Numerical_RSW_arbitrary_domain_time_dependent

March 25, 2022

1 Script to solve RSW adjustment problem for η from a given initial condition.

1.0.1 twnh Sep, Oct, Nov '21, Mar '22

We seek the solution to the (non-dimensional) 1-layer RSW adjustment problem. The problem is to: 1. Determine the $\{\omega, \eta\}$ pairs from the nonlinear (cubic) eigenproblem. The problem is (in weak form)

$$\int_\Omega \omega^3 \psi \eta - \omega \left(\nabla\psi \cdot \nabla\eta + F\psi\eta \right) \, d\Omega - i\sqrt{F} \int_\Gamma \psi \nabla\eta \cdot \mathbf{t} \, d\Gamma = 0, \tag{1}$$

where η is the solution we require and ψ is a test function. * The problem is in a 2D (simply-connected) domain Ω with boundary Γ where $\mathbf{t}$ is the tangent to the boundary running anti-clockwise (to the left of outward unit normal direction $\mathbf{n}$). The domain has characteristic (longest) length L, which is taken in the horizontal direction (without loss of generality). The non-dimensional domain therefore has horizontal length 1. * $F = f^2L^2/(gH)$ is the inverse Froude number. The non-rotating limit is $F = 0$. * Time is non-dimensionalized so that the gravity wave crosses the horizontal domain in 1 time unit. * The deformation radius is therefore given by $1/\sqrt{F}$ in non-dimensional length.

Solving this problem gives $\{\omega_n, \eta_n\}$ eigenvalue, eigenvector pairs which are the elevation modes oscillating at frequency ω_n.

2. Determine the η_∞ steady state solution from the initial PV. The problem is (in weak form)

$$\int_\Omega \nabla\psi \cdot \nabla\eta_\infty + F\psi\eta_\infty \, d\Omega = -\int_\Omega F\psi\eta_i \, d\Omega. \tag{2}$$

   The inital PV is $Q_i = -\eta_i$, which ASSUMES that the initial flow vanishes (see `theory notes.tex` for the more general case).

3. Determine the η_f inertial solution. The problem is (in weak form)

$$\int_\Omega \nabla\psi \cdot \nabla\eta_f \, d\Omega + i \int_\Gamma \psi \nabla\eta_f \cdot \mathbf{t} \, d\Gamma = 0, \tag{3}$$

   with frequency $\omega = \sqrt{F}$.

4. Determine the expansion coefficients from the initial η field (η_i) and the steady solution η_∞.

The problem is to find the coefficients α_n such that

$$\eta_i - \eta_\infty = \Re\Big[\sum_n \alpha_n \eta_n\Big], \tag{4}$$

$$0 = \Re\Big[\sum_n -i\omega_n\, \alpha_n \eta_n\Big], \tag{5}$$

$$\nabla^2 \eta_i = \Re\Big[\sum_n -\omega_n^2\, \alpha_n \eta_n\Big], \tag{6}$$

$$\tag{7}$$

where η_i is the initial elevation field, and ASSUMING that the initial vorticity and divergence of the flow vanish (see `theory notes.tex` for the more general case).

5. Animate the solution. The solution at time t is

$$\eta(t) = \eta_\infty + \Re\Big[\sum_n \alpha_n \eta_n \exp\left(-i\omega_n t\right)\Big] \tag{8}$$

-1. Housekeeping

[2]:
```
using Gridap, GridapGmsh              # FEM package
using LinearAlgebra
using Arpack
using NonlinearEigenproblems          # To solve the cubic eigenproblem

using Interpolations, FileIO          # To read and interpolate results from␣
 ↪Oceananigans runs
using BenchmarkTools                  # For performance testing

using Printf                          # For interface
using GridapMakie
using GLMakie
GLMakie.inline!(true)                 # Makes sure figures appear in notebook␣
 ↪inline.
using Glob                            # For housekeeping
using LaTeXStrings

include("RSW_ModelFunctions.jl")      # Function definitions
rm.(glob("output/eta_*"))             # Clean output directory
writeparaView = false ;               # Write paraView files or not
checkEigenFlag = true ;               # Check if eigensolution satisfies␣
 ↪original nonlinear eigen problem
trimThreshold = 1.0e-6 ;              # Threshold to trim small eigenvalues.␣
 ↪Check figures to make sure this is OK.
```

