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Notes for Mechanical Performance of Materials

Yihui Li

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1 Fatigue of Materials

Often, machine members are found to have failed under the action of repeated or fluctuating stresses. The process of damage and failure due to cyclic loading is called fatigue.

1.1 Definition

1.1.1 Description of Cyclic Loading

In a cycle, mean stress σ_m and stress amplitude σ_a are defined as:

$$\sigma_m = \frac{1}{2}(\sigma_{\max} + \sigma_{\min}); \ \sigma_a = \frac{1}{2}(\sigma_{\max} - \sigma_{\min}); \tag{1}$$

The following ratios are sometimes used:

$$R = \frac{\sigma_{\min}}{\sigma_{\max}}; \tag{2}$$

$$A = \frac{\sigma_a}{\sigma_m}; \tag{3}$$

1.1.2 Nominal stress

Nominal stress is calculated on the basis of the net cross section of a specimen without taking into account the effect of geometric discontinuities. Under axial loading or bending moment, the nominal stress S is calculated by:

$$S = \frac{P}{A}; \text{ (Axial)} \tag{4}$$

$$S = \frac{Mc}{I}; \text{ (Bending)} \tag{5}$$

1.1.3 Loading Condition

All loading conditions are concluded in Table 1:

The value of R	Loading Condition
$R = -1$	Fully reversed loading
$R = 0$	Zero-tension loading
$R = -\infty$	Zero-compression loading
$-\infty < R < 0$	$\sigma_{\min} < 0$, $\sigma_{\max} > 0$
$0 < R < 1$	Tension-tension loading
$R > 1$	Compression-compression loading

Table 1: Loading condition corresponding to R value.

1.1.4 Stress Versus Life (S-N) Curves

If a test specimen of a material or an engineering component is subjected to a sufficiently severe cyclic stress, a fatigue crack or other damage will develop, leading to complete failure of the member. If the test is repeated at a higher stress level, the number of cycles to failure will be smaller. The results of such tests from a number of different stress levels may be plotted to obtain a *stress–life curve*, also called an S-N curve. The amplitude of stress or nominal stress, σ_a or S_a, is commonly plotted versus the number of cycles to failure N_f.

1.1.5 Endurance Limit

Endurance limit is the stress level below which an infinite number of loading cycles can be applied to a material without causing fatigue failure. For steel, we can estimate the endurance limit as:

$$\begin{cases} 0.5S_{ut},\ S_{ut}{\leq}1400\,\mathrm{MPa}; \\ 700\,\mathrm{MPa},\ S_{ut} > 1400\,\mathrm{MPa}; \end{cases} \tag{6}$$

1.2 Stress Versus Life (S-N) Curves under completely reversed loading

If S-N data are found to approximate a straight line on a log-linear plot, Equation (7) can be fitted to obtain a mathematical representation of the curve:

$$\sigma_a = C + D \log N_f; \tag{7}$$

For data approximating a straight line on a log-log plot, Equation (8) can be fitted to obtain a mathematical representation of the curve:

$$\sigma_a = A \cdot {N_f}^B; \tag{8}$$

Also, another form of Equation (8) can be used, which is Basquin's Equation:

$$\sigma_a = \sigma'_f \cdot (2N_f)^b; \tag{9}$$

$$A = 2^b \sigma'_f,\ B = b; \tag{10}$$

1.2.1 Safety Factors for S-N Curves

the stress amplitude A stress level $\hat{\sigma}_a$ and a number of cycles $\hat{N}$ are expected to occur in actual service. Define σ_{a1} as the stress amplitude at the desired service life $\hat{N}$ and N_{f2} as the failure life at the service stress $\hat{\sigma}_a$. Therefore, the safety factors can be defined as:

$$X_S = \frac{\sigma_{a1}}{\hat{\sigma}_a}; \tag{11}$$

$$X_N = \frac{N_{f2}}{\hat{N}_f}; \tag{12}$$

Also, if the value of a load P expected in actual service is denoted $\hat{P}$, and the load factor for this load can be calculated from $P_f = Y_P \hat{P}$.

1.3 Effect of Mean Stress on S-N Curves

1.3.1 Presentation of Mean Stress Data

One procedure used for developing data on mean stress effects is to select several values of mean stress, running tests at various stress amplitudes for each of these. An alternative means of presenting the same information is a constant-life diagram, which is a σ_m-σ_a plot. Also, another procedure often used for developing data on mean stress effect is to choose several values of the stress ratio R and run tests at various stress levels for each of these, plotting a N_f-$\sigma_{\max}$ plot.

1.3.2 Normalized Amplitude-Mean Diagrams

Let the stress amplitude for the particular case of zero mean stress be designated σ_{ar}. We can have three types of normalized amplitude-mean diagrams, which are σ_m-σ_a/σ_{ar} plots. Three expressions of line/curves are included in Table 2, and the example diagram is shown as Figure 1:

Criteria	Equations
Goodman equation	$\frac{\sigma_a}{\sigma_{ar}} + \frac{\sigma_m}{\sigma_{ut}} = 1$
Gerber parabola	$\frac{\sigma_a}{\sigma_{ar}} + \left(\frac{\sigma_m}{\sigma_{ut}}\right)^2 = 1$
J. Morrow Modification on Goodman equation	$\frac{\sigma_a}{\sigma_{ar}} + \frac{\sigma_m}{\tilde{\sigma}_{fB}} = 1$ or $\frac{\sigma_a}{\sigma_{ar}} + \frac{\sigma_m}{\sigma'_f} = 1$

Table 2: Equations for three different criteria. ($\tilde{\sigma}_{fB}$ is the corrected true fracture strength)

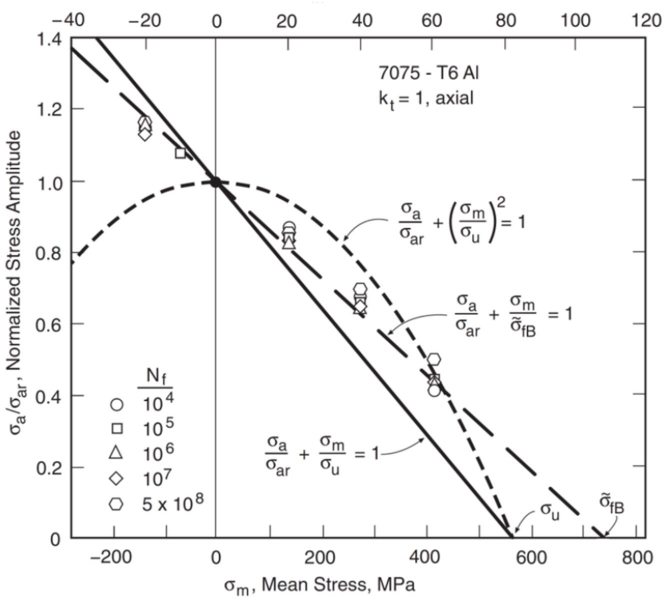


Figure 1: Example of Normalized amplitude-mean diagram.

1.3.3 Smith, Watson and Topper (SWT) equation

To calculate σ_{ar}, two forms of Smith, Watson and Topper (SWT) equation can be used:

$$\sigma_{ar} = \sqrt{\sigma_{\max}\sigma_a} = \sqrt{(\sigma_m + \sigma_a)\sigma_a}, \ (\sigma_{\max} > 0); \tag{13}$$

$$\sigma_{ar} = \sigma_{\max}\sqrt{\frac{1-R}{2}} = (\sigma_m + \sigma_a)\sqrt{\frac{1-R}{2}}, \ (\sigma_{\max} > 0); \tag{14}$$

1.3.4 Walker equation

To calculate σ_{ar}, two forms of Walker equation can also be used:

$$\sigma_{ar} = \sigma_{\max}^{1-\gamma}\sigma_a^{\gamma} = (\sigma_m + \sigma_a)^{1-\gamma}\sigma_a^{\gamma},\ (\sigma_{\max} > 0); \tag{15}$$

$$\sigma_{ar} = \sigma_{\max}\left(\frac{1-R}{2}\right)^{\gamma} = (\sigma_m + \sigma_a)\left(\frac{1-R}{2}\right)^{\gamma},\ (\sigma_{\max} > 0); \tag{16}$$

1.3.5 Life Estimates with Mean Stress

By using the second equation from J. Morrow Modification on Goodman equation, we can have:

$$\sigma_{ar} = \frac{\sigma_a}{1-(\sigma_m)/(\sigma_f')}; \tag{17}$$

Because σ_{ar} can be thought of as an equivalent completely reversed stress amplitude, we can use the value of σ_{ar} as σ_a in Equation (7), (8) and (9).

