Παρακάτω δίνονται οι θεωρητικοί υπολογισμοί των θεμάτων 1,3,4 και 6.

Θέμα 1: Bayes

6ε (1)

Γενικά ισχύει: $P(x|\omega_1)\,P(\omega_1) \underset{R_2}{\overset{R_1}{\gtrless}} P(x|\omega_2)\cdot P(\omega_2)$

$$P(x|\omega_1) = \frac{1}{\sqrt{(2\pi)^2|\Sigma_1|}}\cdot \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu}_1)^T\cdot\Sigma_1^{-1}\cdot(\vec{x}-\vec{\mu}_1)\right)$$

$$P(x|\omega_2) = \frac{1}{\sqrt{(2\pi)^2|\Sigma_2|}}\cdot \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu}_2)^T\cdot\Sigma_2^{-1}\cdot(\vec{x}-\vec{\mu}_2)\right)$$

α) Σύνορο απόφασης:

$$P(x|\omega_1)\cdot P(\omega_2) = P(x|\omega_2)\cdot P(\omega_2) \Rightarrow \ln\left(P(x|\omega_1)\cdot P(\omega_1)\right) = \ln\left(P(x|\omega_2)\,P(\omega_2)\right)$$

$$\Rightarrow \ln\left(P(x|\omega_1)\right) = \ln\left(\frac{P(\omega_2)}{P(\omega_1)}\right) + \ln\left(P(x|\omega_2)\right)$$

$$\Rightarrow \ln\left(\frac{1}{\sqrt{(2\pi)^2\cdot|\Sigma_1|}}\right) - \frac{1}{2}\cdot(\vec{x}-\vec{\mu}_1)^T\cdot\Sigma_1^{-1}(\vec{x}-\vec{\mu}_1) = \ln\left(\frac{1}{\sqrt{(2\pi)^2|\Sigma_2|}}\right) - \frac{1}{2}(\vec{x}-\vec{\mu}_2)\cdot\Sigma_2^{-1}\cdot(\vec{x}-\vec{\mu}_2) + \ln\left(\frac{P(\omega_2)}{P(\omega_1)}\right)$$

$$\Rightarrow \frac{1}{2}(\vec{x}-\vec{\mu}_1)\cdot\Sigma_1^{-1}(\vec{x}-\vec{\mu}_1) = \frac{1}{2}\cdot(\vec{x}-\vec{\mu}_2)^T\cdot\Sigma_2^{-1}\cdot(\vec{x}-\vec{\mu}_2) - \ln\left(\frac{P(\omega_2)}{P(\omega_1)}\right) \quad (1)$$

Αριθμητικοί υπολογισμοί:

$|\Sigma_1| = (1,2)^2 - (0,4)^2 = 1,28$

$|\Sigma_2| = 1,28$

$$\Sigma_1^{-1} = \frac{1}{|\Sigma_1|}\cdot\begin{bmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{12} & \sigma_{11}\end{bmatrix} = \frac{1}{1,28}\begin{bmatrix}1,2 & 0,4\\ 0,4 & 1,2\end{bmatrix} = \frac{1}{1,28}\,\Sigma_2$$

$$\Sigma_2^{-1} = \frac{1}{|\Sigma_2|}\begin{bmatrix}1,2 & -0,4\\ -0,4 & 1,2\end{bmatrix} = \frac{1}{1,28}\,\Sigma_1$$