0. Define problem

[3]:
```
#domainName = "square_384nodes_uneven.msh"
# domainName = "square_803nodes_uneven.msh"
```

```
domainName = "square_1328nodes_uneven.msh"
# domainName = "square_2903nodes_uneven.msh"        # Slow: computing modes takes 2.5 hours
# domainName = "square_258nodes.msh"
# domainName = "square_145nodes.msh"
# domainName = "circle_268nodes.msh"
#domainName = "rectangle_261nodes.msh"

# The non-dimensionalization sets the longest lengthscale of the non-dimensional domain equal to one.
 U,V,Uϖ,Vϖ,Ω,dΩ,dΓ,t_Γ = DefineProblem(domainName)

# Physical parameters: Avoid integer multiples of these parameters (they can throw off the polynomial eigensolver)
#F = 0.0
#F = 0.001
#F = 1.001
#F = 15.98
F = 63.9
#F = 4096.0
f = 1.0e-4      # s^{-1}
g = 9.81        # ms^{-2}
L = 512.0e3     # m
L  = sqrt.(L^2 ./ F)
H = ((f * L ) .^ 2) ./ g
timescale = L ./ sqrt.(g .* H)
@printf("Coriolis parameter                 = [%8.2e] s^{-1}\n",f)
@printf("Gravitational acceleration         = [%8.2f] ms^{-2}\n",g)
@printf("Layer thickness                    = [%8.1f] m\n",H)
@printf("Domain lengthscale                 = [%8.2f] km\n",L/1e3)
@printf("Deformation radius                 = [%8.2f] km\n",L /1e3)
@printf("Characteristic timescale           = [%8.2f] days.\n",L/sqrt(g*H)/86400)
@printf("\nInverse Froude number              F = [%8.2f]\n",F)
@printf("Non-dimensional deformation rad.   = [%8.2f]\n\n",1/sqrt(F))

# Initial condition for \eta (non-dimensional)
function  (x)
    Deltax = 1/8     # Width of the initial step in \eta
      = tanh(x[1]/Deltax)  # tanh step
    return
end ;
```

```
Reading Gmsh discrete model mesh [square_1328nodes_uneven.msh]
Info    : Reading 'square_1328nodes_uneven.msh'…
Info    : 13 entities
Info    : 1328 nodes
```

```
Info    : 2654 elements
Info    : Done reading 'square_1328nodes_uneven.msh'
Coriolis parameter                  = [1.00e-04] s^{-1}
Gravitational acceleration          = [    9.81] ms^{-2}
Layer thickness                     = [     4.2] m
Domain lengthscale                  = [  512.00] km
Deformation radius                  = [   64.05] km
Characteristic timescale            = [    0.93] days.

Inverse Froude number             F = [   63.90]
Non-dimensional deformation rad.    = [    0.13]
```

1. Build polynomial eigenvalue problem and then solve it for the ω, η eigenpairs. Recall, the eigenproblem is:

$$\int_{\Omega} \omega^3 \psi\eta - \omega \left(\nabla\psi \cdot \nabla\eta + F\psi\eta \right) \, d\Omega - i\sqrt{F} \int_{\Gamma} \psi \nabla\eta \cdot \mathbf{t} \, d\Gamma = 0. \tag{9}$$