1.3.6 Safety Factors with Mean Stress

The safety factors with mean stress can be defined as:

$$X_S = \left.\frac{\sigma_{ar1}}{\hat{\sigma}_{ar}}\right|_{N_f=\hat{N}_f}; \tag{18}$$

$$X_N = \left.\frac{N_{f2}}{\hat{N}_f}\right|_{\sigma_{ar}=\hat{\sigma}_{ar}}; \tag{19}$$

A second option is to multiply with load factors Y_a and Y_m as:

$$\sigma_{ar1} = \frac{Y_a\hat{\sigma}_a}{1-(Y_m\hat{\sigma}_m)/(\sigma_f')}\ \text{(Morrow)},\ \sigma_{ar1} = \sqrt{(Y_m\hat{\sigma}_m + Y_a\hat{\sigma}_a)Y_a\hat{\sigma}_a}\ \text{(SWT)}; \tag{20}$$

Also, the safety factor against yielding can be defined as: (S_{eq} is the effective stress)

$$X_o = \frac{\sigma_y}{S_{eq}},\ S_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6({\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{zx}}^2)}; \tag{21}$$

1.3.7 Selection of Mean Stress Equations

Considering all of the mean stress equations given, neither the Goodman nor the Gerber equations are very accurate, with the former often being overly conservative, and the latter often nonconservative. The Morrow relationship is usually reasonably accurate, but suffers from the value of the true fracture strength $\tilde{\sigma}_{fB}$ not always being known. Also, the Morrow relationship considering σ_f' fits data very well for steels, but should be avoided for aluminum alloys and perhaps for other nonferrous metals. The SWT expression is a good choice for general use and fits data particularly well for aluminum alloys. The Walker relationship is the best choice where data exist for fitting the value of γ.

1.4 Multiaxial Fatigue

In engineering components, cyclic loadings that cause complex states of stress are common.

1.4.1 Approach to Multiaxial Fatigue

If all cyclic loads are completely reversed and have the same frequency, we can compute an effective stress amplitude $\tilde{\sigma}_a$ and use the S-N curve for completely reversed uniaxial stress with three principle stress amplitudes σ_{1a}, σ_{2a} and σ_{3a}:

$$\tilde{\sigma}_a = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{1a}-\sigma_{2a})^2+(\sigma_{2a}-\sigma_{3a})^2+(\sigma_{3a}-\sigma_{1a})^2}; \tag{22}$$

If noncyclic loads are present, we can use the following equations:

$$\tilde{\sigma}_a = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{xa}-\sigma_{ya})^2+(\sigma_{ya}-\sigma_{za})^2+(\sigma_{za}-\sigma_{xa})^2+6({\tau_{xya}}^2+{\tau_{yza}}^2+{\tau_{zxa}}^2)}; \tag{23}$$

$$\tilde{\sigma}_m = \sigma_{xm}+\sigma_{ym}+\sigma_{zm}; \tag{24}$$

We can still use the second equation from J. Morrow Modification on Goodman equation:

$$\sigma_{ar} = \frac{\tilde{\sigma}_a}{1-(\tilde{\sigma}_m)/(\sigma'_f)}; \tag{25}$$

For principal stresses, we can determine the values by:

$$\sigma_{1a},\ \sigma_{2a} = \frac{\sigma_{xx,a}+\sigma_{yy,a}}{2}\pm\sqrt{\left(\frac{\sigma_{xx,a}-\sigma_{yy,a}}{2}\right)^2+\tau_{xy,a}^2}; \tag{26}$$

Circumferential stress

If the thin-walled assumption is valid, we can have:

$$\sigma_{\text{hoop}} = 2\sigma_{\text{axial}};\ \text{(Cylinder)} \tag{27}$$

$$\sigma_{\text{hoop}} = \sigma_{\text{axial}};\ \text{(Sphere)} \tag{28}$$

$$\sigma_{\text{axial}} = \frac{pD}{4t},\ D = \frac{OD+ID}{2},\ t = \frac{OD-ID}{2}; \tag{29}$$

1.5 Irregular Loading

1.5.1 The Palmgren–Miner Rule

To apply the Palmgren–Miner rule, we need to first have a table similar to Table 3:

Block j	Number of Cycle N_j	$(\sigma_a)_j$	$(\sigma_m)_j$	$(\sigma_{ar})_j$	$(N_f)_j$	$\frac{N_j}{(N_f)_j}$

Table 3: Table for Life Prediction.

Then, Equation (30) can be used to predict the number of repetitions to failure B_f:

$$B_f \left[\sum_{j=1}^{k} \frac{N_j}{(N_f)_j} \right] = 1; \tag{30}$$

1.5.2 Equivalent Stress Level and Safety Factors

For making life estimates for variable amplitude loading, an alternative procedure is to calculate an equivalent constant amplitude stress level that causes the same life as the variable history if applied for the same number of cycles.

Consider a repeating or sample load history containing $N_B = \sum(N_j)$ rainflow cycles, we can calculate the desired equivalent constant amplitude stress by:

$$\sigma_{aq} = \left[\frac{\sum_{j=1}^{k} N_j (\sigma_{ar})_j^{-1/b}}{N_B} \right]^{-b}; \tag{31}$$

Then, the safety factors in stress and life are calculated by: (N_f are calculated from a certain criteria)

$$X_S = \left. \frac{\sigma_{aq1}}{\hat{\sigma}_{aq}} \right|_{N_f = \hat{N}_f}; \tag{32}$$

$$X_N = \left. \frac{N_{f2}}{\hat{N}_f} \right|_{\sigma_{aq} = \hat{\sigma}_{aq}} = \left. \frac{B_{f2}}{\hat{B}_f} \right|_{\sigma_{aq} = \hat{\sigma}_{aq}}; \tag{33}$$

$$B_f = \frac{N_f}{N_B}; \tag{34}$$

The value of $\hat{\sigma}_{aq}$ can be calculated by using Equation (31). Also, $X_S = X_N^b$.

1.5.3 Load Factor Approach

A load factor approach can also be applied. The entire stress history could be scaled by a single load factor Y, or different components of the loading could be assigned different load factors. The life from the factored stress history must not be less than the desired service life $\hat{N}$. With either the SWT or Walker mean stress equations, a single load factor Y applied to all stresses in the history will always have the same value as the safety factor in stress, which means $Y = X_S$.

1.5.4 Rainflow Cycle Counting

An irregular stress history consists of a series of peaks and valleys, which are points where the direction of loading changes. In performing rainflow cycle counting, a cycle is identified or counted if it meets the criterion illustrated in Figure . A peak-valley-peak or valley-peak-valley combination $X - Y - Z$ in the loading history is considered to contain a cycle if the second range $\Delta\sigma_{YZ}$ is greater than or equal to the first range $\Delta\sigma_{XY}$. If the second range is indeed larger or equal, then a cycle equal to the first range $\Delta\sigma_{XY}$ is counted. If a cycle is counted, this information is recorded, and its peak and valley are assumed not to exist for purposes of further cycle counting.

