

Calculus I  
Notes from TAU Course with Additional Information  
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## Contents

<b>1</b>	<b>Real Analysis</b>	<b>3</b>
1.1	Sequences . . . . .	3
1.2	Cauchy Sequences . . . . .	8
1.3	Series . . . . .	11
1.4	Absolute Convergence . . . . .	15
<b>2</b>	<b>Functions</b>	<b>17</b>
2.1	Basic Definitions . . . . .	17
2.2	Limits and Continuity . . . . .	19
2.3	Classifying Continuity . . . . .	22
2.4	Continuity Theorems . . . . .	24

<b>3</b>	<b>Differential Calculus</b>	<b>27</b>
3.1	Derivatives . . . . .	27
3.2	Mean Value Theorems and Taylor . . . . .	30
3.3	Investigation of Functions . . . . .	34
<b>4</b>	<b>Integral Calculus</b>	<b>37</b>
4.1	Indefinite Integral . . . . .	37
4.2	Definite Integrals . . . . .	43
4.3	Fundamental Theorem of Calculus . . . . .	47

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# 1 Real Analysis

## 1.1 Sequences

**Definition 1.1.1** (Common Sets). *We denote:*

*Natural Numbers:*  $\mathbb{N} = \{1, 2, \dots\}$

*Integers:*  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$

*Rationals:*  $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$

*Reals:*  $\mathbb{R}$

**Definition 1.1.2** (Sequence). *A sequence of real numbers is a set of numbers which are written in some particular order. There are infinitely many terms to the sequence:*

$$a_1, a_2, \dots, a_n, \dots$$

*Denote the entire sequence by:  $\{a_n\}_{n \in \mathbb{N}}$  or just  $\{a_n\}$ . A sequence in  $A \subseteq \mathbb{R}$  is denoted  $\{a_n \in A\}_{n \in \mathbb{N}}$ .*

**Definition 1.1.3** (Bounded Sequences). *A sequence  $\{a_n\}$  is:*

*Bounded from **above** if:  $\exists M \in \mathbb{R} : \forall n \in \mathbb{N}, a_n \leq M$*

*Bounded from **below** if:  $\exists m \in \mathbb{R} : \forall n \in \mathbb{N}, a_n \geq m$*

**Bounded** if it is bounded from both above and below.

*We call every such  $M$  and  $m$  upper and lower bounds, respectively.*

**Lemma 1.1.1.** *A sequence  $\{a_n\}$  is bounded iff:*

$$\exists M > 0 : \forall n \in \mathbb{N}, |a_n| \leq M$$

*Proof.* Let  $A$  and  $a$  be upper and lower bounds of  $\{a_n\}$ , respectively. Choose  $M = \max\{|A|, |a|\}$ .  $\square$

**Definition 1.1.4** (Monotonic Sequences). A sequence  $\{a_n\}$  is:

Monotonic **increasing** if:  $\exists N \in \mathbb{N} : \forall n \geq N, a_n \leq a_{n+1}$

Monotonic **decreasing** if:  $\exists N \in \mathbb{N} : \forall n \geq N, a_n \geq a_{n+1}$

**Stricly** monotonic **increasing** if:  $\exists N \in \mathbb{N} : \forall n \geq N, a_n < a_{n+1}$

**Stricly** monotonic **decreasing** if:  $\exists N \in \mathbb{N} : \forall n \geq N, a_n > a_{n+1}$

Usually, we omit  $N$  and shift the sequence so that it is monotonic  $\forall n \in \mathbb{N}$ .

**Example 1.1.1.**  $\left\{ \frac{n^2}{2^n} \right\}_{n \in \mathbb{N}}$  is stricly monotonically decreasing:

$$a_{n+1} < a_n \Leftrightarrow \frac{(n+1)^2}{2^{n+1}} < \frac{n^2}{2^n} \Leftrightarrow (n+1)^2 < 2n^2 \Leftrightarrow n > \sqrt{2} + 1 > 2.4 \Leftrightarrow n \geq 3$$

Therefore, if we pick  $N = 3$ , the sequence is stricly monotonically decreasing.

**Lemma 1.1.2** (Triangle Inequality).  $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$

**Definition 1.1.5** (Limit of a Sequence). We say  $L$  is the limit of  $\{a_n\}$  if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| < \epsilon$$

We say:  $\lim_{n \rightarrow \infty} a_n = L$  or  $\lim a_n = L$  or even  $a_n \rightarrow L$ . If the limit exists, the sequences **converges** (to  $L$ ), otherwise, it **diverges**. Further, we write the choice of  $N$  as  $N(\epsilon)$ .

**Remark 1.1.1.** If we take a sequence  $\{a_n\}$ , the idea of the limit  $L$  is such that, for any interval  $(L - \epsilon, L + \epsilon)$ , we have a finite number of terms outside the interval and an infinite number of terms inside.

**Theorem 1.1.1** (Uniqueness of Limit). If a sequence  $\{a_n\}$  has a limit  $L$ , then it is unique.

*Proof.* Suppose that there are two limits  $L_1 \neq L_2$ . By definition:

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_n - L_1| < \epsilon$$

$$\forall \epsilon > 0, \exists N_2 \in \mathbb{N} : \forall n \geq N_2, |a_n - L_2| < \epsilon$$

Then, by the triangle inequality:

$$\forall n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}, |L_1 - L_2| \leq |L_1 - a_n| + |a_n - L_2| < 2\epsilon$$

Which is a contradiction for  $\epsilon = \frac{1}{2}|L_1 - L_2| > 0$ .  $\square$

**Theorem 1.1.2** (Limit  $\Rightarrow$  Bounded). *Every convergent sequence is bounded.*

*Proof.* Let  $N = N(1)$  and  $M = \max\{|a_1|, \dots, |a_{N-1}|, 1 + |L|\}$ . Then, by definition of  $a_n \rightarrow L$ :

$$\forall n \geq N, |a_n| \leq |a_n - L| + |L| < 1 + |L|$$

Hence,  $\forall n \in \mathbb{N}, |a_n| \leq M$ .  $\square$

**Lemma 1.1.3** (Linearity of Limits). *Suppose  $\lim a_n = A$ ,  $\lim b_n = B$  and  $C$  is a constant. Then, the following are true:*

1.  $\lim C = C$
2.  $\lim(C \cdot a_n) = C \cdot A$
3.  $\lim(a_n \pm b_n) = A \pm B$

*Proof.* By definition,

1. Pick  $N = 1$ , we have:  $\forall \epsilon > 0, \forall n \geq 1, |c - c| = 0 < \epsilon$

2. Pick  $N = N_A\left(\frac{\epsilon}{|c|}\right)$ , we have:

$$\forall \epsilon > 0, \forall n \geq N, |c \cdot a_n - c \cdot A| = |c| \cdot |a_n - A| < \epsilon$$

3. Let  $N = \max\left\{N_A\left(\frac{\epsilon}{2}\right), N_B\left(\frac{\epsilon}{2}\right)\right\}$  Then, by the triangle inequality:  
 $\forall n \geq N,$

$$|(a_n \pm b_n) - (A \pm B)| = |(a_n - A) \pm (b_n - B)| \leq |a_n - A| + |b_n - B| < \epsilon$$

Hence, it is a linear operation on the convergent sequences.  $\square$

**Remark 1.1.2.** If  $\exists \lim a_n$  and  $\nexists \lim b_n$ , then  $\nexists \lim(a_n + b_n)$ . If  $\nexists \lim a_n$  and  $\exists \lim b_n$ , then there is no rule for  $\lim(a_n + b_n)$ .

**Corollary 1.1.1.**  $\lim a_n = A \Leftrightarrow \lim(a_n - A) = 0$

**Theorem 1.1.3.** Let  $\{b_n\}$  be bounded and  $\lim a_n = 0$ . Then,  $\lim(a_n \cdot b_n) = 0$ .

*Proof.* Let  $M$  be a two-sided bound of  $b_n$ . if  $\lim a_n = 0$ , take  $N = N\left(\frac{\epsilon}{M}\right)$ :

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n| < \frac{\epsilon}{M}$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n \cdot b_n - 0| = |a_n| \cdot |b_n| \leq |a_n| \cdot M < \epsilon$$

that is the definition of the limit being 0. □

**Lemma 1.1.4** (Arithmetic of Limits). Suppose  $\lim a_n = A$ ,  $\lim b_n = B$ . Then, the following are true:

1.  $\lim(a_n \cdot b_n) = A \cdot B$
2. If  $B \neq 0$  and  $b_n \neq 0$ ,  $\lim \frac{a_n}{b_n} = \frac{A}{B}$

*Proof.* We prove each one:

$$1. \lim(a_n \cdot b_n) = \lim \left[ \overbrace{(a_n - A)}^{\rightarrow 0} \cdot \overbrace{b_n}^{\text{bounded}} + A \cdot b_n \right] = 0 + \lim(A \cdot b_n) = A \cdot B$$

$$2. \text{ By the previous result, } \underbrace{\lim b_n}_B \cdot \lim \frac{a_n}{b_n} = \underbrace{\lim a_n}_A$$

□

**Lemma 1.1.5** (Positivity). Let  $\{a_n\}$  be a positive sequence (bounded from below by 0) then  $L = \lim a_n$ , if it exists, is also positive.

*Proof.* By contrary, suppose  $L < 0$ , then:  $\forall n \in \mathbb{N}, |a_n - L| \geq |L|$ . The statement for  $a_n \rightarrow L$  is a contradiction for  $\epsilon = |L|$ . □

**Corollary 1.1.2** (Monotonicity). *If  $M \geq a_n \geq m \Rightarrow M \geq \lim a_n \geq m$*

**Theorem 1.1.4** (Sandwich/Squeeze Theorem). *Given three sequences  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$ . If  $\lim a_n = \lim c_n = L$  and*

$$\exists N_0 \in \mathbb{N} : \forall n \geq N_0, a_n \leq b_n \leq c_n$$

*then,  $\lim b_n = L$ .*

*Proof.* Take  $\epsilon > 0$ :

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, L - \epsilon < a_n < L + \epsilon$$

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, L - \epsilon < c_n < L + \epsilon$$

$$\exists N_0 \in \mathbb{N} : \forall n \geq N_0, a_n \leq b_n \leq c_n$$

Pick  $N_3 = \max\{N_0, N_1, N_2\}$ . Then,  $\forall n \geq N_3$ ,

$$L - \epsilon < c_n \leq b_n \leq a_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon$$

□

**Definition 1.1.6** (Wide Sense). *We say that the sequence  $\{a_n\}$  converges to  $+\infty$  (eq.  $-\infty$ ) if:*

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n \geq N, a_n > M \quad (\text{eq. } a_n < M)$$

*This a wide sense of the limit.*

## 1.2 Cauchy Sequences

**Definition 1.2.1** (Supremum and Infimum). *For a set  $A \subset \mathbb{R}$ , we define:*

*If  $A$  is bounded from above, the supremum of  $A$  is the lowest upper bound (denoted  $\sup A$ ). Otherwise  $\sup A = \infty$ .*

*If  $A$  is bounded from below, the infimum of  $A$  is the greatest lower bound (denoted  $\inf A$ ). Otherwise  $\inf A = -\infty$ .*

**Theorem 1.2.1** (Monotone Convergence). *A bounded monotonic sequence converges. Furthermore:*

*If  $\{a_n\}$  is increasing,  $\lim a_n = \sup\{a_n\}$*

*If  $\{a_n\}$  is decreasing,  $\lim a_n = \inf\{a_n\}$*

*Proof.* We prove for increasing, since decreasing is analogous with  $b_n = -a_n$ . Let  $M = \sup\{a_n\}$ . For every  $\epsilon > 0$ , there exists  $N$  such that  $a_N > M - \epsilon$ , since otherwise  $M - \epsilon$  is an upper bound of  $\{a_n\}$ , which contradicts to the definition of  $M$ . Then, since  $\{a_n\}$  is increasing, and  $M$  is its upper bound, for every  $n \geq N$ , we have  $|a_n - M| = M - a_n \leq M - a_N < \epsilon$ . Hence, by definition, the limit of  $\{a_n\}$  is  $\sup\{a_n\}$ .  $\square$

**Definition 1.2.2** (Subsequence). *Given a sequence  $\{a_n\}$ , a subsequence  $\{b_k\}$  is such that:*

$$b_k = a_{n_k}$$

*where  $\{n_k\}$  is a strictly increasing sequence of natural numbers.*

**Lemma 1.2.1.** *A sequence  $\{a_n\}$  converges to  $L$  in the wide sense if and only if any subsequence of  $\{a_n\}$  converges to the same limit  $L$ .*

**Definition 1.2.3** (Partial Limit). *A real number  $a$  is called a partial limit of the sequence  $\{a_n\}$  if there exists a subsequence of  $\{a_n\}$  which converges to  $a$  in the wide sense.*

**Remark 1.2.1.** *For a partial limit  $a$  of  $\{a_n\}$  for any interval  $(a - \epsilon, a + \epsilon)$ , we are only required to have an infinite number of terms inside. If there are infinitely many outside, there is no one limit  $L$  and if there finitely many outside, it is the one limit  $L = a$ .*



**Theorem 1.2.2** (Bolzano-Weierstrass). *To any bounded sequence, that exists a convergent subsequence, i.e., there exists at least one partial limit.*

*Proof.* By Monotone Convergence, we only need to prove that there exists a monotone subsequence of  $\{a_n\}$ . Call  $a_m$  a "peak" if  $\forall n \geq m, a_m \geq a_n$ .