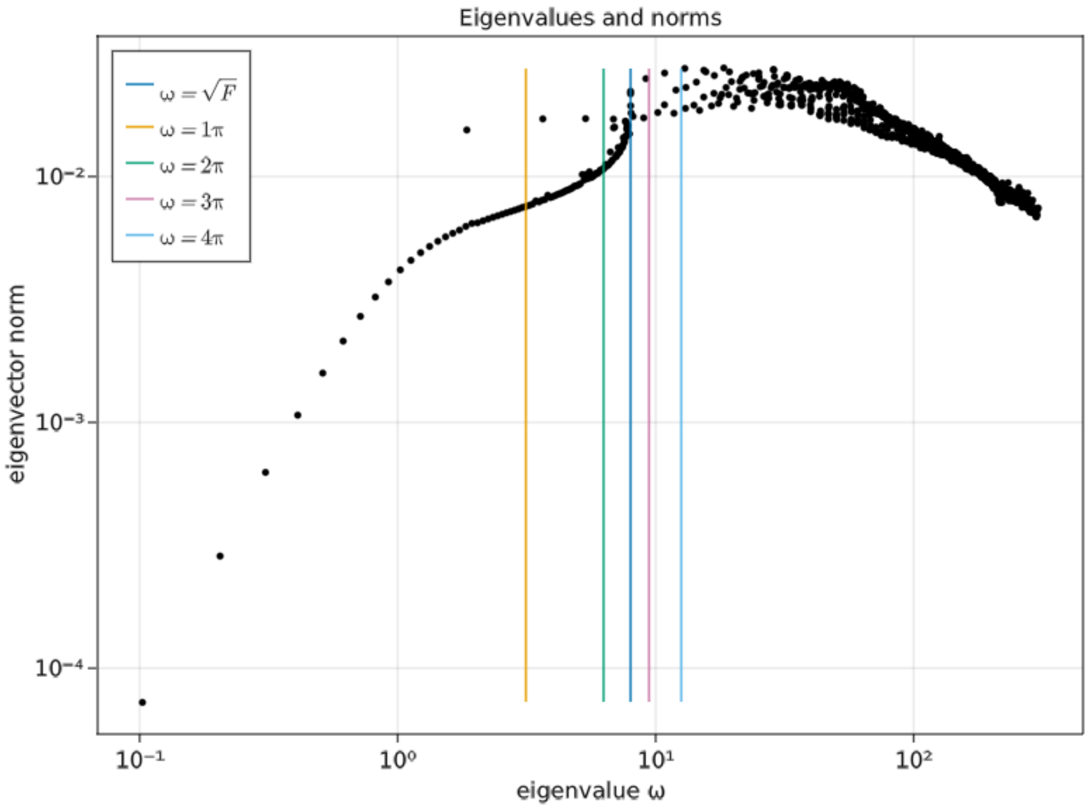
It's hard to assign a particular wavenumber to each spatial mode and thereby compare to the ideal dispersion relations for IGWs and Kelvin waves. Instead, the figure below shows frequency against mode norm.