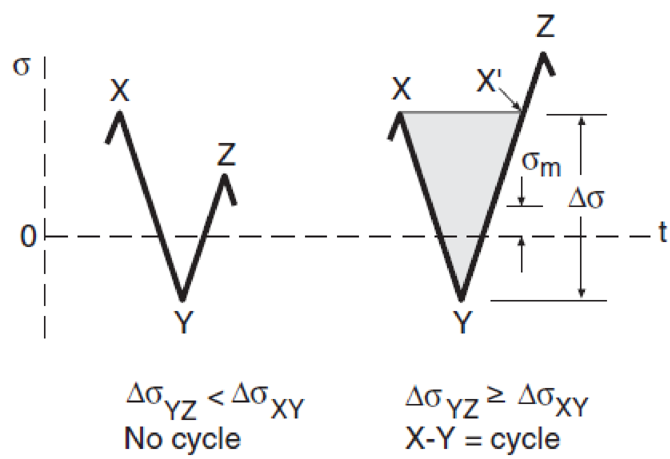


Figure 2: Condition for counting a cycle with the rainflow method.

It is convenient to move a portion of the history to the end so that a sequence is obtained that begins and ends with the highest peak or lowest valley. When doing cycle selection, following the sequence from left to right.

2 Fatigue Behaviour of Notched Members

2.1 Notch Effects at long fatigue lives

2.1.1 Theoretical stress-concentration factor

A theoretical, or geometric, stress-concentration factor K_t is used to relate the actual maximum stress at the discontinuity to the nominal stress. The factor is defined as: (S_0 is the nominal stress calculated by using the elementary stress equations and the net area)

$$K_t = \frac{\sigma_{\max}}{S_0}; \tag{35}$$

Typically,, K_t satisfy:

$$K_{t,\text{axial}} > K_{t,\text{bending}} \neq K_{t,\text{torsion}}; \tag{36}$$

2.1.2 Fatigue stress-concentration factor

Due to stress gradient effect and sampling effect, some materials are not fully sensitive to the presence of notches at long fatigue lives, specifically $N_f = 10^6$ to 10^7 cycles or greater. Therefore, a reduced value of K_f, which is fatigue stress-concentration factor, can be used. The resulting factor, which is formally defined only for completely reversed stresses, is defined as:

$$k_f = \frac{\sigma_{ar}\ \text{(Smooth samples)}}{S_{ar}\ \text{(Notched samples)}}; \tag{37}$$

Notch sensitivity q is defined as:

$$q = \frac{k_f - 1}{k_t - 1}; \tag{38}$$

Values of q can be estimated from empirical material constants that are independent of notch radius:

$$q = \frac{1}{1 + \dfrac{\alpha}{\rho}};\ \text{(Peterson)} \tag{39}$$

ρ: notch radius, α: a material constant having dimensions of length.

$$q = \frac{1}{1 + \sqrt{\dfrac{\beta}{\rho}}};\ \text{(Neuber)} \tag{40}$$

ρ: root notch radius, β: a material constant having dimensions of length.

Also, the fatigue limit S_{er} can be estimated by: (σ_{er}: the fatigue limit for completely reversed loading of unnotched material)

$$S_{er} = \frac{\sigma_{er}}{k_f}; \tag{41}$$

2.2 Notch Effects at intermediate and short fatigue lives

At intermediate and short fatigue lives in ductile materials, the reversed yielding effect becomes increasingly important as higher stresses are considered. For ductile metals, k_f' decreases from k_f at long lives to a value near unity at short lives, as shown in Figure 3:

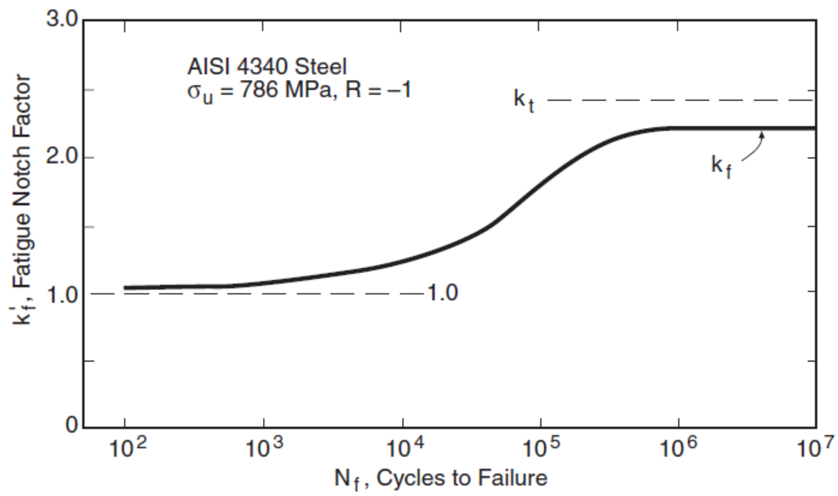


Figure 3: Test data for a ductile metal illustrating variation of k_f' with life.

To find k_f', three situation can be considered: (a) no yielding, (b) local yielding, and (c) full yielding, as illustrated in Figure 4: (σ_o is the yield strength)

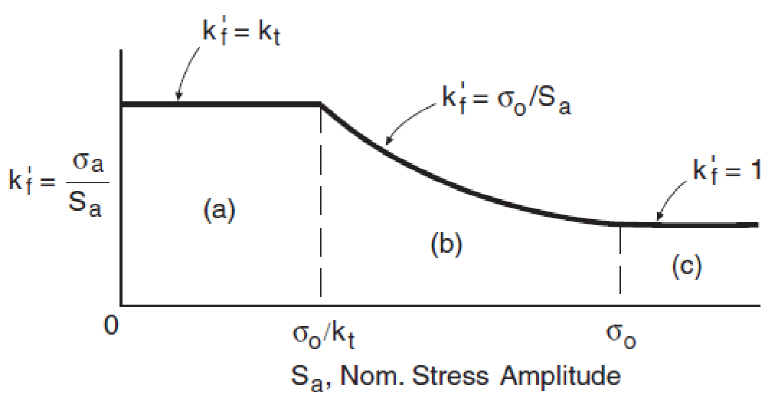


Figure 4: Cyclic yielding for a notched member of an ideal elastic, perfectly plastic material.

Notes: In Figure 3, the reason why the notch effect is small at short life might be that the plastic straining reduces the amplitude.

3 Plastic Deformation Behavior and Models for Materials

3.1 Stress–Strain Curves

3.1.1 Elastic and Perfectly Plastic Relationship

An elastic, perfectly plastic stress–strain relationship is flat beyond yielding, as illustrated in Equation (42):

$$\begin{cases} \sigma = E\varepsilon & \text{if } \sigma \leq \sigma_y \\ \sigma = \sigma_y & \text{if } \varepsilon \geq \dfrac{\sigma_y}{E} \end{cases} \tag{42}$$

Beyond yielding, the strain is the sum of elastic and plastic parts: (ε_p is analogous to the movement of the frictional slider)

$$\varepsilon = \varepsilon_e + \varepsilon_p = \frac{\sigma}{E} + \varepsilon_p \tag{43}$$

3.1.2 Elastic, Linear-Hardening Relationship

Elastic, linear-hardening behavior is useful as a rough approximation for stress–strain curves that rise appreciably following yielding. An equation for the postyield portion can be obtained by taking the slope between any point on this part of the curve and the yield point:

$$\delta E = \frac{\sigma - \sigma_y}{\varepsilon - \varepsilon_y}; \tag{44}$$

Beyond yielding, the strain can also be expressed as:

$$\varepsilon = \frac{\sigma_y}{E} + \frac{\sigma - \sigma_y}{\delta E}; \tag{45}$$

3.1.3 Elastic, Power-Hardening Relationship

Beyond yielding, the strain can be expressed as: (H_1 is a related constant, and n_1 is the strain hardening exponent)

$$\varepsilon = \left(\frac{\sigma}{H_1}\right)^{\frac{1}{n_1}}; \tag{46}$$

3.1.4 Ramberg–Osgood Relationship

The Ramberg–Osgood equation was created to describe the non linear relationship between stress and strain. The equation for strain is:

$$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n = \frac{\sigma}{E} + \left(\frac{\sigma}{H}\right)^{\frac{1}{n}}; \tag{47}$$

