Section 1

# $\text{QG}^{+1}$ Model

The derivation starts from the same set of equations above. We set

$$\epsilon = \text{Ro} = \frac{U}{fL} \qquad \text{Bu} = \left(\frac{NH}{fL}\right)^2$$

The Derivation starts from dimensional form of Eq **??**, **??**, **??** and **??**. The Ertel Potential Vorticity is conserved. In this case, the conserved quantity is the **total Buoyancy** since Eq **??** we define it as

$$b_{tot} = \text{Bu} \cdot z + \text{Ro} \cdot b$$

$$b_{\text{tot}} = N^2 z + b$$

and

$$\boldsymbol{\omega}_a = \nabla_3 \times (u, v, 0) + f\hat{z} = (-v_z, u_z, f + \xi)$$

Then

$$Q = \underbrace{fN^2}_{\text{Background}} + \underbrace{(N^2\xi + fb_z)}_{\text{Linear terms, QGPV}} + \underbrace{(\xi b_z - v_z b_x + u_z b_y)}_{\text{Nonlinear terms}} \tag{1.1}$$

and

$$q_{\text{QG}} = N^2\xi + fb_z$$

Use the vector identity

$$\nabla_3 \cdot (\boldsymbol{\omega} b) = \boldsymbol{\omega} \cdot \nabla_3 b + \underbrace{b(\nabla \cdot \boldsymbol{\omega})}_{=0, \boldsymbol{\omega} = \nabla \times \cdot}$$

We note that

$$\xi b_z - v_z bx + u_z b_y = \boldsymbol{\omega} \cdot \nabla_3 b$$

where

$$\boldsymbol{\omega} = (-v_z, u_z, \xi) = (-v_z, u_z, v_x - u_y)$$

We apply the scalings

$$\xi \sim \frac{U}{L} \quad b \sim \frac{fUL}{H} \quad v_z, u_z \sim \frac{U}{H}$$

Then Eq 1.1 becomes

$$Q = 1 + \epsilon \left(\xi + \frac{b_z}{\text{Bu}} + \frac{\epsilon}{\text{Bu}}(\xi b_z - v_z b_x + u_z b_y)\right) \tag{1.2}$$

or in a more compact way, again we drop the hat for dimensionless variables for simplicity.

$$Q = 1 + \epsilon \left(\xi + \frac{b_z}{\text{Bu}} + \frac{\epsilon}{\text{Bu}} \nabla \cdot (\boldsymbol{\omega} b)\right) \tag{1.3}$$

or express in the form of

$$Q = 1 + \epsilon q$$

where

$$q = \xi + \frac{b_z}{\text{Bu}} + \frac{\epsilon}{\text{Bu}} \nabla \cdot (\boldsymbol{\omega} b) \tag{1.4}$$

**Explanation 1** | Since here $N$ is constant and we assume the Boussinesq Equation, the third term in Eq **??**

becomes,

$$\frac{f_0}{\cancel{\rho}}\cancel{\rho}\frac{\partial b}{\partial z}\left(\frac{fL}{NH}\right)^2 \sim f b_z \frac{1}{\mathrm{Bu}}$$

The only difference here is the third second order ageostrophic quadratic correction we normally ignore in classical QG theories.

SUBSECTION 1.1

## $\mathbf{QG^{+1}}$ Vector Field

In order to facilitate asymptotic approximation, the velocity field $\mathbf{u}$ are writen as the Curl of a vector field $\boldsymbol{A}$. [1]. One convention is writing

[1]Changing three variables in velocity to $\boldsymbol{A}$

$$\boldsymbol{A} = (-G, F, \Phi) \tag{1.5}$$

and

$$\mathbf{u} = \nabla \times \boldsymbol{A}$$
$$u = -\Phi_y - F_z$$
$$v = \Phi_x - G_z$$
$$w = F_x + G_y$$

The Horizontal Vorticity in vector potential form is

$$\xi = v_x - u_y = \Phi_{xx} - G_{zx} + \Phi_{yy} + F_{zy} = \nabla^2\Phi + F_{zy} - G_{zx}$$

However, this $\boldsymbol{A}$ isn't uniquely correspond to a velocity field. Since Gradient is Curl free, $\mathbf{u}$ admits a gauge freedom in that the transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla_3\Gamma$$

Left $\mathbf{u}$ unchange. Now instead of assuming $\boldsymbol{A}$ is divergent free. [2] We fix the buoyancy to be roughly scaled version of $\nabla_3 \cdot \boldsymbol{A}$.

