Equations of Motion for Aircraft: Nonlinear Equations of Motion, General State-Space Formulation, Modified State-Space Formulation and Fully Linear Formulation for Control System Development

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Coordinate and Variable Definitions

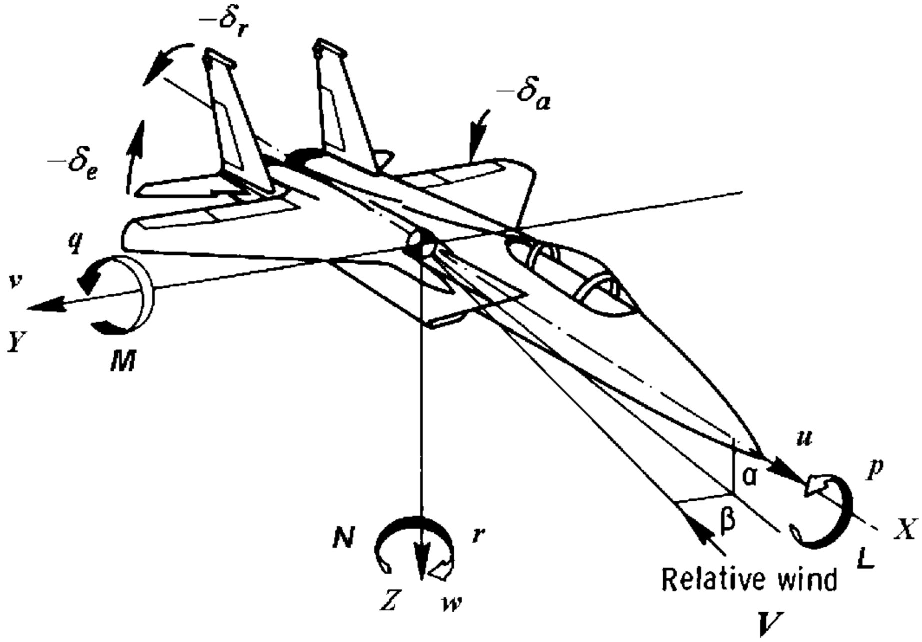

Figure 1: Body Axis Coordinate Definition (from Klein/Morelli, Aircraft and Rotorcraft System Identification)

Stability-Axis Coordinate System: Body-fixed coordinate system rotated by aerodynamic angles α, β; assumed to be initially aligned with trim velocity vector.

Symbol	Description	Units
U	Velocity Along Aircraft x-Body Axis	L/T
V	Velocity Along Aircraft y-Body Axis	L/T
W	Velocity Along Aircraft z-Body Axis	L/T
V_t	Total Aircraft Velocity (as measured by a pitot tube)	L/T
L	Moment About Aircraft x-Body Axis	$\mathrm{M} \cdot \mathrm{L}^2/\mathrm{T}^2$
M	Moment About Aircraft y-Body Axis	$\mathrm{M} \cdot \mathrm{L}^2/\mathrm{T}^2$
N	Moment About Aircraft z-Body Axis	$\mathrm{M} \cdot \mathrm{L}^2/\mathrm{T}^2$
X_{var}	x-Body Axis Force Derivative WRT *var*	[varies]
Y_{var}	y-Body Axis Force Derivative WRT *var*	[varies]
Z_{var}	z-Body Axis Force Derivative WRT *var*	[varies]
g	Gravitational Acceleration, Defined + in $+z$	L/T^2
I_{axes}	Moment of Inertia Along *axes*	$\mathrm{M}\cdot\mathrm{L}^2$
Φ	Roll Euler Angle	rad
Θ	Pitch Euler Angle	rad
Ψ	Yaw Euler Angle	rad
$[\,]_0$	Denotes Reference (Trim Condition, for Example)	[varies]
u	x-Body Axis Perturbed Velocity	L/T
v	y-Body Axis Perturbed Velocity	L/T
w	z-Body Axis Perturbed Velocity	L/T
p	x-Body Axis Perturbed Angular Rate	rad/T
q	y-Body Axis Perturbed Angular Rate	rad/T
r	z-Body Axis Perturbed Angular Rate	rad/T
δ_a	Aileron Deflection, Left Surface + TED	rad
δ_e	Elevator Deflection, + TED	rad
δ_r	Rudder Deflection, + TER	rad
C	Coefficient (of Force or Moment)	[none]
x_a	Axelerometer Offset from Aircraft CG in x direction	L
y_a	Axelerometer Offset from Aircraft CG in y direction	L
z_a	Axelerometer Offset from Aircraft CG in z direction	L
$\mathbf{x}$	State Vector	N/A
$\mathbf{u}$	Input Vector	N/A
$\mathbf{y}$	Output Vector	N/A
E, A, B	State Space State and Control Matrices	N/A
C, H, D	State Space Measurement Matrices	N/A

