_{k=j+1}^{n} \binom{k}{j+1} a_k(x) \cdot y_0^{(k-j-1)}(x)$. Further, $b_{n-1}(x) = a_n(x) \not\equiv 0$, hence the order is $n-1$. □

Corollary 3.3.4. *A n-th order linear ODE has a fundamental set of size n.*

3.4 Constant Coefficients

Definition 3.4.1 (Characteristic Polynomial)**.** *For the homogeneous linear ODE* $\sum_{k=0}^{n} a_k \cdot y^{(k)}(x) \equiv 0$ *with* $a_k \in \mathbb{R}$*, define the polynomial:*

$$\chi(s) = \sum_{k=0}^{n} a_k \cdot s^k$$

We may rewrite it as $L = \sum_{k=0}^{n} a_k \cdot \mathfrak{D}^k = \chi(\mathfrak{D})$ *where* $\mathfrak{D}$ *is the differentiating operator.*

Lemma 3.4.2 (Factoring)**.** *Let* χ *be the characteristic polynomial of* L*, then, if* χ *can be factored into:*

$$\chi(s) = a_n \prod_{i=1}^{N} (s - \lambda_i)^{\mu_i} \Rightarrow L = a_n \prod_{i=1}^{N} (\mathfrak{D} - \lambda_i \cdot \mathrm{id})^{\mu_i}$$

where $\prod$ *in* L *means composition.*

Proof. First, we show $\mathfrak{D} - \lambda_i \cdot \mathrm{id}$ and $\mathfrak{D} - \lambda_j \cdot \mathrm{id}$ commute:

$$\begin{aligned}
&(\mathfrak{D} - \lambda_i \cdot \mathrm{id}) \circ (\mathfrak{D} - \lambda_j \cdot \mathrm{id})[\varphi] = (\mathfrak{D} - \lambda_i \cdot \mathrm{id})[\varphi' - \lambda_j \cdot \varphi] \\
&= \varphi'' - (\lambda_i + \lambda_j) \cdot \varphi' + \lambda_i \lambda_j \cdot \varphi = \varphi'' - (\lambda_j + \lambda_i) \cdot \varphi' + \lambda_j \lambda_i \cdot \varphi \\
&= \cdots = (\mathfrak{D} - \lambda_j \cdot \mathrm{id}) \circ (\mathfrak{D} - \lambda_i \cdot \mathrm{id})[\varphi]
\end{aligned}$$

The rest follows from $L = \chi(\mathfrak{D})$. □

Remark 3.4.3. *By Fundamental Theorem of Algebra, every polynomial can be factored as such.*

Lemma 3.4.4 (Exponential Solutions)**.** *Let* $\gamma(x) \in \ker(\mathfrak{D} - \lambda \cdot \mathrm{id})^{\mu}$*, iff* $e^{-\lambda x} \gamma(x) \in \ker(\mathfrak{D}^{\mu}) = \mathbb{R}_{\mu-1}[x]$*. That is*

$$\ker(\mathfrak{D} - \lambda \cdot \mathrm{id})^{\mu} = \left\{ \wp(x) \cdot e^{\lambda x} \;\middle|\; \wp \in \mathbb{R}_{\mu-1}[x] \right\}$$

Proof. $\mathfrak{D}\left[e^{-\lambda x} \gamma(x)\right] = e^{-\lambda x} \cdot (\mathfrak{D} - \lambda \cdot \mathrm{id})\big[\gamma(x)\big]$ and, by induction, we get: $\mathfrak{D}^{\mu}\left[e^{-\lambda x} \gamma(x)\right] = e^{-\lambda x} \cdot (\mathfrak{D} - \lambda \cdot \mathrm{id})^{\mu}\big[\gamma(x)\big]$ □

Theorem 3.4.5. *For* $\chi(s) = a_n \prod_{i=1}^{N} (s - \lambda_i)^{\mu_i}$*, then*

$$\mathcal{Y} = \left\{ x^k e^{\lambda_i x} \;\middle|\; 0 \le k \le \mu_i - 1 \,,\ 1 \le i \le N \right\}$$

is a fundamental set of $\sum_{k=0}^{n} a_k \cdot y^{(k)}(x) \equiv 0$

Proof. We will show $W_{\mathcal{Y}}(0) \neq 0$ (cf. 3.2.6). It can be shown that $W_{\mathcal{Y}}(0) = \prod_{i<j}(\lambda_i - \lambda_j)^{\mu_i \mu_j}$, which is clearly non-zero, but we'll stick with showing the determinant is non-zero. □

Remark 3.4.6. *We allowed our solution to be $y : \mathbb{R} \to \mathbb{C}$. However, we can express the solution set as purely real function, which are useful since, if the initial conditions and the coefficients are real, then the solutions are real (cf. 3.2.7)*

Lemma 3.4.7 (Complex Roots)**.** *For a complex root $s = \alpha + \beta\, i$, there is another root $s = \alpha - \beta\, i$ and the solutions $\{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$ substitute for $\{e^{(\alpha+\beta\, i)x}, e^{(\alpha-\beta\, i)x}\}$ on 3.4.5.*

