Equations of Motion for Aircraft: General State-Space Formulation, Modified State-Space Formulation for Nonlinear Simulation Applications and Linear Formulation for Control System Development

Thomas V. Greenhill

January 2021

Coordinate and Variable Definitions

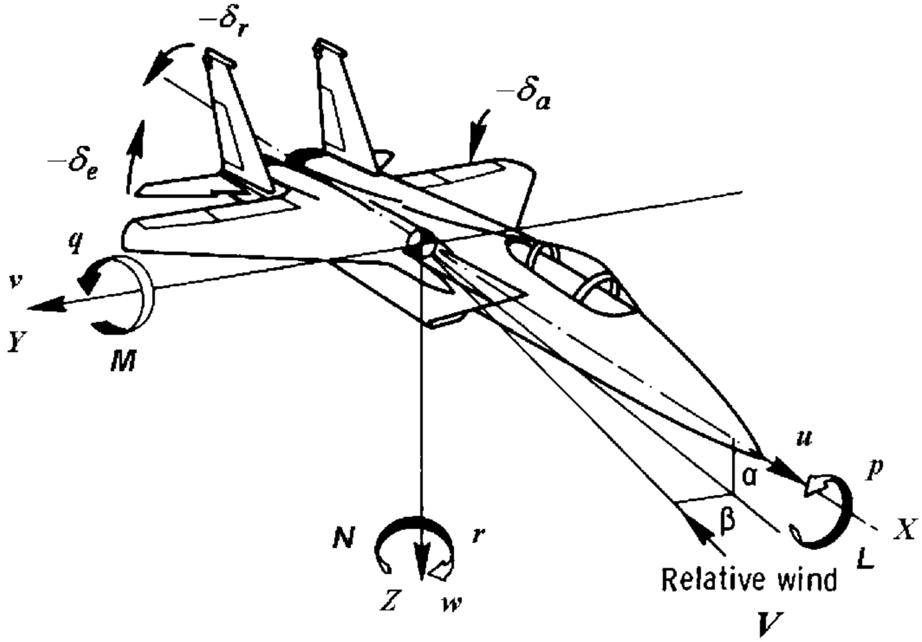

Figure 1: Body Axis Coordinate Definition (from Klein/Morelli, Aircraft and Rotorcraft System Identification)

Stability-Axis Coordinate System: Body-fixed coordinate system rotated by aerodynamic angles α, β; assumed to be initially aligned with trim velocity vector.

Symbol	Description	Units
U	Velocity Along Aircraft x-Body Axis	L/T
V	Velocity Along Aircraft y-Body Axis	L/T
W	Velocity Along Aircraft z-Body Axis	L/T
L	Moment About Aircraft x-Body Axis	$\mathrm{M} \cdot \mathrm{L}^2/\mathrm{T}^2$
M	Moment About Aircraft y-Body Axis	$\mathrm{M} \cdot \mathrm{L}^2/\mathrm{T}^2$
N	Moment About Aircraft z-Body Axis	$\mathrm{M} \cdot \mathrm{L}^2/\mathrm{T}^2$
X_{var}	x-Body Axis Force Derivative WRT var	[varies]
Y_{var}	Y-Body Axis Force Derivative WRT var	[varies]
Z_{var}	z-Body Axis Force Derivative WRT var	[varies]
g	Gravitational Acceleration, Defined + in $+z$	L/T^2
I_{axes}	Moment of Inertia Along $axes$	$\mathrm{M}\cdot\mathrm{L}^2$
Φ	Roll Euler Angle	rad
Θ	Pitch Euler Angle	rad
Ψ	Yaw Euler Angle	rad
$[\,]_0$	Denotes Reference (Trim Condition, for Example)	[varies]
u	x-Body Axis Perturbed Velocity	L/T
v	y-Body Axis Perturbed Velocity	L/T
w	z-Body Axis Perturbed Velocity	L/T
p	x-Body Axis Perturbed Angular Rate	rad/T
q	y-Body Axis Perturbed Angular Rate	rad/T
r	z-Body Axis Perturbed Angular Rate	rad/T
δ_a	Aileron Deflection, Left Surface + TED	rad
δ_e	Elevator Deflection, + TED	rad
δ_r	Rudder Deflection, + TER	rad
x_a	Axelerometer Offset from Aircraft CG in x direction	L
y_a	Axelerometer Offset from Aircraft CG in y direction	L
z_a	Axelerometer Offset from Aircraft CG in z direction	L
$\mathbf{x}$	State Vector	N/A
$\mathbf{u}$	Input Vector	N/A
$\mathbf{y}$	Output Vector	N/A
E, A, B	State Space State and Control Matrices	N/A
C, H, D	State Space Measurement Matrices	N/A