Table 1: Variable Definitions

Nonlinear Equations of Motion

The fully nonlinear equations of motion for an aircraft in 6-DOF flight are provided in this section. The nonlinear equations are very well suited for simulation applications, including the testing of flight control systems.

Note that in the following set of equations, the body axis force and moment coefficients are used without explicitly stating where these come from. For a time-invariant solution, stability derivatives can be used about the trim condition. For a time-variant solution, scheduling of stability derivatives can be used, or a vortex lattice code (such as Drela's AVL) can be used to come up with the aircraft's force and moment coefficients at each time step. Note, however, that if AVL is used, that it does not account for stall so this should be implemented separately.
First, the body-axis velocities are calculated according to the total aircraft velocity V_t (as measured by a pitot tube, and the wind angles α and β.

$$u = V_t \cos\alpha \cos\beta \tag{1}$$

$$v = V_t \sin\beta \tag{2}$$

$$w = V_t \sin\alpha \cos\beta \tag{3}$$

Next, the accelerations are given by:

$$\dot{u} = rv - qw - g\sin\Theta + \frac{\bar{q}SC_x + T}{m} \tag{4}$$

$$\dot{v} = pw - ru + g\cos\Theta\sin\Phi + \frac{\bar{q}SC_y}{m} \tag{5}$$

$$\dot{w} = qu - pv + g\cos\Theta\cos\Phi + \frac{\bar{q}SC_z}{m} \tag{6}$$

The differential equations for the total aircraft velocity and wind angles are as follows.

$$\dot{V}_t = \frac{u\dot{u} + v\dot{v} + w\dot{w}}{V_t} \tag{7}$$

$$\dot{\alpha} = \frac{u\dot{w} - w\dot{u}}{u^2 + w^2} \tag{8}$$

$$\dot{\beta} = \frac{V_t\dot{v} - v\dot{V}_t}{V_t^2\sqrt{1 - \left(\frac{v}{V_t}\right)^2}} \tag{9}$$

Based on the general nonlinear 6-DOF equations of motion, we can come up with the equations which govern the angular acceleration of the aircraft about its body-axes.

$$\dot{p} = (c_1 r + c)2p + c_4 h_{eng})q + \bar{q}Sb(c_3 C_l + c_4 C_n) \tag{10}$$

$$\dot{q} = (c_5 p - c_7 h_{eng})r - c_6(p^2 - r^2) + \bar{q}S\bar{c}c_7 C_m \tag{11}$$

$$\dot{r} = (c_8 p - c_2 r + c_9 h_{eng})q + \bar{q}Sb(c_4 C_l + c_9 C_n) \tag{12}$$