3.2 Three-Dimensional Stress–Strain Relationships

To obtain total strains, we can use the following equations:

$$\varepsilon_x = \varepsilon_{ex} + \varepsilon_{px} = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \frac{1}{E_p}[\sigma_x - 0.5(\sigma_y + \sigma_z)]; \tag{48}$$

$$\varepsilon_y = \varepsilon_{ey} + \varepsilon_{py} = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] + \frac{1}{E_p}[\sigma_y - 0.5(\sigma_x + \sigma_z)]; \tag{49}$$

$$\varepsilon_z = \varepsilon_{ez} + \varepsilon_{pz} = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \frac{1}{E_p}[\sigma_z - 0.5(\sigma_x + \sigma_y)]; \tag{50}$$

$$\gamma_{xy} = \gamma_{exy} + \gamma_{pxy} = \frac{\tau_{xy}}{G} + \frac{3}{E_p}\tau_{xy}; \tag{51}$$

$$\gamma_{yz} = \gamma_{eyz} + \gamma_{pyz} = \frac{\tau_{yz}}{G} + \frac{3}{E_p}\tau_{yz}; \tag{52}$$

$$\gamma_{zx} = \gamma_{ezx} + \gamma_{pzx} = \frac{\tau_{zx}}{G} + \frac{3}{E_p}\tau_{zx}; \tag{53}$$

Where

$$E_p = \frac{\bar{\sigma}}{\bar{\varepsilon}_p} = \frac{\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}}{\frac{\sqrt{2}}{3}\sqrt{(\varepsilon_{p1} - \varepsilon_{p2})^2 + (\varepsilon_{p2} - \varepsilon_{p3})^2 + (\varepsilon_{p3} - \varepsilon_{p1})^2}}; \tag{54}$$

$$\varepsilon_{e1} = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]; \tag{55}$$

$$\varepsilon_{p1} = \frac{1}{E_p}[\sigma_1 - 0.5(\sigma_2 + \sigma_3)]; \tag{56}$$

3.3 Cyclic Stress-Strain Tests and Behavior

Two possible stress responses to a period strain condition with constant amplitude are shown in Figure 5:

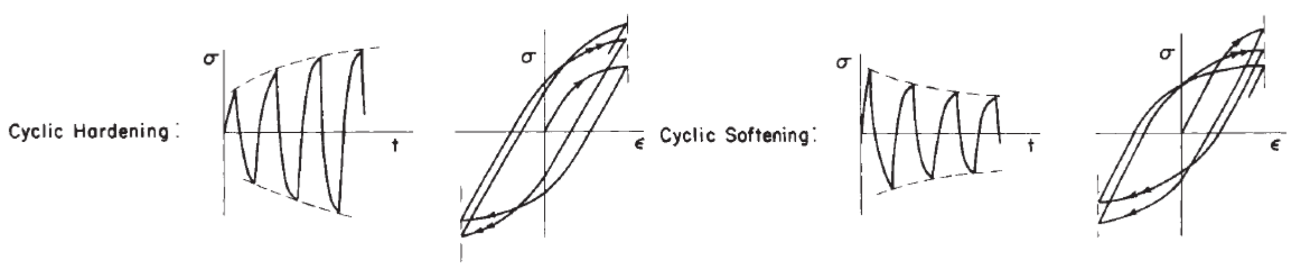


Figure 5: Completely reversed controlled strain test and two possible stress responses.

3.4 Unloading and Cyclic Loading Behavior from Rheological Models

Unloading and reloading behavior for two rheological models is shown in Figure 6:

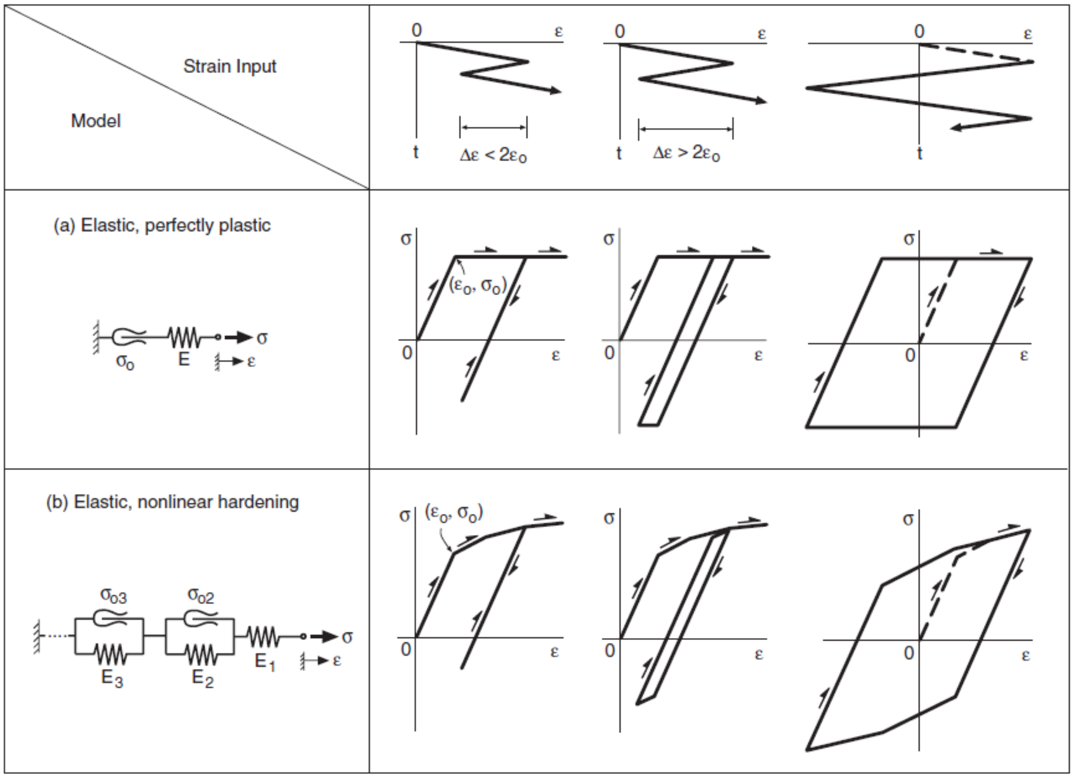


Figure 6: Unloading and reloading behavior for two rheological models.

3.4.1 Cyclic Loading Behavior

During the resulting constant amplitude cycling, the stress-strain loop between $\varepsilon_{\max}$ and $\varepsilon_{\min}$, called a hysteresis loop, Equation (57) and Equation (58) give the two branches of the hysteresis loop for both $|\varepsilon_{\max}| > |\varepsilon_{\min}|$ and $|\varepsilon_{\max}| < |\varepsilon_{\min}|$: $(\varepsilon = f(\sigma))$

$$\varepsilon = \varepsilon_{\max} - 2f\left(\frac{\sigma_{\max} - \sigma}{2}\right); \tag{57}$$

$$\varepsilon = \varepsilon_{\min} + 2f\left(\frac{\sigma - \sigma_{\min}}{2}\right); \tag{58}$$

For cyclic loading biased in either the tensile direction or the compressive direction, Equation (59) and Equation (60) can be applied:

$$\varepsilon_{\max} = f(\sigma_{\max}) \ , \ |\varepsilon_{\max}| > |\varepsilon_{\min}|; \tag{59}$$

$$\varepsilon_{\min} = -f(-\sigma_{\min}) \ , \ |\varepsilon_{\max}| < |\varepsilon_{\min}|; \tag{60}$$

Cyclic loading of engineering metals often follows a Ramberg-Osgood stress-strain relationship.

3.5 Rheological Modeling of Nonlinear Hardening

A stress–strain curve of either the elastic, power-hardening, or Ramberg–Osgood types can be modeled by approximating it as a series of straight line segments, as illustrated in Figure 7. The first segment ends at the yield strength.

