MODULO ℓ DISTINCTION PROBLEMS

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ABSTRACT. This paper concerns the ℓ-modular representations of a connected reductive group G distinguished by a Galois involution, with ℓ an odd prime different from p. We start by proving a general theorem allowing to lift supercuspidal $\overline{\mathbf{F}}_\ell$-representations of $\mathrm{GL}_n(F)$ distinguished by an arbitrary closed subgroup H to a distinguished supercuspidal $\overline{\mathbf{Q}}_\ell$-representation. Given a quadratic field extension E/F and an irreducible $\overline{\mathbf{F}}_\ell$-representation π of $\mathrm{GL}_n(E)$, we verify the Jacquet conjecture in the modular setting that if the Langlands parameter ϕ_π is irreducible and conjugate-self-dual, then π is either $\mathrm{GL}_n(F)$-distinguished or $(\mathrm{GL}_n(F), \omega_{E/F})$-distinguished, but not both, which extends one result of Sécherre to the case $p = 2$. We give another application of our lifting theorem for supercuspidal representations distinguished by a unitary involution, extending one result of Zou to $p = 2$. After that, we give a complete classification of the $\mathrm{GL}_2(F)$-distinguished representations of $\mathrm{GL}_2(E)$. Using this classification we discuss a modular version of the Prasad conjecture for PGL_2. We show that the "classic" Prasad conjecture fails in the modular setting. We propose a solution using non-nilpotent Weil-Deligne representations. Finally, we apply the restriction method of Anandavardhanan and Prasad to classify the $\mathrm{SL}_2(F)$-distinguished modular representations of $\mathrm{SL}_2(E)$.

CONTENTS

1. INTRODUCTION

Let G be a p-adic group. A representation π of a group G is said to be *distinguished* with respect to a subgroup H of G if it admits a non-trivial H-invariant linear form. More generally, if χ is a character of H, we will say that π is (H, χ)-distinguished if there exists a non-trivial linear functional on the space of π on

which H acts via χ, i.e. $f : \pi \to R$ such that

$$f(\pi(h)v) = \chi(h)f(v)$$

for all $h \in H$ and $v \in \pi$.

Distinguished representations are central objects in the study of the *relative Langlands program*. The distinction problem is closely related to the Langlands functorial conjectures and it should be possible to characterize the (H, χ)-distinguished representations as the images with respect to a functorial transfer to G from a third group G' in many cases. There exists a rich literature, such as [AP03, SV17a, Lu18, Lu20], trying to classify all H-distinguished complex representations of G. Furthermore, when θ is the Galois involution of order 2 and $H = G^\theta$, Dipendra Prasad [Pra15] gave a precise conjecture for the multiplicity $\dim_{\mathbb{C}} \operatorname{Hom}_H(\pi, \chi_H)$ in terms of the enhanced Langlands parameter of π, where χ_H is a quadratic character of H.

More recently, mathematicians have been interested in modular representations of p-adic groups, which are smooth representations with coefficients in $\overline{\mathbf{F}}_\ell$, with $\ell \neq p$. The theory of ℓ-modular representations has been initiated by Vignéras in [Vig96]. These works are motivated by a modular local Langlands program and studying congruences between automorphic forms (which have been used to prove many remarkable theorems of arithmetic geometry).

For modular representations, much remains to be done for the distinction problems. Let F be a finite field extension of $\mathbb{Q}_p$. Sécherre and Venketasubramanian have examined the pair $(G, H) = (\operatorname{GL}_n(F), \operatorname{GL}_{n-1}(F))$ in [SV17b]. Sécherre [Sí9] also investigated the pair $(\operatorname{GL}_n(E), \operatorname{GL}_n(F))$ for supercuspidal representations, where E is a quadratic extension of F. Ongoing work by Kurinczuk, Matringe and Sécherre aims at extending the results of the pair $(\operatorname{GL}_n(E), \operatorname{GL}_n(F))$ to all representations (see [KMS]).

In this paper, we consider a quadratic field extension E/F of locally compact non-archimedean local fields of characteristic zero and residual characteristic p. Denote by q_F (resp. q_E) the cardinality of the residue field of F (resp. E). Let ℓ be an odd prime different from p. We are interested in the distinction problems for the pairs $(\operatorname{GL}_n(E), \operatorname{GL}_n(F))$ and $(\operatorname{SL}_2(E), \operatorname{SL}_2(F))$. We also give a modular version of the Prasad conjecture for PGL_2.

We would like to point out that the ongoing work [KMS] of Kurinczuk, Matringe and Sécherre also gives a classification of all irreducible $\operatorname{GL}_2(F)$-distinguished representations of $\operatorname{GL}_2(E)$ (at least for $p \neq 2$). However, our methods and theirs are completely different. They use the gamma factors and the epsilon factors whereas in this article we use Mackey theory.

1.1. **Distinguished supercuspidal representations of** GL_n. Our first result, discussed in Section 3, is the following:

Theorem 1.1 (Theorem 3.4). *Let H be a closed subgroup of $\operatorname{GL}_n(F)$. Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\operatorname{GL}_n(F)$ which is H-distinguished. Then there exists a $\overline{\mathbf{Q}}_\ell$-lift $\tilde{\pi}$ of π such that it is supercuspidal and distinguished by H.*

The proof depends on the type theory and the existence of a projective envelope for supercuspidal types. This is a very general result allowing us to transfer the problem from modular to complex representations. As an immediate consequence, we get:

Corollary 1.2 (Corollary 3.5)**.** *There is no supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$.*

We also give two other applications. The first one, is to study supercuspidal representations of $\mathrm{GL}_n(E)$ distinguished by a Galois involution. For $\ell \neq 2$, we prove the *Dichotomy Theorem* and the *Disjunction Theorem*. Let $\omega_{E/F}$ be the quadratic character of $F^\times$ associated to the field extension E/F by the local class field theory and let σ be the non-trivial F-automorphism of E.

Theorem 1.3 (Theorem 3.14)**.** *Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$. Then π is σ-selfdual if and only if it is either distinguished or $\omega_{E/F}$-distinguished by $\mathrm{GL}_n(F)$, but not both.*

For the complex representations, it is called the Jacquet conjecture proved by Kable in [Kab04]. We first show that we can lift σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representations to σ-selfdual supercuspidal $\overline{\mathbf{Q}}_\ell$-representations. This follows from the fact that since ℓ is odd, the number of inertial classes of such lifts is odd, hence has a fixed point under σ-duality. Then we prove that all these supercuspidal lifts share the same sign due to Schur's Lemma. Combine with the previous theorem, this allows us to deduce the Jacquet conjecture for modular representations from complex representations.

Remark 1.4. When $p \neq 2$, the distinguished supercuspidal representations were studied by Sécherre in [Sí9] (with no restriction on ℓ). Our method is different than the one used in [Sí9] and works for $\ell \neq 2$ but for any p. Therefore, combining the results in this paper and in [Sí9] gives the complete result for distinguished supercuspidal representations for all ℓ and p with $\ell \neq p$.

We also have a characterization using the local Langlands correspondence of Vignéras.

Proposition 1.5 (Proposition 3.15)**.** *Let π be a σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$ and φ its Langlands parameter. Then π is distinguished (resp. $\omega_{E/F}$-distinguished) by $\mathrm{GL}_n(F)$ if and only if φ_π is conjugate-orthogonal (resp. conjugate-symplectic).*

The last application, is for representations distinguished by a unitary involution. Let $\ell \neq 2$ and ς be a *unitary involution* of $GL_n(E)$, that is $\varsigma(g) = \varepsilon \sigma({}^t g^{-1}) \varepsilon^{-1}$ where ε is an hermitian matrix in $M_n(E)$.

Theorem 1.6 (Theorem 3.17)**.** *Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$ and ς a unitary involution of G. Then π is distinguished by $\mathrm{GL}_n(E)^\varsigma$ if and only if $\pi^\sigma \simeq \pi$.*

This extends the result of [Zou21] to the case $p = 2$.

1.2. **Modular distinguished representations for** $(\mathrm{GL}_2(E), \mathrm{GL}_2(F))$**.** In section 4, we complete the classification of all representations of $\mathrm{GL}_2(E)$ distinguished by $\mathrm{GL}_2(F)$.

We use Mackey theory to show that the principal series representation $\pi(\chi_1, \chi_2)$ is distinguished by $\mathrm{GL}_2(F)$ if and only if either $\chi_1\chi_2^\sigma = \mathbf{1}$ or $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$ with $\chi_1 \neq \chi_2$. (See Lemma 4.3 for more details.)

To get a complete classification, we are just missing irreducible subquotients of non-irreducible principal series. We denote by $\nu^{1/2}$ the unramified character of $E^\times$ sending a uniformizer to $q_E^{1/2}$ where the square root $q_E^{1/2}$ of q_E is fixed. Let χ be a character of $E^\times$. When $q_E \not\equiv -1 \pmod{\ell}$, we denote by St_χ the twisted Steinberg representation, that is the unique generic irreducible subquotient of $\pi(\chi\nu^{-1/2}, \chi\nu^{1/2})$. In the case $q_E \equiv -1 \pmod{\ell}$, the generic irreducible subquotient of $\pi(\chi\nu^{-1/2}, \chi\nu^{1/2})$ is the special representation, denoted by Sp_χ.

Theorem 1.7 (Theorem 4.6). *Let χ be a character of $E^\times$.*

(1) *If $\ell \nmid q_E^2 - 1$, then St_χ is $\mathrm{GL}_2(F)$-distinguished if and only if $\chi_{|F^\times} = \omega_{E/F}$.*
(2) *If $\ell \mid q_E + 1$, then Sp_χ is $\mathrm{GL}_2(F)$-distinguished if and only if $\ell \mid q_F + 1$ and $\chi_{|F^\times} = \omega_{E/F}$ or $\nu_{|F^\times}^{1/2}$.*
(3) *If $\ell \mid q_E - 1$, then St_χ is $\mathrm{GL}_2(F)$-distinguished if and only if $\chi_{|F^\times} = \omega_{E/F}$ or $\chi_{|F^\times} = \mathbf{1}$ with $\ell \mid q_F - 1$.*

Remark 1.8. From this theorem, we would like to highlight the fact that we have in the modular case new phenomena that do not appear in the complex setting.

- When $q_F \equiv 1 \pmod{\ell}$, the Steinberg representation St is both $\mathrm{GL}_2(F)$-distinguished and $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished.
- When $q_E \equiv -1 \pmod{\ell}$ and E/F is unramified (and so $q_F \not\equiv 1 \pmod{\ell}$) the special representation Sp is neither $\mathrm{GL}_2(F)$-distinguished nor $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished. This has been mentioned by Vincent Sécherre in [Sí9, Rem. 2.8].
- When $q_E \equiv -1 \pmod{\ell}$ and E/F is ramified (that is $q_F \equiv -1 \pmod{\ell}$) the special representation Sp is $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished and is also $(\mathrm{GL}_2(F), \nu_{|F^\times}^{1/2})$-distinguished.

1.3. **The modulo ℓ Prasad conjecture.** In section 5, we discuss the Prasad conjecture for modular representations. In [Pra15], Dipendra Prasad proposed a conjecture for the multiplicity $\dim_{\mathbb{C}} \mathrm{Hom}_{G(F)}(\pi, \chi_G)$ under the local Langlands conjecture, where G is a quasi-split reductive group defined over F, π is an irreducible smooth representation of $G(E)$ lying in a generic L-packet and χ_G is a quadratic character depending on G and the quadratic extension E/F. Since the local Langlands correspondence for the ℓ-modular representations of $G(F)$ has not been set up in general, except for $G = \mathrm{GL}_n$, we are concerned only with the simplest case where $G = \mathrm{PGL}_2$.

The modulo ℓ local Langlands correspondence for GL_2 has been defined by Vignéras in [Vig01], and is a map $V : \mathrm{Irr}_R(\mathrm{GL}_2(E)) \to \mathrm{Nilp}_R(W_E, \mathrm{GL}_2)$, where $\mathrm{Nilp}_R(W_E, \mathrm{GL}_2)$ denotes the set of isomorphism classes of nilpotent semisimple Weil-Deligne representations. Since the correspondence V sends the central character to the determinant, it induces a map

$$PV : \mathrm{Irr}_R(\mathrm{PGL}_2(E)) \to \mathrm{Nilp}_R(W_E, \mathrm{SL}_2).$$

Using this correspondence, the Prasad conjecture is not valid for ℓ-modular representations (at least in the non-banal case). Actually, it does not work for any map

$L : \mathrm{Irr}_R(\mathrm{PGL}_2(E)) \to \mathrm{Nilp}_R(W_E, \mathrm{SL}_2)$ having the same semisimple part as PV (see Section 5.2).

In order to propose a solution for ℓ-modular representations, we will consider non-nilpotent Weil-Deligne representations as in [KM21]. Let $\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)$ denote the set of isomorphism classes of semisimple Weil-Deligne representation of GL_2. Kurinczuk and Matringe define in [KM21] an equivalence relation $\sim$ on $\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)$, and we denote by $[\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)]$ the quotient set.

Then we define an injection from $\mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$ to $[\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$ (which is not the trivial one in the non-banal case) in the following way. Let χ be a quadratic character of $E^\times$. If $\ell \mid q_E - 1$, we denote by $\Psi_{\mathrm{St},\chi} \in \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$, the Weil-Deligne representation $\Psi_{\mathrm{St},\chi} = (\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, N)$ with $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. And if $\ell \mid q_E + 1$, let $\Psi_{\mathrm{Sp},\chi} := (\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, 0)$. We define an injection

$$P : \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2) \hookrightarrow [\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$$

by

$$P(\Psi) = \begin{cases} [\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] & \text{if } \ell \mid q_E - 1 \text{ and } \Psi = \Psi_{\mathrm{St},\chi} \\ [\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] & \text{if } \ell \mid q_E + 1 \text{ and } \Psi = \Psi_{\mathrm{Sp},\chi} \\ [\Psi] & \text{otherwise.} \end{cases}$$

Then we prove a modulo ℓ version of the Prasad conjecture, using our modified injection P:

Theorem 1.9 (Theorem 5.12). *Let π be an irreducible generic $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{PGL}_2(E)$. Then π is $\omega_{E/F}$-distinguished by $\mathrm{PGL}_2(F)$ if and only if there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim P \circ PV(\pi)$.*

Remark 1.10.   (1) In the banal case i.e., $\ell \nmid q_E^2 - 1$, $[\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)] = \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$ and P is the trivial injection, so we found the "classical" Prasad conjecture.
   (2) When $\ell \mid q_E + 1$, $\ell \mid q_F + 1$ and $\pi = \mathrm{Sp}_\chi$ for a quadratic character χ of $E^\times$ such that $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$, then the representation Ψ_F is not the Langlands parameter of any representation of $\mathrm{PGL}_2(F)$ (nor it is in the image of P).

1.4. **Modular distinguished representations for** $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$. In the last part, Section 6, we use our classification of $\mathrm{GL}_2(F)$-distinguished representation of $\mathrm{GL}_2(E)$ and the restriction method of [AP03] to classify $\mathrm{SL}_2(F)$-distinguished representations for $\mathrm{SL}_2(E)$ in the modular setting.

Let π be an irreducible $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$. We denote by $\mathrm{lg}(\pi)$ the length of $\pi|_{\mathrm{SL}_2(E)}$ and by $\mathrm{lg}_+(\pi)$ the length of $\pi|_{\mathrm{GL}_2^+(E)}$, where $\mathrm{GL}_2^+(E)$ is the subgroup of $\mathrm{GL}_2(E)$, consisting of matrices whose determinants belong to $F^\times E^{\times 2}$.

For supercuspidal representations, we adapt the methods of [AP03] and prove the following theorems.

Theorem 1.11 (Theorem 6.14). *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$ distinguished by $\mathrm{SL}_2(F)$, and π^+ the unique irreducible component of $\pi|_{\mathrm{GL}_2(E)^+}$ that is ψ-generic. Then π^+ is distinguished by $\mathrm{SL}_2(F)$. Furthermore, let τ be an irreducible component of $\pi|_{\mathrm{SL}_2(E)}$, distinguished by $\mathrm{SL}_2(F)$. Then τ is an irreducible component of $\pi^+|_{\mathrm{SL}_2(E)}$.*

Theorem 1.12 (Theorem 6.15). *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$, and τ is an irreducible component of $\pi|_{\mathrm{SL}_2(E)}$. Suppose τ is distinguished by $\mathrm{SL}_2(F)$. Then,*

$$\mathrm{dimHom}_{\mathrm{SL}_2(F)}(\tau, 1) = \begin{cases} 1, & \text{if } \pi|_{\mathrm{SL}_2(E)} \cong \tau; \\ 1, & \text{if } \lg_+(\pi) = 2 \text{ and } \lg(\pi) = 4; \\ 2, & \text{if } \lg_+(\pi) = \lg(\pi) = 2; \\ 4, & \text{if } \lg_+(\pi) = \lg(\pi) = 4. \end{cases}$$

The first case and the last case arise only when $p = 2$.

We also have a criterion for principal series representations. See Theorem 6.16 for more details.

Along the way, we prove the existence of a good lifting for supercuspidal representations of $\mathrm{SL}_2(E)$. Let τ be an irreducible cuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{SL}_2(E)$, and $\tilde{\tau}$ an irreducible cuspidal $\overline{\mathbf{Q}}_\ell$-representation of $\mathrm{SL}_2(E)$, which is ℓ-integral. We say that $\tilde{\tau}$ is a *good $\overline{\mathbf{Q}}_\ell$-lifting* of τ, if the reduction modulo ℓ of $\tilde{\tau}$ is irreducible and isomorphic to τ. We prove in Proposition 6.7 that supercuspidal representations of $\mathrm{SL}_2(E)$ always admit a good $\overline{\mathbf{Q}}_\ell$-lifting.

Acknowledgements

2. Notation

Let E/F be a quadratic extension of locally compact non-archimedean local fields of characteristic zero and residual characteristic p. Denote by q_F (resp. q_E) the cardinality of the residue field of F (resp. E). Let W_F be the Weil group of F. We denote by σ the non-trivial F-automorphism of E.

