Lecture Notes on Surface Quasi-Geostrophic

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Contents

Table 1: Glossary of Variables and Operators

Variables and Operators			
Symbol	**Variable and Operators**	**Symbol**	**Description**
$\mathbf{v}(x,y,z,t) = (u,v,w)$	Full 3-dimensional Velocity	∇	2D Gradient Operator
$\mathbf{u}(x,y,z,t) = (u,v)$	2-dimensional velocity, in x and y direction	∇_3	3D Gradient
$\hat{\cdot}$	Dimensionless Variable	$\frac{\mathrm{D}}{\mathrm{D}t}$	Material Derivative
ψ	Geostrophic Streamfunction defined in Eq	∇^2	2D laplacian operator (zonal and meridional)
$(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$	Unit vector in zonal, meridional and vertical direction	$\widehat{\cdot}$	Wide hat for Fourier Transform

1 | Question for Meetings

1.1 0108

1. So the QG^{+1} model is still geostrphic? How does it improve the accuracy of submesoscale dynamics prediction who is not geostrophic?
2. What is the Physical Meaning of the $\mathcal{L}$ operator acting on Γ which is the Gauge transform is 0.
3. Leading order of the Potential Terms are about Geostrophiic Balanace?
4. Sign Difference?
5. Briefly walk me through how is the code constructed? What are some of the potential **difficulties**? How do we overcome this.

Rewrite all the equation, should be all non-dimensionalized. Only ϵ and Bu should show up.

2 | Literature Review

2.1 Ryan et.al.

Year: 2025

The QG^{+1} model incorporates the first-order corrections that were neglected in the basic QG approximation. It essentially refines the QG equations by accounting for non-geostrophic (ageostrophic) flow components that are dependent on the Rossby number (ϵ).

In **Chapter 2**, the QG^{+1} model is introduced. In this paper,

$$N = f \equiv \text{Constant}^1$$

[1] page 8

To facilitate the asymptotic approximation, a potential field is introduced.

$$\mathbf{A} = (-G, -F, \Phi)$$

By Incompressible condition we have

$$\mathbf{v} = \nabla_3 \times \mathbf{A}^2$$

[2] In this paper ∇_3 is 3D gradient. 2D is just ∇

Some Physical implications of the model

1. Breaking Symmetry of QG model.
2. It Captures Cyclogeostrophic balance.

Cyclogeostrophic balance is a fundamental force balance approximation used in meteorology and physical oceanography to describe the motion of fluids (like air and water) in curved paths, where the Coriolis force is balanced by the pressure gradient force and the centrifugal force. It is an essential extension of the simpler geostrophic balance, which only considers straight flow. This balance is particularly important in systems with high curvature and strong winds, such as tropical cyclones (hurricanes/typhoons), mid-latitude low-pressure systems, and strong ocean eddies. The Governing Equation is

$$\underbrace{f\mathbf{v}}_{\text{Coriolis Force}} + \underbrace{\frac{|\mathbf{v}|^2}{R}}_{\text{Centrifugal Force}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial n}}_{\text{Pressure Gradient Force}} \tag{2.1}$$

Here n is the normal direction pointing toward the center of curvature.

3. Inclusiong of **Frontogenesis** [3].

[3] Generation of Ocean Fronts

In **Chapter 3.** A simulation for QG^{+1} is conducted, showing several features:

1. More Vigorous due to captureing ageostrophic frontogenesis.
2. Since the Ageostrophic effects creates stronger surface velocity. Finer structure can be seen on surface using QG^{+1}. [4].

[4] See Figure 4 in page 24

In summary, this paper provides a very detailed derivation to the QG^{+1} equation which is introduced more detailed in 5. This paper also demonstrate two simulation to show how QG^{+1} model captures balanced submesoscale dynamics and frontogenesis.

Remark 1 In my own derivation following Ryan's work in the later chapter, first half of the calcualtion is based on this paper, then I follow Ryan's note for the rest.

2.1.1 Questions Regrading this Paper

1. In page 10, Equation 15. How does the above derivation implies $F = G = 0$. Does this implies that
$$F^0 + \epsilon F^1 + \cdots = 0$$
Then each order term of F and G is 0. I just don't get the physical meaning here.

2.2 J.Wang et.al. Reconstructing the Ocean's Interior from Surface Data

Year : 2013

In the **Introduction**, the author discussed the current challenge of using SSH and SST [5] measurement to reconstruct subsurface dynamics.

[5] *Surface Sea Height and Surface Sea Temperature*

- Traditional studies assume the signal is dominated by barotropic and first baroclinic modes. However, these modes are typically calculated by **assuming buoyancy anomalies vanish at the surface**.
- SQG theorys works as well. But it normaly assume 0 interior PV.

The author introduced the **Interior plus surface QG** method. It is quasigeostrophic. As introduced in Chapter 2:

1. Surface buoyancy anomoly contributes to the surface part of streamfunction ψ^s :
$$\mathcal{L}\Psi + f_0 + \beta y = Q,$$
$$\mathcal{L} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z} \right), \quad \text{and } -H < z < 0,$$
Governing equation is
$$\mathcal{L}\psi^s = 0$$
Essentially, this is same in the SQG theory where we assume 0 interior PV. With boundary condition :
$$\frac{\partial}{\partial z}\psi^s(\mathbf{x}, z, t) = b(\mathbf{x}, z, t)/f_0 \quad \text{at} \quad z = 0, -H,$$

2. The interior part is governed by
$$\frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\hat{\psi}^i - \kappa^2\hat{\psi}^i = \hat{q}^i \quad \text{with} \quad \frac{d\hat{\psi}^i}{dz} = 0 \quad \text{at} \quad z = 0, -H.$$

However, the interior q^i is not clear.

Remark 2 This is the essential modification of this isQG model. They project the interior induced PV equation onto baroclinic modes and **impose additional boundary conditions to deduce the gravest modes.**

This is the origin of QG^{+1} theory.

3 | Quasi-Geostrophic Equations

In this section, I introduce some basic equations in QG theories. Building a foundation for SQG, eSQG and other variations introduced later. The analysis is already in a **stratified ocean**.

3.1 Governing Equations

Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\frac{\nabla p}{\rho} \tag{3.1}$$

Mass Conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0 \tag{3.2}$$

We are in a **stratified ocean**. Breaking the total state variables into a "hydrostatic reference state" (which depends only on z) and a "dynamic perturbation" (which moves the fluid):

$$\rho = \tilde{\rho}(z) + \rho_1(x, y, z, t) \tag{3.3}$$

and

$$p = p_0(z) + p_1(x, y, z, t) \tag{3.4}$$

Then RHS of Eq 3.1 becomes

$$-\frac{1}{\rho} \nabla p_1 \sim -\frac{1}{\rho_0} \nabla p_1$$

Define the **Kinematic Pressure**

$$\phi = \frac{p_1}{\rho_0} \tag{3.5}$$

Momentum Equation becomes

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\nabla \phi} \tag{3.6}$$

Hydrostatic balance is a state of equilibrium in a fluid where the upward force of pressure exactly balances the downward force of gravity.

$$-g\tilde{\rho} = \frac{\mathrm{d}p_0}{\mathrm{d}z} \tag{3.7}$$

3.1.1 Continuity Equation Approximatio

The genearl continuity equation can always be expressed as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot u = 0$$

In Ocean, we assume that the fluid is **Incompressible** and wrote

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \qquad \nabla \cdot \mathbf{u} = 0$$

In Atmosphere, we use the **Anelastic Assumption** Then the mass conservation yeilds

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot ((\tilde{\rho} + \rho_1)\mathbf{v}) = 0 \quad \Rightarrow \quad \boxed{\nabla \cdot (\tilde{\rho}\mathbf{v}) = 0}$$

Remark 3 Here we drop the $\partial_t \rho_1$ term under the **Anelastic assumption**. Essentially by eliminating this partial derivative, we assume the fluid is anelastic, so sound wave is not supported [6] in the system. However, this approximation is normally used for **deep atmospheric or stellar convertion** where density $\tilde{\rho}$ changes significantly with height.