Case 1:  $\{a_n\}$  has infinitely many peaks. Form a subsequence with these peaks denoted  $a_{n_k}$ . Since each of these terms is a peak and  $n_1 < n_2 < \dots < n_k < \dots$  we have that  $\{a_{n_k}\}_{k \in \mathbb{N}}$  is a monotonic decreasing subsequence of  $\{a_n\}$ .

Case 2:  $\{a_n\}$  has only a finite number of peaks denoted  $a_{n_1}, a_{n_2}, \dots, a_{n_k}$ . Let  $s_1 = n_k + 1$ . Then,  $a_{s_1}$  is not a peak, and so then there exists an  $s_2$  such that  $s_1 < s_2$  and  $a_{s_1} \leq a_{s_2}$ . Also,  $a_{s_2}$  is not a peak and so there exists an  $s_3$  such that  $s_2 < s_3$  and  $a_{s_2} \leq a_{s_3}$ . By induction, we have a subsequence  $a_{s_1} \leq a_{s_2} \leq \dots \leq a_{s_m} \leq \dots$ , and so  $\{a_{s_m}\}_{m \in \mathbb{N}}$  is a monotonic increasing subsequence of  $\{a_n\}$ .

□

**Definition 1.2.4** (Limit Sup and Inf). *The supremum of the set of all partial limits of a sequence is called **upper partial limit** of the sequence  $\{a_n\}$  and is denoted as:*

$$\overline{\lim} a_n \quad \text{or} \quad \limsup a_n$$

*The infimum of the set of all partial limits of a sequence is called **lower partial limit** of the sequence  $\{a_n\}$  and is denoted as:*

$$\underline{\lim} a_n \quad \text{or} \quad \liminf a_n$$

**Theorem 1.2.3** (N&SC for the existence of limit). *If a sequence  $\{a_n\}$  is bounded and  $\liminf a_n = \limsup a_n = a$ , then  $\lim a_n$  exists and is equal to  $a$ .*

*Proof.* Say  $\{a_n\}$  is bounded by the interval  $[m, M]$ . Take an  $\epsilon > 0$ . Looking at the set  $[m, M] \setminus (a - \epsilon, a + \epsilon) = [m, a - \epsilon] \cup [a + \epsilon, M]$ , if there are infinitely many elements of the sequence, by Bolzano-Weierstrass there is a convergent subsequence, which contradicts either  $\limsup a_n = a$  or  $\liminf a_n = a$ . Therefore, there are finitely many elements of the sequence in  $[m, M] \setminus (a - \epsilon, a + \epsilon)$ , hence in  $\mathbb{R} \setminus (a - \epsilon, a + \epsilon)$ , which is the definition of the limit. □

**Definition 1.2.5** (Cauchy). *A sequence  $\{a_n\}$  is called a Cauchy sequence if:*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, |a_m - a_n| < \epsilon$$

*equivalently:*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall p \in \mathbb{N}, |a_{n+p} - a_n| < \epsilon$$

**Lemma 1.2.2** (Cauchy is Bdd). *Every Cauchy sequence is bounded.*

*Proof.* Let  $M = \max\{|a_1|, \dots, |a_{N-1}|, \epsilon + |a_N|\}$ . Then:

$$\forall n \geq N, |a_n| \leq |a_n - a_N| + |a_N| < \epsilon + |a_N|$$

Hence,  $\forall n \in \mathbb{N}, |a_n| \leq M$ . □

**Lemma 1.2.3** (Convergent is Cauchy). *Let  $\lim a_n = L$ , then  $\{a_n\}$  is Cauchy.*

*Proof.* Then, by definition,

$$\begin{aligned} \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |L - a_n| < \frac{\epsilon}{2} \\ \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m \geq N, |a_m - L| < \frac{\epsilon}{2} \end{aligned}$$

Since  $|a_m - a_n| \leq |L - a_n| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ , it is Cauchy. □

**Theorem 1.2.4** (Cauchy Criterion).  *$\{a_n\}$  is convergent  $\Leftrightarrow$  it is Cauchy.*

*Proof.*  $(\Rightarrow)$  is the previous lemma.  $(\Leftarrow)$ : Suppose  $\{a_n\}$  is Cauchy.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - a_N| < \frac{\epsilon}{2}$$

Then:

$$\begin{aligned} |a_n - a_N| < \frac{\epsilon}{2} &\Leftrightarrow a_N - \frac{\epsilon}{2} < a_n < a_N + \frac{\epsilon}{2} \\ a_N - \frac{\epsilon}{2} &\leq \liminf a_n \leq \limsup a_n \leq a_N + \frac{\epsilon}{2} \\ \Rightarrow 0 &\leq \limsup a_n - \liminf a_n \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore,  $\limsup a_n = \liminf a_n$ . By the previous theorems, the sequence is bounded and  $\limsup a_n = \liminf a_n$ , therefore, it converges. □

### 1.3 Series

**Definition 1.3.1** (Infinite Sum). Given a sequence  $\{a_n\}$ , the sum  $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$  (also denoted  $\sum a_n$ ) is called an (infinite) series. Consider partial sums:

$$S_n = \sum_{k=1}^n a_k = a_1 + a_2 + \cdots + a_n$$

If  $\{S_n\}_{n \in \mathbb{N}}$  converges ( $S_n \rightarrow S$ ), then the series is called **convergent**. Otherwise, the series is **divergent**. The number  $S$  is called the sum of the series, that is,  $S = \sum a_k$ .

**Example 1.3.1** (Basel).  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent. Applying Cauchy to the partial sums, we get:

$$\begin{aligned} |S_{n+p} - S_n| &= \sum_{k=n+1}^{n+p} \frac{1}{k^2} < \sum_{k=n+1}^{n+p} \frac{1}{k(k+1)} \\ &= \sum_{k=n+1}^{n+p} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n} < \epsilon \end{aligned}$$

So, choose  $N(\epsilon) = \lceil \frac{1}{\epsilon} \rceil$ .

**Theorem 1.3.1** (Last Element). If  $\sum a_n$  converges, then  $\lim a_n = 0$ .

*Proof.*  $\lim a_n = \lim (S_n - S_{n-1}) = \lim S_n - \lim S_{n-1} = S - S = 0$ , where  $S = \lim S_n$  and  $S_n$  are the partial sums.  $\square$

**Corollary 1.3.1.** If  $\nexists \lim a_n$  or  $\lim a_n = a \neq 0$ , then  $\sum a_n$  diverges.

**Remark 1.3.1.** The converse of the Last Element Theorem is not true.

**Example 1.3.2** (Harmonic).  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent. Applying Cauchy to the partial sums, we get:

$$|S_{n+p} - S_n| = \sum_{k=n+1}^{n+p} \frac{1}{k} \geq \frac{p}{n+p} \stackrel{p=n}{=} \frac{1}{2}$$

So, the sequence is not Cauchy.

**Lemma 1.3.1** (Geometric Series).  $\sum_{n=0}^{\infty} a \cdot q^n$  converges iff  $|q| < 1$ .

*Proof.* Take the partial sums  $S_n = \sum_{k=0}^n a \cdot q^k = a \cdot \frac{1 - q^{n+1}}{1 - q}$ , which clearly converges iff  $|q| < 1$ .  $\square$

**Lemma 1.3.2** (Linearity of Infinite Sums). If  $\sum a_n$  and  $\sum b_n$  converge, then  $\sum(a_n \pm b_n)$  converges (to  $\sum a_n \pm \sum b_n$ ) and  $\sum c \cdot a_n$  converges (to  $c \cdot \sum a_n$ ).

*Proof.* Take  $A_n = \sum_{k=1}^n a_k$  and  $B_n = \sum_{k=1}^n b_k$ . We supposed  $\exists \lim A_n$  and  $\exists \lim B_n$ . Notice  $\sum_{k=1}^n (a_k + b_k) = A_n + B_n$  and  $\sum_{k=1}^n c \cdot a_k = c \cdot A_n$ . By the linearity of the limits, we have the result.  $\square$

**Lemma 1.3.3** (Bounded Series). If  $\exists N \in \mathbb{N} : \forall n \geq N, a_n \geq 0$ , then  $\sum a_n$  converges iff  $\{S_n\}$  is bounded.

*Proof.*  $\exists N \in \mathbb{N} : \forall n \geq N, a_n = S_n - S_{n-1} \geq 0 \Leftrightarrow \{S_n\}$  is monotonically increasing. As a corollary of a previous theorem, a monotonically sequence has a limit iff it is bounded.  $\square$

**Theorem 1.3.2** (Comparison). If  $\exists N \in \mathbb{N} : \forall n \geq N, 0 \leq a_n \leq b_n$  then,  $\sum b_n$  converges  $\Rightarrow \sum a_n$  converges.

*Proof.*  $\forall n \geq N, \sum_{k=N}^n a_k \leq \sum_{k=N}^n b_k$ . By previous lemma,  $\sum b_n$  converges  $\Rightarrow \sum_{k=N}^n b_k$  is bounded from above. Hence,  $\sum_{k=1}^n a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^n a_k$  is bounded from above (and below by 0), so it converges.  $\square$

**Corollary 1.3.2.** If  $\exists N \in \mathbb{N} : \forall n \geq N, a_n \geq b_n \geq 0$  then,  $\sum b_n$  diverges  $\Rightarrow \sum a_n$  diverges.

**Remark 1.3.2.** These follow from the comparison test and the definition of limit.

1. If  $\frac{a_n}{b_n} \rightarrow 0$ , then,  $\sum b_n$  converges  $\Rightarrow \sum a_n$  converges.
2. If  $\frac{a_n}{b_n} \rightarrow \infty$ , then,  $\sum b_n$  diverges  $\Rightarrow \sum a_n$  diverges.

**Theorem 1.3.3** (Limit Comparision). *If  $\exists N \in \mathbb{N} : \forall n \geq N, a_n > 0$  and  $b_n > 0$  and  $\lim \frac{a_n}{b_n} = L > 0$  then,  $\sum b_n$  converges  $\Leftrightarrow \sum a_n$  converges.*

*Proof.*  $\lim \frac{a_n}{b_n} = L > 0 \Rightarrow \exists M > 0 : \forall n \in \mathbb{N}, \frac{a_n}{b_n} \leq M$ . Hence, by linearity  $\sum b_n$  converges  $\Rightarrow \sum M \cdot b_n$  converges, by comparision,  $\Rightarrow \sum a_n$  converges. To prove the converse, take  $\lim \frac{b_n}{a_n} = \frac{1}{L} > 0$ .  $\square$

**Lemma 1.3.4** (Cauchy Condesation). *Let  $a_n$  be a decreasing non-negative sequence. Then,  $\sum_{n=1}^{\infty} a_n$  converges  $\Leftrightarrow \sum_{n=0}^{\infty} 2^n \cdot a_{2^n}$  converges.*

*Proof.* Since  $a_n$  is decreasing, we have the estimate:

$$\forall n \in \mathbb{N}, \sum_{k=1}^{2^n-1} a_k \leq \sum_{k=0}^{n-1} 2^k \cdot a_{2^k} \leq 2 \cdot \sum_{k=1}^{2^n-1} a_k$$

Hence, by the Sandwich Theorem, we are done.  $\square$

**Lemma 1.3.5** (P-Series).  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges iff  $p > 1$ .

*Proof.* If  $p \leq 0$ , it diverges by last element. For  $p > 0$ ,  $a_n = \frac{1}{n^p}$  is a decreasing non-negative sequence. By condensation:  $2^n \cdot a_{2^n} = \frac{2^n}{(2^n)^p} = 2^{n(1-p)} = \left(2^{1-p}\right)^n$ , a geometric series. Hence it converges iff  $2^{1-p} < 1 \Leftrightarrow p > 1$ .  $\square$

**Theorem 1.3.4** (Leibnitz Criteria). *If  $\sum (-1)^{n-1} \cdot a_n$  with  $(a_n > 0)$ , an alternating series, satisfies:*

1.  $a_{n+1} \leq a_n$  (i.e. monotonically decreasing)
2.  $\lim a_n = 0$

*Then, the series converges.*

*Proof.* Consider the parity  $n$  for the partial sums  $S_n$ .