Denote by $\omega_{E/F}$ the quadratic character of $F^\times$ associated to the quadratic field extension E/F by the local class field theory. Let $\mathrm{Nm}_{E/F}$ be the norm from E to F.

For $n \geq 1$, we denote by GL_n the general linear group, and by SL_n the special linear group. Denote the standard Borel subgroup of GL_n by B_n. Let π_i (i=1,2) be an irreducible representation of $\mathrm{GL}_{n_i}(F)$. Denote by $\pi_1 \boxtimes \pi_2$ the tensor product representation of $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$. For a character χ of $F^\times$, we will also denote by χ the character $\chi \circ \det$ of $\mathrm{GL}_n(F)$. Let ν_n be the character $g \mapsto |\det(g)|$ of $\mathrm{GL}_n(F)$.

Let H be a subgroup of G. Let π be an irreducible representation of G. We say that π is H-distinguished if $\mathrm{Hom}_H(\pi, \mathbf{1}) \neq 0$. In case that the subgroup H is clear, we say that π is distinguished sometimes.

In this article, ℓ is an odd prime number different from p (except for Section 3.1, where $\ell = 2$ is also allowed).

3. Distinguished supercuspidal representations of GL_n

In this section, we study the distinguished supercuspidal representation of GL_n. Firstly, we prove a lifting theorem for supercuspidal representations distinguished by an arbitrary closed subgroup H. This theorem allows us to study the supercuspidal representations of $\mathrm{GL}_n(E)$ distinguished by a Galois involution, and extend the result of [Sĺ9] to the case $p = 2$. We also give an application to supercuspidal representations of $\mathrm{GL}_n(E)$ distinguished by a unitary involution.

3.1. **Lifting of modular distinguished representation.** In this section, $G = \mathrm{GL}_n(F)$ and H is an arbitrary closed subgroup of G. In this part only, we do not require the prime ℓ to be odd, just $\ell \neq p$. We want to prove that a supercuspidal H-distinguished $\overline{\mathbf{F}}_\ell$-representation can be lifted to a supercuspidal H-distinguished $\overline{\mathbf{Q}}_\ell$-representation. This allows bringing the modular distinction problems to the complex setting. To do that, we use type theory and the existence of a projective envelope.

We start with a few lemmas. Let K be a finite field extension of F. We denote by ϖ_K (resp. ϖ_F) a uniformizer of K (resp. F). Let $\mathcal{O}_K$ be the ring of integers of K, and for an integer $i \geq 1$, we denote by U_K^i the subgroup of $\mathcal{O}_K^\times$ defined by $U_K^i := \{1 + \varpi_K^i u, u \in \mathcal{O}_K\}$.

Lemma 3.1. *Let $i \geq 1$ and $x \in U_K^i$. Then $x^p \in U_K^{i+1}$.*

Proof. Let us write $x = 1 + \varpi_K^i u$, with $u \in \mathcal{O}_K$. The binomial expansion gives us $x^p = 1 + p\varpi_K^i u + \varpi_K^{2i} u'$, with $u' \in \mathcal{O}_K$. But ϖ_K divides p in $\mathcal{O}_K$ so $x^p \in U_K^{i+1}$. $\square$

Lemma 3.2. *Let $N \in \mathbb{N}^*$. Then there exist two integers $m \geq 1$ and $s \geq 1$ such that $\varpi_K^m \in \varpi_F^s U_K^N$.*

Proof. Let e be the ramification index of K over F. Then $\varpi_K^e = \varpi_F u$, with $u \in \mathcal{O}_K^\times$. Now $\mathcal{O}_K^\times / U_K^1$ is isomorphic to $k_K^\times$ the residue field of K of cardinal $q_K - 1$. Therefore $x := u^{q_K - 1} \in U_K^1$ and from Lemma 3.1, $x^{p^N} \in U_K^N$. We get the result with $m = e(q_K - 1)p^N$ and $s = (q_K - 1)p^N$. $\square$

Let $G := \mathrm{GL}_n(F)$. Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of G. Let us write $\pi = \text{c-Ind}_J^G(\Lambda)$, for an extended maximal simple type (J, Λ). Let J^0 be the unique maximal compact subgroup of J, J^1 its maximal normal pro-p-subgroup and K be the field extension of F associated to (J, Λ), such that $J = K^\times J^0 = \langle \varpi_K, J^0 \rangle$.

Proposition 3.3. *The representation Λ admits a projective envelope in the category of finite length $\overline{\mathbf{Z}}_\ell$-representations of J.*

Proof. Since $F^\times$ is central in J, it acts on Λ by a character χ. Let η be a character of J, trivial on J^1 such that $\eta(\varpi_F) = \chi(\varpi_F)$. Let $\Sigma := \Lambda \eta^{-1}$ such that ϖ_F acts trivially on Σ.

From type theory, there exists an open subgroup H^2 of J^1 such that $H^2 \subseteq \ker(\Sigma_{|J^0})$ and J^0/H^2 is finite. Since H^2 is open in J^0, there exists an integer

$N \geq 1$ such that $U_K^N \subseteq H^2$. By Lemma 3.2, there exist $m, n \geq 1$ such that $\varpi_K^m \in \varpi_F^s U_K^N$. We define a group Q by

$$Q = J/\langle H^2, \varpi_K^m \rangle.$$

The group Q is finite. Moreover, since $\varpi_K^m \in \varpi_F^s H^2$, we have that $\langle H^2, \varpi_K^m \rangle = \langle H^2, \varpi_F^s \rangle$. Therefore Σ is trivial on $\langle H^2, \varpi_K^m \rangle$ and we can consider Σ as a representation of Q.

Since Q is a finite group, we can consider the projective envelope of Σ in the category of $\overline{\mathbf{Z}}_\ell[Q]$-modules, denoted by $\mathcal{P}$. We can regard $\mathcal{P}$ as a representation of J by inflation. Since H^2 is a pro-p-group and ϖ_F^s is central in J, the proof of [DS21, Lem. 4.1] works with $\langle H^2, \varpi_F^s \rangle$. In particular, [DS21, Lem. 4.2] shows that $\mathcal{P}$ is the projective envelope of Σ in the category of finite length $\overline{\mathbf{Z}}_\ell$-representations of J. By [DS21, Lem. 4.3] $\mathcal{P}\eta$ is the projective envelope of $\Sigma\eta = \Lambda$ and we get the result. $\qquad\square$

Using this projective envelope, we can prove the desired theorem.

Theorem 3.4. *Let H be a closed subgroup of $\mathrm{GL}_n(F)$. Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ which is H-distinguished. Then there exists a $\overline{\mathbf{Q}}_\ell$-lift $\tilde{\pi}$ of π such that it is supercuspidal and distinguished by H.*

Proof. Let us write $\pi = \mathrm{c\text{-}Ind}_J^G(\Lambda)$, for an extended maximal simple type (J, Λ). Frobenius reciprocity and Mackey formula give us

$$0 \neq \mathrm{Hom}_H(\pi, 1) \simeq \prod_{g \in J\backslash G/H} \mathrm{Hom}_{J^g \cap H}(\Lambda^g, 1).$$

Thus there exists a $g \in G$ such that $\mathrm{Hom}_{J^g \cap H}(\Lambda^g, 1) \neq 0$. Let $H' := H^{g^{-1}}$ such that $J^g \cap H = (J \cap H')^g$. Therefore

$$0 \neq \mathrm{Hom}_{J^g \cap H}(\Lambda^g, 1) \simeq \mathrm{Hom}_{J \cap H'}(\Lambda, 1).$$

From Proposition 3.3, there exists a projective envelope of Λ in the category of finite length $\overline{\mathbf{Z}}_\ell$-representations of J, denoted by $\mathcal{P}$. Since Λ is a supercuspidal representation of J^0/J^1, by [Vig96, Chapitre III, 2.9] it satisfies the following properties

(1) $\mathcal{P} \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell$ is the projective envelope of Λ in the category of finite length $\overline{\mathbf{F}}_\ell$-representation of J and is indecomposable, with each irreducible component isomorphic to Λ. Since $0 \neq \mathrm{Hom}_{J \cap H'}(\Lambda, 1)$, we get that $\mathrm{Hom}_{\overline{\mathbf{F}}_\ell[J]}(\mathcal{P} \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{F}}_\ell) \neq 0$.

(2) Let $\widetilde{\mathcal{P}} := \mathcal{P} \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{Q}}_\ell$ be a $\overline{\mathbf{Q}}_\ell$-lift of $\mathcal{P}$. Then $\widetilde{\mathcal{P}}$ is isomorphic to the direct sum of all the $\overline{\mathbf{Q}}_\ell$-lifts of Λ.

Since $\mathcal{P}$ is a projective $\overline{\mathbf{Z}}_\ell[Q]$-module, it is a free $\overline{\mathbf{Z}}_\ell$-module. Since $\mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{F}}_\ell$ is also a free $\overline{\mathbf{Z}}_\ell$-module, we have that $\mathrm{Hom}_{\overline{\mathbf{Z}}_\ell[J]}(\mathcal{P}, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{Z}}_\ell)$ is a free $\overline{\mathbf{Z}}_\ell$-module. From (1) we have

$$0 \neq \mathrm{Hom}_{\overline{\mathbf{F}}_\ell[J]}(\mathcal{P} \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{F}}_\ell) \simeq \mathrm{Hom}_{\overline{\mathbf{Z}}_\ell[J]}(\mathcal{P}, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{Z}}_\ell) \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell.$$

Thus

$$\mathrm{Hom}_{\overline{\mathbf{Z}}_\ell[J]}(\mathcal{P}, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{Z}}_\ell) \neq 0$$

and
$$\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[J]}(\widetilde{\mathcal{P}}, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{Q}}_\ell) \simeq \mathrm{Hom}_{\overline{\mathbf{Z}}_\ell[J]}(\mathcal{P}, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{Z}}_\ell) \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{Q}}_\ell \neq 0.$$

From (2) there exists a $\overline{\mathbf{Q}}_\ell$-lifts $\widetilde{\Lambda}$ of Λ such that $\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[J]}(\widetilde{\Lambda}, \mathrm{Ind}_{J \cap H'}^J \overline{\mathbf{Q}}_\ell) \neq 0$. Hence $\mathrm{Hom}_{J \cap H'}(\widetilde{\Lambda}, 1) \neq 0$.

Let $\widetilde{\pi} := \text{c-Ind}_J^G(\widetilde{\Lambda})$. The representation $\widetilde{\pi}$ is then a $\overline{\mathbf{Q}}_\ell$-lift of π and is supercuspidal. We are left to prove that it is H-distinguished. As before

$$\mathrm{Hom}_{J^g \cap H}(\widetilde{\Lambda}^g, 1) \simeq \mathrm{Hom}_{J \cap H'}(\widetilde{\Lambda}, 1) \neq 0$$

and Mackey formula and Frobenius reciprocity give us that $\widetilde{\pi}$ is H-distinguished.
$\square$

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.5. *There is no supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$.*

Proof. It follows from [HR90, Thm. 3.2.2] that there is no complex supercuspidal representation of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. Therefore, the result follows from Theorem 3.4.
$\square$

3.2. **The Langlands correspondence modulo ℓ for GL_n.** In the rest of this paper we will use the modulo ℓ local Langlands correspondence to study distinguished representations. We recall in this section some important results about it.

The first step to define a modulo ℓ local Langlands correspondence for GL_n is to define the correspondence for supercuspidal representations. The supercuspidal representations correspond to irreducible representations of the Weil group. This bijection is defined in the modulo ℓ case using the modulo ℓ reduction. This leads to the semisimple Langlands correspondence using the supercuspidal support of a smooth representation.

Let R be an algebraically closed field of characteristic different from p. We denote by $\mathrm{Irr}_R(\mathrm{GL}_n(F))$ the set of isomorphism classes of smooth irreducible representations of $\mathrm{GL}_n(F)$ over R and by $\mathrm{Scusp}_R(\mathrm{GL}_n(F))$ the subset of supercuspidal R-representations. Let $\mathrm{Irr}_R(\mathrm{GL}_F) := \cup_{n \geq 1} \mathrm{Irr}_R(\mathrm{GL}_n(F))$ and $\mathrm{Scusp}_R(\mathrm{GL}_F) := \cup_{n \geq 1} \mathrm{Scusp}_R(\mathrm{GL}_n(F))$.

Let W_F be the Weil group of F and $\mathrm{Irr}_R(W_F)(n)$ be the set of isomorphism classes of continuous irreducible R-representations of W_F of dimension n. Finally, we denote by $\mathrm{Mod}_R(W_F)$ the set of isomorphism classes of continuous semisimple R-representations of W_F of finite dimension.

In [Vig01] Vignéras defines a bijection using the modulo ℓ reduction of the local Langlands correspondence over $\overline{\mathbf{Q}}_\ell$

Theorem 3.6 ([Vig01, Cor. 2.5]). *There exist unique Langlands bijections on $\overline{\mathbf{F}}_\ell$*

$$\mathrm{Scusp}_{\overline{\mathbf{F}}_\ell}(\mathrm{GL}_n(F)) \leftrightarrow \mathrm{Irr}_{\overline{\mathbf{F}}_\ell}(W_F)(n)$$

which are compatible with modulo ℓ reduction.

Let $\text{MScusp}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F)$ be the set of formal finite sums $\pi_1 + \cdots + \pi_r$ of elements of $\text{Scusp}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F)$. The previous bijections induce a bijection

$$\text{MScusp}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F) \to \text{Mod}_{\overline{\mathbf{F}}_\ell}(W_F).$$

Let $\pi \in \text{Irr}_{\overline{\mathbf{F}}_\ell}(\text{GL}_n(F))$. Then π appears as a subquotient of the parabolic induction of a representation $\pi_1 \boxtimes \cdots \boxtimes \pi_r$, where $\pi_i \in \text{Scusp}_{\overline{\mathbf{F}}_\ell}(\text{GL}_{n_i}(F))$ and $\sum n_i = n$. This defines a surjective map with finite fibers, called the *supercuspidal support*,

$$sc : \text{Irr}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F) \to \text{MScusp}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F)$$

by $sc(\pi) = \pi_1 + \cdots + \pi_r$.

Combining the supercuspidal support and the previous bijections (see [Vig01, Thm. 1.6]) we get a map, that we call the semisimple Langlands correspondence modulo ℓ,

$$V_{ss} : \text{Irr}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F) \to \text{Mod}_{\overline{\mathbf{F}}_\ell}(W_F).$$

The previous map V_{ss} is a surjection but not a bijection. To get a bijection one needs to introduce Weil-Deligne representations.

Let us start by recalling the definition of Weil-Deligne representations.

Definition 3.7. We call a semisimple Weil-Deligne representation of GL_n over R a couple (φ, N) with

(1) $\varphi : W_F \to \text{GL}_n(R)$ that is a continuous homomorphism with image composed of semisimple elements.
(2) $N \in M_n(R)$ is a matrix such for all $w \in W_F$, $\varphi(w)N = \nu(w)N\varphi(w)$.

We denote by $\text{WDRep}_R(W_F, \text{GL}_n)$ the set of isomorphism classes of semisimple Weil-Deligne representation of GL_n. When $R = \overline{\mathbf{Q}}_\ell$, every N as above is nilpotent. However, this is not the case when $R = \overline{\mathbf{F}}_\ell$. Therefore, we denote by $\text{Nilp}_R(W_F, \text{GL}_n)$ the subset composed of the elements (ϕ, N) with N nilpotent.

Theorem 3.8 ([Vig01, Thm. 1.8.2]). *Let $\varphi \in \text{Mod}_{\overline{\mathbf{F}}_\ell}(W_F)$. The set of $\pi \in \text{Irr}_{\overline{\mathbf{F}}_\ell}(\text{GL}_F)$ such that $V_{ss}(\pi) = \varphi$ is in bijection with the nilpotent endomorphisms N of π, up to isomorphism, such that $\varphi(w)N = \nu(w)N\varphi(w)$, for all $w \in W_F$.*

Hence, we get a bijection $\text{Irr}_R(\text{GL}_F) \to \text{Nilp}_R(W_F, \text{GL})$. Vignéras defines the "Langlands" bijection in [Vig01, Section 1.8] using modulo ℓ reduction (in a non-naive way), and we call this bijection the V correspondence :

$$V : \text{Irr}_R(\text{GL}_F) \to \text{Nilp}_R(W_F, \text{GL}).$$

3.3. **Supercuspidal representations of $\text{GL}_n(E)$ distinguished by a Galois involution.** Let E/F be a quadratic extension of locally compact non-archimedean local fields of characteristic zero and residual characteristic p. In this section we are interested in $\text{GL}_n(F)$-distinguished supercuspidal ℓ-modular representations of $\text{GL}_n(E)$. A $\text{GL}_n(F)$-distinguished supercuspidal representation is σ-selfdual, that is $\pi^\vee \simeq \pi^\sigma$ (see [Sí9, Thm. 4.1]). Let us assume that $\ell \neq 2$ and let π be a σ-selfdual supercuspidal representation. The main goal of this section is to prove the *Dichotomy Theorem* that π is either distinguished or $\omega_{E/F}$-distinguished, and the

Disjunction Theorem that π cannot be both distinguished and $\omega_{E/F}$-distinguished. We call it the Jacquet conjecture in the modular setting. For $p \neq 2$ this is proved in [SÍ9, Thm. 10.8]. Using Theorem 3.4, when $\ell \neq 2$, we will give a different proof of this result that also works for $p = 2$.

Remark 3.9. In this section we assume $\ell \neq 2$. When $\ell = 2$ then $p \neq 2$ and the distinguished supercuspidal representations are studied in [SÍ9]. Therefore, combining the results of this section and of [SÍ9] gives a complete result for supercuspidal distinguished representations for all ℓ and p with $\ell \neq p$.

With Theorem 3.4, we know that we can find a distinguished $\overline{\mathbf{Q}}_\ell$-lift of a distinguished supercuspidal $\overline{\mathbf{F}}_\ell$-representation whose Langlands parameter is conjugate-orthogonal in the sense of [GGP12, §7]. With the following lemma, we will show that all the σ-selfdual $\overline{\mathbf{Q}}_\ell$-lift of a σ-selfdual supercuspidal representations are either all distinguished or all $\omega_{E/F}$-distinguished.

