SECTION 1

Quasi-Geostrophic Equations

In this section, I introduce some basic equations in QG theories. Building a foundation for SQG, eSQG and other variations introduced later. The analysis is already in a **stratified ocean**.

SUBSECTION 1.1

Governing Equations

Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\frac{\nabla p}{\rho} \tag{1.1}$$

Mass Conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0 \tag{1.2}$$

We are in a **stratified ocean**. Breaking the total state variables into a "hydrostatic reference state" (which depends only on z) and a "dynamic perturbation" (which moves the fluid):

$$\rho = \tilde{\rho}(z) + \rho_1(x, y, z, t) \tag{1.3}$$

and

$$p = p_0(z) + p_1(x, y, z, t) \tag{1.4}$$

Then RHS of Eq 1.6 becomes

$$-\frac{1}{\rho}\nabla p_1 \sim -\frac{1}{\rho_0}\nabla p_1$$

Define the **Kinematic Pressure**

$$\phi = \frac{p_1}{\rho_0} \tag{1.5}$$

Momentum Equation becomes

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\nabla \phi} \tag{1.6}$$

Hydrostatic balance is a state of equilibrium in a fluid where the upward force of pressure exactly balances the downward force of gravity.

$$-g\tilde{\rho} = \frac{\mathrm{d}p_0}{\mathrm{d}z} \tag{1.7}$$

1.1.1 Continuity Equation Approximatio

The genearl continuity equation can always be expressed as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot u = 0$$

In Ocean, we assume that the fluid is **Incompressible** and wrote

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \qquad \nabla \cdot \mathbf{u} = 0$$

In Atmosphere, we use the **Anelastic Assumption** Then the mass conservation yeilds

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot ((\tilde{\rho} + \rho_1)\mathbf{v}) = 0 \quad \Rightarrow \quad \boxed{\nabla \cdot (\tilde{\rho}\mathbf{v}) = 0}$$

Remark 1 Here we drop the $\partial_t \rho_1$ term under the **Anelastic assumption**. Essentially by eliminating this partial derivative, we assume the fluid is anelastic, so sound wave is not supported [1] in the system. However, this approximation is normally used for **deep atmospheric or stellar convertion** where density $\tilde{\rho}$ changes significantly with height.

[1] *or less important*

For Oceanography, Next we define the Buoyancy :

$$b = -g\frac{\rho}{\bar{\rho}_0} \tag{1.8}$$

Here $\bar{\rho}_0$ is a constant reference density. The the divergent free condition implies

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = 0} \tag{1.9}$$

And thus the Hydrostatic balance equation Eq 1.7 implies

$$\boxed{\frac{\partial \phi}{\partial z} = b} \tag{1.10}$$

All the Boxed Equation together is the **Hydrostatic Anelastic Equations for Stratified Flow**. If we consider the perturbation of Buoyancy

$$b = \tilde{b}(z) + b_1(x, y, z, t)$$

Expand Eq 1.9 can be writen as

$$\boxed{\frac{\mathrm{D}b_1}{\mathrm{D}t} + w\frac{\mathrm{d}b}{\mathrm{d}z} = 0}^2 \tag{1.11}$$

[2] *this is the more familiar buoyancy equation we see in lecture*

In a more familiar form we define

$$N^2 = \frac{\mathrm{d}b}{\mathrm{d}z} = -g\frac{\tilde{\rho}_z}{\bar{\rho}_0}$$

Which is the **Brunt Vasala Frequency**.

Subsection 1.2

Scaling Analysis

To simplify our equation, we introduce some scalings.

$$(x, y) \sim L, \quad (u, v) \sim U, \quad t \sim \frac{L}{U}, \quad z \sim H, \quad f \sim f_0$$

Introduce the **Rosbby Number**:

$$\mathrm{Ro} = \frac{U}{f_0 L} \tag{1.12}$$

Now let $\phi = \tilde{\phi}(z) + \phi_1(x, y, z, t)$. Then since the gradient in Eq 1.6 is horizontal, we can replace ϕ by ϕ_1. Now suppose

$$|\mathbf{f} \times \mathbf{u}| \sim |\nabla \phi_1|$$

From Hydrostatic balance we have

$$b \sim \frac{f_0 U L}{H}$$

Then

$$\frac{(\partial b'/\partial z)}{N^2} \sim \frac{f_0 U L}{(HN)^2} \sim Ro\frac{L^2}{L_d^2}$$

Where we have the deformation radius as a function of z.

$$L_d = \frac{NH}{f_0}$$

Introduce dimensionless variables

$$(\widehat{u}, \widehat{v}) = U^{-1}(u, v) \quad \widehat{w} = \frac{L}{UH}w, \quad \widehat{f} = f_0^{-1}f, \quad \widehat{\phi} = \frac{\phi_1}{f_0UL}, \quad \widehat{b} = \frac{H}{f_0UL}b_1$$