σελ ②

$$(\vec{x}-\vec{\mu}_1)^T \cdot \Sigma_1^{-1} (\vec{x}-\vec{\mu}_1) = \left([\alpha-3,\ \beta-3] \begin{bmatrix} 1{,}2 & 0{,}4 \\ 0{,}4 & 1{,}2 \end{bmatrix} \begin{bmatrix} \alpha-3 \\ \beta-3 \end{bmatrix} \right) \frac{1}{1{,}28}$$

$$= \left([1{,}2 \cdot (\alpha-3) + 0{,}4 \cdot (\beta-3),\ 0{,}4 \cdot (\alpha-3) + 1{,}2(\beta-3)] \cdot \begin{bmatrix} \alpha-3 \\ \beta-3 \end{bmatrix} \right) \cdot \frac{1}{1{,}28}$$

$$= \left(1{,}2 \cdot (\alpha-3)^2 + 0{,}4(\alpha-3)(\beta-3) + 0{,}4(\alpha-3)(\beta-3) + 1{,}2(\beta-3)^2 \right) \cdot \frac{1}{1{,}28}$$

$$= \left(1{,}2 \cdot (\alpha-3)^2 + 1{,}2(\beta-3)^2 + 0{,}8(\alpha-3)(\beta-3) \right) \cdot \frac{1}{1{,}28}$$

$$\cdot (\vec{x}-\vec{\mu}_2)^T \Sigma_2^{-1} (\vec{x}-\vec{\mu}_2) = \left([\alpha-6,\ \beta-6] \cdot \begin{bmatrix} 1{,}2 & -0{,}4 \\ -0{,}4 & 1{,}2 \end{bmatrix} \cdot \begin{bmatrix} \alpha-6 \\ \beta-6 \end{bmatrix} \right) \frac{1}{1{,}28}$$

$$= \left([1{,}2(\alpha-6) - 0{,}4(\beta-6),\ -0{,}4(\alpha-6) + 1{,}2(\beta-6)] \cdot \begin{bmatrix} \alpha-6 \\ \beta-6 \end{bmatrix} \right) \frac{1}{1{,}28}$$

$$= \left(1{,}2(\alpha-6)^2 - 0{,}4(\alpha-6)(\beta-6) - 0{,}4(\alpha-6)(\beta-6) + 1{,}2(\beta-6)^2 \right) \frac{1}{1{,}28}$$

Αντικατάσταση στην (1) :

$$1{,}2(\alpha-3)^2 + 1{,}2(\beta-3)^2 + 0{,}8(\alpha-3)\cdot(\beta-3) = 1{,}2(\alpha-6)^2 + 1{,}2(\beta-6)^2 - 0{,}8(\alpha-6)\cdot(\beta-6) - 2{,}56 \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow 1{,}2\left[(\alpha-3)^2 + (\beta-3)^2 - (\alpha-6) + (\beta-6)^2\right] + 0{,}8\left[(\alpha-3)(\beta-3) + (\alpha-6)(\beta-6)\right] = -2{,}56 \cdot \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$1{,}2\left[\cancel{\alpha^2}-6\alpha+9+\cancel{\beta^2}-6\beta+9-\cancel{\alpha^2}+12\alpha-36-\cancel{\beta^2}+12\beta-36\right]$$

$$+\,0{,}8\left[\alpha\beta-3\alpha-3\beta+9+\alpha\beta-6\alpha-6\beta+36\right]=2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow 1{,}2\cdot(6\alpha+6\beta-54)+0{,}8(2\alpha\beta-9\alpha-9\beta+45)=2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow \cancel{7{,}2\alpha}+\cancel{7{,}2\beta}-64{,}8+1{,}6\alpha\beta-\cancel{7{,}2\alpha}-\cancel{7{,}2\beta}+36=2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow \boxed{1{,}6\alpha\beta-28{,}8=2{,}56\cdot\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)}$$

$$\Rightarrow \alpha=\frac{2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)+28{,}8}{1{,}6\beta}$$

d) $\Sigma_1=\Sigma_2=\begin{pmatrix}1{,}2 & 0{,}4\\ 0{,}4 & 1{,}2\end{pmatrix}$, $\Sigma_1^{-1}=\Sigma_2^{-1}=\frac{1}{1{,}28}\cdot\begin{bmatrix}1{,}2 & -0{,}4\\ -0{,}4 & 1{,}2\end{bmatrix}$