[6] or less important

For Oceanography, Next we define the Buoyancy :

$$b = -g\frac{\rho}{\bar{\rho}_0} \tag{3.8}$$

Here $\bar{\rho}_0$ is a constant reference density. The the divergent free condition implies

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = 0} \tag{3.9}$$

And thus the Hydrostatic balance equation Eq 3.7 implies

$$\boxed{\frac{\partial \phi}{\partial z} = b} \tag{3.10}$$

All the Boxed Equation together is the **Hydrostatic Anelastic Equations for Stratified Flow**. If we consider the perturbation of Buoyancy

$$b = \tilde{b}(z) + b_1(x, y, z, t)$$

Expand Eq 3.9 can be writen as

$$\boxed{\frac{\mathrm{D}b_1}{\mathrm{D}t} + w\frac{\mathrm{d}b}{\mathrm{d}z} = 0}^{7} \tag{3.11}$$

[7] this is the more familiar buoyancy equation we see in lecture

In a more familiar form we define

$$N^2 = \frac{\mathrm{d}b}{\mathrm{d}z} = -g\frac{\tilde{\rho}_z}{\bar{\rho}_0}$$

Which is the **Brunt Vasala Frequency**.

3.2 Scaling Analysis

To simplify our equation, we introduce some scalings.

$$(x, y) \sim L, \quad (u, v) \sim U, \quad t \sim \frac{L}{U}, \quad z \sim H, \quad f \sim f_0$$

Introduce the **Rosbby Number**:

$$\mathrm{Ro} = \frac{U}{f_0 L} \tag{3.12}$$

Now let $\phi = \tilde{\phi}(z) + \phi_1(x, y, z, t)$. Then since the gradient in Eq 3.1 is horizontal, we can replace ϕ by ϕ_1. Now suppose

$$|\mathbf{f} \times \mathbf{u}| \sim |\nabla \phi_1|$$

From Hydrostatic balance we have

$$b \sim \frac{f_0 UL}{H}$$

Then

$$\frac{(\partial b'/\partial z)}{N^2} \sim \frac{f_0 UL}{(HN)^2} \sim Ro\frac{L^2}{L_d^2}$$

Where we have the deformation radius as a function of z.

$$L_d = \frac{NH}{f_0}$$

Introduce dimensionless variables

$$(\widehat{u}, \widehat{v}) = U^{-1}(u, v) \quad \widehat{w} = \frac{L}{UH}w, \quad \widehat{f} = f_0^{-1}f, \quad \widehat{\phi} = \frac{\phi_1}{f_0 UL}, \quad \widehat{b} = \frac{H}{f_0 UL}b_1$$

Remark 4 We then have dimensionless equation of motion for Atmosphere or Ocean. [8]

$$\begin{aligned}
\text{Momentum Equation}: &\quad \mathrm{Ro}\frac{\mathrm{D}\widehat{\mathbf{u}}}{\mathrm{D}t} + \widehat{\mathbf{f}} \times \widehat{\mathbf{u}} = -\nabla\widehat{\phi} && (3.13)\\
\text{Buoyancy Equation}: &\quad \mathrm{Ro}\frac{\mathrm{D}\widehat{b}}{\mathrm{D}t} + \left(\frac{L_d}{L}\right)^2 \widehat{w} = 0 && (3.14)\\
\text{Hydrostatic Balance}: &\quad \frac{\partial\widehat{\phi}}{\partial\widehat{z}} = \widehat{b} && (3.15)\\
\text{Continuity}(\textbf{Atmosphere}): &\quad \widehat{\nabla}\cdot\widehat{\mathbf{u}} + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}\widehat{w}}{\partial\widehat{z}} = 0 && (3.16)\\
\text{Continuity}(\textbf{Oceanography}): &\quad \widehat{\nabla}\cdot\widehat{\mathbf{u}} = 0 && (3.17)
\end{aligned}$$

From now on I will drop the hats.

[8] *often people drop the hat for simplicity. However in the first derivation I keep everything with a hat.*

3.3 Quasi-Geostrophic Potential Vorticity Equation

We now derive the Quasi-Geostrophic Potential Vorticity Equations. Starting from asymptotic expansions [9]

[9] *hat is dropped*

$$\mathbf{u} = (u, v, w) = \mathbf{u}_g + \mathrm{Ro}\mathbf{u}_1 \quad \phi = \phi_0 + \mathrm{Ro}\phi_1 \quad b = b_0 + \mathrm{Ro}b_1$$

Here we consider the β effect.

$$\mathbf{f} = f_0\boldsymbol{k} + \beta y\boldsymbol{k}$$

Let $\epsilon = \mathrm{Ro}$. **Momentum Equation** :

The $O(1)$ momentum equation gives the Geostrophic balance

$$f_0\boldsymbol{k} \times \mathbf{u}_g = -\nabla\phi_0 \tag{3.18}$$

Immediately this implies

$$\nabla \cdot \mathbf{u}_g = 0$$

And $O(\epsilon)$ is

$$\frac{\mathrm{D}_g\mathbf{u}_g}{\mathrm{D}t} + \beta y\boldsymbol{k} \times \mathbf{u}_g + f_0\boldsymbol{k} \times \mathbf{u}_1 = -\nabla\phi_1 \tag{3.19}$$

Here D_g is the geostrophic material derivative

$$\mathrm{D}_g = \partial_t + \mathbf{u}_g \cdot \nabla$$

Mass Equation :
Since geostrophic velocity is divergent free then $O(1)$ mass equations is

$$\frac{\partial \tilde{\rho} w_0}{\partial z} = 0$$

and $O(\epsilon)$,

$$\nabla \cdot \mathbf{u}_1 + \frac{1}{\tilde{\rho}} \left(\frac{\partial \tilde{\rho} w_1}{\partial z} \right) = 0 \tag{3.20}$$

Buoyancy Equation :
$O(1)$:

$$\left(\frac{L_d}{L} \right)^2 w_0 = 0$$

and $O(\epsilon)$:

$$\frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \left(\frac{L_d}{L} \right)^2 w_1 = 0 \tag{3.21}$$

