Written Exam Economics Summer 2025
Household Behavior over the Life Cycle

19 June from 9 AM to 21 June at 9 AM

This exam question consists of 5 pages in total.

Answers only in English.
A take-home exam paper cannot exceed 10 pages – and one page is defined as 2400 keystrokes.

You should hand in a single zip-file with all assignments and the exam. The zip-file should be named after your KU username (e.g. abs123) and have the folder and file structure:

Assignment_1
Assignment_1.pdf - with text and all results
files for reproducing the results

Assignment_2
Assignment_2.pdf - with text and all results
files for reproducing the results

Assignment_3
Assignment_3.pdf - with text and all results
files for reproducing the results

Exam
Exam.pdf - with text and all results
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Use of AI tools is permitted. You must explain how you have used the tools. When text is solely or mainly generated by an AI tool, the tool used must be quoted as a source.

Be careful not to cheat at exams!
Exam cheating is for example if you:

- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text.
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts.
- Reuse parts of a written paper that you have previously submitted and for which you have received a pass grade without making use of quotation marks or source references (self-plagiarism).
- Receive help from others in contrary to the rules in the Faculty of Social Science's common part of the curriculum.

You can read more about the rules on exam cheating on your Study Site and in the Faculty of Social Science's common part of the curriculum.

Exam cheating is always sanctioned by a written warning and expulsion from the exam in question. In most cases, the student will also be expelled from the University for one semester.

Setup and Model Economy

We will study the portfolio allocation of individuals throughout their lives. Baseline code is available through Digital Exam. In each period, individuals choose how much to consume, c_t, and how to allocate resources between a riskless asset, a_t, that earns a guaranteed gross return of R^a between period t and $t+1$ and a risky asset, b_t, that earns a stochastic gross return of R^b_{t+1} between period t to $t+1$, following an AR(1) process. The share in the risky asset is denoted x_t. In each period, households earn a deterministic income of y_t. The state variables are the total amount of savings, $s_t = a_t + b_t \in \mathbb{R}_+$ and the current gross return on the risky asset, $R^b_t \in \mathbb{R}_+$. Consumers are not allowed to borrow such that post-consumption resources must be non-negative, $w_t = s_t + y_t - c_t \geq 0$, and x_t must lie in the unit interval, $x_t \in [0,1]$.

The recursive formulation (Bellman equation) for the model is, for $1 \leq t < T$,

$$V_t(s_t, R^b_t) = \max_{c_t, x_t} U(c_t) + \beta \mathbb{E}_t[V_{t+1}(s_{t+1}, R^b_{t+1})] \tag{1}$$

$$\text{s.t.}$$

$$s_{t+1} = a_{t+1} + b_{t+1} \tag{2}$$

$$a_{t+1} = R^a(1 - x_t)w_t \tag{3}$$

$$b_{t+1} = R^b_{t+1} x_t w_t \tag{4}$$

$$R^b_{t+1} = \begin{cases} R^b_t + \varepsilon_t & \text{if } R^b_t + \varepsilon_t > 0 \\ 0 & \text{else} \end{cases} \tag{5}$$

$$\varepsilon_t \sim iid\mathcal{N}(0, \sigma^2) \tag{6}$$

$$w_t = s_t + y_t - c_t \tag{7}$$

$$c_t \in (0, s_t + y_t] \tag{8}$$

$$x_t \in [0,1] \tag{9}$$

where preferences are Constant Relative Risk Aversion (CRRA),

$$U(c_t) = \frac{c_t^{1-\rho}}{1-\rho}. \tag{10}$$

There is no bequest motive such that for $t = T$,

$$V_T(s_T, R^b_T) = \max_{c_T} U(c_T) \tag{11}$$

$$\text{s.t.}$$

$$0 \leq s_T + y_T - c_T. \tag{12}$$

The baseline parameters are $T = 10$, $\beta = 0.98$, $\rho = 2.0$, $R^a = 1.01$, $y_t = 1\ \forall t$ and $\sigma^2 = 0.03$. When simulating, we draw initial levels of all state variables in the following way: Initial savings are drawn from a uniform distribution on the interval $[0.5, 1.5]$, $s_{i,1} \sim U(0.5, 1.5)$, and the initial level of risky returns are $R_{i,1} = R^a + 0.02$ for all individuals and they then face individual-specific return shocks, $\varepsilon_{i,t}$, throughout their lives. We thus think of the risky asset as individual-specific assets that follows the same distribution but might experience different shocks. We simulate $N = 5,000$ individuals for $T = 10$ periods. Questions can be answered independently but the idea is that you answer question j building on the setup in question $j-1$. Try to avoid copy-pasting code but rather generate one code base that can answer all questions.

Questions

1. Explain why we do not need to keep track of the allocation of savings into the risk-free and risky asset, a_t and b_t, respectively, but only the total amount of savings, s_t. In other words, why is s_t a state variable in our model and not both a_t and b_t?

2. Explain mathematically and in words how the expectation in equation (1) can be approximated when numerically solving the model.
You can draw inspiration from the code.