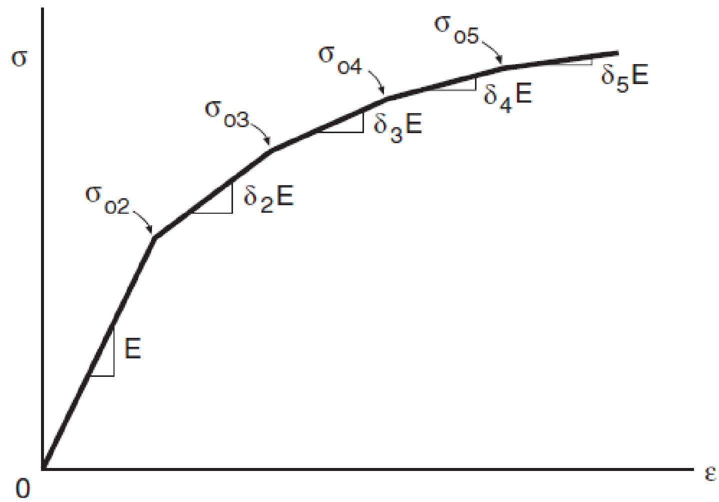


Figure 7: Linearized Representation of $\sigma - \varepsilon$ curve.

For cyclic loading, we need to first determine the initial stage using the piecewise linear elements with the corresponding slope $\delta_j E$ until the strain reaches its maximum/minimum value. After the intial stage, the length is twice the monotonic value and the segment retains the same slope for obtaining the following extreme values. A segment must be skipped if its most recent use was not in the opposite direction of its impending use. Figure 8 illustrates the application mentioned above.

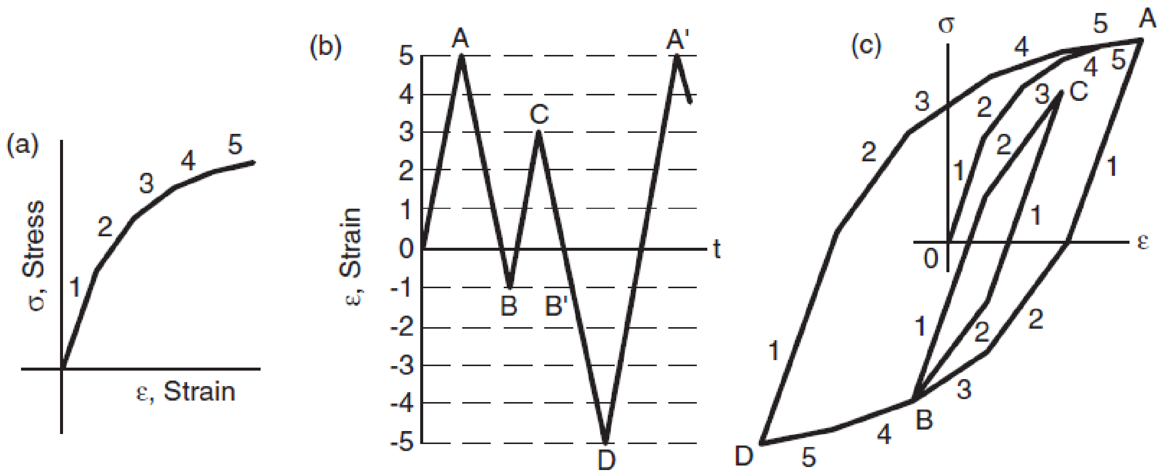


Figure 8: Behavior of a multistage spring–slider rheological model for an cyclic loading.

4 Stress–Strain Analysis of Plastically Deforming Members

4.1 Notched Members

Notched engineering members are often subjected to loads in service that cause localized yielding. As shown in Figure , at low loads, the behavior is everywhere elastic and a simple linear relationship prevails. Localized plastic deformation begins when the stress at the notch exceeds the yield strength of the material. Plastic deformation then spreads over a region of increasing size for increasing load, until the entire cross section of the member has yielded.

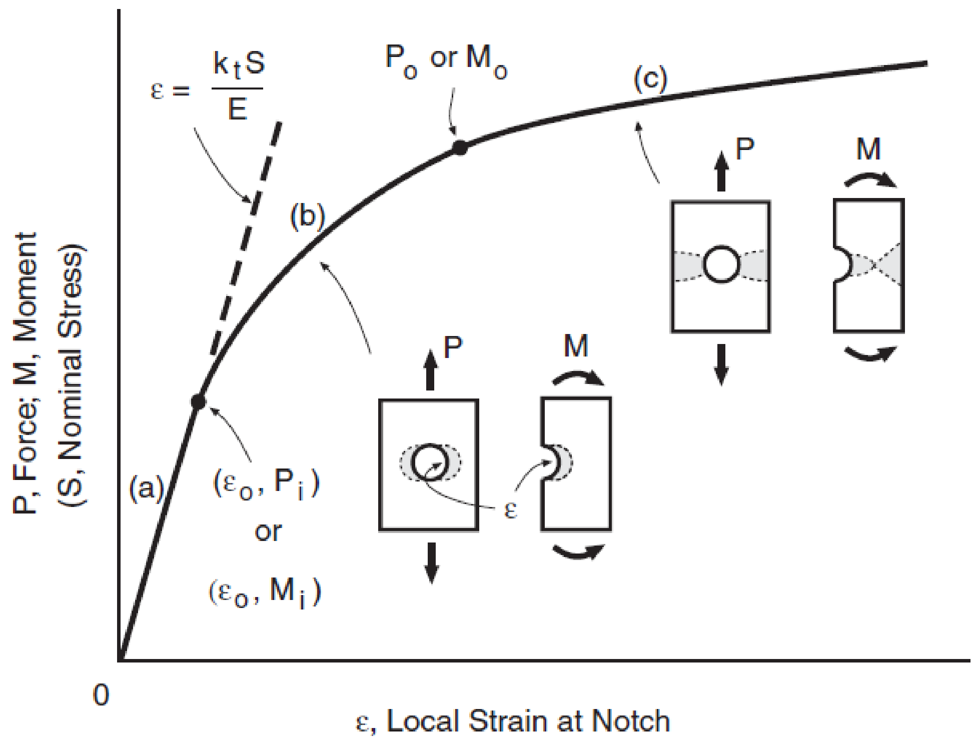


Figure 9: Load versus local strain behavior of a notched member showing three regions of behavior: (a) no yielding, (b) local yielding, and (c) fully plastic yielding.

4.1.1 Elastic Behavior and Initial Yielding

For elastic behavior, the notch stress σ and the corresponding strain ε can be determined from the nominal stress S, the elastic stress concentration factor k_t and Young's Modulus E:

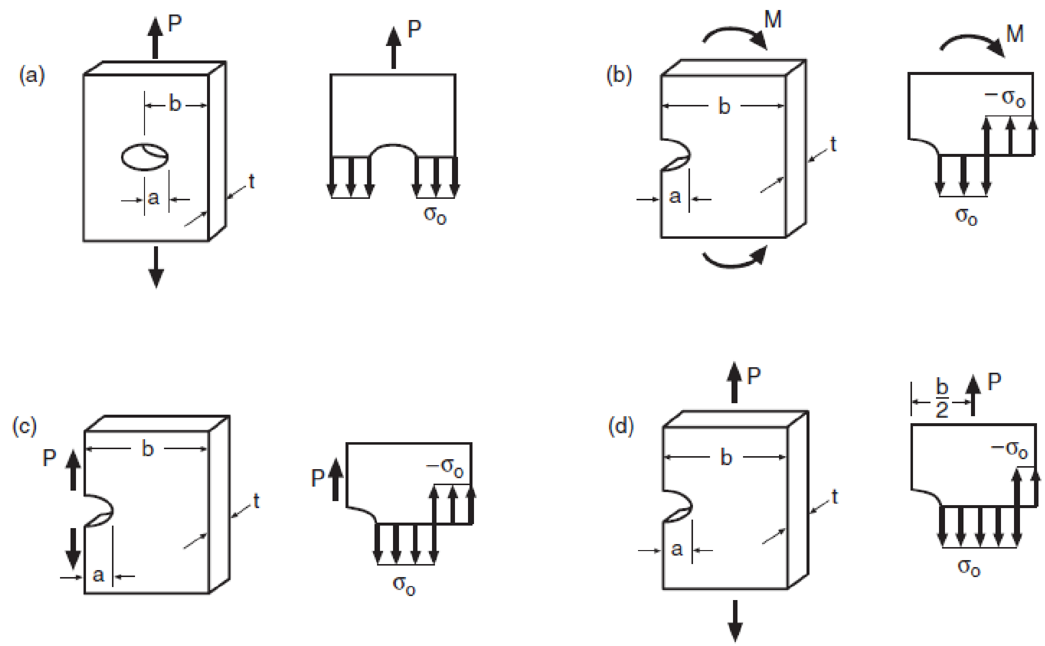
$$\sigma = k_t S; \tag{61}$$

$$\varepsilon = \frac{k_t S}{E} \text{ , } \varepsilon \leq \frac{\sigma_y}{E}; \tag{62}$$

