Modeling a Passive House Using Circuitry

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1 Background

Using ordinary differential equations (ODE's), we can model temperature change over time through heat transfer between different components within a system. One place this method is useful is when designing a house. Although most houses today have air conditioning and heating, this convenience consumes massive amounts of energy in a world where we need to think about every bit of energy we use. The average house in New England consumed more than 17,000 kilowatt hours of energy in space heating machines through all fuels in 2015 alone. To contextualize this value, that much energy could charge a Nissan Leaf car battery, which holds 40 kilowatt hours, approximately 432 times and you could drive for a total of about 65,000 miles. Climate change is a very real issue and we need to reduce our energy consumption to save the planet. Instead of using heating and air conditioning, we can instead design houses that passively heat themselves using only the sun as a source of heat. Most importantly, how can we build a sustainable passive house in the frigid Boston climate and still keep the internal temperature of the house comfortable. Before diving into how we can design and model a home like this, we will outline some of the technical components necessary to understand the model.

1.1 Technical Context

Before we jump into heat flow mathematical models, we must first understand heat flows. The following table shows the common variables with units used in the relevant heat flow equations.

Table 1: Variables in Thermal Model and Units

Variable Used	Significance	Unit
k	Thermal Conductivity	$[W/m-K]$
h	Convective Heat Transfer Coefficient	$[W/m^2-K]$
$A_{Conduction}$	Cross Sectional Area	$[m^2]$
$A_{Convection}$	Surface Area of Material	$[m^2]$
$\dot{Q}$	Heat Transfer Rate	$[W]$
ΔT	Temperature Difference Across a Boundary	$[K]$
m	Mass	$[Kg]$
c	Specific Heat	$[J/Kg-K]$
C	Heat Capacity	[J/K]
t	Time	[s]
U	Internal Energy of a System	[J]

Convection is the transfer of heat through a fluid such as air. If you have felt a nice cool breeze, there was a heat transfer from your body to the air by convection. Heat transfer through convection from the surface of one material to a fluid can be computed with the equation below.

$$hA_{Convection} * |\Delta T| \tag{1}$$

On the other hand, conduction is the transfer of heat through a material. This heat transfer is like when you touch a warm surface. The distinguishing characteristic between convection and conduction is that there is fluid flow in convection; whereas, in conduction there is no material movement. A differential equation expressing the transfer of heat due to conduction is,

$$\frac{kA}{L} * |\Delta T| \tag{2}$$

In order to make a circuit equivalent for a thermal system, there are several analogies that must be made between physical characteristics and electrical components. A heat flow is modeled as a resistor, a specific heat becomes a capacitor, power input is analogous to a current source, and temperature becomes a voltage. In doing this, we are saying the equivalent units are:

Table 2: Unit Analogies for Thermal and Electrical Systems

Thermal Unit	Electrical Unit
Power [W]	Current [A]
Temperature [T]	Voltage [V]
Mass [Kg] * Specific Heat [J/Kg-K]	Capacitance [F]
Internal Energy [J]	Charge [C]

1.1.1 Resistor and Capacitor Equivalent Values

For conductive heat transfer, we use

$$R_{Conductive} = \frac{L}{k * A} \tag{3}$$

to find the equivalent resistor value. In this expression L is the distance over which conduction occurs, k is the conductive heat transfer coefficient, and A is the cross sectional area. Similarly, for conductive heat transfer, the equation for an equivalent resistor becomes,

$$R_{Convective} = \frac{1}{h * A} \tag{4}$$

where h is the convective heat transfer coefficient and A is the surface area of the material. An equivalent capacitor is found using,

$$C = m * c \tag{5}$$

where m is the mass of the material and C is the heat capacity. By utilizing these analogous equations, we can model the thermal heat transfer through a house using current flow through a circuit that represents the house.

2 Design

The goal of the design was to make sure that the internal air temperature stayed in a warm comfortable range between 18 and 25 degrees Celsius. We were able to do this by tuning the thickness of thermal masses and insulation on a fixed size house.

