The Infrared Triangle

Proseminar

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Contents

Introduction

Although gauge theories are a very constrained kind of field theories, they still allow to describe almost all the physics of our universe, including the three interactions of the Standard Model (with gauge group $U(1) \times SU(2) \times SU(3)$) and General Relativity (with changes of coordinates as gauge group). Despite their very different manifestations at our scales, they share a similar structure that is still not fully understood.

In this review, we will present our understanding of the behaviour of gauge theories on the boundary of space-time. Because of gauge invariance, they behave as long-range interactions that do not vanish at infinity, and therefore exhibit additional degrees of freedom at infinity. These degrees of freedom have very different manifestations in different domains, that are unexpectedly linked in non-trivial ways. These phenomena can be schematically represented in the so-called "Infrared Triangle" 2, as well as their relations. The purpose of this report is to give a taste of every aspect of this triangle. The reader looking for deeper details is strongly advised to refer to this book [4] on which this report is based.

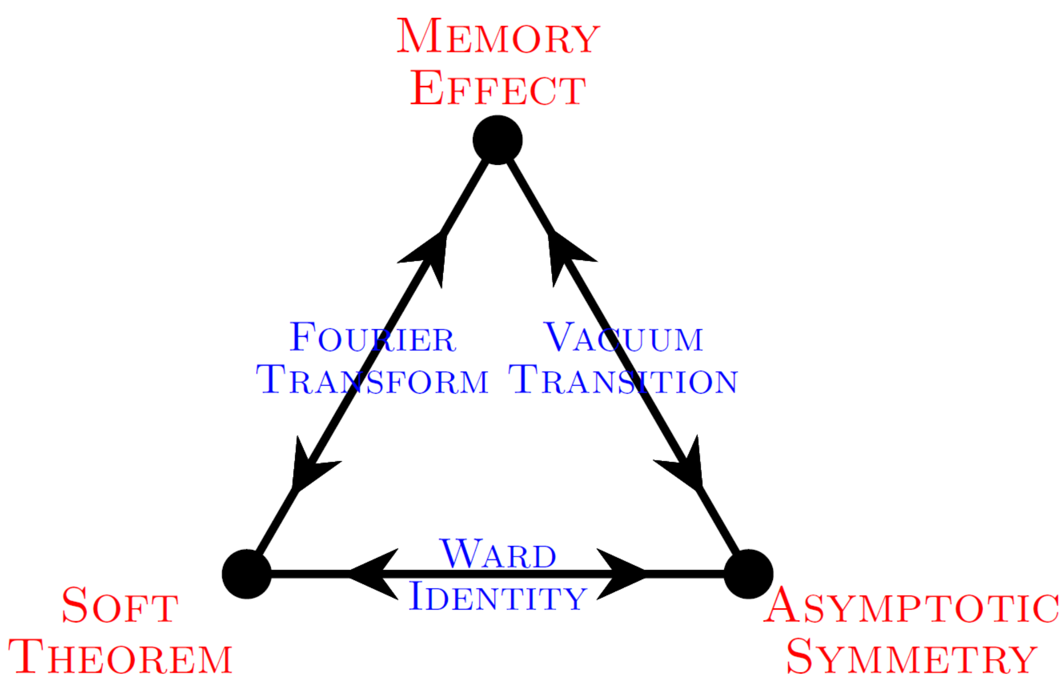


Figure 2: The Infrared Triangle. The vertices are different manifestations of the same phenomenon, and the sides are the physical relations between these phenomena.

Although all the following can be discussed for any gauge theory, we will mainly focus on electrodynamics for which all the technical calculations are simpler. We will then briefly show how the same concepts can be applied to gravity, and provide references concerning the generalizations to more general non abelian gauge theories.

1 Gauge theories

In the Lagrangian formalism, electrodynamics can be described as follows:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_M - j^\mu A_\mu$$

$\mathcal{L}_M$ is a generic term which describes the behaviour of matter, and j^μ is an electromagnetic four-current. A_μ is the four-potential (or photon field at the quantum level), and $F_{\mu\nu}$ is the electromagnetic tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

For this theory to be a "gauge theory", we require gauge invariance, that is, the invariance of the Lagrangian under a gauge transformation (for a generic scalar field α):

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

This must not be understood as a symmetry of our theory, but simply as a redundancy of the description (no additional degrees of freedom are involved in such transformation). Only the global gauge invariance (with uniform α) is a symmetry, which affects only the matter components of the Lagrangian, and is associated through Noether's theorem [1] to a four-current j^μ.