Lemma 3.10. *Let π be a σ-selfdual irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$. Suppose that $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are two σ-selfdual irreducible supercuspidal $\overline{\mathbf{Q}}_\ell$-representations of $\mathrm{GL}_n(E)$ such that both $\tilde{\pi}_i$ have modulo ℓ reduction isomorphic to π. Then $\tilde{\pi}_1$ and $\tilde{\pi}_2$ share the same sign, i.e., $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are either conjugate-symplectic or conjugate-orthogonal at the same time, but not both.*

Proof. Let us denote by φ (resp. $\tilde{\varphi}_i$) the Langlands parameter of π (resp. $\tilde{\pi}_i$). For $i \in \{1, 2\}$, since $\tilde{\pi}_2$ is σ-selfdual, $\tilde{\varphi}_i$ is either conjugate-symplectic or conjugate-orthogonal, and let us denote by B_i the associated bilinear form. We also denote by b_i the sign of B_i.

Since $\tilde{\varphi}_i$ is a $\overline{\mathbf{Q}}_\ell$-lift of φ, reduction modulo ℓ of B_i gives a bilinear form on the space of φ. As $\ell \neq 2$ the sign b_i is preserved by reduction modulo ℓ, and is the sign of the bilinear form on φ. Moreover, π is supercuspidal, so φ is irreducible. Schur's lemma implies that the sign of a bilinear form on φ is unique. Therefore $b_1 = b_2$ and this concludes the proof. $\square$

Theorem 3.4 and the previous lemma implies the following proposition.

Proposition 3.11. *Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$. Then the following assertions are equivalent:*

(1) *π is distinguished.*
(2) *There exists a $\overline{\mathbf{Q}}_\ell$-lift $\tilde{\pi}$ of π which is distinguished.*

Moreover, when these conditions are satisfied, then all the σ-selfdual $\overline{\mathbf{Q}}_\ell$-lifts $\tilde{\pi}$ of π are distinguished.

Proof. The implication $(1) \Rightarrow (2)$ is given by Theorem 3.4 applied with $\mathrm{GL}_n(E)$ and $H = \mathrm{GL}_n(F)$. The other direction $(2) \Rightarrow (1)$ follows from an argument of reduction of invariant linear forms as in [KM20, Thm. 3.4].

To conclude, if π is distinguished, then all the σ-selfdual $\overline{\mathbf{Q}}_\ell$-lifts of π are distinguished by Lemma 3.10. $\square$

We can now prove the disjunction theorem.

Proposition 3.12. *Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$ distinguished by $\mathrm{GL}_n(F)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \omega_{E/F}) = 0.$$

Proof. We prove the result by contradiction. Let us assume that π is $\omega_{E/F}$-distinguished. Let χ be a character of $E^\times$ extending $\omega_{E/F}$. Let $\pi' := \pi(\chi^{-1} \circ \det)$. Then π' is a supercuspidal representation and is distinguished since π is $\omega_{E/F}$-distinguished. Applying Theorem 3.4 to π' we find a lift $\tilde\pi$ of π which is $\widetilde\omega_{E/F}$-distinguished, where $\widetilde\omega_{E/F}$ is the canonical ℓ-adic lift of $\omega_{E/F}$, that is the $\overline{\mathbf{Q}}_\ell$-character of $F^\times$ of kernel $\mathrm{Nm}_{E/F}(F^\times)$.

With Theorem 3.4 we also have a distinguished supercuspidal $\overline{\mathbf{Q}}_\ell$-lift of π. This contradicts Proposition 3.11 since a σ-selfdual $\overline{\mathbf{Q}}_\ell$-representation cannot be both distinguished and $\omega_{E/F}$-distinguished. $\square$

We are left to prove the dichotomy theorem. To do that, we need a theorem analogous to Theorem 3.4 but for σ-selfdual representations.

Proposition 3.13. *Let π be a σ-selfdual supercuspidal representation of $\mathrm{GL}_n(E)$ over $\overline{\mathbf{F}}_\ell$. Then there exists a $\overline{\mathbf{Q}}_\ell$-lift $\tilde\pi$ of π which is σ-selfdual.*

Proof. Let us denote by δ the action of σ-duality on R-representations, that is $V^\delta = (V^\sigma)^\vee$. A representation is σ-selfdual if and only if it is δ invariant.

Let $[\pi]$ be the inertial class of π. Let $\mathcal{I}$ be the set of inertial classes of supercuspidal $\overline{\mathbf{Q}}_\ell$-lifts of π. This is a finite set of cardinal a power of ℓ. In particular, since $\ell \neq 2$, this cardinal is odd. The action of δ induces an action on inertial classes of representations and since π is δ-invariant, δ induces an action on $\mathcal{I}$. But the cardinal of $\mathcal{I}$ is odd and δ is an involution, therefore there exists an inertial class which is δ-invariant, that is there exists $\tilde\pi$ a supercuspidal $\overline{\mathbf{Q}}_\ell$-lift of π such that $[\tilde\pi^\delta] = [\tilde\pi]$. Let $\tilde\chi$ be an unramified character such that $\tilde\pi^\sigma = \tilde\pi^\vee \otimes \tilde\chi$.

Since $\tilde\chi$ is unramified, it is trivial on the kernel of the norm map, that is $\tilde\chi = \tilde\chi^\sigma$. Therefore, there exists an unramified character $\tilde\eta$ such that $\tilde\chi = \tilde\eta\tilde\eta^\sigma = \tilde\eta^2$. Let $\tilde\pi' := \tilde\pi \otimes \tilde\eta^{-1}$ such that $\tilde\pi'$ is σ-selfdual. Let us denote by $\pi' = \pi \otimes \eta^{-1}$ its reduction modulo ℓ where η is the reduction modulo ℓ of $\tilde\eta$. Since $\tilde\pi'$ is σ-selfdual so is π'. The representations π and $\pi' = \pi \otimes \eta^{-1}$ are both σ-selfdual thus $\pi = \pi \otimes \eta^{-2}$.

Let $X_u(\pi)$ be the set of unramified characters ξ such that $\pi = \pi \otimes \xi$. It is contained in the cyclic group of order n composed of the character ξ such that $\xi^n = 1$ and is characterized by its order $f(\pi)$. Let $o(\eta)$ be the order of η. We have $o(\eta) \mid 2f(\pi)$. Let $\tilde\rho$ be the canonical $\overline{\mathbf{Q}}_\ell$-lift of η. In particular $o(\tilde\rho) = o(\eta) \mid 2f(\pi)$. Let $\tilde\pi'' := \tilde\pi' \otimes \tilde\rho$. It is a supercuspidal $\overline{\mathbf{Q}}_\ell$-lift of π. To finish the proof, we are left to prove that it is σ-selfdual. The representation $\tilde\pi'$ being σ-selft dual, we just need to prove that $\tilde\pi'' = \tilde\pi'' \otimes \tilde\rho^2$, that is that $o(\tilde\rho) \mid 2f(\tilde\pi')$. The explicit formulas for $f(\tilde\pi')$ and $f(\pi)$ show that $f(\tilde\pi') = f(\pi)\ell^k$ for some integer k (see for instance [MS14, Rem. 3.21]). Thus $2f(\pi) \mid 2f(\tilde\pi')$ and $o(\tilde\rho) \mid 2f(\tilde\pi')$. Therefore $\tilde\pi''$ is σ-selfdual. $\square$

Theorem 3.14. *Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$. Then π is σ-selfdual if and only if it is either distinguished or $\omega_{E/F}$-distinguished, but not both.*

Proof. Assume that π is σ-selfdual. Then there exists a σ-selfdual lift $\tilde\pi$ by Proposition 3.13. From the dichotomy theorem in the complex case (see [Kab04]), $\tilde\pi$ is either distinguished or $\widetilde\omega_{E/F}$-distinguished. Since the reduction modulo ℓ preserves distinction, it is also true for π. The disjunction is Proposition 3.12 and the reciprocity is Theorem 4.1. $\square$

Proposition 3.15. *Let π be a σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$ and φ_π its Langlands parameter. Then π is distinguished if and only if φ_π is conjugate-orthogonal and π is $\omega_{E/F}$-distinguished if and only if φ_π is conjugate-symplectic.*

Proof. A σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation is either distinguished or $\omega_{E/F}$-distinguished by Theorem 3.14. By Proposition 3.11 we can lift a distinguished representation to a distinguished representation. Hence the result follows from the complex case in [Kab04]. $\qquad\square$

3.4. **Supercuspidal representations of $\mathrm{GL}_n(E)$ distinguished by a unitary involution.** Another example of application of Theorem 3.4 is for supercuspidal representations distinguished by a unitary involution. Let $G = \mathrm{GL}_n(E)$ and assume $\ell \neq 2$. Let ε be an hermitian matrix in $M_n(E)$ ($\sigma({}^t\varepsilon) = \varepsilon$). We denote by ς the *unitary involution* on G defined by $\varsigma(g) = \varepsilon\sigma({}^tg^{-1})\varepsilon^{-1}$. In this section, we are interested in supercuspidal $\overline{\mathbf{F}}_\ell$-representations of G distinguished by G^ς. In particular, we are going to prove that a supercuspidal representation π of G is distinguished by G^ς if and only if it is σ-invariant. When $p \neq 2$, this is proved by Zou in [Zou21].

Proposition 3.16. *Let π be a σ-invariant supercuspidal representation of $\mathrm{GL}_n(E)$ over $\overline{\mathbf{F}}_\ell$. Then there exists a $\overline{\mathbf{Q}}_\ell$-lift $\tilde\pi$ of π which is σ-invariant.*

Proof. We start with the same argument as in the proof of Proposition 3.13. The number of inertial classes of supercuspidal $\overline{\mathbf{Q}}_\ell$-lifts of π is a power of ℓ, hence is odd. The involution σ permutes its elements, so there exists an inertial class $[\tilde\pi]$ which is σ-invariant. Let $\tilde\chi$ be an unramified character such that $\tilde\pi^\sigma = \tilde\pi \otimes \tilde\chi$.

Let us prove that $\tilde\pi$ is in fact σ-invariant. As in the proof of Proposition 3.13, let $X_u(\tilde\pi)$ (resp. $X_u(\pi)$) be the set of unramified $\overline{\mathbf{Q}}_\ell$-characters $\tilde\xi$ (resp. unramified $\overline{\mathbf{F}}_\ell$-characters ξ) such that $\tilde\pi = \tilde\pi \otimes \tilde\xi$ (resp. $\pi = \pi \otimes \xi$). It is characterized by its order $f(\tilde\pi)$ (resp. $f(\pi)$). Let $o(\tilde\chi)$ be the order of $\tilde\chi$. To finish the proof, we need to prove that $o(\tilde\chi)$ divides $f(\tilde\pi)$. Indeed, this implies that $\tilde\chi \in X_u(\tilde\pi)$ and $\tilde\pi^\sigma \simeq \tilde\pi \otimes \tilde\chi \simeq \tilde\pi$.

Let χ be the modulo ℓ reduction of $\tilde\chi$ of order $o(\chi)$. By taking the modulo ℓ reduction we have that $\pi \simeq \pi \otimes \chi$, thus $\chi \in X_u(\pi)$ and $o(\chi) \mid f(\pi)$. Also there exists an integer $a \in \mathbb{N}$ such that $o(\tilde\chi) = o(\chi)\ell^a$. The explicit formulas for $f(\tilde\pi)$ and $f(\pi)$ show that there exists an integer $b \in \mathbb{N}$ such that $f(\tilde\pi) = f(\pi)\ell^b$ and $f(\pi)$ is prime to ℓ.

Since $\tilde\chi$ is unramified, it is trivial on the kernel of the norm map and $\tilde\chi = \tilde\chi^\sigma$. In particular, $\tilde\pi \simeq (\tilde\pi \otimes \tilde\chi)^\sigma \simeq \tilde\pi \otimes \tilde\chi^2$ and $o(\tilde\chi) \mid 2f(\tilde\pi)$. Since ℓ is odd, we get that $a \leq b$. At the end, $o(\chi) \mid f(\pi)$ and $\ell^a \mid \ell^b$ thus $o(\tilde\chi) \mid f(\tilde\pi)$ and this finishes the proof.

$\qquad\square$

Theorem 3.17. *Let π be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $G = \mathrm{GL}_n(E)$ and ς a unitary involution of G. Then π is distinguished by G^ς if and only if $\pi^\sigma \simeq \pi$.*

Proof. Since the modulo ℓ reduction preserves σ-invariance and the property of being distinguished for supercuspidal representations ([KM20, Thm. 3.4]), the result follows from the complex case (see [HM02, Section 4, Corollary]) by Theorem 3.4 and Proposition 3.16. $\qquad\square$

Remark 3.18. The implication, π distinguished by G^ς implies π-invariant, is also proved in [Zou21, Thm. 4.1] similarly.

4. The $\mathrm{GL}_2(F)$-distinguished representations

This section is devoted to the study of $\mathrm{GL}_2(F)$-distinguished ℓ-modular representations of $\mathrm{GL}_2(E)$. The case of supercuspidal representations has been dealt with in Section 3.3. Using Mackey theory, we will describe non-supercuspidal distinguished representations, hence giving a complete classification of all $\mathrm{GL}_2(F)$-distinguished representations.

Before we start, let us recall a result of Sécherre about $\mathrm{GL}_2(F)$-distinguished representations of $\mathrm{GL}_2(E)$.

Theorem 4.1 ([Sí9, Thm. 4.1]). *Let π be a distinguished irreducible representation of $\mathrm{GL}_2(E)$. Then:*

(1) *The central character of π is trivial on $F^\times$.*
(2) *The contragredient representation $\pi^\vee$ is distinguished.*
(3) *The space $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, 1)$ has dimension 1.*
(4) *The representation π is σ-selfdual, that is $\pi^\vee \simeq \pi^\sigma$.*

4.1. **Non-supercuspidal representations.** In Section 3.3, we studied when supercuspidal representations are distinguished. In this section, we will deal with non-supercuspidal representations including the principal series representations.

Let us start with an easy lemma that is needed for the rest of this section.

Lemma 4.2. *Let χ be a character of $E^\times$. Then $\chi\chi^\sigma = \mathbf{1}$ if and only if $\chi_{|F^\times} = \mathbf{1}$ or $\chi_{|F^\times} = \omega_{E/F}$.*

Proof. The condition $\chi\chi^\sigma = \mathbf{1}$ is equivalent to χ being trivial on the norm group $\mathrm{Nm}_{E/F}(E^\times)$. By the local class field theory the only two characters of $F^\times$ trivial on $\mathrm{Nm}_{E/F}(E^\times)$ are $\mathbf{1}$ and $\omega_{E/F}$. $\qquad\square$

Let χ_1, χ_2 be two characters of $E^\times$. We denote by $\pi(\chi_1, \chi_2)$ the principal series representation of $\mathrm{GL}_2(E)$ induced from (χ_1, χ_2), that is, $\pi(\chi_1, \chi_2)$ is the normalized parabolic induction of $\chi_1 \boxtimes \chi_2$ from the standard Borel subgroup to $\mathrm{GL}_2(E)$.

Lemma 4.3. *Let $\pi = \pi(\chi_1, \chi_2)$ be a principal series representation of $\mathrm{GL}_2(E)$. Then π is distinguished by $\mathrm{GL}_2(F)$ if and only if either $\chi_1\chi_2^\sigma = \mathbf{1}$ or $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$ with $\chi_1 \neq \chi_2$.*

Proof. Recall that $B_2(E)$ is the standard Borel subgroup of $\mathrm{GL}_2(E)$. Note that $\mathrm{GL}_2(E) = B_2(E)\,\mathrm{GL}_2(F) \sqcup B_2(E)\eta\,\mathrm{GL}_2(F)$ with $\eta = \begin{pmatrix} 1 & \delta \\ 1 & -\delta \end{pmatrix}$. Applying Mackey Theory, one has a short exact sequence

$$0 \to \mathrm{Hom}_{F^\times \times F^\times}(\chi_1\boxtimes\chi_2, \mathbf{1}) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}_2) \to \mathrm{Hom}_{E^\times}(\chi_1\chi_2^\sigma, \mathbf{1}) \to \mathrm{Ext}^1_{F^\times \times F^\times}(\chi_1\boxtimes\chi_2, \mathbf{1})$$

where $\mathrm{Ext}^1_{F^\times \times F^\times}$ denotes the Ext functor in the category of the finite dimensional representations of $F^\times \times F^\times$. Thanks to [DS21, Prop. 8.4] that $\mathrm{Ext}^1_{F^\times}(\chi, \mathbf{1}) \neq 0$ if and only if χ is trivial, one has that $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}) \neq 0$ if and only if $\chi_1\chi_2^\sigma = \mathbf{1}$ or $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$. If $\chi_1\chi_2^\sigma = \mathbf{1}$ and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$, then $\chi_1 = \chi_2$. This finishes the proof. $\qquad\square$

Moreover, when $\pi = \pi(\chi_1, \chi_2)$ is distinguished, we can compute exactly the dimension of $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}_2)$.

Lemma 4.4. *Let $\pi = \pi(\chi_1, \chi_2)$ be a principal series representation of $\mathrm{GL}_2(E)$ with $\chi_1 \chi_2^\sigma = \mathbf{1}$ or $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}_2) = \begin{cases} 2, & \text{if } \ell \mid q_F - 1, \chi_1 = \chi_2 \text{ and } \chi_1|_{F^\times} = \mathbf{1}; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. If $\chi_1 \neq \chi_2$, only one of the conditions $\chi_1 \chi_2^\sigma = \mathbf{1}$ or $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$ can be true. We then see from the proof of Lemma 4.3 that $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}_2) = \mathrm{Hom}_{F^\times \times F^\times}(\chi_1 \boxtimes \chi_2, \mathbf{1})$ or $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}_2) = \mathrm{Hom}_{E^\times}(\chi_1 \chi_2^\sigma, \mathbf{1})$. Therefore

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbf{1}_2) = 1.$$

We are left to study the case $\chi_1 = \chi_2$ with $\chi_1|_{F^\times} = \mathbf{1}$. Let $\chi := \chi_1 = \chi_2$. Note that if $q_E \not\equiv 1 \pmod \ell$ then $\pi(\chi, \chi)$ is irreducible. Thus by Theorem 4.1, $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi, \chi), \mathbf{1}_2) = 1$. So we assume that $q_E \equiv 1 \pmod \ell$. We have two cases:

(1) Suppose $q_F \equiv -1 \pmod \ell$. In this case $(\nu^{-1/2})_{|F^\times} \neq \mathbf{1}$. The representation $\pi(\chi, \chi)$ is a direct sum of two irreducible representations : the character $(\chi \nu^{-1/2}) \circ \det$ and a twisted Steinberg $\mathrm{St}_{\chi \nu^{-1/2}}$. As $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi \nu^{-1/2}) \circ \det, \mathbf{1}_2) = 0$ and $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{St}_{\chi \nu^{-1/2}}, \mathbf{1}_2) \leq 1$ by Theorem 4.1, we get that $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi, \chi), \mathbf{1}_2) = 1$.