Where $h_{eng} = I_{eng}\omega_{eng}$ is the angular momentum vector for the rotating mass of the engine (assumed to be alligned with the positive x-body axis. The constants used in the preceding three relations are defined by:

$$c_1 = \frac{(I_{yy} - I_{zz})I_{zz} - I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2} \tag{13}$$

$$c_2 = \frac{(I_{xx} - I_{yy} + I_{zz})I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} \tag{14}$$

$$c_3 = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2} \tag{15}$$

$$c_4 = \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} \tag{16}$$

$$c_5 = \frac{I_{zz} - I_{xx}}{I_{yy}} \tag{17}$$

$$c_6 = \frac{I_{xz}}{I_{yy}} \tag{18}$$

$$c_7 = \frac{1}{I_{yy}} \tag{19}$$

$$c_8 = \frac{(I_{xx} - I_{yy})I_{xx} - I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2} \tag{20}$$

$$c_9 = \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2} \tag{21}$$

The navigation equations (in the Earth Axis coordinate frame) are given by:

$$\dot{\Phi} = p + \tan\Theta(q\sin\Phi + r\cos\Phi) \tag{22}$$

$$\dot{\Theta} = q\cos\Phi - r\sin\Phi \tag{23}$$

$$\dot{\Psi} = \frac{q\sin\Phi + r\cos\Phi}{\cos\Theta} \tag{24}$$

$$\dot{x_E} = u\cos\Psi cos\Theta + v(\cos\Psi\sin\Theta\sin\Phi - \sin\Psi\cos\Phi) + w(\cos\Psi\sin\Theta\cos\Phi + \sin\Psi\sin\Phi) \tag{25}$$

$$\dot{y_E} = u\sin\Psi\cos\Theta + v(\sin\Psi\sin\Theta\sin\Phi + \cos\Psi\cos\Phi) + w(\sin\Psi\sin\Theta\cos\Phi - \cos\Psi\sin\Phi) \tag{26}$$

$$\dot{z_E} = u\sin\Theta - v\cos\Theta\sin\Phi - w\cos\Theta\cos\Phi \tag{27}$$

Linearized Equations of Motion

The six equations of motions linearized about the trim condition with Taylor Series Expansion (such that stability derivatives can be used) for six-DOF flight are given by the following relations in the stability-axis coordinate system. There should be as many force and moment derivatives as states, though naturally some aircraft are more or less coupled than others, so many often go to zero (hence the ...'s in the following equations). Similarly, depending on the number of control inputs available, there are a different number of control deflection derivatives.

$$\dot{u} = -W_0 q + V_0 r - (g\cos\Theta_0)\theta + X_u u + X_v v + ... + X_q q + ... + X_{\delta_a}\delta_a + X_{\delta_e}\delta_e + ... \tag{28}$$

$$\dot{v} = -U_0 r + W_0 p + (g\cos\Theta_0)\phi + Y_u u + Y_v v + Y_{\dot{v}}\dot{v} + ... + Y_q q + ... + Y_{\delta_a}\delta_a + Y_{\delta_e}\delta_e + ... \tag{29}$$

$$\dot{w} = -V_0 p + U_0 q - (g\sin\Theta_0)\theta + Z_u u + Z_v v + ... + Z_q q + ... + Z_{\delta_a}\delta_a + Z_{\delta_e}\delta_e + ... \tag{30}$$

$$\dot{p} - \frac{I_{xz}}{I_{xx}}\dot{r} = L_u u + L_v v + ... + L_p p + ... + L_{\delta_a}\delta_a + L_{\delta_e}\delta_e + ... \tag{31}$$

$$\dot{q} = M_u u + M_v v + ... + M_{\dot{w}}\dot{w} + M_q q + ... + M_{\delta_a}\delta_a + M_{\delta_e}\delta_e + ... \tag{32}$$

$$\dot{r} - \frac{I_{xz}}{I_{zz}}\dot{p} = N_u u + N_v v + ... + N_r r + ... + N_{\delta_a}\delta_a + N_{\delta_e}\delta_e + ... \tag{33}$$

The aerodynamic angles are given by the following relations, with an offset for the nose boom (x_{nb}, y_{nb}, z_{nb} are zero if measurements are desired at the aircraft's CG).