However, one surprising aspect of electrodynamics (and other gauge theories) is that the interaction Hamiltonian does not vanish at infinity [2]. This can be understood from the Gauss equation, which states that the electric flux over an arbitrary large closed surface will not vanish. This is a clue that some additional degrees of freedom could exist on the boundary of space-time, and therefore could induce new physical symmetries.

With that idea in mind, let's start with the usual electric charge Q expressed as the flux of the electric field on the boundary, and see to what extents it can be modified. We need to compute the radial electric field at infinity, which can be done with the Liénard-Wiechert solution [6]. Let's write it in the simple case of a set of a punctual charges in uniform motion:

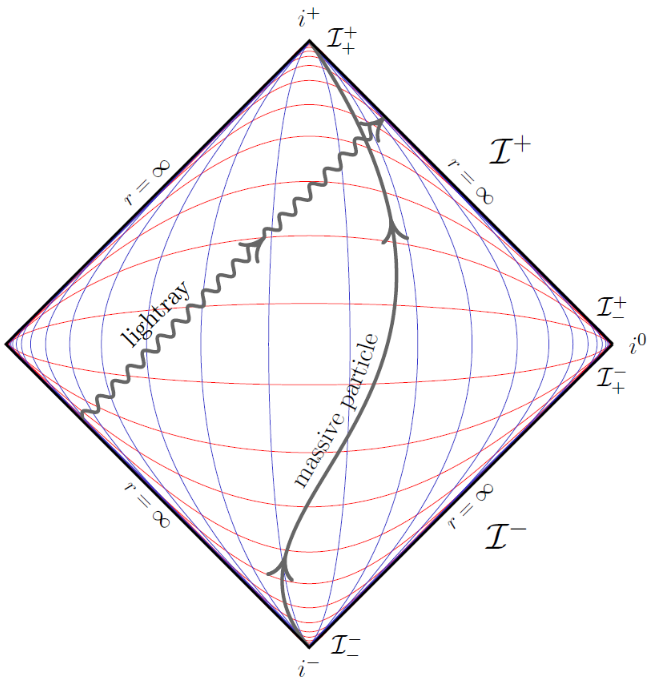
$$E_r(\vec{r}, t) = \frac{e^2}{4\pi} \sum_s \frac{Q_s \gamma_s (r - t\vec{n} \cdot \vec{\beta}_s)}{|\gamma_s^2 (t - r\vec{n} \cdot \vec{\beta}_s)^2 - t^2 + r^2|^{3/2}}$$

where the s index labels the sources, $\vec{r} = r\vec{n}$, γ_s and β_s are the Lorentz factor and velocity of each source, and Q_s is the charge of the source.

We can then try to evaluate this expression at spatial infinity $r \to \infty$, in order to compute the flux, but unfortunately, it turns out that this limit is ill-defined. More precisely, the limit is not always the same depending on how t varies with r in the limit. To understand this phenomenon and patch the limit, we then need to analyse in more details the structure of the boundary of space-time.

2 Structure of the boundary of space-time

A useful picture to represent space-time and its boundary is a Penrose diagram (represented and explained in Figure 3). In particular, it highlights the singular points in the boundary, especially i^0, which is the point where all the limits $r \to \infty$ with $t = o(r)$ end. Since this point is singular, it explains why the limit is ill-defined.



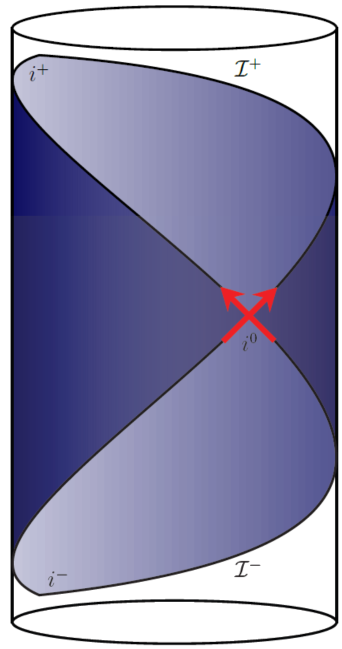


Figure 3: **Left:** Penrose diagram. Diagonal lines (not represented on the diagram) are lines of constant $u = t - r$ or constant $v = t + r$, while the red lines represented on the diagram are the lines of constant r (for the "horizontal" ones) or constant t (for the "vertical" ones). **Right:** Schematical representation of the antipodal matching at i^0.

