Calculating the Angle of Vanishing Stability

Boat: Up 'n Gybe

Thomas Jagielski and Sander Miller

Quantitative Engineering Analysis
Module 1: Boats

February 19, 2019

1 Proposed Design

Our boat is defined by the expressions,

$$y = (\frac{x}{1.599})^2 \tag{1}$$

where $-3.198 \leq x \leq 3.198$.

$$y = \frac{9 * z^4}{20000} \tag{2}$$

where $-10 \leq z \leq 10$ and $0 \leq y \leq 4$.

To add a transom to the boat,

$$y < \frac{3}{4}x + 9.25 \tag{3}$$

Each of these equations were adjusted to fit to a 1 inch coordinate system. Thus, the dimensions for the boat can be defined by a length of 20 inches, a beam of 6.396 inches, and a height of 4 inches. Figure 1 shows a 3D model of the hull defined by these expressions.

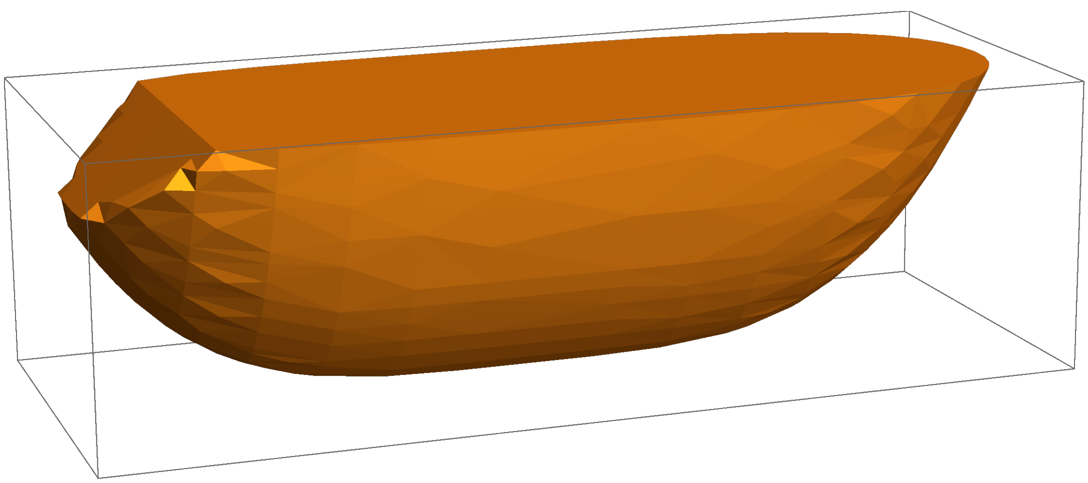


Figure 1: A 3D image of the shape of our hull generated in Mathematica.

When determining the shape for the boat, we took into consideration the design requirements for the project. Our requirements were to make a boat that,

- Floats with a ballast of between 700 g and 1000 g
- Floats flat (within 5° heel) when the boat is fully loaded
- The angle of vanishing stability is between 120° and 140°

- Have a maximum computed righting moment of at least 0.2 Nm
- The hull should be as fast as possible

The inspiration for our boat came from a combination of sailboat characteristics that would help us meet the design requirements for this project. In order to make the boat stable, yet have an angle of vanishing stability of 130°, we took the flat-bottomed style of a scow and combined it with the narrow beam of an FJ or J-boat. The flat nature of the scow will help the boat float flat, and the narrow hull will decrease drag and allow for the maximum speed. We added the transom in order to make the boat more visually appealing. The effect the transom has on the boat is the center of mass moving towards the stern. At high heel angles, there is a pitch that results in the center of buoyancy moving towards the bow. However, the effect on the center of mass is nearly negligible, and can be easily counteracted with imbalance of the ballast in the bow and stern.

Using the Mathematica model we created, the center of mass would need to be at $(0, 1.95, 0)$. When creating our SolidWorks assembly, we determined the center of mass was too high without ballast. Therefore, we added the 1000 g of ballast to the lowest point possible. This lowered the center of mass to 1.7 inches (in the y-direction). Therefore, we distributed the ballast higher along the ribs of the boat. This will increase the waterline, and thus, the waterline length of the boat. This is instrumental when considering hull speed and how fast our boat can travel. The wave created from the bow produces a large drag as the rest of the boat rides up the wave. Hull speed is increased when the waterline length grows.

