Dynamics of opinion polarization

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The Problem: How do people form opinions?

- The rise of online social media has played a big role on the formation of peoples' opinion.
- The provide an algorithm personalization which reinforces cognitive biases, reducing discomfort experienced when exposed to opposite opinions and thus, creates **echo chambers**.
- Whether this is the reason of polarized opinions is still debatable in literature. Some argue that some inherent characteristics of social nets and human interaction are predominant than algorithmic filtering.

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- It's a scientific challenge, since only a few models can describe opinion dynamics while being both realistic and mathematically tractable at the same time.
- This paper's focus is the investigation of opinion dynamics through the **Friedkin-Johnsen (FJ) model**, which incorporates individual stubbornness (resistance to change of their opinion).
- Explores whether this model can actually explain polarization and under what conditions.

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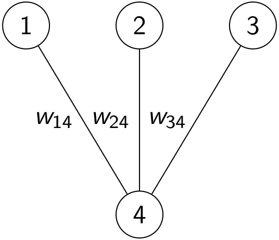
Related Work: DeGroot's model

Opinion of a node at each (discrete) time step is the average opinion of it's neighbor nodes, weighted by the strength of their social influence.



$$z_4(t+1) = \sum_j w_{4j} z_j(t)$$

where $z_i(t)$ is the opinion of node i at timestep t, w_{ij} is the influence of node i to node j.
When it converges all nodes have the same opinion (consensus) and no polarization $\implies$ not realistic.

Related Work on Friedkin - Johnsen model

Generalization of DeGroot's model. Introducing λ_i - **susceptibility**, i.e. willingness of a node to accept new opinions.

$$z_4(t+1) = (1 - \lambda_4) z_4(0) + \lambda_4 \sum_j w_{4j} z_j(t)$$

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This paper derives the conditions under which FJ model yields polarization.

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Problem formulation

- The paper focuses on undirected graphs representing social networks. Two types of graphs with the same vertices $\mathcal{V}$ and edges $\mathcal{E}$:
  1. Social graph, $\mathcal{S}$ where each edge weight is the number of social interactions.
  2. Influence graph, $\mathcal{I}$ where each edge weight w_{ij} is the influence of node i to node j.

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- Weights are normalized: $w_{ij} = \frac{\hat{w}_{ij}}{\sum_{j=1}^{n} \hat{w}_{ij}}$
- $W = (w_{ij})$ is the influence matrix. It's row stochastic (i.e. rows sum to 1)

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- $W = (w_{ij})$ is the influence matrix. It's row stochastic (i.e. rows sum to 1)
- Time is discrete.
- Opinion of node i at time k: $z_i(k)$. In general $z_i(k) \in \mathbb{R}$ but in this paper is assumed $z_i(k) \in [-1, 1]$.

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- Convergence: $\forall i z_i(k+1) \to z_i$ as $k \to \infty$
- Consensus: $\forall i z_i(k+1) \to z$ as $k \to \infty$
- A **choice shift** occurs when the mean attitude of the group at the end is different from the mean attitude at the beginning:

$$\sum_i z_i \neq \sum s_i$$

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FJ family of models

Table: The FJ family of models

Model	Update Equation
Generalized FJ (gFJ)	$z_i(k+1) = (1-\lambda_i)s_i + \lambda_i \sum_{j \in \{i\} \cup N(i)} w_{ij} z_j(k)$

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Variational FJ (vFJ)	$z_i(k+1) = \dfrac{\hat{w}_{ii} s_i + \sum_{j \in N(i)} \hat{w}_{ij} z_j(k)}{\hat{w}_{ii} + \sum_{j \in N(i)} \hat{w}_{ij}}$
Restricted FJ (rFJ)	$z_i(k+1) = \dfrac{s_i + \sum_{j \in N(i)} \hat{w}_{ij} z_j(k)}{1 + \sum_{j \in N(i)} \hat{w}_{ij}}$

- **vFJ** excludes the agent's own current opinion from the update.
- **rFJ** is the simplified version of FJ models. It sets the weight of the internal opinion to 1 ($w_{ii} = 1$), allowing only indirect control over susceptibility through the social weights, and is widely chosen in literature for its mathematical tractability.

