Lecture Notes on Surface Quasi-Geostrophic

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Contents

Table 1: Glossary of Variables and Operators

Variables and Operators			
Symbol	**Variable and Operators**	**Symbol**	**Description**
$\mathbf{v}(x, y, z, t) = (u, v, w)$	Full 3-dimensional Velocity	∇	2D Gradient Operator
$\mathbf{u}(x, y, z, t) = (u, v)$	2-dimensional velocity, in x and y direction	∇_3	3D Gradient
$\widehat{\cdot}$	Dimensionless Variable	$\frac{\mathrm{D}}{\mathrm{D}t}$	Material Derivative
ψ	Geostrophic Streamfunction defined in Eq	∇^2	2D laplacian operator (zonal and meridional)
$(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$	Unit vector in zonal, meridional and vertical direction		

SECTION 1

Literature Review

SUBSECTION 1.1

Ryan et.al.

Year: 2025

The QG^{+1} model incorporates the first-order corrections that were neglected in the basic QG approximation. It essentially refines the QG equations by accounting for non-geostrophic (ageostrophic) flow components that are dependent on the Rossby number (ϵ).

In **Chapter 2**, the QG^{+1} model is introduced. In this paper,

$$N = f \equiv \text{Constant}^1$$

[1] *page 8*

To facilitate the asymptotic approximation, a potential field is introduced.

$$\mathbf{A} = (-G, -F, \Phi)$$

By Incompressible condition we have

$$\mathbf{v} = \nabla_3 \times \mathbf{A}^2$$

[2] *In this paper ∇_3 is 3D gradient. 2D is just ∇*

Some Physical implications of the model

1. Breaking Symmetry of QG model.
2. It Captures Cyclogeostrophic balance.

Cyclogeostrophic balance is a fundamental force balance approximation used in meteorology and physical oceanography to describe the motion of fluids (like air and water) in curved paths, where the Coriolis force is balanced by the pressure gradient force and the centrifugal force. It is an essential extension of the simpler geostrophic balance, which only considers straight flow. This balance is particularly important in systems with high curvature and strong winds, such as tropical cyclones (hurricanes/typhoons), mid-latitude low-pressure systems, and strong ocean eddies. The Governing Equation is

$$\underbrace{f\mathbf{v}}_{\text{Coriolis Force}} + \underbrace{\frac{|\mathbf{v}|^2}{R}}_{\text{Centrifugal Force}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial n}}_{\text{Pressure Gradient Force}} \tag{1.1}$$

Here n is the normal direction pointing toward the center of curvature.

3. Inclusiong of **Frontogenesis** [3].

[3] *Generation of Ocean Fronts*

In **Chapter 3.** A simulation for QG^{+1} is conducted, showing several features:

1. More Vigorous due to captureing ageostrophic frontogenesis.
2. Since the Ageostrophic effects creates stronger surface velocity. Finer structure can be seen on surface using QG^{+1}. [4].

[4] *See Figure 4 in page 24*

In summary, this paper provides a very detailed derivation to the QG^{+1} equation which is introduced more detailed in 4. This paper also demonstrate two simulation to show how QG^{+1} model captures balanced submesoscale dynamics and frontogenesis.

SUBSECTION 1.2

J.Wang et.al. Reconstructing the Ocean's Interior from Surface Data

Year : 2013

In the **Introduction**, the author discussed the current challenge of using SSH and SST [5] measurement to reconstruct subsurface dynamics.

[5] *Surface Sea Height and Surface Sea Temperature*

- Traditional studies assume the signal is dominated by barotropic and first baroclinic modes. However, these modes are typically calculated by **assuming buoyancy anomalies vanish at the surface**.
- SQG theorys works as well. But it normaly assume 0 interior PV.

The author introduced the **Interior plus surface QG** method. It is quasigeostrophic

SECTION 2

Quasi-Geostrophic Equations

In this section, I introduce some basic equations in QG theories. Building a foundation for SQG, eSQG and other variations introduced later. The analysis is already in a **stratified ocean**.

