Probabilistic Object and Label Association Algorithms for Distributed Multiobject Tracking: Supplementary Material

Thomas Kropfreiter and Franz Hlawatsch

August 19, 2024

This manuscript supplements the associated manuscript, 'Probabilistic Object and Label Association Algorithms for Distributed Multiobject Tracking' by the same authors [3]. The presented material comprises additional derivations related to the fusion algorithms for the JPDA, JIPDA, and LMB filters; a discussion related to the Gaussian implementation of these fusion algorithms; and an extension of pairwise fusion to distributed networkwide fusion. Basic definitions, notation, and assumptions are given in [3] and will not be repeated here.

1 JPDA Filter – Proof of [3, Eqs. (5)–(8)]

The material presented in this section supplements [3, Sec. II-C] by proving [3, Eqs. (5)–(8)]. Inserting [3, Eq. (2)] for $s = 1, 2$ into [3, Eq. (4)] yields

$$\tilde{f}_{\boldsymbol{a}}(\boldsymbol{x}|\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}) = \frac{1}{B_{\boldsymbol{a}}}\bigg(\prod_{i\in\mathcal{I}^{(1)}} f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(1)})\bigg)^{\omega}\bigg(\prod_{i'\in\mathcal{I}^{(1)}} f^{(2)}(\boldsymbol{x}_{a_{i'}}^{(2)}\!=\boldsymbol{x}_{i'}|\boldsymbol{z}^{(2)})\bigg)^{1-\omega}$$

$$= \frac{1}{B_{\boldsymbol{a}}}\prod_{i\in\mathcal{I}^{(1)}} \big(f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(1)})\big)^{\omega}\big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(2)})\big)^{1-\omega}. \tag{1}$$

Here, as a result of using the modified Chernoff fusion rule defined in [3, Eq. (4)], the posterior multiobject pdf $\prod_{i\in\mathcal{I}^{(2)}} f^{(2)}(\boldsymbol{x}_i^{(2)}|\boldsymbol{z}^{(2)})$ of [3, Eq. (2)] is replaced by $\prod_{i\in\mathcal{I}^{(1)}} f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}|\boldsymbol{z}^{(2)})$. The normalization constant in (1) is given by

$$B_{\boldsymbol{a}} = \int\bigg(\prod_{i\in\mathcal{I}^{(1)}} \big(f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(1)})\big)^{\omega}\big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(2)})\big)^{1-\omega}\bigg)\mathrm{d}\boldsymbol{x}$$

$$= \prod_{i\in\mathcal{I}^{(1)}} \int \big(f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(1)})\big)^{\omega}\big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(2)})\big)^{1-\omega}\mathrm{d}\boldsymbol{x}_i\,,$$

or equivalently

$$B_{\boldsymbol{a}} = \prod_{i\in\mathcal{I}^{(1)}} \beta_{i,a_i}\,, \tag{2}$$

with

$$\beta_{i,a_i} \triangleq \int \big(f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(1)})\big)^{\omega}\big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\boldsymbol{x}_i|\boldsymbol{z}^{(2)})\big)^{1-\omega}\mathrm{d}\boldsymbol{x}_i\,. \tag{3}$$

This work was supported in part by the Austrian Science Fund (FWF) under grants P 32055-N31 and J 4726-N. Parts of this paper were previously presented at FUSION 2020 [1] and ICASSP 2024 [2].

T. Kropfreiter is with the Scripps Institution of Oceanography and the Department of Electrical and Computer Engineering, University of California San Diego, La Jolla, CA, USA (e-mail: tkropfreiter@ucsd.edu).

F. Hlawatsch is with the Institute of Telecommunications, TU Wien, Vienna, Austria (e-mail: franz.hlawatsch@tuwien.ac.at).

Here, expressions (2) and (3) are equal to [3, Eq. (8)] and [3, Eq. (7)], respectively. Finally, inserting (2) into (1) yields

$$\tilde{f}_{\boldsymbol{a}}(\boldsymbol{x}|\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}) = \prod_{i \in \mathcal{I}^{(1)}} f_{i,a_i}(\boldsymbol{x}_i)\,, \tag{4}$$

with

$$f_{i,a_i}(\boldsymbol{x}_i) = \frac{1}{\beta_{i,a_i}} \big(f^{(1)}(\boldsymbol{x}_i^{(1)} \!=\! \boldsymbol{x}_i|\boldsymbol{z}^{(1)})\big)^{\omega} \big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)} \!=\! \boldsymbol{x}_i|\boldsymbol{z}^{(2)})\big)^{1-\omega}. \tag{5}$$

Here, expressions (4) and (5) are equal to [3, Eq. (5)], and [3, Eq. (6)], respectively.

2 Gaussian Fusion for the JPDA Filter

The material presented in this section supplements the discussion in [3, Sec. II-E]. For Gaussian posterior pdfs at the two local JPDA filters, i.e., $f^{(s)}(\boldsymbol{x}_i^{(s)}|\boldsymbol{z}^{(s)}) = \mathcal{N}\big(\boldsymbol{x}_i^{(s)}; \boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)$ for $i \in \mathcal{I}^{(s)}$ and $s \in \{1,2\}$, it was shown in [4] that also $f_{i,a_i}(\boldsymbol{x}_i)$ in [3, Eq. (6)] is Gaussian, i.e.,

$$f_{i,a_i}(\boldsymbol{x}_i) = \mathcal{N}\big(\boldsymbol{x}_i; \boldsymbol{\mu}_{i,a_i}, \boldsymbol{\Sigma}_{i,a_i}\big)\,, \tag{6}$$

where $\boldsymbol{\mu}_{i,a_i}$ and $\boldsymbol{\Sigma}_{i,a_i}$ are obtained from $\boldsymbol{\mu}_i^{(1)}$, $\boldsymbol{\Sigma}_i^{(1)}$ and $\boldsymbol{\mu}_{a_i}^{(2)}$, $\boldsymbol{\Sigma}_{a_i}^{(2)}$ according to [3, Eqs. (21) and (22)]. Furthermore, according to [3, Eq. (18)], the fused spatial pdf $f_i(\boldsymbol{x}_i)$ is a linear combination of the pdfs $f_{i,a_i}(\boldsymbol{x}_i)$. Because of (6), it is actually a Gaussian mixture pdf. We approximate it by a single Gaussian as stated in [3, Sec. II-E], i.e., $\bar{f}_i(\boldsymbol{x}_i) \triangleq \mathcal{N}(\boldsymbol{x}_i; \bar{\boldsymbol{\mu}}_i, \bar{\boldsymbol{\Sigma}}_i)$, $i \in \mathcal{I}^{(1)}$, with the mean vector $\bar{\boldsymbol{\mu}}_i$ and covariance matrix $\bar{\boldsymbol{\Sigma}}_i$ chosen equal to those of $f_i(\boldsymbol{x}_i)$. Using [3, Eq. (18)] and (6), and following the derivation in [5], one obtains for $\bar{\boldsymbol{\mu}}_i$ and $\bar{\boldsymbol{\Sigma}}_i$ the expressions given in [3, Eqs. (19) and (20)].