Table: Matrix H for the different FJ models

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Generalized FJ (gFJ)	$H_g = (I - \Lambda W)^{-1}(I - \Lambda)$
Variational FJ (vFJ)	$H_v = (D + \tilde{A} - A)^{-1}\tilde{A}$
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- In each model the final opinion vector $\mathbf{z}$ can be calculated as $\mathbf{z} = H\mathbf{s}$ where H is a matrix that varies depending on the specific FJ version considered (see table 2).
- The equation of H_g holds and gFJ model is convergent and its unique stationary point $\mathbf{z}$ (i.e., the steady-state opinion vector) iff ΛW is stable (i.e., it has eigenvalues inside the open unit circle $\{z \in \mathbb{C} : |z| < 1\}$)

Polarization metrics

Polarization

For a polarization index Φ, we say that an opinion formation model $\mathcal{M}$ is Φ-polarizing (or simply polarizing for Φ) if there exists at least one initial opinion vector $\mathbf{s}$ such that the corresponding final opinion vector $\mathbf{z}$ satisfies:

$$\Phi(\mathbf{z}) > \Phi(\mathbf{s}).$$

The induced polarization is measured by the polarization shift:

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Polarization indices

Let $\mathbf{x} = (x_i) \in [-1, 1]^n$ be an opinion vector.

$$\mathrm{NDI}(\mathbf{x}) = \sum_{(i,j)\in E} w_{ij}(x_i - x_j)^2 \qquad \mathrm{GDI}(\mathbf{x}) = \sum_{i<j}(x_i - x_j)^2$$

$$P_1(\mathbf{x}) = \sum_i (x_i - \bar{x})^2 = \|\mathbf{x} - \bar{x}\|_2^2 \qquad P_2(\mathbf{x}) = \frac{1}{|V|}\sum_i x_i^2 = \frac{1}{|V|}\|\mathbf{x}\|_2^2$$

$$P_3(\mathbf{x}) = \sum_i x_i^2 = \|\mathbf{x}\|_2^2 \qquad P_4(\mathbf{x}) = \sum_i |x_i| = \|\mathbf{x}\|_1$$

Polarization invariants

Many polarization metrics exist, but they are often treated in isolation. The lemma 1-3 in the paper reveals how they relate:

- Global Disagreement Index (GDI) and P_1 are equivalent: $GDI(x) = |\mathcal{V}|P_1(x)$
- P_3 and P_2 are equivalent: $P_3(x) = |\mathcal{V}|P_2(x)$
- Polarization invariant: $P_1(\mathbf{x}) \geq P_3(\mathbf{x}) - \frac{P_4(\mathbf{x})^2}{|\mathcal{V}|}$

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Table: Classes of Polarization

Type	**What is Captured**	**Indices**
Local	Opinion spread among neighboring nodes	NDI
Dispersion	Opinion spread among all nodes	GDI, P_1
Absolute (Quadratic)	Closeness to extreme opinions (squared)	P_2, P_3
Absolute (Linear)	Total deviation from neutrality	P_4

gFJ is globally polarizing but locally depolarizing

- (Theorem 2) gFJ model is **NDI-depolarizing** for any initial opinion vector. Intuition of the proof: the expressed opinion $(z_i)_i$ is the Nash Equilibrium of cost function. i.e. z_i minimizes $f_i, \forall i$, so that $f_i(z_i) \leq f_i(s_i), \forall i$.

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- (Theorem 3) gFJ is **globally polarizing** iff the matrix $H_g = (I - \Lambda W)^{-1}(I - \Lambda)$ is **not doubly stochastic**. If there are any **naive nodes** ($\lambda = 1$), polarization occurs. Otherwise, polarization depends on imbalance in influence/susceptibility.