Table 1: Variable Definitions

Longitudinal Equations of Motion

The six equations of motions for six-DOF flight are given by the following relations in the stability-axis coordinate system. There should be as many force and moment derivatives as states, though naturally some aircraft are more or less coupled than others, so many often go to zero (hence the ...'s in the following equations). Similarly, depending on the number of control inputs available, there are a different number of control deflection derivatives.

$$\dot{u} = -W_0 q + V_0 r - (g\cos\Theta_0)\theta + X_u u + X_v v + ... + X_q q + ... + X_{\delta_a}\delta_a + X_{\delta_e}\delta_e + ... \quad (1)$$

$$\dot{v} = -U_0 r + W_0 p + (g\cos\Theta_0)\phi + Y_u u + Y_v v + Y_{\dot{v}}\dot{v} + ... + Y_q q + ... + Y_{\delta_a}\delta_a + Y_{\delta_e}\delta_e + ... \quad (2)$$

$$\dot{w} = -V_0 p + U_0 q - (g\sin\Theta_0)\theta + Z_u u + Z_v v + ... + Z_q q + ... + Z_{\delta_a}\delta_a + Z_{\delta_e}\delta_e + ... \quad (3)$$

$$\dot{p} - \frac{I_{xz}}{I_{xx}}\dot{r} = L_u u + L_v v + ... + L_p p + ... + L_{\delta_a}\delta_a + L_{\delta_e}\delta_e + ... \quad (4)$$

$$\dot{q} = M_u u + M_v v + ... + M_{\dot{w}}\dot{w} + M_q q + ... + M_{\delta_a}\delta_a + M_{\delta_e}\delta_e + ... \quad (5)$$

$$\dot{r} - \frac{I_{xz}}{I_{zz}}\dot{p} = N_u u + N_v v + ... + N_r r + ... + N_{\delta_a}\delta_a + N_{\delta_e}\delta_e + ... \quad (6)$$

The aerodynamic angles are given by the following relations, with an offset for the nose boom (x_{nb}, y_{nb}, z_{nb} are zero if measurements are desired at the aircraft's CG).

$$\alpha = \arctan\left(\frac{w + p y_{nb} - q x_{nb}}{U}\right) \quad (7)$$

$$\beta = \arcsin\left(\frac{v - p z_{nb} + r x_{nb}}{V_{total}}\right) \quad (8)$$

Assuming $V_0 \approx 0$ and $W_0 \approx 0$ the total velocity is given by:

$$V_{total} = \left((U_0 + u)^2 + v^2 + w^2\right)^{1/2} \quad (9)$$

Coupled State-Space Representation

NOTE: In this section, the effects of aerodynamic angles are neglected and reintroduced in the following section depending on whether the application requires linearization.

Assuming a plane of symmetry $I_{xy} = 0$, the coupled state space representation is formulated. The general form of the state space follows.

$$E\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{10}$$

$$\mathbf{y} = C\mathbf{x} + H\dot{\mathbf{x}} + D\mathbf{u} \tag{11}$$

We have the following state vector (neglect the position and rotation states x, y, z, ϕ, θ, ψ if you don't care about them):

$$\mathbf{x} = \begin{bmatrix} x \\ u \\ y \\ v \\ z \\ w \\ \phi \\ p \\ \theta \\ q \\ \psi \\ r \end{bmatrix} \tag{12}$$

For a typical airplane, we have the following control surfaces:

1. Ailerons
2. Elevator
3. Rudder

The control input vector is therefore given by:

$$\mathbf{u} = \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix} \tag{13}$$

The E, A and B matrices are formulated accordingly.