- Even:  $S_{2(m+1)} = S_{2m} + (a_{2m+1} - a_{2m+2}) \geq S_{2m} \Rightarrow \{S_{2m}\}$  is monotonically increasing. On the other hand,

$$S_{2m} = a_1 - a_{2m} - \sum_{k=1}^{m-1} (a_{2k} - a_{2k+1}) \leq a_1 - a_{2m} < a_1$$

hence  $S_{2m}$  is bounded from above. Therefore,  $\lim_{m \rightarrow \infty} S_{2m} = S$ .

- Odd:  $S_{2m+1} = S_{2m} + a_{2m+1} \Rightarrow \lim_{m \rightarrow \infty} S_{2m+1} = \lim_{m \rightarrow \infty} S_{2m} + \lim_{m \rightarrow \infty} a_{2m+1} = \lim_{m \rightarrow \infty} S_{2m} = S$ .

Therefore,  $\lim_{m \rightarrow \infty} S_{2m+1} = \lim_{m \rightarrow \infty} S_{2m} = S \Rightarrow \lim_{n \rightarrow \infty} S_n = S$ .  $\square$

## 1.4 Absolute Convergence

**Definition 1.4.1** (Abs Convergence). *A series  $\sum a_n$  converges absolutely if the series  $\sum |a_n|$  converges.*

**Theorem 1.4.1** (Abs Convergence is Stronger). *If a series converges absolutely, then it converges.*

*Proof.*  $-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$ . By comparison test, let  $b_n = a_n + |a_n|$ , then  $\sum a_n = \sum (b_n - |a_n|)$  converges.  $\square$

**Theorem 1.4.2** (Riemann Rearrangement). *If the series  $\sum a_n$  absolutely converges and the sequence  $b_n$  is obtained by a permutation of  $a_n$ , then  $\sum b_n$  also absolutely converges and  $\sum b_n = \sum a_n$ .*

**Theorem 1.4.3** (d’Alambert Criteria). *Given a sequence  $\{a_n\}$ :*

1. *If  $\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$  then  $\sum a_n$  converges absolutely.*
2. *If  $\lim \left| \frac{a_{n+1}}{a_n} \right| > 1$  (including wide sense) then  $\sum a_n$  diverges.*

*Proof.* By definition of limit,

1.  $\exists N \in \mathbb{N} : \forall n \geq N, |a_{n+1}| \leq r|a_n|$ , where  $r < 1$ . By induction,  $\forall n \geq N, |a_n| \leq r^{n-N}|a_N|$ . By comparison test with  $r^{n-N}|a_N|$ ,  $\sum |a_n|$  converges.
2.  $\exists N \in \mathbb{N} : \forall n \geq N, |a_{n+1}| \geq |a_n| > 0 \Rightarrow \lim a_n \neq 0 \Rightarrow \sum a_n$  diverges.

$\square$

**Lemma 1.4.1.** *Given a sequence  $\{a_n\}$ :*

1. *If  $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$  then  $\sum a_n$  converges absolutely.*

2. If  $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$  (including wide sense) then  $\sum a_n$  diverges.

**Theorem 1.4.4** (Cauchy Criteria). *Given a sequence  $\{a_n\}$ :*

1. If  $\limsup \sqrt[n]{|a_n|} < 1$  then  $\sum a_n$  converges absolutely.
2. If  $\limsup \sqrt[n]{|a_n|} > 1$  (including wide sense) then  $\sum a_n$  diverges.

*Proof.* By definition of limit superior, for infinitely many  $n$  :

1.  $\sqrt[n]{|a_n|} \leq r$ , that is,  $|a_n| \leq r^n$ , where  $r < 1$ . By comparison with  $r^n$ ,  $\sum |a_n|$  converges.
2.  $|a_n| \geq 1 \Rightarrow \lim a_n \neq 0 \Rightarrow \sum a_n$  diverges.

□

**Definition 1.4.2** (Power Series). *A series  $\sum_{n=0}^{\infty} a_n(x - a)^n$  in the variable  $x$  is called a power series.*

**Lemma 1.4.2** (Radius of Convergence). *Define*

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

*if the RHS is finite and 0 otherwise. Then,*

$$\forall x \in (a - R, a + R), \sum_{n=0}^{\infty} a_n(x - a)^n \text{ converges}$$

$$\sum_{n=0}^{\infty} a_n(x - a)^n \text{ converges} \Rightarrow x \in [a - R, a + R]$$

*Proof.* It is a direct result of Cauchy's Criteria. The second proposition takes into account when the test is inconclusive. □

**Remark 1.4.1.** *We have to check the endpoints  $x = a \pm R$  separately.*



## 2 Functions

### 2.1 Basic Definitions

**Definition 2.1.1** (Functions). Let  $\mathcal{D}$  and  $\mathcal{R}$  be two subsets of  $\mathbb{R}$ . A function from  $\mathcal{D}$  to  $\mathcal{R}$  is a well-defined law which, to each  $x \in \mathcal{D}$ , there is a unique number  $y \in \mathcal{R}$ . The set  $\mathcal{D}$  is called the domain of  $f$  and the set  $\mathcal{R}$  is called the range of  $f$ . Denote  $f : \mathcal{D} \rightarrow \mathcal{R}$  or  $y = f(x)$ . The variable  $x$  is called the **independent variable** and  $y$  is a **dependent variable**. The variable  $x$  is also called the **origin** of  $y$  and  $y$  is the **image** of  $x$ .

A set of points  $\{(x, f(x)) \mid x \in \mathcal{D}\}$  in the plane  $\mathbb{R}^2$  is called a graph of a function  $y = f(x)$  (denoted  $G(f)$  or only  $f$ ).

**Definition 2.1.2** (Image and Domain). For  $f : \mathcal{D} \rightarrow \mathcal{R}$ , the image is:

$$\text{Im}(f) = \{y \in \mathcal{R} \mid \exists x \in \mathcal{D} : y = f(x)\} = \{f(x) \mid x \in \mathcal{D}\}$$

i.e. the smallest possible range. We always have:  $\text{Im}(f) \subseteq \mathcal{R}$ .

The biggest possible domain of  $f$  is called the **existence domain** of  $f$ .

**Definition 2.1.3** (Parity). Given  $f : \mathcal{D} \rightarrow \mathcal{R}$  such that  $\mathcal{D}$  is symmetric, that is,  $\forall x \in \mathcal{D}, -x \in \mathcal{D}$ , the function is called:

- Even:  $\forall x \in \mathcal{D}, f(-x) = f(x)$
- Odd:  $\forall x \in \mathcal{D}, f(-x) = -f(x)$

**Definition 2.1.4** (Periodicity). Given  $f : \mathcal{D} \rightarrow \mathcal{R}$  is called periodical if:

$$\exists T \neq 0 : \forall x \in \mathcal{D}, x + T \in \mathcal{D} \text{ and } f(x + T) = f(x)$$

Furthermore,  $T$  is called a period of  $f$ . The smallest such  $T > 0$  (if it exists) is called **the** period of  $f$ .

**Definition 2.1.5** (Monotonicity). Given  $f : \mathcal{D} \rightarrow \mathcal{R}$  is called (eq. strictly) monotonic increasing if:

$$\forall x, y \in \mathcal{D}, x < y \Rightarrow f(x) \leq f(y) \text{ (eq. } f(x) < f(y))$$

And is called (eq. stricly) monotonic decreasing if:

$$\forall x, y \in \mathcal{D}, x < y \Rightarrow f(x) \geq f(y) \text{ (eq. } f(x) > f(y))$$

**Definition 2.1.6** (Injectivity). Given  $f : \mathcal{D} \rightarrow \mathcal{R}$  is called injective or one-to-one if:

$$\forall x, y \in \mathcal{D}, x \neq y \Rightarrow f(x) \neq f(y)$$

equivalently, if:

$$\forall y \in \text{Im}(f), \exists ! x \in \mathcal{D} : y = f(x)$$

**Lemma 2.1.1** (Str Monotone is Inj).  $f : \mathcal{D} \rightarrow \mathcal{R}$  is strictly monotonic, then it is injective.

*Proof.* WLOG,  $f$  is stricly monotonic increasing. If  $f(x) = f(y)$ , then, we have three cases. If  $x > y$ ,  $f(x) > f(y)$ . If  $y > x$ ,  $f(y) > f(x)$ . None of these are true, so were left with  $x = y$ .  $\square$

**Definition 2.1.7** (Surjectivity). Given  $f : \mathcal{D} \rightarrow \mathcal{R}$  is called surjective or onto if:  $\mathcal{R} = \text{Im}(f)$

**Definition 2.1.8** (Inverse Function). Given  $f : \mathcal{D}(f) \rightarrow \text{Im}(f)$  is one-to-one, one can define a function  $g : \text{Im}(f) \rightarrow \mathcal{D}(f)$  by  $g(y) = x$ , where  $x$  is the unique value such that  $y = f(x)$ . Therefore,  $g(f(x)) = x$ . The function  $g$  is called **inverse function** of  $f$ . Notation:  $g = f^{-1}$ . Notice that:  $\mathcal{D}(f^{-1}) = \text{Im}(f)$  and  $\text{Im}(f^{-1}) = \mathcal{D}(f)$ .

The graph of  $f$  and  $f^{-1}$  are symmetric with respect to the line  $y = x$ .

**Definition 2.1.9** (Composition). Let  $f : \mathcal{D}(f) \rightarrow \mathcal{R}$  and  $g : \mathcal{D}(g) \rightarrow \mathcal{S}$ , the composition  $h = g \circ f : \mathcal{D}(h) \rightarrow \mathcal{S}$  is defined as follows:

$$\begin{aligned} \mathcal{D}(h) &= \{x \in \mathcal{D}(f) \mid f(x) \in \mathcal{D}(g)\} \\ \forall x \in \mathcal{D}(f), h(x) &= g(f(x)) \end{aligned}$$

where  $\mathcal{D}(h) \neq \emptyset$ .

## 2.2 Limits and Continuity

**Definition 2.2.1** (Limit of a Function). *Let  $f$  be defined in an open interval  $\mathcal{U}$  about  $a$ , except possibly at  $a$  itself ("except"). A number  $L$  is called the limit of  $f$  at point  $a$  if:*

$$(Heine) \forall \{x_n \in \mathcal{U}\}_{n \in \mathbb{N}}, x_n \rightarrow a \Rightarrow f(x_n) \rightarrow L$$

$$(Cauchy) \forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathcal{U}, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

We denote  $L = \lim_{x \rightarrow a} f(x)$ . Notice, by Heine, the limit is unique, if it exists.

**Lemma 2.2.1** (H&C). *The Heine definition and the Cauchy definition of the limit are equivalent.*

*Proof.* We prove each direction:

( $\Rightarrow$ ) By contrary, suppose  $\exists \epsilon > 0 : \forall \delta > 0, \exists x \in \mathcal{U} : 0 < |x - a| < \delta \Rightarrow |f(x) - L| \geq \epsilon$ . Define the sequence  $\{x_n\} \in \mathcal{U}$  by picking  $x_n$  such that  $|x_n - a| < \frac{1}{n} \Rightarrow |f(x_n) - L| \geq \epsilon$ . Hence  $x_n \rightarrow a$  since  $a - \frac{1}{n} < x_n < a + \frac{1}{n}$ , by sandwich theorem. Therefore  $f(x_n) \not\rightarrow L$ , by definition, so Heine does not hold.

( $\Leftarrow$ ) By contrary, suppose  $\exists \{x_n \in \mathcal{U}\}_{n \in \mathbb{N}} : x_n \rightarrow a$ , but  $f(x_n) \not\rightarrow L$ . By definition of the limits:

$$\forall \delta > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - a| < \delta$$

$$\exists \epsilon > 0 : \forall N \in \mathbb{N}, \exists n_0 \geq N : |f(x_{n_0}) - L| \geq \epsilon$$

Hence  $|x_{n_0} - a| < \delta \not\Rightarrow |f(x_{n_0}) - L| < \epsilon$ , so Cauchy does not hold.