2.1 Physical Design

We decided to build a house (shown in Figure 1) using dimensions of 5 meters long, 5.1 meters wide, and 3 meters tall. The roof has an overhang of 0.9 meters to ensure that during the Summer when the sun is higher in the sky and shines towards the house at a 72 degree angle no sunlight reaches the house, but during the winter, when the sunlight comes at a 25 degree angle, the entire tiled floor is covered with sunlight. The window is 2.6 meters tall and the entire 5 meter length of the house. The flooring is a 15 centimeter thick tile, and the walls which cover every wall and the ceiling are a 25 centimeter thick layer of bricks. The massive thermal capacity for that much bricks allows the house to hold heat very well and reduce the fluctuation amplitude over the course of 24 hours. Then outside of the bricks is a 3 centimeter thick layer of insulation. We modeled the house as if it was on stilts to make the heat transfer through the floor easier to model as going into the air. We came to these values through plugging them into our model described in the next section so that the air temperature would vary from about 20 to 25 degrees Celsius. We found that to reduce the fluctuation in the house's temperature range, we had to make our thermal masses very large and to get the temperature right we had to make sure that enough heat was escaping through the insulation, which is why our insulation is relatively thin and our flooring and walls are relatively thick.

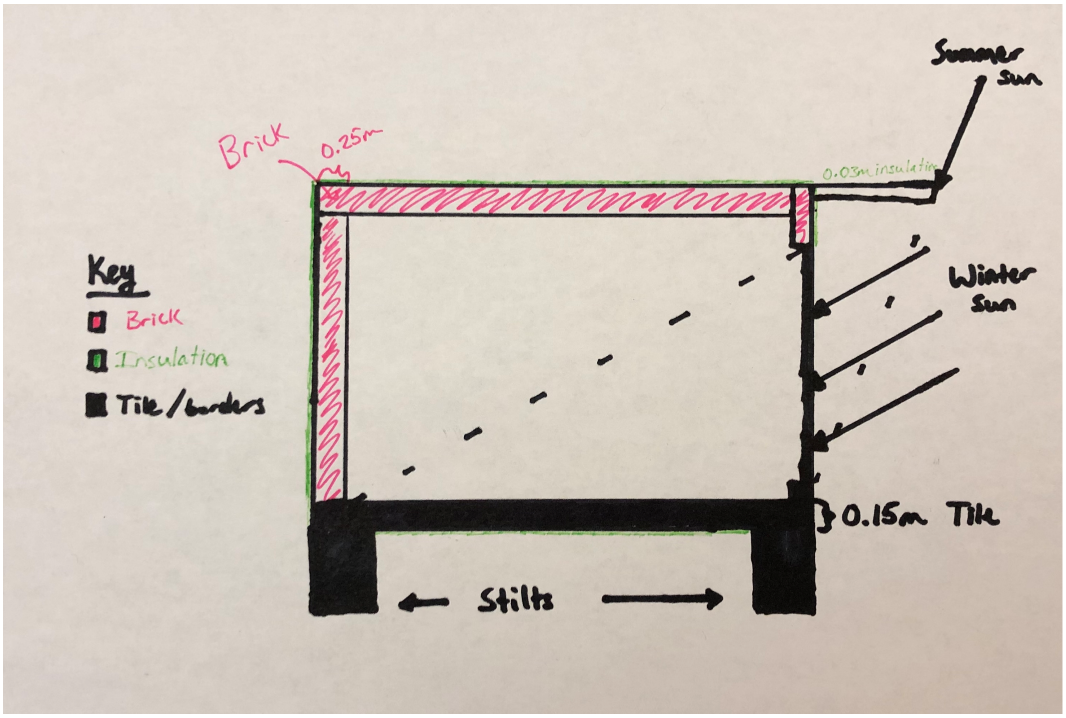

Figure 1: Diagram of the house we are modeling.

3 Modeling

When creating any model, we must consider the imperfections and factors that we can't always account for. Before diving into the system of ODE's we used to model the house temperature, we must first outline the goal of the model and then all the assumptions made.

3.1 Goal of the Model

By creating this model, we want to be able to find the temperature of the air given values for an input solar flux and outside temperature.