The expressions [3, Eqs. (19) and (20)] involve besides the means $\boldsymbol{\mu}_{i,a_i}$ and covariance matrices $\boldsymbol{\Sigma}_{i,a_i}$ the marginal association probabilities $p(a_i)$. Calculating the $p(a_i)$ via [3, Eqs. (16) and (13)] requires the factors β_{i,a_i} given by [3, Eq. (7)]. For Gaussian $f^{(s)}(\boldsymbol{x}_i^{(s)}|\boldsymbol{z}^{(s)})$, it was shown in [4] that

$$\beta_{i,a_i} = \gamma_i \kappa_{a_i} \mathcal{N}\Big(\boldsymbol{\mu}_i^{(1)}; \boldsymbol{\mu}_{a_i}^{(2)}, \frac{1}{\omega}\boldsymbol{\Sigma}_i^{(1)} + \frac{1}{1-\omega}\boldsymbol{\Sigma}_{a_i}^{(2)}\Big)\,, \tag{7}$$

where

$$\gamma_i \triangleq \sqrt{\det\Big(\frac{2\pi}{\omega}\boldsymbol{\Sigma}_i^{(1)}\Big) \big/ \big(\det(2\pi\boldsymbol{\Sigma}_i^{(1)})\big)^{\omega}}\,, \tag{8}$$

$$\kappa_{a_i} \triangleq \sqrt{\det\Big(\frac{2\pi}{1-\omega}\boldsymbol{\Sigma}_{a_i}^{(2)}\Big) \big/ \big(\det(2\pi\boldsymbol{\Sigma}_{a_i}^{(2)})\big)^{1-\omega}}\,, \tag{9}$$

and $\mathcal{N}\big(\boldsymbol{\mu}_i^{(1)}; \boldsymbol{\mu}_{a_i}^{(2)}, \frac{1}{\omega}\boldsymbol{\Sigma}_i^{(1)} + \frac{1}{1-\omega}\boldsymbol{\Sigma}_{a_i}^{(2)}\big)$ denotes the positive number obtained by evaluating the Gaussian pdf $\mathcal{N}\big(\boldsymbol{x}; \boldsymbol{\mu}_{a_i}^{(2)}, \frac{1}{\omega}\boldsymbol{\Sigma}_i^{(1)} + \frac{1}{1-\omega}\boldsymbol{\Sigma}_{a_i}^{(2)}\big)$ at $\boldsymbol{x} = \boldsymbol{\mu}_i^{(1)}$.

3 JIPDA Filter – Proof of [3, Eqs. (27)–(32)]

The material presented in this section supplements [3, Sec. III-B] by proving [3, Eqs. (27)–(32)]. Inserting [3, Eq. (25)] for $s = 1, 2$ into [3, Eq. (26)] yields

$$\begin{aligned}&\tilde{f}_{\boldsymbol{a}}(\boldsymbol{x}, \boldsymbol{e}|\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}) \\ &= \frac{1}{C_{\boldsymbol{a}}}\Bigg(\prod_{i \in \mathcal{I}^{(1)}} P_i^{(1)}(e_i) f^{(1)}(\boldsymbol{x}_i^{(1)} \!=\! \boldsymbol{x}_i|e_i^{(1)} \!=\! e_i, \boldsymbol{z}^{(1)})\Bigg)^{\omega} \Bigg(\prod_{i' \in \mathcal{I}^{(1)}} P_{a_{i'}}^{(2)}(e_{i'}) f^{(2)}(\boldsymbol{x}_{a_{i'}}^{(2)} \!=\! \boldsymbol{x}_{i'}|e_{a_{i'}}^{(2)} \!=\! e_{i'}, \boldsymbol{z}^{(2)})\Bigg)^{1-\omega}\end{aligned}$$

$$= \frac{1}{C_{\boldsymbol{a}}} \prod_{i\in\mathcal{I}^{(1)}} \big(P_i^{(1)}(e_i) f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\!\boldsymbol{x}_i|e_i^{(1)}\!=\!e_i, \boldsymbol{z}^{(1)})\big)^{\omega} \big(P_{a_i}^{(2)}(e_i) f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\!\boldsymbol{x}_i|e_{a_i}^{(2)}\!=\!e_i, \boldsymbol{z}^{(2)})\big)^{1-\omega}, \tag{10}$$

with $P_i^{(s)}(e) \triangleq p^{(s)}(e_i^{(s)}\!=\!e|\boldsymbol{z}^{(s)})$. Analogously to the JPDA case in Section 1, due to the use of the modified Chernoff fusion rule in [3, Eq. (26)], the posterior multiobject pdf $\prod_{i\in\mathcal{I}^{(2)}} f^{(2)}(\boldsymbol{x}_i^{(2)}|e_i^{(2)}, \boldsymbol{z}^{(2)})\, p^{(2)}(e_i^{(2)}|\boldsymbol{z}^{(2)})$ in [3, Eq. (25)] is replaced by $\prod_{i\in\mathcal{I}^{(1)}} f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}|e_{a_i}^{(2)}, \boldsymbol{z}^{(2)})\, p^{(2)}(e_{a_i}^{(2)}|\boldsymbol{z}^{(2)})$. The normalization constant in (10) is given as

$$\begin{aligned} C_{\boldsymbol{a}} &= \sum_{\boldsymbol{e}\in\{0,1\}^{I^{(1)}}} \int \bigg(\prod_{i\in\mathcal{I}^{(1)}} \big(P_i^{(1)}(e_i) f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\!\boldsymbol{x}_i|e_i^{(1)}\!=\!e_i, \boldsymbol{z}^{(1)})\big)^{\omega} \\ &\qquad\qquad \times \big(P_{a_i}^{(2)}(e_i) f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\!\boldsymbol{x}_i|e_{a_i}^{(2)}\!=\!e_i, \boldsymbol{z}^{(2)})\big)^{1-\omega} \bigg) \mathrm{d}\boldsymbol{x} \\ &= \prod_{i\in\mathcal{I}^{(1)}} \sum_{e_i=\{0,1\}} \int \big(P_i^{(1)}(e_i) f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\!\boldsymbol{x}_i|e_i^{(1)}\!=\!e_i, \boldsymbol{z}^{(1)})\big)^{\omega} \\ &\qquad\qquad \times \big(P_{a_i}^{(2)}(e_i) f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\!\boldsymbol{x}_i|e_{a_i}^{(2)}\!=\!e_i, \boldsymbol{z}^{(2)})\big)^{1-\omega} \mathrm{d}\boldsymbol{x}_i\,, \end{aligned}$$

or equivalently

$$C_{\boldsymbol{a}} = \prod_{i\in\mathcal{I}^{(1)}} C_{i,a_i}\,, \tag{11}$$

with

$$\begin{aligned} C_{i,a_i} \triangleq \sum_{e_i=\{0,1\}} \big(P_i^{(1)}(e_i)\big)^{\omega} \big(P_{a_i}^{(2)}(e_i)\big)^{1-\omega} &\int \big(f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\!\boldsymbol{x}_i|e_i^{(1)}\!=\!e_i, \boldsymbol{z}^{(1)})\big)^{\omega} \\ &\times \big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\!\boldsymbol{x}_i|e_{a_i}^{(2)}\!=\!e_i, \boldsymbol{z}^{(2)})\big)^{1-\omega} \mathrm{d}\boldsymbol{x}_i\,. \end{aligned} \tag{12}$$