□

**Definition 2.2.2** (Infinite Limit of Functions). *Let  $f$  be defined in an open interval  $\mathcal{U}$  about  $a$ , "except". We say  $f(x) \rightarrow \infty$  (eq.  $-\infty$ ) at  $x = a$  if:*

$$(Cauchy) \forall M > 0, \exists \delta > 0 : \forall x \in \mathcal{U}, 0 < |x - a| < \delta \Rightarrow f(x) > M \text{ (eq. } f(x) < M)$$

$$(Heine) \forall \{x_n \in \mathcal{U}\}_{n \in \mathbb{N}}, x_n \rightarrow a \Rightarrow f(x_n) \rightarrow \infty \text{ (eq. } -\infty)$$

**Definition 2.2.3** (Limit at Infinity). *Let  $f$  be defined in an unbounded open interval  $(a, \infty)$  or  $(-\infty, a)$ . A number  $L$  is called the limit of  $f$  at  $\infty$  (eq.  $-\infty$ ) if: (Cauchy)*

$$\forall \epsilon > 0, \exists M > 0 : \forall x \in \mathcal{U}, x > M \text{ (eq. } x < M) \Rightarrow |f(x) - L| < \epsilon$$

$$\text{(Heine)} \forall \{x_n \in \mathcal{U}\}_{n \in \mathbb{N}}, x_n \rightarrow \infty \text{ (eq. } -\infty) \Rightarrow f(x_n) \rightarrow L$$

**Lemma 2.2.2.** *A periodic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has no limit at  $\pm\infty$ .*

*Proof.* Let  $T$  be the period of  $f$ . Since  $f$  is non-constant,  $\exists x, y \in \mathbb{R} : f(x) \neq f(y)$ . Take  $x_n = x + nT$  and  $y_n = y + nT$ , we get  $\forall n \in \mathbb{N}, f(x_n) = f(x) \neq f(y) = f(y_n)$ . Hence  $x_n, y_n \rightarrow \infty$ , but  $f(x_n) \rightarrow f(x) \neq f(y) \leftarrow f(y_n)$   $\square$

**Theorem 2.2.1** (Sandwich/Squeeze Theorem). *Given three functions defined on an open interval  $\mathcal{U}$  about  $a$ , "except". If  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$  and  $\forall x \in \mathcal{U} \setminus \{a\}, f(x) \leq g(x) \leq h(x)$ , then,  $\lim_{x \rightarrow a} g(x) = L$*

*Proof.* Heine definition and the Sandwich Theorem for sequences.  $\square$

**Definition 2.2.4** (Continuity). *Let  $f$  be defined on an open interval  $\mathcal{U}$  around  $a$ . We say that  $f$  is continuous at point  $a$  if:*

$$\lim_{x \rightarrow a} f(x) = f(a)$$

*If  $\forall a \in \mathcal{U}, f$  is continuous at  $a$ , then  $f$  is continuous at  $\mathcal{U}$ .*

**Lemma 2.2.3** (Arithmetic of Limits of functions). *Suppose  $\lim_{x \rightarrow a} f(x) = F$ ,  $\lim_{x \rightarrow a} g(x) = G$  and  $C$  is a constant:*

1.  $\lim_{x \rightarrow a} c = c, \lim_{x \rightarrow a} (c \cdot f(x)) = c \cdot F$
2.  $\lim_{x \rightarrow a} (f(x) \pm g(x)) = F \pm G$
3.  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = F \cdot G$
4. *If  $G \neq 0$  and  $g(x) \neq 0$ ,  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$*

*Proof.* Heine definition and the previous results for sequences.  $\square$

**Corollary 2.2.1** (Arithmetic of Continuity). *If  $f$  and  $g$  are continuous, then:  $c \cdot f(x)$ ,  $f(x) \pm g(x)$ ,  $f(x) \cdot g(x)$  and  $\frac{f(x)}{g(x)}$  are continuous. Further  $f \circ g$  is continuous.*

**Definition 2.2.5** (Limit from the Side). *We say  $x_n \rightarrow a^+$  iff  $x_n \rightarrow a$  and  $\forall n \in \mathbb{N}, x_n > a$ . Analogously,  $x_n \rightarrow a^-$  iff  $x_n \rightarrow a$  and  $\forall n \in \mathbb{N}, x_n < a$ .*

**Definition 2.2.6** (One-Sided Limits). *We define:*

(From the right)  $\lim_{x \rightarrow a^+} f(x) = L$  iff: (Cauchy)

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathcal{U}, a < x < a + \delta \Rightarrow |f(x) - L| < \epsilon$$

(Heine)  $\forall \{x_n \in \mathcal{U}\}_{n \in \mathbb{N}}, x_n \rightarrow a^+ \Rightarrow f(x_n) \rightarrow L$

(From the left)  $\lim_{x \rightarrow a^-} f(x) = L$  iff: (Cauchy)

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathcal{U}, a - \delta < x < a \Rightarrow |f(x) - L| < \epsilon$$

(Heine)  $\forall \{x_n \in \mathcal{U}\}_{n \in \mathbb{N}}, x_n \rightarrow a^- \Rightarrow f(x_n) \rightarrow L$

**Remark 2.2.1.** *The proof of the equivalency of the definitions is the same.*

**Definition 2.2.7** (Continuity on Closed Sets). *We say  $f$  is continuous on  $[a, b]$  if it is continuous on  $\mathcal{U} = (a, b)$  and both  $\lim_{x \rightarrow a^+} f(x) = f(a)$  and  $\lim_{x \rightarrow b^-} f(x) = f(b)$ .*

**Theorem 2.2.2** (Two-Sided Limit Criteria). *Let  $f$  be defined on an open interval  $\mathcal{U}$  around  $a$ , "except". Then:*

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

*Proof.* The  $(\Rightarrow)$  direction is trivial. For the  $(\Leftarrow)$  let  $x_n \rightarrow a$ , define  $\{x_n^+\}$  as the (largest) subsequence of  $x_n$  such that  $\forall n \in \mathbb{N}, x_n^+ > a$ . Define  $x_n^-$  analogously. Then,  $x_n^+ \rightarrow a^+$  and  $x_n^- \rightarrow a^-$ . By Heine,  $f(x_n^+) \rightarrow L$  and  $f(x_n^-) \rightarrow L$ . Since  $\{x_n\} = \{x_n^+\} \cup \{x_n^-\}$ , we get:  $f(x_n) \rightarrow L$ .  $\square$

## 2.3 Classifying Continuity

**Definition 2.3.1** (Uniform Continuity). *A function  $f : \mathcal{U} \rightarrow \mathbb{R}$  is uniformly continuous iff: (Cauchy)*

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in \mathcal{U}, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

**Remark 2.3.1.** *The main difference between uniform continuity and regular continuity is that the  $\delta$  is chosen independent of  $x$  and  $y$ , whereas the  $\delta$  in regular continuity is, in general, dependent on the choice of  $a$ .*

**Theorem 2.3.1** (UC  $\Rightarrow C^0$ ). *If  $f$  is uniformly continuous on  $\mathcal{U}$ , then it is continuous on  $\mathcal{U}$ .*

*Proof.*  $a \in \mathcal{U} \Rightarrow \forall \epsilon > 0, \exists \delta > 0 : \forall x \in \mathcal{U}, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ , hence, by definition  $\lim_{x \rightarrow a} f(x) = f(a)$ .  $\square$

**Theorem 2.3.2** (Heine-Cantor). *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then it is uniformly continuous.*

*Proof.* By contrary, suppose

$$\exists \epsilon > 0 : \forall \delta > 0, \exists x, y \in [a, b] : |x - y| < \delta \Rightarrow |f(x) - f(y)| \geq \epsilon$$

Define  $\{x_n\}$  and  $\{y_n\}$  s.t.  $\forall n \in \mathbb{N}, |x_n - y_n| < \frac{1}{n}$  and  $|f(x_n) - f(y_n)| \geq \epsilon$ . Since  $x_n$  is bounded, there is a convergent subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  with limit we denote  $x_0 \in [a, b]$ . By Triangle Inequality,

$$|y_{n_k} - x_0| \leq |x_{n_k} - x_0| + |x_{n_k} - y_{n_k}| \leq |x_{n_k} - x_0| + \frac{1}{n_k}$$

hence  $y_{n_k} \rightarrow x_0$ . Since  $f$  is continuous,  $\lim_{k \rightarrow \infty} f(x_{n_k}) = f(x_0) = \lim_{k \rightarrow \infty} f(y_{n_k})$ . But  $\forall k \in \mathbb{N}, |f(x_{n_k}) - f(y_{n_k})| \geq \epsilon > 0$ . By taking  $k \rightarrow \infty$ , there is a contradiction.  $\square$

**Lemma 2.3.1.** *If  $f : [a, \infty) \rightarrow \mathbb{R}$  is continuous and  $\lim_{x \rightarrow \infty} f(x)$  exists and is finite, then,  $f$  is uniformly continuous.*

*Proof.* Let  $\epsilon > 0$ . By definition,

$$\exists M \in \mathbb{R} : \forall x \in \mathcal{U}, x > M \Rightarrow |f(x) - L| < \frac{\epsilon}{3}$$

Then,  $\forall x, y > M, |f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \frac{2\epsilon}{3} < \epsilon$ . Since  $[a, M]$  is compact,  $f$  is UC there, so,

$$\exists \delta > 0 : \forall x, y \in [a, M], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$$

$\forall x \in [a, M], \forall y > M, |x - y| < \delta \Rightarrow |x - M| < \delta$  and  $|y - M| < \delta \Rightarrow |f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| < \epsilon$ .  $\square$

**Definition 2.3.2** (Lipschitz Continuity). *Let  $f : \mathcal{U} \rightarrow \mathbb{R}$  is Lipschitz continuous (on  $\mathcal{U}$ ) if:*

$$\exists K > 0 : \forall x, y \in \mathcal{U}, |f(x) - f(y)| < K|x - y|$$

**Theorem 2.3.3** (LpC  $\Rightarrow$  UC). *Let  $f : \mathcal{U} \rightarrow \mathbb{R}$  is Lipschitz continuous (on  $\mathcal{U}$ ), then  $f$  is uniformly continuous on  $\mathcal{U}$ .*

*Proof.* Choose  $\delta(\epsilon) = \frac{\epsilon}{K}$  on the definition.  $\square$

**Remark 2.3.2.** *The converse is not true. Counterexample:  $f(x) = \sqrt{|x|}$  is uniformly continuous on  $\mathbb{R}$ , but not Lipschitz continuous (cusp at  $x = 0$ ).*

**Definition 2.3.3** (Types of Discontinuity). *Let  $f$  be defined on an open interval about  $a$ , "except".*

*The point  $a$  is a **removable** discontinuity point of  $f$  if:*

- (a)  $\exists \lim_{x \rightarrow a} f(x)$
- (b)  $\lim_{x \rightarrow a} f(x) \neq f(a)$  or  $f(a)$  is not defined.

*The point  $a$  is a **jump/first kind** discontinuity point of  $f$  if:*

- (a)  $\exists \lim_{x \rightarrow a^-} f(x)$  and  $\exists \lim_{x \rightarrow a^+} f(x)$
- (b)  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$

*The point  $a$  is a **essential/second kind** discontinuity point of  $f$  if:  
At least one of  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  does not exist.*

## 2.4 Continuity Theorems

**Definition 2.4.1** (Boundedness). *A function  $f$  is bounded on  $\mathcal{D}$  if*

$$\exists M > 0 : \forall x \in \mathcal{D}, |f(x)| \leq M$$

*or equivalently,  $\exists m, M \in \mathbb{R} : \forall x \in \mathcal{D}, m \leq f(x) \leq M$*

**Theorem 2.4.1.** *Suppose that  $g(x)$  is bounded and  $\lim_{x \rightarrow a} f(x) = 0$ . Then,*  

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = 0.$$

*Proof.* Heine definition and the previous lemma for sequences. □

**Lemma 2.4.1** (Boundedness Theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, then  $f$  is bounded.*

*Proof.* By contrary, suppose  $f$  is not bounded from above on  $[a, b]$ . Then,  $\forall n \in \mathbb{N}, \exists x_n \in [a, b] : f(x_n) > n$ . This defines a sequence  $\{x_n\}_{n \in \mathbb{N}}$ . Because  $[a, b]$  is bounded, by Bolzano-Weierstrass, there exists a convergent subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  with limit  $x_0 \in [a, b]$ . Since  $f$  is continuous,  $f(x_{n_k}) \rightarrow f(x_0)$ . But  $\forall k \in \mathbb{N}, f(x_{n_k}) > n_k \geq k$  which implies that  $f(x_{n_k}) \rightarrow \infty$ , contradiction. Therefore,  $f$  is bounded from above on  $[a, b]$ . Analogously, to prove it is bounded from below. □

**Theorem 2.4.2** (Weierstrass/Extreme Value Theorem). *Let  $f$  be continuous on the closed interval  $[a, b]$ . Then  $f$  gets its maximal and minimal values on  $[a, b]$ .*

*Proof.* We prove for the maximal, since the minimal is analogous. Since  $f$  is bounded, take  $M = \sup \{f(x) \mid x \in [a, b]\}$ . We just need to show  $\exists x_0 \in [a, b] : f(x_0) = M$ .

