Section 1

Quasi-Geostrophic Equations

In this section, I introduce some basic equations in QG theories. Building a foundation for SQG, eSQG and other variations introduced later. The analysis is already in a **stratified ocean**.

Subsection 1.1

Governing Equations

Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\frac{\nabla p}{\rho} \tag{1.1}$$

Mass Conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0 \tag{1.2}$$

We are in a **stratified ocean**. Breaking the total state variables into a "hydrostatic reference state" (which depends only on z) and a "dynamic perturbation" (which moves the fluid):

$$\rho = \tilde{\rho}(z) + \rho_1(x, y, z, t) \tag{1.3}$$

and

$$p = p_0(z) + p_1(x, y, z, t) \tag{1.4}$$

Then RHS of Eq 1.6 becomes

$$-\frac{1}{\rho}\nabla p_1 \sim -\frac{1}{\rho_0}\nabla p_1$$

Define the **Kinematic Presure**

$$\phi = \frac{p_1}{\rho_0} \tag{1.5}$$

Momentum Equation becomes

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\nabla \phi} \tag{1.6}$$

Hydrostatic balance is a state of equilibrium in a fluid where the upward force of pressure exactly balances the downward force of gravity.

$$-g\tilde{\rho} = \frac{\mathrm{d}p_0}{\mathrm{d}z} \tag{1.7}$$

In Ocean, we assume the velocity field is divergent free. THen the mass conservation yeilds

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot ((\tilde{\rho} + \rho_1)\mathbf{v}) = 0 \quad \Rightarrow \quad \boxed{\nabla \cdot (\tilde{\rho}\mathbf{v}) = 0}$$

Remark 1 Here we drop the $\partial_t \rho_1$ term under the **Anelastic assumption**. Essentially by eliminating this partial derivative, we assume the fluid is anelastic, so sound wave is not supported [1] in the system.

[1] *or less important*

Next we define the Buoyancy :

$$b = -g\frac{\rho}{\bar{\rho}_0} \tag{1.8}$$

Here $\bar{\rho}_0$ is a constant reference density. The the divergent free condition implies

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = 0} \tag{1.9}$$

And thus the Hydrostatic balance equation Eq 1.7 implies

$$\boxed{\frac{\partial \phi}{\partial z} = b} \tag{1.10}$$

All the Boxed Equation together is the **Hydrostatic Anelastic Equations for Stratified Flow**. If we consider the perturbation of Buoyancy

$$b = \tilde{b}(z) + b_1(x, y, z, t)$$

Expand Eq 1.9 can be writen as

$$\boxed{\frac{\mathrm{D}b_1}{\mathrm{D}t} + w\frac{\mathrm{d}b}{\mathrm{d}z} = 0}\,[2] \tag{1.11}$$

[2] *this is the more familiar buoyancy equation we see in lecture*

In a more familiar form we define

$$N^2 = \frac{\mathrm{d}b}{\mathrm{d}z} = -g\frac{\tilde{\rho}_z}{\bar{\rho}_0}$$

Which is the **Brunt Vasala Frequency**.

Subsection 1.2

Scaling Analysis

To simplify our equation, we introduce some scalings.

$$(x, y) \sim L, \quad (u, v) \sim U, \quad t \sim \frac{L}{U}, \quad z \sim H, \quad f \sim f_0$$

Introduce the **Rosbby Number**:

$$\mathrm{Ro} = \frac{U}{f_0 L} \tag{1.12}$$

Now let $\phi = \tilde{\phi}(z) + \phi_1(x, y, z, t)$. Then since the gradient in Eq 1.6 is horizontal, we can replace ϕ by ϕ_1. Now suppose

$$|\mathbf{f} \times \mathbf{u}| \sim |\nabla \phi_1|$$

From Hydrostatic balance we have

$$b \sim \frac{f_0 U L}{H}$$

Then

$$\frac{(\partial b'/\partial z)}{N^2} \sim Ro\frac{L^2}{L_d^2}$$

Where we have the deformation radius as a function of z.

$$L_d = \frac{NL}{f_0}$$

Introduce dimensionless variables

$$(\widehat{u}, \widehat{v}) = U^{-1}(u, v) \quad \widehat{w} = \frac{L}{UH}w, \quad \widehat{f} = f_0^{-1}f, \quad \widehat{\phi} = \frac{\phi'}{f_0 U L}, \quad \widehat{b} = \frac{H}{f_0 U L}b'$$

Remark 2 We then have dimensionless equation of motion for Anelastic Assumption [3]

$$\text{Momentum Equation}: \quad \text{Ro}\frac{\mathrm{D}\widehat{\mathbf{u}}}{\mathrm{D}t} + \widehat{\mathbf{f}} \times \widehat{\mathbf{u}} = -\nabla\widehat{\phi} \tag{1.13}$$

$$\text{Buoyancy Equation}: \quad \text{Ro}\frac{\mathrm{D}\widehat{b}}{\mathrm{D}t} + \left(\frac{L_d}{L}\right)^2 \widehat{w} = 0 \tag{1.14}$$

$$\text{Hydrostatic Balance}: \quad \frac{\partial\widehat{\phi}}{\partial\widehat{z}} = \widehat{b} \tag{1.15}$$

$$\text{Continuity}: \quad \widehat{\nabla} \cdot \widehat{\mathbf{u}} + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}\widehat{w}}{\partial\widehat{z}} = 0 \tag{1.16}$$

[3] *often people drop the hat for simplicity. However in the first derivation I keep everything with a hat.*

Subsection 1.3

Quasi-Geostrophic Potential Vorticity Equation

Section 2

Surface Quasi-Geostrophic Balance

Section 3

QG^{+1} Model