Here, expression (11) is equal to [3, Eq. (32)]. Inserting (11) into (10) yields

$$\tilde{f}_{\boldsymbol{a}}(\boldsymbol{x}, \boldsymbol{e}|\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}) = \prod_{i\in\mathcal{I}^{(1)}} f_{i,a_i}(\boldsymbol{x}_i, e_i)\,, \tag{13}$$

with

$$f_{i,a_i}(\boldsymbol{x}_i, e_i) = \frac{1}{C_{i,a_i}} \big(P_i^{(1)}(e_i) f^{(1)}(\boldsymbol{x}_i^{(1)}\!=\!\boldsymbol{x}_i|e_i^{(1)}\!=\!e_i, \boldsymbol{z}^{(1)})\big)^{\omega} \big(P_{a_i}^{(2)}(e_i) f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}\!=\!\boldsymbol{x}_i|e_{a_i}^{(2)}\!=\!e_i, \boldsymbol{z}^{(2)})\big)^{1-\omega}. \tag{14}$$

Next, we formally factor the pdf $f_{i,a_i}(\boldsymbol{x}_i, e_i)$ in (13) as

$$f_{i,a_i}(\boldsymbol{x}_i, e_i) = f_{i,a_i}(\boldsymbol{x}_i|e_i)\, p_{i,a_i}(e_i) \tag{15}$$

(which, upon insertion into (13), gives [3, Eq. (27)]), where $f_{i,a_i}(\boldsymbol{x}_i|e_i)$ and $p_{i,a_i}(e_i)$ remain to be determined. By way of preparation, we first derive the expressions of C_{i,a_i} and C'_{i,a_i} in [3, Eq. (30)] and [3, Eq. (31)], respectively. This is done by replacing in (12) $f^{(s)}(\boldsymbol{x}_i^{(s)}\!=\!\boldsymbol{x}_i|e_i\!=\!0, \boldsymbol{z}^{(s)})$ with $f_{\mathrm{D}}(\boldsymbol{x}_i)$ and using the fact that $\int \big(f_{\mathrm{D}}(\boldsymbol{x}_i)\big)^{\omega} \big(f_{\mathrm{D}}(\boldsymbol{x}_i)\big)^{1-\omega} \mathrm{d}\boldsymbol{x}_i = \int f_{\mathrm{D}}(\boldsymbol{x}_i) \mathrm{d}\boldsymbol{x}_i = 1$. We thereby obtain the expression of C_{i,a_i} in [3, Eq. (30)] along with the expression of C'_{i,a_i} in [3, Eq. (31)].

We first derive expressions of $f_{i,a_i}(\boldsymbol{x}_i|e_i)$ and $p_{i,a_i}(e_i)$ for $e_i = 0$. Evaluating (14) at $e_i = 0$ and again replacing $f^{(s)}(\boldsymbol{x}_i^{(s)}\!=\!\boldsymbol{x}_i|e_i\!=\!0, \boldsymbol{z}^{(s)})$ with $f_{\mathrm{D}}(\boldsymbol{x}_i)$, we obtain

$$f_{i,a_i}(\boldsymbol{x}_i, e_i\!=\!0) = \frac{f_{\mathrm{D}}(\boldsymbol{x}_i)}{C_{i,a_i}} \big(P_i^{(1)}(0)\big)^{\omega} \big(P_{a_i}^{(2)}(0)\big)^{1-\omega}.$$

A comparison with (15) evaluated at $e_i\!=\!0$ then yields $f_{i,a_i}(\boldsymbol{x}_i|e_i\!=\!0) = f_{\mathrm{D}}(\boldsymbol{x}_i)$ (as in [3, Eq. (29)]) and

$$p_{i,a_i}(e_i\!=\!0) = \frac{1}{C_{i,a_i}} \big(P_i^{(1)}(0)\big)^{\omega} \big(P_{a_i}^{(2)}(0)\big)^{1-\omega}$$

(as in [3, Eq. (28)]). Finally, we derive expressions of $f_{i,a_i}(\boldsymbol{x}_i|e_i)$ and $p_{i,a_i}(e_i)$ for $e_i\!=\!1$. We have

$$p_{i,a_i}(e_i{=}1) \,=\, 1-p_{i,a_i}(e_i{=}\,0) \,=\, 1-\frac{1}{C_{i,a_i}}\big(P_i^{(1)}(0)\big)^{\omega}\big(P_{a_i}^{(2)}(0)\big)^{1-\omega} \,=\, \frac{C_{i,a_i}-\big(P_i^{(1)}(0)\big)^{\omega}\big(P_{a_i}^{(2)}(0)\big)^{1-\omega}}{C_{i,a_i}}\,. \tag{16}$$

From [3, Eq. (30)], we conclude that $C_{i,a_i} - \big(P_i^{(1)}(0)\big)^{\omega}\big(P_{a_i}^{(2)}(0)\big)^{1-\omega} = \big(P_i^{(1)}(1)\big)^{\omega}\big(P_{a_i}^{(2)}(1)\big)^{1-\omega}C'_{i,a_i}$. Inserting this identity into (16) yields

$$p_{i,a_i}(e_i{=}1) = \frac{\big(P_i^{(1)}(1)\big)^{\omega}\big(P_{a_i}^{(2)}(1)\big)^{1-\omega}C'_{i,a_i}}{C_{i,a_i}}\,, \tag{17}$$

which equals the expression of $p_{i,a_i}(e_i{=}1)$ in [3, Eq. (28)]. Evaluating (15) at $e_i=1$ and inserting expression (17) for $p_{i,a_i}(e_i{=}1)$ yields

$$f_{i,a_i}(\boldsymbol{x}_i, e_i{=}1) = f_{i,a_i}(\boldsymbol{x}_i|e_i{=}1)\,\frac{C'_{i,a_i}}{C_{i,a_i}}\big(P_i^{(1)}(1)\big)^{\omega}\big(P_{a_i}^{(2)}(1)\big)^{1-\omega}.$$

Comparing this expression with (14) evaluated at $e_i=1$, we conclude that $f_{i,a_i}(\boldsymbol{x}_i|e_i{=}1)$ is given by

$$f_{i,a_i}(\boldsymbol{x}_i|e_i{=}1) = \frac{1}{C'_{i,a_i}}\big(f^{(1)}(\boldsymbol{x}_i^{(1)}{=}\boldsymbol{x}_i|e_i^{(1)}{=}1,\boldsymbol{z}^{(1)})\big)^{\omega}\big(f^{(2)}(\boldsymbol{x}_{a_i}^{(2)}{=}\boldsymbol{x}_i|e_{a_i}^{(2)}{=}1,\boldsymbol{z}^{(2)})\big)^{1-\omega},$$

which equals the respective expression in [3, Eq. (29)].