3.2 Assumptions

We can start by outlining the environmental assumptions we made. The first assumption we made was to model the average solar flux, $\dot{Q}_{in}$, over the course of a day through a south facing window with the equation defined below.

$$\dot{Q}_{in} = A_{window} * (-0.361 * \cos(t * \pi/(12 * 3600)) + 0.2243 * \cos(t * \pi/(6 * 3600)) + 0.2097) \left[\frac{kW}{m^2}\right] \quad (6)$$

In this function, derived as a curve of best fit for solar flux, t represents the time since the beginning of a day in seconds and A_{window} represents the area of the window. Likewise, we modeled the outside air throughout the day using the equation,

$$T_{outside} = 270 + 6 * \sin\left[\frac{(2 * \pi * t)}{(24 * 3600)}\right] + \frac{3\pi}{4} \quad [K] \tag{7}$$

We will also assume that the terrain is not blocking heat from entering through the window at any point in time. In addition, we are assuming that the sun is only heating the objects within the house and not any of the exterior pieces such as the walls of the house. Finally, we are assuming that the house is basically floating and the heat that would be lost to the ground is modeled as heat transferred through the insulation into the outside air like the other walls. While in reality materials will slowly warm up and may warm quicker at one physical location than another, we are assuming that all materials in the system are individually at uniform temperature. The insulation could be considered as a heat storage, but we are going to assume that it has no storage and heat is transferred through it to the outside air. Now that we have established our assumptions for the house, we can look into the physical model.

3.3 Physical Model

The heat transfer through our house is as follows

- Solar flux comes from the sun and heats the tile within the house
- The tile will lose heat to the inside air through convection and through the outside air through the insulation on which it rests.
- The air will lose heat to the brick walls and through the window to the outside air through convection
- The brick walls will lose heat through the insulation to the outside air.

Using the flow of heat we can write three differential equations to represent the rate of change of the tile, the inside air, and the bricks. The rate of change of temperature in the tile is given by,

$$m_{tile}c_{tile}\frac{\partial T_{tile}}{\partial t} = \dot{Q}_{in} - hA_{tile}\left(T_{tile} - T_{air}\right) - \frac{K_{ins} * A_{tile}}{L} * \left(T_{tile} - T_{out}\right) \tag{8}$$

Similarly, the rate of change of temperature for the inside air is given by,

$$m_{air}c_{air}\frac{\partial T_{air}}{\partial t} = hA_{tile}\left(T_{tile} - T_{air}\right) - hA_{window}\left(T_{air} - T_{out}\right) - hA_{brick}\left(T_{air} - T_{out}\right) \tag{9}$$

Finally, the temperature of the bricks is given by,

$$m_{brick}c_{brick}\frac{\partial T_{brick}}{\partial t} = hA_{brick}\left(T_{air} - T_{brick}\right) - \frac{K_{ins} * A_{ins}}{L} * \left(T_{brick} - T_{out}\right) \tag{10}$$

In these equations, T_{air} is the temperature of the inside air and T_{out} is the temperature of the outside air. Using Mathematica to solve the system of ODE's, we were able to tune the constant values such as the thickness of the tile, brick, and insulation to reach a temperature range we were comfortable with. Figures 2-4 show the temperatures of the three thermal capacitors we modeled as a function of time over 2 days.

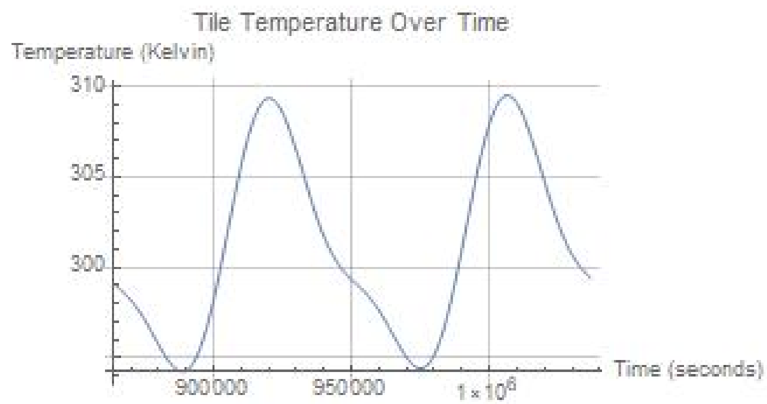

Figure 2: Tile temperature from day 10 through day 12 of the model.