Proof. Since $\chi \in \mathbb{R}[s]$, taking the conjugate of $\chi(s) = 0$ shows $s \in \mathbb{C}$ is a root iff $\bar{s} \in \mathbb{C}$. Further, Euler's Formula shows:

$$\begin{aligned} y(x) &= f(x)\, e^{(\alpha+\beta\, i)x} + g(x)\, e^{(\alpha-\beta\, i)x} \\ &= e^{\alpha x} \Big[(f(x) + g(x)) \cos(\beta x) + i(f(x) - g(x)) \sin(\beta x) \Big] \\ &= \tilde{f}(x)\, e^{\alpha x} \cos(\beta x) + \tilde{g}(x)\, e^{\alpha x} \sin(\beta x) \end{aligned}$$

□

4 Linear Systems

4.1 General Analysis

Definition 4.1.1 (First Order System)**.** *Given a function $\underline{F} : U \subset \mathbb{R}^{1+n} \to \mathbb{R}^n$, an (explicit) first-order system is a vector equation of the following form:*

$$\underline{y}' = \underline{F}(x, \underline{y})$$

for a function $\underline{y} : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}^n$ differentiable. The system is linear, if there are $\mathcal{A} : \mathbb{R} \to M_n(\mathbb{R})$ (cf. Linear Algebra) and $\underline{b} : \mathbb{R} \to \mathbb{R}^n : \underline{F}(x, \underline{y}) = \mathcal{A}(x)\,\underline{y} + \underline{b}(x)$, hence:

$$\underline{y}' = \mathcal{A}(x)\,\underline{y} + \underline{b}(x)$$

Lemma 4.1.2 (Phase Transformation)**.** *Let $F : U \subseteq \mathbb{R}^{1+n} \to \mathbb{R}$, then, there is a function $\underline{F} : U \subseteq \mathbb{R}^{1+n} \to \mathbb{R}^n$, so the ODE $y^{(n)} = F(x, y, \cdots, y^{(n-1)})$ becomes $\underline{y}' = \underline{F}(x, \underline{y})$ which is first order on $\underline{y} = \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix}$.*

Proof. Define: $\underline{F} : U \subseteq \mathbb{R}^{1+n} \to \mathbb{R}^n$ s.t. $\underline{F}\left(x, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = \begin{bmatrix} y_2 \\ \vdots \\ y_n \\ F(x, y_1, \cdots, y_n) \end{bmatrix}$.

Then, the ODE wil take the form:

$$\underline{F}(x, \underline{y}) = \underline{F}\left(x, \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix}\right) = \begin{bmatrix} y' \\ \vdots \\ y^{(n-1)} \\ F(x, y, \cdots, y^{(n-1)}) \end{bmatrix} = \begin{bmatrix} y' \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{bmatrix} = \underline{y}'$$

which is what we were looking for. □

Corollary 4.1.3. *For the equation $y^{(n)} + \sum_{i=0}^{n-1} a_k(x) \cdot y^{(k)}(x) = b(x)$, we get:*

$$\underline{y}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(x) & -a_1(x) & -a_2(x) & \cdots & -a_{n-1}(x) \end{bmatrix} \underline{y} + b(x)\,\underline{e}_n$$

Definition 4.1.4. *A set* $\{\underline{y}_1, \underline{y}_2, \cdots, \underline{y}_n\} \subset C^1(\mathcal{I}, \mathbb{R}^n)$ *is linearly dependent if:* $\exists\, (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n \setminus \{\underline{0}\} : \sum_{k=1}^n a_k \cdot \underline{y}_k(x) \equiv \underline{0}$. *It is linear independent otherwise. For that set,* $\mathcal{Y} = (\underline{y}_1, \cdots, \underline{y}_n)$ *ordered as a sequence, we define the fundamental matrix:*

$$M_{\mathcal{Y}}(x) = \begin{bmatrix} | & | & & | \\ \underline{y}_1 & \underline{y}_2 & \cdots & \underline{y}_n \\ | & | & & | \end{bmatrix}$$

and the Wronskian: $W_{\mathcal{Y}}(x) = \det(M_{\mathcal{Y}}(x))$. *This first with 3.2.1 due to 4.1.2.*

Remark 4.1.5. *The ODE* $\underline{y}' = \mathcal{A}(x)\,\underline{y} + \underline{b}(x)$ *can be expressed as* $L[\underline{y}](x) = \underline{b}(x)$. *We may rewrite it as:* $L = \mathfrak{D} - \mathcal{A}(x)$ *where* $\mathfrak{D}$ *is tuple differentiation.*

Definition 4.1.6. *For a linear ODE* $L[\underline{y}](x) = \underline{b}(x)$, *a linear independent set of solutions for the homogeneous equation is called a fundamental set.*

Theorem 4.1.7 (Liouville Formula)**.** *Let* $\mathcal{Y} = (\underline{y}_1, \underline{y}_2, \cdots \underline{y}_n)$ *be solutions to the homogeneous ODE* $\underline{y}' = \mathcal{A}(x)\,\underline{y}$, *then:*

$$W_{\mathcal{Y}}(x) = C\,\exp\left[\int \operatorname{tr}(\mathcal{A}(x))\,dx\right] = W_{\mathcal{Y}}(a)\cdot\exp\left[\int_a^x \operatorname{tr}(\mathcal{A}(t))\,dt\right]$$