[2]if $\boldsymbol{A}$ is divergent free, then this is adding an additioal information to the system and $\boldsymbol{A}$ can be uniquely determined.

$$\nabla_3 \cdot \boldsymbol{A} = -G_x + F_y - \Phi_z \quad \Rightarrow \quad b = \Phi_z + \mathrm{Bu}(G_x - F_y)$$

The advantage can be seen by calculating the QGPV Eq 1.3 (**Dimensionless form**) :

$$q_{\mathrm{QG}} = \xi + \frac{b_z}{\mathrm{Bu}} = \nabla^2\Phi + F_{zy} - G_{zx} + \frac{1}{\mathrm{Bu}}\Phi_{zz} + (G_{xz} - F_{yz}) = (\Phi_{xx} + \Phi_{yy}) + \frac{\Phi_{zz}}{\mathrm{Bu}} = \mathcal{L}\Phi \tag{1.6}$$

Where

$$\mathcal{L} = \nabla^2 + \frac{1}{\mathrm{Bu}}\partial_{zz} \tag{1.7}$$

Then $\Phi$ is the only component related to the QGPV compare to dependence on all three component of velocity in the classical QGPV. When is the Buoyancy unchanged?

**Explanation 2** Suppose we add a gradient to the original Vector Potential

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla_3\Gamma$$

Then

$$G \to G - \Gamma_x \quad F \to F + \Gamma_y \to \Phi \to \Phi - \Gamma_z$$

And

$$b_{new} = \underbrace{\left[\Phi_z + \mathrm{Bu}(G_x - F_y)\right]}_{b_{origin}} - \underbrace{\left[\Gamma_{zz} + \mathrm{Bu}(\Gamma_{xx} + \Gamma_{yy})\right]}_{b_{change}}$$

So in order to have $b_{new} = b_{origin}$. We have

$$b_{change} = \Gamma_{zz} + \mathrm{Bu}(\Gamma_{xx} + \Gamma_{yy}) \Rightarrow \boxed{\mathcal{L}(\Gamma) = 0} \tag{1.8}$$

Some Discussions

1. **In a triply-periodic domain** :
   Eq 1.8 implies that $\Gamma$ is constant in the domain.
2. **Rigid Lid and flat bottom** :
   since $w = 0$ at upper and lower boundary, we have

$$F_x + G_y = 0$$

SUBSECTION 1.2

## Evolution and Inversion Equations of QG$^{+1}$

Assume [3]

$$G^0 = F^0 = 0$$

and

$$\epsilon \ll 1 \qquad \mathrm{Bu} \sim O(1)$$

Then expand $\boldsymbol{A}$ asymptotically we have

$$u = -\Phi_y^0 - \epsilon(\Phi_y^1 + F_z^1) \tag{1.9a}$$
$$v = \Phi_x^0 + \epsilon(\Phi_x^1 - G_z^1) \tag{1.9b}$$
$$w = 0 + \epsilon(F_x^1 + G_y^1) \tag{1.9c}$$
$$b = \Phi_z^0 + \epsilon\left(\Phi_z^1 + \mathrm{Bu}(G_x^1 - F_y^1)\right) \tag{1.9d}$$

Evaluate Eq **??** we have up to $O(\epsilon)$.

$$\begin{aligned} q &= \underbrace{(\nabla^2\Phi^0 + \nabla^2\Phi^1 - G_{zx}^1 + F_{zy}^1)}_{\text{This is } \xi} + \left[\frac{1}{\mathrm{Bu}}\Phi_{zz}^0 + \frac{\epsilon}{\mathrm{Bu}}\left(\Phi_{zz}^1 + \mathrm{Bu}(G_{xz}^1 - F_{yz}^1)\right)\right] \\ &+ \frac{\epsilon}{\mathrm{Bu}}\underbrace{\left[-\Phi_{xz}^0, -\Phi_{yz}^0, \nabla^2\Phi^0\right]}_{\text{This is } \boldsymbol{\omega}=(-v_z,u_z,\xi)} \cdot \underbrace{\left[\Phi_{zx}^0, \Phi_{zy}^0, \Phi_{zz}^0\right]}_{\text{this is } \nabla_3 b} \\ &= \boxed{\mathcal{L}(\Phi^0) + \epsilon\mathcal{L}(\Phi^1) + \frac{\epsilon}{\mathrm{Bu}}\left(-|\nabla\Phi_z^0|^2 + \Phi_{zz}^0\nabla^2\Phi^0\right) + O(\epsilon^2)} \end{aligned} \tag{1.10}$$