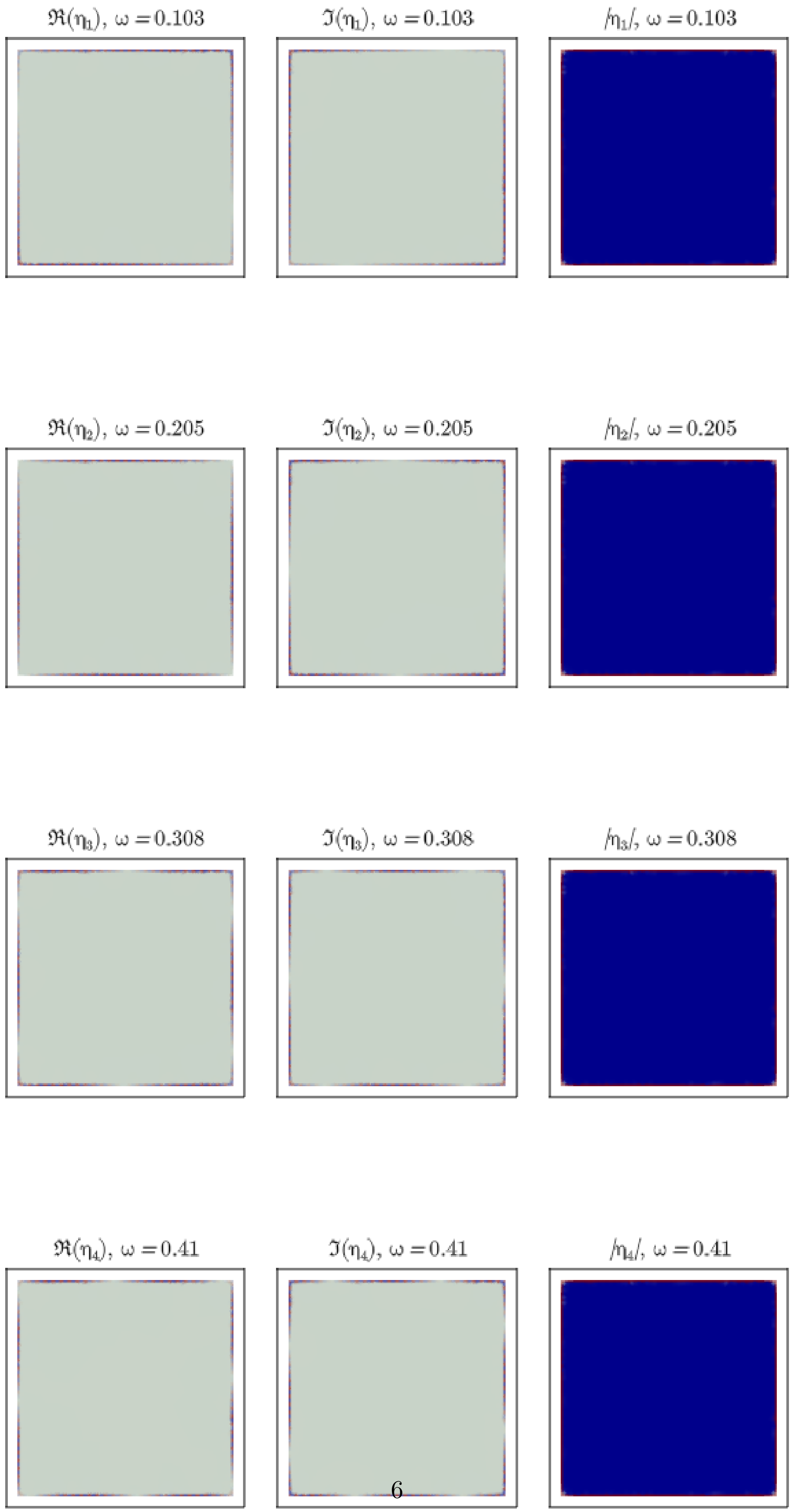
[4]:
```
# Find modes from cubic eigenproblem
 raw,  matraw,  raw,  normraw = solveModesProblem(F) ;
```

```
Solving modal problem for ( , ) eigenpairs with F = [ 63.9000]:
Done in [246.16]s.

Matrix problem size                 = [  1328 x   1328]
Number of eigenvalues               = [  3984]
Eigenvalue range                    = [1.42e-20 -- 3.04e+02]
Size of eigenvector matrix          = [  3984 x      1]
Maximum residual for eigenvalue nn = [  3937],   = [-277.518 +  0.000im], max
residual = [4.776e-08]
```

[5]:
```
# Trim insignificant eigenvalues and plot
# All modes with frequency > 0
 ,  mat,  ,  norm = trimEigen( raw, matraw, raw, normraw,x -> real(x) >␣
 ↪trimThreshold)
plotModes( ,  ,  norm,"eta_modes")
```


Eigenvalues and norms
$\omega = \sqrt{F}$
$\omega = 1\pi$
$\omega = 2\pi$
$\omega = 3\pi$
$\omega = 4\pi$
eigenvector norm
10^{-2}
10^{-3}
10^{-4}
10^{-1}
10^{0}
10^{1}
10^{2}
eigenvalue ω


$\Re(\eta_1)$, $\omega = 0.103$
$\Im(\eta_1)$, $\omega = 0.103$
$/\eta_1/$, $\omega = 0.103$
$\Re(\eta_2)$, $\omega = 0.205$
$\Im(\eta_2)$, $\omega = 0.205$
$/\eta_2/$, $\omega = 0.205$
$\Re(\eta_3)$, $\omega = 0.308$
$\Im(\eta_3)$, $\omega = 0.308$
$/\eta_3/$, $\omega = 0.308$
$\Re(\eta_4)$, $\omega = 0.41$
$\Im(\eta_4)$, $\omega = 0.41$
$/\eta_4/$, $\omega = 0.41$