Now we take the **Curl** of Eq 3.19, note that

$$\nabla \times (\boldsymbol{k} \times \mathbf{u}_1) = \boldsymbol{k} \nabla \cdot \mathbf{u}_1 - \underbrace{u_1 \nabla \cdot \boldsymbol{k}}_{=0} + \underbrace{(\mathbf{u}_1 \cdot \nabla)\boldsymbol{k}}_{=0} - \underbrace{(\boldsymbol{k} \cdot \nabla)\mathbf{u}_1}_{=0} = \boldsymbol{k} \nabla \cdot \mathbf{u}_1$$

Define the geostrophic vorticity :

$$\xi_g = \nabla \times \mathbf{u}_g$$

Then Eq 3.19 becomes

$$\frac{\mathrm{D}_g \xi_g}{\mathrm{D}t} + \beta v_0 = -f_0 \nabla \cdot \mathbf{u}_1$$ [10]

Plug in Eq 3.20,

$$= \frac{f_0}{\tilde{\rho}} \frac{\partial \tilde{\rho} w_1}{\partial z}$$

Plug in Eq 3.21 to replace w_1

$$= -\frac{f_0}{\tilde{\rho}} \underbrace{\frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d} \right)^2 \frac{\mathrm{D}_g b_0}{\mathrm{D}t} \right)}_{\equiv I}$$

Now we examine I, normally in QG theory, we assume L_d is a constant. Thought from its definition, N cound actually depends on z. Since $\nabla \tilde{\rho} = 0$, we can put the first two terms into the material derivative. [11]

$$I = \partial_z \left(\tilde{\rho} \left(\frac{L}{L_d} \right)^2 \right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho} \left(\frac{L}{L_d} \right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} \equiv I_1 + I_2$$

[10] *this equation is already in* $\boldsymbol{k}$ *direction so the unit vector is dropped*

[11] *Here we use the fact that N is constant*

Let's go back to the Hydrostatic balance equation, [12] For $O(1)$:

[12] we haven't use it yet.

$$\frac{\partial \phi_0}{\partial z} = b_0 \quad + \quad f_0 \boldsymbol{k} \times \mathbf{u}_g = -\nabla \phi_0 \quad \Rightarrow \quad \boldsymbol{k} \times \frac{\partial \mathbf{u}_g}{\partial z} = -\frac{\nabla b_0}{f_0}$$

Then

$$I_2 = \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} = \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \left[\frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} + \underbrace{\partial_z \mathbf{u}_g \cdot \nabla b_0}_{=0}\right]$$

Therefore

$$I = \partial_z \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2\right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho} \left(\frac{L}{L_d}\right)^2 \frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} = \frac{\mathrm{D}_g}{\mathrm{D}t} \left[\frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2 b_0\right)\right]$$

Then evantually we have

$$\frac{\mathrm{D}_g}{\mathrm{D}t} \left[\zeta_g + f + \frac{f_0}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2 b_0\right)\right] = 0 \tag{3.22}$$

We can rewrite this equation using Streamfunction in a more simple form. Recall Eq 3.9, we have

$$b_0 = \frac{\partial \phi_0}{\partial z}$$

From Eq 3.18, the Kinematic Pressure can be expressed in terms of geostrophic streamfunction

$$u_g = -\partial_y \psi_g \quad v_g = \partial_x \psi_g \qquad \text{where } \boxed{\phi_0 = f_0 \psi_g} \quad \Rightarrow \quad \zeta_g = \nabla^2 \psi_g$$

Then Eq 3.22 becomes

$$\frac{\mathrm{D}_g}{\mathrm{D}t} \left[\nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d}\right)^2 \frac{\partial \psi_g}{\partial z}\right)\right] = 0 \tag{3.23}$$

Restore the dimensions

$$\frac{\mathrm{D}g}{\mathrm{D}t} \left[\nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z}\right)\right] = 0 \tag{3.24}$$

3.4 Ertel PV Conservation

Theorem 3.1 The Ertel PV, denoted usually as q or Q, is defined as:

$$Q = \frac{\boldsymbol{\omega}_a \cdot \nabla \psi}{\rho} \tag{3.25}$$

Where:

1. $\boldsymbol{\omega}_a = \nabla \times \mathbf{u} + 2\boldsymbol{\Omega}$ is the absolute vorticity.
2. ψ is a conserved scalar (like potential temperature θ or density).
3. ρ is the density.

The derivation starts from the Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p = -2\boldsymbol{\Omega} \times \mathbf{u}$$

Use the vector identity then take Curl

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}^2) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

We have

$$\frac{\partial \boldsymbol{\omega}_a}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}_a) = \nabla \times \left(-\frac{1}{\rho}\nabla p\right)$$

Then apply the vector identity

$$\nabla \times (\frac{1}{\rho}\nabla p) = \frac{1}{\rho^2}\nabla\rho \times \nabla p$$

We get

$$\frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = (\boldsymbol{\omega}_a \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}_a(\nabla \cdot \mathbf{u}) + \frac{\nabla\rho \times \nabla p}{\rho^2} \tag{3.26}$$

4 | Surface Quasi-Geostrophic Equations

The surface Quasi-Geostrophic Equation takes the problem to the next step, how could we retrive the interior motion from surface measurements such as SSH and SST. Recall the Buoyancy Equation

$$\frac{\mathrm{D}b_1}{\mathrm{D}t} + wN^2 = 0 \tag{4.1}$$

At the surface, $z = \eta$, the boundary condition yeilds that $w = 0$. We denote the surface buoyancy as b_s and surface velocity $\mathbf{u}_s$ Then

$$\frac{\partial b_s}{\partial t} + \mathbf{u}_s \cdot \nabla b_s = 0$$

and

$$b_s = f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0}$$

Explanation 1

The critical principle of SQG is to view surface buoyancy as a PV sheet. Since

$$\int_0^{\epsilon} f + \nabla^2\psi_g + \frac{f_0^2}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\frac{\tilde{\rho}}{N^2}\frac{\partial \psi}{\partial z}\right) \mathrm{d}z = 0$$

Then we impose a boundary condition

$$\frac{\partial \psi}{\partial z}\Big|_{z=\epsilon} = \frac{b_s}{f_0} \quad \Rightarrow \quad \int_0^{\epsilon} \frac{b_s}{f_0} = \frac{\partial \psi}{\partial z}\Big|_0^{\epsilon}$$

Compare the latter witht the integral, by defining

$$q_{\mathrm{SQG}} = \nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) + \frac{b_s}{f_0} \delta(z)$$

We have

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \frac{\partial \psi}{\partial z} = 0$$

Then the surface buoyancy appears in the QGPV equation naturally, it is as if adding an additional PV sheet at the surface. This inspires us to seperate the surface induced dynamics and interior dynamics.