We made our boat short to increase speed and stability. If the boat was taller, the center of mass would be raised. Thus, we decided on a height of 4 inch ribs, but a total height of 4.5 inches (including the keel). In doing this, we were able to position our center of mass at 1.95 inches in the y-direction. Another attempt to make our boat stable was flattening the bottom, we used a z^4 expression to define the length of our boat in the y-z plane. We then found the coefficient of $\frac{9}{20000}$ by setting the expression $a * z^4 = 4.5$ at $z = 10$, so the total length would be 20 inches. This length yields enough volume for the boat to float at a reasonable level, resulting in the center of buoyancy being higher in the hull. When evalutating this expression, we get $\frac{9}{20000}$. Over the same bounds, if $\frac{9}{20000}$ is greater the boat will be taller, and shorter if the value is smaller. The expression $(\frac{x}{1.599})^2$ represents the shape of our hull in the x-y plane (the ribs). We found the value 1.599 through qualitative inspection of various values in the function. When this value increases the curve will be less steep, and it will be more steep if the value decreases. We were looking for a curve that was not too steep at the keel to maximize stability, but we were also looking for a beam of close to 6 inches. Once we decided on 1.599, the function yields the bounds of -3.198 and 3.198 for x. Thus, the total beam of our boat is 6.396 inches. A graphical representation of the expressions are shown in Figure 2 and Figure 3.

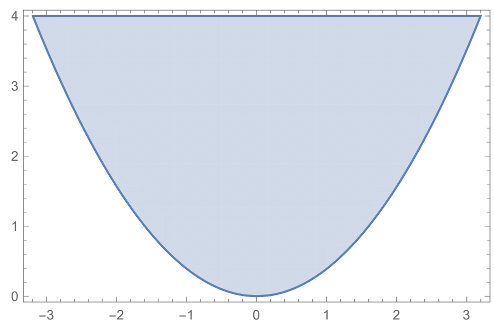


Figure 2: This is an xy-plane cross-section of our boat. This is generated using the expression $(\frac{x}{1.599})^2$

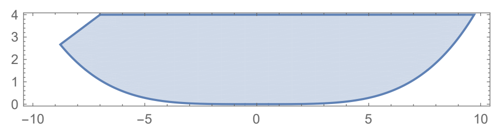


Figure 3: This is an zy-plane cross-section of our boat. This is generated using the expression $\frac{9*z^4}{20000}$

We were able to determine the mass and center of mass of the boat using the SolidWorks "Mass Properties" feature. To do this, we assigned a density of $990\frac{kg}{m^3}$ to the fiber board, a density of $2710\frac{kg}{m^3}$ to the mast, and used a cube with a mass of 1000 g to simulate the ballast in the SolidWorks assembly. The assembly is shown in Figure 4. According to this feature, our total weight is 3.04 lbs and the center of mass is at the point $(0, 1.95, -0.09)$. However, we also checked the total mass by massing the boat when it is fully constructed including the ballast, which yielded 1349 g or 2.97 lbs.

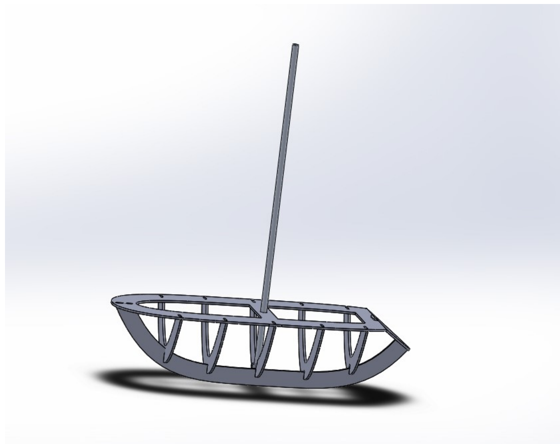


Figure 4: SolidWorks assembly of Up 'n Gybe.

2 Design Justification

2.1 Theory

In order to assess how safe a boat is, we need to determine the angle of vanishing stability. This is the point at which a boat will capsize. In order to make this computation, we must consider the moment arm created between the center of mass and center of buoyancy at particular heel angles. A free body diagram of the system is shown in Figure 5.