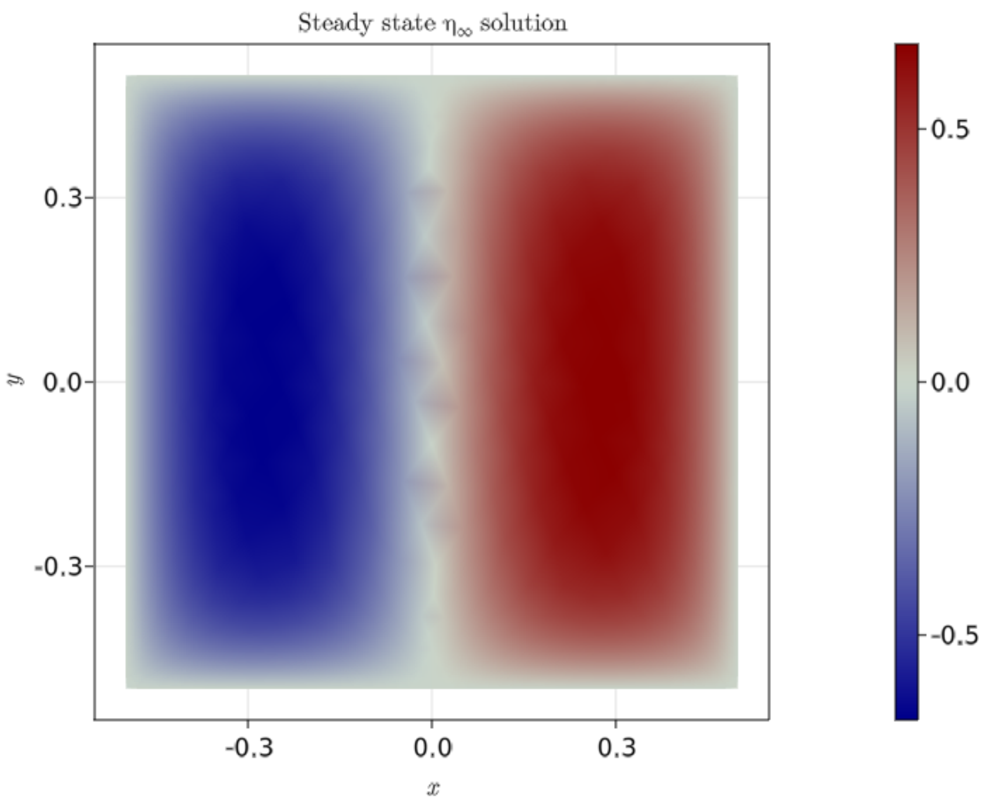
```
Trimmed [2549] eigenvalues to leave [1435 = 2 x 718] significant eigenvalues
with [1328] cells.
Trimmed eigenvalue range            = [1.03e-01 -- 3.04e+02]
```

2. Find steady solution η_∞ Recall, the problem is:

$$\int_\Omega \nabla\psi \cdot \nabla\eta_\infty + F\psi\eta_\infty \, d\Omega = -\int_\Omega F\psi\eta_i \, d\Omega. \tag{10}$$

The inital PV is $Q_i = -\eta_i$, which ASSUMES that the initial flow vanishes (see `theory notes.tex` for the more general case).