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- (Theorem 4) The most polarizing vectors (i.e. initial opinion vectors **s** which lead to polarization) can be chosen with **concordant signs** (all entries ≥ 0 or ≤ 0). This means that cooperation among like-minded nodes increases polarization more effectively than conflict.

gFJ (cntd.)

- (Theorems 5 & 6) methods of finding polarization vectors in gFJ

Metric	Polarizing Vector	Brief Explanation

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- (Corollary 7) The rFJ model on **undirected** social graphs is **never polarizing**, in any polarization metrics, for any initial opinion vector.

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Paper Results - gFJ on the Karate network dataset

	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}	Δ_{NDI}	Δ_{GDI}
$\lambda_i \propto \mathrm{P}_i$						
s_{unif}	-8.8e-1	-1.7e-2	-5.7e-1	3.1e-1	-1.03	-29.93
$s_{B_2(1)}$	-6.4e-2	2.2e-3	7.6e-2	4.2e-1	-3.1e-1	-2.18
$s_{B_2(t)}$	-1.15	4.0e-2	1.36	1.77	-5.61	-38.94
$s_{V_{>1}}$	-1.18	4.1e-2	1.40	1.80	-5.67	-40.28
$s^{heu}_{V_{>1}}$	-1.18	4.1e-2	1.40	1.80	-5.67	-40.28
$s_{\max P_{2,3}}$	-1.81	6.0e-2	2.05	2.05	-6.94	-61.48
$s_{B_1(1)}$	-2.0e-1	-5.4e-3	-1.9e-1	1.6e-1	-1.3e-1	-6.63
$s_{\max P_4}$	-3.57	3.4e-2	1.16	2.65	-10.65	-121.421
$\lambda_i \propto \mathrm{P}_i^{-1}$						
s_{unif}	-1.62	-4.8e-2	-1.63	-1.1e-2	-1.41	-55.22
$s_{B_2(1)}$	6.8e-1	4.3e-2	1.45	3.32	-3.1e-1	23.19
$s_{B_2(t)}$	7.6e-1	4.3e-2	1.63	3.52	-3.5e-1	25.97
$s_{V_{>1}}$	3.0e-1	1.3e-1	4.48	7.50	-5.08	10.07
$s^{heu}_{V_{>1}}$	3.1e-1	1.1e-1	3.61	6.74	-3.72	10.45
$s_{\max P_{2,3}}$	-2.30	2.01e-1	6.86	6.86	-3.98	-78.17
$s_{B_1(1)}$	3.1e-1	2.2e-2	7.5e-1	2.98	-1.95	10.67
$s_{\max P_4}$	-4.27	1.3e-1	4.59	10.04	-9.36	-145.22
$\lambda_i = 0.8$						
s_{unif}	-2.78	-7.7e-2	-2.62	1.7e-1	-2.45	-94.67
$s_{B_2(1)}$	-3.3e-1	1.5e-1	5.2e-1	2.44	-1.22	-11.30
$s_{B_2(t)}$	-1.43	6.6e-2	2.23	5.06	-5.22	-48.49
$s_{\max P_{2,3}}$	-1.73	1.3e-1	4.57	4.57	-3.33	-58.73
$s_{B_1(1)}$	-8.2e-1	-1.5e-2	-5.2e-1	2.34	-5.60	-27.90
$s_{\max P_4}$	-6.86	-1.7e-1	-1.7e-1	8.10	-13.32	-233.38

