Simulation of Mercury's Perihelion Precession using Symplectic Integrators

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October 30, 2025

1 Introduction

The precession of Mercury's perihelion is a classic problem in celestial mechanics that cannot be fully explained by Newtonian gravity. The observed precession is slightly faster than predicted by classical mechanics, a discrepancy that was famously explained by Albert Einstein's theory of General Relativity.

This report details a numerical simulation of Mercuryś orbit to study this precession. We use a symplectic integrator to solve the equations of motion, including a correction term to the Newtonian potential that models the relativistic effects. The goal is to numerically verify the relationship between the strength of this correction and the rate of precession, and to reproduce the observed precession of Mercury.

2 Numerical Method

To simulate the orbit, we need to solve the equations of motion for a body orbiting the Sun under a modified gravitational potential. The Hamiltonian of the system is given by:

$$H = \frac{p_x^2 + p_y^2}{2m} - \frac{GMm}{r}\left(1 + \frac{\alpha}{r^2}\right) \tag{1}$$

where $r = \sqrt{x^2 + y^2}$, m is the mass of Mercury, M is the mass of the Sun, G is the gravitational constant, and α is a parameter that represents the strength of the relativistic correction.

The equations of motion are:

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{m} \tag{2}$$

$$\dot{\mathbf{p}} = -\nabla V(\mathbf{r}) \tag{3}$$

where $V(\mathbf{r})$ is the potential energy part of the Hamiltonian.

For our simulation, we use a 4th order symplectic integrator as described by Neri. This is a type of partitioned Runge-Kutta method that is well-suited for Hamiltonian systems as it preserves the symplectic structure of the phase space, which ensures the long-term stability of the simulation.

The integration algorithm is implemented as a sequence of updates to the position and velocity. For each time step Δt, we perform four stages of updates:

$$\begin{aligned} x_{k+1} &= x_k + a_k v_x \Delta t \\ y_{k+1} &= y_k + a_k v_y \Delta t \\ v_{x,k+1} &= v_{x,k} + b_k F_x(x_{k+1}, y_{k+1})\Delta t \\ v_{y,k+1} &= v_{y,k} + b_k F_y(x_{k+1}, y_{k+1})\Delta t \end{aligned}$$

for $k = 1, 2, 3, 4$. The coefficients a_k and b_k are given by:

$$a_1 = \frac{1}{2(2 - 2^{1/3})}, \quad a_2 = \frac{1 - 2^{1/3}}{2(2 - 2^{1/3})}, \quad a_3 = a_2, \quad a_4 = a_1$$

$$b_1 = \frac{1}{2 - 2^{1/3}}, \quad b_2 = \frac{-2^{1/3}}{2 - 2^{1/3}}, \quad b_3 = b_1, \quad b_4 = 0$$

These coefficients are chosen to ensure the method is of 4th order and symplectic.

3 Kepler Orbit

First, we test our integrator with $\alpha = 0$. This corresponds to the classical Kepler problem. The ellipse is almost closed, because here we examine the behavior of the trajectory for time equal to the 0.95 of Mercury period. The simulation result is shown in Figure 1.

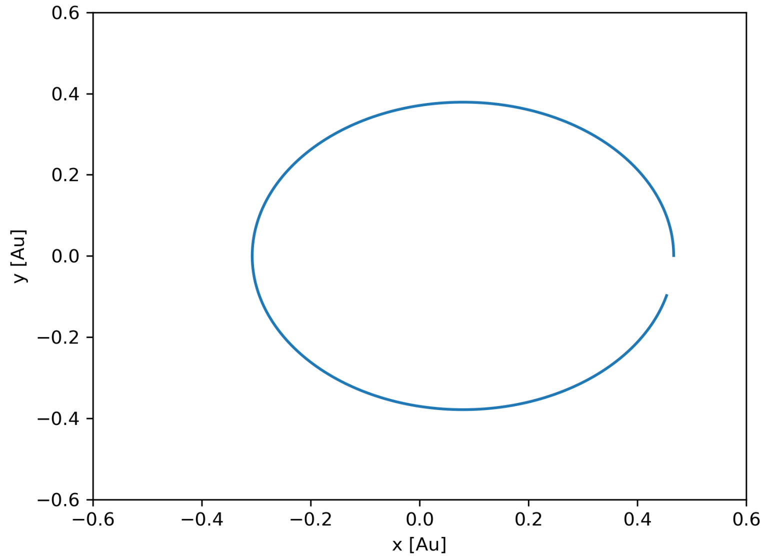

Figure 1: Simulated trajectory of Mercury with $\alpha = 0$. The orbit is a stable, closed ellipse as expected from Newtonian mechanics.

4 Convergence Test

To ensure the accuracy of our simulation, we perform a convergence test by running the simulation with much higer $t_{max} = T_M \cdot 4$. The resulting trajectory is shown in Figure 2. The stability of the orbit even with different time steps demonstrates the robustness of the symplectic integrator.