Interior Dynamics

$$\begin{cases} q & = \nabla^2 \psi + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) \\ f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0} & = 0 \\ \frac{\mathrm{D}q}{\mathrm{D}t} & = 0 \end{cases}$$

4.1 Surface Buoyancy Induced Dynamics

Surface Dynamics, this is the Surface Quasi-Geostrophic Dynamics:

$$\begin{cases} q & = \nabla^2 \psi + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) = 0^{13} \\ f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0} & = b_s \\ \frac{\mathrm{D}b_s}{\mathrm{D}t} & = 0 \end{cases}$$

[13] *no interior Potential vorticity*

The key assumption for SQG theories is that all the Potential vorticity is injected into the system by surface buoyancy. [14]

[14] *surface buoyancy is actually first order, so it is also called surface buoyancy anomoly in some context.*

The horizontal Fourier coefficients of the solution is

$$\hat{\Phi}^0(k, l, z) = A e^{\mu z}$$

Where

$$\mu = \frac{NK}{f} \sqrt{\mathrm{Bu}} \qquad K^2 = k^2 + l^2$$

4.2 Retriving Vertical Velocity

5 | QG^{+1} Model

The derivation starts from the same set of equations above. We set

$$\epsilon = \mathrm{Ro} = \frac{U}{fL} \qquad \mathrm{Bu} = \left(\frac{NH}{fL}\right)^2$$

The Derivation starts from dimensional form of Eq 3.14, 3.17, 3.15 and 3.13. The Ertel Potential Vorticity is conserved. In this case, the conserved quantity is the **total Buoyancy** since Eq 3.14 we define it as

$$b_{tot} = \mathrm{Bu} \cdot z + \mathrm{Ro} \cdot b$$

$$b_{\text{tot}} = N^2 z + b$$

and

$$\boldsymbol{\omega}_a = \nabla_3 \times (u, v, 0) + f\hat{z} = (-v_z, u_z, f + \zeta)$$

Then

$$Q = \underbrace{fN^2}_{\text{Background}} + \underbrace{(N^2\zeta + fb_z)}_{\text{Linear terms, QGPV}} + \underbrace{(\zeta b_z - v_z b_x + u_z b_y)}_{\text{Nonlinear terms}} \tag{5.1}$$

and

$$q_{\text{QG}} = N^2\zeta + fb_z$$

Use the vector identity

$$\nabla_3 \cdot (\boldsymbol{\omega} b) = \boldsymbol{\omega} \cdot \nabla_3 b + \underbrace{b(\nabla \cdot \boldsymbol{\omega})}_{=0, \boldsymbol{\omega} = \nabla \times \cdot}$$

We note that

$$\zeta b_z - v_z bx + u_z b_y = \boldsymbol{\omega} \cdot \nabla_3 b$$

where

$$\boldsymbol{\omega} = (-v_z, u_z, \zeta) = (-v_z, u_z, v_x - u_y)$$

We apply the scalings

$$\zeta \sim \frac{U}{L} \quad b \sim \frac{fUL}{H} \quad v_z, u_z \sim \frac{U}{H}$$

Then Eq 5.1 becomes

$$Q = 1 + \epsilon \left(\zeta + \frac{b_z}{\mathrm{Bu}} + \frac{\epsilon}{\mathrm{Bu}}(\zeta b_z - v_z b_x + u_z b_y)\right) \tag{5.2}$$

or in a more compact way, again we drop the hat for dimensionless variables for simplicity.

$$Q = 1 + \epsilon \left(\zeta + \frac{b_z}{\mathrm{Bu}} + \frac{\epsilon}{\mathrm{Bu}} \nabla \cdot (\boldsymbol{\omega} b)\right) \tag{5.3}$$

or express in the form of

$$Q = 1 + \epsilon q$$

where

$$q = \zeta + \frac{b_z}{\mathrm{Bu}} + \frac{\epsilon}{\mathrm{Bu}} \nabla \cdot (\boldsymbol{\omega} b) \tag{5.4}$$

Explanation 2

Since here N is constant and we assume the Boussinesq Equation, the third term in Eq 3.22 becomes,

$$\frac{f_0}{\rho}\rho\frac{\partial b}{\partial z}\left(\frac{fL}{NH}\right)^2 \sim f b_z \frac{1}{\mathrm{Bu}}$$

The only difference here is the third second order ageostrophic quadratic correction we normally ignore in classical QG theories.

5.1 QG^{+1} Vector Field

In order to facilitate asymptotic approximation, the velocity field $\mathbf{u}$ are writen as the Curl of a vector field $\boldsymbol{A}$. [15]. One convention is writing

$$\boldsymbol{A} = (-G, F, \Phi) \tag{5.5}$$

[15] Changing three variables in velocity to $\boldsymbol{A}$

and

$$\begin{aligned} \mathbf{u} &= \nabla \times \boldsymbol{A} \\ u &= -\Phi_y - F_z \\ v &= \Phi_x - G_z \\ w &= F_x + G_y \end{aligned}$$

The Horizontal Vorticity in vector potential form is

$$\zeta = v_x - u_y = \Phi_{xx} - G_{zx} + \Phi_{yy} + F_{zy} = \nabla^2\Phi + F_{zy} - G_{zx} \tag{5.6}$$

However, this $\boldsymbol{A}$ isn't uniquely correspond to a velocity field. Since Gradient is Curl free, $\mathbf{u}$ admits a gauge freedom in that the transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla_3\Gamma$$

Left $\mathbf{u}$ unchange. Now instead of assuming $\boldsymbol{A}$ is divergent free. [16] We fix the buoyancy to be roughly scaled version of $\nabla_3 \cdot \boldsymbol{A}$.

[16] if $\boldsymbol{A}$ is divergent free, then this is adding an additioal information to the system and $\boldsymbol{A}$ can be uniquely determined.

$$\nabla_3 \cdot \boldsymbol{A} = -G_x + F_y - \Phi_z \quad \Rightarrow \quad b = \Phi_z + \mathrm{Bu}(G_x - F_y)$$

The advantage can be seen by calculating the QGPV Eq 5.3 (**Dimensionless form**) :

$$q_{\mathrm{QG}} = \zeta + \frac{b_z}{\mathrm{Bu}} = \nabla^2\Phi + F_{zy} - G_{zx} + \frac{1}{\mathrm{Bu}}\Phi_{zz} + (G_{xz} - F_{yz}) = (\Phi_{xx} + \Phi_{yy}) + \frac{\Phi_{zz}}{\mathrm{Bu}} = \mathcal{L}\Phi \tag{5.7}$$

Where

$$\mathcal{L} = \nabla^2 + \frac{1}{\mathrm{Bu}}\partial_{zz} \tag{5.8}$$

Then Φ is the only component related to the QGPV compare to dependence on all three component of velocity in the classical QGPV. When is the Buoyancy unchanged?