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$s_{B_2(1)}$	−6.4e−2	2.2e−3	7.6e−2	4.2e−1	−3.1e−1	−2.18
$s_{B_2(t)}$	−1.15	4.0e−2	1.36	1.77	−5.61	−38.94
$s_{V_{>1}}$	−1.18	4.1e−2	1.40	1.80	−5.67	−40.28
$s^{heu}_{V_{>1}}$	−1.18	4.1e−2	1.40	1.80	−5.67	−40.28
$s_{\max P_{2,3}}$	−1.81	6.0e−2	2.05	2.05	−6.94	−61.48
$s_{B_1(1)}$	−2.0e−1	−5.4e−3	−1.9e−1	1.6e−1	−1.3e−1	−6.63
$s_{\max P_4}$	−3.57	3.4e−2	1.16	2.65	−10.65	−121.421
$\lambda_i \propto \mathrm{P}_i^{-1}$						
s_{unif}	−1.62	−4.8e−2	−1.63	−1.1e−2	−1.41	−55.22
$s_{B_2(1)}$	6.8e−1	4.3e−2	1.45	3.32	−3.1e−1	23.19
$s_{B_2(t)}$	7.6e−1	4.3e−2	1.63	3.52	−3.5e−1	25.97
$s_{V_{>1}}$	3.0e−1	1.3e−1	4.48	7.50	−5.08	10.07
$s^{heu}_{V_{>1}}$	3.1e−1	1.1e−1	3.61	6.74	−3.72	10.45
$s_{\max P_{2,3}}$	−2.30	2.01e−1	6.86	6.86	−3.98	−78.17
$s_{B_1(1)}$	3.1e−1	2.2e−2	7.5e−1	2.98	−1.95	10.67
$s_{\max P_4}$	−4.27	1.3e−1	4.59	10.04	−9.36	−145.22
$\lambda_i = 0.8$						
s_{unif}	−2.78	−7.7e−2	−2.62	1.7e−1	−2.45	−94.67
$s_{B_2(1)}$	−3.3e−1	1.5e−1	5.2e−1	2.44	−1.22	−11.30
$s_{B_2(t)}$	−1.43	6.6e−2	2.23	5.06	−5.22	−48.49
$s_{\max P_{2,3}}$	−1.73	1.3e−1	4.57	4.57	−3.33	−58.73
$s_{B_1(1)}$	−8.2e−1	−1.5e−2	−5.2e−1	2.34	−5.60	−27.90
$s_{\max P_4}$	−6.86	−1.7e−1	−1.7e−1	8.10	−13.32	−233.38