(2) Suppose $q_F \equiv 1 \pmod \ell$. This time $(\nu^{-1/2})_{|F^\times} = \mathbf{1}$. Therefore,

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi, \chi), \mathbf{1}_2) = \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(V(\mathbf{1}, \mathbf{1}), \mathbf{1}_2)$$

where $V(\mathbf{1}, \mathbf{1})$ denotes the non-normalized induction.

Recall that

$$\mathrm{GL}_2(E) = B_2(E) \, \mathrm{GL}_2(F) \sqcup B_2(E) \eta \, \mathrm{GL}_2(F)$$

where η represents the open orbit in double coset $B_2(E) \backslash \mathrm{GL}_2(E) / \mathrm{GL}_2(F)$. Let l_1 be a linear functional defined on a subset consisting of functions in $V(\mathbf{1}, \mathbf{1})$ supported on $B_2(E) \, \mathrm{GL}_2(F)$ given by

$$l_1(f) = \int_{B_2(F) \backslash \mathrm{GL}_2(F)} f(x) dx$$

for $f \in V(\mathbf{1}, \mathbf{1})$ supported on $B_2(E) \, \mathrm{GL}_2(F)$. There is a zero extension of l_1 to the whole space $V(\mathbf{1}, \mathbf{1})$, still denoted by l_1. Then l_1 gives a $\mathrm{GL}_2(F)$-invariant linear functional on π. Note that there is a $\mathrm{GL}_2(E)$-invariant distribution on $P^1(E) = B_2(E) \backslash \mathrm{GL}_2(E)$ if and only if $\delta_{B(E)} = 1$ i.e., $q_E \equiv 1 \pmod \ell$ (see [Bou04, Chapter VII.6, Thm 3]). Denote by $d\mu$ the $\mathrm{GL}_2(E)$-invariant measure on $P^1(E)$ and so it is $\mathrm{GL}_2(F)$-invariant. Then the restriction of $d\mu$ on $P^1(F) = B_2(F) \backslash \mathrm{GL}_2(F)$ is zero since $P^1(F)$ has measure zero with respect to $d\mu$. Therefore $d\mu$ and ℓ_1 generates two different $\mathrm{GL}_2(F)$-invariant linear functionals on $V(\mathbf{1}, \mathbf{1})$. Thus $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(V(\mathbf{1}, \mathbf{1}), \mathbf{1}_2) = 2$ (the dimension is bounded by 2 by the proof of Lemma 4.3). $\qquad\square$

Let us come back to a criterion for being distinguished for irreducible non-super-cuspidal representation. By Lemma 4.3, we have the result for all irreducible principal series. Note that $\pi(\chi_1, \chi_2)$ is reducible if and only if $\chi_1 = \nu \chi_2$ or $\chi_2 = \nu \chi_1$.

Hence let χ be a character of $E^\times$ and we are left to study the irreducible sub-quotients of $\pi(\chi\nu^{-1/2}, \chi\nu^{1/2})$. If $q_E \not\equiv -1 \pmod{\ell}$, then $\pi(\chi\nu^{-1/2}, \chi\nu^{1/2})$ has length 2 with irreducible sub-quotients χ and St_χ. If $q_E \equiv -1 \pmod{\ell}$, the length is 3 and the sub-quotients are χ, $\chi\nu_2$ and Sp_χ. Denote by $\mathbf{1}_2$ the trivial character of $\mathrm{GL}_2(F)$.

Lemma 4.5. *Let χ be a quadratic character of $F^\times$. Then $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\mathbf{1}_2, \chi) = 0$ where $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}$ is the Ext functor in the category of the smooth representations of $\mathrm{GL}_2(F)$ with trivial central character.*

Proof. Suppose that $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\mathbf{1}_2, \chi)$ has the form

$$g \mapsto \begin{pmatrix} 1 & \alpha(g) \\ 0 & \chi(g) \end{pmatrix}$$

for $g \in \mathrm{GL}_2(F)$. Then $\alpha(g_1 g_2) = \alpha(g_1) + \chi(g_1)\alpha(g_2)$ for $g_i \in \mathrm{GL}_2(F)$ and $\alpha(z) = 0$ for every element z in the center of $\mathrm{GL}_2(F)$. It is obvious that $\alpha(g) = 0$ for all g in $\mathrm{SL}_2(F)$ since $\mathrm{SL}_2(F)$ is a perfect group. Thus α is a cocycle from $F^\times$ to $\overline{\mathbf{F}}_\ell$. Then $\alpha(g) = c \cdot (\chi(g) - 1)$ for some constant c in $\overline{\mathbf{F}}_\ell$ which is a coboundary. Therefore $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\mathbf{1}_2, \chi) = 0$. $\qquad\square$

Theorem 4.6. *Let χ be a character of $E^\times$.*
   (1) *If $q_E^2 \not\equiv 1 \pmod{\ell}$, then St_χ is distinguished if and only if $\chi_{|F^\times} = \omega_{E/F}$.*
   (2) *If $q_E \equiv -1 \pmod{\ell}$, then Sp_χ is distinguished if and only if $q_F \equiv -1 \pmod{\ell}$ and $\chi_{|F^\times} = \omega_{E/F}$ or $\nu_{|F^\times}^{1/2}$.*
   (3) *If $q_E \equiv 1 \pmod{\ell}$, then St_χ is distinguished if and only if $\chi_{|F^\times} = \omega_{E/F}$ or $\chi_{|F^\times} = \mathbf{1}$ with $q_F \equiv 1 \pmod{\ell}$.*

Proof. Suppose that $q_E \equiv -1 \pmod{\ell}$. If Sp_χ is distinguished then by Theorem 4.1 (2), Sp_χ has trivial central character and so $\chi_{|F^\times}^2 = 1$. Thus we may assume that $\chi_{|F^\times}^2 = \mathbf{1}$. The principal series $\pi(\chi\nu^{1/2}, \chi\nu^{-1/2})$ has length 3. There are two exact sequences

$$0 \to J_\chi \to \pi(\chi\nu^{1/2}, \chi\nu^{-1/2}) \to \chi \to 0$$

and

$$0 \to \chi\nu_2 \to J_\chi \to \mathrm{Sp}_\chi \to 0$$

of $\mathrm{GL}_2(E)$-modules. Taking the functor $\mathrm{Hom}_{\mathrm{GL}_2(F)}(-, \mathbf{1}_2)$, we have

$$0 \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) \to \mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2)$$

where $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}$ is the Ext functor in the category of the smooth representations of $\mathrm{GL}_2(F)$ with trivial central character. By Lemma 4.5 $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\mathbf{1}_2, \chi) = 0$. If $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) = 0$ then $\mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) = 0$ and $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{Sp}_\chi, \mathbf{1}_2) = 0$. Hence Sp_χ is not distinguished.

Therefore we can assume that $\pi(\chi\nu^{1/2}, \chi\nu^{-1/2})$ is distinguished. By Lemma 4.3 we get that $\chi\chi^\sigma = \mathbf{1}$ (that is $\chi_{|F^\times} = \mathbf{1}$ or $\chi_{|F^\times} = \omega_{E/F}$ by Lemma 4.2) or $\chi_{|F^\times}\nu_{|F^\times}^{1/2} = \chi_{|F^\times}\nu_{|F^\times}^{-1/2} = \mathbf{1}$ (which can only happen if $\nu_{|F^\times} = \mathbf{1}$ that is if $q_F \equiv -1 \pmod{\ell}$). Moreover, in this case, Lemma 4.4 gives us

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) = 1.$$

We have three cases to study.

- Suppose $\chi_{|F^\times} = \mathbf{1}$. In this case $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) = 0$ and so Sp_χ is not distinguished.
- Suppose $\chi_{|F^\times} = \omega_{E/F}$. Now $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2) = 0$ and so $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) = 1$. If $q_F \equiv -1 \pmod{\ell}$, then $\nu_2|_{\mathrm{GL}_2(F)} = \mathbf{1}_2$. Furthermore, the exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{Sp}_\chi, \mathbf{1}_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi\nu_2, \mathbf{1}_2) = 0$$

  implies that $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{Sp}_\chi, \mathbf{1}_2) = \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) = 1$. If $q_F^2 \equiv -1 \pmod{\ell}$, i.e., E/F is unramified, then $\nu_2|_{\mathrm{GL}_2(F)} = \omega_{E/F}$. In this case $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi\nu_2, \mathbf{1}_2) = 1$ and so $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{Sp}, \omega_{E/F}) = 0$.
- Suppose $\chi_{|F^\times} = \nu_{|F^\times}^{1/2}$ and $q_F \equiv -1 \pmod{\ell}$. In the same way, $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2) = 0$ and so $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(J_\chi, \mathbf{1}_2) = 1$. Since $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi\nu_2, \mathbf{1}_2) = 0$ we get that Sp_χ is distinguished.

Suppose $q_E \equiv 1 \pmod{\ell}$. In this case $\pi(\chi\nu^{-1/2}, \chi\nu^{-1/2})$ is reducible and semisimple: $\pi(\chi\nu^{-1/2}, \chi\nu^{-1/2}) = \chi \oplus \mathrm{St}_\chi$. If St_χ is distinguished then so is $\pi(\chi\nu^{-1/2}, \chi\nu^{-1/2})$. By Lemma 4.3, $\chi\chi^\sigma = \mathbf{1}$ so $\chi_{|F^\times} = \mathbf{1}$ or $\chi_{|F^\times} = \omega_{E/F}$ (Lemma 4.2). If $\chi_{|F^\times} = \omega_{E/F}$, then $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2) = 0$. Thanks to Lemma 4.3 $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{-1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) \geq 1$ and so St_χ is distinguished. Now if $\chi_{|F^\times} = \mathbf{1}$, $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2) = 1$ and from Lemma 4.4

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{-1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) = \begin{cases} 2 \text{ if } q_F \equiv 1 \pmod{\ell} \\ 1 \text{ if } q_F \equiv -1 \pmod{\ell} \end{cases}.$$

Thus St_χ is distinguished if and only if $q_F \equiv 1 \pmod{\ell}$.

The last case is $q_E^2 \not\equiv 1 \pmod{\ell}$. We will do a similar argument as $q_E \equiv -1 \pmod{\ell}$. We have an exact sequence

$$0 \to \mathrm{St}_\chi \to \pi(\chi\nu^{1/2}, \chi\nu^{-1/2}) \to \chi \to 0$$

of $\mathrm{GL}_2(E)$-modules. If St_χ is distinguished then so is $\pi(\chi\nu^{1/2}, \chi\nu^{-1/2})$, so $\chi_{|F^\times} = \mathbf{1}$ or $\chi_{|F^\times} = \omega_{E/F}$. Taking the functor $\mathrm{Hom}_{\mathrm{GL}_2(F)}(-, \mathbf{1}_2)$, in the category of the smooth representations of $\mathrm{GL}_2(F)$ with trivial central character, we have

$$0 \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{St}_\chi, \mathbf{1}_2) \to \mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\chi, \mathbf{1}_2).$$

By Lemma 4.5 $\mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\mathbf{1}_2, \chi) = 0$. By Lemma 4.4 $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\chi\nu^{1/2}, \chi\nu^{-1/2}), \mathbf{1}_2) = 1$. Therefore, St_χ is distinguished if and only if $\chi_{|F^\times} = \omega_{E/F}$. $\square$

Remark 4.7. It can be seen from this theorem that we have in the modular case new phenomena that do not appear in the complex setting.

- When $q_F \equiv 1 \pmod{\ell}$, the Steinberg representation St is both $\mathrm{GL}_2(F)$-distinguished and $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished.
- When $q_E \equiv -1 \pmod{\ell}$ and E/F is unramified (that is $q_F \not\equiv 1 \pmod{\ell}$) the special representation Sp is neither $\mathrm{GL}_2(F)$-distinguished nor $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished. This has been mentioned by Vincent Sécherre in [S19, Rem. 2.8].
- When $q_E \equiv -1 \pmod{\ell}$ and E/F is ramified (and so $q_F \equiv -1 \pmod{\ell}$) the special representation Sp is $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished and is also $(\mathrm{GL}_2(F), \nu_{|F^\times}^{1/2})$-distinguished.

5. The Prasad conjecture for ℓ-modular representations of PGL_2

In [Pra15], Dipendra Prasad proposed a conjecture for the multiplicity $\dim \mathrm{Hom}_{G(F)}(\pi, \chi_G)$ under the local Langlands conjecture, where G is a quasi-split reductive group defined over F, π is an irreducible smooth representation of $G(E)$ and χ_G is a quadratic character depending on G and the quadratic extension E/F. Since the local Langlands correspondence for the ℓ-modular representations of $G(F)$ has not been set up in general, except for $G = \mathrm{GL}_n$, we are concerned only with the simplest case where $G = \mathrm{PGL}_2$.

First, we will show that the Prasad conjecture is not valid in the ℓ-modular setting (when ℓ is non-banal). Then, we provide a potential solution. To do that we define a non-trivial injection P from nilpotent Weil-Deligne representations $\mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$ to equivalence classes of non-nilpotent one $[\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$. Composing the local Langlands correspondence of Vignéras PV with P gives us a modular version of the Prasad conjecture. That is, an irreducible generic $\overline{\mathbf{F}}_\ell$-representation π of $\mathrm{PGL}_2(E)$ is $\omega_{E/F}$-distinguished if and only if there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim P \circ PV(\pi)$.

5.1. The Langlands correspondence for PGL_2. Firstly, we recall how to get a Langlands correspondence for PGL_2 using the correspondence for GL_2.

Let $R = \overline{\mathbf{Q}}_\ell$ or $\overline{\mathbf{F}}_\ell$. We denote by $\mathrm{WDRep}_R(W_E, \mathrm{SL}_2)$ the subset of $\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)$ composed of the elements (φ, N) such that $\mathrm{Im}(\varphi) \subseteq \mathrm{SL}_2(R)$ and $\mathrm{tr}(N) = 0$. Let $\mathrm{Nilp}_R(W_E, \mathrm{SL}_2) := \mathrm{WDRep}_R(W_E, \mathrm{SL}_2) \cap \mathrm{Nilp}_R(W_E, \mathrm{GL}_2)$.

Remark 5.1. Let (φ_1, N_1) and (φ_2, N_2) be two semisimple Weil-Deligne representations with $\mathrm{Im}(\varphi_1) \subseteq \mathrm{SL}_2(R)$ and $\mathrm{Im}(\varphi_2) \subseteq \mathrm{SL}_2(R)$. If (φ_1, N_1) and (φ_2, N_2) are isomorphic in GL_2 then they are also isomorphic in SL_2. Indeed, let $A \in \mathrm{GL}_2(R)$ such that $A^{-1}\varphi_1 A = \varphi_2$ and $A^{-1}N_1 A = N_2$. Since R is algebraically closed, let $\alpha \in R$ such that $\alpha^2 = \det(A)^{-1}$. Let $B := \alpha A$. Then $B \in \mathrm{SL}_2(R)$ and $B^{-1}\varphi_1 B = \varphi_2$, $B^{-1}N_1 B = N_2$.

Let L be any map

$$L : \mathrm{Irr}_R(\mathrm{GL}_2(E)) \to \mathrm{Nilp}_R(W_E, \mathrm{GL}_2)$$

such that L sends the central character to the determinant. Then L induces a map

$$PL : \mathrm{Irr}_R(\mathrm{PGL}_2(E)) \to \mathrm{Nilp}_R(W_E, \mathrm{SL}_2)$$

making the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Irr}_R(\mathrm{PGL}_2(E)) & \xrightarrow{\ PL\ } & \mathrm{Nilp}_R(W_E, \mathrm{SL}_2) \\
\downarrow & & \downarrow \\
\mathrm{Irr}_R(\mathrm{GL}_2(E)) & \xrightarrow{\ L\ } & \mathrm{Nilp}_R(W_E, \mathrm{GL}_2)
\end{array}
$$

where the vertical map on the left-hand side is given by the projection $\mathrm{GL}_2(E) \to \mathrm{PGL}_2(E)$ and the vertical map on the right-hand side is the inclusion $\mathrm{Nilp}_R(W_E, \mathrm{SL}_2) \subseteq \mathrm{Nilp}_R(W_E, \mathrm{GL}_2)$.

In particular, since the V correspondence sends the central character to the determinant ([KM21, Lem. 6.4]) we get a map $PV : \mathrm{Irr}_R(\mathrm{PGL}_2(E)) \to \mathrm{Nilp}_R(W_E, \mathrm{SL}_2)$.

5.2. **Problem with the modular Prasad Conjecture.** We want to show that the Prasad conjecture does not hold for ℓ-modular representations with the V correspondence. One can think that it might work with a different choice of bijection in Theorem 3.8. We will show in this subsection that such kind of bijection does not exist; see Corollary 5.3.

Let L be a map

$$L : \mathrm{Irr}_{\overline{\mathbf{F}}_\ell}(\mathrm{PGL}_2(E)) \to \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$$

such that the semisimple part is given by V_{ss}.

When $q_E \equiv -1 \pmod \ell$, the image under L of the special representation Sp, has $\nu^{-1/2} \oplus \nu^{1/2}$ for the semisimple part.