The force or moment where yielding first occurs is called the initial yielding force P_i or initial yielding moment M_i.

4.1.2 Fully Plastic Yielding

Beyond the point of yielding, the local notch strains are larger than would be estimated from elastic analysis. Yielding is initially confined to a relatively small volume of material, but a larger volume yields as loads increase. When yielding spreads to the entire cross-sectional area, the situation is described as fully plastic yielding. For an elastic, perfectly plastic material, no further increase in load is possible, and the load versus strain curve becomes flat. The fully plastic force P_o and the fully plastic moment M_o can be determined from Figure 10:



Fully plastic force or moment for given $\alpha = a/b$:

(a) $P_o = 2bt\sigma_o(1-\alpha)$ (b) $M_o = \dfrac{b^2 t\sigma_o}{4}(1-\alpha)^2$

(c) $P_o = bt\sigma_o\left[-\alpha - 1 + \sqrt{2\left(1+\alpha^2\right)}\right]$ (d) $P_o = bt\sigma_o\left[-\alpha + \sqrt{2\alpha^2 - 2\alpha + 1}\right]$

Crack length at fully plastic yielding for given load, where, for (c) and (d), $P' = P/(bt\sigma_o)$:

(a) $a_o = b\left[1 - \dfrac{P}{2bt\sigma_o}\right]$ (b) $a_o = b\left[1 - \dfrac{2}{b}\sqrt{\dfrac{M}{t\sigma_o}}\right]$

(c) $a_o = b\left[P' + 1 - \sqrt{2P'(P'+2)}\right]$ (d) $a_o = b\left[P' + 1 - \sqrt{2P'(P'+1)}\right]$

Figure 10: Freebody diagrams and resulting equations for fully plastic forces or moments.

4.1.3 Local Yielding

Numerical analysis, as by finite elements, can be used, but nonlinear elasto-plastic stress-strain relationships complicate such analysis and increase costs compared with linear-elastic analysis. Although nonlinear numerical analysis is sometimes necessary, various approximate methods for estimating notch stresses and strains have also been developed. Neuber's rule is the most widely used one.

Neuber's rule states that the geometric mean of the stress and strain concentration factors k_σ and k_ε remains equal to k_t during plastic deformation:

$$k_t = \sqrt{k_\sigma k_\varepsilon}; \tag{63}$$

$$k_\sigma = \frac{\sigma}{S}; \tag{64}$$

$$k_\varepsilon = \frac{\varepsilon}{e}; \tag{65}$$

With Equation (63), (64) and (65),we can obtain: (S can be $S_{\max}$ or ΔS)

$$\sigma\varepsilon = \frac{(k_t S)^2}{E}; \tag{66}$$

If Equation (46) is used, we can have:

$$\sigma = H_1 \left[\frac{(k_t S)^2}{E H_1}\right]^{\frac{n_1}{n_1+1}}; \tag{67}$$

$$\varepsilon = \left[\frac{(k_t S)^2}{E H_1}\right]^{\frac{1}{n_1+1}}; \tag{68}$$

If a Ramberg–Osgood stress–strain curve is used, S and σ can be solved by:

$$S = \frac{1}{k_t}\sqrt{\sigma^2 + \sigma E\left(\frac{\sigma}{H}\right)^{1/n}}; \tag{69}$$

4.1.4 Residual Stresses and Strains at Notches

If a notched member is loaded sufficiently for local yielding to occur, and then the load is removed, residual stresses will remain. Residual stresses and strains at notches can be estimated by extending the application of Neuber's rule to the unloading event:

$$\Delta\sigma\Delta\varepsilon = \frac{(k_t S)^2}{E}; \tag{70}$$

When combined with the stress-strain curve for unloading, $\Delta\sigma$ and $\Delta\varepsilon$ can be determined. For example, if we have the monotonic response $\varepsilon = f(\sigma)$, the unloading response is:

$$\frac{\Delta\varepsilon}{2} = f\left(\frac{\Delta\sigma}{2}\right); \tag{71}$$

The concept of stress-strain unloading and reloading behavior consistent with a spring and slider rheological model is shown in Figure 11:

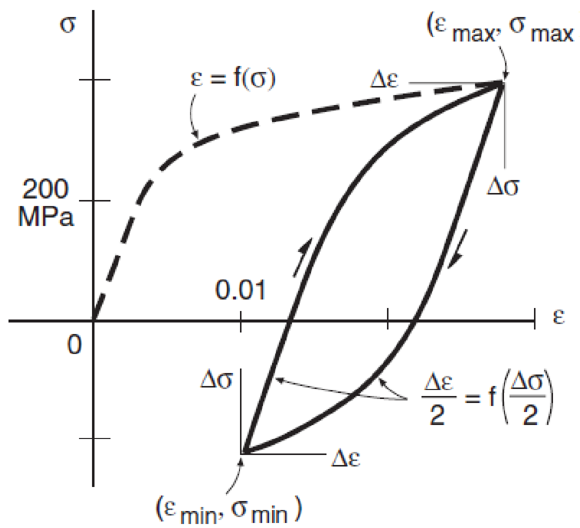


Figure 11: Stress–strain unloading and reloading behavior under the rheological model.

With the maximum stress and strain σ' and ε', we can have the residual stress ans strain:

$$\sigma_r = \sigma' - \Delta\sigma; \tag{72}$$

$$\varepsilon_r = \varepsilon' - \Delta\varepsilon; \tag{73}$$

4.2 Cyclic Loading

4.2.1 Generalized Methodology

With $\varepsilon = g(S)$, the stress-strain response can be estimated by: (Equation (75) and (76) are equivalent)

$$\begin{cases} \varepsilon_{\max} = g(S_{\max}) = f(\sigma_{\max}) \ , \ R \geq \ -1; \\ \varepsilon_{\min} = g(S_{\min}) = -f(-\sigma_{\min}) \ , \ R < -1; \end{cases} \tag{74}$$

$$\varepsilon_a = g(S_a) = f(\sigma_a); \tag{75}$$

$$\frac{\Delta\varepsilon}{2} = g\left(\frac{\Delta S}{2}\right) = f\left(\frac{\Delta\sigma}{2}\right); \tag{76}$$

Note: The same model should be applied for the entire analysis, such as a spring and slider rheological model.