4 JIPDA Filter – Proof of [3, Eqs. (38)–(40)]

The material presented in this section supplements [3, Sec. III-C2] by proving [3, Eqs. (38)–(40)]. Splitting the product $\prod_{i\in\mathcal{I}^{(1)}}$ in [3, Eq. (35)] into the product over i with $e_i\!=0$ and the product over i with $e_i\!=1$, and inserting the expressions of $p_{i,a_i}(e_i\!=0)$ and $p_{i,a_i}(e_i{=}1)$ from [3, Eq. (28)], we obtain

$$w_{\boldsymbol{e},\boldsymbol{a}} = \frac{1}{C}\Bigg(\prod_{i\in\mathcal{I}^{(1)}:\,e_i=0} Q_i(0)\Bigg)\prod_{i'\in\mathcal{I}^{(1)}:\,e_{i'}=1} Q_{i'}(1)\,C'_{i',a_{i'}}\,, \tag{18}$$

for $\boldsymbol{a}\in\mathcal{A}$, where we have used the short notations $Q_i(0) \triangleq \big(P_i^{(1)}(0)\big)^{\omega}\big(P_{a_i}^{(2)}(0)\big)^{1-\omega}$ and $Q_i(1) \triangleq \big(P_i^{(1)}(1)\big)^{\omega}$ $\times\big(P_{a_i}^{(2)}(1)\big)^{1-\omega}$. Here, $C = \sum_{\boldsymbol{a}\in\mathcal{A}}\prod_{i\in\mathcal{I}^{(1)}} C_{i,a_i}$ (see [3, Sec. III-C1]). Multiplying and dividing expression (18) by $\prod_{j\in\mathcal{I}^{(1)}} Q_j(0)$ gives

$$\begin{aligned} w_{\boldsymbol{e},\boldsymbol{a}} &= \frac{\prod_{j\in\mathcal{I}^{(1)}} Q_j(0)}{\prod_{j'\in\mathcal{I}^{(1)}} Q_{j'}(0)}\,\frac{1}{C}\Bigg(\prod_{i\in\mathcal{I}^{(1)}:\,e_i=0} Q_i(0)\Bigg)\prod_{i'\in\mathcal{I}^{(1)}:\,e_{i'}=1} Q_{i'}(1)\,C'_{i',a_{i'}} \\ &= \frac{1}{C'}\,\frac{1}{\prod_{j'\in\mathcal{I}^{(1)}} Q_{j'}(0)}\Bigg(\prod_{i\in\mathcal{I}^{(1)}:\,e_i=0} Q_i(0)\Bigg)\prod_{i'\in\mathcal{I}^{(1)}:\,e_{i'}=1} Q_{i'}(1)\,C'_{i',a_{i'}}\,, \end{aligned} \tag{19}$$

with

$$C' \triangleq \frac{C}{\prod_{j\in\mathcal{I}^{(1)}} Q_j(0)} = \frac{\sum_{\boldsymbol{a}\in\mathcal{A}}\prod_{i\in\mathcal{I}^{(1)}} C_{i,a_i}}{\prod_{j\in\mathcal{I}^{(1)}} Q_j(0)} = \sum_{\boldsymbol{a}\in\mathcal{A}}\prod_{i\in\mathcal{I}^{(1)}}\frac{C_{i,a_i}}{Q_i(0)}\,. \tag{20}$$

By using the factorization $\prod_{j\in\mathcal{I}^{(1)}} Q_j(0) = \big(\prod_{j\in\mathcal{I}^{(1)}:\,e_j=0} Q_i(0)\big)\prod_{j'\in\mathcal{I}^{(1)}:\,e_{j'}=1} Q_{j'}(0)$ in the denominator of (19), we obtain further

$$w_{\boldsymbol{e},\boldsymbol{a}} = \frac{1}{C'} \frac{\left(\prod_{i\in\mathcal{I}^{(1)}:e_i=0} Q_i(0)\right) \prod_{i'\in\mathcal{I}^{(1)}:e_{i'}=1} Q_{i'}(1)\, C'_{i',a_{i'}}}{\left(\prod_{j\in\mathcal{I}^{(1)}:e_j=0} Q_j(0)\right) \prod_{j'\in\mathcal{I}^{(1)}:e_{j'}=1} Q_{j'}(0)} = \frac{1}{C'} \prod_{i\in\mathcal{I}^{(1)}:e_i=1} \frac{Q_i(1)\, C'_{i,a_i}}{Q_i(0)} \,. \tag{21}$$

Inserting our definitions for $Q_i(1)$ and $Q_i(0)$ into (21) and rewriting the resulting expression in terms of $\boldsymbol{a}'$ as introduced in Section III-C2, we finally obtain

$$w_{\boldsymbol{a}'} = \frac{1}{C'} \prod_{i\in\mathcal{I}^{(1)}:a'_i\in\mathcal{I}^{(2)}} \frac{\big(P_i^{(1)}(1)\big)^{\omega} \big(P_{a'_i}^{(2)}(1)\big)^{1-\omega} C'_{i,a'_i}}{\big(P_i^{(1)}(0)\big)^{\omega} \big(P_{a'_i}^{(2)}(0)\big)^{1-\omega}} \,, \tag{22}$$

for $\boldsymbol{a}' \in \mathcal{A}'$. Expression (22) is seen to be equivalent to [3, Eqs. (38) and (39)] for the case $\boldsymbol{a}' \in \mathcal{A}'$. Furthermore, the fact that $\beta'_{i,0} = 1$ in [3, Eq. (39)] accounts for the fact that the product (22) does not contain a factor for $a'_i = 0$. Finally, C' in (20) can be equivalently written in terms of $\boldsymbol{a}'$ as

$$C' \triangleq \sum_{\boldsymbol{a}'\in\mathcal{A}'} \prod_{i\in\mathcal{I}^{(1)}:a'_i\in\mathcal{I}^{(2)}} \frac{C_{i,a'_i}}{\big(P_i^{(1)}(0)\big)^{\omega} \big(P_{a'_i}^{(2)}(0)\big)^{1-\omega}} \,,$$

which is seen to be equal to [3, Eq. (40)].

5 Gaussian Fusion for the JIPDA Filter

The material presented in this section supplements the discussion in [3, Sec. III-D]. For Gaussian posterior pdfs at the two local JIPDA filters, i.e., $f^{(s)}(\boldsymbol{x}_i^{(s)}|e_i^{(s)}=1, \boldsymbol{z}^{(s)}) = \mathcal{N}\big(\boldsymbol{x}_i^{(s)}; \boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)$ for $i \in \mathcal{I}^{(s)}$ and $s \in \{1,2\}$, it was shown in [4] that also the spatial pdf $f'_{i,a'_i}(\boldsymbol{x}_i)$ in [3, Eq. (37)] is Gaussian, i.e.,

$$f'_{i,a'_i}(\boldsymbol{x}_i) = \mathcal{N}\big(\boldsymbol{x}_i; \boldsymbol{\mu}_{i,a'_i}, \boldsymbol{\Sigma}_{i,a'_i}\big) \,. \tag{23}$$