$$\alpha = \arctan\left(\frac{w + p y_{nb} - q x_{nb}}{U}\right) \tag{34}$$

$$\beta = \arcsin\left(\frac{v - p z_{nb} + r x_{nb}}{V_{total}}\right) \tag{35}$$

Assuming $V_0 \approx 0$ and $W_0 \approx 0$ the total velocity is given by:

$$V_{total} = \left((U_0 + u)^2 + v^2 + w^2\right)^{1/2} \tag{36}$$

Coupled State-Space Representation

NOTE: In this section, the effects of aerodynamic angles are neglected and reintroduced in the following section depending on whether the application requires linearization.

Assuming a plane of symmetry $I_{xy} = 0$, the coupled state space representation is formulated. The general form of the state space follows.

$$E\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{37}$$

$$\mathbf{y} = C\mathbf{x} + H\dot{\mathbf{x}} + D\mathbf{u} \tag{38}$$

We have the following state vector (neglect the position and rotation states x, y, z, ϕ, θ, ψ if you don't care about them):

$$\mathbf{x} = \begin{bmatrix} x \\ u \\ y \\ v \\ z \\ w \\ \phi \\ p \\ \theta \\ q \\ \psi \\ r \end{bmatrix} \tag{39}$$

For a typical airplane, we have the following control surfaces:

1. Ailerons
2. Elevator
3. Rudder

The control input vector is therefore given by:

$$\mathbf{u} = \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix} \tag{40}$$

The E, A and B matrices are formulated accordingly.

$$E = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -X_{\dot{v}} & 0 & -X_{\dot{w}} & 0 & -X_{\dot{p}} & 0 & -X_{\dot{q}} & 0 & -X_{\dot{r}} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -Y_{\dot{u}} & 0 & 1 & 0 & -Y_{\dot{w}} & 0 & -Y_{\dot{p}} & 0 & -Y_{\dot{q}} & 0 & -Y_{\dot{r}} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -Z_{\dot{u}} & 0 & -Z_{\dot{v}} & 0 & 1 & 0 & -Z_{\dot{p}} & 0 & -Z_{\dot{q}} & 0 & -Z_{\dot{r}} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{\dot{u}} & 0 & -L_{\dot{v}} & 0 & -L_{\dot{w}} & 0 & 1 & 0 & -L_{\dot{q}} & 0 & -L_{\dot{r}} - \frac{I_{xz}}{I_{xx}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -M_{\dot{u}} & 0 & -M_{\dot{v}} & 0 & -M_{\dot{w}} & 0 & -M_{\dot{p}} & 0 & 1 & 0 & -M_{\dot{r}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -N_{\dot{u}} & 0 & -N_{\dot{v}} & 0 & -N_{\dot{w}} & 0 & -N_{\dot{p}} - \frac{I_{xz}}{I_{zz}} & 0 & -N_{\dot{q}} & 0 & 1
\end{bmatrix} \tag{41}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_u & 0 & X_v & 0 & X_w & 0 & X_p & -g \cdot \cos\Theta_0 & X_q - W_0 & 0 & X_r + V_0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_u & 0 & Y_v & 0 & Y_w & g \cdot \cos\Theta_0 & Y_p + W_0 & 0 & Y_q & 0 & Y_r - U_0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z_u & 0 & Z_v & 0 & Z_w & 0 & Z_p - V_0 & -g \cdot \sin\Theta_0 & Z_q + U_0 & 0 & Z_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & L_u & 0 & L_v & 0 & L_w & 0 & L_p & 0 & L_q & 0 & L_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & M_u & 0 & M_v & 0 & M_w & 0 & M_p & 0 & M_q & 0 & M_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & N_u & 0 & N_v & 0 & N_w & 0 & N_p & 0 & N_q & 0 & N_r \end{bmatrix} \tag{42}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ X_{\delta_a} & X_{\delta_e} & X_{\delta_r} \\ 0 & 0 & 0 \\ Y_{\delta_a} & Y_{\delta_e} & Y_{\delta_r} \\ 0 & 0 & 0 \\ Z_{\delta_a} & Z_{\delta_e} & Z_{\delta_r} \\ 0 & 0 & 0 \\ L_{\delta_a} & L_{\delta_e} & L_{\delta_r} \\ 0 & 0 & 0 \\ M_{\delta_a} & M_{\delta_e} & M_{\delta_r} \\ 0 & 0 & 0 \\ N_{\delta_a} & N_{\delta_e} & N_{\delta_r} \end{bmatrix} \tag{43}$$