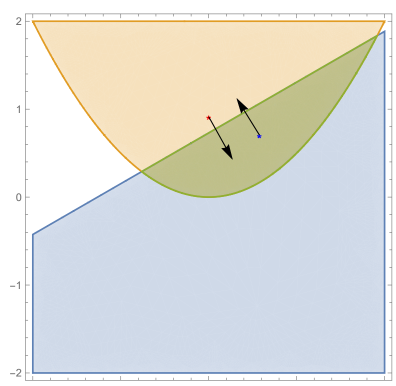


Figure 5: Free body diagram of boat with 30° of heel. The buoyant force is shown with the arrow from the blue point (under the waterline), and the force of gravity is expressed as the arrow extending from the red point. A moment arm is created between them resulting in either a righting or destabilizing torque depending on the particular angle of heel.

2.1.1 Center of Mass

The center of mass is a weighted average of the mass distribution throughout the hull,

$$\vec{r}_{COM} = \frac{\sum_{i=1} m_i \vec{r}_i}{M_{total}} \tag{4}$$

where m_i is the effective mass of a slice, $\vec{r}_i$ is the distance from the origin to the mass representing the slice expressed by m_i, and M_{total} is the total mass of the object. The total mass can be expressed by the equation,

$$M_{total} = \sum_{i=1} m_i \tag{5}$$

Further, it can be separated for the x, y, and z components of center of mass as,

$$x_{COM} = \frac{\sum_{i=1} m_i x_i}{M_{total}} \tag{6}$$

For the y and z components, x would be replaced with the respective component.

When represented continuously, the expression becomes,

$$x_{COM} = \frac{1}{M_{total}} \iiint_R x\rho \, dV \tag{7}$$

where R is the region you are integrating over, ρ is the density of the object, and M_{Total} is the total mass expressed as,

$$COM = \iiint_R \rho \, dV \tag{8}$$

2.1.2 Center of Buoyancy

Similar to gravity, buoyancy is a distributed force. Thus, there is an effective point at which the force is acting. This point is considered the center of buoyancy. For the boat to be in static equilibrium, we know,

$$\sum Forces = 0 \tag{9}$$

In the case of a boat,

$$\begin{aligned} F_{G_{Boat}} &= F_{B_{Boat}} \\ M_{Boat} * g &= M_{DisplacedWater} * g \\ M_{Boat} &= M_{DisplacedWater} \end{aligned} \tag{10}$$

where F_G is the force of gravity, F_B is the buoyant force, g is the force of gravity, and M is the mass.

2.1.2.1 Waterline

As shown above (in the Center of Buoyancy section), the waterline depends on the mass of the boat. However, it also depends on the heel of the boat. In order to maintain the same volume submerged, the waterline will change based on the heel angle of the boat,

$$w(x) = \tan\theta * x + D \tag{11}$$

where θ is the heel angle and D is the draft of the boat.

We know the mass of the boat must equal the mass of the fluid displaced. If we assume the boat is constant density,

$$M_{Boat} = \iiint_S \rho_{Water} * g(x, y, z)\, dx\, dy\, dz \tag{12}$$

where M_{boat} is the total mass of the boat, $g(x, y, z)$ is a function expressing the submerged volume, and the region S is the submerged volume.

As $g(x, y, z)$ is dependent on the waterline, we can solve for the draft of the boat. Once we know the draft, we are able to determine the submerged volume by using draft as an upper limit of integration.

2.1.2.2 Center of Buoyancy

We can solve for the center of buoyancy by finding the weighted average of the hydrostatic pressures on the hull. This is similar to finding the center of mass of the hull, but it differs in that you are integrating over the submerged region. Thus, it can be computed through,

$$x_{COB} = \frac{1}{M_{region}} \iiint_R x\rho\, dV \tag{13}$$

where the only differences are the region you are integrating (expressed as R) and ρ is the density of water. This can then be repeated for the y and z components.

2.1.3 Righting Moment

The vector between the center of buoyancy and center of mass create a moment arm. This vector can be expressed by,

$$\vec{r} = COB - COM \tag{14}$$

The force of buoyancy will always act perpendicular to the waterline (see Figure 5); and thus, its direction becomes,

$$\hat{F_B} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{15}$$

where, $\hat{F_B}$ is a unit vector in the direction of the force of buoyancy($\vec{F_B}$). The force of buoyancy vector can be represented by,

$$\vec{F_B} = |F_b| * \hat{F_B} \tag{16}$$

We can then find the resulting torque by,

$$\vec{\tau} = \vec{r} \times \vec{F}_B \tag{17}$$

2.2 Mathematical Representation of Our Boat Design

The hull of the boat modelled is represented by the equations,

$$y = (\frac{x}{1.599})^2 \tag{18}$$

where $-3.198 \leq x \leq 3.198$.

$$y = \frac{9 * z^4}{20000} \tag{19}$$

where $-10 \leq z \leq 10$ and $0 \leq y \leq 4$.