Explanation 3

Suppose we add a gradient to the original Vector Potential

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla_3\Gamma$$

Then

$$G \to G - \Gamma_x \quad F \to F + \Gamma_y \to \Phi \to \Phi - \Gamma_z$$

And

$$b_{new} = \underbrace{[\Phi_z + \mathrm{Bu}(G_x - F_y)]}_{b_{origin}} - \underbrace{[\Gamma_{zz} + \mathrm{Bu}(\Gamma_{xx} + \Gamma_{yy})]}_{b_{change}}$$

So in order to have $b_{new} = b_{origin}$. We have

$$b_{change} = \Gamma_{zz} + \mathrm{Bu}(\Gamma_{xx} + \Gamma_{yy}) \Rightarrow \boxed{\mathcal{L}(\Gamma) = 0} \tag{5.9}$$

Some Discussions

1. **In a triply-periodic domain** :
   Eq 5.9 implies that Γ is constant in the domain.
2. **Rigid Lid and flat bottom** :
   since $w = 0$ at upper and lower boundary, we have

$$F_x + G_y = 0$$

5.2 Evolution and Inversion Equations of QG^{+1}

Assume [17]

$$G^0 = F^0 = 0$$

and

$$\epsilon \ll 1 \qquad \mathrm{Bu} \sim O(1)$$

Then expand A asymptotically we have

$$u = -\Phi^0_y - \epsilon(\Phi^1_y + F^1_z) \tag{5.10a}$$
$$v = \Phi^0_x + \epsilon(\Phi^1_x - G^1_z) \tag{5.10b}$$
$$w = 0 + \epsilon(F^1_x + G^1_y) \tag{5.10c}$$
$$b = \Phi^0_z + \epsilon\left(\Phi^1_z + \mathrm{Bu}(G^1_x - F^1_y)\right) \tag{5.10d}$$

Evaluate Eq 3.23 we have up to $O(\epsilon)$.

$$\begin{aligned} q &= \underbrace{(\nabla^2\Phi^0 + \nabla^2\Phi^1 - G^1_{zx} + F^1_{zy})}_{\text{This is } \zeta} + \left[\frac{1}{\mathrm{Bu}}\Phi^0_{zz} + \frac{\epsilon}{\mathrm{Bu}}\left(\Phi^1_{zz} + \mathrm{Bu}(G^1_{xz} - F^1_{yz})\right)\right] \\ &+ \frac{\epsilon}{\mathrm{Bu}}\underbrace{\left[-\Phi^0_{xz}, -\Phi^0_{yz}, \nabla^2\Phi^0\right]}_{\text{This is } \omega=(-v_z,u_z,\zeta)} \cdot \underbrace{\left[\Phi^0_{zx}, \Phi^0_{zy}, \Phi^0_{zz}\right]}_{\text{this is } \nabla_3 b} \\ &= \boxed{\mathcal{L}(\Phi^0) + \epsilon\mathcal{L}(\Phi^1) + \frac{\epsilon}{\mathrm{Bu}}\Big(-|\nabla\Phi^0_z|^2 + \Phi^0_{zz}\nabla^2\Phi^0\Big) + O(\epsilon^2)} \end{aligned} \tag{5.11}$$

The surface Buoyancy from

$$b^t = \Phi^0_z\Big|_{z=0} + \Phi^1_z\Big|_{z=0} + O(\epsilon^2) \tag{5.12a}$$
$$b^b = \Phi^0_z\Big|_{z=-H} + \Phi^1_z\Big|_{z=-H} + O(\epsilon^2) \tag{5.12b}$$

[17] *Why? First order is geostrophic balance?*

To complete the inversion, we need to relate ageostrophic vertical streamfunctions F^1 and G^1 to Φ^0. We start with the first order primitive equations

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv^1 = -p^1_x \tag{5.13a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu^1 = -p^1_y \tag{5.13b}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + \mathrm{Bu}N^2 w^1 = 0 \tag{5.13c}$$

$$p^1_z = b^1 \tag{5.13d}$$

Take the difference of z-derivative of f times Eq 5.13b and x-derivetive Eq 5.13c we derive

$$\mathcal{L}(F^1) = \frac{2}{\mathrm{Bu}} J(\Phi^0_z, \Phi^0_x) \tag{5.14}$$

and similarly

$$\mathcal{L}(G^1) = \frac{2}{\mathrm{Bu}} J(\Phi^0_z, \Phi^0_y) \tag{5.15}$$

6 | SQG^{+1} Model

From the Geostrophic Balance Equation we have

$$\nabla^2 p^0 = \zeta^0$$

To obtain the next order balance equation, we use Eq 5.10d, 5.14 and 5.15.

$$\begin{aligned}
\nabla^2 b^1 - \zeta_z^1 &= \left(\nabla^2 \Phi_z + \mathrm{Bu}\nabla^2 (G_x - F_y)\right) - \nabla^2 \Phi_z - \partial_{zz}(G_x - F_y) \\
&= \mathrm{Bu} \cdot \mathcal{L}(G_x - F_y) \\
&\quad \text{(Here apply Eq 5.14 and Eq 5.15)} \\
&= 2\partial_z J(\Phi_x^0, \Phi_y^0)
\end{aligned} \tag{6.1}$$

Recall that $p_z = b$ by definition then we have

$$\boxed{\nabla^2 p^1 - f\zeta^1 = 2J(\Phi_x^0, \Phi_y^0)} \tag{6.2}$$

Explanation 4

> The RHS is called the cyclogeostrophic term because it represents the centrifugal force generated by the curvature of the fluid flow. However, real ocean/atmospheric eddies are curved (swirling). When fluid moves in a curve, it experiences a Centrifugal Force (V^2/R).Gradient Wind Balance (Cyclogeostrophic): Coriolis Force + Centrifugal Force ≈ Pressure Gradient Force.

This model captures the ageostrophic at first order and the term on the right hand side represents the **cyclogeostrophic correction**. In SQG, interior potential vorticity is 0. Then from Eq 5.11, since $q^1 = 0$. Then

$$\mathcal{L}(\Phi^1) = \frac{1}{\mathrm{Bu}}\left(|\nabla\Phi_z^0|^2 - \Phi_{zz}^0 \nabla^2 \Phi^0\right)$$

Using $\mathcal{L}(\Phi^0) = 0$. We have

$$\mathcal{L}(\Phi^1) = \frac{1}{\mathrm{Bu}}\left(|\nabla\Phi_z^0|^2 + \frac{1}{\mathrm{Bu}}\Phi_{zz}^0 \Phi_{zz}^0\right) \tag{6.3}$$

Together with Eq 5.14, Eq 5.15 and boundary conditions.

$$\Phi_z^1 = F^1 = G^1 = 0 \qquad \text{at } z = 0$$

Further we assume all potential vanish at infinity.

6.1 Φ^0 Inversion

To get the Φ^0 from SSH data, we do the standard inversion as in the Quasi-Geostrophic Balance Model.

The first order potential is geostrophic. And in the Surface Geostrophic Model, the interior

PV is 0. So we have the Governing equation

$$\mathcal{L}(\Phi^0) = 0 \qquad f\Phi^0_z = b$$

In the Fourier space, since Φ vanish as $z \to -\infty$. Then the Horizontal Fourier modes are

$$\widehat{\Phi}^0 = \frac{\hat{b}^{0,t}}{f\mu} e^{\mu z}$$

With constant

$$\mu = \sqrt{\mathrm{Bu}}\frac{NK}{f} \qquad K = \sqrt{m^2 + l^2}$$

6.2 Higher Order Potential Inversion

Now the Φ^0 is known already.

Explanation 5

For this system, the goal is to know Φ^1, G^1 and F^1. We recall that first order potential $G^0 = F^0 = 0$. And Φ^0 can be obtain from geostrophic balance. This is a system of decoupled linear elliptic (Poisson) equations.