$$E = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -X_{\dot{v}} & 0 & -X_{\dot{w}} & 0 & -X_{\dot{p}} & 0 & -X_{\dot{q}} & 0 & -X_{\dot{r}} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -Y_{\dot{u}} & 0 & 1 & 0 & -Y_{\dot{w}} & 0 & -Y_{\dot{p}} & 0 & -Y_{\dot{q}} & 0 & -Y_{\dot{r}} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -Z_{\dot{u}} & 0 & -Z_{\dot{v}} & 0 & 1 & 0 & -Z_{\dot{p}} & 0 & -Z_{\dot{q}} & 0 & -Z_{\dot{r}} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{\dot{u}} & 0 & -L_{\dot{v}} & 0 & -L_{\dot{w}} & 0 & 1 & 0 & -L_{\dot{q}} & 0 & -L_{\dot{r}} - \frac{I_{xz}}{I_{xx}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -M_{\dot{u}} & 0 & -M_{\dot{v}} & 0 & -M_{\dot{w}} & 0 & -M_{\dot{p}} & 0 & 1 & 0 & -M_{\dot{r}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -N_{\dot{u}} & 0 & -N_{\dot{v}} & 0 & -N_{\dot{w}} & 0 & -N_{\dot{p}} - \frac{I_{xz}}{I_{zz}} & 0 & -N_{\dot{q}} & 0 & 1
\end{bmatrix} \tag{14}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_u & 0 & X_v & 0 & X_w & 0 & X_p & -g \cdot \cos\Theta_0 & X_q - W_0 & 0 & X_r + V_0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_u & 0 & Y_v & 0 & Y_w & g \cdot \cos\Theta_0 & Y_p + W_0 & 0 & Y_q & 0 & Y_r - U_0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z_u & 0 & Z_v & 0 & Z_w & 0 & Z_p - V_0 & -g \cdot \sin\Theta_0 & Z_q + U_0 & 0 & Z_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & L_u & 0 & L_v & 0 & L_w & 0 & L_p & 0 & L_q & 0 & L_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & M_u & 0 & M_v & 0 & M_w & 0 & M_p & 0 & M_q & 0 & M_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & N_u & 0 & N_v & 0 & N_w & 0 & N_p & 0 & N_q & 0 & N_r \end{bmatrix} \tag{15}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ X_{\delta_a} & X_{\delta_e} & X_{\delta_r} \\ 0 & 0 & 0 \\ Y_{\delta_a} & Y_{\delta_e} & Y_{\delta_r} \\ 0 & 0 & 0 \\ Z_{\delta_a} & Z_{\delta_e} & Z_{\delta_r} \\ 0 & 0 & 0 \\ L_{\delta_a} & L_{\delta_e} & L_{\delta_r} \\ 0 & 0 & 0 \\ M_{\delta_a} & M_{\delta_e} & M_{\delta_r} \\ 0 & 0 & 0 \\ N_{\delta_a} & N_{\delta_e} & N_{\delta_r} \end{bmatrix} \tag{16}$$

In the context of aircraft control and system identification, the translational and rotational states typically are not of interest. Accelerations typically are of interest, however, so we define out output vector, $\mathbf{y}$ accordingly.

$$\mathbf{y} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \\ a_x \\ a_y \\ a_z \end{bmatrix} \tag{17}$$

The relationships for accelerations as functions of state variables and state rates follow immediately below. Note that in these relationships, x_a, y_a, z_a are the accelerometer offsets from the aircraft center of gravity.

$$a_x = \dot{u} + W_0 q - V_0 r + (g\cos\Theta_0)\theta + z_a\dot{q} - y_a\dot{r} \tag{18}$$

$$a_y = \dot{v} + U_0 r - W_0 p - (g\cos\Theta_0)\phi - z_a\dot{p} + x_a\dot{r} \tag{19}$$

$$a_z = \dot{w} + V_0 p - U_0 q + (g\sin\Theta_0)\theta + y_a\dot{p} - x_a\dot{q} \tag{20}$$

The measurement matrices follow.

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g\cos\Theta_0 & W_0 & 0 & -V_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g\cos\Theta_0 & -W_0 & 0 & 0 & 0 & U_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_0 & g\sin\Theta_0 & -U_0 & 0 & 0 \end{bmatrix} \tag{21}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_a & 0 & -y_a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -z_a & 0 & 0 & 0 & x_a \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & y_a & 0 & -x_a & 0 & 0 \end{bmatrix} \tag{22}$$

Note that we could compile all of the information in C and H into one matrix to be multiplied by $\dot{\mathbf{x}}$ since our state vector includes translational and rotational position states, but to conform with the conventional form of the state space introduced previously, they are left separate. Since none of the control inputs map directly to the outputs, we define the D matrix as a matrix of zeros with appropriate dimensions.

$$D = \text{zeros(9,3)} \tag{23}$$

Linear and Nonlinear Formulations

Nonlinear Formulation

For simulation models which deviate significantly from the trim conditions, a nonlinear formulation is typically desired. At each time step, the aerodynamic angles should be calculated and their effects appended to the state-space. Recall the equations for the aerodynamic angles.