Proof. By the product rule,

$$\begin{aligned}
W_{\mathcal{Y}}'(x) &= \begin{vmatrix} y_{1,1}' & y_{2,1}' & \cdots & y_{n,1}' \\ y_{1,2} & y_{2,2} & \cdots & y_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,n} & y_{2,n} & \cdots & y_{n,n} \end{vmatrix} + \cdots + \begin{vmatrix} y_{1,1} & y_{2,1} & \cdots & y_{n,1} \\ y_{1,2} & y_{2,2} & \cdots & y_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,n}' & y_{2,n}' & \cdots & y_{n,n}' \end{vmatrix} \\
&= \begin{vmatrix} \sum_{k=1}^n a_{1,k}\cdot y_{1,k} & \sum_{k=1}^n a_{1,k}\cdot y_{2,k} & \cdots & \sum_{k=1}^n a_{1,k}\cdot y_{n,k} \\ y_{1,2} & y_{2,2} & \cdots & y_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,n} & y_{2,n} & \cdots & y_{n,n} \end{vmatrix} + \\
&\cdots + \begin{vmatrix} y_{1,1} & y_{2,1} & \cdots & y_{n,1} \\ y_{1,2} & y_{2,2} & \cdots & y_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{n,k}\cdot y_{1,k} & \sum_{k=1}^n a_{n,k}\cdot y_{2,k} & \cdots & \sum_{k=1}^n a_{n,k}\cdot y_{n,k} \end{vmatrix} \\
&= \begin{vmatrix} a_{1,1}\cdot y_{1,1} & \cdots & a_{1,1}\cdot y_{n,1} & \\ y_{1,2} & y_{2,2} & \cdots & y_{n,2} \\ \vdots & \ddots & \vdots & \\ y_{1,n} & \cdots & y_{n,n} & \end{vmatrix} + \cdots + \begin{vmatrix} y_{1,1} & \cdots & y_{n,1} \\ y_{1,2} & \cdots & y_{n,2} \\ \vdots & \ddots & \vdots \\ a_{n,n}\cdot y_{1,n} & \cdots & a_{n,n}\cdot y_{n,n} \end{vmatrix} \\
&= (a_{1,1} + \cdots + a_{n,n})\cdot W_{\mathcal{Y}}(x) = \operatorname{tr}(\mathcal{A}(x))\cdot W_{\mathcal{Y}}(x)
\end{aligned}$$

The result follows by 1.2.1. ∎

Lemma 4.1.8. *For a linear system* $\underline{y}' = \mathcal{A}(x)\,\underline{y} + \underline{b}(x)$ *and* $\mathcal{Y} = (\underline{y}_1, \underline{y}_2, \cdots, \underline{y}_n)$ *be fundamental solutions* $(L[\underline{y}] \equiv \underline{0})$ *, then* $\ker(L) = \mathrm{Span}(\mathcal{Y})$.

Proof. Analogous to 3.2.7. □

Corollary 4.1.9. *The solution to the ODE* $L[\underline{y}] \equiv \underline{0}$ *with an initial values at* x_0 *is given by:*

$$y_h(x) = M_{\mathcal{Y}}(x)\ M_{\mathcal{Y}}^{-1}(x_0)\ \underline{y}(x_0)$$

where $\mathcal{Y} = (\underline{y}_1, \underline{y}_2, \cdots, \underline{y}_n)$ *is a fundamental set.*

Remark 4.1.10. *The fundamental matrix satisfies:* $M'_{\mathcal{Y}}(x) = \mathcal{A}(x)\, M_{\mathcal{Y}}(x)$.

Lemma 4.1.11 (Variation of Parameters)**.** *A particular solution to* $L[\underline{y}](x) = \underline{y}' - \mathcal{A}(x)\,\underline{y} = \underline{b}(x)$ *is given by:*

$$\underline{y}_p(x) = M_{\mathcal{Y}}(x) \int_a^x M_{\mathcal{Y}}^{-1}(t)\ \underline{b}(t)\, dt$$

where $\mathcal{Y} = (y_1, \cdots, y_{n-1}, y_n)$ *is a fundamental set.*

Proof. By direct calculation (Leibnitz rules):

$$\underline{y}'_p(x) = M'_{\mathcal{Y}}(x) \int_a^x M_{\mathcal{Y}}^{-1}(t)\ \underline{b}(t)\, dt + M_{\mathcal{Y}}(x)\ M_{\mathcal{Y}}^{-1}(x)\ \underline{b}(x) = \mathcal{A}(x)\,\underline{y}_p(x) + \underline{b}(x)$$

since $M'_{\mathcal{Y}}(x) = \mathcal{A}(x)\, M_{\mathcal{Y}}(x)$. □

4.2 Jordanization of Exponential Matrix

Definition 4.2.1 (Matrix Exponential)**.** *For $A \in M_n(\mathbb{C})$, define:*

$$\exp(A\,x) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (A\,x)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot A^k$$

(considering $A^n = \overbrace{A \times \cdots \times A}^{n \text{ times}}$ and $A^0 = I$) which converges entrywise to the matrix $\exp(A\,x) \in \mathrm{GL}_n(\mathbb{C})$ (cf. Linear Algebra) for every $A \in M_n(\mathbb{C})$ and $x \in \mathbb{C}$.