Take  $f(x_n) = M - \frac{1}{n}$ , so that  $f(x_n) \rightarrow M$ .  $\{x_n\}$  is bounded, so by Bolzano-Weierstrass, there is  $\{x_{n_k}\}_{k \in \mathbb{N}}$  with limit  $x_0 \in [a, b]$ . Since  $f$  is continuous,  $f(x_{n_k}) \rightarrow f(x_0) = M$ . □



**Theorem 2.4.3** (Cauchy/Intermediate Value Theorem). *Let  $f$  be continuous on  $[a, b]$  and let  $y \in \mathbb{R} : f(a) < y < f(b)$  or  $f(a) > y > f(b)$ . Then*

$$\exists c \in (a, b) : f(c) = y$$

*Proof.* We will prove for  $f(a) < y < f(b)$ . Let  $S = \{x \in [a, b] \mid f(x) \leq y\}$ . Then  $S$  is non-empty since  $a \in S$ , and  $S$  is bounded above by  $b$ . Let  $c = \sup S$  so that  $c \in (a, b)$ . Let's prove that  $f(c) = y$ .

Let  $\epsilon > 0$ . Since  $f$  is continuous,  $\exists \delta > 0 : \forall x \in [a, b], |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ . That is,  $\forall x \in (c - \delta, c + \delta), f(x) - \epsilon < f(c) < f(x) + \epsilon$ . By the properties of the supremum, there exists some  $\alpha \in (c - \delta, c]$  that is contained in  $S$ , and so  $f(c) < f(\alpha) + \epsilon \leq y + \epsilon$ . Picking  $\beta \in (c, c + \delta)$ , we know that  $\beta \notin S$  because  $c$  is the supremum of  $S$ . This means that  $f(c) > f(\beta) - \epsilon > y - \epsilon$ . Since,  $\forall \epsilon > 0, y - \epsilon < f(c) \leq y + \epsilon$ , we get  $f(c) = y$ .  $\square$

**Corollary 2.4.1.** *Let  $f$  be continuous on  $[a, b]$ . If  $f(a) \cdot f(b) < 0$ , then  $\exists c \in (a, b) : f(c) = 0$*

**Theorem 2.4.4** (Brouwer's Fixed Point Theorem).  *$f : [a, b] \rightarrow [a, b]$  continuous, then  $\exists x_0 \in [a, b] : f(x_0) = x_0$*

*Proof.* Let  $g(x) = f(x) - x$ . Then, we get:  $g(b) = f(b) - b \leq b - b = 0$  and  $g(a) = f(a) - a \geq a - a = 0$ . Then, either  $g(b) = 0$ , or  $g(a) = 0$  or  $g(b) < 0 < g(a)$ , so by IVT,  $\exists c \in (a, b) : g(c) = 0$ . Either way,  $\exists c \in [a, b] : g(c) = 0$ .  $\square$

**Corollary 2.4.2.** *Let  $f$  be continuous on  $[a, b]$ . Then  $\text{Im}(f)$  is a closed interval  $[m, M]$ , where  $M$  is the maximum of  $f$  and  $m$  is the minimum.*

**Definition 2.4.2** (Intermediate Value Property).  *$f : [a, b] \rightarrow \mathbb{R}$  has intermediate value property if, for any  $y \in \mathbb{R} : f(a) < y < f(b)$  or  $f(a) > y > f(b)$ , then*

$$\exists c \in (a, b) : f(c) = y$$

**Theorem 2.4.5** (Darboux). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable on  $(a, b)$ , then  $f'$  has the intermediate value property in  $[\alpha, \beta] \subset (a, b)$ .*

*Proof.* Suppose  $f'(\alpha) > y > f'(\beta)$ . Let  $g : [a, b] \rightarrow \mathbb{R}$  such that  $g(x) = f(x) - yx$ . Since  $g$  is continuous on  $[\alpha, \beta]$ , by Weierstrass (EVT),  $g$  attains its maximum in  $[\alpha, \beta]$ .

Because  $g'(\alpha) = f'(\alpha) - y > 0$ , we know  $g$  cannot attain its maximum value at  $\alpha$ . Likewise, because  $g'(\beta) = f'(\beta) - y < 0$ , we know  $g$  cannot attain its maximum value at  $\beta$ .

Therefore,  $g$  must attain its maximum value at some point  $x \in (a, b)$ . Hence, by Fermat's theorem,  $g'(x) = 0$ , i.e.  $f'(x) = y$   $\square$

**Corollary 2.4.3.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ , then there are no removable or jump discontinuity. Hence if  $\exists \lim_{x \rightarrow c^-} f'(x)$  and  $\exists \lim_{x \rightarrow c^+} f'(x)$  for  $c \in (a, b)$ , then  $\lim_{x \rightarrow c^+} f'(x) = \lim_{x \rightarrow c^-} f'(x) = f'(c)$ .*

## 3 Differential Calculus

### 3.1 Derivatives

**Definition 3.1.1** (Derivative). *Given a function  $f : \mathcal{D} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , the derivative at a point  $a \in \mathcal{D}$ , so that there is an open interval in  $\mathcal{D}$  around  $a$ , is defined as:*

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

*If the limit exists,  $f$  is differentiable at  $a$ . We can also take the derivative at each point and get a function  $f' : \mathcal{D}' \subseteq \mathcal{D} \rightarrow \mathbb{R}$  such that:*

$$f'(x) = \lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta}$$

$\mathcal{D}'$  is the domain of differentiability.

**Remark 3.1.1.** *The line tangent to the curve of the graph of  $f$  has slope  $f'(a)$  at point  $a$ . Actually the tangent line has formula  $y = f'(a) \cdot (x - a) + f(a)$ . Furthermore, the normal line to the curve ( if  $f'(a) \neq 0$  ) has equation  $y = -\frac{1}{f'(a)} \cdot (x - a) + f(a)$*

**Lemma 3.1.1** (Infinitesimal Function). *Let  $f$  be defined on an open interval  $\mathcal{U}$  about  $a$ . Then,  $f$  is differentiable at  $a$  iff  $\exists L \in \mathbb{R}$  and  $\exists \varphi : \mathcal{U} \rightarrow \mathbb{R}$  with  $\lim_{x \rightarrow a} \varphi(x) = 0$ , such that:  $\frac{f(x) - f(a)}{x - a} = L + \varphi(x)$ . In that case,  $L = f'(a)$ .*

**Theorem 3.1.1.** *If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .*

*Proof.* By the previous lemma,

$$f(x) = f(a) + \left( f'(a) + \varphi(x) \right) \cdot (x - a) \Rightarrow \lim_{x \rightarrow a} f(x) = f(a)$$

□

**Remark 3.1.2.** *The derivative function might not be continuous. Example:  $f(x) = x^2 \sin(1/x)$  is differentiable everywhere,  $f'(x)$  is not continuous at 0.*

**Lemma 3.1.2** (Product Rule and Arithmetic of Limits). *If  $\exists f'(x)$  and  $\exists g'(x)$ :*

1.  $(c \cdot f(x))' = c \cdot f'(x)$
2.  $(f(x) + g(x))' = f'(x) + g'(x)$
3. (Product Rule)  $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. (Quotient Rule)  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$

*Proof.* The first two follow from linearity. We will only show the product rule.

$$\begin{aligned} (f \cdot g)'(x) &= \lim_{\delta \rightarrow 0} \frac{f(x + \delta) \cdot g(x + \delta) - f(x) \cdot g(x)}{\delta} = \\ &= \lim_{\delta \rightarrow 0} \left[ f(x + \delta) \cdot \frac{g(x + \delta) - g(x)}{\delta} + \frac{f(x + \delta) - f(x)}{\delta} \cdot g(x) \right] \\ &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \end{aligned}$$

□

**Corollary 3.1.1** (Power Rule).  $p(x) = \sum_{k=0}^n a_k \cdot x^k \Rightarrow p'(x) = \sum_{k=1}^n k \cdot a_k \cdot x^{k-1}$

**Lemma 3.1.3.** *Let  $f$  be an invertible and continuous in an open interval about  $a$ . If  $\exists f'(a)$  and  $f'(a) \neq 0$ , then, the inverse function  $f^{-1}$  is differentiable at  $b = f(a)$  and:*

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

*Proof.* By definition of the derivative:

$$\begin{aligned} (f^{-1})'(b) &= \lim_{y \rightarrow b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \lim_{y \rightarrow b} \frac{f^{-1}(y) - f^{-1}(b)}{f(f^{-1}(y)) - f(f^{-1}(b))} \\ &= \left( \lim_{y \rightarrow b} \frac{f(f^{-1}(y)) - f(f^{-1}(b))}{f^{-1}(y) - f^{-1}(b)} \right)^{-1} \end{aligned}$$

Since  $f$  is bijective and continuous,  $f^{-1}(y) \rightarrow f^{-1}(b)$  as  $y \rightarrow b$ . Then, since  $x = f^{-1}(y)$  and  $a = f^{-1}(b)$ :

$$(f^{-1})'(b) = \left( \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \right)^{-1} = (f'(a))^{-1} = (f'(f^{-1}(b)))^{-1}$$

□

**Theorem 3.1.2** (Chain Rule). *Let  $g$  be differentiable in an open interval about  $a$  and  $f$  be differentiable in an open interval about  $b = g(a)$ . Then,  $f \circ g$  is differentiable at  $a$  and:*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a) = (f' \circ g)(a) \cdot g'(a)$$

*Proof.* We can represent:

$$\begin{aligned} \frac{g(x) - g(a)}{x - a} &= g'(a) + \alpha(x) \\ \frac{f(y) - f(a)}{y - b} &= f'(b) + \beta(y) \end{aligned}$$

Putting it together, we get:

$$\begin{aligned} f(g(x)) - f(g(a)) &= (f'(g(a)) + \beta(g(x))) \cdot (g(x) - g(a)) \\ &= (f'(g(a)) + \beta(g(x))) \cdot (g'(a) + \alpha(x)) \cdot (x - a) \end{aligned}$$

By definition of the derivative:  $(f \circ g)'(a) =$

$$= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \rightarrow a} (f'(g(a)) + \beta(g(x))) \cdot (g'(a) + \alpha(x))$$

Since  $\lim_{x \rightarrow a} \alpha(x) = 0$  and  $\lim_{x \rightarrow a} \beta(g(x)) = \lim_{y \rightarrow b} \beta(y) = 0$ , we get:

$$\left\{ \begin{aligned} \lim_{x \rightarrow a} (f'(g(a)) + \beta(g(x))) &= f'(g(a)) \\ \lim_{x \rightarrow a} (g'(a) + \alpha(x)) &= g'(a) \end{aligned} \right\} \Rightarrow (f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

□

## 3.2 Mean Value Theorems and Taylor

**Theorem 3.2.1** (Fermat). *Let  $f$  be defined on an open interval  $\mathcal{U}$  and let  $f$  be differentiable at  $x_0 \in \mathcal{U}$ . If  $f$  gets its maximal (or minimal) value at  $x_0$ , then:  $f'(x_0) = 0$*

*Proof.* WLOG  $f(x_0)$  is the maximal value of  $f$  at  $\mathcal{U} = (a, b)$ . Then:

$$\forall \delta \in (a - x_0, b + x_0), f(x_0 + \delta) \leq f(x_0)$$

- $\delta > 0$  :  $\frac{f(x_0 + \delta) - f(x_0)}{\delta} \leq 0 \Rightarrow \lim_{\delta \rightarrow 0^+} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \leq 0$
- $\delta < 0$  :  $\frac{f(x_0 + \delta) - f(x_0)}{\delta} \geq 0 \Rightarrow \lim_{\delta \rightarrow 0^-} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \geq 0$

Hence, since the derivate exists, the one sided limits are equal:

$$0 \leq \lim_{\delta \rightarrow 0^-} \frac{f(x_0 + \delta) - f(x_0)}{\delta} = f'(x_0) = \lim_{\delta \rightarrow 0^+} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \leq 0$$

$$\Rightarrow f'(x_0) = 0. \quad \square$$

**Theorem 3.2.2** (Rolle). *Let  $f$  be defined on an closed interval  $[a, b]$ . If  $f$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$  and  $f(a) = f(b)$ , then,  $\exists c \in (a, b) : f'(c) = 0$ .*

*Proof.* Since  $[a, b]$  is compact (closed and bounded),  $f$  is bounded. Using Weierstrass Theorem,  $f$  gets its maximum  $M$  and minimum  $m$  on  $[a, b]$ . Consider the following possibilities:

- $m = M$  :  $\Rightarrow f(x) = \text{const.} \Rightarrow \forall c \in (a, b), f'(x) = 0$ .
- $m < M$  :  $\Rightarrow$  at least one of  $\{m, M\}$  are obtained in  $(a, b)$ , otherwise  $f(a) \neq f(b)$ . Therefore, by Fermat's Theorem, let  $c$  be the value for minimum or maximum, we have:  $f'(c) = 0$ .