However, taking the limit $r \to \infty$ at u fixed or v fixed, one would expect to have a well-defined result, eventually depending on u or v. Actually, when we perform this limit for the radial electric field, we even find a u-independent (respectively v-independent) result:

$$E_r(r \to \infty, \vec{n}, u) = \frac{e^2}{4\pi r^2} \sum_s \frac{Q_s}{\gamma_s^2 (1 - \vec{n} \cdot \vec{\beta}_s)^2}$$

$$E_r(r \to \infty, \vec{n}, v) = \frac{e^2}{4\pi r^2} \sum_s \frac{Q_s}{\gamma_s^2 (1 + \vec{n} \cdot \vec{\beta}_s)^2}$$

Furthermore, the only difference between these two expressions is $\vec{n} \longleftrightarrow -\vec{n}$. This leads us toward the solution to patch the singularity at i^0. If we actually perform the matching at i^0 not with $\vec{n}$ fixed, but antipodally (that is, changing $\vec{n}$ to $-\vec{n}$ when going from $\mathcal{I}^-$ to $\mathcal{I}^0$), then the expression for $E_r(r \to \infty)$ becomes continuous on the whole boundary, and independent on t (or equivalently on u or v). The antipodal matching is schematically represented on Figure 3.

Thus, in all the following, antipodal matching will be assumed in all expressions, that is:

$$E_r^{(2)}(\vec{n})|_{\mathcal{I}^+} = \lim_{r \to \infty} r^2 E_r(r, \vec{n}, u)$$

$$E_r^{(2)}(\vec{n})|_{\mathcal{I}^-} = \lim_{r \to \infty} r^2 E_r(r, -\vec{n}, v)$$

So that $E_r^{(2)}(\vec{n})|_{\mathcal{I}^+} = E_r^{(2)}(\vec{n})|_{\mathcal{I}^-}$

3 Asymptotic symmetries

Now with a well-defined limit for the radial electric field, we can compute the electric charge as the flux on the boundary of space, which gives:

$$Q = \int d\Omega \, E_r^{(2)}(\vec{n})$$

Since the result does not depend on t, the charge is conserved as expected. However, one can see that we have the freedom here to define a more general charge while keeping this time independence, thus creating an infinite number of conserved charges parameterized by a function ε on the sphere:

$$Q_\varepsilon = \int d\Omega \, \varepsilon(\vec{n}) E_r^{(2)}(\vec{n})$$

By Noether's theorem, we can expect an infinite number of physical symmetries associated to these charges, which should affect degrees of freedom on the boundary of space-time. In order to figure out what this symmetry is, we can compute the Poisson bracket of these charges with the four-potential. This needs first to transform a little bit the expression thanks to the equations of motion $\partial^\mu F_{\mu\nu} = j^\nu$ and integration by parts on $\mathcal{I}^+$:

$$Q_\varepsilon|_{\mathcal{I}^+_-} = \int_{\mathcal{I}^+} du \, d^2\Omega \, \left[\varepsilon(\vec{n}) j_u(u, \vec{n}) - D^\Omega \varepsilon(\vec{n}) E_\Omega^{(0)}(u, \vec{n}) \right]$$

E_Ω is the spherical component of E and D^Ω is the covariant derivative on the sphere. We see two contributions to this charge: the so-called "hard" charge, proportional to the four-current, is similar to the usual charge (actually the same is we take $\varepsilon(\vec{n}) = 1$). The second contribution, called "soft" charge, involves the spherical components of the electric field on the boundary, and vanishes in the usual case $\varepsilon(\vec{n}) = 1$.