To add a transom to the boat,

$$y < \frac{3}{4}x + 9.25 \tag{20}$$

Each of these equations were adjusted to fit to a 1 inch coordinate system. Thus, the dimensions for the boat can be defined by a length of 20 inches, a beam of 6.396 inches, and a height of 4 inches. A 3D representation of the hull is shown in Figure 1.

2.2.1 Modeling

In order to demonstrate the calculations, the heel angle of $30°$ is used.

2.2.1.1 Assumptions

- There is no pitch as heel increases
- The center of mass is at point $(0, 1.95, 0)$
- The effective torque is given by the z-component of $\tau = \vec{r} \times \vec{F}_B$
- The density of the hardboard is roughly 990 $\frac{g}{m^3}$

2.2.1.2 Volume and Floating

We can calculate the volume of the hull with,

$$V_{hull} = \left[\int_{-3.198}^{3.198} \int_{-6\frac{2}{3}}^{10} \int_{[(\frac{9}{20000} * z^4)] + |\frac{x}{1.599}|^2]}^{4} dy\, dz\, dx) \right] + \left[\int_{-3.198}^{3.198} \int_{-10}^{-6\frac{2}{3}} \int_{[(\frac{9}{20000} * z^4) + |\frac{x}{1.599}|^2]}^{\frac{3}{4}x + 9.25} dy\, dz\, dx) \right] \tag{21}$$

which is equal to 242.31 inches^3

Mathematica was then used to find the volume of the submerged region. The expression computed is,

$$V_{displaced} = \left[\int_{-6\frac{2}{3}}^{10} \int_{[1.2784(-\tan\theta - \sqrt{-1.56445*d+\tan^2\theta})]}^{[1.2784(-\tan\theta + \sqrt{-1.56445*d+\tan^2\theta})]} \int_{[(\frac{3}{5000}*z^4)]+|\frac{x}{1.599}|^2]}^{6} dy\,dx\,dz) \right]$$

$$+ \iiint_{Transom} dy\,dx\,dz,$$

$$Transom = \{(x,y,z)| -3.198 \le x \le 3.198, [(\frac{9}{20000} * z^4) + \frac{x}{1.599}^2] \le y \le \frac{3}{4}x + 9.25,$$

$$-10 \le z \le -6\frac{2}{3}\} \tag{22}$$

where d is the draft of the boat. The boat will float when the mass of the boat is equal to the mass of water displaced (Equation 10). Therefore, it is found by equating the $V_{displaced}$ expression to the mass of the boat. When the boat is floating the volume displaced is,

$$V_{displaced} = 3.04lbs * (\frac{0.453592kg}{1lbs}) * (\frac{m^3}{1000kg}) * (\frac{61023.7in^3}{1m^3}) \tag{23}$$

which yields $V_{displaced}$=84.1468 inches^3. This is less than the total volume of the boat; and therefore, the boat will float. For the case we were considering, 30° heel, the draft is 1.94439 inches and the volume displaced remains 84.1468 inches^3.

2.2.1.3 Center of Buoyancy

We can find the center of buoyany (Equation 13) by finding the weighted average in the x, y, and z directions. This is shown through,

$$x_{COB} = \frac{1}{V_{displaced}} \left[\int_{-6\frac{2}{3}}^{10} \int_{[1.2784(-\tan\theta - \sqrt{-1.56445*d+\tan^2\theta})]}^{[1.2784(-\tan\theta + \sqrt{-1.56445*d+\tan^2\theta})]} \int_{[(\frac{9}{20000}*z^4)]+|\frac{x}{1.599}|^2]}^{4} x\,dy\,dx\,dz) \right]$$

$$+ \iiint_{Transom} x\,dy\,dx\,dz \tag{24}$$

$$y_{COB} = \frac{1}{V_{displaced}} \left[\int_{-6\frac{2}{3}}^{10} \int_{[1.2784(-\tan\theta - \sqrt{-1.56445*d+\tan^2\theta})]}^{[1.2784(-\tan\theta + \sqrt{-1.56445*d+\tan^2\theta})]} \int_{[(\frac{9}{20000}*z^4)]+|\frac{x}{1.599}|^2]}^{4} y\,dy\,dx\,dz) \right]$$