In the context of aircraft control and system identification, the translational and rotational states typically are not of interest. Accelerations typically are of interest, however, so we define out output vector, $\mathbf{y}$ accordingly.

$$\mathbf{y} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \\ a_x \\ a_y \\ a_z \end{bmatrix} \tag{44}$$

The relationships for accelerations as functions of state variables and state rates follow immediately below. Note that in these relationships, x_a, y_a, z_a are the accelerometer offsets from the aircraft center of gravity.

$$a_x = \dot{u} + W_0 q - V_0 r + (g\cos\Theta_0)\theta + z_a\dot{q} - y_a\dot{r} \tag{45}$$

$$a_y = \dot{v} + U_0 r - W_0 p - (g\cos\Theta_0)\phi - z_a\dot{p} + x_a\dot{r} \tag{46}$$

$$a_z = \dot{w} + V_0 p - U_0 q + (g\sin\Theta_0)\theta + y_a\dot{p} - x_a\dot{q} \tag{47}$$

The measurement matrices follow.

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g\cos\Theta_0 & W_0 & 0 & -V_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g\cos\Theta_0 & -W_0 & 0 & 0 & 0 & U_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_0 & g\sin\Theta_0 & -U_0 & 0 & 0 \end{bmatrix} \tag{48}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_a & 0 & -y_a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -z_a & 0 & 0 & 0 & x_a \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & y_a & 0 & -x_a & 0 & 0 \end{bmatrix} \tag{49}$$

Note that we could compile all of the information in C and H into one matrix to be multiplied by $\dot{\mathbf{x}}$ since our state vector includes translational and rotational position states, but to conform with the conventional form of the state space introduced previously, they are left separate. Since none of the control inputs map directly to the outputs, we define the D matrix as a matrix of zeros with appropriate dimensions.

$$D = \text{zeros}(9,3) \tag{50}$$

Formulations

Formulation Without Small Angle Approximation

For simulation models which deviate significantly from the trim conditions but still within the region of linearity about the trim condition, a formulation which does not use small angle approximations is typically desired. At each time step, the aerodynamic angles should be calculated and their effects appended to the state-space. Recall the equations for the aerodynamic angles.

$$\alpha = \arctan\left(\frac{w + py_{nb} - qx_{nb}}{U}\right) \tag{51}$$

$$\beta = \arcsin\left(\frac{v - pz_{nb} + rx_{nb}}{V_{total}}\right) \tag{52}$$

The nonlinear part is appended to the state-space formulation.

$$E\dot{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ -X_{\dot\alpha} & -X_{\dot\beta} \\ 0 & 0 \\ -Y_{\dot\alpha} & -Y_{\dot\beta} \\ 0 & 0 \\ -Z_{\dot\alpha} & -Z_{\dot\beta} \\ 0 & 0 \\ -L_{\dot\alpha} & -L_{\dot\beta} \\ 0 & 0 \\ -M_{\dot\alpha} & -M_{\dot\beta} \\ 0 & 0 \\ -N_{\dot\alpha} & -N_{\dot\beta} \end{bmatrix} \begin{bmatrix} \dot\alpha \\ \dot\beta \end{bmatrix} = A\mathbf{x} + B\mathbf{u} + \begin{bmatrix} 0 & 0 \\ X_\alpha & X_\beta \\ 0 & 0 \\ Y_\alpha & Y_\beta \\ 0 & 0 \\ Z_\alpha & Z_\beta \\ 0 & 0 \\ L_\alpha & L_\beta \\ 0 & 0 \\ M_\alpha & M_\beta \\ 0 & 0 \\ N_\alpha & N_\beta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{53}$$

Where E, A and B are those introduced in (41), (42), (43).