Thanks to this formula, we can then compute the action of the associated symmetry on the four-potential, and find:

$$\{Q_\varepsilon, A_\Omega(u, \vec{n})\} = D_\Omega \varepsilon(\vec{n})$$
$$\{Q_\varepsilon, A_r(u, \vec{n})\} = 0$$
$$\{Q_\varepsilon, A_u(u, \vec{n})\} = 0$$

We can thus interpret this symmetry as a "large gauge" symmetry, with a pure spherical gauge ε. While we usually consider that a pure gauge transformation has no physical effect, we see in this case that it actually affects the boundary degrees of freedom of our theory. This is called an asymptotic symmetry.

4 Soft theorems

After discovering the existence of asymptotic symmetries, it is natural to wonder how they act at the quantum level. Since A_μ is the photon field, we expect somehow an impact on the behaviour of photons. To highlight this effect, let's impose the asymptotic symmetries on a scattering process, writing the so-called Ward identity:

$$\langle out | \, [Q_\varepsilon, \mathcal{S}] \, | in \rangle = 0$$

$|in\rangle$ and $|out\rangle$ are asymptotic states of finite number of particles with well-defined momenta, and $\mathcal{S}$ is the S-matrix which describes the evolution between in and out states.

Now, in order to figure out how the charge acts on $|in\rangle$ and $|out\rangle$ states, we will use the decomposition $Q_\varepsilon = Q_\varepsilon^{\text{hard}} + Q_\varepsilon^{\text{soft}}$ introduced in previous section. As mentioned above, the hard charge is the exact equivalent of the usual charge, simply with an angular weight. Therefore, we can write as usual:

$$\langle out|\,[Q_\varepsilon^{\text{hard}}, \mathcal{S}]\,|in\rangle = -\left(\sum_{j=0}^{N} \eta_j\, Q_j\, \varepsilon(\vec{n}_j)\right) \langle out|\,\mathcal{S}\,|in\rangle$$

where j labels the particles in $|in\rangle$ and $|out\rangle$ states, $\eta_j = -1$ for incoming particles and $\eta_j = +1$ for outgoing particles, and $\vec{n}_j$ is the position of the asymptotic particle on the celestial sphere.

Concerning the soft charge, we will rewrite it as follows:

$$Q_\varepsilon^{\text{soft}} = \int d^2\Omega D^\Omega \varepsilon(\vec{n}) \int du\, E_\Omega(u, \vec{n})$$

In that form, we figure out that the integral over u actually selects the $\omega = 0$ mode of E_Ω in the Fourier space (one can add a factor e^{-i0u} in the integral). To manage the angular part and simplify the integral over the celestial sphere, we will choose a specific basis of ε (that actually generate the whole space of ε functions):

$$\varepsilon(\vec{n}) = \frac{n \cdot \epsilon}{n \cdot k}$$

n is the four-vector $(1, \vec{n})$, k is a unit four-vector and ϵ is a four-vector orthogonal to k. As we have seen, this integral deals with the $\omega = 0$ mode of the photon field, that is the "soft" photon, and we will see just after that ϵ can be interpreted as the polarization of the soft photon and k as its direction of propagation, that we can thus rewrite as $\lim_{\omega \to 0} \dfrac{q}{\omega}$ with q the four-momentum of the soft photon.

With all these considerations, we can finally rewrite the hard and soft contributions as:

$$\langle out|\,[Q_\varepsilon^{\text{hard}}, \mathcal{S}]\,|in\rangle = -\lim_{\omega \to 0} \left(\sum_{j=0}^{N} \eta_j\, Q_j\, \omega \frac{p_j \cdot \epsilon}{p_j \cdot q}\right) \langle out|\,\mathcal{S}\,|in\rangle$$

$$\langle out|\,[Q_\varepsilon^{\text{soft}}, \mathcal{S}]\,|in\rangle = \lim_{\omega \to 0} \Big(\omega\, \langle out|\, a_{out}^\dagger(q, \epsilon) \mathcal{S}\,|in\rangle\Big)$$

Using the Ward identity and dividing by ω, we thus derive the Soft Theorem (to be understood in the limit $\omega \to 0$:

$$\langle out|\, a_{out}^\dagger(q, \epsilon) \mathcal{S}\,|in\rangle = \left(\sum_{j=0}^{N} \eta_j\, Q_j\, \frac{p_j \cdot \epsilon}{p_j \cdot q}\right) \langle out|\,\mathcal{S}\,|in\rangle + \mathcal{O}(\omega)$$

This tells us that the amplitude of emitting an additional soft photon with given momentum and polarization in a scattering process is controlled by the soft factor $\sum_{j=0}^{N} \eta_j \, Q_j \, \frac{p_j \cdot \epsilon}{p_j \cdot q}$. Therefore, at the quantum level, the degrees of freedom living on the boundary of space-time that we identified in the previous sections are actually the soft modes of the gauge field, and their behaviour is well constrained by the asymptotic symmetries, a manifestation of which is the Soft Theorems.