Remark 2 We then have dimensionless equation of motion for Atmosphere or Ocean. [3]

$$\text{Momentum Equation}: \quad \text{Ro}\frac{\text{D}\widehat{\mathbf{u}}}{\text{D}t} + \widehat{\mathbf{f}} \times \widehat{\mathbf{u}} = -\nabla\widehat{\phi} \tag{1.13}$$

$$\text{Buoyancy Equation}: \quad \text{Ro}\frac{\text{D}\widehat{b}}{\text{D}t} + \left(\frac{L_d}{L}\right)^2 \widehat{w} = 0 \tag{1.14}$$

$$\text{Hydrostatic Balance}: \quad \frac{\partial\widehat{\phi}}{\partial\widehat{z}} = \widehat{b} \tag{1.15}$$

$$\text{Continuity}(\textbf{Atmosphere}): \quad \widehat{\nabla} \cdot \widehat{\mathbf{u}} + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}\widehat{w}}{\partial\widehat{z}} = 0 \tag{1.16}$$

$$\text{Continuity}(\textbf{Oceanography}): \quad \widehat{\nabla} \cdot \widehat{\mathbf{u}} = 0 \tag{1.17}$$

From now on I will drop the hats.

[3] *often people drop the hat for simplicity. However in the first derivation I keep everything with a hat.*

Subsection 1.3

Quasi-Geostrophic Potential Vorticity Equation

We now derive the Quasi-Geostrophic Potential Vorticity Equations. Starting from asymptotic expansions [4]

$$\mathbf{u} = (u, v, w) = \mathbf{u}_g + \text{Ro}\mathbf{u}_1 \quad \phi = \phi_0 + \text{Ro}\phi_1 \quad b = b_0 + \text{Ro}b_1$$

[4] *hat is dropped*

Here we consider the β effect.

$$\mathbf{f} = f_0\boldsymbol{k} + \beta y\boldsymbol{k}$$

Let $\epsilon = \text{Ro}$. **Momentum Equation** :

The $O(1)$ momentum equation gives the Geostrophic balance

$$f_0\boldsymbol{k} \times \mathbf{u}_g = -\nabla\phi_0 \tag{1.18}$$

Immediately this implies

$$\nabla \cdot \mathbf{u}_g = 0$$

And $O(\epsilon)$ is

$$\frac{\text{D}_g\mathbf{u}_g}{\text{D}t} + \beta y\boldsymbol{k} \times \mathbf{u}_g + f_0\boldsymbol{k} \times \mathbf{u}_1 = -\nabla\phi_1 \tag{1.19}$$

Here D_g is the geostrophic material derivative

$$\text{D}_g = \partial_t + \mathbf{u}_g \cdot \nabla$$

Mass Equation :

Since geostrophic velocity is divergent free then $O(1)$ mass equations is

$$\frac{\partial\tilde{\rho}w_0}{\partial z} = 0$$

and $O(\epsilon)$,

$$\nabla \cdot \mathbf{u}_1 + \frac{1}{\tilde{\rho}} \left(\frac{\partial \tilde{\rho} w_1}{\partial z} \right) = 0 \tag{1.20}$$

Buoyancy Equation :

$O(1)$:

$$\left(\frac{L_d}{L} \right)^2 w_0 = 0$$

and $O(\epsilon)$:

$$\frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \left(\frac{L_d}{L} \right)^2 w_1 = 0 \tag{1.21}$$

Now we take the **Curl** of Eq 1.19, note that

$$\nabla \times (\boldsymbol{k} \times \mathbf{u}_1) = \boldsymbol{k} \nabla \cdot \mathbf{u}_1 - \underbrace{u_1 \nabla \cdot \boldsymbol{k}}_{=0} + \underbrace{(\mathbf{u}_1 \cdot \nabla) \boldsymbol{k}}_{=0} - \underbrace{(\boldsymbol{k} \cdot \nabla) \mathbf{u}_1}_{=0} = \boldsymbol{k} \nabla \cdot \mathbf{u}_1$$

Define the geostrophic vorticity :

$$\xi_g = \nabla \times \mathbf{u}_g$$

Then Eq 1.19 becomes

$$\frac{\mathrm{D}_g \xi_g}{\mathrm{D}t} + \beta v_0 = -f_0 \nabla \cdot \mathbf{u}_1{}^5$$

[5] *this equation is already in* $\boldsymbol{k}$ *direction so the unit vector is dropped*