Here, $\boldsymbol{\mu}_{i,a'_i}$ and $\boldsymbol{\Sigma}_{i,a'_i}$ are obtained from $\boldsymbol{\mu}_i^{(1)}$, $\boldsymbol{\Sigma}_i^{(1)}$ and $\boldsymbol{\mu}_{a'_i}^{(2)}$, $\boldsymbol{\Sigma}_{a'_i}^{(2)}$ according to [3, Eqs. (21) and (22)] with a_i replaced by a'_i. Furthermore, using (23), the fused spatial pdf $f_i(\boldsymbol{x}_i|e_i = 1)$ in [3, Eq. (50)] becomes a Gaussian mixture pdf. We approximate it by a Gaussian pdf as stated in [3, Sec. III-D], i.e., $\bar{f}_i(\boldsymbol{x}_i|e_i = 1) \triangleq \mathcal{N}\big(\boldsymbol{x}_i; \bar{\boldsymbol{\mu}}'_i, \bar{\boldsymbol{\Sigma}}'_i\big)$, $i \in \mathcal{I}^{(1)}$, with the mean vector $\bar{\boldsymbol{\mu}}'_i$ and covariance matrix $\bar{\boldsymbol{\Sigma}}'_i$ chosen equal to those of $f_i(\boldsymbol{x}_i|e_i = 1)$. Using [3, Eq. (50)] and (23), and following the derivation in [5], one obtains the expressions of $\bar{\boldsymbol{\mu}}'_i$ and $\bar{\boldsymbol{\Sigma}}'_i$ given in [3, Eqs. (52) and (53)].

The expressions [3, Eqs. (52) and (53)] also involve the marginal association probabilities $p(a'_i)$. Calculating them via [3, Eqs. (46) and (43)] requires the factors β'_{i,a'_i} given by [3, Eq. (39)]. For Gaussian $f^{(s)}(\boldsymbol{x}_i^{(s)}|e_i^{(s)} = 1, \boldsymbol{z}^{(s)})$, it was shown in [4] that the C'_{i,a'_i} involved in [3, Eq. (39)] are given by expression (7) with a_i replaced by a'_i, i.e.,

$$C'_{i,a'_i} = \gamma_i \kappa_{a'_i} \mathcal{N}\Big(\boldsymbol{\mu}_i^{(1)}; \boldsymbol{\mu}_{a'_i}^{(2)}, \frac{1}{\omega}\boldsymbol{\Sigma}_i^{(1)} + \frac{1}{1-\omega}\boldsymbol{\Sigma}_{a'_i}^{(2)}\Big) \,. \tag{24}$$

Here, γ_i is given by (8) and $\kappa_{a'_i}$ is given by (9) with a_i replaced by a'_i. The desired expression of β'_{i,a'_i} is thus obtained by inserting (24) into [3, Eq. (39)].

6 LMB Filter – Proof of [3, Eqs. (62)–(68)]

The material presented in this section supplements [3, Sec. IV-C] by proving [3, Eqs. (62)–(68)]. Inserting [3, Eq. (58)] for $s = 1, 2$ into [3, Eq. (61)] yields

$$
\begin{aligned}
\tilde{f}_{\bar{\boldsymbol{a}}}(X|Z^{(1)},Z^{(2)}) &= \frac{1}{D_{\bar{\boldsymbol{a}}}}\bigg(\Delta(X)\prod_{l\in\mathbb{L}^{(1)*}} f_l^{(1)}\big(X_l^{(1)}\!=\!X_l\big)\bigg)^{\omega}\bigg(\Delta(X_{\bar{\boldsymbol{a}}})\prod_{l'\in\mathbb{L}^{(1)*}} f_{\bar{a}_{l'}}^{(2)}\big(X_{\bar{a}_{l'}}^{(2)}\!=\!X_{l'}\big)\bigg)^{1-\omega}\\
&= \frac{1}{D_{\bar{\boldsymbol{a}}}}\,\Delta(X)\prod_{l\in\mathbb{L}^{(1)*}}\big(f_l^{(1)}\big(X_l^{(1)}\!=\!X_l\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!X_l\big)\big)^{1-\omega}.
\end{aligned}
\tag{25}
$$

Here, we used the identity $\big(\Delta(X)\big)^{\omega}\big(\Delta(X_{\bar{\boldsymbol{a}}})\big)^{1-\omega}=\Delta(X)$ and the fact that, because of the use of the modified GCI fusion rule in [3, Eq. (61)], the LMB posterior multiobject pdf $\prod_{l\in\mathbb{L}^{(2)*}} f_l^{(2)}(X_l^{(2)})$ in [3, Eq. (58)] is replaced by $\prod_{l\in\mathbb{L}^{(1)*}} f_{\bar{a}_l}^{(2)}(X_{\bar{a}_l}^{(2)})$. The normalization constant in (25) is given by

$$
\begin{aligned}
D_{\bar{\boldsymbol{a}}} &= \int\bigg(\prod_{l\in\mathbb{L}^{(1)*}}\big(f_l^{(1)}\big(X_l^{(1)}\!=\!X_l\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!X_l\big)\big)^{1-\omega}\bigg)\delta X\\
&= \prod_{l\in\mathbb{L}^{(1)*}}\int\big(f_l^{(1)}\big(X_l^{(1)}\!=\!X_l\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!X_l\big)\big)^{1-\omega}\delta X_l\,,
\end{aligned}
$$

or equivalently

$$
D_{\bar{\boldsymbol{a}}} = \prod_{l\in\mathbb{L}^{(1)*}} D_{l,\bar{a}_l}\,,
\tag{26}
$$

with

$$
\begin{aligned}
D_{l,\bar{a}_l} &\triangleq \int\big(f_l^{(1)}\big(X_l^{(1)}\!=\!X_l\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!X_l\big)\big)^{1-\omega}\delta X_l\\
&= \big(f_l^{(1)}\big(X_l^{(1)}\!=\!\emptyset\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!\emptyset\big)\big)^{1-\omega}+\int\big(f_l^{(1)}\big(X_l^{(1)}\!=\!\{(\boldsymbol{x},l)\}\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!\{(\boldsymbol{x},\bar{a}_l)\}\big)\big)^{1-\omega}\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Here, the last expression was obtained by using the definition of the set integral [6]. Expression (26) is equal to [3, Eq. (68)]. Finally, inserting the expressions for $f_l^{(1)}\big(X_l^{(1)}\big)$ and $f_l^{(2)}\big(X_l^{(2)}\big)$ given by [3, Eq. (59)] yields

$$
D_{l,\bar{a}_l} = \big(1-r_l^{(1)}\big)^{\omega}\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega}+\big(r_l^{(1)}\big)^{\omega}\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}\int\big(f_l^{(1)}(\boldsymbol{x})\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}(\boldsymbol{x})\big)^{1-\omega}\mathrm{d}\boldsymbol{x}\,,
$$

which is equal to [3, Eqs. (66), (67)].