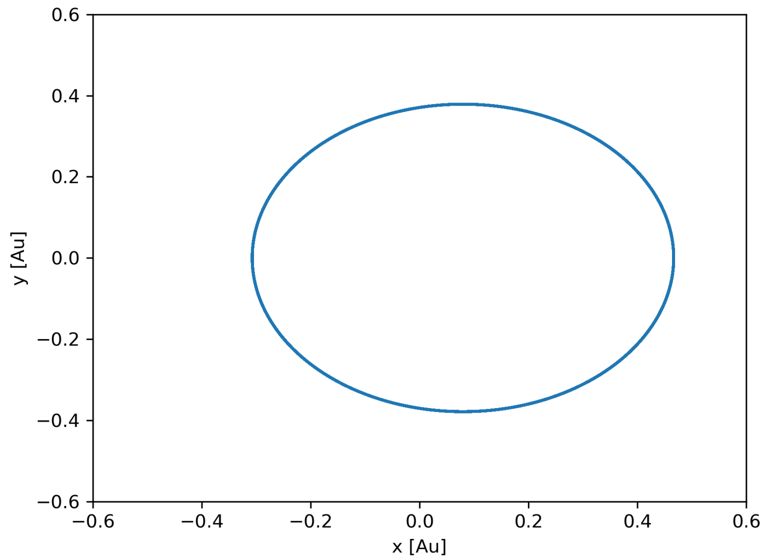

Figure 2: Simulated trajectory with a smaller time step. The orbit remains stable.

5 Precession of Perihelion

Next, we introduce the relativistic correction by setting $\alpha = 0.01$. As shown in Figure 3, the orbit is no longer a closed ellipse. The perihelion (the point of closest approach to the Sun) precesses, or rotates, over time. The plot shows the trajectory over a long period, with the lines indicating the changing positions of the perihelion and aphelion.

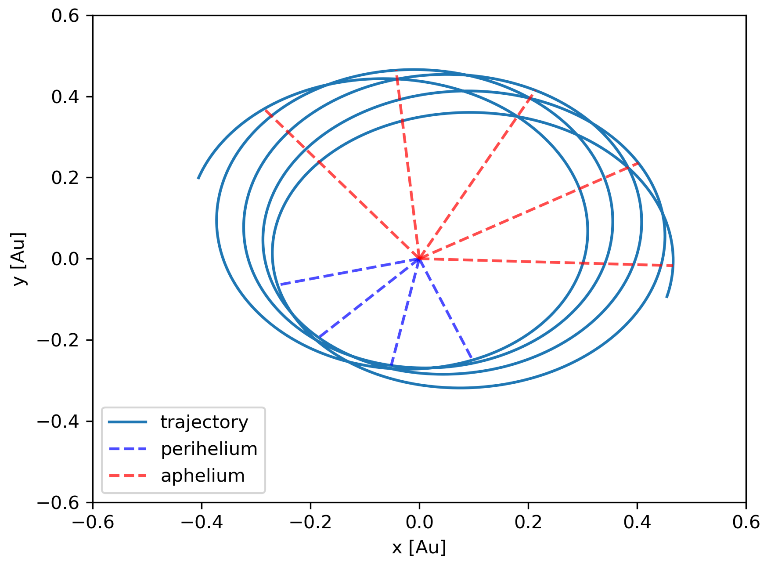

Figure 3: Simulated trajectory with $\alpha = 0.01$. The lines show the precession of the perihelion (blue) and aphelion (red).

6 Precession Rate vs. Alpha

To quantify the precession, we measure the rate of precession ω for different values of α. The theory predicts a linear relationship between ω and α. Figure 4 shows a plot of the measured ω as a function of α.

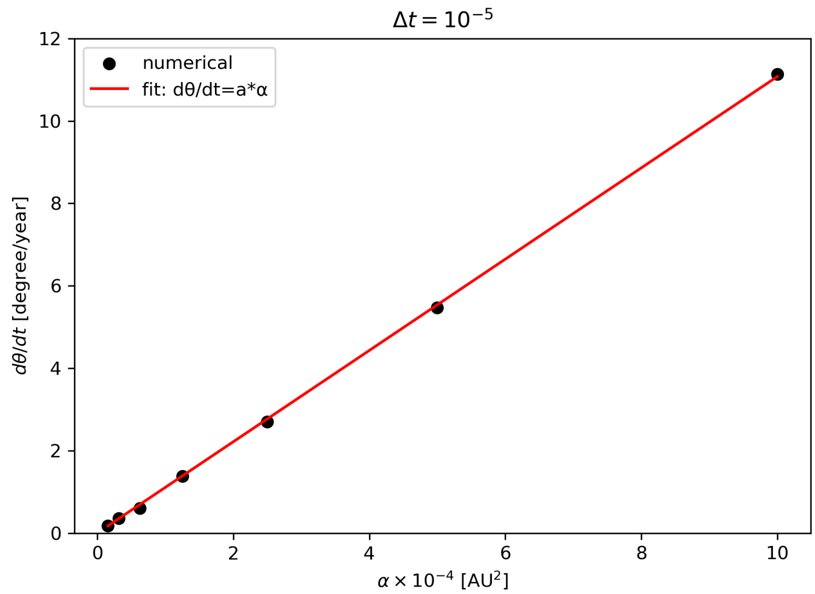

Figure 4: Rate of precession ω as a function of the relativistic parameter α. The numerical data (black dots) are well-described by a linear fit (red line).

By performing a linear fit to the data, we find the slope of the relationship. Using this slope, we can predict the precession for the actual value of α for Mercury, which is approximately 1.1×10^{-8} AU^2. The simulation predicts a precession of about 43 arcseconds per century, which is in excellent agreement with the observed value of 42.98 arcseconds per century.

7 Conclusion

This simulation successfully modeled the precession of Mercurý's perihelion using a 4th order symplectic integrator. We have shown that:

- The integrator produces stable orbits for the classical Kepler problem.
- The introduction of a $1/r^3$ term in the potential leads to a precession of the orbitś perihelion.
- The rate of precession is linearly proportional to the strength of this correction term.
- The simulation quantitatively reproduces the observed precession of Mercury with high accuracy.

This demonstrates the power of numerical simulations in verifying predictions from fundamental physical theories like General Relativity.