$$+\iiint_{Transom} y\,dy\,dx\,dz \tag{25}$$

$$z_{COB} = \frac{1}{V_{displaced}}\left[\int_{-6\frac{2}{3}}^{10}\int_{[1.2784(-\tan\theta-\sqrt{-1.56445*d+\tan^2\theta})]}^{[1.2784(-\tan\theta+\sqrt{-1.56445*d+\tan^2\theta})]}\int_{[(\frac{9}{20000}*z^4)]+|\frac{x}{1.599}|^2]}^{4} z\,dy\,dx\,dz)\right]$$

$$+\iiint_{Transom} z\,dy\,dx\,dz \tag{26}$$

In Mathematica, we are able to use the RegionCentroid function to compute the center of buoyancy. In the case we are using, 30° heel, the point is $(0.74, 1.59, 0)$.

2.2.1.4 Righting Torque

We can then compute the moment arm (refer to Equation 14), which in this case is $(0.0187314, -0.0091296, 0)$. Similarly, we can compute $||F_{Buoyancy}||$ as $M_{WaterDisplaced} * gravity$ (refer to Equation 10) which evaluates to $13.5134N$. We can then use (Equation 15),

$$\hat{F_B} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{27}$$

to compute the direction of $\hat{F}_{Buoyancy}$. This evaluates to $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ when the heel is 30 °. When multiplying the direction vecotr by $||F_{Buoyancy}||$, $\vec{F}_{Buoyancy}$ becomes $(-6.75671, 11.703, 0)$. This vector can then be crossed with the moment arm (Equation 17),

$$\tau = \vec{r} \times \vec{F}_B \tag{28}$$

which yields, $(0, 0, 0.15741)$. This means the $\tau_{effective}$ is 0.15741 Nm.

2.2.1.5 AVS Curve

When this process is repeated for a range of heel angles, the angle of vanishing stability can be found. This is the point at which the righting torque will become a destabilizing torque, which will consequently capsize the boat. For the particular boat examined above, the angle of vanishing stability plot is shown in Figure 6. The angle at which the boat will capsize for this particular case is roughly 129.557 °.

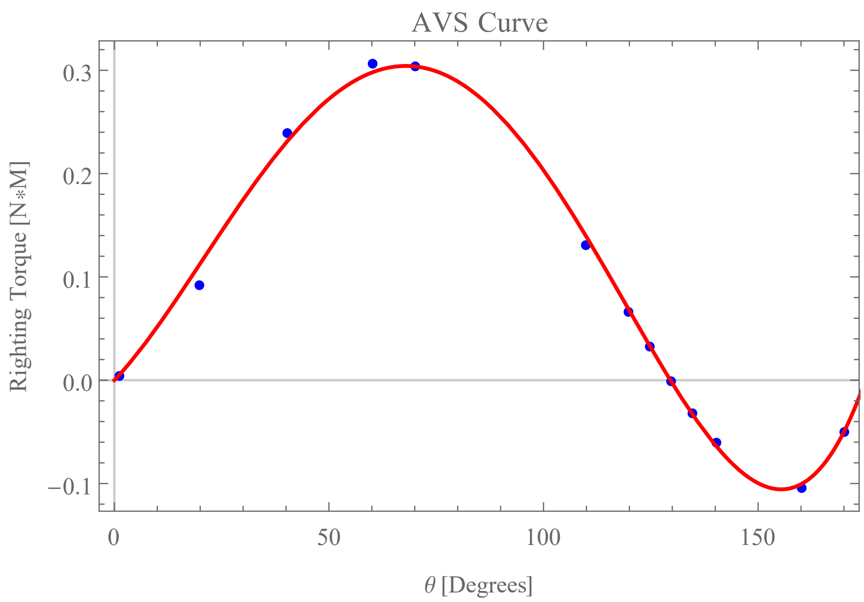


Figure 6: The AVS for this particular boat shape is rouhgly 129.557°.

2.2.1.6 Waterline, COM, and COB Diagrams

Figures 7-12 show the effect different heel angles have on the center of mass, center of buoyancy, and waterline. The center of mass is shown as the blue point in the figures, and the center of buoyancy is shown as the red point.