Lemma 5.2. *Let E be a quadratic extension of F. Let us assume that $q_E \equiv -1$ (mod ℓ). Then there exists $\Phi_F \in \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Phi_{F|W_E} = L(\mathrm{Sp})$.*

Proof. There are 3 elements in $\mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$ with semisimple part $\nu^{-1/2} \oplus \nu^{1/2}$: $(\nu^{-1/2} \oplus \nu^{1/2}, 0)$, $(\nu^{-1/2} \oplus \nu^{1/2}, N)$ and $(\nu^{1/2} \oplus \nu^{3/2}, N)$, with $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The corresponding Weil-Deligne representations with ν_F a lift of ν gives a lifting. $\square$

Corollary 5.3. *In the non banal case, the Prasad conjecture is not true for any map L as above.*

Proof. If $q_E \equiv -1 \pmod \ell$ and E/F is unramified, then by Theorem 4.6, the special representation Sp is not $\omega_{E/F}$-distinguished. However, the semisimple part of $L(\mathrm{Sp})$ is $\nu^{-1/2} \oplus \nu^{1/2}$. By Lemma 5.2 the Langlands parameter $L(\mathrm{Sp})$ of Sp can be lifted to W_F. $\square$

5.3. **Non-nilpotent Weil-Deligne representations.** To fix the issue with the Prasad conjecture in the modular setting discussed in the previous paragraph, we want to modify the V correspondence. In [KM21], Kurinczuk and Matringe modify the local Langlands of Vignéras using non-nilpotent Weil-Deligne representations. We will do something similar to solve our problem.

Kurinczuk and Matringe [KM21, Def. 4.8] defined an equivalence relation $\sim$ on $\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)$. We recall the definition here. Let (Φ, U) and (Φ', U') be two Weil-Deligne representations (up to isomorphism) in $\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)$. Then

(1) If (Φ, U) and (Φ', U') are indecomposable (as Weil-Deligne representations), we say that $(\Phi, U) \sim (\Phi', U')$ if there exists $\lambda \in R^\times$ such that

$$(\Phi', U') \simeq (\Phi, \lambda U).$$

(2) In the general case, $(\Phi, U) \sim (\Phi', U')$ if one can decompose $(\Phi, U) = \bigoplus_{i=1}^r (\Phi_i, U_i)$ and $(\Phi', U') = \bigoplus_{i=1}^r (\Phi_i', U_i')$ with indecomposable summands such that $(\Phi_i, U_i) \sim (\Phi_i', U_i')$.

We denote by $[(\Phi, U)]$ the equivalence class of (Φ, U) and by

$$[\mathrm{WDRep}_R(W_E, \mathrm{GL}_2)] := \mathrm{WDRep}_R(W_E, \mathrm{GL}_2)/\sim.$$

We define also an equivalence relation $\sim$ on $\mathrm{WDRep}_R(W_E, \mathrm{SL}_2)$ through the inclusion $\mathrm{WDRep}_R(W_E, \mathrm{SL}_2) \subseteq \mathrm{WDRep}_R(W_E, \mathrm{GL}_2)$ and we denote by $[\mathrm{WDRep}_R(W_E, \mathrm{SL}_2)] := \mathrm{WDRep}_R(W_E, \mathrm{SL}_2)/\sim$.

Let χ be a quadratic character of $E^\times$. If $\ell \mid q_E - 1$, we denote by $\Psi_{\text{St},\chi} \in \text{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2)$, the Weil-Deligne representation $\Psi_{\text{St},\chi} = (\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, N)$ with $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. And if $\ell \mid q_E + 1$, we denote by $\Psi_{\text{Sp},\chi} \in \text{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2)$, the Weil-Deligne representation $\Psi_{\text{Sp},\chi} = (\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, 0)$. We define an injection

$$P : \text{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2) \hookrightarrow [\text{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2)]$$

by

$$P(\Psi) = \begin{cases} [\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] & \text{if } \ell \mid q_E - 1 \text{ and } \Psi = \Psi_{\text{St},\chi} \\ [\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] & \text{if } \ell \mid q_E + 1 \text{ and } \Psi = \Psi_{\text{Sp},\chi} \\ [\Psi] & \text{otherwise.} \end{cases}$$

Since $\text{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2) = [\text{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2)]$ due to [KM21, Prop. 4.11], P is clearly an injection.

Remark 5.4.   (1) When ℓ is banal, that is $\ell \nmid q_E^2 - 1$, P is just the identity, as we have $[\text{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2)] = \text{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \text{SL}_2)$.
   (2) The map P is not exactly the CV map of [KM21]. The image is different for non banal supercuspidal representation and the Steinberg representation when $\ell \mid q_E - 1$.

5.4. **Lifting for non-irreducible Weil-Deligne representations.** In Section 4.1 we have described which of the non-supercuspidal irreducible representations are distinguished. To prove the Prasad conjecture, we also need to inquire when Weil-Deligne representations of W_E can be lifted to W_F. This is what we do in this section.

We begin by giving a criterion to lift the semisimple part of a Weil-Deligne representation.

Lemma 5.5. *Let χ be a character of $E^\times$. Let $\Psi : W_E \to \text{SL}_2(\overline{\mathbf{F}}_\ell)$ defined by $\Psi = \chi\nu^{-1/2} \oplus \chi^{-1}\nu^{1/2}$. Then there exists a semisimple $\Psi_F : W_F \to \text{SL}_2(\overline{\mathbf{F}}_\ell)$ such that $\Psi_{F|W_E} = \Psi$ if and only if $\chi = \chi^\sigma$ or $\chi \neq \chi^\sigma$ and $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$.*

Moreover, if $\chi = \chi^\sigma$ then $\Psi_F = \eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}$, with η a character of $F^\times$ such that $\chi = \eta \circ \text{Nm}_{E/F}$. And if $\chi \neq \chi^\sigma$ and $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$ then $\Psi_F = \text{Ind}_{W_E}^{W_F}(\chi\nu^{-1/2})$ (which is irreducible).

Proof. We have two cases if there is a lift to W_F: the two-dimensional representation of W_F is irreducible or it is the sum of two characters.

Let us first study when there is a lift which is the sum of two characters. In this case we can lift χ to W_F. This is equivalent to $\chi = \eta \circ \text{Nm}_{E/F}$ with η a character of $F^\times$ also equivalent to $\chi = \chi^\sigma$. If this condition is satisfied then the lift is $\Psi_F = \eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}$.

Now we deal with the case where there is an irreducible lift. Then this lift must be $\Psi_F = \text{Ind}_{W_E}^{W_F}(\chi\nu^{-1/2})$. Let $\mu = \chi\nu^{-1/2}$. The representation Ψ_F is irreducible if and only if $\mu \neq \mu^\sigma$ if and only if $\chi \neq \chi^\sigma$. So let us assume that $\chi \neq \chi^\sigma$. Also if Ψ_F is a lift of Ψ then $\mu^\sigma = \chi^{-1}\nu^{1/2}$ that is $\mu\mu^\sigma = 1$ or by Lemma 4.2 $\mu_{|F^\times} = \mathbf{1}$ or $\omega_{E/F}$. If these conditions are satisfied then $\text{Ind}_{W_E}^{W_F}(\mu)$ is an irreducible lift of Ψ_F in

$\mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$. We are left to prove at which condition it is in $\mathrm{SL}_2(\overline{\mathbf{F}}_\ell)$. By the previous condition we already have $\mu\mu^\sigma = 1$, so for $w \in W_E$, $\mathrm{Ind}_{W_E}^{W_F}(\mu)(w) \subseteq \mathrm{SL}_2(\overline{\mathbf{F}}_\ell)$. Let $s \in W_F \setminus W_E$. Then

$$\mathrm{Ind}_{W_E}^{W_F}(\mu)(s) = \begin{pmatrix} 0 & \mu(s^2) \\ 1 & 0 \end{pmatrix}$$

Therefore, $\mathrm{Ind}_{W_E}^{W_F}(\mu)(s) \in \mathrm{SL}_2(\overline{\mathbf{F}}_\ell)$ if and only if $\mu(s^2) = -1$ if and only if $\mu_{|F^\times} = \omega_{E/F}$. And this finish the proof. $\qquad\square$

Remark 5.6. Note that, when χ is quadratic, the case where $\Psi_F = \mathrm{Ind}_{W_E}^{W_F}(\chi\nu^{-1/2})$ is irreducible can only happen when $q_E \equiv q_F \equiv -1 \pmod{\ell}$. Indeed, we have $\chi^\sigma = \chi\nu$ and $\chi \neq \chi^\sigma$. Therefore $\nu \neq 1$ and $q_E \not\equiv 1 \pmod{\ell}$. Also since $\chi^\sigma = \chi\nu$, we get that $\nu_{|F^\times} = 1$ and so that $q_F^2 \equiv 1 \pmod{\ell}$. Hence $q_E \equiv q_F \equiv -1 \pmod{\ell}$.

Now we can examine when a Weil-Deligne representation (Ψ, N) can be lifted. The N will be chosen such that these representations correspond under $P \circ PV$ to an irreducible generic representation of PGL_2.

Lemma 5.7. *Let $q_E \equiv -1 \pmod{\ell}$ and χ be a quadratic character of $E^\times$. Let $\Psi \in [\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$ defined by $\Psi = [\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, N]$, with $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim \Psi$ in $[\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$ if and only if $q_F \equiv -1 \pmod{\ell}$ and $\chi_{|F^\times} = \mathbf{1}$ or $\omega_{E/F}\nu_{|F^\times}^{1/2}$.*

Proof. If we have a lift Ψ_F then we also have a lift of the semisimple part of Ψ. By Lemma 5.5 this is possible only if $\chi = \chi^\sigma$ (which is equivalent by Lemma 4.2 to $\chi_{|F^\times} = \mathbf{1}$ or $\omega_{E/F}$) or $\chi \neq \chi^\sigma$ and $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$.

If $\chi_{|F^\times} = \mathbf{1}$, then by Lemma 5.5, the semisimple part of Ψ_F should be $\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}$, with η a character of $F^\times$ such that $\chi = \eta \circ \mathrm{Nm}_{E/F}$. Note that $\chi_{|F^\times} = \mathbf{1}$ implies that $\eta^2 = 1$. If $q_F \equiv -1 \pmod{\ell}$ then we can take $\Psi_F = [\eta\nu^{-1/2} \oplus \eta\nu^{1/2}, N]$. If $q_F \not\equiv -1 \pmod{\ell}$ then $q_F^2 \not\equiv 1 \pmod{\ell}$ (E is a quadratic extension of F). In this case, $[\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)] = \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$. Since N is not nilpotent, Ψ cannot be the restriction of an element of $\mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$.

If $\chi_{|F^\times} = \omega_{E/F}$, again by Lemma 5.5, the semisimple part of Ψ_F should be $\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}$. This time $\eta^2 = \omega_{E/F} \neq \mathbf{1}$. Thus the only Weil-Deligne representation $[\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}, M]$ is with $M = 0$ and is not a lift of Ψ.

Suppose $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$. We get that $\nu_{|F^\times} = 1$ so $q_F \equiv -1 \pmod{\ell}$. This time Lemma 5.5 tells us that the semisimple part of Ψ_F should be $\mathrm{Ind}_{W_E}^{W_F}(\mu)$, with $\mu = \chi\nu^{-1/2}$. Let $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. First, let us show that $\Psi_F := (\mathrm{Ind}_{W_E}^{W_F}(\mu), M) \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$. Let $s \in W_F \setminus W_E$. For $w \in W_E$, we get that $\mathrm{Ind}_{W_E}^{W_F}(\mu)(w) = \begin{pmatrix} \mu(w) & 0 \\ 0 & \mu^s(w) \end{pmatrix}$. Since $\mu^\sigma = \mu\nu$ we get

$$\begin{pmatrix} \mu(w) & 0 \\ 0 & \mu^s(w) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \nu(w) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mu(w) & 0 \\ 0 & \mu^s(w) \end{pmatrix}.$$

Since $\mu(s^2) = -1$, we also have $\mathrm{Ind}_{W_E}^{W_F}(\mu)(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As we are in the case where $q_F \equiv -1 \pmod{\ell}$, the extension E/F is ramified. The element s must then be in the inertia subgroup $s \in I_F$ and therefore $\nu(s) = 1$. This gives us:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \nu(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have checked that $\Psi_F = (\mathrm{Ind}_{W_E}^{W_F}(\mu), M) \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$. To finish the proof we need to show that $\Psi_{F|W_E} \sim \Psi$. We have $\Psi_{F|W_E} = (\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, M)$. And $(\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, M) \sim \Psi$ by [KM21, Lem. 4.23]. $\qquad\square$

Lemma 5.8. *Let $q_E \equiv 1 \pmod{\ell}$ and χ be a quadratic character of $E^\times$. Let $\Psi \in [\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$ defined by $\Psi = [\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, N]$, with $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim \Psi$ if and only if $\chi_{|F^\times} = \mathbf{1}$ or $q_F \equiv 1 \pmod{\ell}$ and $\chi_{|F^\times} = \omega_{E/F}$.*

Proof. By Lemma 5.5 the semisimple part of Ψ can be lifted to W_F if and only if $\chi_{|F^\times} = \mathbf{1}$ or $\omega_{E/F}$ ($\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$ implies that $\chi = \chi^\sigma$). Moreover, this lift is $\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}$, with η a character of $F^\times$ such that $\chi = \eta \circ \mathrm{Nm}_{E/F}$.

If $\chi_{|F^\times} = \mathbf{1}$, then $\eta^2 = 1$. We can take $\Psi_F = (\eta\nu^{-1/2} \oplus \eta\nu^{1/2}, N)$ (since $\nu_F^2 = 1$) and $\Psi_{F|W_E} \sim \Psi$.

If $\chi_{|F^\times} = \omega_{E/F}$, this time $\eta^2 = \omega_{E/F} \neq \mathbf{1}$. If $q_F \equiv -1 \pmod{\ell}$ then $\nu_F = \omega_{E/F}$. In this case, $\eta^{-1} = \eta\nu_F$. Thus $\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2} = \eta\nu^{-1/2} \oplus \eta\nu^{-1/2}$. The only Weil-Deligne representation with this semisimple part is $(\eta\nu^{-1/2} \oplus \eta\nu^{-1/2}, 0)$ which is not a lift of Ψ. If $q_F \equiv 1 \pmod{\ell}$ let $\Psi_F := (\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$. We are left to prove that $\Psi_{F|W_E} \sim \Psi$. Let us remark that N is diagonalizable. Therefore, $(\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, N)$ is isomorphic to $(\chi\nu^{-1/2} \oplus \chi\nu^{-1/2}, N')$ with $N' = diag(1, -1)$.

Hence $\Psi_{F|W_E} \sim \Psi$ and we get the result. $\qquad\square$

Lemma 5.9. *Let $q_E^2 \not\equiv 1 \pmod{\ell}$ and χ be a quadratic character of $E^\times$. Let $\Psi \in [\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$ defined by $\Psi = [\chi\nu^{-1/2} \oplus \chi\nu^{1/2}, N]$, with $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim \Psi$ if and only if $\chi_{|F^\times} = \mathbf{1}$.*

Proof. By Lemma 5.5 we can lift the semisimple part if and only if $\chi_{|F^\times} = \mathbf{1}$ or $\omega_{E/F}$ and this lift is $\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}$, with η a character of $F^\times$ such that $\chi = \eta \circ \mathrm{Nm}_{E/F}$. If $\chi_{|F^\times} = \mathbf{1}$, then $\eta^2 = 1$. We can take $\Psi_F = (\eta\nu^{-1/2} \oplus \eta\nu^{1/2}, N)$ and $\Psi_{F|W_E} \sim \Psi$. If $\chi_{|F^\times} = \omega_{E/F}$, $\eta^2 = \omega_{E/F} \neq \mathbf{1}$. If $(\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}, M)$ is a Weil-Deligne representation then $M = 0$ and this is not a lift of Ψ. $\qquad\square$

Lemma 5.10. *Let χ be a character of $E^\times$. Let $\Psi \in [\mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)]$ defined by $\Psi = [\chi\nu^{-1/2} \oplus \chi^{-1}\nu^{1/2}, N]$, with $N = 0$. Then there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim \Psi$ if and only if $\chi = \chi^\sigma$ or $\chi \neq \chi^\sigma$ and $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$.*

Proof. If $\chi = \chi^\sigma$, take η a character of $F^\times$ such that $\chi = \eta \circ Nm_{E/F}$. Then $\Psi_F = [\eta\nu^{-1/2} \oplus \eta^{-1}\nu^{1/2}, N]$ is a lift. And if $\chi \neq \chi^\sigma$, Lemma 5.5 tells us that if there is a lift then $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$. In this case, we can take $\Psi_F = [\mathrm{Ind}_{W_E}^{W_F}(\chi\nu^{-1/2}), N]$. $\qquad\square$

5.5. **A modulo ℓ Prasad conjecture for** PGL_2. Now we can gather together all the results of the previous sections to prove a "modified" Prasad conjecture for PGL_2.

Let PV be the correspondence induced by the Vignéras correspondence

$$PV : \mathrm{Irr}_{\overline{\mathbf{F}}_\ell}(\mathrm{PGL}_2(E)) \to \mathrm{Nilp}_{\overline{\mathbf{F}}_\ell}(W_E, \mathrm{SL}_2)$$

Let us start with the supercuspidal representations.