SUBSECTION 2.1

Governing Equations

Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\frac{\nabla p}{\rho} \tag{2.1}$$

Mass Conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2.2}$$

We are in a **stratified ocean**. Breaking the total state variables into a "hydrostatic reference state" (which depends only on z) and a "dynamic perturbation" (which moves the fluid):

$$\rho = \tilde{\rho}(z) + \rho_1(x, y, z, t) \tag{2.3}$$

and

$$p = p_0(z) + p_1(x, y, z, t) \tag{2.4}$$

Then RHS of Eq 2.6 becomes

$$-\frac{1}{\rho}\nabla p_1 \sim -\frac{1}{\rho_0}\nabla p_1$$

Define the **Kinematic Pressure**

$$\phi = \frac{p_1}{\rho_0} \tag{2.5}$$

Momentum Equation becomes

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\nabla \phi} \tag{2.6}$$

Hydrostatic balance is a state of equilibrium in a fluid where the upward force of pressure exactly balances the downward force of gravity.

$$-g\tilde{\rho} = \frac{\mathrm{d}p_0}{\mathrm{d}z} \tag{2.7}$$

In Ocean, we assume the velocity field is divergent free. THen the mass conservation yeilds

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot ((\tilde{\rho} + \rho_1)\mathbf{v}) = 0 \quad \Rightarrow \quad \boxed{\nabla \cdot (\tilde{\rho}\mathbf{v}) = 0}$$

Remark 1 Here we drop the $\partial_t \rho_1$ term under the **Anelastic assumption**. Essentially by eliminating this partial derivative, we assume the fluid is anelastic, so sound wave is not supported [6] in the system.

[6] *or less important*

Next we define the Buoyancy :

$$b = -g\frac{\rho}{\bar{\rho}_0} \tag{2.8}$$

Here $\bar{\rho}_0$ is a constant reference density. The the divergent free condition implies

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = 0} \tag{2.9}$$

And thus the Hydrostatic balance equation Eq 2.7 implies

$$\boxed{\frac{\partial \phi}{\partial z} = b} \tag{2.10}$$

All the Boxed Equation together is the **Hydrostatic Anelastic Equations for Stratified Flow**. If we consider the perturbation of Buoyancy

$$b = \tilde{b}(z) + b_1(x, y, z, t)$$

Expand Eq 2.9 can be writen as

$$\boxed{\frac{\mathrm{D}b_1}{\mathrm{D}t} + w\frac{\mathrm{d}b}{\mathrm{d}z} = 0}\,[7] \tag{2.11}$$

[7] *this is the more familiar buoyancy equation we see in lecture*

In a more familiar form we define

$$N^2 = \frac{\mathrm{d}b}{\mathrm{d}z} = -g\frac{\tilde{\rho}_z}{\bar{\rho}_0}$$

Which is the **Brunt Vasala Frequency**.

SUBSECTION 2.2

Scaling Analysis

To simplify our equation, we introduce some scalings.

$$(x, y) \sim L, \quad (u, v) \sim U, \quad t \sim \frac{L}{U}, \quad z \sim H, \quad f \sim f_0$$

Introduce the **Rosbby Number**:

$$\mathrm{Ro} = \frac{U}{f_0 L} \tag{2.12}$$

Now let $\phi = \tilde{\phi}(z) + \phi_1(x, y, z, t)$. Then since the gradient in Eq 2.6 is horizontal, we can replace ϕ by ϕ_1. Now suppose

$$|\mathbf{f} \times \mathbf{u}| \sim |\nabla \phi_1|$$

From Hydrostatic balance we have

$$b \sim \frac{f_0 U L}{H}$$

Then

$$\frac{(\partial b'/\partial z)}{N^2} \sim Ro\frac{L^2}{L_d^2}$$

Where we have the deformation radius as a function of z.

$$L_d = \frac{NL}{f_0}$$

Introduce dimensionless variables

$$(\widehat{u}, \widehat{v}) = U^{-1}(u, v) \quad \widehat{w} = \frac{L}{UH}w, \quad \widehat{f} = f_0^{-1}f, \quad \widehat{\phi} = \frac{\phi_1}{f_0 U L}, \quad \widehat{b} = \frac{H}{f_0 U L}b_1$$

Remark 2 We then have dimensionless equation of motion for Anelastic Assumption [8]

$$\text{Momentum Equation}: \quad \mathrm{Ro}\frac{\mathrm{D}\widehat{\mathbf{u}}}{\mathrm{D}t} + \widehat{\mathbf{f}} \times \widehat{\mathbf{u}} = -\nabla\widehat{\phi} \tag{2.13}$$

$$\text{Buoyancy Equation}: \quad \mathrm{Ro}\frac{\mathrm{D}\widehat{b}}{\mathrm{D}t} + \left(\frac{L_d}{L}\right)^2 \widehat{w} = 0 \tag{2.14}$$

$$\text{Hydrostatic Balance}: \quad \frac{\partial\widehat{\phi}}{\partial\widehat{z}} = \widehat{b} \tag{2.15}$$

$$\text{Continuity}: \quad \widehat{\nabla}\cdot\widehat{\mathbf{u}} + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}\widehat{w}}{\partial\widehat{z}} = 0 \tag{2.16}$$

From now on I will drop the hats.