$$\alpha = \arctan\left(\frac{w + py_{nb} - qx_{nb}}{U}\right) \tag{24}$$

$$\beta = \arcsin\left(\frac{v - pz_{nb} + rx_{nb}}{V_{total}}\right) \tag{25}$$

The nonlinear part is appended to the state-space formulation.

$$E\dot{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ -X_{\dot{\alpha}} & -X_{\dot{\beta}} \\ 0 & 0 \\ -Y_{\dot{\alpha}} & -Y_{\dot{\beta}} \\ 0 & 0 \\ -Z_{\dot{\alpha}} & -Z_{\dot{\beta}} \\ 0 & 0 \\ -L_{\dot{\alpha}} & -L_{\dot{\beta}} \\ 0 & 0 \\ -M_{\dot{\alpha}} & -M_{\dot{\beta}} \\ 0 & 0 \\ -N_{\dot{\alpha}} & -N_{\dot{\beta}} \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = A\mathbf{x} + B\mathbf{u} + \begin{bmatrix} 0 & 0 \\ X_{\alpha} & X_{\beta} \\ 0 & 0 \\ Y_{\alpha} & Y_{\beta} \\ 0 & 0 \\ Z_{\alpha} & Z_{\beta} \\ 0 & 0 \\ L_{\alpha} & L_{\beta} \\ 0 & 0 \\ M_{\alpha} & M_{\beta} \\ 0 & 0 \\ N_{\alpha} & N_{\beta} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{26}$$

Where E, A and B are those introduced in (14), (15), (16).

Linear Formulation

If the application requires a linear state space (for control system development, for example), we linearize the aerodynamic angles about the trim condition using a small angle approximation and by assuming small perturbed velocities ($U \approx U_0, V_{total} \approx \sqrt{U_0^2 + V_0^2 + W_0^2}$).

$$\alpha \approx \frac{w + py_{nb} - qx_{nb}}{U_0} \tag{27}$$

$$\beta \approx \frac{v - pz_{nb} + rx_{nb}}{\sqrt{U_0^2 + V_0^2 + W_0^2}} \tag{28}$$

The aerodynamic angular rates are therefore approximated by:

$$\dot{\alpha} \approx \frac{1}{U_0}\dot{w} + \frac{y_{nb}}{U_0}\dot{p} - \frac{x_{nb}}{U_0}\dot{q} \tag{29}$$

$$\dot{\beta} \approx \frac{1}{\sqrt{U_0^2 + V_0^2 + W_0^2}}\dot{v} - \frac{z_{nb}}{\sqrt{U_0^2 + V_0^2 + W_0^2}}\dot{p} + \frac{x_{nb}}{\sqrt{U_0^2 + V_0^2 + W_0^2}}\dot{r} \tag{30}$$

Clearly, the linearized aerodynamic angles and rates are linearly dependent on our other state variables. The A and E matrices of the state-space are therefore modified to implement the effects of aerodynamic angles.

$$
E = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -X_{\dot{v}}-X_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -X_{\dot{w}}-\frac{X_{\dot{\alpha}}}{U_0} & 0 & -X_{\dot{p}}-\frac{X_{\dot{\alpha}}y_{nb}}{U_0}- X_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -X_{\dot{q}}+\frac{X_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -X_{\dot{r}}-X_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -Y_{\dot{u}} & 0 & 1-Y_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -Y_{\dot{w}}-\frac{Y_{\dot{\alpha}}}{U_0} & 0 & -Y_{\dot{p}}-\frac{Y_{\dot{\alpha}}y_{nb}}{U_0}- Y_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -Y_{\dot{q}}+\frac{Y_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -Y_{\dot{r}}- Y_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -Z_{\dot{u}} & 0 & -Z_{\dot{v}}-Z_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & 1-\frac{Z_{\dot{\alpha}}}{U_0} & 0 & -Z_{\dot{p}}-\frac{Z_{\dot{\alpha}}y_{nb}}{U_0}- Z_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -Z_{\dot{q}}+\frac{Z_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -Z_{\dot{r}}- Z_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{\dot{u}} & 0 & -L_{\dot{v}}-L_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -L_{\dot{w}}-\frac{L_{\dot{\alpha}}}{U_0} & 0 & 1-\frac{L_{\dot{\alpha}}y_{nb}}{U_0}- L_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -L_{\dot{q}}+\frac{L_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -L_{\dot{r}}-\frac{I_{xz}}{I_{xx}}- L_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -M_{\dot{u}} & 0 & -M_{\dot{v}}-M_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -M_{\dot{w}}-\frac{M_{\dot{\alpha}}}{U_0} & 0 & -M_{\dot{p}}-\frac{M_{\dot{\alpha}}y_{nb}}{U_0}- M_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & 1+\frac{M_{\dot{\alpha}}x_{nb}}{U_0} & 0 & -M_{\dot{r}}- M_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -N_{\dot{u}} & 0 & -N_{\dot{v}}-N_{\dot{\beta}}\frac{1}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -N_{\dot{w}}-\frac{N_{\dot{\alpha}}}{U_0} & 0 & -N_{\dot{p}}-\frac{I_{xz}}{I_{zz}}-\frac{N_{\dot{\alpha}}y_{nb}}{U_0}- N_{\dot{\beta}}\frac{z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & -N_{\dot{q}}+\frac{N_{\dot{\alpha}}x_{nb}}{U_0} & 0 & 1- N_{\dot{\beta}}\frac{x_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}
\end{bmatrix} \tag{31}
$$