Proposition 5.11. *Let π be an irreducible supercuspidal representation of $\mathrm{PGL}_2(E)$ over $\overline{\mathbf{F}}_\ell$. Then π is $\omega_{E/F}$-distinguished if and only if its Langlands parameter $PV(\pi)$ can be lifted to W_F.*

Proof. Let π be an irreducible supercuspidal representation of $\mathrm{PGL}_2(E)$ over $\overline{\mathbf{F}}_\ell$. Let $\varphi := PV(\pi)$ be the Langlands parameter of W_E associated to π. By Proposition 3.11 π is $\omega_{E/F}$-distinguished if and only if there exists $\tilde{\pi}$ a $\overline{\mathbf{Q}}_\ell$-lift of π which is supercuspidal and $\tilde{\omega}_{E/F}$-distinguished. By the Prasad conjecture, this happens if and only if the Langlands parameter $\tilde{\varphi}$ of $\tilde{\pi}$, can be extended to W_F. From the definition of the modulo ℓ Langlands correspondence, $\tilde{\varphi}$ is a $\overline{\mathbf{Q}}_\ell$-lift of φ. Thus $\tilde{\varphi}$ can be extended to W_F if and only if φ can be extended to W_F. $\square$

Now let us examine the irreducible generic representations of $\mathrm{PGL}_2(E)$. Let π be such a representation and denote by $PV(\pi) = (\Psi, N)$ its Langlands parameter. We can classify the irreducible generic representations as follows:

(1) π is supercuspidal. In this case, Ψ is irreducible and $N = 0$.
(2) π is an irreducible principal series. Here $\pi = \pi(\chi\nu^{-1/2}, \chi^{-1}\nu^{1/2})$ with $\chi^2 \neq 1$. We have $\Psi = \chi\nu^{-1/2} \oplus \chi^{-1}\nu^{1/2}$ and $N = 0$
(3) Suppose that π is the unique generic subquotient of a reducible principal series. Let χ be a quadratic character of $E^\times$ such that π is a subquotient of $\pi(\chi\nu^{-1/2}, \chi\nu^{1/2})$.
    (a) If $q_E \not\equiv 1 \pmod{\ell}$, then $\pi = \mathrm{St}_\chi$. In this case, $\Psi = \chi\nu^{-1/2} \oplus \chi\nu^{1/2}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
    (b) If $q_E \equiv -1 \pmod{\ell}$, this time $\pi = \mathrm{Sp}_\chi$, $\Psi = \chi\nu^{-1/2} \oplus \chi\nu^{1/2}$ and $N = 0$.

We can summarize the previous results of this article to prove the Prasad conjecture in the modular case.

Theorem 5.12. *Let π be an irreducible generic representation of $\mathrm{PGL}_2(E)$ over $\overline{\mathbf{F}}_\ell$. Then π is $\omega_{E/F}$-distinguished if and only if there exists $\Psi_F \in \mathrm{WDRep}_{\overline{\mathbf{F}}_\ell}(W_F, \mathrm{SL}_2)$ such that $\Psi_{F|W_E} \sim P \circ PV(\pi)$.*

Proof. For supercuspidal representations, the result follows from Proposition 5.11. For irreducible principal series representations it follows from Lemmas 4.3 and 5.10. And for the Steinberg representations or the special representations, it follows from Theorem 4.6 and Lemmas 5.7, 5.8 and 5.9 (depending on the order of q_E modulo ℓ). $\square$

Remark 5.13. When $q_E \equiv -1 \pmod{\ell}$, $q_F \equiv -1 \pmod{\ell}$ and χ is a quadratic character of $E^\times$ such that $\chi_{|F^\times} = \omega_{E/F}\nu_{|F^\times}^{1/2}$. We have proved that $P \circ PV(\mathrm{Sp}_\chi)$

admits a lift to W_F which is $\Psi_F = (\mathrm{Ind}_{W_E}^{W_F}(\chi\nu^{-1/2}), M)$ with $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The semisimple part of Ψ_F is irreducible and M is non-zero, and so this lift is not the Langlands parameter of any representation of $\mathrm{PGL}_2(F)$ (nor it is in the image of P).

6. The $\mathrm{SL}_2(F)$-distinguished representations

In this section, we classify all the representations of $\mathrm{SL}_2(E)$ distinguished by $\mathrm{SL}_2(F)$. For supercuspidal representations, we will use the restriction method of [AP03]. Similar to the case $\mathrm{GL}_2(E)$, we will deal with principal series representations using Mackey theory.

6.1. **Modulo ℓ representations of SL_2.** We start by recalling some general facts about modulo ℓ representations of SL_2.

Recall that E is a quadratic extension of F. Let $\mathfrak{p}_E$ be a uniformizer of E, and $\mathfrak{o}_E$ be the ring of integers of E. Let π be an irreducible cuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(E)$. Thanks to [Cui20, Prop. 2.35], we have that the restricted representation $\pi|_{\mathrm{SL}_n(E)}$ is semisimple with finite length and multiplicity-free. Denote by $\mathrm{lg}(\pi)$ the length of $\pi|_{\mathrm{SL}_n(E)}$. Let

$$Y(\pi) = \{\chi : \pi \otimes \chi \circ \det \cong \pi\}.$$

We call $Y(\pi)$ *the twist isomorphism set of π.* By Corollary 3.8 of [Cui20], the cardinality $|Y(\pi)|$ is an integer prime to ℓ, and we deduce, from the fact that $\pi|_{\mathrm{SL}_n(F)}$ being multiplicity-free, an equation that

$$(6.1) \qquad |Y(\pi)| = \mathrm{lg}(\pi)_{\ell'},$$

where for any $m \in \mathbb{N}^*$, we denote by $m_{\ell'}$ the largest divisor of m which is coprime to ℓ.

Lemma 6.1. *When $n = 2$, the equation holds, i.e.,*

$$|Y(\pi)| = \mathrm{lg}(\pi).$$

Proof. By (6.1), it is sufficient to prove that the length $\mathrm{lg}(\pi)$ is coprime to ℓ. Let $Z_{E^\times}$ be the center of $\mathrm{GL}_2(E)$. The length of $\pi|_{Z_{E^\times} \mathrm{SL}_2(E)}$ is equal to the length $\mathrm{lg}(\pi)$, which is a divisor of the cardinality $|\mathrm{GL}_2(E) : Z_{E^\times} \mathrm{SL}_2(E)|$, and the latter is equal to the cardinality $|E^\times : E^{\times 2}|$.

When $p \neq 2$, by Hensel's lemma we have $[E^\times : E^{\times 2}]$ is equal to 4. When $p = 2$, by the binomial expansion formula, we know that when N is big enough, an element in $1 + \mathfrak{p}_E^N \mathfrak{o}_E$ is a square of an element in $\mathfrak{o}_E^\times$, which means $\mathfrak{o}_E^\times / \mathfrak{o}_E^{\times 2}$ is finite, hence the order of the quotient is a power of 2 from the fact that each element is order 2. We deduce that $\mathrm{lg}(\pi)$ is a power of 2, and we have the desired equation under our assumption $\ell \neq 2$. $\qquad\square$

Definition 6.2. Define $\mathrm{GL}_2^+(E)$ to be a subgroup of $\mathrm{GL}_2(E)$, consisting of matrices whose determinant belongs to $F^\times E^{\times 2}$. We have $\mathrm{GL}_2^+(E) = Z_{E^\times} \mathrm{SL}_2(E) \mathrm{GL}_2(F)$ where $Z_{E^\times}$ denotes the center of $\mathrm{GL}_2(E)$.

Corollary 6.3. *Let $\mathrm{lg}_+(\pi)$ be the length of $\pi|_{\mathrm{GL}_2^+(E)}$, and $Y_+(\pi) = \{\chi : \pi \otimes \chi \circ \det \cong \pi, \ \chi|_{F^\times} = 1\}$. Then we have an equation*

$$\mathrm{lg}_+(\pi) = |Y_+(\pi)|.$$

Proof. From the proof of Lemma 6.1 the quotient group $\mathrm{GL}_2(E)/\mathrm{GL}_2^+(E)$ is finite abelian, and the order is a power of 2 which is prime to ℓ. We deduce that the dual group $(\mathrm{GL}_2(E)/\mathrm{GL}_2^+(E))^\wedge$ is equivalent to $Y_+(\pi)$. Then the method of Corollary 3.8 of [Cui20] can be applied. In particular, Corollary 3.8 of [Cui20] studies the length of $\pi|_{\mathrm{SL}_2(E)}$. Then we consider the characters of $\mathrm{GL}_2(E)/\mathrm{SL}_2(E) \cong E^\times$ and we have the equation that the length of $\pi|_{\mathrm{SL}_2(E)}$ is equal to the cardinality of the twist isomorphism set. To study the length of $\pi|_{\mathrm{GL}_2^+(E)}$, we need to consider the characters in $(\mathrm{GL}_2(E)/\mathrm{GL}_2^+(E))^\wedge$. Hence by applying the proof of Corollary 3.8 of [Cui20], we obtain the desired equation. $\qquad\square$

To compute the length $\mathrm{lg}(\pi)$ we will need to use the local Langlands correspondence.

6.2. **Local Langlands for supercuspidal representations of** SL_2**.** In this section, we use the local Langlands correspondence for GL_2 to define a correspondence modulo ℓ for supercuspidal representations of SL_2. As in the complex case, for SL_2, this correspondence is not a bijection, we give a description of the L-packet.

Let τ be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{SL}_2(E)$. Let π be a *lift* to $\mathrm{GL}_2(E)$ i.e., $\tau \subset \pi|_{\mathrm{SL}_2(E)}$. To π we associate by the local Langlands correspondence of Vignéras its Langlands parameter $\varphi_\pi : W_E \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$. Let γ be the projection $\gamma : \mathrm{GL}_2(\overline{\mathbf{F}}_\ell) \to \mathrm{PGL}_2(\overline{\mathbf{F}}_\ell)$. Then we define $\varphi_\tau : W_E \to \mathrm{PGL}_2(\overline{\mathbf{F}}_\ell)$ by $\varphi_\tau := \gamma \circ \varphi_\pi$. The parameter φ_τ does not depend on the choice of the lift π since two lifts differ by a character and so are their Langlands parameters.

Let $\varphi : W_E \to \mathrm{PGL}_2(\overline{\mathbf{F}}_\ell)$ be the parameter of a representation of $\mathrm{SL}_2(E)$. Denote by $S_\varphi := C_{\mathrm{PGL}_2(\overline{\mathbf{F}}_\ell)}(\varphi(W_E))$ the centralizer in $\mathrm{PGL}_2(\overline{\mathbf{F}}_\ell)$ of the image of φ.

Proposition 6.4. *Let τ (resp. π) be a supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{SL}_2(E)$ (resp. $\mathrm{GL}_2(E)$) and $\tau \subset \pi|_{\mathrm{SL}_2(F)}$. Then we have an isomorphism*

$$S_{\varphi_\tau} \simeq Y(\pi)$$

Proof. We follow the strategy of [GK82, Thm. 4.3]. To simplify the notation here, we will simply denote φ_τ by φ. From the definition of φ, we have that $\varphi = \gamma \circ \varphi_\pi$. Let $s \in S_\varphi$ and $\tilde{s} \in \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ such that $\gamma(\tilde{s}) = s$. We define a function $\chi_s : W_E \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ by

$$\chi_s(w) := \tilde{s}\varphi_\pi(w)\tilde{s}^{-1}\varphi_\pi(w)^{-1}, \text{ for } w \in W_F.$$

This definition is independent of the choice of $\tilde{s}$. Moreover, since $\gamma(\chi_s(w)) = 1$, $\chi_s(w)$ is a scalar times the identity. We will denote this scalar again $\chi_s(w)$. Hence, we have

$$\tilde{s}\varphi_\pi(w)\tilde{s}^{-1} = \chi_s(w)\varphi_\pi(w).$$

Let $w_1, w_2 \in W_E$. Then

$$
\begin{aligned}
\chi_s(w_1 w_2)\varphi_\pi(w_1 w_2) &= \tilde{s}\varphi_\pi(w_1 w_2)\tilde{s}^{-1} = \tilde{s}\varphi_\pi(w_1)\varphi_\pi(w_2)\tilde{s}^{-1} \\
&= \tilde{s}\varphi_\pi(w_1)\tilde{s}^{-1}\tilde{s}\varphi_\pi(w_2)\tilde{s}^{-1} \\
&= \chi_s(w_1)\varphi_\pi(w_1)\chi_s(w_2)\varphi_\pi(w_2) \\
&= \chi_s(w_1)\chi_s(w_2)\varphi_\pi(w_1)\varphi_\pi(w_2) = \chi_s(w_1)\chi_s(w_2)\varphi_\pi(w_1 w_2)
\end{aligned}
$$

Thus $\chi_s(w_1 w_2) = \chi_s(w_1)\chi_s(w_2)$ and χ_s is a character.

Since $\varphi_\pi \simeq \chi_s \varphi_\pi$ we have $\pi \simeq \pi \otimes (\chi_s \circ \det)$ and $\chi_s \in Y(\pi)$. This defines a morphism $S_\varphi \to Y(\pi)$, $s \mapsto \chi_s$.

This morphism is surjective. Indeed, if $\omega \in Y(\pi)$ then $\pi \simeq \pi \otimes (\omega \circ \det)$ and thus $\varphi_\pi \simeq \omega \varphi_\pi$. If $\tilde{A}$ implements the equivalence then $\tilde{A}\varphi_\pi(w)\tilde{A}^{-1} = \omega(w)\varphi_\pi(w)$. Let $A := \gamma(\tilde{A})$. Then $A \in S_\varphi$ and $\omega = \chi_A$.

We are left to prove that $S_\varphi \to Y(\pi)$ is injective. Let $s \in S_\varphi$ such that $\chi_s = 1$. This implies that $\tilde{s}$ centralizes the image of φ_π. Since φ_π is irreducible, Schur's lemma tells us that $\tilde{s}$ is a scalar. Hence $s = 1$ and we have the injectivity. $\qquad\square$

6.3. **Explicit computation of the length.** By Proposition 6.4, we compute the length $\lg(\pi)$ by considering the cardinality of S_{φ_τ}. The method in [She79] can be generalised to the case when ℓ is positive, based on which we also obtain the existence of good lifting.

Definition 6.5. Let τ be an irreducible cuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{SL}_2(E)$, and $\tilde{\tau}$ an irreducible cuspidal $\overline{\mathbf{Q}}_\ell$-representation of $\mathrm{SL}_2(E)$, which is ℓ-integral. We say

- $\tilde{\tau}$ is a $\overline{\mathbf{Q}}_\ell$-lifting of τ, if τ is a direct factor of the reduction modulo ℓ of $\tilde{\tau}$;
- $\tilde{\tau}$ is a good $\overline{\mathbf{Q}}_\ell$-lifting of τ, if the reduction modulo ℓ of $\tilde{\tau}$ is irreducible and isomorphic to τ.

For the case of $\mathrm{GL}_n(E)$, an irreducible cuspidal $\overline{\mathbf{F}}_\ell$-representation π of $\mathrm{GL}_n(E)$ always has a $\overline{\mathbf{Q}}_\ell$-lifting $\tilde{\pi}$, and every $\overline{\mathbf{Q}}_\ell$-lifting is a good $\overline{\mathbf{Q}}_\ell$-lifting. The latter property is not true for SL_2. However, we will show the existence of a good $\overline{\mathbf{Q}}_\ell$-lifting of an irreducible cuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{SL}_2(E)$.

Proposition 6.6. *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$.*

(1) *When φ_π is dihedral, the length $\lg(\pi)$ is 2 or 4;*
(2) *When φ_π is not dihedral, then it must be tetrahedral or octahedral, and the length $\lg(\pi)$ is 1.*

Proof. Let $\tilde{\pi}$ be a $\overline{\mathbf{Q}}_\ell$-lifting of π. Then the length $\lg(\tilde{\pi})$ divides the length $\lg(\pi)$. Suppose φ_π is dihedral. Then $\varphi_{\tilde{\pi}}$ is dihedral as well, which implies that in the first case 2 divides $\lg(\pi)$. Now we write φ_π as $\mathrm{ind}_{W_K}^{W_E} \chi$, and let η be a non-trivial $\overline{\mathbf{F}}_\ell$-quasicharacter of $E^\times$ such that $\eta \in Y(\pi)$. We deduce that either $\eta = \omega_{K/E}$ or $\eta|_{W_K} \cong \chi^{-1}\chi^s$, where s is the non-trivial element in $\mathrm{Gal}(K/E)$. Suppose that $\eta \neq \omega_{K/E}$. Since $\chi^{-1}\chi^s = \eta \circ \mathrm{Nm}_{K/E}$ is quadratic, i.e. $\chi^2 = (\chi^2)^s$, there exists another $\overline{\mathbf{F}}_\ell$-quasicharacter η_E of $E^\times$ such that $\chi^2 \cong \eta_E \circ \mathrm{Nm}_{K/E}$, which implies that $\chi|_{E^\times}$ is isomorphic either to $\eta_E \otimes \eta$ or to $\eta_E \otimes \eta \otimes \omega_{K/E}$. We conclude that $Y(\pi)$ consists of $\chi|_{E^\times} \otimes \eta_E^{-1}, \chi|_{E^\times} \otimes \eta_E^{-1} \otimes \omega_{K/E}, \omega_{K/E}$ and the trivial character. In other words, the length $\lg(\pi)$ is either 2 or 4.

Suppose $p = 2$ and φ_π is not dihedral. Then by Class Field Theory and Clifford Theory, we know that the set $Y(\pi)$ must be a singleton. In this case, the image of a $\overline{\mathbf{Q}}_\ell$-lifting $\tilde{\pi}$ under the projection from $\mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ to $\mathrm{PGL}_2(\overline{\mathbf{Q}}_\ell)$ is either isomorphic to S_4 or A_4. For the second case, since any two subgroups of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_\ell)$ being isomorphic to A_4 are conjugate to each other, we can choose $\varphi_{\tilde{\pi}}$ such that its image in $\mathrm{PGL}_2(\overline{\mathbf{Q}}_\ell)$ will be $N \rtimes C$, where

$$N = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}, C = \left\{ I, \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix} \right\}.$$

Since $\ell \neq 2$, after reduction modulo ℓ the image of φ_π is isomorphic to $N \rtimes C \cong S_4$ as well. For the case of S_4, it is isomorphic to $\langle N \rtimes C, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 1 \end{pmatrix} \rangle$. Then repeat the similar argument to the case of A_4. $\qquad\square$

Proposition 6.7. *Let π be as in Proposition 6.6 and $\tilde{\pi}$ be as above. Let τ be an irreducible component of $\pi|_{\mathrm{SL}_2(E)}$ and $\tilde{\tau}$ an irreducible component of $\tilde{\pi}|_{\mathrm{SL}_2(E)}$.*

(1) *Suppose that φ_π is tetrahedral or octahedral. Then the same is true for $\tilde{\pi}$, and the reduction modulo ℓ of $\tilde{\pi}|_{\mathrm{SL}_2(E)}$ is irreducible and isomorphic to τ. The representation $\tilde{\pi}|_{\mathrm{SL}_2(E)}$ is a good $\overline{\mathbf{Q}}_\ell$-lifting of τ.*

(2) *Suppose that φ_π is dihedral.*

   (a) *If the cardinality of $S_{\varphi_{\tilde\tau}}$ is 4, then the reduction modulo ℓ of $\tilde{\tau}$ is irreducible. In particular, there exists one component $\tilde{\tau}$ whose reduction modulo ℓ is isomorphic to τ, and it is a good $\overline{\mathbf{Q}}_\ell$-lifting of τ.*

   (b) *If the cardinality of $S_{\varphi_{\tilde\tau}}$ is 2, then the reduction modulo ℓ of $\tilde{\tau}$ may be reducible. If it is irreducible, there exists one component $\tilde{\tau}$ whose reduction modulo ℓ is isomorphic to τ, and it is a good $\overline{\mathbf{Q}}_\ell$-lifting of τ. If it is reducible, then there exists another $\overline{\mathbf{Q}}_\ell$-lifting $\tilde{\pi}'$ of π, such that the reduction modulo ℓ of each irreducible component $\tilde{\tau}'$ of $\tilde{\pi}'|_{\mathrm{SL}_2(F)}$ is irreducible, and the case (a) implies that one of such component is a good $\overline{\mathbf{Q}}_\ell$-lifting of τ.*

Proof. Part (1) is clear from Proposition 6.6(2). For part (2), assume that the cardinality of $S_{\varphi_{\tilde\tau}}$ is 4. We switch the order of restriction to SL_2 and reduction modulo ℓ. Then Proposition 6.6(1) implies that the length $\mathrm{lg}(\pi)$ is 4, and every irreducible component of $\tilde{\pi}|_{\mathrm{SL}_2}$ is ℓ-integral and with irreducible reduction modulo ℓ. The unicity of Jordan-Hölder components implies the existence of a good lifting $\tilde{\tau}$.