The surface Buoyancy from

$$b^t = \Phi_z^0\Big|_{z=0} + \Phi_z^1\Big|_{z=0} + O(\epsilon^2) \tag{1.11a}$$
$$b^b = \Phi_z^0\Big|_{z=-H} + \Phi_z^1\Big|_{z=-H} + O(\epsilon^2) \tag{1.11b}$$

[3] *Why? First order is geostrophic balance?*

To complete the inversion, we need to relate ageostrophic vertical streamfunctions $F^1$ and $G^1$ to $\Phi^0$. We start with the first order primitive equations

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv^1 = -p_x^1 \tag{1.12a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu^1 = -p_y^1 \tag{1.12b}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + \mathrm{Bu}N^2 w^1 = 0 \tag{1.12c}$$

$$p_z^1 = b^1 \tag{1.12d}$$

Take the difference of $z$-derivative of $f$ times Eq 1.12b and $x$-derivetive Eq 1.12c we derive

$$\mathcal{L}(F^1) = \frac{2}{\mathrm{Bu}} J(\Phi_z^0, \Phi_x^0) \tag{1.13}$$

and similarly

$$\mathcal{L}(G^1) = \frac{2}{\mathrm{Bu}} J(\Phi_z^0, \Phi_y^0) \tag{1.14}$$

SECTION 2

# SQG$^{+1}$ Model

From the Geostrophic Balance Equation we have

$$\nabla^2 p^0 = \xi^0$$

To obtain the next order balance equation, we use Eq 1.9d, 1.13 and 1.14.

$$\begin{aligned}\nabla^2 b^1 - \xi^1_z &= \left(\nabla^2\Phi_z + \mathrm{Bu}\nabla^2(G_x - F_y)\right) - \nabla^2\Phi_z - \partial_{zz}(G_x - F_y)\\ &= \mathrm{Bu}\cdot\mathcal{L}(G_x - F_y)\\ &\quad\text{(Here apply Eq 1.13 and Eq 1.14)}\\ &= 2\partial_z J(\Phi^0_x, \Phi^0_y)\end{aligned} \tag{2.1}$$

This is some modification Recall that $p_z = b$ by definition then we have

$$\boxed{\nabla^2 p^1 - f\xi^1 = 2J(\Phi^0_x, \Phi^0_y)} \tag{2.2}$$

**Explanation 3** The RHS is called the cyclogeostrophic term because it represents the centrifugal force generated by the curvature of the fluid flow. However, real ocean/atmospheric eddies are curved (swirling). When fluid moves in a curve, it experiences a Centrifugal Force ($V^2/R$).Gradient Wind Balance (Cyclogeostrophic): Coriolis Force + Centrifugal Force ≈ Pressure Gradient Force.

This model captures the ageostrophic at first order and the term on the right hand side represents the **cyclogeostrophic correction**. In SQG, interior potential vorticity is 0. Then from Eq 1.10, since $q^1 = 0$. Then

$$\mathcal{L}(\Phi^1) = \frac{1}{\mathrm{Bu}}\left(|\nabla\Phi^0_z|^2 - \Phi^0_{zz}\nabla^2\Phi^0\right)$$

Using $\mathcal{L}(\Phi^0) = 0$. We have

$$\mathcal{L}(\Phi^1) = \frac{1}{\mathrm{Bu}}\left(|\nabla\Phi^0_z|^2 + \frac{1}{\mathrm{Bu}}\Phi^0_{zz}\Phi^0_{zz}\right) \tag{2.3}$$

Together with Eq 1.13, Eq 1.14 and boundary conditions.

$$\Phi^1_z = F^1 = G^1 = 0 \qquad \text{at } z = 0$$

Further we assume all potential vanish at infinity.

SUBSECTION 2.1

## $\Phi^0$ Inversion

To get the $\Phi^0$ from SSH data, we do the standard inversion as in the Quasi-Geostrophic Balance Model.

The first order potential is geostrophic. And in the Surface Geostrophic Model, the interior

PV is 0. So we have the Governing equation

$$\mathcal{L}(\Phi^0) = 0 \qquad f\Phi^0_z = b$$

In the Fourier space, since $\Phi$ vanish as $z \to -\infty$. Then the Horizontal Fourier modes are

$$\widehat{\Phi}^0 = \frac{\hat{b}^{0,t}}{f\mu} e^{\mu z}$$

With constant

$$\mu = \sqrt{\mathrm{Bu}} \frac{NK}{f} \qquad K = \sqrt{m^2 + l^2}$$

SUBSECTION 2.2

## Higher Order Potential Inversion

Now the $\Phi^0$ is known already.