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X_u & 0 & X_v+\frac{X_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & X_w+\frac{X_\alpha}{U_0} & 0 & X_p+\frac{X_\alpha y_{nb}}{U_0}-\frac{X_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & -g\cdot\cos\Theta_0 & X_q-W_0-\frac{X_{nb}}{U_0} & 0 & X_r+V_0+\frac{X_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Y_u & 0 & Y_v+\frac{Y_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & Y_w+\frac{Y_\alpha}{U_0} & g\cdot\cos\Theta_0 & Y_p+W_0+\frac{Y_\alpha y_{nb}}{U_0}-\frac{Y_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & Y_q-\frac{Y_{nb}}{U_0} & 0 & Y_r-U_0+\frac{Y_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Z_u & 0 & Z_v+\frac{Z_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & Z_w+\frac{Z_\alpha}{U_0} & 0 & Z_p-V_0+\frac{Z_\alpha y_{nb}}{U_0}-\frac{Z_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & -g\cdot\sin\Theta_0 & Z_q+U_0-\frac{Z_{nb}}{U_0} & 0 & Z_r+\frac{Z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & L_u & 0 & L_v+\frac{L_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & L_w+\frac{L_\alpha}{U_0} & 0 & L_p+\frac{L_\alpha y_{nb}}{U_0}-\frac{L_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & L_q-\frac{L_{nb}}{U_0} & 0 & L_r+\frac{L_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & M_u & 0 & M_v+\frac{M_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & M_w+\frac{M_\alpha}{U_0} & 0 & M_p+\frac{M_\alpha y_{nb}}{U_0}-\frac{M_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & M_q-\frac{M_{nb}}{U_0} & 0 & M_r+\frac{M_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & N_u & 0 & N_v+\frac{N_\beta}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & N_w+\frac{N_\alpha}{U_0} & 0 & N_p+\frac{N_\alpha y_{nb}}{U_0}-\frac{N_\beta z_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}} & 0 & N_q-\frac{N_{nb}}{U_0} & 0 & N_r+\frac{N_{nb}}{\sqrt{U_0^2+V_0^2+W_0^2}}
\end{bmatrix} \tag{32}
$$

The B, C, H and D matrices remain unchanged.

$$
B = \begin{bmatrix}
0 & 0 & 0 \\
X_{\delta_a} & X_{\delta_e} & X_{\delta_r} \\
0 & 0 & 0 \\
Y_{\delta_a} & Y_{\delta_e} & Y_{\delta_r} \\
0 & 0 & 0 \\
Z_{\delta_a} & Z_{\delta_e} & Z_{\delta_r} \\
0 & 0 & 0 \\
L_{\delta_a} & L_{\delta_e} & L_{\delta_r} \\
0 & 0 & 0 \\
M_{\delta_a} & M_{\delta_e} & M_{\delta_r} \\
0 & 0 & 0 \\
N_{\delta_a} & N_{\delta_e} & N_{\delta_r}
\end{bmatrix} \tag{33}
$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g\cos\Theta_0 & W_0 & 0 & -V_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g\cos\Theta_0 & -W_0 & 0 & 0 & 0 & U_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_0 & g\sin\Theta_0 & -U_0 & 0 & 0 \end{bmatrix} \tag{34}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_a & 0 & -y_a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -z_a & 0 & 0 & 0 & x_a \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & y_a & 0 & -x_a & 0 & 0 \end{bmatrix} \tag{35}$$

$$D = \text{zeros(9,3)} \tag{36}$$