Με αντικατάσταση στην (1):

$$-0{,}8(\alpha-3)(\beta-3)+0{,}8(\alpha-6)(\beta-6)=2{,}56\cdot\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow 0{,}8\left(\alpha\beta-6\alpha-6\beta+36-\alpha\beta+3\alpha+3\beta-9\right)=2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow 0{,}8\cdot(-3\alpha-3\beta+27)=2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_1)}\right)\Rightarrow -2{,}4\alpha-2{,}4\beta+21{,}6=2{,}56\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

$$\Rightarrow \alpha+\beta-9=-1{,}0666\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)\Rightarrow \boxed{\alpha=-\beta+9-1{,}0666\cdot\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)}$$

ΘΕΜΑ 3

Για $x>0$:

$$\lambda_{12} P(x_0|\omega_1) \cancel{P(\omega_1)} = \lambda_{21} P(x_0|\omega_2) \cancel{P(\omega_2)}$$

$$\Rightarrow \lambda_{12} P(x_0|\omega_1) = \lambda_{21} P(x_0|\omega_2) \Rightarrow \lambda_{12} \cdot \frac{x_0}{\sigma_1^2} e^{-\frac{x_0^2}{2\sigma_1^2}} = \lambda_{21} \frac{x_0}{\sigma_2} e^{-\frac{x_0^2}{2\sigma_2^2}}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{x_0}{1} e^{-\frac{x_0^2}{2}} = \frac{x_0}{4} e^{-\frac{x_0^2}{8}} \Rightarrow \cancel{x} e^{-\frac{x_0^2}{2}} = \frac{\cancel{x}}{2} e^{-\frac{x_0^2}{8}}$$

$$\Rightarrow 2e^{-\frac{x_0^2}{2}} = e^{-\frac{x_0^2}{8}} \Rightarrow \ln(2) + \left(-\frac{x_0^2}{2}\right) = -\frac{x_0^2}{8} \Rightarrow \frac{x_0^2}{2} - \frac{x_0^2}{8} = \ln(2)$$

$$\Rightarrow \frac{3x_0^2}{8} = \ln(2) \Rightarrow x_0^2 = \frac{8\ln(2)}{3} \Rightarrow x_0 = \pm\sqrt{\frac{8\ln(2)}{3}} \overset{x>0}{=\!=} \sqrt{\frac{8}{3}\ln(2)} \simeq 1{,}359.$$

ΘΕΜΑ 4

$$X = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$X = U\Sigma V^T$: Δύο εξ: i) $X^T X = V \Sigma^2 V^T$ ii) $XX^T = U\Sigma^2 U^T$

1) $$X^T X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(X^T X - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = 0 \Rightarrow (2-\lambda)\cdot(1-\lambda)^2 - (-1)\left((-1)(1-\lambda) - 0\right) + 1\cdot(0-(1-\lambda)) = 0$$

$$\Rightarrow (2-\lambda)\cdot(1-\lambda)^2 - (1-\lambda) - (1-\lambda) = 0 \Rightarrow (2-\lambda)(1-\lambda)^2 - 2\cdot(1-\lambda)$$

$$\Rightarrow (1-\lambda)\cdot\left[(2-\lambda)(1-\lambda) - 2\right] = (1-\lambda)\left(\cancel{2} - 2\lambda - \lambda + \lambda^2 - \cancel{2}\right) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda) = 0 \Rightarrow \lambda(1-\lambda)(\lambda-3) = 0$$

Ιδιοτιμές: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$

$$\cancel{X^T X V_1 = \lambda_1 V_1 \Rightarrow X^T X V_1 = 0 \Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Για τα ιδιοδιανύσματα:

- $(X^T X - \lambda_1 I)\vec{V}_1 = 0 \Rightarrow X^T X \vec{V}_1 = 0 \Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\left.\begin{aligned} &\Rightarrow 1x + 1z = 0 \Rightarrow x = -z \\ &-x + \cancel{1}y = 0 \Rightarrow y = x \end{aligned}\right\} \Rightarrow x = y = -z$$