5 Memory effect

Finally, once we figured out what the additional boundary degrees of freedom are thanks to the quantum treatment, we can go back to the classical level to determine their impact on a classical evolution. Let's consider again the same scattering process as in the previous section, and take the classical limit for the soft modes of the gauge field (we multiply by ω in order to have finite quantities, expressed in terms of the direction of propagation k):

$$\lim_{\omega \to 0} \left(\omega \, A_\mu^\perp(\omega, \vec{k}) \right) = \lim_{\omega \to 0} \frac{\langle out | \, a^\dagger_{out,\mu}(q) \mathcal{S} \, | in \rangle}{\langle out | \, \mathcal{S} \, | in \rangle} = \left(\sum_{j=0}^{N} \eta_j \, Q_j \, \frac{p_{j,\mu}}{p_j \cdot k} \right)^\perp$$

On the other hand, let's write the Fourier transform $A_\mu^\perp(\omega, k)$ in the stationary phase approximation [5]:

$$A_\mu^\perp(\omega, \vec{k}) = 4\pi i \lim_{r \to \infty} r \int du e^{i\omega u} A_\mu^\perp(u, r\vec{k})$$

which, after taking the time derivative, gives:

$$-i\omega A_\mu^\perp(\omega, \vec{k}) = 4\pi i \lim_{r \to \infty} r \int du e^{i\omega u} \partial_u A_\mu^\perp(u, r\vec{k})$$

Now taking finite large r_0 and $\omega \to 0$, we can compute the integral simply as a difference between late and early times, and reorder the terms:

$$\Delta A_\mu^\perp(\vec{k}) = -\frac{1}{4\pi r_0} \lim_{\omega \to 0} \left(\omega A_\mu^\perp(\omega, \vec{k}) \right)$$

$\Delta A_\mu^\perp(\vec{k})$ is the shift of the potential between late and early times, that is the shift due to the scattering process that occurred. Combining this equation with the one derived from the soft theorems, we thus get:

$$\Delta A_\mu^\perp(\vec{k}) = -\frac{1}{4\pi r_0} \left(\sum_{j=0}^{N} \eta_j \, Q_j \, \frac{p_{j,\mu}}{p_j \cdot k} \right)^\perp$$

This tells us that any scattering process happening in the bulk will cause a permanent shift of the transverse potential at the boundary, which is a measurable effect! Actually, one could show that this is a shift between different vacua inequivalent under asymptotic symmetries. We thus recover the physical effect of vacuum transitions, drawing a direct link between memory effect and asymptotic symmetries.

6 Gravity

As mentioned in the introduction, everything we have developed here for electrodynamics can be generalized for any gauge theory. In particular, the case of gravity is particularly interesting, as the gauge field (the perturbation of the metric) is directly observable. Indeed, the metric can be perturbed around the flat metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The gauge transformations are the change of coordinates, which in terms of the perturbation $h_{\mu\nu}$ can be written as:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

ξ_μ is a Killing field, which describes how space-time is locally parameterized (it plays a similar role as the gauge α in electrodynamics). Now, as we did before, we can derive here an asymptotic symmetry affecting the degrees of freedom on the boundaries, that is, in this case, a group which preserves the asymptotic flatness of the metric (the actual condition is more precise but we will stay generic in this discussion). This group is called the Bondi–Metzner–Sachs (BMS) group [3]. It contains obviously the Poincaré group, but but also much more general transformations such as *supertranslations* or *superrotations*. Let's take the particular case of supertranslations, they can be parameterized as follow:

$$\xi = f\partial_u - \frac{1}{r}\tilde{D}^\Omega f \partial_\Omega + D_\Omega^2 f \partial_r + \cdots$$

f is a function of the celestial sphere. Figure 4 represents the $2D2$ equivalent of supertranslations, for the first circular harmonics ($\sin(n\theta)$ and $\cos(n\theta)$ for $n \in \mathbb{N}$). We see in particular that usual spatial translations are contained in supertranslations, while time translations is just $f = 1$.