Theorem 4.2.2. *For the ODE $\underline{y}' = A\,\underline{y}$, the solution is $\underline{y} = \exp(A\,x)\,\underline{y}(0)$. Moreover, $\ker(L) = \mathrm{cols}(\exp(A\,x))$ for $L[\underline{y}] = \underline{y}' - A\,\underline{y}$.*

Proof. Calculating: $\underline{y}(x) = \exp(A\,x)\,\underline{y}(\vec{0}) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot A^k\,\underline{y}(0)$

$$\begin{aligned}\underline{y}'(x) &= \sum_{k=0}^{\infty} \frac{k \cdot x^{k-1}}{k!} \cdot A^k\,\underline{y}(0) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} \cdot A^k\,\underline{y}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot A^{k+1}\,\underline{y}(0) = A\,\exp(A\,x)\,\underline{y}(0) = A\,\underline{y}(x)\end{aligned}$$

Further, it can be rewritten as: $\underline{y}(x) = \exp[A\,(x - x_0)]\,\underline{y}(x_0)$, which is the unique solution for any given $\underline{y}(x_0)$. □

Corollary 4.2.3. *A fundamental set $\mathcal{Y}$ for $\underline{y}' = A\underline{y}$ is: $M_{\mathcal{Y}}(x) = \exp(A\,x)$.*

Lemma 4.2.4. *For $A, B \in M_n(\mathbb{C})$, if $A \sim B$ (cf. Linear Algebra), that is, $\exists\, P \in \mathrm{GL}_n(\mathbb{C}) : A = P\,B\,P^{-1}$, then: $\forall\, x \in \mathbb{C}\,,\ \exp(A\,x) = P\,\exp(B\,x)\,P^{-1}$, and so, $\exp(A\,x) \sim \exp(B\,x)$.*

Proof. Calculating: $\exp(A\,x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot A^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot (P\,B\,P^{-1})^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot P\,B^k\,P^{-1} = P\,\left[\sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot B^k\right]\,P^{-1} = P\,\exp(B\,x)\,P^{-1}$ □

Corollary 4.2.5. *If A is diagonalizable (cf. Linear Algebra), then the exponential is: $\exp(A\,x) = P\,\exp(\Lambda\,x)\,P^{-1}$, for $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_k)$ and $\exp(\Lambda x) = \mathrm{diag}(e^{\lambda_1 x}, \cdots, e^{\lambda_k x})$. Moreover,*

$$\mathrm{cols}(\exp(A\,x)) = \mathrm{Span}(\underline{v}_1\,e^{\lambda_1 x}, \cdots, \underline{v}_k\,e^{\lambda_k x})$$

where $\underline{v}_i$ are eigenvectors with eigenvalue λ_i.

Lemma 4.2.6. *For $A, B \in M_n(\mathbb{R})$, if $AB = BA$, then* $\exp(A\,x)\,\exp(B\,x) = \exp((A+B)\,x)$.

Proof. If follows from the same arithmetics that shows $e^{ax} \cdot e^{bx} = e^{(a+b)x}$. $\square$

Definition 4.2.7. *A chain of generalized eigenvectors, also called a Jordan chain, of $A \in M_n(\mathbb{C})$ with eigenvalue λ is sequence $\mathcal{C}_m(\lambda) = (\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_m) \in (\mathbb{C}^n \setminus \{\underline{0}\})^m$ such that:*

$$A\,\underline{v}_1 = \lambda \cdot \underline{v}_1 \quad \textit{and} \quad A\,\underline{v}_k = \lambda \cdot \underline{v}_k + \underline{v}_{k-1} \quad \textit{for } 2 \le k \le m$$

Lemma 4.2.8. *A Jordan chain of A, $\mathcal{C}_m(\lambda)$, is a linear independent sequence in* $\operatorname{sols}(A - \lambda I)^m$ *(cf. Linear Algebra).*

Proof. Notice $\forall\, 1 \le k \le m\,,\ (A - \lambda I)^k\,\underline{v}_k = \underline{0}$ and $(A - \lambda I)^{k-1}\underline{v}_k = \underline{v}_{k-1} \neq \underline{0}$. Hence, they are in $\operatorname{sols}(A - \lambda I)^m$. To prove linear independence, let $\sum_{k=1}^{m} \alpha_k \cdot \underline{v}_k = \underline{0}$. Applying $(A - \lambda I)^{m-1}$ to both sides gives $\alpha_m \cdot \underline{v}_{m-1} = \underline{0} \Rightarrow \alpha_m = 0$. By induction on applying $(A - \lambda I)^k$ and getting $\alpha_k = 0$, we get, $\forall\, 1 \le k \le m\,,\ \alpha_k = 0$. Therefore, the chain is linearly independent. $\square$

Theorem 4.2.9. *For any $A \in M_n(\mathbb{R})$ there is a set of Jordan chains of A whose union is a basis of $\mathbb{R}^n$.*

Proof. By induction on n:

- Base Case: A is a multiplication operator, so it has a unique eigenvector and eigenvalue.
- Take an eigenvalue λ of A, $\dim \operatorname{cols}(A - \lambda\, I) = n - \dim \operatorname{sols}(A - \lambda\, I) \le n - 1$. By induction, there is a set of Jordan chains of A whose union is a basis of $\operatorname{cols}(A - \lambda\, I)$.
  - If $\operatorname{cols}(A - \lambda\, I) \cap \operatorname{sols}(A - \lambda\, I) = \{\underline{0}\}$, then there are $\operatorname{am}(\lambda) = \operatorname{sols}(A - \lambda\, I)$ linearly independent eigenvectors.
  - Then, there is a chain of λ in $\operatorname{cols}(A - \lambda\, I)$. Take one of those chains $\mathcal{C}_{m-1}(\lambda) = (\underline{v}_1, \cdots, \underline{v}_{m-1})$. Since $\underline{v}_{m-1} \in \operatorname{cols}(A - \lambda\, I)$, $\exists\, \underline{v}_m \in \mathbb{R}^n : (A - \lambda\, I)\,\underline{v}_m = \underline{v}_{m-1}$. Hence, the chain increases by 1: $\mathcal{C}_m(\lambda) = (\underline{v}_1, \cdots, \underline{v}_m)$. So, every eigenvector is in a chain, which may include just itself $\mathcal{C}_1(\lambda) = (\underline{u}_1)$.