$$w_{ij} = \frac{1}{\deg(i)}$$

Paper Results - gFJ on the Karate network dataset

	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}	Δ_{NDI}	Δ_{GDI}
$\lambda_i \propto \mathrm{P}_i$						
$\boldsymbol{s}_{unif}$	−8.8e−1	−1.7e−2	−5.7e−1	3.1e−1	−1.03	−29.93
$\boldsymbol{s}_{B_2(1)}$	−6.4e−2	2.2e−3	7.6e−2	4.2e−1	−3.1e−1	−2.18
$\boldsymbol{s}_{B_2(t)}$	−1.15	4.0e−2	1.36	1.77	−5.61	−38.94
$\boldsymbol{s}_{V_{>1}}$	−1.18	4.1e−2	1.40	1.80	−5.67	−40.28
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$\boldsymbol{s}_{\max P_{2,3}}$	−1.81	6.0e−2	2.05	2.05	−6.94	−61.48
$\boldsymbol{s}_{B_1(1)}$	−2.0e−1	−5.4e−3	−1.9e−1	1.6e−1	−1.3e−1	−6.63
$\boldsymbol{s}_{\max P_4}$	−3.57	3.4e−2	1.16	2.65	−10.65	−121.421
$\lambda_i \propto \mathrm{P}_i^{-1}$						
$\boldsymbol{s}_{unif}$	−1.62	−4.8e−2	−1.63	−1.1e−2	−1.41	−55.22
$\boldsymbol{s}_{B_2(1)}$	6.8e−1	4.3e−2	1.45	3.32	−3.1e−1	23.19
$\boldsymbol{s}_{B_2(t)}$	7.6e−1	4.3e−2	1.63	3.52	−3.5e−1	25.97
$\boldsymbol{s}_{V_{>1}}$	3.0e−1	1.3e−1	4.48	7.50	−5.08	10.07
$\boldsymbol{s}^{heu}_{V_{>1}}$	3.1e−1	1.1e−1	3.61	6.74	−3.72	10.45
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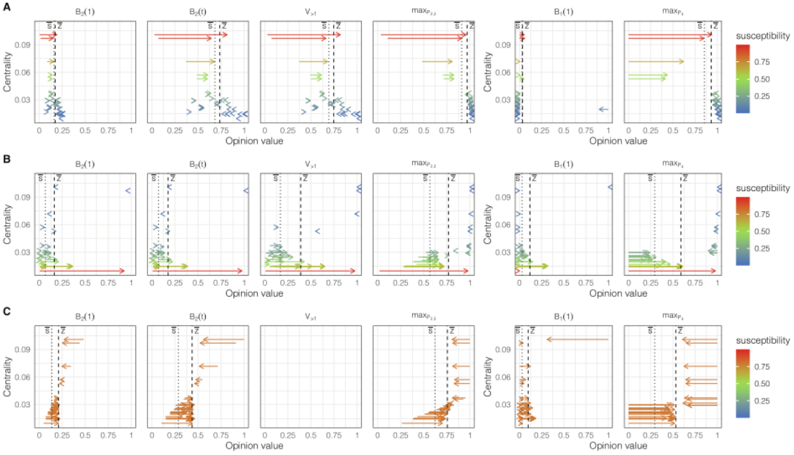
$$w_{ij} = \frac{1}{\deg(i)} \qquad \lambda_i \propto C_i = \mathrm{PageRank}(G)_i$$

Paper Results - gFJ on the Karate network dataset (cntd.)



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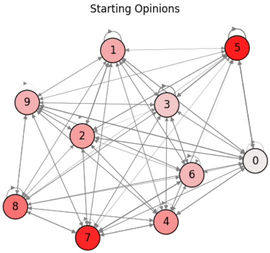
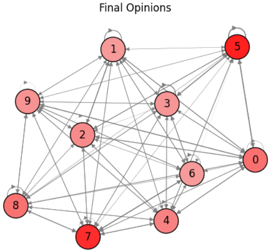
Reproduction of results with random graph

s	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-62.89	-9.19	-4.19	-0.29	-4.34	-4.68
$s_{B_2(1)}$	-2.05	-0.28	-0.14	0.01	0.14	0.16
$s_{B_2(t)}$	-2.05	-0.28	-0.14	0.01	0.14	0.16
$s_{max}^{P_2 P_3}$	-1.74	-0.23	-0.12	0.01	0.15	0.15
$s_{B_1(1)}$	-13.51	-2.13	-0.90	-0.06	-0.88	0.17
$s_{max}^{P_4}$	-53.81	-7.29	-3.59	-0.19	-2.81	0.70

Table: gFJ in random graph, reproduction of paper results.



(a) Random network, $s_{B_2(1)}$



(b) Random network, z_{final}

Karate Club Dataset

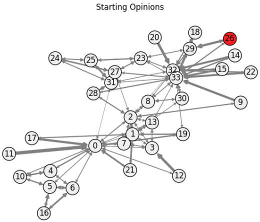
$\lambda = 0.8$, λ proportional to centrality, λ reverse proportional to centrality respectively.