Plug in Eq 1.20,

$$= \frac{f_0}{\tilde{\rho}} \frac{\partial \tilde{\rho} w_1}{\partial z}$$

Plug in Eq 1.21 to replace w_1

$$= -\frac{f_0}{\tilde{\rho}} \underbrace{\frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d} \right)^2 \frac{\mathrm{D}_g b_0}{\mathrm{D}t} \right)}_{\equiv I}$$

Now we examine I, normally in QG theory, we assume L_d is a constant. Thought from its definition, N cound actually depends on z. Since $\nabla \tilde{\rho} = 0$, we can put the first two terms into the material derivative. [6]

[6] *Here we use the fact that* N *is constant*

$$I = \partial_z \left(\tilde{\rho} \left(\frac{L}{L_d} \right)^2 \right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho} \left(\frac{L}{L_d} \right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} \equiv I_1 + I_2$$

Let's go back to the Hydrostatic balance equation, [7] For $O(1)$:

[7] *we haven't use it yet.*

$$\frac{\partial \phi_0}{\partial z} = b_0 \quad + \quad f_0 \boldsymbol{k} \times \mathbf{u}_g = -\nabla \phi_0 \quad \Rightarrow \quad \boldsymbol{k} \times \frac{\partial \mathbf{u}_g}{\partial z} = -\frac{\nabla b_0}{f_0}$$

Then

$$I_2 = \tilde{\rho} \left(\frac{L}{L_d} \right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} = \tilde{\rho} \left(\frac{L}{L_d} \right)^2 \left[\frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} + \underbrace{\partial_z \mathbf{u}_g \cdot \nabla b_0}_{=0} \right]$$

Therefore

$$I = \partial_z \left(\tilde{\rho} \left(\frac{L}{L_d} \right)^2 \right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho} \left(\frac{L}{L_d} \right)^2 \frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} = \frac{\mathrm{D}_g}{\mathrm{D}t} \left[\frac{\partial}{\partial_z} \left(\tilde{\rho} \left(\frac{L}{L_d} \right)^2 b_0 \right) \right]$$

Then evantually we have

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\xi_g + f + \frac{f_0}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 b_0\right)\right] = 0 \tag{1.22}$$

We can rewrite this equation using Streamfunction in a more simple form. Recall Eq 1.9, we have

$$b_0 = \frac{\partial \phi_0}{\partial z}$$

From Eq 1.18, the Kinematic Pressure can be expressed in terms of geostrophic streamfunction

$$u_g = -\partial_y \psi_g \quad v_g = \partial_x \psi_g \qquad \text{where } \boxed{\phi_0 = f_0 \psi_g} \quad \Rightarrow \quad \xi_g = \nabla^2 \psi_g$$

Then Eq 1.22 becomes

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\nabla^2\psi_g + f + \frac{f_0^2}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 \frac{\partial \psi_g}{\partial z}\right)\right] = 0 \tag{1.23}$$

Restore the dimensions

$$\frac{\mathrm{D}g}{\mathrm{D}t}\left[\nabla^2\psi_g + f + \frac{f_0^2}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\frac{\tilde{\rho}}{N^2}\frac{\partial \psi}{\partial z}\right)\right] = 0 \tag{1.24}$$

SUBSECTION 1.4

Ertel PV Conservation

Theorem 1.1 The Ertel PV, denoted usually as q or Q, is defined as:

$$Q = \frac{\boldsymbol{\omega}_a \cdot \nabla\psi}{\rho} \tag{1.25}$$

Where:

1. $\boldsymbol{\omega}_a = \nabla \times \mathbf{u} + 2\boldsymbol{\Omega}$ is the absolute vorticity.
2. ψ is a conserved scalar (like potential temperature θ or density).
3. ρ is the density.

The derivation starts from the Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p = -2\boldsymbol{\Omega} \times \mathbf{u}$$

Use the vector identity then take Curl

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}^2) - \mathbf{u}\times(\nabla\times\mathbf{u})$$

We have

$$\frac{\partial \boldsymbol{\omega}_a}{\partial t} - \nabla \times (\mathbf{u}\times\boldsymbol{\omega}_a) = \nabla \times \left(-\frac{1}{\rho}\nabla p\right)$$

Then apply the vector identity

$$\nabla \times (\frac{1}{\rho}\nabla p) = \frac{1}{\rho^2}\nabla\rho \times \nabla p$$

We get

$$\frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = (\boldsymbol{\omega}_a \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}_a(\nabla \cdot \mathbf{u}) + \frac{\nabla\rho \times \nabla p}{\rho^2} \tag{1.26}$$