Formulation Using Small Angle Approximation

If the application requires a fully linear state space (for control system development, for example), we linearize the aerodynamic angles about the trim condition using a small angle approximation and by assuming small perturbed velocities ($U \approx U_0, V_{total} \approx \sqrt{U_0^2 + V_0^2 + W_0^2}$).

$$\alpha \approx \frac{w + py_{nb} - qx_{nb}}{U_0} \tag{54}$$

$$\beta \approx \frac{v - pz_{nb} + rx_{nb}}{\sqrt{U_0^2 + V_0^2 + W_0^2}} \tag{55}$$

The aerodynamic angular rates are therefore approximated by:

$$\dot\alpha \approx \frac{1}{U_0}\dot w + \frac{y_{nb}}{U_0}\dot p - \frac{x_{nb}}{U_0}\dot q \tag{56}$$

$$\dot\beta \approx \frac{1}{\sqrt{U_0^2 + V_0^2 + W_0^2}}\dot v - \frac{z_{nb}}{\sqrt{U_0^2 + V_0^2 + W_0^2}}\dot p + \frac{x_{nb}}{\sqrt{U_0^2 + V_0^2 + W_0^2}}\dot r \tag{57}$$

Clearly, the linearized aerodynamic angles and rates are linearly dependent on our other state variables. The A and E matrices of the state-space are therefore modified to implement the effects of aerodynamic angles.

$$
E=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -X_{\dot{v}}-X_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -X_{\dot{w}}-\frac{X_{\dot{\alpha}}}{U_0} & 0 & -X_{\dot{p}}-\frac{X_{\dot{\alpha}}y_{nb}}{U_0}-X_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -X_{\dot{q}}+\frac{X_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -X_{\dot{r}}-X_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -Y_{\dot{u}} & 0 & 1-Y_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -Y_{\dot{w}}-\frac{Y_{\dot{\alpha}}}{U_0} & 0 & -Y_{\dot{p}}-\frac{Y_{\dot{\alpha}}y_{nb}}{U_0}-Y_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -Y_{\dot{q}}+\frac{Y_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -Y_{\dot{r}}-Y_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -Z_{\dot{u}} & 0 & -Z_{\dot{v}}-Z_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & 1-\frac{Z_{\dot{\alpha}}}{U_0} & 0 & -Z_{\dot{p}}-\frac{Z_{\dot{\alpha}}y_{nb}}{U_0}-Z_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -Z_{\dot{q}}+\frac{Z_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -Z_{\dot{r}}-Z_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -L_{\dot{u}} & 0 & -L_{\dot{v}}-L_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -L_{\dot{w}}-\frac{L_{\dot{\alpha}}}{U_0} & 0 & 1-\frac{L_{\dot{\alpha}}y_{nb}}{U_0}-L_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -L_{\dot{q}}+\frac{L_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -L_{\dot{r}}-\frac{I_{xz}}{I_{xx}}-L_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -M_{\dot{u}} & 0 & -M_{\dot{v}}-M_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -M_{\dot{w}}-\frac{M_{\dot{\alpha}}}{U_0} & 0 & -M_{\dot{p}}-\frac{M_{\dot{\alpha}}y_{nb}}{U_0}-M_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & 1+\frac{M_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -M_{\dot{r}}-M_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & -N_{\dot{u}} & 0 & -N_{\dot{v}}-N_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -N_{\dot{w}}-\frac{N_{\dot{\alpha}}}{U_0} & 0 & -N_{\dot{p}}-\frac{I_{xz}}{I_{zz}}-\frac{N_{\dot{\alpha}}y_{nb}}{U_0}-N_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -N_{\dot{q}}+\frac{N_{\dot{\alpha}}x_{nb}}{U_0} & 0 & 1-N_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}
\end{bmatrix} \tag{58}
$$