[6]:
```
ꝏ = solveSteadyProblem()
plotSteadySoln("eta_steady")
```



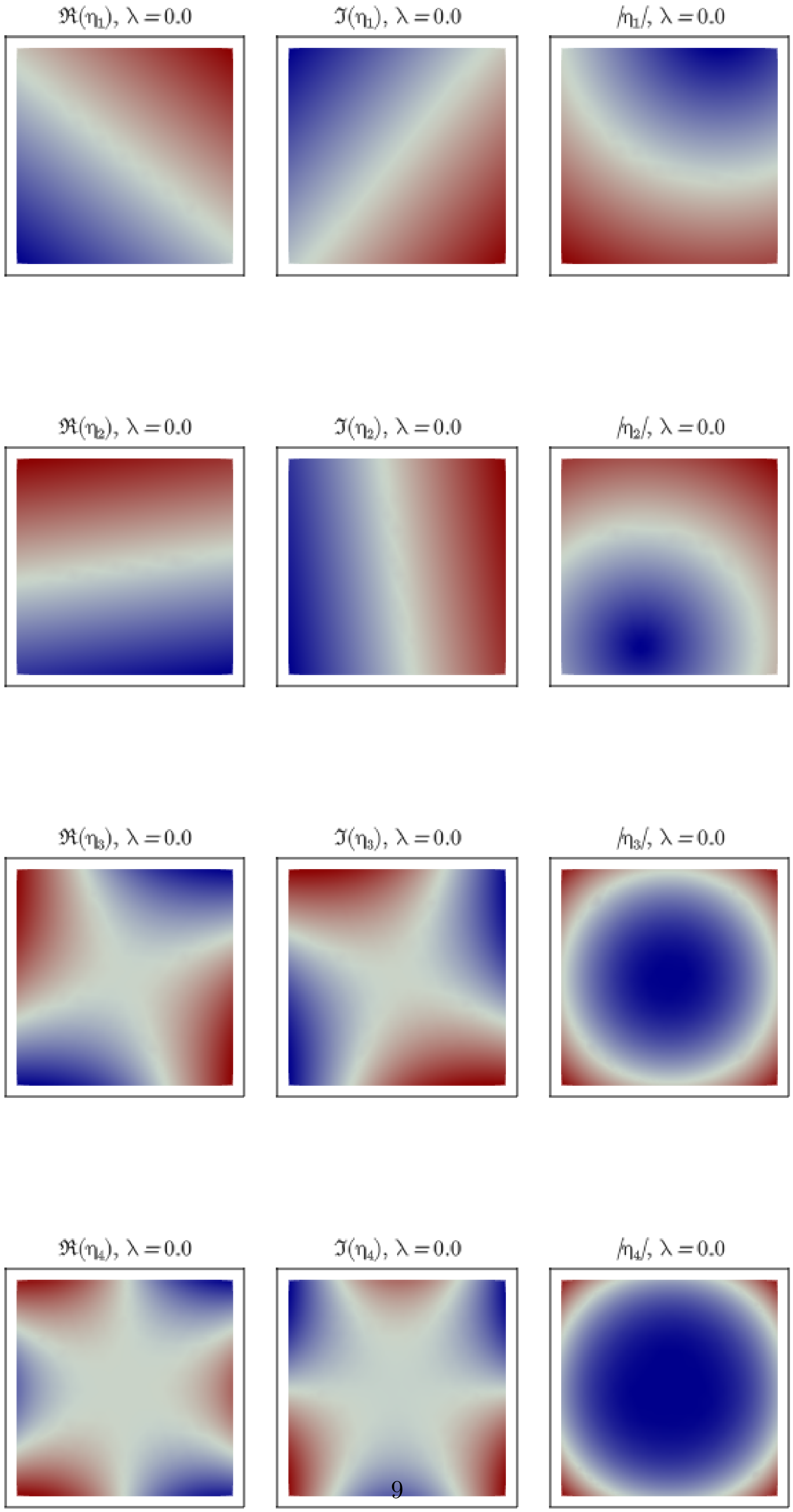

```
Solving steady problem for  ꝏ.
Done in [  4.97]s.
```

2. Find inertial solution η_f Recall, the problem is

$$\int_\Omega \nabla\psi \cdot \nabla\eta_f \, d\Omega + i \int_\Gamma \psi \nabla\eta_f \cdot \mathbf{t} \, d\Gamma = 0, \tag{11}$$

with frequency $\omega = \sqrt{F}$.

[7]:
```
_, mat, _, norm = solveInertialProblem()
plotInertialSoln("inertial")
```


$\Re(\eta_1)$, $\lambda = 0.0$
$\Im(\eta_1)$, $\lambda = 0.0$
$|\eta_1|$, $\lambda = 0.0$
$\Re(\eta_2)$, $\lambda = 0.0$
$\Im(\eta_2)$, $\lambda = 0.0$
$|\eta_2|$, $\lambda = 0.0$
$\Re(\eta_3)$, $\lambda = 0.0$
$\Im(\eta_3)$, $\lambda = 0.0$
$|\eta_3|$, $\lambda = 0.0$
$\Re(\eta_4)$, $\lambda = 0.0$
$\Im(\eta_4)$, $\lambda = 0.0$
$|\eta_4|$, $\lambda = 0.0$

```
Solving inertial problem for   (this mode has frequency   = √F, but a mode
structure independent of F).
Done in [  2.77]s.

Singular value (should be zero): [6.6640e-18 + 4.3960e-33 im].
Singular value (should be zero): [2.3830e-16 + 5.2529e-31 im].
Singular value (should be zero): [1.5587e-09 + 8.4031e-19 im].
Singular value (should be zero): [1.1757e-08 + 1.9415e-16 im].
```