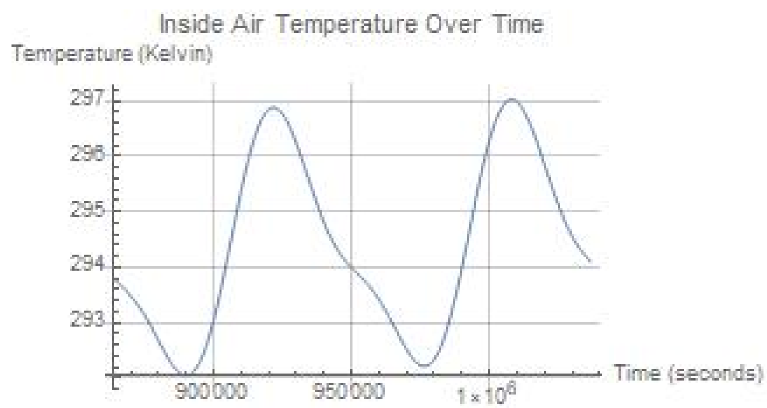

Figure 3: Air temperature from day 10 through day 12 of the model.

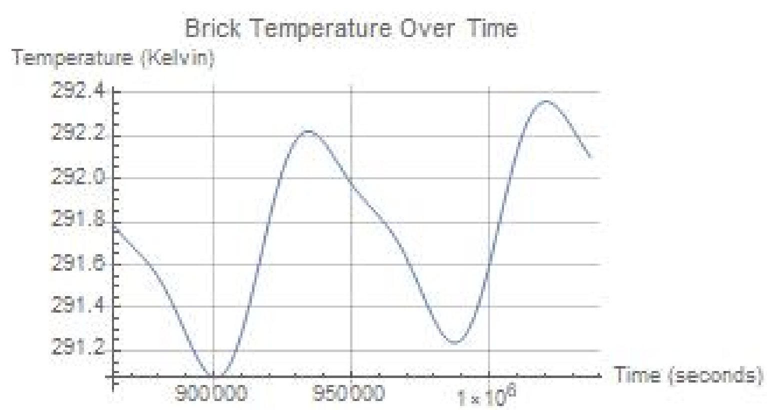

Figure 4: Brick temperature from day 10 through day 12 of the model.

We can see that the air temperature oscillates from about 292 Kelvin (19 degrees Celsius) to 297 Kelvin (24 degrees Celsius) and fits our model goal. When increasing the tile and brick thickness, it was increasing the thermal mass and reducing the fluctuation in temperature over the course of the day while also slightly increasing the average temperature of the house. While changing the insulation thickness, we were changing how much heat left the system, so increasing the thickness drastically increased the average temperature inside the house. The final thicknesses are given in the Physical Design section.

3.4 Electrical Model

After we tuned all of the wall thicknesses and found a comfortable model using ODE's, we decided to experimentally model the house with a circuit. As we mentioned in the *Technical Context*, using equations 3-5, we can model components with heat capacity as capacitors and heat flows as resistors. However, there were some issues to modeling the house to achieve a purpose of modeling many days very quickly while maintaining the physics of the system.

3.4.1 Scaling the Physical Model to Working Electrical Values

An important idea in applied math and modeling is scaling equations. With our model, we would have to run a circuit simulation for 24 hours just to get a days worth of temperature data and the voltage output representing outside temperature would be above 200V assuming a 1:1 relationship between temperature in Kelvin and volts. Furthermore, some of the capacitor values calculated would have to be over 1,000,000 farads. To avoid this issue, we scaled the differential equations representing the house to a circuit. Our goal in doing this was to scale the time to represent each day as a shorter duration. This allowed us to model several days of the thermal model in a couple of seconds. Similarly, we scaled the temperature range to a voltage between 0 and 5 volts. For our system to maintain the same physics, we needed to ensure that the $R * C$ value (time constant) remained the same after scaling. Mathematically, we can represent time, t, as a new time, t_{new}, multiplied by some scaling constant as shown by

$$t_{new} = \alpha * t \tag{11}$$

Similarly, we can scale temperature (T) to a smaller range such that we can feasibly generate that value as a voltage (V). That relationship is represented with

$$V = \beta * T \tag{12}$$

Therefore, for a simple thermal model with a single $\dot{Q}_{in}$ and a single $\dot{Q}_{out}$ such a tile that loses heat through convection to the air, you can take the ODE,