$$
A=\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & X_u & 0 & X_v+\frac{X_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & X_w+\frac{X_\alpha}{U_0} & 0 & X_p+\frac{X_\alpha y_{nb}}{U_0}-\frac{X_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & -g\cdot\cos\Theta_0 & X_q-W_0-\frac{X_\alpha x_{nb}}{U_0} & 0 & X_r+V_0+\frac{X_\beta x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & Y_u & 0 & Y_v+\frac{Y_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & Y_w+\frac{Y_\alpha}{U_0} & g\cdot\cos\Theta_0 & Y_p+W_0+\frac{Y_\alpha y_{nb}}{U_0}-\frac{Y_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & Y_q-\frac{Y_\alpha x_{nb}}{U_0} & 0 & Y_r-U_0+\frac{Y_\beta x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & Z_u & 0 & Z_v+\frac{Z_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & Z_w+\frac{Z_\alpha}{U_0} & 0 & Z_p-V_0+\frac{Z_\alpha y_{nb}}{U_0}-\frac{Z_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & -g\cdot\sin\Theta_0 & Z_q+U_0-\frac{Z_\alpha x_{nb}}{U_0} & 0 & Z_r+\frac{Z_\beta x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & L_u & 0 & L_v+\frac{L_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & L_w+\frac{L_\alpha}{U_0} & 0 & L_p+\frac{L_\alpha y_{nb}}{U_0}-\frac{L_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & L_q-\frac{L_\alpha x_{nb}}{U_0} & 0 & L_r+\frac{L_\beta x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & M_u & 0 & M_v+\frac{M_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & M_w+\frac{M_\alpha}{U_0} & 0 & M_p+\frac{M_\alpha y_{nb}}{U_0}-\frac{M_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & M_q-\frac{M_\alpha x_{nb}}{U_0} & 0 & M_r+\frac{M_\beta x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & N_u & 0 & N_v+\frac{N_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & N_w+\frac{N_\alpha}{U_0} & 0 & N_p+\frac{N_\alpha y_{nb}}{U_0}-\frac{N_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & N_q-\frac{N_\alpha x_{nb}}{U_0} & 0 & N_r+\frac{N_\beta x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}
\end{bmatrix} \tag{59}
$$

The B, C, H and D matrices remain unchanged.

$$
B=\begin{bmatrix}
0 & 0 & 0\\
X_{\delta_a} & X_{\delta_e} & X_{\delta_r}\\
0 & 0 & 0\\
Y_{\delta_a} & Y_{\delta_e} & Y_{\delta_r}\\
0 & 0 & 0\\
Z_{\delta_a} & Z_{\delta_e} & Z_{\delta_r}\\
0 & 0 & 0\\
L_{\delta_a} & L_{\delta_e} & L_{\delta_r}\\
0 & 0 & 0\\
M_{\delta_a} & M_{\delta_e} & M_{\delta_r}\\
0 & 0 & 0\\
N_{\delta_a} & N_{\delta_e} & N_{\delta_r}
\end{bmatrix} \tag{60}
$$

$$
C = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g\cos\Theta_0 & W_0 & 0 & -V_0 \\
0 & 0 & 0 & 0 & 0 & 0 & -g\cos\Theta_0 & -W_0 & 0 & 0 & 0 & U_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & V_0 & g\sin\Theta_0 & -U_0 & 0 & 0
\end{bmatrix} \tag{61}
$$

$$
H = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_a & 0 & -y_a \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -z_a & 0 & 0 & 0 & x_a \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & y_a & 0 & -x_a & 0 & 0
\end{bmatrix} \tag{62}
$$

$$
D = \text{zeros}(9,3) \tag{63}
$$