We will next show [3, Eqs. (62)–(65)]. Inserting (26) into (25) yields

$$
\tilde{f}_{\bar{\boldsymbol{a}}}(X|Z^{(1)},Z^{(2)}) = \Delta(X)\prod_{l\in\mathbb{L}^{(1)*}} f_{l,\bar{a}_l}(X_l)\,,
\tag{27}
$$

with

$$
f_{l,\bar{a}_l}(X_l) = \frac{1}{D_{l,\bar{a}_l}}\big(f_l^{(1)}\big(X_l^{(1)}\!=\!X_l\big)\big)^{\omega}\big(f_{\bar{a}_l}^{(2)}\big(X_{\bar{a}_l}^{(2)}\!=\!X_l\big)\big)^{1-\omega}.
\tag{28}
$$

Note that (27) is equal to [3, Eq. (62)]. Since X_l in (28) is either equal to $\emptyset$ or $\{(\boldsymbol{x},l)\}$, we can represent $f_{l,\bar{a}_l}(X_l)$ as a labeled Bernoulli pdf according to [3, Eq. (63)]. In the following, we derive expressions of the existence probability $r_{l,\bar{a}_l}$ and spatial pdf $f_{l,\bar{a}_l}(\boldsymbol{x})$ involved in [3, Eq. (63)]. To this end, we equate [3, Eq. (63)] and (28) and consider the resulting equation for the two cases in [3, Eq. (63)]. For the first case, i.e., $X_l=\emptyset$, expression [3, Eq. (63)] gives $f_{l,\bar{a}_l}(X_l\!=\!\emptyset)=1-r_{l,\bar{a}_l}$ while (28) gives $f_{l,\bar{a}_l}(X_l\!=\!\emptyset)=\frac{1}{D_{l,\bar{a}_l}}\big(1-r_l^{(1)}\big)^{\omega}\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega}$, and thus we obtain $1-r_{l,\bar{a}_l}=\frac{1}{D_{l,\bar{a}_l}}\big(1-r_l^{(1)}\big)^{\omega}\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega}$ and further

$$
r_{l,\bar{a}_l} = 1-\frac{1}{D_{l,\bar{a}_l}}\big(1-r_l^{(1)}\big)^{\omega}\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega} = \frac{D_{l,\bar{a}_l}-\big(1-r_l^{(1)}\big)^{\omega}\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega}}{D_{l,\bar{a}_l}}\,.
\tag{29}
$$

From [3, Eq. (66)], we conclude that $D_{l,\bar{a}_l}-\big(1-r_l^{(1)}\big)^{\omega}\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega}=\big(r_l^{(1)}\big)^{\omega}\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}D'_{l,\bar{a}_l}$. Inserting this identity into (29) yields

$$r_{l,\bar{a}_l} = \frac{\big(r_l^{(1)}\big)^{\omega}\,\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}\,D'_{l,\bar{a}_l}}{D_{l,\bar{a}_l}}\,,$$

which is equal to [3, Eq. (64)]. For the second case, i.e., $X_l = \{(\boldsymbol{x},l)\}$, expression [3, Eq. (63)] gives $f_{l,\bar{a}_l}\big(X_l\!=\!\{(\boldsymbol{x},l)\}\big) = r_{l,\bar{a}_l}\,f_{l,\bar{a}_l}(\boldsymbol{x})$ while (28) gives $f_{l,\bar{a}_l}\big(X_l\!=\!\{(\boldsymbol{x},l)\}\big) = \frac{1}{D_{l,\bar{a}_l}}\,\big(r_l^{(1)}\big)^{\omega}\big(f_l^{(1)}(\boldsymbol{x})\big)^{\omega}\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}$ $\times\big(f_{\bar{a}_l}^{(2)}(\boldsymbol{x})\big)^{1-\omega}$, and thus we obtain $r_{l,\bar{a}_l}\,f_{l,\bar{a}_l}(\boldsymbol{x}) = \frac{1}{D_{l,\bar{a}_l}}\,\big(r_l^{(1)}\big)^{\omega}\big(f_l^{(1)}(\boldsymbol{x})\big)^{\omega}\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}\big(f_{\bar{a}_l}^{(2)}(\boldsymbol{x})\big)^{1-\omega}$ and further

$$f_{l,\bar{a}_l}(\boldsymbol{x}) = \frac{1}{D_{l,\bar{a}_l}\,r_{l,\bar{a}_l}}\,\big(r_l^{(1)}\big)^{\omega}\,\big(f_l^{(1)}(\boldsymbol{x})\big)^{\omega}\,\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}\,\big(f_{\bar{a}_l}^{(2)}(\boldsymbol{x})\big)^{1-\omega}.$$

By inserting expression [3, Eq. (64)] for $r_{l,\bar{a}_l}$, we finally obtain

$$f_{l,\bar{a}_l}(\boldsymbol{x}) = \frac{1}{D'_{l,\bar{a}_l}}\,\big(f_l^{(1)}(\boldsymbol{x})\big)^{\omega}\,\big(f_{\bar{a}_l}^{(2)}(\boldsymbol{x})\big)^{1-\omega},$$

which is equal to [3, Eq. (65)].

7 LMB Filter – Proof of [3, Eqs. (75)–(77)]

The material presented in this section supplements [3, Sec. IV-D2] by proving [3, Eqs. (75)–(77)]. We first insert expression [3, Eq. (64)] for $r_{l,\bar{a}_l}$ into [3, Eq. (73)], which yields

$$w'_{\bar{\boldsymbol{a}}}(L_X) = \frac{1}{D'}\Bigg(\prod_{l\in L_X}\big(r_l^{(1)}\big)^{\omega}\,\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}\,D'_{l,\bar{a}_l}\Bigg)\prod_{l'\in\mathbb{L}^{(1)*}\setminus L_X}\Big(D_{l',\bar{a}_{l'}} - \big(r_{l'}^{(1)}\big)^{\omega}\,\big(r_{\bar{a}_{l'}}^{(2)}\big)^{1-\omega}\,D'_{l',\bar{a}_{l'}}\Big)\,.$$

Inserting expression [3, Eq. (66)] for $D_{l,\bar{a}_l}$ yields further

$$w'_{\bar{\boldsymbol{a}}}(L_X) = \frac{1}{D'}\Bigg(\prod_{l\in L_X} R_l(1)\,D'_{l,\bar{a}_l}\Bigg)\prod_{l'\in\mathbb{L}^{(1)*}\setminus L_X} R_{l'}(0)\,, \tag{30}$$

for $\bar{\boldsymbol{a}}\in\bar{\mathcal{A}}$, where we used the short notations $R_l(0) \triangleq \big(1-r_l^{(1)}\big)^{\omega}\,\big(1-r_{\bar{a}_l}^{(2)}\big)^{1-\omega}$ and $R_l(1) \triangleq \big(r_l^{(1)}\big)^{\omega}\,\big(r_{\bar{a}_l}^{(2)}\big)^{1-\omega}$. Here, $D' = \sum_{\bar{\boldsymbol{a}}\in\bar{\mathcal{A}}}\prod_{l\in\mathbb{L}^{(1)*}} D_{l,\bar{a}_l}$ (see [3, Sec. IV-D1]). Multiplying and dividing expression (30) by $\prod_{l\in\mathbb{L}^{(1)*}} R_l(0)$ gives