[8] *often people drop the hat for simplicity. However in the first derivation I keep everything with a hat.*

Subsection 2.3

Quasi-Geostrophic Potential Vorticity Equation

We now derive the Quasi-Geostrophic Potential Vorticity Equations. Starting from asymptotic expansions [9]

$$\mathbf{u} = (u, v, w) = \mathbf{u}_g + \mathrm{Ro}\mathbf{u}_1 \quad \phi = \phi_0 + \mathrm{Ro}\phi_1 \quad b = b_0 + \mathrm{Ro}b_1$$

[9] *hat is dropped*

Here we consider the β effect.

$$\mathbf{f} = f_0\boldsymbol{k} + \beta y\boldsymbol{k}$$

Let $\epsilon = \mathrm{Ro}$. **Momentum Equation** :

The $O(1)$ momentum equation gives the Geostrophic balance

$$f_0\boldsymbol{k} \times \mathbf{u}_g = -\nabla\phi_0 \tag{2.17}$$

Immediately this implies

$$\nabla\cdot\mathbf{u}_g = 0$$

And $O(\epsilon)$ is

$$\frac{\mathrm{D}_g\mathbf{u}_g}{\mathrm{D}t} + \beta y\boldsymbol{k} \times \mathbf{u}_g + f_0\boldsymbol{k} \times \mathbf{u}_1 = -\nabla\phi_1 \tag{2.18}$$

Here D_g is the geostrophic material derivative

$$\mathrm{D}_g = \partial_t + \mathbf{u}_g\cdot\nabla$$

Mass Equation :

Since geostrophic velocity is divergent free then $O(1)$ mass equations is

$$\frac{\partial\tilde{\rho}w_0}{\partial z} = 0$$

and $O(\epsilon)$,

$$\nabla\cdot\mathbf{u}_1 + \frac{1}{\tilde{\rho}}\left(\frac{\partial\tilde{\rho}w_1}{\partial z}\right) = 0 \tag{2.19}$$

Buoyancy Equation :

$O(1)$:

$$\left(\frac{L_d}{L}\right)^2 w_0 = 0$$

and $O(\epsilon)$:

$$\frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \left(\frac{L_d}{L}\right)^2 w_1 = 0 \tag{2.20}$$

Now we take the **Curl** of Eq 2.18, note that

$$\nabla \times (\boldsymbol{k} \times \mathbf{u}_1) = \boldsymbol{k}\nabla \cdot \mathbf{u}_1 - \underbrace{u_1 \nabla \cdot \boldsymbol{k}}_{=0} + \underbrace{(\mathbf{u}_1 \cdot \nabla)\boldsymbol{k}}_{=0} - \underbrace{(\boldsymbol{k} \cdot \nabla)\mathbf{u}_1}_{=0} = \boldsymbol{k}\nabla \cdot \mathbf{u}_1$$

Define the geostrophic vorticity :

$$\xi_g = \nabla \times \mathbf{u}_g$$

Then Eq 2.18 becomes

$$\frac{\mathrm{D}_g \xi_g}{\mathrm{D}t} + \beta v_0 = -f_0 \nabla \cdot \mathbf{u}_1{}^{10}$$

[10] *this equation is already in $\boldsymbol{k}$ direction so the unit vector is dropped*

Plug in Eq 2.19,

$$= \frac{f_0}{\tilde{\rho}} \frac{\partial \tilde{\rho} w_1}{\partial z}$$

Plug in Eq 2.20 to replace w_1

$$= -\frac{f_0}{\tilde{\rho}} \underbrace{\frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2 \frac{\mathrm{D}_g b_0}{\mathrm{D}t}\right)}_{\equiv I}$$

Now we examine I, normally in QG theory, we assume L_d is a constant. Thought from its definition, N cound actually depends on z. Since $\nabla \tilde{\rho} = 0$, we can put the first two terms into the material derivative. [11]

[11] *Here we use the fact that N is constant*

$$I = \partial_z \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2\right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} \equiv I_1 + I_2$$

Let's go back to the Hydrostatic balance equation, [12] For $O(1)$:

[12] *we haven't use it yet.*

$$\frac{\partial \phi_0}{\partial z} = b_0 \quad + \quad f_0 \boldsymbol{k} \times \mathbf{u}_g = -\nabla \phi_0 \quad \Rightarrow \quad \boldsymbol{k} \times \frac{\partial \mathbf{u}_g}{\partial z} = -\frac{\nabla b_0}{f_0}$$

Then

$$I_2 = \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} = \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \left[\frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} + \underbrace{\partial_z \mathbf{u}_g \cdot \nabla b_0}_{=0}\right]$$

Therefore

$$I = \partial_z \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2\right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} = \frac{\mathrm{D}_g}{\mathrm{D}t} \left[\frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2 b_0\right)\right]$$

Then evantually we have

$$\frac{\mathrm{D}_g}{\mathrm{D}t} \left[\xi_g + f + \frac{f_0}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2 b_0\right)\right] = 0 \tag{2.21}$$

We can rewrite this equation using Streamfunction in a more simple form. Recall Eq 2.9, we have

$$b_0 = \frac{\partial \phi_0}{\partial z}$$

From Eq 2.17, the Kinematic Pressure can be expressed in terms of geostrophic streamfunction

$$u_g = -\partial_y \psi_g \quad v_g = \partial_x \psi_g \qquad \text{where} \quad \boxed{\phi_0 = f_0 \psi_g} \quad \Rightarrow \quad \xi_g = \nabla^2 \psi_g$$

Then Eq 2.21 becomes

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 \frac{\partial \psi_g}{\partial z}\right)\right] = 0 \tag{2.22}$$

Restore the dimensions

$$\frac{\mathrm{D}g}{\mathrm{D}t}\left[\nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z}\left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z}\right)\right] = 0 \tag{2.23}$$

SECTION 3

Surface Quasi-Geostrophic Equations

The surface Quasi-Geostrophic Equation takes the problem to the next step, how could we retrive the interior motion from surface measurements such as SSH and SST. Recall the Buoyancy Equation

$$\frac{\mathrm{D}b_1}{\mathrm{D}t} + wN^2 = 0 \tag{3.1}$$

At the surface, $z = \eta$, the boundary condition yeilds that $w = 0$. We denote the surface buoyancy as b_s and surface velocity $\mathbf{u}_s$ Then

$$\frac{\partial b_s}{\partial t} + \mathbf{u}_s \cdot \nabla b_s = 0$$

and

$$b_s = f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0}$$

Explanation 1 The critical principle of SQG is to view surface buoyancy as a PV sheet. Since

$$\int_0^{\epsilon} f + \nabla^2 \psi_g + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) \mathrm{d}z = 0$$

Then we impose a boundary condition

$$\frac{\partial \psi}{\partial z}\Big|_{z=\epsilon} = \frac{b_s}{f_0} \quad \Rightarrow \quad \int_0^{\epsilon} \frac{b_s}{f_0} = \frac{\partial \psi}{\partial z}\Big|_0^{\epsilon}$$

Compare the latter witht the integral, by defining

$$q_{\mathrm{SQG}} = \nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) + \frac{b_s}{f_0} \delta(z)$$

We have

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \frac{\partial \psi}{\partial z} = 0$$

Then the surface buoyancy appears in the QGPV equation naturally, it is as if adding an additional PV sheet at the surface. This inspires us to seperate the surface induced dynamics and interior dynamics.

Interior Dynamics

$$\begin{cases} q & = \nabla^2 \psi + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) \\ f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0} & = 0 \\ \frac{\mathrm{D}q}{\mathrm{D}t} & = 0 \end{cases}$$

SUBSECTION 3.1

Surface Buoyancy Induced Dynamics

Surface Dynamics, this is the Surface Quasi-Geostrophic Dynamics:

$$\begin{cases} q & = \nabla^2\psi + f + \dfrac{f_0^2}{\tilde{\rho}}\dfrac{\partial}{\partial_z}\left(\dfrac{\tilde{\rho}}{N^2}\dfrac{\partial\psi}{\partial z}\right) = 0^{13} \\ f_0\dfrac{\partial\psi}{\partial z}\Big|_{z=0} & = b_s \\ \dfrac{\mathrm{D}b_s}{\mathrm{D}t} & = 0 \end{cases}$$

The key assumption for SQG theories is that all the Potential vorticity is injected into the system by surface buoyancy. [14]

[13] *no interior Potential vorticity*

[14] *surface buoyancy is actually first order, so it is also called surface buoyancy anomoly in some context.*

Section 4

QG^{+1} Model