□

**Theorem 3.2.3** (Lagrange/Mean Value Theorem). *Let  $f$  be defined on an closed interval  $[a, b]$ . If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then,*

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

*Proof.* Let  $g(x) = f(a) - f(x) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$ , so that  $g(a) = g(b) = 0$ . Also, notice that  $g'(x) = -f'(x) + \frac{f(b) - f(a)}{b - a}$ . Then, by Rolle's Theorem,  $\exists c \in (a, b)$  :

$$0 = g'(c) = -f'(c) + \frac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

**Theorem 3.2.4** (Cauchy Mean Value Theorem). *Let  $f$  and  $g$  be defined on an closed interval  $[a, b]$ . If  $f$  and  $g$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then,*

$$\exists c \in (a, b) : f'(c) \cdot (g(b) - g(a)) = g'(c) \cdot (f(b) - f(a))$$

*Proof.* Let  $h(x) = (f(x) - f(a)) \cdot (g(b) - g(a)) - (f(b) - f(a)) \cdot (g(x) - g(a))$ , so that  $h(a) = h(b) = 0$ . Also, notice that  $h'(x) = f'(x) \cdot (g(b) - g(a)) - (f(b) - f(a)) \cdot g'(x)$ . Then, by Rolle's Theorem,  $\exists c \in (a, b)$  :

$$0 = h'(c) = f'(c) \cdot (g(b) - g(a)) - (f(b) - f(a)) \cdot g'(c)$$

$$\Rightarrow f'(c) \cdot (g(b) - g(a)) = g'(c) \cdot (f(b) - f(a))$$

□

**Definition 3.2.1** (Higher Order Derivatives). *Suppose that  $f$  is differentiable. If  $f'$  is differentiable, we say that  $f$  is **twice** differentiable. Notation:  $f''(x) = (f')'(x)$ . It is called the second derivative of  $f$ . Similarly, we define the  $n$ -th derivative, denoted  $f^{(n)}(x)$ .*

**Definition 3.2.2** ( $C^k$  classes). *We define  $f \in C^0$  if  $f$  is continuous. Further, we define  $f \in C^k \Leftrightarrow f' \in C^{k-1}$ . If  $f$  is infinitely differentiable, we write  $C^\infty$ .*

**Theorem 3.2.5** (Taylor's Formula). *Let  $f$  be  $n+1$  times differentiable (with  $n \in \mathbb{N}_0$ ) on an open interval  $\mathcal{U}$  about  $a$  and let  $x \in \mathcal{U}$ . Then,  $\exists c$  between  $a$  and  $x$ , which might depend on  $x$ , such that:*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(x)$$

With  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$ , called the Lagrange Remainder.

*Proof.* Denote  $T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$ . Fixing  $x$ , let:

$$g(t) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (x-t)^k - \left[ f(x) - T_n(x) \right] \frac{(x-t)^{n+1}}{(x-a)^{n+1}}$$

Notice that  $g$  is continuous between  $a$  and  $x$  (on  $[a, x]$  or  $[x, a]$ ). Also,  $g(a) = g(x) = 0$ . Moreover,  $g$  is differentiable between  $a$  and  $x$  with:

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!} (x-t)^n + (n+1) \left[ f(x) - T_n(x) \right] \frac{(x-t)^n}{(x-a)^{n+1}}$$

Hence, by Rolle's Theorem,  $\exists c$  between  $a$  and  $x$  such that:

$$\begin{aligned} 0 = g'(c) &= -\frac{f^{(n+1)}(c)}{n!} (x-c)^n + (n+1) \left[ f(x) - T_n(x) \right] \frac{(x-c)^n}{(x-a)^{n+1}} \\ \Leftrightarrow f(x) - T_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} = R_n(x) \end{aligned}$$

□

**Remark 3.2.1.** *In order of stronger to weaker: Taylor > Lagrange > Rolle.*

**Definition 3.2.3** (Analytical Function). *If  $\forall x \in \mathcal{U}$ ,  $\lim_{n \rightarrow \infty} R_n(x) = 0$  and  $f \in C^\infty$ , we have:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

*the Taylor Series. If  $a = 0$ , the series is called MacLaurin Series. Then, we call  $f$  an analytical function (denoted  $f \in C^\omega$ ), that is, it's Taylor Series converges pointwise to  $f$  in some open interval.*



**Remark 3.2.2** (Error). *If we want to calculate with an error of less than  $\varepsilon$  we only need to find  $n$  such that:*

$$\left| R_n(x) \right| = \frac{|f^{(n+1)}(c)| \cdot |x|^{n+1}}{(n+1)!} < \varepsilon$$

**Theorem 3.2.6** (L'Hôpital's Rule). *Suppose that  $f$  and  $g$  are continuous on a closed interval  $[a, b]$ , and are differentiable on the open interval  $(a, b)$ . Suppose that  $g'(x)$  is never zero on  $(a, b)$ , and that  $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$  exists, and that  $f(a) = g(a) = 0$ . Then,*

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$$

*It is also true for  $x \rightarrow a^-$ , so also true for  $x \rightarrow a$ .*

*Proof.*  $\forall x \in (a, b)$ ,  $f$  and  $g$  are continuous on  $[a, x]$  and differentiable on  $(a, x)$ . By Cauchy's MVT  $\exists c \in (a, x) : f'(c) \cdot g(x) = f(x) \cdot g'(c)$ . Then:

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(c)}{g'(c)} = \lim_{c \rightarrow a^+} \frac{f'(c)}{g'(c)}$$

Since  $c \rightarrow a^+$  as  $x \rightarrow a^+$ . □

**Remark 3.2.3.** *We can continue the theorem to the  $n$ -th derivative:*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$$

*It is also valid for  $a = \pm\infty$  and the limit in the wide sense.*

**Lemma 3.2.1** (SC for LpC). *If  $f : \mathcal{U} \rightarrow \mathbb{R}$  is differentiable at  $\mathcal{U}$  and the derivative is bounded, then  $f$  is Lipschitz continuous.*

*Proof.* Let  $K = \sup |f'(x)|$  then, by Lagrange's  $\forall x, y \in \mathcal{U}, \exists c \in (x, y) :$   
 $\frac{f(x) - f(y)}{x - y} = f'(c) \Rightarrow |f(x) - f(y)| \leq K|x - y|$  □

### 3.3 Investigation of Functions

**Definition 3.3.1** (Extremum Point). *Let  $f$  be defined on an open interval  $\mathcal{U}$  about  $a$ . We say that  $f$  has a **local minimum** (eq. **maximum**) at  $a$  if exists an open interval  $\mathcal{I} \subseteq \mathcal{U}$  about  $a$  such that:*

$$\forall x \in \mathcal{I}, f(x) \geq f(a) \text{ (eq. } f(x) \leq f(a) \text{)}$$

*The point  $a$  which is either a local minimum or local maximum point is called a **local extremum point**.*

**Definition 3.3.2** (Critical Point). *We say that  $a$  is a **critical point** of  $f$  if either  $f'(a) = 0$  or  $\nexists f'(a)$ .*

**Lemma 3.3.1** (NC for Local Extremum).  *$a$  is a local extremum  $\Rightarrow a$  is a critical point.*

*Proof.* If  $f$  is differentiable at  $a$ , we use Fermat's Theorem. Otherwise,  $\nexists f'(a)$ . Either way,  $a$  is a critical point.  $\square$

**Lemma 3.3.2** (SC I for Local Extremum). *If  $\exists f'$  and  $\exists f''$  which are on continuous on an open interval about  $a$  and  $f'(a) = 0$  and  $f''(a) \neq 0$ . Then,  $a$  is local extremum and:*

- *If  $f''(a) < 0$ ,  $a$  is local maximum.*
- *If  $f''(a) > 0$ ,  $a$  is local minimum.*

*Proof.* If  $f''(a) > 0$  and  $f''$  is continuous, then  $f''(a) > 0$  in some open interval around  $a$ . From Taylor's Formula we get:

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(c)(x - a)^2 \geq f(a) + f'(a)(x - a) = f(a)$$

since  $\frac{1}{2}f''(c)(x - a)^2 \geq 0$ . Analogously, for local maximum.  $\square$

**Lemma 3.3.3** (SC II for Local Extremum). *Let  $a$  is critical point of  $f$ .  $f$  is continuous at  $a$  and differentiable on an open interval about  $a$ , "except".*

- If  $f'$  changes sign from negative to positive at  $a$ , then  $a$  is a local minimum.
- If  $f'$  changes sign from positive to negative at  $a$ , then  $a$  is a local maximum.
- Else,  $a$  is not a local extremum point.

**Definition 3.3.3.** Let  $f$  be defined on a domain  $\mathcal{D}$ . We say that  $f$  has a **global minimum (eq. maximum)** at  $a$  if

$$\forall x \in \mathcal{D}, f(x) \geq f(a) \left( \text{eq. } f(x) \leq f(a) \right)$$

**Theorem 3.3.1.** Let  $f$  be continuous on  $[a, b]$ , then  $f$  has at least one global maximum and at least one global minimum. Further, if  $c$  is a global extremum, then it is either an endpoint or a critical point.

*Proof.* The first part is exactly Weierstrass' Theorem. The second part follow from: if  $c$  is a global extremum, then it is either a local extremum on  $(a, b)$ , in which case it is a critical point. Else,  $c = a$  or  $c = b$ .  $\square$

**Definition 3.3.4** (Convexity). Let  $f$  be differentiable at  $a$ . We say that  $f$  is **convex upward (eq. downward)** at  $a$  if exists an open interval about  $a$  in which the graph of the function is situated under (eq. above) the tangent line to  $f$  at  $(a, f(a))$ , i.e.

$$\begin{aligned} \exists \delta > 0 : \forall x \in (a - \delta, a + \delta), f(x) \leq f(a) + f'(a)(x - a) \\ \left( \text{eq. } f(x) \geq f(a) + f'(a)(x - a) \right) \end{aligned}$$

In an interval,  $f$  is **convex upward (eq. downward)** if, for every point in the interval, it is convex upward (eq. downward).