  Hence, we added $\dim \operatorname{sols}(A - \lambda\, I)$ vectors to the chains. Therefore, we have a set of Jordan chains of A whose union is a basis of $\mathbb{R}^n$, since we have n linearly independent vectors.

$\square$

Corollary 4.2.10. *For each eigenvalue, there is a set of Jordan chains of A, each with same eigenvalue λ, with* $\mathrm{am}(\lambda)$ *(cf. Linear Algebra) vectors in the union.*

Theorem 4.2.11. *Let $\mathcal{C}_m(\lambda) = (\underline{v}_1, \cdots, \underline{v}_m)$ be a Jordan chain of A. Then, the following are linearly independent solutions to $\underline{y}' = A\,\underline{y}$, for $1 \leq k \leq m$:*

$$\underline{y}_k(x) = e^{\lambda x} \sum_{j=0}^{k-1} \frac{x^j}{j!} \cdot \underline{v}_{k-j}$$

Proof. Sufficient to notice $\underline{y}_k(x) = \exp(A\,x)\,\underline{v}_k = e^{\lambda\,x} \exp((A - \lambda\,I)\,x)\,\underline{v} = e^{\lambda x} \sum_{j=0}^{k-1} \frac{x^j}{j!} \cdot \underline{v}_{k-j}$, since $(A - \lambda\,I)^j\,\underline{v}_k = \underline{v}_{k-j}$, for $j < k$, and $(A - \lambda\,I)^k\,\underline{v}_k = \underline{0}$, which is a solution due to 4.2.2. $\square$

5 Integral and Series Methods

5.1 Power Series

Definition 5.1.1 (Classification of points)**.** *For $y'' + p(x) \cdot y' + q(x) \cdot y = 0$, a point x_0 is called an ordinary point of the ODE if $p(x)$ and $q(x)$ are analytic functions at x_0, otherwise, it is a singular point. Moreover, a point x_0 is called a regular singular point if $x \cdot p(x)$ and $x^2 \cdot q(x)$ are analytic functions at x_0.*

Remark 5.1.2. *The radius of convergence of $y(x) = \sum_{n\geq 0} a_n (x - x_0)^n$ is $\dfrac{1}{R_{x_0}(y)} = \limsup \sqrt[n]{|a_n|}$ (cf. Calculus I).*

Lemma 5.1.3 (Cauchy product)**.**

$$\left[\sum_{n\geq 0} a_n (x - x_0)^n\right] \cdot \left[\sum_{n\geq 0} b_n (x - x_0)^n\right] = \sum_{n\geq 0} \left[\sum_{k=0}^{n} a_{n-k} \cdot b_k\right] (x - x_0)^n$$

Proof. Direct calculation: $c_n = \sum_{i+j=n} a_i \cdot b_j$. ☐

Theorem 5.1.4. *If x_0 is an ordinary point of $y'' + p(x) \cdot y' + q(x) \cdot y = 0$, then there is a solution y that is analytic at x_0. Furthermore, $R_{x_0}(y) \geq$* $\min\{R_{x_0}(p), R_{x_0}(q)\}$

Proof. Take $y(x) = \sum_{n\geq 0} a_n (x - x_0)^n$. We get:

$$0 = y'' + p(x) \cdot y' + q(x) \cdot y = y'' + y' \cdot \sum_{n\geq 0} p_n (x - x_0)^n + y \cdot \sum_{n\geq 0} q_n (x - x_0)^n$$

$$= \sum_{n\geq 0} \left[(n+2)(n+1)\, a_{n+2} + \sum_{k=0}^{n} \Big(p_{n-k} \cdot (k+1) a_{k+1} + q_{n-k} \cdot a_k\Big)\right] (x - x_0)^n$$

$$\Leftrightarrow a_{n+2} = -\frac{1}{(n+2)(n+1)} \sum_{k=0}^{n} \Big(p_{n-k} \cdot (k+1) a_{k+1} + q_{n-k} \cdot a_k\Big)$$

Hence we get all the coefficients by induction, so y (uniquely) defined, given a_0 and a_1. ☐

Definition 5.1.5 (Indicial Polynomial)**.** *For $(x - x_0)^2 \cdot y'' + (x - x_0) \cdot p(x) \cdot y' + q(x) \cdot y = 0$ where p and q are analytic at x_0, hence x_0 is a regular singular point. Define $\iota(s) = s(s-1) + p(x_0) \cdot s + q(x_0)$.*