5 Strain-Based Approach to Fatigue

5.1 Strain-Life Tests and Equations

Strain-life curves are derived from fatigue tests under completely reversed ($R = -1$) cyclic loading between constant strain limits. Coffin-Manson relationship can be used to represent the curves:

$$\varepsilon_a = \frac{\sigma'_f}{E}(2N_f)^b + \varepsilon'_f(2N_f)^c; \tag{77}$$

5.2 Mean Stress Effects

Mean stress effects need to be evaluated in applying the strain-based approach. The following equations can be used:

Mean Stress Equation of Morrow

$$\varepsilon_a = \frac{\sigma'_f}{E}\left(1 - \frac{\sigma_m}{\sigma'_f}\right)(2N_f)^b + \varepsilon'_f\left(1 - \frac{\sigma_m}{\sigma'_f}\right)^{c/b}(2N_f)^c; \tag{78}$$

Modified Morrow Approach (at relatively long lives)

$$\varepsilon_a = \frac{\sigma'_f}{E}\left(1 - \frac{\sigma_m}{\sigma'_f}\right)(2N_f)^b + \varepsilon'_f(2N_f)^c; \tag{79}$$

Smith, Watson, and Topper (SWT) Parameter

$$\sigma_{\max}\varepsilon_a = \frac{(\sigma'_f)^2}{E}(2N_f)^{2b} + \sigma'_f\varepsilon'_f(2N_f)^{b+c}; \tag{80}$$

Walker Mean Stress Equation

$$\varepsilon_a = \frac{\sigma'_f}{E}\left(\frac{1-R}{2}\right)^{(1-\gamma)}(2N_f)^b + \varepsilon'_f\left(\frac{1-R}{2}\right)^{c(1-\gamma)/b}(2N_f)^c; \tag{81}$$

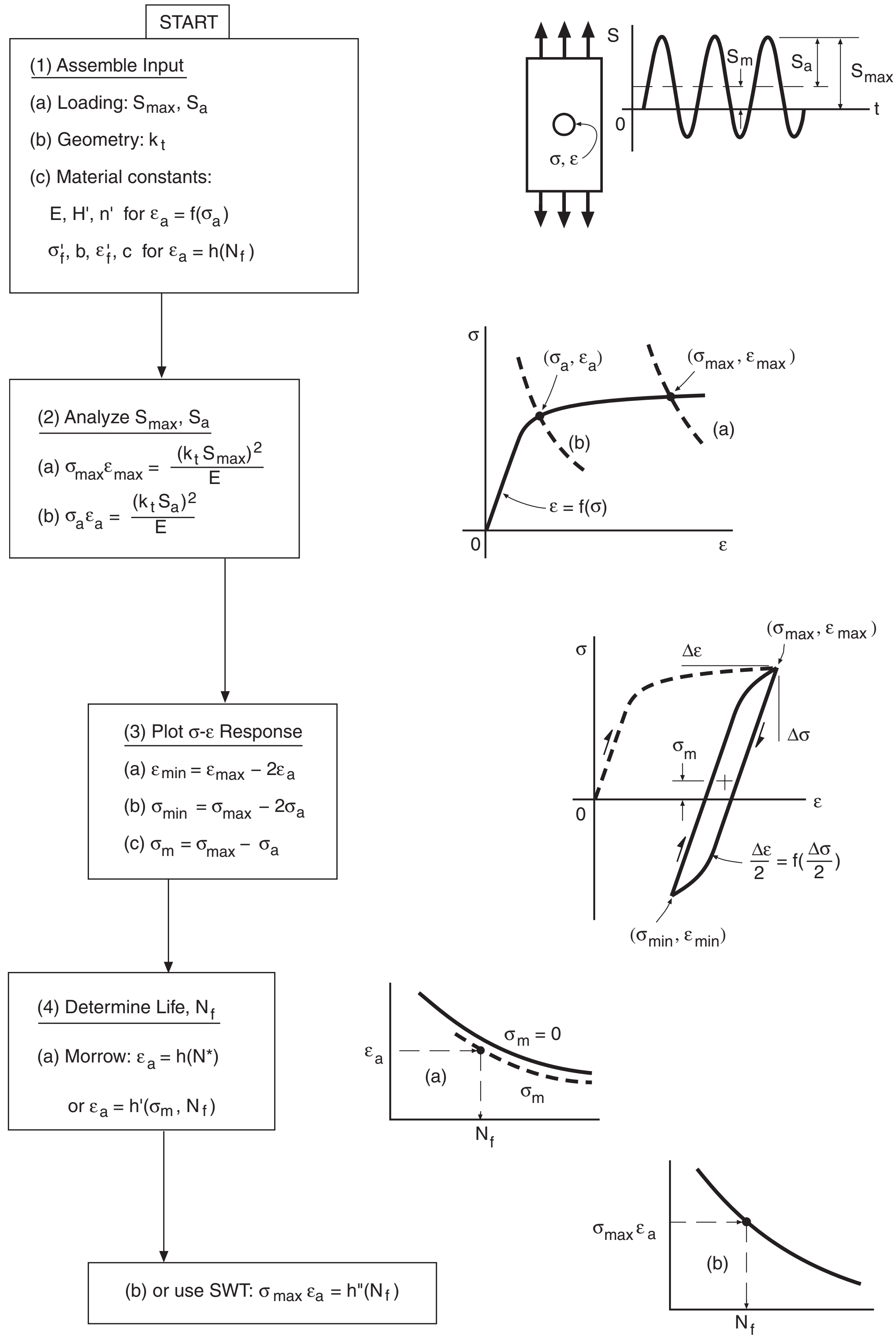


Figure 14.15 Steps required in strain-based life prediction for a notched member under constant amplitude loading.

6 Fracture of Cracked Members

6.1 General Nature of Cracks

6.1.1 Effects of Cracks on Strength

If the load applied to a member containing a crack is too high, the crack may suddenly grow and cause the member to fail by fracturing in a brittle manner—that is, with little plastic deformation. From the theory of fracture mechanics, a useful quantity called the *stress intensity factor*, K, can be defined.

A given material can resist a crack without brittle fracture occurring as long as this K is below a critical value K_c, called the fracture toughness. Values of K_c vary widely for different materials and are affected by temperature and loading rate, and secondarily by the thickness of the member. Thicker members have lower K_c values until a worst-case value is reached, which is denoted K_{Ic} and called the plane strain fracture toughness. Hence, K_{Ic} is a measure of a given material's ability to resist fracture in the presence of a crack.

If the crack is located in the center of a wide plate, K can be determined by:

$$K = S\sqrt{\pi a}; \tag{82}$$

6.1.2 Effects of Cracks on Brittle Versus Ductile Behavior

The transition crack length can be calculated by:

$$a_t = \frac{1}{\pi}\left(\frac{K_c}{\sigma_y}\right)^2; \tag{83}$$

Cracks longer than this transition crack length will cause the strength to be limited by brittle fracture, rather than by yielding.

6.1.3 Mode I loading

For any case of Mode I loading, the stresses near the crack tip can be expressed as: (For plane stress, $\sigma_{zz} = \tau_{yz} = \tau_{zx} = 0$)

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right] + \cdots; \tag{84}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right] + \cdots; \tag{85}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right) + \cdots; \tag{86}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y) \quad \text{(Plane Stress)} \tag{87}$$

Table 8.1 Fracture Toughness and Corresponding Tensile Properties for Representative Metals at Room Temperature

Material	Toughness K_{Ic}	Yield σ_o	Ultimate σ_u	Elong. $100\varepsilon_f$	Red. Area $\%RA$
	MPa$\sqrt{\text{m}}$ (ksi$\sqrt{\text{in}}$)	MPa (ksi)	MPa (ksi)	%	%
(a) Steels					
AISI 1144	66 (60)	540 (78)	840 (122)	5	7
ASTM A470-8 (Cr-Mo-V)	60 (55)	620 (90)	780 (113)	17	45
ASTM A517-F	187 (170)	760 (110)	830 (121)	20	66
AISI 4130	110 (100)	1090 (158)	1150 (167)	14	49
18-Ni maraging air melted	123 (112)	1310 (190)	1350 (196)	12	54
18-Ni maraging vacuum melted	176 (160)	1290 (187)	1345 (195)	15	66
300-M 650°C temper	152 (138)	1070 (156)	1190 (172)	18	56
300-M 300°C temper	65 (59)	1740 (252)	2010 (291)	12	48
(b) Aluminum and Titanium Alloys (L-T Orientation)					
2014-T651	24 (22)	415 (60)	485 (70)	13	—
2024-T351	34 (31)	325 (47)	470 (68)	20	—
2219-T851	36 (33)	350 (51)	455 (66)	10	—
7075-T651	29 (26)	505 (73)	570 (83)	11	—
7475-T7351	52 (47)	435 (63)	505 (73)	14	—
Ti-6Al-4V annealed	66 (60)	925 (134)	1000 (145)	16	34

Sources: Data in [Barsom 87] p. 172, [Boyer 85] pp. 6.34, 6.35, and 9.8, [MILHDBK 94] pp. 3.10–3.12 and 5.3, and [Ritchie 77].