Assume the cardinality of $S_{\varphi_{\tilde\tau}}$ is 2. Then there exists a field extension K/E and an ℓ-integral $\overline{\mathbb{Q}}_\ell$-quasicharacter $\tilde{\theta}$ of $K^\times$, such that $\varphi_{\tilde\tau} \cong \mathrm{Ind}_{W_K}^{W_E} \tilde{\theta}$, and $(\tilde{\theta}^s/\tilde{\theta})^2 \neq 1$, where $\mathrm{Gal}(K/E) = \langle s \rangle$. Let θ be the reduction modulo ℓ of $\tilde{\theta}$. Then $\varphi_\pi \cong \mathrm{Ind}_{W_K}^{W_E} \theta$. Suppose $(\theta^s/\theta)^2 \neq 1$. Then the cardinality of S_{φ_τ} is 2 as well. In this case, we apply the same argument as in the first case of part (2), and deduce the existence of a good lifting $\tilde{\tau}$. If $(\theta^s/\theta)^2 = 1$, the cardinality of S_{φ_τ} is 4, which implies that the length of the reduction modulo ℓ of each irreducible component in $\tilde{\pi}|_{\mathrm{SL}_2}$ is 2. It means that none of them is a good lifting of τ. Now fix a group embedding from $\overline{\mathbf{F}}_\ell^\times$ to $\overline{\mathbf{Z}}_\ell^\times$ by sending an element of $\overline{\mathbf{F}}_\ell^\times$ to its Teichmuller representative, which gives a natural $\overline{\mathbf{Q}}_\ell$-lifting of θ, denoted by $\tilde{\theta}_0$. We deduce that $(\tilde{\theta}_0^s/\tilde{\theta}_0)^2 = 1$. Let $\tilde{\pi}'$ be an irreducible $\overline{\mathbf{Q}}_\ell$-representation of $\mathrm{GL}_2(E)$ corresponding to $\varphi_0 = \mathrm{Ind}_{W_K}^{W_E} \tilde{\theta}_0$, which is ℓ-integral with reduction modulo ℓ isomorphic to π, and the cardinality of $Y(\tilde{\pi}')$ is 4. We apply the argument of the case (a) of part (2), and deduce that there exists an irreducible component $\tilde{\tau}'$ of $\tilde{\pi}'|_{\mathrm{SL}_2(F)}$, which is a good $\overline{\mathbf{Q}}_\ell$-lifting of τ. $\qquad\square$

6.4. **Representations of $\mathrm{GL}_2(E)$ distinguished by $\mathrm{SL}_2(F)$.** In the next two subsections, we follow the restriction method of [AP03]. The main strategy of [AP03] works. However in the modular setting, some modifications are needed in the proofs. For convenience, we state the results which are required for further use.

For an irreducible $\overline{\mathbf{F}}_\ell$-representation π of $\mathrm{GL}_2(E)$, we denote by $X(\pi)$ the following set

$$X(\pi) = \{\chi : \overline{\mathbf{F}}_\ell\text{-quasicharacter of } F^\times, \pi \text{ is } (\mathrm{GL}_2(F), \chi)\text{-distinguished}\}$$

Proposition 6.8. [AP03, Prop. 4.1] *Let π be an irreducible $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi, 1) = |X(\pi)|.$$

Proof. Assume that π is $\mathrm{SL}_2(F)$-distinguished. Let $Z_{F^\times}$ be the center of $\mathrm{GL}_2(F)$ and $Z'_{F^\times}$ be the center of $\mathrm{SL}_2(F)$. The group $Z'_{F^\times}$ is the kernel of determinant that maps $Z_{F^\times}$ into $F^\times$. The central character ω_π of π is trivial on $Z'_{F^\times}$ and so there exists a character χ_F of $F^\times$ such that $\chi_F^2 = \omega_\pi|_{Z_{F^\times}}$. Since each smooth $\overline{\mathbf{F}}_\ell$-character of $F^\times$ can be extended to a smooth $\overline{\mathbf{F}}_\ell$-character of $E^\times$, we obtain that after twisting by a smooth $\overline{\mathbf{F}}_\ell$-character of $E^\times$, the central character ω_π of π is trivial on $Z_{F^\times}$. On the other hand, if π is $(\mathrm{GL}_2(F), \chi)$-distinguished for an $\overline{\mathbf{F}}_\ell$-character χ of $F^\times$, then π is $\mathrm{SL}_2(F)$-distinguished, and ω_π is trivial on $Z_{F^\times}$ after twisting by a smooth $\overline{\mathbf{F}}_\ell$-character of $E^\times$ as explained above. We obtain the fact that if ω_π is never trivial on $Z_{F^\times}$ after twisting by a smooth $\overline{\mathbf{F}}_\ell$-character of $E^\times$, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi, 1) = |X(\pi)| = 0$.

Now assume that the central character ω_π is trivial on $Z_{F^\times}$. As in the proof of Proposition 4.1 of [AP03], we consider the $\overline{\mathbf{F}}_\ell$-space $\mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi, 1)$, which has a $\mathrm{GL}_2(F)$-module structure and $F^\times \cdot \mathrm{SL}_2(F)$ acts trivially. Since $F^\times/F^{\times 2}$ is a finite abelian group whose order is a power of 2, our assumption that $\ell \neq 2$ implies that $\mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi, 1)$ can decompose into a direct sum of $\overline{\mathbf{F}}_\ell$-characters of $F^\times/F^{\times 2}$, hence a direct sum of $\overline{\mathbf{F}}_\ell$-characters of $\mathrm{GL}_2(F)$. In particular, from the definition, for an $\overline{\mathbf{F}}_\ell$-character χ of $\mathrm{GL}_2(F)$ which appears in the direct sum above, we have that π is χ-distinguished with respect to $\mathrm{GL}_2(F)$. Due to Theorem 4.1(3), the direct sum above is multiplicity-free as $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(F)$. Hence we obtain the equation. $\qquad\square$

6.5. **Supercuspidal case.** This subsection focuses on the distinction problems for supercuspidal representations.

Denote by U the subgroup of $\mathrm{GL}_2(E)$ consisting of the upper triangular matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then $\mathrm{U} \cong E$. Fix an $\overline{\mathbf{F}}_\ell$-character ψ_0 of E, which is non-trivial on $\mathfrak{o}_E$ and is trivial on both $\mathfrak{p}_E$ and F. Let π be an irreducible infinite-dimensional $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$. Then π is ψ_0-generic, and π has a unique Whittaker model $\mathcal{W}(\pi, \psi_0)$.

Lemma 6.9. *Let π_0 be an irreducible $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2^+(E)$ which is of infinite dimension and distinguished by $\mathrm{SL}_2(F)$. Then it has a Whittaker model with respect to ψ_0.*

Proof. A similar method as in [AP03, Lem. 3.1] can be applied here, and we add some preliminaries at first. We give a proof with details for the convenience of the reader.

Let π be an irreducible $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$ such that $\pi|_{\mathrm{GL}_2^+(E)} \supset \pi_0$. Up to twisting by an $\overline{\mathbf{F}}_\ell$-quasicharacter of $F^\times$ on π_0, we can assume that π_0 is distinguished with respect to $\mathrm{GL}_2(F)$. Note that each $\overline{\mathbf{F}}_\ell$-quasicharacter of $F^\times$ can

be extended to $E^\times$. Hence we can assume that both π_0 and π are distinguished by $\mathrm{GL}_2(F)$. Recall that ψ_0 is an additive $\overline{\mathbf{F}}_\ell$-character of E which is trivial on F, and denote by $\mathcal{W}(\pi, \psi_0)$ the ψ_0-Whittaker model of π. Let $W \in \mathcal{W}(\pi, \psi_0)$. By [KM20, §8.2] we know that the unique $\mathrm{GL}_2(F)$-invariant linear form can be realized as an integration form

$$l(W) = \int_{F^\times} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})d^\times a,$$

where $d^\times a$ is a Haar measure on $F^\times$, which is non-zero (in [KM20] this integration form is denoted by $\mathcal{P}_\pi$) and $\mathrm{GL}_2(F)$-invariant. There is a unique irreducible component of the restricted representation $\pi|_{\mathrm{GL}_2^+(E)}$ that is ψ_0-generic. Let π_1 be a non ψ_0-generic irreducible component of $\mathcal{W}(\pi, \psi_0)|_{\mathrm{GL}_2^+(E)}$, and $W \in \mathcal{W}(\pi, \psi_0)$ a function belonging to π_1. The truncation of W to $\mathrm{GL}_2^+(E)$ must be zero, otherwise it will induce a non-trivial morphism from π_1 to the space of ψ_0-Whittaker functions on $\mathrm{GL}_2^+(E)$, which contradicts the assumption that π_1 is not ψ_0-generic. Therefore, the $\mathrm{GL}_2(F)$-invariant linear form can be non-zero only on the ψ_0-generic part of $\pi|_{\mathrm{GL}_2^+(E)}$, and we conclude that π_0 is ψ_0-generic. $\qquad\square$

Lemma 6.10. [AP03, Lem. 3.2] *Let π be an irreducible representation of $\mathrm{GL}_2^+(E)$. The restricted representation $\pi|_{\mathrm{SL}_2(E)}$ is semisimple with finite length and all irreducible components are conjugate to each other under the action of $\mathrm{GL}_2(F)$. Hence the dimension $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, 1)$ is independent of the choice of irreducible component τ of $\pi|_{\mathrm{SL}_2(E)}$.*

Proposition 6.11. [AP03, Prop. 4.2] *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$, which is distinguished with respect to $\mathrm{SL}_2(F)$. Then we have an equation*

$$|X(\pi)| = |Y_+(\pi)|.$$

Assume further that π is distinguished with respect to $\mathrm{GL}_2(F)$. Then composition with the norm map $N_{E/F}$ induced a bijection from $X(\pi)$ to $Y_+(\pi)$.

Proof. The strategy of [AP03, Prop. 4.2] can be applied, and we write the proof here for the convenience of the reader. By Proposition 6.8, after twisting π by an $\overline{\mathbf{F}}_\ell$-character of $E^\times$, we assume that π is distinguished with respect to $\mathrm{GL}_2(F)$. As in the proof of [AP03, Prop. 4.2], we give a bijection from $X(\pi)$ to $Y_+(\pi)$ given by the composition with the norm map.

Let $\mathrm{Nm}_{E/F}$ be the norm map from $E^\times$ to $F^\times$ and $\chi \in X(\pi)$. Then $\mathcal{N} : \chi \mapsto \chi \circ \mathrm{Nm}_{E/F}$ maps $X(\pi)$ to $Y_+(\pi)$. Indeed, let $\chi \in X(\pi)$ and $\tilde{\chi}$ be a character of E^* restricting to χ. By the last part of Theorem 4.1, π and $\pi \otimes \tilde{\chi}^{-1}$ are σ-selfdual. Therefore, $\pi \simeq \pi \otimes \chi \circ \mathrm{Nm}_{E/F}$, and $\chi \circ \mathrm{Nm}_{E/F} \in Y_+(\pi)$.

Conversely, let $\mu \in Y_+(\pi)$. Since $\mu^2 = 1$ and $\mu|_{F^\times} = 1$, by Hilbert 90th Theorem, it is trivial on the kernel of $\mathrm{Nm}_{E/F}$. Hence there exists an $\overline{\mathbf{F}}_\ell$-character η of $F^\times$, such that $\mu \cong \eta \circ \mathrm{Nm}_{E/F}$. Let $\tilde{\eta}$ be an extension of η to $E^\times$. Then $\mu \cong \tilde{\eta} \cdot \tilde{\eta}^\sigma$. Since π is distinguished with respect to $\mathrm{GL}_2(F)$, we have

$$(\pi \otimes \tilde{\eta})^\vee \cong (\pi \otimes \tilde{\eta})^\sigma.$$

On the other hand, let $Z_{F^\times}$ be the center of $\mathrm{GL}_2(F)$, then $\omega_{\pi \otimes \tilde{\eta}}|_{Z_{F^\times}}$ is trivial where $\omega_{\pi \otimes \tilde{\eta}}$ is the central character of $\pi \otimes \tilde{\eta}$. By Theorem 3.14, $\pi \otimes \tilde{\eta}$ is either distinguished or $\omega_{E/F}$-distinguished by $\mathrm{GL}_2(F)$ but not both. We map μ to η or

$\eta \otimes \omega_{E/F}$ accordingly. This gives a map from $Y_+(\pi)$ to $X(\pi)$ which is the inverse of $\mathcal{N}$, hence we complete the proof. $\qquad\square$

Corollary 6.12. [AP03, Cor. 4.3] *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$, distinguished by $\mathrm{SL}_2(F)$. The number of $\mathrm{SL}_2(F)$-invariant linear functionals on π is equal to the length of $\pi|_{\mathrm{GL}_2^+(E)}$ and both are $\mathrm{lg}_+(\pi)$.*

Proof. It follows from Corollary 6.3, Proposition 6.8 and Proposition 6.11. $\qquad\square$

Recall that $\mathrm{lg}(\pi)$ is the length of $\pi|_{\mathrm{SL}_2(E)}$ and $\mathrm{lg}_+(\pi)$ is the length of $\pi|_{\mathrm{GL}_2^+(E)}$.

Proposition 6.13. [AP03, Prop. 4.4] *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$ such that $\mathrm{lg}(\pi)$ is different from 1. Suppose that π is distinguished by $\mathrm{SL}_2(F)$. Then $\mathrm{lg}_+(\pi)$ is different from 1, and the only irreducible component of $\pi|_{\mathrm{GL}_2^+(E)}$ which is distinguished by $\mathrm{SL}_2(F)$ is the one that is ψ_0-generic (see the beginning of Section 6.5 for the definition of ψ_0).*

Proof. The proof of Proposition 4.4 of [AP03] can be applied, and we write the proof here for completeness. Suppose that $\mathrm{lg}_+(\pi) = 1$ and denote $\pi|_{\mathrm{GL}_2^+(E)}$ by π^+. Under this assumption, we have that $\pi^+|_{\mathrm{SL}_2(E)}$ is not irreducible. By Lemma 6.10 we have that the number of $\mathrm{SL}_2(F)$-invariant linear forms is strictly bigger than one. However by Corollary 6.12, there is only one $\mathrm{SL}_2(F)$-invariant linear form, which is a contradiction. Hence $\mathrm{lg}_+(\pi) \neq 1$, and the result follows from Lemma 6.9 and Lemma 6.10. $\qquad\square$

Theorem 6.14. [AP03, Thm 1.1] *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$ distinguished by $\mathrm{SL}_2(F)$, and π^+ the unique irreducible component of $\pi|_{\mathrm{GL}_2(E)^+}$ that is ψ_0-generic. Then π^+ is distinguished by $\mathrm{SL}_2(F)$. Furthermore, let τ be an irreducible component of $\pi|_{\mathrm{SL}_2(E)}$, distinguished by $\mathrm{SL}_2(F)$. Then τ is an irreducible component of $\pi^+|_{\mathrm{SL}_2(E)}$.*

Proof. It follows from Lemma 6.9 and Lemma 6.10 directly. $\qquad\square$

Theorem 6.15. *Let π be an irreducible supercuspidal $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_2(E)$ and τ an irreducible component of $\pi|_{\mathrm{SL}_2(E)}$. Suppose τ is distinguished by $\mathrm{SL}_2(F)$. Then,*

$$\mathrm{dimHom}_{\mathrm{SL}_2(F)}(\tau, 1) = \begin{cases} 1, & \text{if } \pi|_{\mathrm{SL}_2(E)} \cong \tau; \\ 1, & \text{if } \mathrm{lg}_+(\pi) = 2 \text{ and } \mathrm{lg}(\pi) = 4; \\ 2, & \text{if } \mathrm{lg}_+(\pi) = \mathrm{lg}(\pi) = 2; \\ 4, & \text{if } \mathrm{lg}_+(\pi) = \mathrm{lg}(\pi) = 4. \end{cases}$$

The first case and the last case arises only when $p = 2$.

Proof. Let $\pi^+ \subset \pi|_{\mathrm{GL}_2^+(E)}$ be irreducible and ψ_0-generic. Let $\mathrm{lg}(\pi)'$ be the length of $\pi^+|_{\mathrm{SL}_2(E)}$. Since the components of $\pi|_{\mathrm{SL}_2(E)}$ are $\mathrm{GL}_2(E)$-conjugate, we deduce that $\mathrm{lg}(\pi)' = \mathrm{lg}(\pi)/\mathrm{lg}_+(\pi)$. By Lemma 6.10, Corollary 6.12 and Theorem 6.14, we summarize that $\mathrm{dimHom}_{\mathrm{SL}_2(F)}(\pi, 1) = \mathrm{lg}_+(\pi)$, and

$$\mathrm{dimHom}_{\mathrm{SL}_2(F)}(\pi', 1) = \mathrm{lg}_+(\pi)/\mathrm{lg}(\pi)' = \mathrm{lg}_+(\pi)^2/\mathrm{lg}(\pi).$$

We obtain the result by a direct computation.