Theorem 5.1.6 (Frobenius Method)**.** *Let x_0 be a regular singular point for the ODE: $(x-x_0)^2 \cdot y'' + (x-x_0) \cdot p(x) \cdot y' + q(x) \cdot y = 0$ and λ be a root of the indicial polynomial. Then, there is a solution $y(x) = (x-x_0)^\lambda \cdot z(x)$, where z is analytic at x_0.*

Proof. Take $y(x) = \sum_{n\geq 0} a_n \, (x-x_0)^{n+\lambda}$. We get:

$$
\begin{aligned}
0 &= (x-x_0)^2 y'' + p(x) \cdot (x-x_0) y' + q(x) \cdot y \\
&= \sum_{n\geq 0} \left[(n+\lambda)(n+\lambda-1)\, a_n + \sum_{k=0}^{n} \Big(p_{n-k}\,(n+\lambda) + q_{n-k} \Big) a_k \right] (x-x_0)^{n+\lambda} \\
&\Leftrightarrow a_n = -\frac{1}{n(n+2\lambda-1+p_0)} \sum_{k=0}^{n-1} \Big(p_{n-k}\,(n+\lambda) + q_{n-k} \Big) a_k
\end{aligned}
$$

Hence we get all the coefficients by induction, so y (uniquely) defined, given a_0 and a_1. Observe there is some difficulty when $2\lambda - 1 + p_0 = \iota'(\lambda)$ is a negative integer. □

5.2 Laplace Transform

Definition 5.2.1 (Exponential Type)**.** *A function $f : [0\infty) \to \mathbb{R}$ is of exponential type if: $\exists\, K > 0, a \in \mathbb{R} : \forall\, t > 0\,,\, |f(t)| \leq K\, e^{at}$.*

Definition 5.2.2 (Laplace Transform)**.** *Given a function $f : [0, \infty) \to \mathbb{R}$ of exponential type, we define the Laplace transform: $F : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ as:*

$$F(s) = \mathcal{L}\left\{f(t)\right\}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Obviously, this operation is linear.

Theorem 5.2.3 (Formulae)**.** *For $F(s) = \mathcal{L}\left\{f(t)\right\}(s)$:*

(i) $\mathcal{L}\left\{f(t) \cdot e^{at}\right\}(s) = F(s - a)$

(ii) $\mathcal{L}\left\{f(at)\right\}(s) = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$ *for $a > 0$.*

(iii) $\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n F(s) - \sum_{k=1}^{n} s^{n-k} \cdot f^{(k-1)}(0^+)$

(iv) $\mathcal{L}\left\{f(t) \cdot t^n\right\}(s) = (-1)^n \cdot F^{(n)}(s)$

Proof. We prove each one:

(i) $\mathcal{L}\left\{f(t) \cdot e^{at}\right\}(s) = \int_0^\infty f(t)\, e^{at}\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-(s-a)t}\, dt = F(s-a)$ by definition.

(ii) $\mathcal{L}\left\{f(at)\right\}(s) = \int_0^\infty f(at)\, e^{-st}\, dt = \int_0^\infty f(\tau)\, e^{-\frac{s}{a}\tau}\, \frac{1}{a}\, d\tau = \frac{1}{a} F\left(\frac{s}{a}\right)$

(iii) By integration by parts, and applying induction:

$$\begin{aligned}
\mathcal{L}\left\{f'(t)\right\}(s) &= \int_0^\infty f'(t)\, e^{-st}\, dt = \Big[f(t)\, e^{-st}\Big]_0^\infty + s\int_0^\infty f(t)\, e^{-st}\, dt \\
&= s\, F(s) - f(0^+) \\
\Rightarrow \mathcal{L}\left\{f^{(n)}(t)\right\}(s) &= s\mathcal{L}\left\{f^{(n-1)}(t)\right\}(s) - f^{(n-1)}(0) \\
&= s \cdot s^{n-1} F(s) - s\sum_{k=1}^{n-1} s^{n-1-k} \cdot f^{(k-1)}(0^+) - f^{(n-1)}(0^+) \\
&= s^n F(s) - \sum_{k=1}^{n} s^{n-k} \cdot f^{(k-1)}(0^+)
\end{aligned}$$

(iv) Using Leibniz Rule for Integration (cf. Calculus I)

$$\mathcal{L}\left\{f(t)\cdot t^n\right\}(s) = \int_0^\infty f(t)\, t^n\, e^{-st}\, dt = (-1)^n \int_0^\infty \frac{\partial^n}{\partial s^n}\left[f(t)\, e^{-st}\right] dt$$
$$= (-1)^n \frac{d^n}{ds^n}\int_0^\infty f(t)\, e^{-st}\, dt = (-1)^n \cdot F^{(n)}(s)$$

□

Remark 5.2.4. *We defined $f(0^+) = \lim_{t\to 0^+} f(t)$. If f is C^n, we can ignore it on formula (iii) and instead take $f(0)$.*

Corollary 5.2.5. *Calculating: $\mathcal{L}\left\{1\right\}(s) = \int_0^\infty e^{-st}\, dt = \frac{1}{s}$ for $s > 0$. Hence,*

(i) $\mathcal{L}\left\{e^{at}\right\}(s) = \frac{1}{s-a}$ *for $s > a$*

(ii) $\mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}}$ *for $s > 0$*

(iii) $\mathcal{L}\left\{t^n\, e^{at}\right\}(s) = \frac{n!}{(s-a)^{n+1}}$ *for $s > a$*