1. Decoupled: You can solve for Φ^1, F^1, and G^1 independently of each other.
2. Linear/Poisson: Each equation takes the form $\mathcal{L}(\text{Potential}) = \text{Source Term}$.
3. Dependence: The "Source Terms" on the right-hand side (RHS) are known quantities derived entirely from the zeroth-order solution Φ^0 (which is obtained from the observed SSH/buoyancy)

The inversion is carried on by decomposing potential into interior and surface part. The interior part satisfies the main equations but ignore the boundary condition. The surface part solves the homogeneous problem and but corrects the boundary conditions. For Φ^1, we first notice that

$$\Phi^1_{int} = \frac{1}{2\mathrm{Bu}}\Phi^0_z\Phi^0_z \tag{6.4}$$

Is an analytic solution to Eq 6.3. Then it remains to solve the surface part given by

$$\mathcal{L}(\Phi^1_{sur}) = 0 \tag{6.5}$$

$$\Phi^{1,t}_{sur,z} = C_b - \partial_z\Phi^{1,t}_{int} = C_b - \frac{1}{\mathrm{Bu}}\Phi^{0,t}_z\Phi^{0,t}_{zz} \tag{6.6}$$

This problem is much more familiar, exactly the same problem when solving first order potential Φ^0. The Fourier modes are [18]

[18] C_b is added

$$\widehat{\Phi}^1_{sur} = \left(C_b - \partial_z\Phi^{1,t}_{int,z}\right)\frac{1}{\mu}e^{\mu z} = \left(C_b - \frac{1}{\mathrm{Bu}}\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}\right)\frac{1}{\mu}e^{\mu z}$$

The analytic solution of Φ_{int} is given by Eq 6.4 already. So the total Fourier mode of Φ is

$$\widehat{\Phi}^1 = \widehat{\Phi}^1_{sur} + \widehat{\Phi}^1_{int} \tag{6.7}$$

Similarly, by setting

$$F^1_{int} = \frac{1}{\mathrm{Bu}}\Phi^0_y\Phi^0_z$$

We see that

$$
\begin{aligned}
\mathcal{L}(F^1) &= \frac{1}{\text{Bu}}\left(\underbrace{\mathcal{L}(\Phi^0_y)\Phi^0_z}_{=0 \text{ since } \mathcal{L}(\Phi_0)=0} + \underbrace{\mathcal{L}(\Phi^0_z)\Phi^0_y}_{\text{same}} + 2(\nabla\Phi^0_y \cdot \nabla\Phi^0_z) + \underbrace{\frac{2}{\text{Bu}}\Phi_{yz}\Phi_{zz}}_{\text{rewrite using } \mathcal{L}(\Phi^0)=0}\right) \\
&= \frac{1}{\text{Bu}}\Big(2(\Phi^0_{xy}\Phi^0_{xz} + \Phi^0_{yy}\Phi^0_{zy}) - 2\Phi_{yz}(\Phi^0_{xx} + \Phi^0_{yy})\Big) \\
&= \frac{2}{\text{Bu}} J(\Phi^0_z, \Phi^0_x)
\end{aligned}
$$

This match with Eq 5.14. Then similar as Φ^1. The total Fourier mode is the sum of interior part and surface part. Where surface part is set to match the boundary condition. THe equations are

$$
\mathcal{L}(F^1_{int}) = \frac{2f}{\text{Bu}} J(\Phi^0_z, \Phi^0_x) \quad \mathcal{L}(F^1_{sur}) = 0 \ \& \ F^0_{sur}(z=0) = -F_{int}(z=0)
$$

$$
\widehat{F^1} = \frac{1}{\text{Bu}}\left(\widehat{\Phi^0_y\Phi^0_z} - \widehat{\Phi^{0,t}_y\Phi^{0,t}_z} e^{\mu z}\right) \tag{6.8}
$$

Similarly we can set [19]

[19] *Do the calculation for G here later.*

$$
G^1_{int} = -\frac{1}{\text{Bu}}\Phi^0_x\Phi^0_z
$$

and

$$
\widehat{G^1} = -\frac{1}{\text{Bu}}\left(\widehat{\Phi^0_x\Phi^0_z} - \widehat{\Phi^{0,t}_x\Phi^{0,t}_z} e^{\mu z}\right) \tag{6.9}
$$

We don't see the μ here on the denominator since the surface part for F and G has Dirichlet Boundary condition [20] while Φ has Neumann Boundary Condition.

[20] *condition on the function*

The inversion of surface horizontal velocity can be calculate as follow :

The derivative is taken before the Fourier Transform, since *int* part has analytic solution.

$$
\begin{aligned}
u &= \sum\Big(-\hat{\Phi}^{0,t}_y - \epsilon(\hat{\Phi}^{1,t}_y + \hat{F}^{1,t}_z)\Big)e^{i(kx+ly)} \\
&= -\epsilon \cdot \frac{1}{\text{Bu}}\left[\partial_y\left(\underbrace{\frac{\Phi^{0,t}_z\Phi^{0,t}_z}{2}}_{\Phi^{1,t}_{int}}\right) + \partial_z\left(\underbrace{\Phi^{0,t}_y\Phi^{0,t}_z}_{F^1_{int}}\right)\right] \\
&+ \sum\left[-\hat{\Phi}^{0,t}_y e^{i(kx+ly)} - \epsilon \cdot \frac{1}{\text{Bu}}\partial_y\left(\left(\underbrace{-\frac{\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}}{\mu}e^{\mu z}}_{\Phi^1_{sur}}\right)e^{i(kx+ly)}\right) - \epsilon \cdot \frac{1}{\text{Bu}}\partial_z\left(-\underbrace{\widehat{\Phi^{0,t}_y\Phi^{0,t}_z}e^{\mu z}}_{F^1_{sur}}\right)e^{i(kx+ly)}\right]
\end{aligned}
$$

Then take the derivative, then take the Fourier Transform we have

$$
\begin{aligned}
\hat{u}^t &= \hat{u}(z=0) \\
&= -\Phi^{0,t}_y - \epsilon \cdot \frac{1}{\text{Bu}}\left[\underbrace{\widehat{\Phi^{0,t}_y\Phi^{0,t}_{zz}} + 2\widehat{\Phi^{0,t}_{yz}\Phi^{0,t}_z}}_{\text{Interior Part}} - \underbrace{\mu\widehat{\Phi^{0,t}_y\Phi^{0,t}_{zz}} - \frac{il}{\mu}\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}}_{\text{Surface Part}}\right]
\end{aligned} \tag{6.10}
$$

Similarly

$$\begin{aligned} \hat{v}^t &= \hat{v}(z=0) \\ &= \Phi_x^{0,t} + \epsilon \cdot \frac{1}{\mathrm{Bu}} \left[\underbrace{2\widehat{\Phi_{xz}^{0,t}\Phi_z^{0,t}} + \widehat{\Phi_x^{0,t}\Phi_{zz}^{0,t}}}_{\text{Interior Part}} - \underbrace{\frac{ik}{\mu}\widehat{\Phi_z^{0,t}\Phi_{zz}^{0,t}} - \mu\widehat{\Phi_x^{0,t}\Phi_z^{0,t}}}_{\text{Surface Part}} \right]^{21} \end{aligned} \tag{6.11}$$