When p is odd, $E^\times/F^\times E^{\times 2}$ is of order 2. Since $\mathrm{lg}_+(\pi) = |Y_+(\pi)|$ and the latter is a subset of $\overline{\mathbf{F}}_\ell$-characters of $E^\times/F^\times E^{\times 2}$, the case $\mathrm{lg}_+(\pi) = 4$ can exist only when p is equal to 2.

If $\pi|_{\mathrm{SL}_2(E)}$ is irreducible which implies that π is primitive, then $p = 2$. $\qquad\square$

6.6. **The principal series representations.** In this subsection, we will use Mackey Theory to prove the following theorem, following [Lu18].

Theorem 6.16. *Let $I(\chi)$ be a principal series representation of $\mathrm{SL}_2(E)$.*

(1) *Let τ be an irreducible principal series representation of $\mathrm{SL}_2(E)$.*
  (a) *If $\tau = I(\chi)$ with $\chi|_{F^\times} = \mathbf{1}$ and $\chi \neq \mathbf{1}$, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 1$.*
  (b) *If $\tau = I(\chi)$ with $\chi^\sigma = \chi$, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 2$.*

(2) *Let τ be an irreducible subrepresentation of $I(\chi)$ distinguished by $\mathrm{SL}_2(F)$ with $q_F \not\equiv 1 \pmod{\ell}$.*
  (a) *If $\chi = \nu^{\pm 1}$ and $\ell \nmid q_E^2 - 1$, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 1$.*
  (b) *If $\chi^2 = \mathbf{1}, \chi = \chi_F \circ \mathrm{Nm}_{E/F} \neq \mathbf{1}$ and $\ell \nmid q_E^2 - 1$, then*

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = \begin{cases} 3, & \text{if } \chi_F^2 = \mathbf{1}; \\ 1, & \text{if } \chi_F^2 = \omega_{E/F}. \end{cases}$$

  (c) *If $\ell \mid q_E + 1$ and $\chi = \nu$, then τ is the trivial character and $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 1$. In this case, there are two cuspidal (not supercuspidal) representations inside the Jordan-Holder series of $I(\nu)$ which are not distinguished by $\mathrm{SL}_2(F)$ if E/F is unramified. If E/F is ramified, then only one of two cuspidal representations is distinguished by $\mathrm{SL}_2(F)$ with multiplicity two.*

  (d) *If $\ell \mid q_E - 1$ and $\ell \mid q_F + 1$, then*

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = \begin{cases} 2, & \text{if } \tau \text{ is the Steinberg representation;} \\ 3, & \text{if } \chi = \chi_F \circ \mathrm{Nm}_{E/F} \neq \mathbf{1} \text{ with } \chi_F^2 = \mathbf{1}; \\ 1, & \text{if } \chi = \chi_F \circ \mathrm{Nm}_{E/F} \text{ with } \chi_F^2 = \omega_{E/F}. \end{cases}$$

Let $B(E)$ be the standard Borel subgroup of $\mathrm{SL}_2(E)$ and $B(E) \backslash \mathrm{SL}_2(E) \cong P^1(E)$. Recall that there are two F-rational $\mathrm{GL}_2(F)$-orbits in $P^1(E)$ which are $P^1(F)$ and $P^1(E) - P^1(F)$. Moreover, the open orbit $P^1(E) - P^1(F)$ is isomorphic to $E^\times \backslash \mathrm{GL}_2(F)$. There is an exact sequence

$$1 \to E^1 \backslash \mathrm{SL}_2(F) \to E^\times \backslash \mathrm{GL}_2(F) \to N_{E/F} E^\times \backslash F^\times \to 1$$

where $E^1 = \{e \in E^\times : N_{E/F}(e) = 1\}$. Thus $P^1(E)$ decomposes into 3 F-rational $\mathrm{SL}_2(F)$-orbits: one closed orbit $P^1(F)$ and two open orbits corresponding to $F^\times / \mathrm{Nm}_{E/F} E^\times = \{\pm 1\}$.

Lemma 6.17. *Let $I(\chi)$ be a principal series representation of $\mathrm{SL}_2(E)$. Then $\mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\chi), \mathbf{1}) \neq 0$ if and only if either $\chi|_{F^\times} = \mathbf{1}$ or $\chi|_{E^1} = \mathbf{1}$.*

Proof. Applying Mackey Theory, one has an exact sequence

$$0 \to \mathrm{Hom}_{F^\times}(\chi, \mathbf{1}) \to \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\chi), \mathbf{1}) \to \mathrm{Hom}_{E^1}(\chi, \mathbf{1}) \oplus \mathrm{Hom}_{E^1}(\chi^{-1}, \mathbf{1}) \to \mathrm{Ext}^1_{F^\times}(\chi, \mathbf{1})$$

of $\overline{\mathbf{F}}_\ell$-vector spaces. If $\mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\chi), \mathbf{1})$ is nonzero, then either $\chi|_{F^\times}$ or $\chi|_{E^1}$ is trivial. Conversely, it suffices to show that $\chi|_{F^\times} \neq \mathbf{1}$ and $\chi|_{E^1} = \mathbf{1}$ imply $\mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\chi), \mathbf{1}) \neq 0$. Note that $\mathrm{Ext}^1_{F^\times}(\chi, \mathbf{1}) = 0$ if and only if $\chi|_{F^\times}$ is nontrivial; see [DS21, Prop. 8.4]. By the exact sequence, one have

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\chi), \mathbf{1}) = 2 \dim \mathrm{Hom}_{E^1}(\chi, \mathbf{1})$$

when $\chi|_{F^\times} \neq \mathbf{1}$. This finishes the proof. $\qquad\qquad\square$

Lemma 6.18. *Suppose that $\ell \nmid q_F - 1$. Assume that π (resp. τ) is a principal series representation of $\mathrm{GL}_2(E)$ distinguished by $\mathrm{GL}_2(F)$ (resp. $\mathrm{SL}_2(F)$) and $\tau \subset \pi|_{\mathrm{SL}_2(E)}$. Set $\pi|_{\mathrm{GL}_2^+(E)} = \oplus_i \pi_i$. Then $\tau \subset \pi_1|_{\mathrm{SL}_2(E)}$ if and only if π_1 is ψ_0-generic where ψ_0 is a non-degenerate character on $N(E)/N(F)$.*

Proof. Suppose that $\pi = \pi(\chi_1, \chi_2)$ with $\chi_1 \chi_2$ trivial on $F^\times$. Then the $\mathrm{GL}_2(F)$-invariant linear functional on π, up to a constant, is given by

$$l(W) = \int_{F^\times} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) d^\times a$$

where W is the unique right $\mathrm{GL}_2(\mathcal{O}_E)$-invariant ψ_0-Whittaker function on π. Since there is a unique ψ_0-Whittaker functional on the space of π, exactly one constituent of the restriction of π to $\mathrm{GL}_2^+(E)$ is ψ_0-generic. Thus the $\mathrm{GL}_2(F)$-invariant functional can be non-zero only on the ψ_0-generic part of the restriction of π to $\mathrm{GL}_2^+(E)$. Therefore $\tau \subset \pi_1|_{\mathrm{SL}_2(E)}$ i.e., π_1 is $\mathrm{SL}_2(F)$-distinguished if and only if π_1 is ψ_0-generic. $\square$

Remark 6.19. Let ϖ_F denote the uniformizer of the ring $\mathfrak{o}_F$ of integers of F. Nadir Matringe pointed out that when $q_F \equiv 1 \pmod{\ell}$, π is distinguished by $\mathrm{GL}_2(F)$ but

$$\int_{F^\times} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) da = \frac{Vol(\mathcal{O}_F^\times)}{(1 - \chi_1(\varpi_F))(1 - \chi_2(\varpi_F))} = 0.$$

Thus, this method does not work when we try to determine the multiplicity for the irreducible constituent of $I(\chi)$ with $\chi^2 = \mathbf{1}$ when $q_F \equiv 1 \pmod{\ell}$.

Now we are ready to prove Theorem 6.16.

Proof of Theorem 6.16. (1) Note that if $\chi \neq \nu^{\pm 1}$ and $\chi^2 \neq \mathbf{1}$, then $I(\chi)$ is irreducible. Then it follows from Lemma 6.17 except that $\chi = \mathbf{1}$ and $\ell \nmid q_E - 1$. It is enough to show that $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\mathbf{1}), \mathbf{1}) = 2$ when $\ell \nmid q_E - 1$. Note that $\pi(\mathbf{1}, \mathbf{1})$ is both $\mathrm{GL}_2(F)$-distinguished and $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished with multiplicity one. Thus $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\mathbf{1}), \mathbf{1}) = 2$ by Proposition 6.8.

(2) If τ is a trivial character, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 1$. Let τ be an infinite-dimensional subrepresentation of $I(\chi)$.

(a) If $\chi = \nu^{-1}$, then τ is the trivial character. If $\chi = \nu$ and τ is the Steinberg representation St, then there exists a short exact sequence

$$0 \to \mathrm{St} \to I(\nu) \to \mathbf{1} \to 0$$

of $\mathrm{SL}_2(E)$-representations. Taking the functor $\mathrm{Hom}_{\mathrm{SL}_2(F)}(-, \mathbf{1})$, one has an exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathbf{1}, \mathbf{1}) \to \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\nu), \mathbf{1}) \to \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{St}, \mathbf{1}) \to \mathrm{Ext}^1_{\mathrm{SL}_2(F)}(\mathbf{1}, \mathbf{1}).$$

Note that $\mathrm{Ext}^1_{\mathrm{SL}_2(F)}(\mathbf{1}, \mathbf{1}) = 0$. Thanks to Lemma 6.17, we have

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\nu), \mathbf{1}) = 2.$$

Therefore, $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{St}, \mathbf{1}) = 2 - 1 = 1$.

(b) If $\chi = \chi_F \circ \mathrm{Nm}_{E/F}$ with $\chi_F^2 = \mathbf{1}$, then the irreducible principal series representation $\pi(\mathbf{1}, \chi)$ is distinguished by $\mathrm{GL}_2(F)$. Furthermore,

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\mathbf{1}, \chi), \chi_F) = 1 = \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi(\mathbf{1}, \chi), \chi_F \omega_{E/F}).$$

There is only one subrepresentation in $I(\chi)$ which is (N, ψ_0)-generic, which implies that only one constituent in $I(\chi)$ is distinguished by $\mathrm{SL}_2(F)$. Thus $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 3$.

If $\chi = \chi_F \circ \mathrm{Nm}_{E/F}$ with $\chi_F^2 = \omega_{E/F}$, then $\pi(\mathbf{1}, \chi)$ is both $(\mathrm{GL}_2(F), \chi_F)$-distinguished and $(\mathrm{GL}_2(F), \chi_F^{-1})$-distinguished. Thus

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(\chi), \mathbf{1}) = 2.$$

Note that two constituents in $I(\chi)$ are (N, ψ_0)-generic. Thus each one in $I(\chi)$ is distinguished by $\mathrm{SL}_2(F)$ with multiplicity one.

(c) This case is trivial.

If $\ell \mid q_E + 1$ and τ is a cuspidal nonsupercuspidal representation of $\mathrm{SL}_2(E)$, then there is a special representation Sp of $\mathrm{GL}_2(E)$ such that $\tau \subset \mathrm{Sp}\vert_{\mathrm{SL}_2(E)}$. If E/F is unramified, then Sp is neither $\mathrm{GL}_2(F)$-distinguished nor $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished. (See [SÍ9, Rem. 2.8].) Thus $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{Sp}\vert_{\mathrm{SL}_2(E)}, \mathbf{1}) = 0$ by Proposition 6.8. If E/F is ramified, then Sp is not $\mathrm{GL}_2(F)$-distinguished but $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished. It is also $(\mathrm{GL}_2(F), \nu\vert_{F^\times}^{1/2})$-distinguished; see Theorem 4.6(2). Due to Proposition 6.8, $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{Sp}\vert_{\mathrm{SL}_2(E)}, \mathbf{1}) = 2$. Note that there is only one constituent in $\mathrm{Sp}\vert_{\mathrm{SL}_2(E)}$ which is (N, ψ_0)-generic, denoted by τ. Thus

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbf{1}) = 2.$$

This finishes the proof.

(d) If $\ell \mid q_E - 1$ and $\ell \mid q_F + 1$, then E/F is an unramified field extension. In this case, $I(\mathbf{1}) = \mathbf{1} \oplus \mathrm{St}$. Thanks to Theorem 4.6(3),

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{St}, \mathbf{1}) = 2.$$

If $\tau \subset I(\chi_F \circ \mathrm{Nm}_{E/F})$ with $\chi_F^2 = \omega_{E/F}$ or $\mathbf{1}$, the case is similar to case (b).

$\square$

Remark 6.20. If $\ell \mid q_F - 1$, one can still prove that St is distinguished by $\mathrm{SL}_2(F)$ with multiplicity two due to Theorem 4.6 and that the constituents in $I(\chi_F \circ \mathrm{Nm}_{E/F})$ with $\chi_F^2 = \omega_{E/F}$ are both distinguished by $\mathrm{SL}_2(F)$ with multiplicity one. But we can not determine the situation for $\tau \subset I(\chi_F \circ \mathrm{Nm}_{E/F})$ with $\chi_F^2 = \mathbf{1}$.

REFERENCES

[AP03] UK Anandavardhanan and Dipendra Prasad. Distinguished representations for SL(2). *Math. Res. Lett.*, 10(6):867–878, 2003. 2, 5, 24, 27, 28, 29, 30

[Bou04] Nicolas Bourbaki. *Integration. II. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Translated from the 1963 and 1969 French originals by Sterling K. Berberian. 15

[Cui20] Peiyi Cui. Modulo ℓ-representations of p-adic groups $\mathrm{SL}_n(F)$: maximal simple k-types. *arXiv e-prints*, page arXiv:2012.07492, December 2020. 24, 25

[DS21] Bastien Drevon and Vincent Sécherre. Block decomposition of the category of l-modular smooth representations of finite length of GL(m,D). *accepted by Ann. Inst. Fourier, arXiv*, page arXiv:2102.05898, February 2021. 8, 14, 31

[GGP12] W. T. Gan, B. H. Gross, and D. Prasad. Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. *Astérisque*, (346):1–109, 2012. Sur les conjectures de Gross et Prasad. I. 11

[GK82] S. S. Gelbart and A. W. Knapp. L-indistinguishability and R groups for the special linear group. *Adv. in Math.*, 43(2):101–121, 1982. 25

[HM02] Jeffrey Hakim and Fiona Murnaghan. Globalization of distinguished supercuspidal representations of $GL(n)$. *Canad. Math. Bull.*, 45(2):220–230, 2002. 13

[HR90] Michael J. Heumos and Stephen Rallis. Symplectic-Whittaker models for Gl_n. *Pacific J. Math.*, 146(2):247–279, 1990. 9

[Kab04] Anthony C. Kable. Asai L-functions and Jacquet's conjecture. *Amer. J. Math.*, 126(4):789–820, 2004. 3, 12, 13

[KM20] Robert Kurinczuk and Nadir Matringe. Characterisation of the poles of the ℓ-modular Asai L-factor. *Bull. Soc. Math. France*, 148(3):481–514, 2020. 11, 13, 29

[KM21] Robert Kurinczuk and Nadir Matringe. The ℓ-modular local langlands correspondence and local constants. *J. Inst. Math. Jussieu*, 20(5):1585–1635, 2021. 5, 18, 19, 20, 22

[KMS] Robert Kurinczuk, Nadir Matringe, and Vincent Sécherre. Galois distinguished ℓ-modular representations of p-adic GL_n. *in preparation.* 2

[Lu18] Hengfei Lu. Theta correspondence and the Prasad conjecture for $SL(2)$. *Pacific J. Math.*, 295(2):477–498, 2018. 2, 31

[Lu20] Hengfei Lu. The Prasad conjectures for GSp_4 and $PGSp_4$. *Algebra Number Theory*, 14(9):2417–2480, 2020. 2

[MS14] Alberto Mínguez and Vincent Sécherre. Types modulo ℓ pour les formes intérieures de GL_n sur un corps local non archimédien. *Proc. Lond. Math. Soc. (3)*, 109(4):823–891, 2014. With an appendix by Vincent Sécherre et Shaun Stevens. 12

[Pra15] Dipendra Prasad. A 'relative' local Langlands correspondence. *arXiv preprint arXiv:1512.04347*, 2015. 2, 4, 18

[SÍ9] Vincent Sécherre. Supercuspidal representations of $GL_n(F)$ distinguished by a Galois involution. *Algebra Number Theory*, 13(7):1677–1733, 2019. 2, 3, 4, 7, 10, 11, 14, 17, 33

[She79] D. Shelstad. Notes on L-indistinguishability (based on a lecture of R. P. Langlands). In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 193–203. Amer. Math. Soc., Providence, R.I., 1979. 26

[SV17a] Yiannis Sakellaridis and Akshay Venkatesh. Periods and harmonic analysis on spherical varieties. *Astérisque*, (396):viii+360, 2017. 2

[SV17b] Vincent Sécherre and Coolimuttam G. Venketasubramanian. Modular representations of $GL(n)$ distinguished by $GL(n-1)$ over a p-adic field. *Int. Math. Res. Not. IMRN*, (14):4435–4492, 2017. 2

[Vig96] Marie-France Vignéras. *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$*, volume 137 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1996. 2, 8

[Vig01] Marie-France Vignéras. Correspondance de Langlands semi-simple pour $GL(n, F)$ modulo $\ell \neq p$. *Invent. Math.*, 144(1):177–223, 2001. 4, 9, 10

[Zou21] Jiandi Zou. Supercuspidal representations of $GL_n(F)$ distinguished by a unitary involution. *Bull. Soc. Math. Fr.*, https://doi.org/10.24033/bsmf.2848, 2021. 3, 13, 14

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