3. Solve and simulate the baseline model. Also solve and simulate a no-risk version of the model, in which there is no uncertainty about the risky asset return and the initial level of risky returns are still $R_{i,1} = R^a + 0.02$ for all individuals.

   (a) Be explicit about which parameter(s) you change to construct this alternative model.

   (b) Plot the average age profiles of the portfolio allocation, x_t, for both models jointly and total savings, s_t, for both models jointly.

   (c) Explain the observed differences between the baseline and no-risk versions of the model, through the economic incentives in the model. Recall that the last period portfolio choice is set to $x_T = 0 \, \forall s_T, R^b_T$ as the value does not matter in the terminal period.

Use the baseline parametrization with $\sigma^2 = 0.03$ to answer the following questions.

4. Calculate the realized return rate on the portfolio, $R_{i,t+1} = R^a(1 - x_{i,t}) + R^b_{i,t+1}x_{i,t}$, and store it in "sim".

   (a) Plot the average age profile of the realized return rate.

   (b) Calculate and report the correlation between simulated total savings in the final period of life, $s_{i,T}$, and the simulated realized return in period two, $R_{i,2}$.

   (c) What does this correlation suggest about the role of capital returns in shaping wealth inequality?

   (d) Discuss if the correlation could stem from differences in initial wealth. To add to this discussion, you could consider simulating from a model in which all start with, e.g. $s_{i,1} = 1$ and re-calculate the correlation.

5. Introduce a tax on all capital gains, $\tau = 0.15$.
   Be specific about which equations from (1) to (12) that are influenced by this modification and write out all the new equations.

6. Implement the capital gain tax above, solve and simulate the new model.
   (a) Plot average age profiles of total wealth, the portfolio allocation, and the realized return from the original model together with the same output simulated from the new model.
   (b) Calculate the correlation between total savings in the final period of life, $s_{i,T}$, and the realized return in period two, $R_{i,2}$.
   Compare with the original correlation from above and discuss the potential difference.

7. How much additional initial wealth, $s_0^\star$, should each individual be compensated with in the beginning of life to be indifferent between the regime without a tax on capital gains and one with a tax on capital gains?
   This analysis follows the idea of Borella, De Nardi and Yang (2023), discussed in the course. The initial wealth of an individual in the economy with a capital gains tax is thus $s_0^\star$ higher than in the economy without a capital gains tax ($\tau = 0$).
   (a) Report $s_0^\star$ and be precise with how you found it.
   (b) Report and discuss average savings profiles from
   i) the model with $\tau = 0$,
   ii) the model with $\tau = 0.15$ (without compensation) and
   iii) the model with $\tau = 0.15$ with compensation $s_0^\star$.

Use the model with the capital gains tax *without* the compensation (i.e. set $s_0^\star = 0$) to answer the following questions (if nothing else is stated).

8. We now want to estimate the constant relative risk aversion coefficient (CRRA), ρ, and the discount factor, β. We are told that
   i) the average total savings in period five is $\frac{1}{N}\sum_{i=1}^{N} s_{i,5} = 0.3$ and
   ii) the average share of total wealth in risky assets in period five is $\frac{1}{N}\sum_{i=1}^{N} x_{i,5} = 0.7$
   Use these two moments to estimate ρ and β using simulated method of moments.
   Be mathematical precise about how this estimator is formulated and explain how you implemented it in Python.
   Good starting values could be $\rho = 2.6$ and $\beta = 0.96$.

9. Discuss why the two moments above should be informative about the values of ρ and β.

10. Introduce a new state variable $r_t \in \{0, 1\}$ that denotes if the economy is in a recession ($r_t = 1$) or not ($r_t = 0$).
    The likelihood of being in a recession in the next period is 0.1 if the economy is currently not in a recession. If the economy is currently in a recession, the likelihood of also being in

a recession in the next period is 0.8.

Formulate mathematically the transition equation for the new state variable.

11. The risky return now depends on the state of the economy as the return is scaled by a factor $\gamma(r_t)$

$$R^b_{t+1} = \begin{cases} \gamma(r_t)R^b_t + \varepsilon_t & \text{if } \gamma(r_t)R^b_t + \varepsilon_t > 0 \\ 0 & \text{else} \end{cases}$$

where

$$\gamma(r_t) = \begin{cases} \gamma & \text{if } r_t = 0 \\ 0 & \text{if } r_t = 1. \end{cases}$$

Let $\gamma = 0.8$. Solve this extended model in Python. Be explicit about which parameter values you use.

Show the optimal consumption function and optimal portfolio choice in the first period for $R^b_1 = 1.02$ (the 9th point in the grid of potential values of the risky return state) across values of $s_1 \in [0, 8]$ for $r_1 = 0$ and $r_1 = 1$.

12. Simulate the new model from question 11, where all individuals experience the same realization of the economy and the economy starts with $r_1 = 0$. Be explicit about how you changed the simulation module.

Plot the average age profiles of all state variables and choices in this model.

Show the simulations together with a version of this model with $\gamma = 1$ and discuss the differences.

References

Borella, M., M. De Nardi and F. Yang (2023): "Are Marriage-Related Taxes and Social Security Benefits Holding Back Female Labor Supply?," *Review of Economic Studies*, 90(1), 102–131.