	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-321.39	-14.71	-9.45	-0.26	-8.69	-8.96
$s_{B_2}(1)$	-53.42	-8.88	-1.57	0.07	2.52	5.46
$s_{B_2}(t)$	-53.42	-8.88	-1.57	0.07	2.52	5.46
$s_{max}^{P_2,P_3}$	-85.65	-7.08	-2.52	0.16	5.56	5.56
$s_{B_1}(1)$	-27.65	-6.48	-0.81	-0.02	-0.54	2.23
$s_{max}^{P_4}$	-252.99	-23.11	-7.44	0.04	1.42	9.10

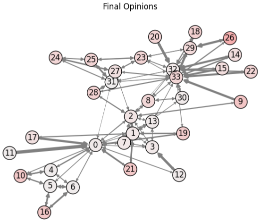
	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-96.98	-10.77	-2.85	-0.08	-2.84	-2.19
$s_{B_2}(1)$	-1.58	-0.21	-0.05	0.00	0.05	0.05
$s_{B_2}(t)$	-1.58	-0.21	-0.05	0.00	0.05	0.05
$s_{max}^{P_2,P_3}$	-1.49	-0.22	-0.04	0.00	0.05	0.05
$s_{B_1}(1)$	-9.35	-0.52	-0.28	-0.01	-0.27	0.04
$s_{max}^{P_4}$	-75.81	-9.10	-2.23	-0.05	-1.87	0.29

	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-125.52	-6.87	-3.69	-0.11	-3.68	-2.48
$s_{B_2}(1)$	0.84	-1.70	0.02	0.02	0.59	1.54
$s_{B_2}(t)$	0.84	-1.70	0.02	0.02	0.59	1.54
$s_{max}^{P_2,P_3}$	-26.93	-2.40	-0.79	0.06	2.01	2.01
$s_{B_1}(1)$	-0.81	-1.37	-0.02	0.00	0.07	1.07
$s_{max}^{P_4}$	-85.82	-9.11	-2.52	0.01	0.31	3.74

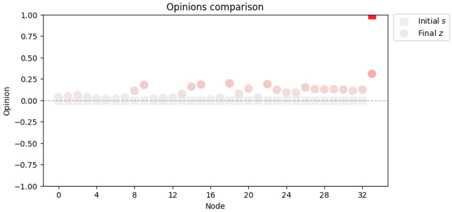
Karate Club dataset plots for $\lambda = 0.8$ and $s_{B_1(1)}$



(a) Karate club network, $s_{B_1(1)}$

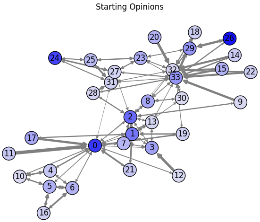


(b) Karate club network, z_{final}



(c) Karate club network, initial and final opinion vectors.

Karate Club dataset plots for $\lambda = 0.8$ and $s_{B_2(1)}$



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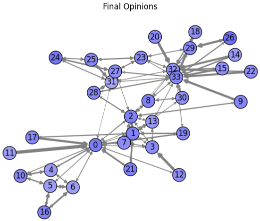
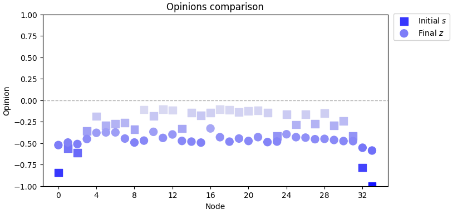


(b) Karate club network, z_{final}



(c) Karate club network, initial and final opinion vectors.

Barabasi Albert network

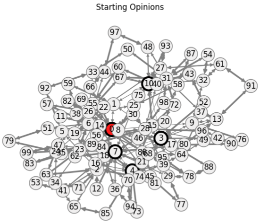
$\lambda = 0.8$, λ proportional to centrality, λ reverse proportional to centrality respectively.