$$m * c * \frac{\partial T_{tile}}{\partial t} = \dot{Q} - hA(T_{tile} - T_{air})$$

and scale it such that

$$\hat{T}_{tile} = \frac{T_{tile}}{T_{air}}$$

and

$$\hat{t} = \frac{t}{\alpha}$$

where $\hat{T}_{tile}$ is the new scaled temperature and $\hat{t}$ is the new scaled time. Therefore, we have

$$\frac{T_{air}}{\alpha} * \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\dot{Q}}{mc} - \frac{hAT_{air}}{mc}(\hat{T} - 1)$$

which can be simplified to

$$\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\alpha \dot{Q}}{mcT_{air}} - \frac{\alpha hA}{mc}(\hat{T} - 1)$$

which can be simplified to

$$\frac{\partial \hat{T}}{\partial \hat{t}} = a - b(\hat{T} - 1)$$

where $a = \frac{\alpha \dot{Q}}{mcT_{air}}$ and $b = \frac{ahA}{mc}$.

On the other hand, the circuit equivalent becomes

$$C\frac{\partial V_{tile}}{\partial t} = I - \frac{(V_{tile} - V_{air})}{R}$$

and scale it such that

$$\hat{V}_{tile} = \frac{V_{tile}}{V_{air}}$$

and

$$\hat{t} = \frac{t}{\beta}$$

Therefore, we have

$$\frac{V_{out}}{\beta} * \frac{\partial \hat{V}}{\partial \hat{t}} = \frac{I}{C} - \frac{V_{out}}{RC}(\hat{V} - 1)$$

which we can rearrange to become

$$\frac{\partial \hat{V}}{\partial \hat{t}} = \frac{\beta I}{CV_{air}} - \frac{\beta}{RC}(\hat{V} - 1)$$

This can be simplified to

$$\frac{\partial \hat{T}}{\partial \hat{t}} = a - b(\hat{V} - 1)$$

where $a = \frac{\alpha I}{CV_{air}}$ and $b = \frac{\beta}{RC}$.

The important part of this is showing that the thermal and circuit differential equations can be expressed in the same form. In the scaling of this system, we begin by choosing a value for α. We scale it such that $\hat{t}$ becomes a desired scaled time. For example, if $\alpha = \frac{1}{86400}$, then every second in the model corresponds to 1 day in the physical world. Similarly, β is set to create a scaled temperature where we can feasibly use a voltage to represent the temperature. By setting the a and b from the thermal and circuit systems equal, we can find the new input current. We used this idea to scale all of the ODE's for the circuit model.