**Theorem 3.3.2.** Let  $f$  be twice differentiable on the interval  $(a, b)$ .

1. If  $\forall x \in (a, b), f''(x) > 0$ ,  $f$  is convex downward on  $(a, b)$
2. If  $\forall x \in (a, b), f''(x) < 0$ ,  $f$  is convex upward on  $(a, b)$

*Proof.*  $\forall x \in (a, b)$ , by Taylor's formula, for  $n = 1$ ,  $\exists c \in (a, x)$  :

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(c) (x - a)^2$$

Then, we have either  $f''(c) > 0 \Rightarrow f(x) \geq f(a) + f'(a)(x - a)$  or, on the other case,  $f''(c) < 0 \Rightarrow f(x) \leq f(a) + f'(a)(x - a)$   $\square$

**Definition 3.3.5** (Inflection Point). *Let  $f$  be continuous on an open interval about  $a$  and is differentiable at  $a$  in a wide sense. We say  $a$  is an **inflection point** of  $f$  if exists an interval  $(a - \delta, a + \delta)$  such that the function changes its convexity passing through  $a$ .*

**Remark 3.3.1.** *It may happen that at an inflection point  $a$  the derivative equals 0 (ex.  $f(x) = x^3$ ,  $a = 0$ ) or  $\infty$  (ex.  $f(x) = \sqrt[3]{x}$ ,  $a = 0$ ) or any value  $L$  (ex.  $f(x) = L \sin x$ ,  $a = 0$ )*

**Definition 3.3.6** (Asymptotes). *Let  $f$  be defined (for some  $\delta > 0$ ) on either  $(a - \delta, a)$  or  $(a, a + \delta)$  or  $(a - \delta, a + \delta)$  "except". If at least one of the one-sided limits of  $f$  is equal to  $\pm\infty$ , then we say that the straight line  $x = a$  is a **vertical asymptote** of  $f$ .*

*The straight line  $y = ax + b$  is called an **oblique asymptote** of  $f$  at  $+\infty$  (eq.  $-\infty$ ) if:*

$$\lim_{x \rightarrow +\infty} [f(x) - (ax + b)] = 0 \quad \left( \text{eq. } \lim_{x \rightarrow -\infty} [f(x) - (ax + b)] = 0 \right)$$

*In the case  $a = 0$ , the asymptote is also called a **horizontal asymptote**.*

**Lemma 3.3.4.** *The asymptote at  $\pm\infty$  (if it exists) is unique.*

*Proof.* Let  $a_1x + b_1$  and  $a_2x + b_2$  be asymptote of  $f$  at  $\infty$ .

$$\left\{ \begin{array}{l} \lim_{x \rightarrow \infty} [f(x) - (a_1x + b_1)] = 0 \\ \lim_{x \rightarrow \infty} [(a_2x + b_2) - f(x)] = 0 \end{array} \right\} \Rightarrow \lim_{x \rightarrow \infty} [(a_2x + b_2) - (a_1x + b_1)] = 0$$

$\Rightarrow a_2 = a_1$  and  $b_1 = b_2$ . Analogously, for  $-\infty$ .  $\square$

**Theorem 3.3.3** (Calculating Asymptotes). *Let  $f$  be defined on  $(c, \infty)$ . If there exists  $a = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$  and  $b = \lim_{x \rightarrow \infty} [f(x) - ax]$ , then the straight line  $y = ax + b$  is a unique oblique asymptote of  $f$  at  $+\infty$ . Analogously, for  $-\infty$ .*

## 4 Integral Calculus

### 4.1 Indefinite Integral

**Definition 4.1.1** (Indefinite Integral). A function  $F$  is called an antiderivative of  $f$  in the domain  $\mathcal{D}$  if  $\forall x \in \mathcal{D}, F'(x) = f(x)$ . If  $F$  is an antiderivative of  $f$ , then  $F(x) + C$  also is. The set of all antiderivatives of  $f$ , i.e.  $F(x) + C$  is called the **indefinite integral** of  $f$  and is denoted by:

$$\int f(x) dx = F(x) + C$$

**Remark 4.1.1.** We have the following properties:

1. If  $\exists f'$ , then  $\int f'(x) dx = f(x) + C$
2. The integration is linear:  $\int [a \cdot f(x) + b \cdot g(x)] dx = a \cdot \int f(x) dx + b \cdot \int g(x) dx$

**Example 4.1.1.** We have:

1.  $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$ , if  $\alpha \neq -1$
2.  $\int \frac{dx}{x} = \ln|x| + C$
3.  $\int [a \cdot \cos x + b \cdot \sin x] dx = a \cdot \sin x - b \cdot \cos x + C$
4.  $\int \frac{dx}{1+x^2} = \arctan x + C$
5.  $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

**Lemma 4.1.1** (Substitution). *The chain rule states:*

$$[f(u(x))]' = f'(u(x)) \cdot u'(x) \Rightarrow \int f(u) \cdot u' dx = \int f(u) du$$

We also write:  $\int f(x) dx = \left\{ \begin{array}{l} x = g(t) \\ dx = g'(t) dt \end{array} \right\} = \int f(g(t)) g'(t) dt$

**Lemma 4.1.2** (Integration by Parts). *The product rule states:*

$$\begin{aligned} (u \cdot v)'(x) &= u'(x) \cdot v(x) + u(x) \cdot v'(x) \Rightarrow u \cdot v' = (u \cdot v)' - u' \cdot v \\ &\Rightarrow \int u \cdot v' dx = uv - \int u' \cdot v dx \end{aligned}$$

**Lemma 4.1.3** (Integration of Trigonometric Functions). *We have:*

1. *Prosthaphaeresis:*

$$\begin{aligned} \sin(\alpha) \cdot \sin(\beta) &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos(\alpha) \cdot \cos(\beta) &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin(\alpha) \cdot \cos(\beta) &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta)) \\ \cos(\alpha) \cdot \sin(\beta) &= \frac{1}{2} (\sin(\alpha + \beta) - \sin(\alpha - \beta)) \end{aligned}$$

2.  $\int \sin^{2k+1}(x) \cdot \cos^m(x) dx = \int (1 - \cos^2(x))^k \cdot \sin(x) \cdot \cos^m(x) dx =$   
 $\left\{ \begin{array}{l} u = \cos x \\ u' = -\sin x \end{array} \right\} = - \int (1 - u^2)^k \cdot u^m du$

3. *We use recursively:*  $\cos^2(x) = \frac{1 + \cos(2x)}{2} \cdot \int \sin^{2k}(x) \cdot \cos^{2m}(x) dx =$   
 $\int \left( \frac{1 - \cos(2x)}{2} \right)^k \left( \frac{1 + \cos(2x)}{2} \right)^m dx$

**Example 4.1.2.** *We have:*

$$\begin{aligned}
1. \quad & \int \sin(3x) \cos(7x) dx = \frac{1}{2} \int \left[ \sin(10x) - \sin(4x) \right] dx = \frac{1}{2} \left[ -\frac{\cos(10x)}{10} + \frac{\cos(4x)}{4} \right] + C = -\frac{\cos(10x)}{20} + \frac{\cos(4x)}{8} + C \\
2. \quad & \int \sin^3(x) \cos^2(x) dx = \left\{ \begin{array}{l} u = \cos x \\ u' = -\sin x \end{array} \right\} = - \int (1 - u^2) \cdot u^2 du = \\
& \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C \\
3. \quad & \int \cos^4(x) dx = \frac{1}{4} \int [1 + 2\cos(2x) + \cos^2(2x)] dx = \frac{1}{8} \int [3 + 4\cos(2x) + \\
& \cos(4x)] dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

**Lemma 4.1.4** (Substitutions). *We get:*

Expression	Substitution	Differential	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$ $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$dx = a \cos \theta d\theta$	$\sqrt{1 - \sin^2 \theta} = \cos \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$ $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$dx = a \sec^2 \theta d\theta$	$\sqrt{1 + \tan^2 \theta} = \sec \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$ $\theta \in \left[0, \frac{\pi}{2}\right)$	$dx = a \tan \theta \sec \theta d\theta$	$\sqrt{\sec^2 \theta - 1} = \tan \theta$

**Example 4.1.3.** *We get:*

$$\begin{aligned}
1. \quad & \int \frac{\sqrt{9 - x^2}}{x^2} dx = \left\{ \begin{array}{l} x = 3 \sin \theta \\ dx = 3 \cos \theta d\theta \end{array} \right\} = \int \cot^2 \theta d\theta = -\cot \theta - \theta + \\
& C = -\frac{\sqrt{9 - x^2}}{x} - \arcsin \frac{x}{3} + C \\
2. \quad & \int \frac{x}{\sqrt{x^2 + 4}} dx = \left\{ \begin{array}{l} x = 2 \tan \theta \\ dx = 2 \sec^2 \theta d\theta \end{array} \right\} = 2 \int \tan \theta \cdot \sec \theta d\theta = 2 \sec \theta + \\
& C = \sqrt{x^2 + 4} + C
\end{aligned}$$

**Definition 4.1.2** (Simple Rational Function). A rational function is a function  $f(x) = \frac{P(x)}{Q(x)}$ , where  $P$  and  $Q$  are polynomials. A rational function  $f(x) = \frac{P(x)}{Q(x)}$  is called **simple** if  $\deg P < \deg Q$ .

**Lemma 4.1.5** (Simplifying RFs). If a rational function is not simple, it can be written as the sum of a polynomial and a simple rational function.

*Proof.* By polynomial long division, we have:  $P(x) = Q(x) \cdot S(x) + R(x)$  with  $\deg R < \deg Q$ , then  $f(x) = S(x) + \frac{R(x)}{Q(x)}$ , so  $\frac{R(x)}{Q(x)}$  is simple.  $\square$

**Definition 4.1.3** (Basic RF). We identify four basic simple rational functions:

$f(x)$	$\int f(x) dx (+C)$
$\frac{A}{x - \alpha}$	$A \ln  x - \alpha $
$\frac{A}{(x - \alpha)^n}$	$A \frac{(x - \alpha)^{1-n}}{1 - n}$
$\frac{Ax + B}{x^2 + 2px + q}$	$\frac{A}{2} \ln  x^2 + 2px + q  + \frac{B - Ap}{\sqrt{q - p^2}} \arctan \left( \frac{x + p}{\sqrt{q - p^2}} \right)$
$\frac{Ax + B}{(x^2 + 2px + q)^n}$	$\frac{A}{2} \frac{(x^2 + 2px + q)^{1-n}}{1 - n} + \frac{B - Ap}{(q - p^2)^{n-\frac{1}{2}}} \mathcal{I}_n \left( \frac{x + p}{\sqrt{q - p^2}} \right)$

so that  $p^2 - q < 0$  and  $n \in \mathbb{N}_{>1}$ .

$$\begin{aligned}
1. \int \frac{Ax + B}{x^2 + 2px + q} dx &= \int \frac{Ax + B}{(x + p)^2 + \sqrt{q - p^2}^2} = \left\{ \begin{array}{l} a = \sqrt{q - p^2} \\ t = x + p \\ dt = dx \end{array} \right\} = \\
&\int \frac{At dt}{t^2 + a^2} + (B - Ap) \int \frac{dt}{t^2 + a^2} = \frac{A}{2} \ln |t^2 + a^2| + \frac{B - Ap}{a} \arctan \left( \frac{t}{a} \right) + \\
C &= \frac{A}{2} \ln |x^2 + px + q| + \frac{B - Ap}{\sqrt{q - p^2}} \arctan \left( \frac{x + p}{\sqrt{q - p^2}} \right) + C
\end{aligned}$$



$$\begin{aligned}
2. \int \frac{Ax + B}{(x^2 + 2px + q)^n} dx &= \left\{ \begin{array}{l} a = \sqrt{q - p^2} \\ t = x + p \\ dt = dx \end{array} \right\} = A \int \frac{t dt}{(t^2 + a^2)^n} + (B - \\
&Ap) \int \frac{dt}{(t^2 + a^2)^n} = \frac{A}{2} \frac{(t^2 + a^2)^{1-n}}{1-n} + (B - Ap) \frac{1}{a^{2n-1}} \mathcal{I}_n\left(\frac{t}{a}\right) + C = \\
&\frac{A}{2} \frac{(x^2 + 2px + q)^{1-n}}{1-n} + \frac{(B - Ap)}{\sqrt{q - p^2}^{2n-1}} \mathcal{I}_n\left(\frac{x + p}{\sqrt{q - p^2}}\right) + C
\end{aligned}$$

Where  $\mathcal{I}_n(u) = \int \frac{du}{(u^2 + 1)^n}$  is given by a recursive formula.

**Theorem 4.1.1** (Partial Fraction Decomposition). *Every simple rational function can be written as a sum of basic rational functions.*

*Proof.* Let  $f(x) = \frac{R(x)}{Q(x)}$

1. If  $Q(x) = A(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$ :

$$\frac{R(x)}{Q(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_k}{x - \alpha_k} = \sum_{i=1}^k \frac{A_i}{x - \alpha_i}$$

where they are given explicitly by:  $A_i = \lim_{x \rightarrow \alpha_i} (x - \alpha_i) f(x)$ .

2. If  $Q(x) = A(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_k)^{m_k}$ ,  $m_i \in \mathbb{N}_{>1}$ :

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^k \sum_{j=1}^{m_i} \frac{A_{i,j}}{(x - \alpha_i)^j}$$

where they are given by:  $A_{i,j} = \frac{1}{(m_i - j)!} \lim_{x \rightarrow \alpha_i} \frac{d^{m_i-j}}{dx^{m_i-j}} (x - \alpha_i)^{m_i} f(x)$ .