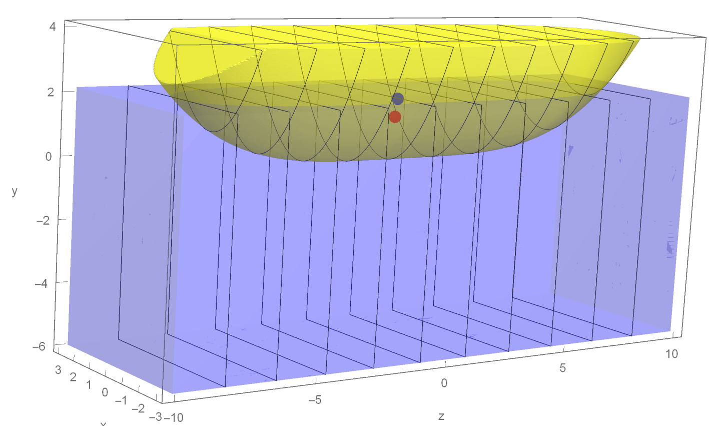


Figure 7: For 0° heel, the center of mass and center of buoyancy are in a line vertically, which yields a torque of zero.

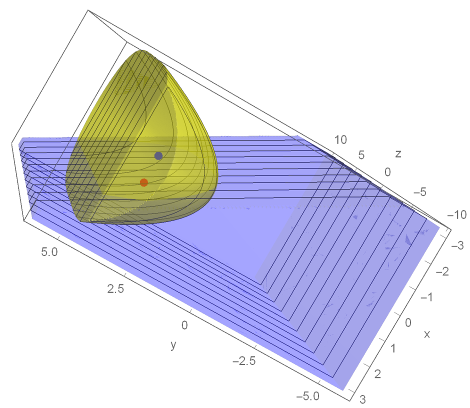


Figure 8: For 60° heel, the offset of the center of buoyancy and center of mass will create a positive righting torque.

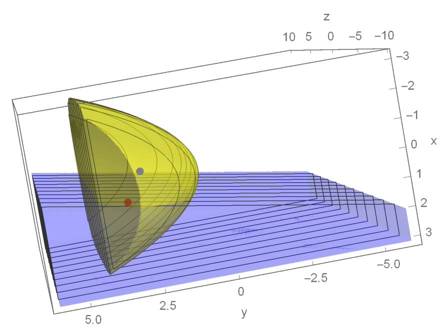


Figure 9: For 100° heel, the offset of the center of buoyancy and center of mass will create a positive righting torque.

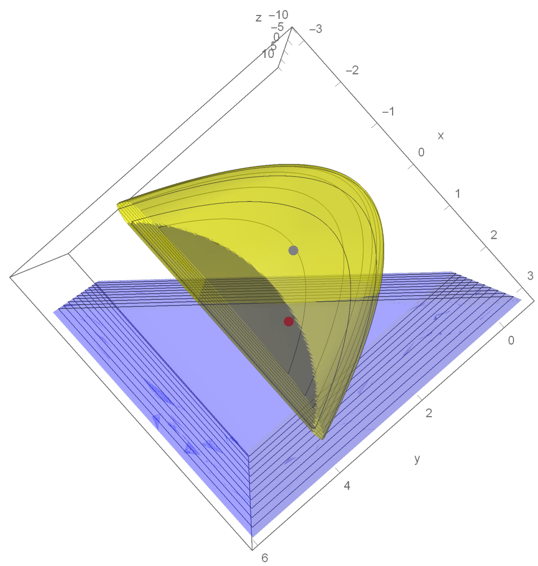


Figure 10: For 130° heel, the center of mass and center of buoyancy are nearly aligned vertically, which yields no torque .

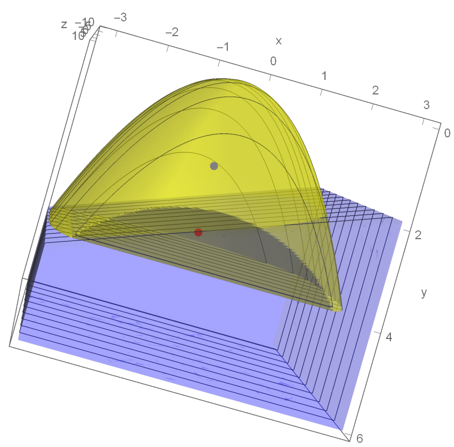


Figure 11: For 160° heel, the moment arm created between the center of mass and center of buoyancy creates a destabilitizing torque.