Corollary 5.2.6. *Taking real and imaginary parts, evaluating the integral as usual:*

(i) $\mathcal{L}\left\{\cos(at)\right\}(s) = \frac{s}{s^2+a^2}$

(ii) $\mathcal{L}\left\{\sin(at)\right\}(s) = \frac{a}{s^2+a^2}$

Lemma 5.2.7. *Let $u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$ be the Heaviside step function. Then: $\mathcal{L}\left\{f(t-\tau)\cdot u(t-\tau)\right\}(s) = e^{-\tau s}\, F(s)$.*

Proof. Calculate: $\mathcal{L}\left\{f(t-\tau)\cdot u(t-\tau)\right\}(s) = \int_0^\infty f(t-\tau)\cdot u(t-\tau)\, e^{-st}\, dt = \int_\tau^\infty f(t-\tau)\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-s(t+\tau)}\, dt = e^{-\tau s}\, F(s)$ □

Theorem 5.2.8 (Convolution)**.** *Given $f, g : [0, \infty) \to \mathbb{R}$ be functions of exponential type. Define $(f * g) : [0, \infty) \to \mathbb{R}$ s.t.:*

$$(f * g)(t) = \int_0^t f(\tau) \cdot g(t - \tau)\, d\tau$$

which is a commutative and associative operation, called the convolution. Then, $\mathcal{L}\{f * g\}(s) = F(s) \cdot G(s)$.

Proof. By changing the boundaries of integration and using the previous result:

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty (f * g)(t)\, e^{-st}\, dt = \int_0^\infty \int_0^t f(\tau) \cdot g(t - \tau)\, e^{-st}\, d\tau\, dt$$
$$= \int_0^\infty \int_t^\infty f(\tau) \cdot g(t - \tau)\, e^{-st}\, dt\, d\tau = \int_0^\infty f(\tau) \cdot \mathcal{L}\{g(t - \tau)\, u(t - \tau)\}(s)\, d\tau$$
$$= \int_0^\infty f(\tau) \cdot e^{-\tau s}\, G(s)\, d\tau = F(s) \cdot G(s)$$

□

Theorem 5.2.9 (Transfer Function)**.** *For the linear ODE $\sum_{k=0}^n a_k \cdot y^{(k)}(x) = f(x)$ with $a_k \in \mathbb{R}$, the particular solution is given by:*

$$y_p(x) = (f * K)(x) = \int_0^x f(t) \cdot K(x - t)\, dt$$

where $K(t) = \mathcal{L}^{-1}\left\{\frac{1}{\chi(s)}\right\}(t)$ is called the transfer function.

Proof. Since $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$, taking Laplace transform of both sides, we get:

$$\sum_{k=0}^n a_k\, s^k \cdot Y(s) = \chi(s) \cdot Y(s) = F(s) \Rightarrow Y(s) = F(s) \cdot \frac{1}{\chi(s)}$$

it follows from the previous theorem. □

Corollary 5.2.10. *By partial fraction decomposition of $\frac{1}{\chi(s)}$, we get the solution of 3.3.1 with fundemental set as in 3.4.5.*

Theorem 5.2.11 (Initial and Final Value)**.**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s \cdot F(s) \ \textit{and} \ \lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$$

Proof. By substitution, we get:

$$s \cdot F(s) = \int_0^\infty f\left(\frac{t}{s}\right) e^{-t}\, dt$$

the results follow by dominated convergence theorem. □

5.3 Sturm Liouville Theory

Definition 5.3.1 (SL problem)**.** *A Sturm-Liouville (boundary value) problem is a triple:*

(i) $\Big(p(x) \cdot y'(x)\Big)' - q(x) \cdot y(x) + \lambda \cdot r(x) \cdot y(x) \equiv 0$

(ii) $\alpha_1 \cdot y(a) + \alpha_2 \cdot y'(a) = 0$ *where* $(\alpha_1, \alpha_2) \neq (0, 0)$.

(iii) $\beta_1 \cdot y(b) + \beta_2 \cdot y'(b) = 0$ *where* $(\beta_1, \beta_2) \neq (0, 0)$.

Define the linear operator:

$$L[y](x) = \frac{1}{r(x)} \left[-\Big(p(x) \cdot y'(x)\Big)' + q(x) \cdot y(x) \right]$$

and let $B_a[y] = \alpha_1 \cdot y(a) + \alpha_2 \cdot y'(a)$, $B_b[y] = \beta_1 \cdot y(b) + \beta_2 \cdot y'(b)$. *Then the problem becomes:* $L[y] = \lambda \cdot y$ *and* $B_a[y] = B_b[y] = 0$.

Definition 5.3.2 (Inner Product on Continuous Functions)**.** *Define the following inner products on continuous complex-valued functions on an interval* $\mathcal{I} = [a, b]$. *For function* $r > 0$*:*

$$\langle f, g \rangle = \int_a^b f(x) \cdot \overline{g(x)} \cdot r(x)\, dx$$

which obeys every property of inner product (cf. Linear Algebra).

Theorem 5.3.3 (Lagrange's Identity)**.** $\langle L[f], g\rangle - \langle f, L[g]\rangle = p(x) \cdot W_{(f,\overline{g})}(x)\Big|_a^b$.