$$\begin{aligned} w'_{\bar{\boldsymbol{a}}}(L_X) &= \frac{\prod_{\lambda\in\mathbb{L}^{(1)*}} R_\lambda(0)}{\prod_{\lambda'\in\mathbb{L}^{(1)*}} R_{\lambda'}(0)}\,\frac{1}{D'}\Bigg(\prod_{l\in L_X} R_l(1)\,D'_{l,\bar{a}_l}\Bigg)\prod_{l'\in\mathbb{L}^{(1)*}\setminus L_X} R_{l'}(0) \\ &= \frac{1}{D''}\,\frac{1}{\prod_{\lambda'\in\mathbb{L}^{(1)*}} R_{\lambda'}(0)}\Bigg(\prod_{l\in L_X} R_l(1)\,D'_{l,\bar{a}_l}\Bigg)\prod_{l'\in\mathbb{L}^{(1)*}\setminus L_X} R_{l'}(0)\,, \end{aligned} \tag{31}$$

with

$$D'' \triangleq \frac{D'}{\prod_{\lambda\in\mathbb{L}^{(1)*}} R_\lambda(0)} = \frac{\sum_{\bar{\boldsymbol{a}}\in\bar{\mathcal{A}}}\prod_{l\in\mathbb{L}^{(1)*}} D_{l,\bar{a}_l}}{\prod_{\lambda\in\mathbb{L}^{(1)*}} R_\lambda(0)} = \sum_{\bar{\boldsymbol{a}}\in\bar{\mathcal{A}}}\prod_{l\in\mathbb{L}^{(1)*}}\frac{D_{l,\bar{a}_l}}{R_l(0)}\,. \tag{32}$$

By using the factorization $\prod_{\lambda\in\mathbb{L}^{(1)*}} R_\lambda(0) = \big(\prod_{\lambda\in L_X} R_\lambda(0)\big)\prod_{\lambda'\in\mathbb{L}^{(1)*}\setminus L_X} R_{\lambda'}(0)$ in the denominator of (31), we obtain further

$$w'_{\bar{\boldsymbol{a}}}(L_X) = \frac{1}{D''}\,\frac{\big(\prod_{l\in L_X} R_l(1)\,D'_{l,\bar{a}_l}\big)\prod_{l'\in\mathbb{L}^{(1)*}\setminus L_X} R_{l'}(0)}{\big(\prod_{\lambda\in L_X} R_\lambda(0)\big)\prod_{\lambda'\in\mathbb{L}^{(1)*}\setminus L_X} R_{\lambda'}(0)} = \frac{1}{D''}\prod_{l\in L_X}\frac{R_l(1)\,D'_{l,\bar{a}_l}}{R_l(0)}\,. \tag{33}$$

Inserting our definitions for $R_l(1)$ and $R_l(0)$ into (33) and rewriting the resulting expression in terms of $\bar{\boldsymbol{a}}'$ as introduced in Section III-C2, we finally obtain

$$w_{\bar{\boldsymbol{a}}'} = \frac{1}{D''} \prod_{l \in \mathbb{L}^{(1)*}:\, \bar{a}'_l \in \mathbb{L}^{(2)*}} \frac{\big(r_l^{(1)}\big)^{\omega} \big(r_{\bar{a}'_l}^{(2)}\big)^{1-\omega} D'_{l,\bar{a}'_l}}{\big(1-r_l^{(1)}\big)^{\omega} \big(1-r_{\bar{a}'_l}^{(2)}\big)^{1-\omega}}\,, \tag{34}$$

for $\bar{\boldsymbol{a}}' \in \bar{\mathcal{A}}'$. This expression is equivalent to [3, Eqs. (75) and (76)] for the case $\bar{\boldsymbol{a}}' \in \bar{\mathcal{A}}'$. Furthermore, the fact that $\bar{\beta}_{l,0} = 1$ in [3, Eq. (76)] accounts for the fact that the product (34) does not contain a factor for $\bar{a}'_l = 0$. Finally, D'' in (32) can be equivalently written in terms of $\bar{\boldsymbol{a}}'$ as

$$D'' \triangleq \sum_{\bar{\boldsymbol{a}}' \in \bar{\mathcal{A}}'} \prod_{l \in \mathbb{L}^{(1)*}:\, \bar{a}'_l \in \mathbb{L}^{(2)*}} \frac{D_{l,\bar{a}'_l}}{\big(1-r_l^{(1)}\big)^{\omega} \big(1-r_{\bar{a}'_l}^{(2)}\big)^{1-\omega}}\,,$$

which is seen to be equal to [3, Eq. (77)].

8 Gaussian Fusion for the LMB Filter

The material presented in this section supplements the discussion in [3, Sec. IV-E]. When the posterior pdfs at the two local LMB filters are Gaussian, i.e., $f_l^{(s)}(\boldsymbol{x}) = \mathcal{N}\big(\boldsymbol{x}; \boldsymbol{\mu}_l^{(s)}, \boldsymbol{\Sigma}_l^{(s)}\big)$ for $l \in \mathbb{L}^{(s)*}$ and $s \in \{1,2\}$, then, as shown in [4], also $f_{l,\bar{a}'_l}(\boldsymbol{x})$ in [3, Eq. (65)] (with $\bar{a}_l$ replaced by $\bar{a}'_l$) is Gaussian, i.e.,

$$f_{l,\bar{a}'_l}(\boldsymbol{x}) = \mathcal{N}\big(\boldsymbol{x}; \boldsymbol{\mu}_{l,\bar{a}'_l}, \boldsymbol{\Sigma}_{l,\bar{a}'_l}\big)\,. \tag{35}$$

Here, $\boldsymbol{\mu}_{l,\bar{a}'_l}$ and $\boldsymbol{\Sigma}_{l,\bar{a}'_l}$ are obtained from $\boldsymbol{\mu}_l^{(1)}$, $\boldsymbol{\Sigma}_l^{(1)}$ and $\boldsymbol{\mu}_{\bar{a}'_l}^{(2)}$, $\boldsymbol{\Sigma}_{\bar{a}'_l}^{(2)}$ according to [3, Eqs. (21) and (22)] with i replaced by l and a_i replaced by $\bar{a}'_l$. Furthermore, inserting (35) into [3, Eq. (88)] shows that the fused spatial pdf $f_l(\boldsymbol{x})$ is a Gaussian mixture pdf. We approximate it by a Gaussian pdf as stated in [3, Sec. IV-E], i.e., $\bar{f}_l(\boldsymbol{x}) \triangleq \mathcal{N}\big(\boldsymbol{x}; \bar{\boldsymbol{\mu}}_l, \bar{\boldsymbol{\Sigma}}_l\big)$, $l \in \mathbb{L}^{(1)*}$, with the mean $\bar{\boldsymbol{\mu}}_l$ and covariance matrix $\bar{\boldsymbol{\Sigma}}_l$ chosen equal to those of $f_l(\boldsymbol{x})$. With [3, Eq. (88)] and (35), and following the derivation in [5], one obtains the expressions of $\bar{\boldsymbol{\mu}}_l$ and $\bar{\boldsymbol{\Sigma}}_l$ given in [3, Eqs. (89) and (90)].