[21] *The sign here? Should be both minus sign perhaps?*

For the Vertical Velocity : Start in the primitive space

$$\begin{aligned} w &= \epsilon(F_x^1 + G_y^1) \\ &= \epsilon \frac{1}{\mathrm{Bu}} \sum \partial_x \left[\left(\widehat{\Phi_y^0\Phi_z^0} - \widehat{\Phi_y^{0,t}\Phi_z^{0,t}} e^{\mu z} \right) e^{i(kx+ly)} \right] - \partial_y \left[\left(\widehat{\Phi_x^0\Phi_z^0} - \widehat{\Phi_x^{0,t}\Phi_z^{0,t}} e^{\mu z} \right) e^{i(kx+ly)} \right] \end{aligned}$$

The Fourier Transform aren't function of x or y

$$= \epsilon \frac{1}{\mathrm{Bu}} \sum \left(ik\widehat{\Phi_y^0\Phi_z^0} - il\widehat{\Phi_x^0\Phi_z^0} \right) e^{i(kx+ly)} - \left(ik\widehat{\Phi_y^{0,t}\Phi_z^{0,t}} - il\widehat{\Phi_x^{0,t}\Phi_z^{0,t}} \right) e^{i(kx+ly)}$$

Taking x derivative to Fourier Component pulls down an additional fact of ik. Then

$$\hat{w} = \epsilon \frac{1}{\mathrm{Bu}} \tag{6.12}$$

7 | Simulation Set Up

In this section, I briefly describe the simulation set up from the most trivial case all the way up to the true problem. We start with a brief introduction about how the inversion process proceed.

7.1 Inversion Process (Primal Space)

From hydrostatic balance, the surface pressure is propotional to the surface sea height (SSH).

$$\eta = \frac{p\{z=0\}}{g} \tag{7.1}$$

In all the QG models, the pressure is expanded asymptotically in the form of

$$p_{total} \sim p^0 + \epsilon p^1 \tag{7.2}$$

The first order pressure is directly related to the first order potential Φ^0 by the formula

$$p^0 = f\Phi^0$$

As from geostrophic balance we have

$$p^0_y = -fu^0 = -f\partial_y \Phi^0 \qquad p^0_x = fv^0 = f\partial_x \Phi^0$$

Now we have

$$\eta^0 = \frac{f\Phi^0}{g}$$

Similarly from the first order correction we have

$$\eta^1 = \frac{p^1}{g}$$

So in general, considering the first order correction to the hydrostatic balance Eq 7.1, we have

$$\eta \sim \eta^0 + \epsilon\eta^1 = \frac{f\Phi^0}{g} + \epsilon\frac{p^1}{g} \tag{7.3}$$

This is the equation 40 in Ryan's note.

Now remember our goal is to use η to invert for Φ^0. What is the relationship between p^1 and Φ^0? The relationship is given by eq 6.2. We copy it here

$$\boxed{\nabla^2 p^1 - f\zeta^1 = 2J(\Phi^0_x, \Phi^0_y)}$$

Where J is the Jacobian operator. This is a non-linear relationship, recall that ζ^1 is related to the potentials via Eq 5.6. So

$$\nabla^2 p^1 = f\left(\nabla^2\Phi^1 + F^1_{zy} - G^1_{zx}\right) + 2J(\Phi^0_x, \Phi^0_y) \quad \rightarrow \quad \Phi^{0,s} + \epsilon\mathcal{N}(\Phi^{0,s}) = \eta(x,y)$$

Where $\mathcal{N}$ is a non-linear operator. Here the relationship between Φ^1, F^1 and G^1 are given by Eq 6.7, 6.8 and 6.9. Then the whole inversion problem is clear. Given $\eta(x,y)$ we wish to find a $\Phi^{0,s}$ [22]

[22] *given $\Phi^{0,s}$ we can reconstruct the 3D Φ already.*

7.2 Inversion Process (Spectral Space)

The method introduced in the previous section isn't very efficient. As all the problem is solved mainlyin the physical space, however, the inversion would be much easier if we do it in the spectral space as all the calculation here would be 2D and thereform much easier.

The key equation is still Eq 6.2.

$$\nabla^2 p^1 - f\zeta^1 = 2J(\Phi_x^0, \Phi_y^0)$$

We wish to obtain p^1 in the spectral space, then use Eq 7.3 to ge the surface SSH.

$$\eta \sim \frac{f}{g}\Phi^{0,s} + \epsilon\frac{1}{g}p^{1,s}$$

Since $p^{1,s}$ is related to the surface potential $\Phi^{0,s}$ nonlinearly, so we can write [23]

[23] *I use the upper-index s and t interchangeably all for surface data. I should change this in the future.*

$$\eta(x,y)^s = \frac{f}{g}\Phi^{0,s} + \frac{\epsilon}{g}\mathcal{N}(\Phi^{0,s})$$

No we just need a why to get this $\mathcal{N}$ operator.

7.2.1 First Order Potential

To get Φ^1, we use Eq 6.7.

$$\boxed{\hat{\Phi}^1 = \underbrace{\frac{1}{2\mathrm{Bu}}\widehat{\Phi_z^0\Phi_z^0}}_{\text{Fourier transform of interior part}} - \underbrace{\frac{1}{\mathrm{Bu}}\widehat{\Phi_z^{0,t}\Phi_{zz}^{0,t}}\frac{e^{\mu z}}{\mu} + \frac{C_b}{\mu}e^{\mu z}}_{\text{Fourier transform of surface part}}}$$

and to get G^1, similarly we use Eq 6.9.

$$\boxed{\widehat{G^1} = -\frac{1}{\mathrm{Bu}}\left(\widehat{\Phi_x^0\Phi_z^0} - \widehat{\Phi_x^{0,t}\Phi_z^{0,t}}e^{\mu z}\right)}$$

and to get F^1, use Eq 6.8 directly.

$$\boxed{\widehat{F^1} = \frac{1}{\mathrm{Bu}}\left(\widehat{\Phi_y^0\Phi_z^0} - \widehat{\Phi_y^{0,t}\Phi_z^{0,t}}e^{\mu z}\right)}$$

7.2.2 Surface Ageostrophic Vorticity

We will invert p^1 in the spectral space following Eq 6.2. The philosophy is that fourier transform is a linear operator, so whenever we see a product of two variables in the physical space, we need to complete this operation in the physical space first then do the fourier transform.

From Eq 5.6 we have

$$\begin{aligned}\widehat{\zeta^1} &= \widehat{\nabla^2\Phi^1} + \widehat{F^1_{yz}} - \widehat{G^1_{xz}}\,^{24} \\ &= -K^2\widehat{\Phi^1} + ik_y\partial_z\widehat{F^1} - ik_x\partial_z\widehat{G^1}\end{aligned}$$

[24] *∇^2 here is 2D Laplacian operator*

where $K^2 = k_x^2 + k_y^2$. We sepearate the derivation into interior part and surface part.