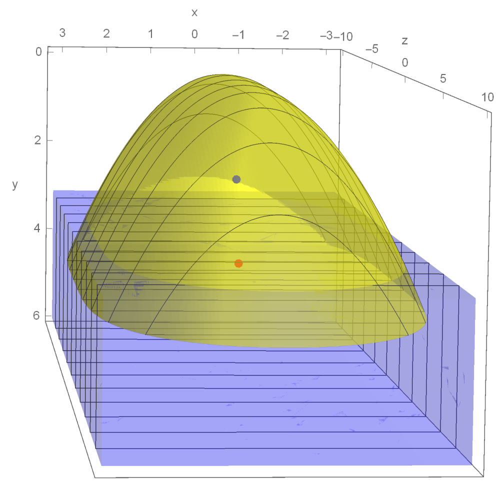


Figure 12: For 180° heel, the center of mass and center of buoyancy are aligned vetically and will yield no torque.

When considering Figures 7-12 and the stability curve, we establish at 0°, 129.557°, and 180° heel there is no torque, so the boat is in equilibrium. When the boat is heeled to angles between 0° and 129.557°, the resulting torque will right the boat; whereas, at angles between 129.557° and 180° the resulting torque is a destabilizing torque, and the boat will capsize.

3 Expected Performance

3.1 Stability

When looking at the stability curve in Figure 6, we can see there is no torque when the boat has no heel. However, when the boat begins to heel, the righting torque increases, which will bring the boat back to center. With this in mind, the boat will float flat, as this is the point where there will be no righting torque. Considering the system in static equilibrium, the sum of the torques and forces on the boat will be zero when the boat is flat. This means the boat will be in equilibrium when it is flat, and will tend towards flat when it is slightly heeled.

3.2 Angle of Vanishing Stability

We are able to generate an angle of vanishing stability curve (shown in Figure 6) by finding the effective torque at various angles of heel, and then generating a line of best fit. For our particular boat design, the line of best fit is expressed by the function,

$$f(x) = 2.718*10^{-12}*x^5+6.158*10^{-9}*x^4-1.87*10^{-6}*x^3+\frac{x^2}{10000}+0.004344*x \tag{29}$$

When solving for the angle of vanishing stability, we are able to find the zero within the range of 120° and 140°. For the case of this function, the zero is at 129.557. Which means this is the angle we anticipate the boat will capsize at. However, our model accounts for the heel but not pitch of the boat. In practice, the boat will pitch as it heels at higher angles. It will bring the center of buoyancy forward in the boat creating a longer moment arm. The result is a larger righting torque acting on the boat.

3.3 Speed

. The maximum speed of our boat will be determined by its "hull speed". This is the point at which the waves created from the bow slicing through the water and the stern with its displacement contstructively interfere to create a large drag force. At this point, the boat's speed increase will be negligible. Hull speed can be given by[1],

$$V_{hull} = 1.34 * \sqrt{w} \tag{30}$$

where V_{hull} is the hull speed in knots and w is the waterline length (which is the length of the boat at the waterline) in feet. To find the waterline length,

$$2.15601 = (\frac{9 * z^4}{20000}) \tag{31}$$

where 2.15601 is the draft of the boat at 0 degrees heel and $y = (\frac{9*z^4}{20000})$ is the expression that defines our boat in the y-z plane. This yields (for real solutions) $z_1 = -8.31974$ and $z_2 = 8.31974$. Therefore, the waterline length is 16.6395 inches or 1.35566667 feet. Used in the hull speed expression, we determine the anticipated speed for our boat to be 1.57792 knots or 0.81175 $\frac{m}{s}$.

4 Mathematica Program

The Mathematica code used to generate the angle of vanishing stability plot is shown in Figures 13 and 14.