These expressions also involve the association probabilities $p(\bar{a}'_l)$. To calculate them according to [3, Eqs. (83) and (80)], we need the factors $\bar{\beta}_{l,\bar{a}'_l}$, which are in turn given by [3, Eq. (76)]. For Gaussian $f_l^{(s)}(\boldsymbol{x})$, it was shown in [4] that the $D'_{l,\bar{a}'_l}$ involved in [3, Eq. (76)] are given by expression (7) with i replaced by l and a_i replaced by $\bar{a}'_l$, i.e.,

$$D'_{l,\bar{a}'_l} = \gamma_l\, \kappa_{\bar{a}'_l}\, \mathcal{N}\Big(\boldsymbol{\mu}_l^{(1)}; \boldsymbol{\mu}_{\bar{a}'_l}^{(2)}, \frac{1}{\omega}\boldsymbol{\Sigma}_l^{(1)} + \frac{1}{1-\omega}\boldsymbol{\Sigma}_{\bar{a}'_l}^{(2)}\Big)\,, \tag{36}$$

where γ_l is given by (8) with i replaced by l and $\kappa_{\bar{a}'_l}$ is given by (9) with a_i replaced by $\bar{a}'_l$. The desired expression of $\bar{\beta}_{l,\bar{a}'_l}$ is thus obtained by inserting (36) into [3, Eq. (76)].

9 Distributed Networkwide Fusion

The pairwise fusion methods for the JPDA, JIPDA, and LMB filters proposed in [3, Secs. II–IV] can be extended to distributed networkwide multisensor fusion by repeating them in a recursive manner. Let us first consider the JPDA filter. In our formulation of pairwise JPDA fusion in [3, Sec. II], sensor $s = 1$ was considered as the "reference sensor" for fusing the posterior pdfs of sensors $s = 1$ and $s = 2$, and thus $p(a_i)$ in [3, Eq. (16)] or $\tilde{p}(a_i)$ in [3, Eq. (96)], $\bar{\boldsymbol{\mu}}_i$ in [3, Eq. (19)], and $\bar{\boldsymbol{\Sigma}}_i$ in [3, Eq. (20)] were defined for $i \in \mathcal{I}^{(1)}$. In a distributed implementation of pairwise fusion, each sensor $s \in \{1,2\}$ has its own local JPDA parameter set $\big\{\big(\boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ and runs its own instance of the pairwise fusion method, thereby incorporating the JPDA parameter set of the respective other sensor. We will denote by $\big\{\big(\bar{\boldsymbol{\mu}}_i^{(s)}, \bar{\boldsymbol{\Sigma}}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ the fused JPDA parameter set calculated at sensor $s \in \{1,2\}$. This calculation presupposes that the original local JPDA parameter set of the respective other sensor is available at sensor s. Thus, each sensor s has to transmit its original local JPDA parameter set $\big\{\big(\boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ to its fusion partner.

To achieve *networkwide* fusion in a connected network of $S \geq 2$ sensors $s \in \{1, \ldots, S\}$, we use the pairwise fusion scheme within the following iterative procedure [4]. Let $N_s \subseteq \{1, \ldots, S\} \backslash \{s\}$ denote the set of "neighbor" sensors with which sensor s is able to communicate. In the first iteration, each sensor s transmits its local JPDA parameter set $\big\{\big(\boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ to its neighbors $s' \in N_s$, and thus also receives their parameter sets. Next, each sensor s performs pairwise fusion sequentially (recursively) with each of its neighbors, in arbitrary order. More specifically, the local parameter set is fused with that of an arbitrary neighbor $s' \in N_s$, the parameter set resulting from this pairwise fusion is fused with that of another neighbor $s'' \in N_s \backslash \{s'\}$, etc. Let $\big\{\big(\bar{\boldsymbol{\mu}}_i^{(s,1)}, \bar{\boldsymbol{\Sigma}}_i^{(s,1)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$, where the superscript "1" indicates the first iteration, denote the parameter set resulting from this sequence of $|N_s|$ successive pairwise fusion steps. In the second iteration, the sequence of $|N_s|$ pairwise fusion steps is repeated but with the original local parameter set $\big\{\big(\boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ replaced by $\big\{\big(\bar{\boldsymbol{\mu}}_i^{(s,1)}, \bar{\boldsymbol{\Sigma}}_i^{(s,1)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ and the original local parameter sets of the other sensors replaced by the fusion results of the other sensors; this requires another round of parameter transmissions between neighboring sensors. This procedure of pairwise fusions with, and parameter transmissions to, the neighbors is repeated in all subsequent iterations.

The pairwise JIPDA and LMB fusion methods can be extended to networkwide fusion in a similar way; we only have to formally replace the JPDA parameters $\big\{\big(\boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ by the JIPDA parameters $\big\{\big(p^{(s)}(e_i^{(s)} = 1|\boldsymbol{z}^{(s)}), \boldsymbol{\mu}_i^{(s)}, \boldsymbol{\Sigma}_i^{(s)}\big)\big\}_{i \in \mathcal{I}^{(s)}}$ (see [3, Secs. III-A and III-B]) or the (Gaussian) LMB parameters $\big\{\big(r_l^{(s)}, \boldsymbol{\mu}_l^{(s)}, \boldsymbol{\Sigma}_l^{(s)}\big)\big\}_{l \in \mathbb{L}^{(s)*}}$ (see [3, Secs. IV-B and IV-E]).

An alternative to this recursive extension of pairwise fusion is based on a gossip algorithm [7]. In each iteration of the gossip algorithm, the current JPDA, JIPDA, or LMB parameter sets of two randomly chosen neighboring sensors are fused using the respective pairwise fusion method. This recursion, too, is initialized by the original local parameter sets.

References

[1] T. Kropfreiter and F. Hlawatsch, "A probabilistic label association algorithm for distributed labeled multi-Bernoulli filtering," in *Proc. FUSION-20*, Rustenburg, South Africa, Jul. 2020.

[2] T. Kropfreiter, F. Meyer, and F. Hlawatsch, "A distributed joint integrated probabilistic data association (JIPDA) filter with soft object association," in *Proc. IEEE ICASSP-24*, Seoul, Republic of Korea, Apr. 2024, pp. 12 906–12 910.

[3] T. Kropfreiter and F. Hlawatsch, "Probabilistic object and label association algorithms for distributed multiobject tracking," *IEEE Trans. Signal Process.*, submitted, 2024.

[4] G. Battistelli, L. Chisci, C. Fantacci, A. Farina, and A. Graziano, "Consensus CPHD filter for distributed multitarget tracking," *IEEE J. Sel. Top. Signal Process.*, vol. 7, no. 3, pp. 508–520, Jun. 2013.

[5] D. P. Malladi and J. L. Speyer, "A new approach to multiple model adaptive estimation," in *Proc. IEEE CDC-97*, San Diego, CA, USA, Dec. 1997, pp. 3460–3467.

[6] R. P. S. Mahler, *Advances in Statistical Multisource-Multitarget Information Fusion*. Boston, MA, USA: Artech House, 2014.

[7] A. G. Dimakis, S. Kar, J. M. F. Moura, M. G. Rabbat, and A. Scaglione, "Gossip algorithms for distributed signal processing," *Proc. IEEE*, vol. 98, no. 11, pp. 1847–1864, Nov. 2010.