6.2 Stress Intensity Factor and its Application

Stress intensity factor K_I, which characterizes the intensity of the stresses in the vicinity of an ideally sharp crack tip in a linear-elastic and isotropic material, based on the opening mode can be determined by:

$$K_I = FS\sqrt{\pi a}; \tag{88}$$

6.2.1 Safety Factors

If S_g and a are the stress and crack length that are expected to occur in actual service, the safety factor against brittle fracture X_K is:

$$X_K = \frac{K_{Ic}}{K} = \frac{K_{Ic}}{FS_g\sqrt{\pi a}} = \frac{F_c S_g\sqrt{\pi a_c}}{FS_g\sqrt{\pi a}}; \tag{89}$$

Also, the safety factor on crack length X_a is:

$$X_a = \frac{a_c}{a} = \left(\frac{F}{F_c}X_K\right); \tag{90}$$

The safety factor against fully plastic yielding is: (no crack is presented)

$$X_o = \frac{\sigma_o}{S_g}; \tag{91}$$

If crack is presented, the safety factor against yielding is:

$$X_o' = \frac{P_o}{P}; \tag{92}$$

Finally, the overall (controlling) safety factor is:

$$X = \min\{X_K, X_o\}; \tag{93}$$

6.3 Plastic Zone Size

6.3.1 Plastic Zone Size for Plane Stress

Based on Mode I loading, we can have the distance ahead of the crack tip where the elastic stress distribution exceeds the yield criterion for plane stress as:

$$r_{o\sigma} = \frac{1}{2\pi}\left(\frac{K}{\sigma_o}\right)^2; \tag{94}$$

7 Fracture Mechanics Approach ti Fatigue

In defining K, the material is assumed to behave in a linear-elastic manner so that the approach being discussed is based on linear-elastic fracture mechanics (LEFM).

7.1 Fatigue Crack Growth Rate Testing

In fatigue crack growth, we can assume that Δa per cycle $= f(\Delta K)$. Then, to obtain growth rates from crack length versus cycles data, a simple and generally suitable approach is to calculate straight-line slopes between the data points, which is:

$$\left(\frac{da}{dN}\right)_j \approx \left(\frac{\Delta a}{\Delta N}\right) = \frac{a_j - a_{j-1}}{N_j - N_{j-1}}; \tag{95}$$

The corresponding ΔK is calculated from the average crack length during the interval:

$$\Delta K_j = F \Delta S \sqrt{\pi a_{\text{avg}}}; \tag{96}$$

$$\Delta K_j = F_P \frac{\Delta P}{t\sqrt{b}}; \tag{97}$$

Also, the fatigue crack growth behavior of materials can be described as:

$$\frac{da}{dN} = C(\Delta K)^m; \tag{98}$$

7.2 Effects of R on Fatigue Crack Growth

To consider the effect of R, let C_0 be the constant when $R = 0$ and we can have:

$$\overline{\Delta K} = \frac{\Delta K}{(1-R)^{1-\gamma}}; \tag{99}$$

$$\frac{da}{dN} = C_0(\Delta K)^m; \tag{100}$$

$$\frac{da}{dN} = \frac{C_0}{(1-R)^{m(1-\gamma)}}(\Delta K)^m; \tag{101}$$

$$\log\left(\frac{da}{dN}\right) = m\log(\Delta K) - m(1-\gamma)\log(1-R) + \log(C_0); \tag{102}$$

7.2.1 The Forman Equation

Another proposed generalization to include R effects is:

$$\frac{da}{dN} = \frac{C_2(\Delta K)^{m_2}}{(1-R)K_c - \Delta K}; \tag{103}$$

Constants for the Forman equation are given in Figure 12:

Material	Yield σ_o	Toughness K_{Ic}	Forman Equation C_2	C_2	m_2	K_c
	MPa (ksi)	$\text{MPa}\sqrt{\text{m}}$ $(\text{ksi}\sqrt{\text{in}})$	$\frac{\text{mm/cycle}}{(\text{MPa}\sqrt{\text{m}})^{m_2-1}}$	$\frac{\text{in/cycle}}{(\text{ksi}\sqrt{\text{in}})^{m_2-1}}$		$\text{MPa}\sqrt{\text{m}}$ $(\text{ksi}\sqrt{\text{in}})$
17-4 PH steel (H1025)	1145 (166)	—	1.40×10^{-6}	6.45×10^{-8}	2.65	132 (120)
Inconel 718 (Fe-Ni-base, aged)	1172 (170)	132 (120)	4.29×10^{-6}	2.00×10^{-7}	2.79	132 (120)
2024-T3 Al[1]	353 (51.2)	34 (31)	2.31×10^{-6}	1.14×10^{-7}	3.38	110 (100)
7075-T6 Al[1]	523 (75.9)	29 (26)	5.29×10^{-6}	2.56×10^{-7}	3.21	78.7 (71.6)

Figure 12: Constants for the Forman Equation for Several Metals.

7.2.2 The Walker Equation

$$\overline{\Delta K} = K_{\max}(1 - R)^{\gamma}; \tag{104}$$

7.3 Life Estimates for Constant Amplitude Loading

With Equation (105), we can have Equation (106): (N_{if} is the number of elapsed cycles)

$$\frac{da}{dN} = f(\Delta K, R); \tag{105}$$

$$N_{if} = \int_{a_i}^{a_f} \frac{da}{f(\Delta K, R)}; \tag{106}$$

7.3.1 Closed-Form Solutions

If we use Equation (98) and Equation (96) to express $\frac{\Delta a}{\Delta N}$ and ΔK, we can have:

$$N_{if} = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C(F\Delta S\sqrt{\pi})^m(1 - m/2)} \qquad (m \neq 2) \tag{107}$$

7.3.2 Crack Length at Failure

When $K_{\max}$ reaches the fracture toughness K_c for the material and thickness of interest, failure is expected at the length a_c:

$$a_c = \frac{1}{\pi}\left(\frac{K_c}{FS_{\max}}\right)^2; \tag{108}$$

7.3.3 Solutions by Numerical Integration

If F changes excessively between the initial and final crack lengths, the closed-form integration equation cannot be used.

To do the numerical integration, we need to do the following:

Step 1: Determine n (n should be an even integer):

$$a_f = r^n a_i \qquad (r \approx 1.1)$$

Step 2: Determine the expressions of F, δK and $y = \dfrac{\Delta N}{\Delta a}$.

Step 3: Solve ΔN_{j+2} by:

$$\Delta N_{j+2} = \int_{a_j}^{a_j+2} y da$$

$$\Delta N_{j+2} = \frac{a_j(r^2-1)}{6r}[y_j r(2-r) + y_{j+1}(r+1)^2 + y_{j+2}(2r-1)]$$

Step 4: Determin N_{if} by summing all ΔN_{j+2} .

7.4 Life Estimates for Variable Amplitude Loading

The equivalent zero-to-tension stress level can be defined:

$$\Delta S_q = \left[\frac{\sum_{j=1}^{n} N_i(\overline{\Delta S_j})^m}{N_B}\right]^{1/m}$$

$$\overline{\Delta S_j} = (S_{\max})_j(1-R_j)^\gamma$$

Then, the number of repetition can be determined by:

$$N_{if} = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C_0(F\Delta S_q\sqrt{\pi})^m(1-m/2)} \tag{109}$$

$$B_{if} = \frac{N_{if}}{N_B} \tag{110}$$