**Explanation 4** For this system, the goal is to know $\Phi^1, G^1$ and $F^1$. We recall that first order potential $G^0 = F^0 = 0$. And $\Phi^0$ can be obtain from geostrophic balance. This is a system of decoupled linear elliptic (Poisson) equations.

1. Decoupled: You can solve for $\Phi^1$, $F^1$, and $G^1$ independently of each other.
2. Linear/Poisson: Each equation takes the form $\mathcal{L}(\text{Potential}) = \text{Source Term}$.
3. Dependence: The "Source Terms" on the right-hand side (RHS) are known quantities derived entirely from the zeroth-order solution $\Phi^0$ (which is obtained from the observed SSH/buoyancy)

The inversion is carried on by decomposing potential into interior and surface part. The interior part satisfies the main equations but ignore the boundary condition. The surface part solves the homogeneous problem and but corrects the boundary conditions. For $\Phi^1$, we first notice that

$$\Phi^1_{int} = \frac{1}{2\mathrm{Bu}} \Phi^0_z \Phi^0_z \tag{2.4}$$

Is an analytic solution to Eq 2.3. Then it remains to solve the surface part given by

$$\mathcal{L}(\Phi^1_{sur}) = 0 \tag{2.5}$$

$$\Phi^{1,t}_{sur,z} = C_b - \partial_z \Phi^{1,t}_{int} = C_b - \frac{1}{\mathrm{Bu}} \Phi^{0,t}_z \Phi^{0,t}_{zz} \tag{2.6}$$

This problem is much more familiar, exactly the same problem when solving first order potential $\Phi^0$. The Fourier modes are [4]

[4] *$C_b$ is added*

$$\widehat{\Phi}^1_{sur} = \left( C_b - \partial_z \Phi^{1,t}_{int,z} \right) \frac{1}{\mu} e^{\mu z} = \left( C_b - \frac{1}{\mathrm{Bu}} \widehat{\Phi^{0,t}_z \Phi^{0,t}_{zz}} \right) \frac{1}{\mu} e^{\mu z}$$

The analytic solution of $\Phi_{int}$ is given by Eq 2.4 already. So the total Fourier mode of $\Phi$ is

$$\widehat{\Phi}^1 = \widehat{\Phi}^1_{sur} + \widehat{\Phi}^1_{int}$$

Similarly, by setting

$$F^1_{int} = \frac{1}{\mathrm{Bu}} \Phi^0_y \Phi^0_z$$

We see that

$$
\begin{aligned}
\mathcal{L}(F^1) &= \frac{1}{\text{Bu}}\left(\underbrace{\mathcal{L}(\Phi^0_y)\Phi^0_z}_{=0 \text{ since } \mathcal{L}(\Phi_0)=0} + \underbrace{\mathcal{L}(\Phi^0_z)\Phi^0_y}_{\text{same}} + 2(\nabla\Phi^0_y\cdot\nabla\Phi^0_z) + \underbrace{\frac{2}{\text{Bu}}\Phi_{yz}\Phi_{zz}}_{\text{rewrite using } \mathcal{L}(\Phi^0)=0}\right)\\
&= \frac{1}{\text{Bu}}\Big(2(\Phi^0_{xy}\Phi^0_{xz} + \Phi^0_{yy}\Phi^0_{zy}) - 2\Phi_{yz}(\Phi^0_{xx} + \Phi^0_{yy})\Big)\\
&= \frac{2}{\text{Bu}}J(\Phi^0_z, \Phi^0_x)
\end{aligned}
$$

This match with Eq 1.13. Then similar as $\Phi^1$. The total Fourier mode is the sum of interior part and surface part. Where surface part is set to match the boundary condition. THe equations are

$$
\mathcal{L}(F^1_{int}) = \frac{2f}{\text{Bu}}J(\Phi^0_z, \Phi^0_x) \quad \mathcal{L}(F^1_{sur}) = 0 \;\&\; F^0_{sur}(z=0) = -F_{int}(z=0)
$$

$$
\widehat{F^1} = \frac{1}{\text{Bu}}\left(\widehat{\Phi^0_y\Phi^0_z} - \widehat{\Phi^{0,t}_y\Phi^{0,t}_z}e^{\mu z}\right) \tag{2.7}
$$

Similarly we can set [5]

[5] *Do the calculation for G here later.*

$$
G^1_{int} = -\frac{1}{\text{Bu}}\Phi^0_x\Phi^0_z
$$

and

$$
\widehat{G^1} = -\frac{1}{\text{Bu}}\left(\widehat{\Phi^0_x\Phi^0_z} - \widehat{\Phi^{0,t}_x\Phi^{0,t}_z}e^{\mu z}\right) \tag{2.8}
$$

We don't see the $\mu$ here on the denominator since the surface part for $F$ and $G$ has Dirichlet Boundary condition [6] while $\Phi$ has Neumann Boundary Condition.