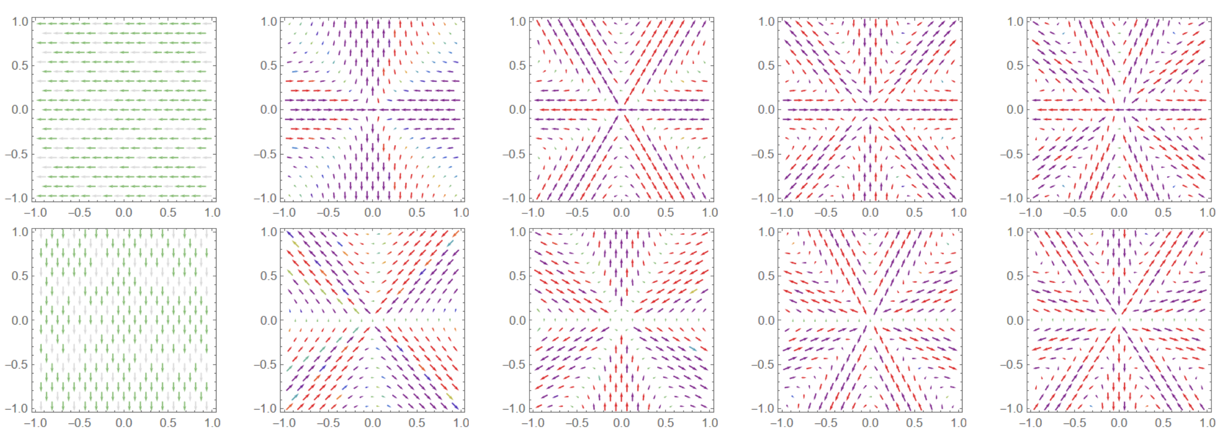


Figure 4: Supertranslations in 2 dimensions for the first circular harmonics. The arrows represent the spatial components of the Killing field, while the color represents the time component of the Killing field.

Now, at the quantum level, just as we did for electrodynamics, we can write a soft graviton theorem which takes the exact same form, but replacing the electric charge Q by the momentum p of the particles:

$$\langle out| \, h^{\dagger}_{out}(q,\epsilon)\mathcal{S}\,|in\rangle = \left(\sum_{j=1}^{N} \eta_j \frac{p_j^{\mu} p_j^{\nu} \epsilon_{\mu\nu}}{p_j \cdot q}\right) \langle out| \, \mathcal{S}\,|in\rangle + o(\omega)$$

And similarly, we can derive a gravitational memory effect:

$$\Delta h^{\perp}_{\mu\nu}(\vec{k}) = -\frac{1}{4\pi r_0}\left(\sum_{j=1}^{N} \eta_j \frac{p_{j\mu} p_{j\nu}}{p_j \cdot q}\right)^{\perp}$$

This means that, for instance, if a collision of massive objects takes place at some point in space, and we are able to observe the gravitational waves emitted during the collision with plates in free fall (such as the LISA project), then we should observe a permanent change in the distance between the plates, according to this formula. This would be the first observational confirmation of all this discussion!

Conclusion

To conclude, we saw that a general feature of gauge theories is that their Hamiltonian do not vanish at infinity, which leads to additional degrees of freedom on the boundary of space-time. A fine analysis of the structure of this boundary allows us to properly project the fields of our theory on the boundary and identify an asymptotic symmetry affecting these degrees of freedom. Although this symmetry takes a similar form as a pure gauge symmetry, it actually is a physical symmetry regarding the boundary, and is associated to an infinite number of conserved charges.

These conserved charges, as every Noether charge, have an impact on scattering processes at quantum level. The ward identity allows to determine this impact, and leads to the Soft Theorems, revealing that the boundary degrees of freedom are related to the soft modes of the gauge fields, and that their emission during scattering processes is very constrained due to asymptotic symmetries. In the end, taking the classical limit and a time Fourier transform in this analysis reveals that this soft modes behave as permanent shifts in the potential, known as Memory Effect. These shifts are transitions between inequivalent vacua related by asymptotic symmetries. This finally explains all the aspects of the infrared triangle 2.