3. If  $Q(x) = A(x^2 + 2p_1x + q_1) \cdots (x^2 + 2p_kx + q_k)(x - \alpha_1) \cdots (x - \alpha_l)$ , where  $p_i^2 - q_i < 0$ :

$$\frac{R(x)}{Q(x)} = \frac{P_1x + Q_1}{x^2 + 2p_1x + q_1} + \cdots + \frac{P_kx + Q_k}{x^2 + 2p_kx + q_k} + \frac{A_1}{x - \alpha_1} + \cdots + \frac{A_l}{x - \alpha_l}$$

4. If  $Q(x) = A(x^2 + 2p_1x + q_1)^{m_1}(x^2 + 2p_2x + q_2)^{m_2} \cdots (x^2 + 2p_kx + q_k)^{m_k}$ ,  
where  $m_i \in \mathbb{N}_{>1}$  and  $p_i^2 - q_i < 0$ :

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^k \sum_{j=1}^{m_i} \frac{P_{i,j}x + Q_{i,j}}{(x^2 + 2p_ix + q_i)^j}$$

□

**Corollary 4.1.1.** *Every rational function has an antiderivative in closed form, that is, there is an expression using only elementary functions for its antiderivative.*

## 4.2 Definite Integrals

**Definition 4.2.1** (Riemann Sums and Integral). *Let  $f$  be defined on the closed interval  $[a, b]$ . Let  $n \in \mathbb{N}$  and  $T$  is a partition for  $[a, b]$  on  $n$  sub-intervals.*

$$T : a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

Denote  $\Delta x_i = x_i - x_{i-1}$  and we denote  $\|T\| = \max_{i=1,2,\dots,n} \Delta x_i$ , the norm of the partition.

Choose a point  $c_i \in [x_{i-1}, x_i]$ . Then, the sum  $\sum_{i=1}^n f(c_i) \Delta x_i$  is called the **Riemann sum** of  $T$ .

A function  $f(x)$  is called Riemann integrable over  $[a, b]$  if there exists the limit  $I = \lim_{\|T\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$  and the limit is independent on the choice of  $T$  and  $c_i$ . This limit is denoted by  $I = \int_a^b f(x) dx$  and is called the **definite integral** (or the Riemann integral).

**Remark 4.2.1.** *The Riemann sum approximates the area under the curve  $y = f(x)$  and the  $x$ -axis by rectangles of base  $\Delta x_i$  and height  $f(c_i)$ . Therefore, the Riemann integral is exactly that area.*

**Lemma 4.2.1.** *We have:*

1.  $\int_a^a f(x) dx = 0$
2.  $\int_a^b f(x) dx = - \int_b^a f(x) dx$
3.  $\int_b^a c dx = c(b - a)$
4. *If  $f$  is not bounded on  $[a, b]$ , then  $f$  is not Riemann integrable on  $[a, b]$ .*

**Example 4.2.1.** We define the following bounded function:

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

It is not Riemann integrable on any  $[a, b]$ .

*Proof.* Take any partition  $T$  on  $[a, b]$  and choose:

- $c_i \in \mathbb{Q} : \sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n \Delta x_i = b - a$
- $c_i \notin \mathbb{Q} : \sum_{i=1}^n f(c_i) \Delta x_i = 0$

Since  $b - a \neq 0$ ,  $\nexists \lim_{\|T\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$ , since it depends on the choice of  $c_i$ .  $\square$

**Definition 4.2.2** (Darboux Upper and Lower integrals). Let  $f$  be defined and bounded on the closed interval  $[a, b]$ . Let  $n \in \mathbb{N}$  and  $T$  is a partition for  $[a, b]$  on  $n$  sub-intervals.

$$T : a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

Denote  $\Delta x_i = x_i - x_{i-1}$  and we denote  $\|T\| = \max_{i=1,2,\dots,n} \Delta x_i$ , the norm of the partition. Define  $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$  and  $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ .

Then, the sum  $L(T) = \sum_{i=1}^n m_i \Delta x_i$  is called the **lower Darboux sum** of  $T$

and the sum  $U(T) = \sum_{i=1}^n M_i \Delta x_i$  is called the **upper Darboux sum** of  $T$ .

Denoting  $M = \sup_{x \in [a, b]} f(x)$  and  $m = \inf_{x \in [a, b]} f(x)$ , we notice:

$$\begin{aligned} m(b-a) &= \sum_{i=1}^n m \Delta x_i \leq \sum_{i=1}^n m_i \Delta x_i = L(T) \leq A \\ A &\leq U(T) = \sum_{i=1}^n M_i \Delta x_i \leq \sum_{i=1}^n M \Delta x_i = M(b-a) \end{aligned}$$

We define:  $\sup_T L(T) = L$  the **lower Darboux integral** and  $\inf_T U(T) = U$  the **upper Darboux integral** denoted

$$\int_a^b f(x) dx = L \quad \text{and} \quad U = \overline{\int_a^b f(x) dx}$$

If  $U = L$ , the function is said to be **Darboux integrable**.

**Lemma 4.2.2** (N&SC for Darboux Integrability).  $f$  is Darboux integrable iff  $\forall \epsilon > 0, \exists T : U(T) - L(T) < \epsilon$

*Proof.* That condition is equivalent to  $\lim_{\|T\| \rightarrow 0} (U(T) - L(T)) = 0$   $\square$

**Lemma 4.2.3** (DI  $\Leftrightarrow$  RI). Any Riemann sum  $R(T, c) = \sum_{i=1}^n f(c_i) \Delta x_i$  is between the Darboux sums:  $U(T) \geq R(T, c) \geq L(T)$ . If  $\lim_{\|T\| \rightarrow 0} U(T) = \lim_{\|T\| \rightarrow 0} L(T) = I$ , by Sandwich,  $\lim_{\|T\| \rightarrow 0} R(T) = I \Rightarrow f$  is Riemann integrable

over  $[a, b]$ . That is,  $\int_a^b f(x) dx = \overline{\int_a^b f(x) dx} = \int_a^b f(x) dx$

**Theorem 4.2.1.** Every continuous function is Riemann Integrable.

*Proof.* Since  $f$  is defined on a closed interval  $[a, b]$ , it is uniformly continuous. So,  $\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in [a, b], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ . Then, given  $\epsilon > 0$ , choose the constant partition  $\|T_n\| = \frac{1}{n} < \min\{1, \delta\}$ . Therefore,  $M_i - m_i < \epsilon$ , so  $U(T_n) - L(T_n) < \sum_{i=1}^n \epsilon \cdot \frac{1}{n} = \epsilon$ .  $\square$

**Definition 4.2.3** (Piecewise Continuity). *A function  $f$  which is defined and bounded on  $[a, b]$  is called piecewise continuous if it has at most a finite number of discontinuity points and all of them are removable or of the first kind.*

**Theorem 4.2.2.** *Every piecewise continuous function is Riemann Integrable.*

*Proof.* By linearity of the integral, split the interval so that we are left with the integral of only continuous functions.  $\square$

**Theorem 4.2.3.** *Every bounded monotonic function is Riemann Integrable.*

*Proof.* WLOG, let  $f$  be increasing on  $[a, b]$ . Given  $\epsilon > 0$ , choose the constant partition  $\|T_n\| = \frac{1}{n} < \frac{\epsilon}{f(b) - f(a)}$ . Further  $M_i = f(x_i)$  and  $m_i = f(x_{i-1})$ . Therefore,  $U(T) - L(T) = \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \frac{1}{n} = \frac{f(b) - f(a)}{n} < \epsilon$   $\square$

**Theorem 4.2.4** (Intermediate Integral Theorem). *Let  $f$  be continuous on  $[a, b]$ , then  $\exists c \in [a, b]$  :*

$$\int_a^b f(x) dx = f(c) (b - a)$$

*Proof.* Take  $g(t) = f(t) \cdot (b - a) - \int_a^b f(x) dx$ , which is continuous, since  $f$  is continuous. By Weierstrass' theorem,  $f$  achieves a maximum  $M$  and a minimum  $m$  on  $[a, b]$ . By Darboux Integration,  $m \cdot (b - a) \leq \int_a^b f(x) dx \leq M \cdot (b - a)$ , hence, by IVT,  $\exists c \in [a, b] : g(c) = 0$ .  $\square$

### 4.3 Fundamental Theorem of Calculus

**Theorem 4.3.1** (FTC I). *Let  $f$  be continuous on  $(a, b)$  and  $c \in (a, b)$ . Then, the function  $F(x) = \int_c^x f(t) dt$  is an antiderivative function of  $f$  on  $(a, b)$ , i.e.  $\forall x \in (a, b), F'(x) = f(x)$ .*

*Proof.* Given  $\delta > 0$ , for any  $x \in (a, b)$  such that  $x + \delta \in (a, b)$ , by Intermediate Integral Theorem,  $\exists \tau \in [x, x + \delta]$  :

$$\begin{aligned} F(x + \delta) - F(x) &= \int_c^{x+\delta} f(t) dt - \int_c^x f(t) dt = \int_x^{x+\delta} f(t) dt = f(\tau) \delta \\ \Rightarrow F'(x) &= \lim_{\delta \rightarrow 0} \frac{F(x + \delta) - F(x)}{\delta} = \lim_{\delta \rightarrow 0} f(\tau) = f(x) \end{aligned}$$

since  $\tau \in [x, x + \delta]$ . □

**Remark 4.3.1.** *The theorem is also true for a piecewise continuous function, with the caveat  $F'(x) = f(x)$  for  $x$  a continuity point.*

**Theorem 4.3.2** (Newton-Leibnitz / FTC II). *Let  $f$  be continuous on  $[a, b]$  and let  $F$  be an arbitrary antiderivative function of  $f$  on  $(a, b)$ . Then:*

$$\int_a^b f(x) dx = F(b) - F(a)$$

*Proof.* Let  $c \in (a, b)$ . By FTC,  $G(x) = \int_c^x f(t) dt$  is an antiderivative of  $f$ . Taking any other antiderivative  $F(x) = G(x) + C$ . Calculate:

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) = G(b) - G(a) \\ &= \int_c^b f(t) dt - \int_c^a f(t) dt = \int_a^b f(t) dt \end{aligned}$$

□

*Notation:*  $\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$ .

**Remark 4.3.2.** If  $f$  is piecewise continuous then:

$$\int_a^b f(x) dx = F(b^-) - F(a^+)$$

where  $F(b^-) = \lim_{x \rightarrow b^-} F(x)$  and  $F(a^+) = \lim_{x \rightarrow a^+} F(x)$ .

**Lemma 4.3.1** (Leibnitz Formula).

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t, x) dt = f(b(x), x) \cdot b'(x) - f(a(x), x) \cdot a'(x) + \int_{a(x)}^{b(x)} \partial_x f(t, x) dt$$

where  $\partial_x f(t, x) = \lim_{\delta \rightarrow 0} \frac{f(t, x + \delta) - f(t, x)}{\delta}$ .

**Definition 4.3.1** (Improper Integral). An integral is said to be improper if either (or both) bounds are infinite or the function is unbounded in the interval of integration. We calculate by the following:

$$\begin{aligned} \int_a^\infty f(x) dx &= \lim_{R \rightarrow \infty} \int_a^R f(x) dx \\ \int_{-\infty}^b f(x) dx &= \lim_{R \rightarrow \infty} \int_{-R}^b f(x) dx \\ \int_a^b f(x) dx &= \lim_{\epsilon \rightarrow 0} \left[ \int_{c+\epsilon}^b f(x) dx + \int_a^{c-\epsilon} f(x) dx \right] \end{aligned}$$

**Example 4.3.1.**

$$\begin{aligned} \int_1^\infty x^{-\frac{4}{3}} dx &= \lim_{R \rightarrow \infty} \int_1^R x^{-\frac{4}{3}} dx = \lim_{R \rightarrow \infty} -3x^{-\frac{1}{3}} \Big|_1^R = 3 \\ \int_{-\infty}^{-1} x^{-\frac{4}{3}} dx &= \lim_{R \rightarrow \infty} \int_{-R}^{-1} x^{-\frac{4}{3}} dx = \lim_{R \rightarrow \infty} -3x^{-\frac{1}{3}} \Big|_{-R}^{-1} = 3 \\ \int_1^{-1} x^{-\frac{1}{3}} dx &= \lim_{\epsilon \rightarrow 0} \left[ \int_\epsilon^1 x^{-\frac{1}{3}} dx + \int_{-1}^{-\epsilon} x^{-\frac{1}{3}} dx \right] \end{aligned}$$

**Theorem 4.3.3** (Integral Test). Let  $f : [N, \infty) \rightarrow \mathbb{R}$  be an integrable function that is monotone decreasing.

$$\sum_{n=N}^{\infty} f(n) \text{ converges} \Leftrightarrow \int_N^{\infty} f(x) dx \text{ converges}$$



*Proof.* We have the following estimate:  $\forall M \in \mathbb{N}$ ,

$$\int_N^{M+1} f(x) dx \leq \sum_{n=N}^M f(n) \leq f(N) + \int_N^M f(x) dx$$

Taking limit of both sides, we get the result. □