1. Interior part:
   (a) For Φ^1 its interior part is $\propto \Phi^0_z\Phi^0_z/2$. So applying Laplacian to it we have
   $$\|\nabla\Phi^0_z\|^2 + \Phi^0_z\nabla^2\Phi^0_z$$
   (b) From $F^1_{yz} - G^1_{xz}$, the interior part of F^1 is $\propto -\Phi^0_y\Phi^0_z$ and the interior part of G^1 is $\propto \Phi^0_x\Phi^0_z$.
   $$\nabla\Phi^0\cdot\nabla\Phi^0_{zz} + \nabla^2\Phi^0\cdot\Phi^0_{zz} + \underbrace{\|\nabla\Phi^0_z\|^2 + \Phi^0_z\nabla^2\Phi^0_z}_{\text{matches with interior part of } \Phi^1}$$
2. Surface part : The surface part is already an exponential decay in the vertical direction starting from the surface first order potential fourier transform.
   (a) For Φ^1. The fourier transform of the surface part is $\propto -\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}e^{\mu z}/\mu$. So taking the laplacian is equivalent to mulfiply by $-K^2$ in the spectral space. Evaluate at $z=0$, the exponential decay term is 1.
   $$\frac{1}{\mathrm{Bu}}\frac{K^2}{\mu}\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}}$$
   (b) For $F^1_{yz} - G^1_{xz}$, the surface part of $\hat{F}_1$ is $\propto -\widehat{\Phi^{0,t}_y\Phi^{0,t}_z}e^{\mu z}$ and $\propto \widehat{\Phi^{0,t}_x\Phi^{0,t}_z}e^{\mu z}$. For $\hat{G}_1$. Then the surface part evaluate at $z=0$ is
   $$-\left(ik_y\mu\widehat{\Phi^{0,t}_y\Phi^{0,t}_z} + ik_x\mu\widehat{\Phi^{0,t}_x\Phi^{0,t}_z}\right)$$

Formula 7.3.1

So to sum up we have
$$\boxed{\zeta^{1,s} = \frac{1}{\mathrm{Bu}}\left[I_1 + I_2 - I_3\right]} \tag{7.4}$$
where the three terms are defined as follows:
$$I_1 = \nabla\Phi^{0,t}\cdot\nabla\Phi^{0,t}_{zz} + \nabla^2\Phi^{0,t}\cdot\Phi^{0,t}_{zz} + 2\|\nabla\Phi^{0,t}_z\|^2 + 2\Phi^{0,t}_z\nabla^2\Phi^{0,t}_z \tag{7.5}$$
$$I_2 = \frac{K^2}{\mu}\widehat{\Phi^{0,t}_z\Phi^{0,t}_{zz}} \tag{7.6}$$
$$I_3 = ik_y\mu\widehat{\Phi^{0,t}_y\Phi^{0,t}_z} + ik_x\mu\widehat{\Phi^{0,t}_x\Phi^{0,t}_z} \tag{7.7}$$
Everything here is evaluated at the surface $z=0$.

7.2.3 Surface Cyclogeostrophic Correction

We also need the cyclogeostrophic correction, evaluate at the surface $z=0$.

Formula 7.7.1

The result is simple, since derivative and fourier transform commute, we have
$$\boxed{2\widehat{J(\Phi^{0,t}_x, \Phi^{0,t}_y)}} \tag{7.8}$$

7.3 Pre-Defined Velocity Fields

We start with a very simple case to play around with this model.

7.3.1 Toy Model Setup

The toy model start with a pre-defined potential Φ^0. I use a random generator here and make the spectrum follow a power law with slope -3, which is typical for 3D turbulence.

```
rng(42); % Set seed for reproducibility
% We produce a random $\Phi^0$ here.
k_peak = 4;
slope = -3; % This slope matches the energy cascade in 3D turbulance
phase = rand(N, N) * 2 * pi;
amplitude = (K ./ k_peak).^(slope) .* exp(-(K./k_peak).^2);

amplitude(1,1) = 0;
% Compute the hat
phi0_hat = amplitude .* exp(1i * phase);
% Inverse transform to get to the physical space
phi0_surf = real(ifft2(phi0_hat));
% Normalize
phi0_surf = phi0_surf / std(phi0_surf(:));
```

7.3.2 Forward Process to get the true velocity fields

Then some functions are defined for different purposes.

1. This function compute the 3D potential Φ^0 from the surface data. The fourier components has an exponential decay in the vertical direction.

   ```
       phi0_3d_true = derive_phi0_3d(phi0_surf, K, z, Bu);
   ```

2. This function compute all the other first order potential F^1, G^1 and Φ^1 using Eq 6.7, 6.8 and 6.9. A possion problem is solved in the spectral space.

   ```
       [F1_true, G1_true, Phi1_true] =
           calculate_higher_order(phi0_3d_true, K, kx, ky, z, Bu, N,
           nz);
   ```

3. This function computes the first order pressure p^1 using Eq 6.2.

   ```
       p1_true = solve_p1(f, dx, dz, kx, ky, z, Bu, Ro,
           phi0_3d_true, F1_true, G1_true, Phi1_true);
   ```

4. With all the data above, we can compute the true surface sea height using Eq 7.3.

   ```
       p1_surf = p1_true(:, :, end);
   ssh_true = phi0_surf + Ro * p1_surf;
   ```

   Here I use end because the z corredinate starts from the bottom to the top.

Now we have the surface sea height. This is where the inversion starts.

7.3.3 Inversion Process solving the Optimization Problem

The inversion process starts with η. We make an initial guess for Φ^0, in my code I use a zero initial Φ^0.

```
phi0_guess_flat = zeros(N, N);
```

Then use the function

```
cost_func = @(phi0_flat) sqg_cost_function(phi0_flat, f, ssh_true, K,
    kx, ky, z, Bu, Ro, N, nz, dx, dz);
```

To compute the cost. In this `sqg_cost_function` basically do the same thing as the forward process, compute the 3D potential Φ^0 first, then solve the Possion equation to get F^1, G^1, Φ^1, then compute p^1 and finally compute the SSH. The difference is that at the final step, we compute the difference between the computed SSH and the true SSH to obtain the cost.

```
    % Cost
    difference = ssh_guess - ssh_obs;
    cost = sum(difference(:).^2);
```

This is the returned value of that function. I then use a built in Matlab toolbox to solve the optimization problem.

```
num_iteration = 20;
options = optimoptions('fminunc', 'Display', 'iter', 'Algorithm',
    'quasi-newton', 'MaxIterations', num_iteration);

% Compute the cost function
cost_func = @(phi0_flat) sqg_cost_function(phi0_flat, f, ssh_true, K,
    kx, ky, z, Bu, Ro, N, nz, dx, dz);

% Run Optimization
tic;
try
    [phi0_opt_flat, fval] = fminunc(cost_func, phi0_guess_flat,
        options);
    phi0_surf_opt = reshape(phi0_opt_flat, N, N);
    disp('Optimization Complete.');
catch ME
    disp('Optimization failed or interrupted.');
    disp(ME.message);
    phi0_surf_opt = reshape(phi0_guess_flat, N, N); % Fallback
end
toc;
```

Then plot the results. [25]

add something here

[25] *The plot part is writen by Gemini*