3. Find coefficients of eigenmodes to match the initial condition and steady state Recall, the problem is to find the coefficients α_n such that

$$\eta_i - \eta_\infty = \Re\Big[\sum_n \alpha_n \eta_n\Big], \tag{12}$$

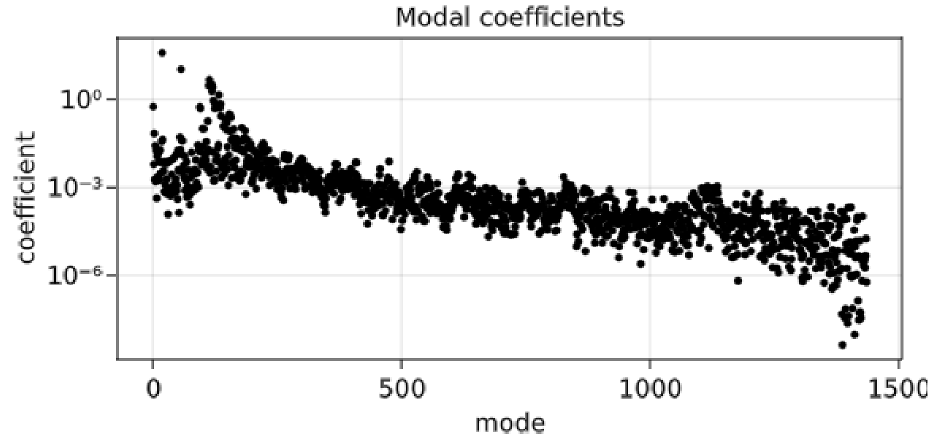
$$0 = \Re\Big[\sum_n -i\omega_n\, \alpha_n \eta_n\Big], \tag{13}$$

$$\nabla^2 \eta_i = \Re\Big[\sum_n -\omega_n^2\, \alpha_n \eta_n\Big], \tag{14}$$

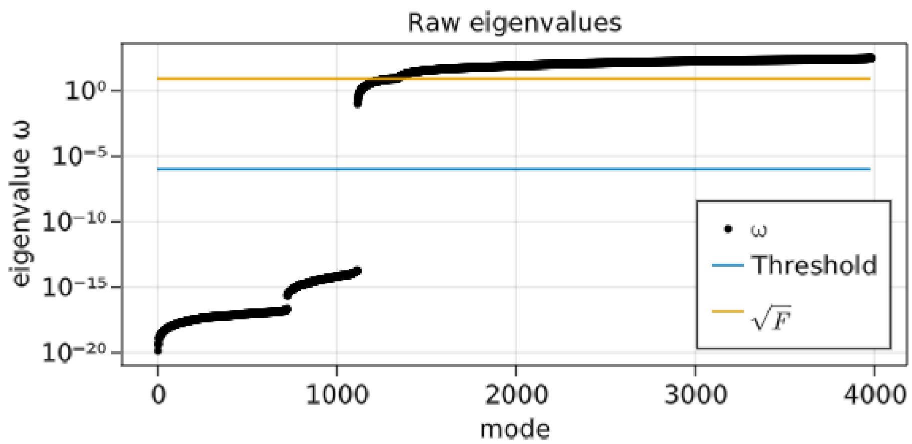
$$\tag{15}$$

where η_i is the initial elevation field, and ASSUMING that the initial vorticity and divergence of the flow vanish (see `theory notes.tex` for the more general case).

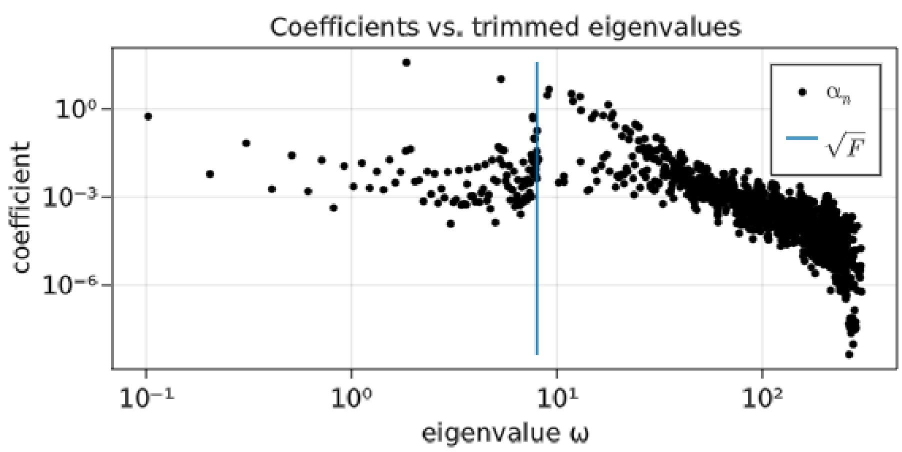
[8]:
```
  = findExpansionCoefficients( mat) ;
```