[1]https://www.sailboat-cruising.com/hullspeed.html

```
In[◦]:= rightingmoment[theta_] := Module[{},
   boat = ImplicitRegion[((9/20000 * z^4) + ((x/1.599)^2)) ≤ y ≤ 4 && y ≤ 0.75 * z + 9.25,
     {{x, -3.198, 3.198}, {y, 0, 4}, {z, -10, 10}}];
   (*massboat=0.05*RegionMeasure[boat,Method→NIntegrate, WorkingPrecision→3];
   (*Density of Hardboard*)*)
   massboat = 3.04; (*lbs*)
   angle = theta Degree;
   water =
    ImplicitRegion[If[angle ≤ 90 Degree, y ≤ Tan[angle] * x + d, y ≥ Tan[angle] * x + d] &&
      -3.198 ≤ x ≤ 3.198 && -6 ≤ y ≤ 6 && -10 < z < 10, {x, y, z}];
   under = RegionIntersection[boat, water];
   Quiet[waterline = FindRoot[
      0.0361273 * RegionMeasure[under, Method → NIntegrate, WorkingPrecision → 3] == massboat,
      {d, -6, 6}, AccuracyGoal → 3, PrecisionGoal → 3]]; (*Density of Water*)
   draft = N[d /. waterline[[1]]];
   cob = RegionCentroid[(under /. {d → draft}), WorkingPrecision → 3];
   (*Geometric Center because water has constant density*)
   com = {0, 1.95, 0};
   rvector = 0.0254 * (cob - com); (*Inches to Meters*)
   Fb = massboat * 0.453592 * 9.8; (*Lbs to Kg*)
   fvector = Fb * {-Sin[angle], Cos[angle], 0};
   torque = Cross[rvector, fvector];
   efftorque = torque[[3]]
  ]

In[◦]:= AVS = Table[{theta, rightingmoment[theta]},
   {theta, {1, 20, 40, 60, 70, 110, 120, 125, 130, 135, 140, 160, 170}}]

Out[◦]= {{1, 0.00422778}, {20, 0.0925063}, {40, 0.237948}, {60, 0.305722}, {70, 0.302703},
  {110, 0.130871}, {120, 0.0650667}, {125, 0.0318604}, {130, -0.000582155},
  {135, -0.0314244}, {140, -0.0595096}, {160, -0.104048}, {170, -0.0491263}}
```

Figure 13: A module is used to wrap the calculations for finding the righting moment at a particular heel angle. The module is then run with various heel angles to create the list "AVS" which gives the resultant torque at these angles.

```
In[∘]:= (*model=a*x^5+b*x^4+c*x^3+d*x^2+e*x+f*)
        model = a * x^5 + b * x^4 + c * x^3 + 1 * 10^-4 * x^2 + f * x
        params = {{a}, {b}, {c}, {d}, {f}}
        bestparams = FindFit[AVS, model, params, x]
        bestmodel = model /. bestparams
        bestmodel /. {x → 180}
        Show[ListPlot[AVS, FrameLabel → {"θ [Degrees]", "Righting Torque [N*M]"},
          PlotLabel → "AVS Curve", PlotStyle → Blue, PlotTheme → "Scientific"]
          (*ListPlot[AVS,FrameLabel→{θ Degrees,Righting Torque},PlotLabel→"AVS Curve",
          PlotTheme→"Scientific",Joined→True ]*), Plot[bestmodel, {x, 0, 180}, PlotStyle → Red]]
        Root[bestmodel,
         3]
```

Out[∘]= $f\,x + \frac{x^2}{10\,000} + c\,x^3 + b\,x^4 + a\,x^5$

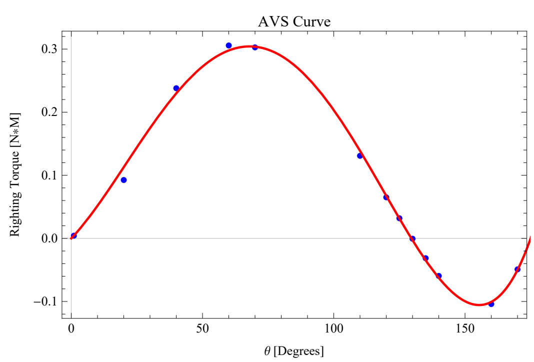
Out[∘]= {{a}, {b}, {c}, {d}, {f}}

Out[∘]= $\{a \to 2.71812 \times 10^{-12},\ b \to 6.15775 \times 10^{-9},\ c \to -1.87413 \times 10^{-6},\ d \to 0.,\ f \to 0.00434496\}$

Out[∘]= $0.00434496\,x + \frac{x^2}{10\,000} - 1.87413 \times 10^{-6}\,x^3 + 6.15775 \times 10^{-9}\,x^4 + 2.71812 \times 10^{-12}\,x^5$

Out[∘]= 0.0699624

Out[∘]=



Out[∘]= 129.557

Figure 14: Once we have found the righting moment at various points, we can then generate a line of best fit. The anticipated angle of vanishing stability is where this function is equal to zero within the range we are considering. For our boat it is 129.557°.