	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-2730.45	-46.01	-27.30	-0.27	-27.32	-32.37
$s_{B_2}(1)$	-258.03	-14.92	-2.58	0.04	4.03	11.60
$s_{B_2}(t)$	-258.03	-14.92	-2.58	0.04	4.03	11.60
$s_{max}^{P_2,P_3}$	-426.52	-15.74	-4.27	0.12	11.98	11.98
$s_{B_1}(1)$	-83.35	-7.06	-0.83	-0.01	-0.67	3.18
$s_{max}^{P_4}$	-2097.50	-68.86	-20.97	-0.01	-1.46	22.54

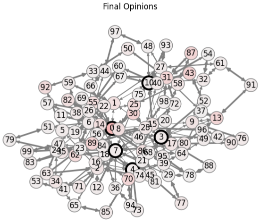
	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-2097.50	-68.86	-20.97	-0.01	-1.46	22.54
$s_{B_2}(1)$	-1.86	-0.11	-0.02	0.00	0.02	0.02
$s_{B_2}(t)$	-1.86	-0.11	-0.02	0.00	0.02	0.02
$s_{max}^{P_2,P_3}$	-1.79	-0.11	-0.02	0.00	0.02	0.02
$s_{B_1}(1)$	-20.76	-0.55	-0.21	-0.00	-0.21	0.02
$s_{max}^{P_4}$	-468.57	-21.01	-4.69	-0.04	-4.41	0.24

	Δ_{GDI}	Δ_{NDI}	Δ_{P_1}	Δ_{P_2}	Δ_{P_3}	Δ_{P_4}
s_{unif}	-545.31	-12.65	-5.45	-0.05	-5.41	-4.21
$s_{B_2}(1)$	0.51	-0.87	0.01	0.00	0.30	1.28
$s_{B_2}(t)$	0.51	-0.87	0.01	0.00	0.30	1.28
$s_{max}^{P_2,P_3}$	-96.59	-3.35	-0.97	0.03	2.62	2.62
$s_{B_1}(1)$	-1.58	-0.82	-0.02	0.00	0.00	0.66
$s_{max}^{P_4}$	-510.59	-16.82	-5.11	-0.01	-0.99	5.46

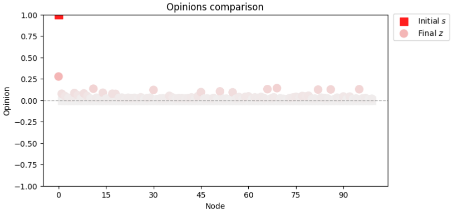
Barabassi Albert network plots with $\lambda = 0.8$ and $s_{B_1(1)}$



(a) Barabassi Albert network, $s_{B_1(1)}$

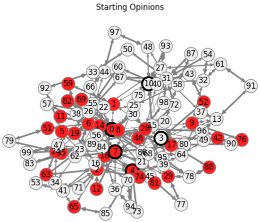


(b) Barabassi Albert network, z_{final}



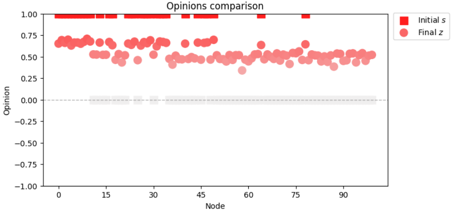
(c) Barabassi Albert network, initial and final opinion vectors.

Barabassi Albert network plots with $\lambda = 0.8$ and s_{max}^{P4}



(a) Barabassi Albert network, s_{max}^{P4}

(b) Barabassi Albert network, z_{final}



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Class-related Takeaway

- Class theoretical tools used in paper:
  - Graphs Notation
  - Social Networks: Notation, **Barabasi-Albert** model.
  - **PageRank centrality** used for susceptibility values for the paper results.
  - Occurrence of a non-Convex and a linear **optimization problem** (Theorems 5 and 6 respectively).

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- Other theoretical tools:
  - **Eigenvalue decomposition** used in Theorem 5 for analytical construction of some polarizing initial vectors **s**.
  - Definitions from analysis (l_2-Ball, subspaces).
- Estimated difficulty:
  - Despite the complex and massive formulation, the core concepts and ideas are easy to understand and elegant.
  - Proofs of the theorems are a bit harder to understand (as usual) but don't prevent from understanding the points of the paper.

Thank you for your attention![1]

[1] Dynamics of opinion polarization, E. Biondi https://arxiv.org/abs/2206.06134