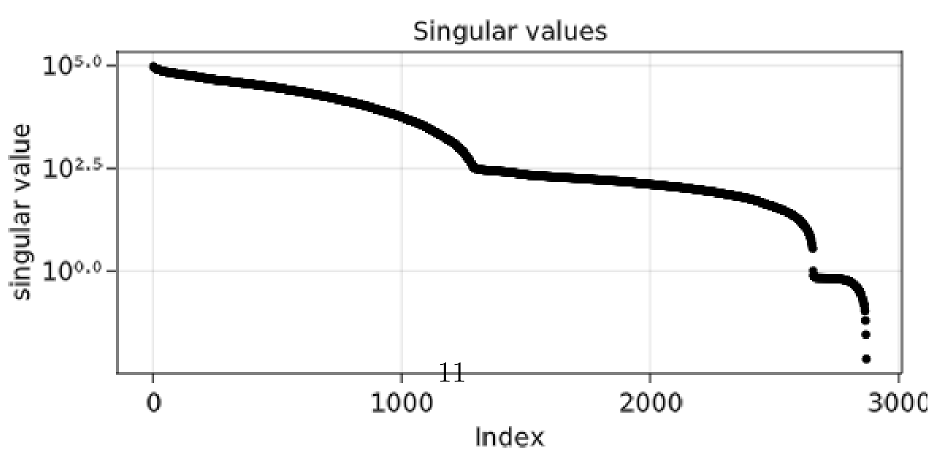
Modal coefficients
coefficient
10⁰
10⁻³
10⁻⁶
0
500
1000
1500
mode



Raw eigenvalues
eigenvalue ω
10⁰
10⁻⁵
10⁻¹⁰
10⁻¹⁵
10⁻²⁰
ω
Threshold
$\sqrt{F}$
0
1000
2000
3000
4000
mode



Coefficients vs. trimmed eigenvalues
coefficient
10⁰
10⁻³
10⁻⁶
α_n
$\sqrt{F}$
10⁻¹
10⁰
10¹
10²
eigenvalue ω



Singular values
singular value
10⁵·⁰
10²·⁵
10⁰·⁰
0
1000
2000
3000
Index

```
Finding expansion coefficients with [1435] modes on a mesh with [1328] nodes:
This ASSUMES the initial divergence and vorticity vanish.
Computing eigenvector expansion of initial condition using SVD. Solves y = Re[
E*x ]:
size(E) = [3984 x 1435], size(y) = [3984] => size(x) = [1435]
get_particular_svd_soln: Using [2870] vectors to construct solution with [0]
nullspace vectors.
Residual of (y - real(E*x))'*(y - real(E*x)) = [2.449e-03].
Done in [ 17.99]s.
```

4. Make animation of the time dependent solution $\eta(t)$. Recall, the solution at time t is

$$\eta(t) = \eta_\infty + \Re\Big[\sum_n \alpha_n \eta_n \exp\left(-i\omega_n t\right)\Big]. \tag{16}$$

Compare this with the Oceananighans DNS solution from `RSW_channel_adjustment.ipynb` (see `RSW_adjustment_movie.key`).

[9]:
```
# In \Delta t the numerator is in seconds and matches that from the␣
↪Oceananighans notebook, except Oceananighans only plots every 4th (or 8th)␣
↪timestep.
Δt = sqrt(F)*62.451210/timescale
nsteps = 512
Tf = nsteps*Δt
animateSolution(nsteps,Δt,Tf, , mat, ,"eta") ;
```

```
Making animation of time dependent solution:
Writing output to [output/eta_results.mp4] with [512] steps of length [0.006245]
and final time = [3.1975].
Done in [ 42.23]s.
```

[10]:
```

```