This analysis has many applications in every gauge theory, and this is still a very active research field. Figure 5 represents some of these applications and the sides that are already revealed for each of them. As the reader can see, many holes remain, and much work still has to be done to fully understand the Infrared Triangle.

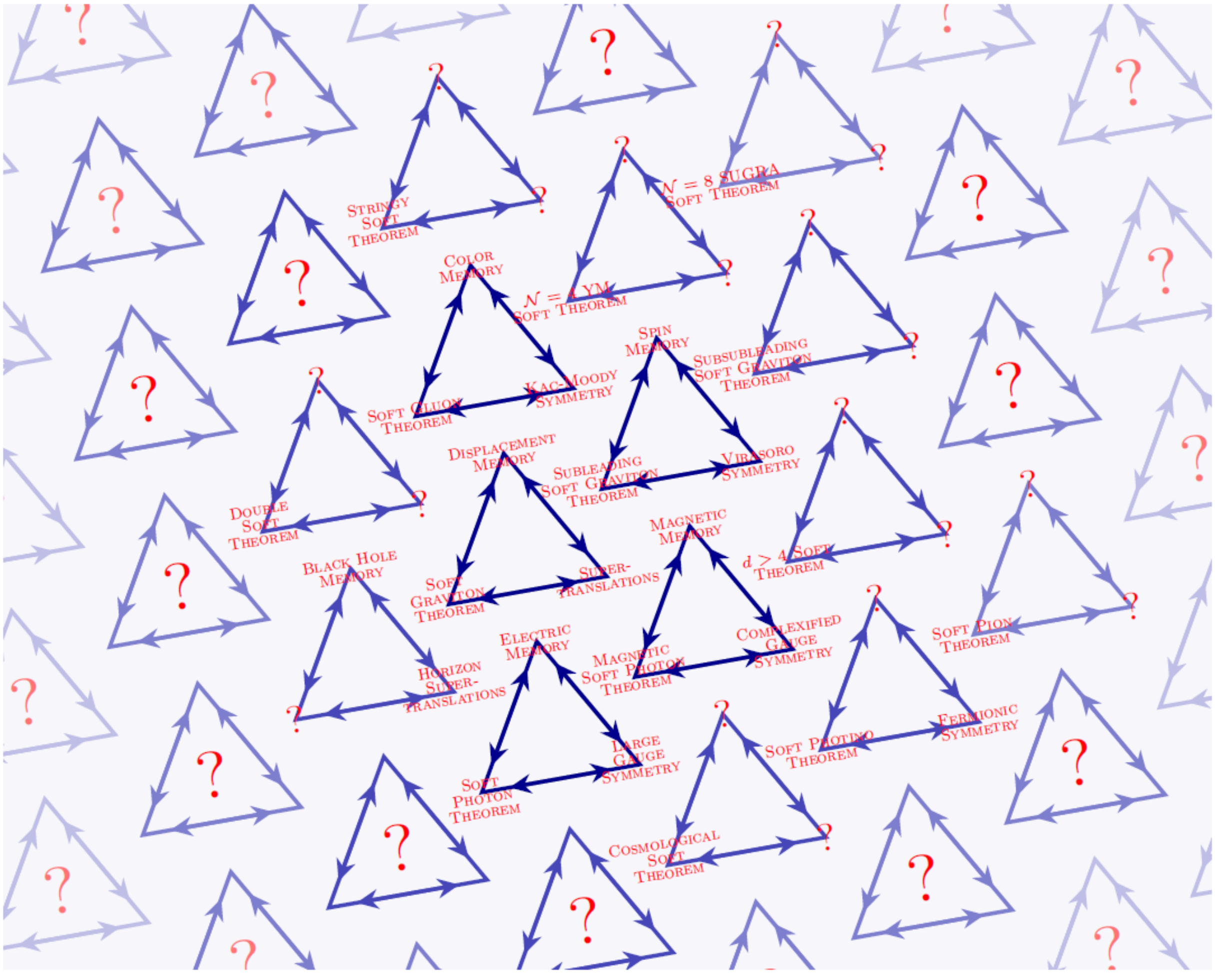


Figure 5: Different applications of the Infrared Triangle.

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