Proof.

$$\begin{aligned}
\langle L[f], g\rangle &= \int_a^b \left[-\Big(p(x) \cdot f'(x)\Big)' + q(x) \cdot f(x) \right] \cdot \overline{g(x)}\, dx \\
&= -\int_a^b \Big(p(x) \cdot f'(x)\Big)' \cdot \overline{g(x)}\, dx + \int_a^b q(x) \cdot f(x) \cdot \overline{g(x)}\, dx \\
&= -\, p(x) \cdot f'(x) \cdot \overline{g(x)}\Big|_a^b + \int_a^b p(x) \cdot f'(x) \cdot \overline{g'(x)}\, dx + \int_a^b q(x) \cdot f(x) \cdot \overline{g(x)}\, dx \\
&= p(x) \cdot W_{(f,\overline{g})}(x)\Big|_a^b - \int_a^b f(x) \cdot \overline{\Big(p(x) \cdot g'(x)\Big)'}\, dx + \int_a^b f(x) \cdot \overline{q(x) \cdot g(x)}\, dx \\
&= p(x) \cdot W_{(f,\overline{g})}(x)\Big|_a^b + \langle f, L[g]\rangle
\end{aligned}$$

□

Lemma 5.3.4. *For two functions f, g: $B_a[f] = B_a[g] = 0 \Leftrightarrow W_{(f,g)}(a) = 0$, the same for b.*

Proof. $B_a[f] = B_a[g] = 0$ are two homogeneous linear equations with non-trivial solution. Hence, the Wronskian is zero. $\square$

Corollary 5.3.5. *If f and g obey the boundary conditions: $\langle L[f], g\rangle = \langle f, L[g]\rangle$, that is, L is a self-adjoint operator on the set of piecewise continuous functions that obey the boundary conditions. It is* ***actually*** *self adjoint since the two domains match.*

Remark 5.3.6. *If we define $L[y](x) = -\Big(p(x)\cdot y'(x)\Big)' + q(x)\cdot y(x)$, then it is self adjoint of the inner product with $r \equiv 1$. We, however, shall use the other one since the weighted inner product appears much more often and the SL problem becomes exactly an eigenvalue problem.*

Lemma 5.3.7 (Real Eigenvalues)**.** *All eigenvalues of L are real.*

Proof. Let $L[y] = \lambda \cdot y$, then, by 5.3.5: $\lambda \cdot \langle y, y\rangle = \langle L[y], y\rangle = \langle y, L[y]\rangle = \overline{\lambda}\cdot\langle y, y\rangle \Rightarrow (\lambda - \overline{\lambda})\cdot\langle y, y\rangle = 0 \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$. $\square$

Theorem 5.3.8 (Sturm-Liouville)**.** *Let y_n and y_m be eigenfunctions with distinct eigenvalues λ_n and λ_m, respectively, that is, $\lambda_n \neq \lambda_m$. Then, they are orthogonal (with respect to the weighted inner product). Further, the multiplicity of each eigenvalue is one.*

Proof. By 5.3.5 and 5.3.7, we get: $\lambda_n \cdot \langle y_n, y_m\rangle = \langle L[y_n], y_m\rangle = \langle y_n, L[y_m]\rangle = \lambda_m \cdot \langle y_n, y_m\rangle \Rightarrow (\lambda_n - \lambda_m)\cdot\langle y_n, y_m\rangle = 0 \Rightarrow \langle y_n, y_m\rangle = 0$. Suppose there are two linearly independent solutions y_n and $\tilde{y}_n$. However, then by 2.2.7 every solution of the ODE obeys the boundary conditions, contraction. $\square$

Theorem 5.3.9 (Generalized Fourier Series)**.** *For f piecewise continuous in $[a, b]$, and $\{y_n\}_{n=0}^{\infty}$ be eigenfunction of a SL problem (L, B_a, B_b), then* $\exists\,\{a_n \in \mathbb{R}\}_{n=0}^{\infty} : \forall\, x \in [a, b]\,,$

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=0}^{\infty} a_n \cdot y_n(x)$$

Where $f(x^{\pm}) = \lim\limits_{x' \to x^{\pm}} f(x')$. Of course, if f is continuous at x, then the LHS equals $f(x)$. In particular, $a_n = \dfrac{\langle f, y_n\rangle}{\langle y_n, y_n\rangle}$.

Proof. Pythagoras: $\left\|f - \sum_{n=0}^{N} a_n \cdot y_n\right\|^2 = \|f\|^2 - \sum_{n=0}^{N} |a_n|^2 \cdot \|y_n\|^2 \Rightarrow \|f\|^2 \geq \sum_{n=0}^{N} |a_n|^2 \cdot \|y_n\|^2$ (Bessel), hence it converges uniformly by Weierstrass (M-test and Boundedness). $\square$

Lemma 5.3.10 (Rayleigh Quotient)**.** *For each pair* (λ_n, y_n)*:* $\lambda_n = \dfrac{\langle L[y_n], y_n\rangle}{\langle y_n, y_n\rangle}$, *where* y_n *is its respective eigenfunction.*

Proof. The quotient follows by: $L[y_n] = \lambda_n \cdot y_n \Rightarrow \langle L[y_n], y_n\rangle = \lambda_n \cdot \langle y_n, y_n\rangle$. $\square$

Theorem 5.3.11 (Min-Max Theorem)**.** *In view of 5.3.7, 5.3.5 and 5.3.10, the eigenvalues are ordered:* $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$

Proof. Ommited here. $\square$