3.4.2 Electrical Circuit of the System

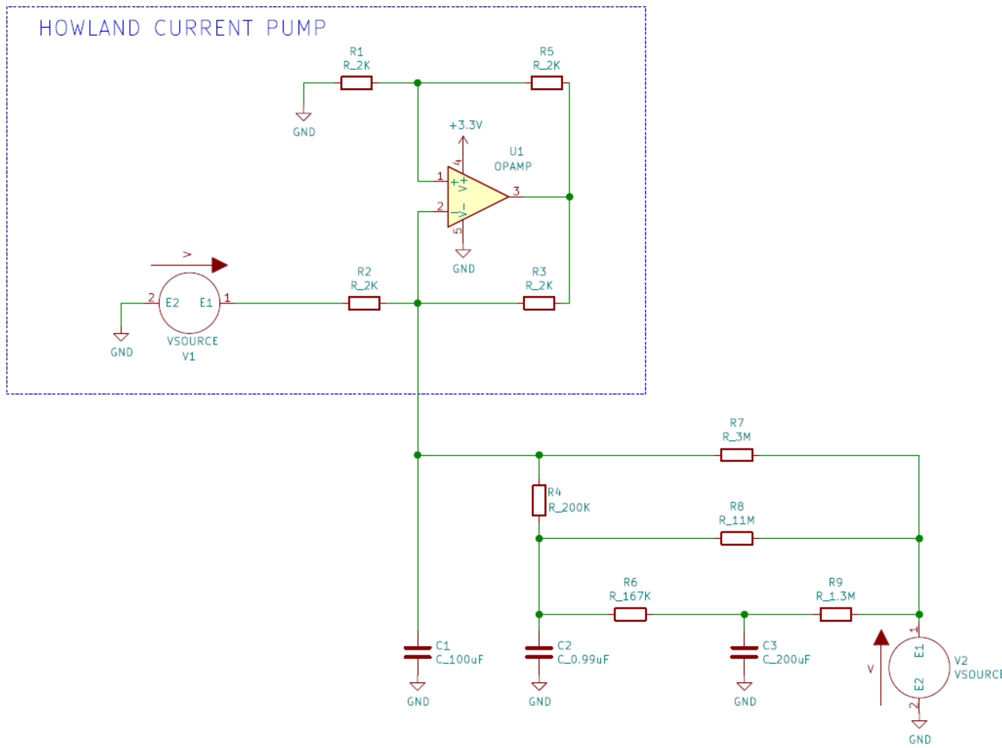

Figure 5: Electrical circuit equivalent of the heat transfer throughout the home.

By utilizing a Howland current pump design with an op-amp and a voltage input (VSOURCE V1) oscillating as a function, we were able to create the scaled $\dot{Q}_{in}$ current entering the circuit representing the scaled house. C1 is the heat capacity of the tile, while R4 is the heat transfer into the inside air and R7 is the heat transfer through the insulation to the outside air. From R4, C2 begins to fill up which represents the heat capacity of the inside air. R6 is the heat transfer from the inside air into the bricks represented by C3 and R8 is the heat transfer through the window into the outside air. R9 is the heat transfer from the bricks through the insulation to the outside air. The outside air is modeled by the equation (7) as VSOURCE V2 in the schematic. Thus we have the completed circuit and a scaled model where we can measure the voltage at any point to scale it back to temperature.

To find the values of resistors needed in the Howland current pump, we used the equation

$$I_{out} = \frac{VSOURCE}{R_2} - \frac{V_{load}}{\frac{R_3}{\frac{R_3}{R_2} - \frac{R_5}{R_1}}} \tag{13}$$

where V_{load} is the voltage across whatever is connected to the output of the current source (load). To ensure the current is independent of the voltage across the load, we know $\frac{V_{load}}{\frac{R_3}{\frac{R_3}{R_2} - \frac{R_5}{R_1}}} \to \infty$. Therefore $\frac{R_3}{R_2} = \frac{R_5}{R_1}$. In our system, these values were all 2000 Ω as it satisfies this condition and allows us to set the input voltage to a reasonable value (between 0 and 5 volts). The actual circuit we built is shown below in Figure 6.

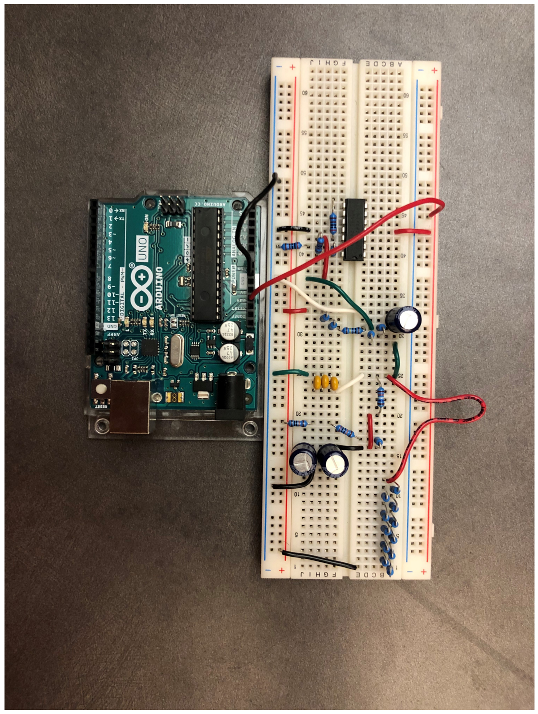

Figure 6: This is the physical circuit we built and used to collect data as represented by the schematic in Figure 5.