$2x - y + z = 0$

Άρα για $x = 1$: $y = 1$, $z = -1$ άρα $\vec{V}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

- $(X^T X - \overset{=1}{\lambda_2} I)\vec{V}_2 = 0 \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vec{V}_2 = 0 \Rightarrow \left.\begin{aligned} &x - y + z = 0 \\ &-x = 0 \Rightarrow x = 0 \\ &\cancel{x} = 0 \end{aligned}\right\} \Rightarrow \vec{V}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

- $(X^T X - \overset{=3}{\lambda_3} I)\vec{V}_2 = 0 \Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \vec{V}_2 = 0 \Rightarrow \left.\begin{aligned} &-x - y + z = 0 \\ &-x - 2y = 0 \\ &x - 2z = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &x = -2y \\ &x = 2z \end{aligned}$

Για $x = 2$ $y = -1$, $z = 1$: $\vec{V}_3 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$$\cancel{V = \begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 0 & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}} , \quad 2) \quad \Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \quad V = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix}$$

$$\cancel{ii)\; XV = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 0 & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4/\sqrt{6} \\ 0 & 0 & -3/\sqrt{6} \end{bmatrix} =}$$

3) $XX^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\det(XX^T - \lambda \mathbb{I}) = 0 \Rightarrow \left| \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \right| = 0 \Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 4 - 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = 3 \quad , \quad \lambda_2 = 1$

$\lambda_2 = 1$

$(XX^T - \lambda_1 I)\vec{u}_2 = 0 \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \left.\begin{matrix} x - y = 0 \\ -x + y = 0 \end{matrix}\right\} \Rightarrow x = y \quad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

$(XX^T - \lambda_2 \mathbb{I}) \cdot \vec{u}_2 = 0 \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x = -y \quad u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Άρα $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Άρα $X = U \cdot \Sigma \cdot V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$

$X = \sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i$

4) Rank-1 προσέγγιση:

$\hat{X} = \sigma_1 \vec{u}_1 \vec{v}_1^T = \sqrt{3} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \sqrt{3} \cdot \begin{bmatrix} 2/\sqrt{12} & [illegible] & [illegible] \\ -2/\sqrt{12} & 1/\sqrt{12} & [illegible] \end{bmatrix}$

$\Rightarrow \hat{X} = \begin{bmatrix} 1 & -1/2 & 1 \\ -1 & 1/2 & -1 \end{bmatrix}$

ΘΕΜΑ 6

Global mean: $\mu_0 = \frac{1}{2}(\mu_1 + \mu_2) = \begin{bmatrix} (-5+10)/2 \\ (10+15)/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 35/2 \end{bmatrix}$ διότι οι δύο κλάσεις είναι ισοπιθ.

$$S_w = \sum_{i=1}^{c} P_i \Sigma_i = \frac{1}{2}(\Sigma_1 + \Sigma_2) = \frac{1}{2} \begin{pmatrix} 12 & 9 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 6 & 9/2 \\ 9/2 & 6 \end{pmatrix}$$

$$S_w^{-1} = \frac{1}{13^2 - 9^2} \begin{bmatrix} 13 & -9 \\ -9 & 13 \end{bmatrix} = \frac{1}{88} \begin{bmatrix} 13 & -9 \\ -9 & 13 \end{bmatrix}$$

2-κλάσεις οπότε:

$$w = S_w^{-1}(\mu_1 - \mu_2) = \frac{1}{88} \begin{bmatrix} 13 & -9 \\ -9 & 13 \end{bmatrix} \cdot \begin{bmatrix} -15 \\ -10 \end{bmatrix} = \frac{1}{88} \begin{bmatrix} -105 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{105}{88} \\ \frac{5}{88} \end{bmatrix}$$

Κανονικοποίηση: $\|w\| = \sqrt{\frac{105^2 + 5^2}{88^2}} = \frac{105,11898}{88} = 1,19$

Άρα $\boxed{w_{norm} = \frac{w}{1,19}}$