[6] *condition on the function*

The inversion of surface horizontal velocity can be calculate as follow :

The derivative is taken before the Fourier Transform, since *int* part has analytic solution.

$$
\begin{aligned}
u &= \sum\Big(-\hat{\Phi}^{0,t}_y - \epsilon(\hat{\Phi}^{1,t}_y + \hat{F}^{1,t}_z)\Big)e^{i(kx+ly)}\\
&= -\epsilon\cdot\frac{1}{\text{Bu}}\left[\partial_y\left(\underbrace{\frac{\Phi^{0,t}_z\Phi^{0,t}_z}{2}}_{\Phi^{1,t}_{int}}\right) + \partial_z\left(\underbrace{\Phi^{0,t}_y\Phi^{0,t}_z}_{F^1_{int}}\right)\right]\\
&+ \sum\left[-\hat{\Phi}^{0,t}_y e^{i(kx+ly)} - \epsilon\cdot\frac{1}{\text{Bu}}\partial_y\left(\left(\underbrace{-\frac{\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}}{\mu}e^{\mu z}}_{\Phi^1_{sur}}\right)e^{i(kx+ly)}\right) - \epsilon\cdot\frac{1}{\text{Bu}}\partial_z\left(-\underbrace{\widehat{\Phi^{0,t}_y\Phi^{0,t}_z}e^{\mu z}}_{F^1_{sur}}\right)e^{i(kx+ly)}\right]
\end{aligned}
$$

Then take the derivative, then take the Fourier Transform we have

$$
\begin{aligned}
\hat{u}^t &= \hat{u}(z=0)\\
&= -\Phi^{0,t}_y - \epsilon\cdot\frac{1}{\text{Bu}}\left[\underbrace{\widehat{\Phi^{0,t}_y\Phi^{0,t}_{zz}} + 2\widehat{\Phi^{0,t}_{yz}\Phi^{0,t}_z}}_{\text{Interior Part}} - \underbrace{\mu\widehat{\Phi^{0,t}_y\Phi^{0,t}_{zz}} - \frac{il}{\mu}\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}}_{\text{Surface Part}}\right]
\end{aligned} \tag{2.9}
$$

Similarly

$$\begin{aligned}\hat{v}^t &= \hat{v}(z=0)\\ &= \Phi_x^{0,t} + \epsilon \cdot \frac{1}{\text{Bu}}\left[\underbrace{2\widehat{\Phi_{xz}^{0,t}\Phi_z^{0,t}} + \widehat{\Phi_x^{0,t}\Phi_{zz}^{0,t}}}_{\text{Interior Part}} \underbrace{- \frac{ik}{\mu}\widehat{\Phi_z^{0,t}\Phi_{zz}^{0,t}} - \mu\widehat{\Phi_x^{0,t}\Phi_z^{0,t}}}_{\text{Surface Part}}\right]\end{aligned} \tag{2.10}$$ [7]

*[7] The sign here? Should be both minus sign perhaps?*

For the Vertical Velocity : Start in the primitive space

$$\begin{aligned}w &= \epsilon(F_x^1 + G_y^1)\\ &= \epsilon\frac{1}{\text{Bu}}\sum\partial_x\left[\left(\widehat{\Phi_y^0\Phi_z^0} - \widehat{\Phi_y^{0,t}\Phi_z^{0,t}}e^{\mu z}\right)e^{i(kx+ly)}\right] - \partial_y\left[\left(\widehat{\Phi_x^0\Phi_z^0} - \widehat{\Phi_x^{0,t}\Phi_z^{0,t}}e^{\mu z}\right)e^{i(kx+ly)}\right]\end{aligned}$$

The Fourier Transform aren't function of $x$ or $y$

$$= \epsilon\frac{1}{\text{Bu}}\sum\left(ik\widehat{\Phi_y^0\Phi_z^0} - il\widehat{\Phi_x^0\Phi_z^0}\right)e^{i(kx+ly)} - \left(ik\widehat{\Phi_y^{0,t}\Phi_z^{0,t}} - il\widehat{\Phi_x^{0,t}\Phi_z^{0,t}}\right)e^{i(kx+ly)}$$

Taking $x$ derivative to Fourier Component pulls down an additional fact of $ik$. Then

$$\hat{w} = \epsilon\frac{1}{\text{Bu}} \tag{2.11}$$