3.4.3 Experimental Scaled Simulation

To scale the system, we chose $\beta = 21600$ to scale one day to four seconds. Similarly, we divided all temperatures by 270 to normalize them and scale them to a reasonable range of voltage values that we could use. Using Equations 3-5, we can find values for R and C. When doing this, we must scale these values to a reasonable range, while keeping the time constant (τ or $R * C$) to preserve the physics of the initial model. We found that we must divide the capacitance by some scale factor and multiply the resistor by the same scaling factor in order to reduce the capacitance to a reasonable value, increase the resistance to a reasonable value, and maintain τ. In this case, we had to use a scale multiple of 100,000,000. Finally, we had to also scale the power in and the temperature outside to the same time and voltage range. The scaled equations are shown below.

$$\dot{Q}_{scaled} = \frac{1}{100000000} * A_{window} * (-0.361 * \cos(21600 * t * \pi/(12 * 3600)) + 0.2243 * \cos(21600 * t * \pi/(6 * 3600)) + 0.2097) \left[\frac{kW}{m^2}\right] \tag{14}$$

To find the input voltage that corresponds with the computed current, we multiply by $\dot{Q}_{scaled}$ by 2000 as explained by the Howland current pump calculations shown above.

$$T_{outside} = \frac{270 + 6 * \sin\left[\frac{(2*\pi*t*21600)}{(24*3600)}\right] + \frac{3\pi}{4}}{270} \quad [K] \tag{15}$$

4 Experimental Model Results

After running the circuit simulation and collecting the voltage representing the air temperature over time, we plotted them in Figure 7 and saw a very similar shape to the physical model, but the actual values were slightly off.

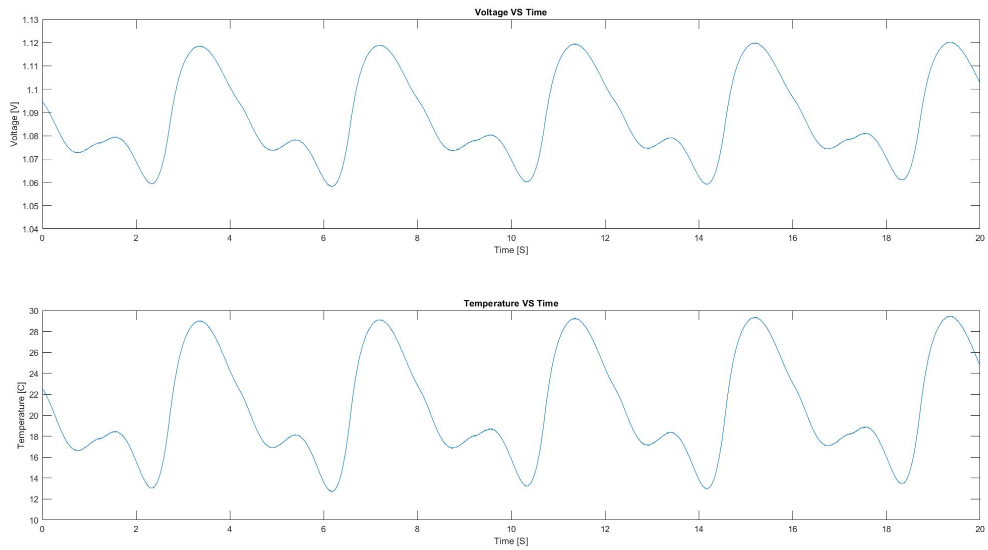

Figure 7: Measured voltage and computed temperature (in C) for the air temperature.

By measuring the voltage and converting it to an equivalent temperature in Celsius, we find that the electrical model shows a temperature oscillation of 14 degrees to 28 degrees. When actually using resistors and capacitors, some of the values had to be rounded due to physical constraints in finding the components. Although this should not change the system much, we are computing the inside air temperature based on changes on the scale of a few millivolts, meaning the difference between the maximum and minimum voltage was less than 0.01 volts. The rounding of these values, therefore, significantly contributed to the inaccuracy of the model. However, the model served its purpose to take a large complex system and scale it into a quick simulation to estimate what will happen to the temperature at different components and at different times.

5 Practicality

Now that we have analyzed the system and modeled it, we can dive into the practicality of living in this house. Compared to our homes, the maximum temperature is pretty warm, and the minimum is slightly on the cold end. One potential fix to this is to incorporate a solar cell system to power a small heater or fans to minimize the fluctuation in temperature over the course of 24 hours. Otherwise, the house is pretty comfortable to live in, though it is pretty small.

6 Sources

https://www.eia.gov/consumption/residential/data/2015/ce/pdf/ce6.1.pdf
https://electricityplans.com/kwh-kilowatt-hour-can-power/
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