SECTION 2

Surface Quasi-Geostrophic Equations

The surface Quasi-Geostrophic Equation takes the problem to the next step, how could we retrive the interior motion from surface measurements such as SSH and SST. Recall the Buoyancy Equation

$$\frac{\mathrm{D}b_1}{\mathrm{D}t} + wN^2 = 0 \tag{2.1}$$

At the surface, $z = \eta$, the boundary condition yeilds that $w = 0$. We denote the surface buoyancy as b_s and surface velocity $\mathbf{u}_s$ Then

$$\frac{\partial b_s}{\partial t} + \mathbf{u}_s \cdot \nabla b_s = 0$$

and

$$b_s = f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0}$$

Explanation 1 The critical principle of SQG is to view surface buoyancy as a PV sheet. Since

$$\int_0^{\epsilon} f + \nabla^2 \psi_g + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) \mathrm{d}z = 0$$

Then we impose a boundary condition

$$\frac{\partial \psi}{\partial z}\Big|_{z=\epsilon} = \frac{b_s}{f_0} \quad \Rightarrow \quad \int_0^{\epsilon} \frac{b_s}{f_0} = \frac{\partial \psi}{\partial z}\Big|_0^{\epsilon}$$

Compare the latter witht the integral, by defining

$$q_{\mathrm{SQG}} = \nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) + \frac{b_s}{f_0} \delta(z)$$

We have

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \frac{\partial \psi}{\partial z} = 0$$

Then the surface buoyancy appears in the QGPV equation naturally, it is as if adding an additional PV sheet at the surface. This inspires us to seperate the surface induced dynamics and interior dynamics.

Interior Dynamics

$$\begin{cases} q & = \nabla^2 \psi + f + \dfrac{f_0^2}{\tilde{\rho}} \dfrac{\partial}{\partial_z} \left(\dfrac{\tilde{\rho}}{N^2} \dfrac{\partial \psi}{\partial z} \right) \\ f_0 \dfrac{\partial \psi}{\partial z}\Big|_{z=0} & = 0 \\ \dfrac{\mathrm{D}q}{\mathrm{D}t} & = 0 \end{cases}$$

SUBSECTION 2.1

Surface Buoyancy Induced Dynamics

Surface Dynamics, this is the Surface Quasi-Geostrophic Dynamics:

$$\begin{cases} q & = \nabla^2\psi + f + \dfrac{f_0^2}{\tilde{\rho}}\dfrac{\partial}{\partial_z}\left(\dfrac{\tilde{\rho}}{N^2}\dfrac{\partial\psi}{\partial z}\right) = 0^8 \\ f_0\dfrac{\partial\psi}{\partial z}\Big|_{z=0} & = b_s \\ \dfrac{\mathrm{D}b_s}{\mathrm{D}t} & = 0 \end{cases}$$

[8] *no interior Potential vorticity*

The key assumption for SQG theories is that all the Potential vorticity is injected into the system by surface buoyancy. [9]

[9] *surface buoyancy is actually first order, so it is also called surface buoyancy anomoly in some context.*

Subsection 2.2

Retriving Vertical Velocity

SECTION 3

QG^{+1} Model

The derivation starts from the same set of equations above. We set

$$\epsilon = \text{Ro} = \frac{U}{fL} \qquad \text{Bu} = \left(\frac{NH}{fL}\right)^2$$

The Derivation starts from Eq 1.14, 1.17, 1.15 and 1.13. The Ertel Potential Vorticity is conserved. In this case, the conserved quantity is the **total Buoyancy** since Eq 1.14 we define it as

$$b_{\text{tot}} = N^2 z + b$$

and

$$\boldsymbol{\omega}_a = \nabla_3 \times (u, v, 0) + f\hat{z} = (-v_z, u_z, f + \xi)$$

Then

$$Q = \underbrace{fN^2}_{\text{Background}} + \underbrace{(N^2\xi + fb_z)}_{\text{Linear terms}} + \underbrace{(\xi b_z - v_z b_x + u_z b_y)}_{\text{Nonlinear terms}}$$

Use the vector identity

$$\nabla_3 \cdot (\boldsymbol{\omega}_a b_{\text{tot}}) = \boldsymbol{\omega} \cdot \nabla_3 b_{\text{tot}} + \underbrace{b_{\text{tot}}(\nabla \cdot \boldsymbol{\omega})}_{=0, \boldsymbol{\omega}=\nabla\times\cdot}$$

Then after scaling anlysis we have

$$Q = N^2 f + \epsilon q$$

where

$$q = N^2\xi + \frac{1}{\text{Bu}} fb_z + \frac{\epsilon}{\text{Bu}} \nabla_3(\boldsymbol{\omega} b_{\text{tot}})$$

The first two term is the calssical Quasi Geostrophic Potential Vorticity.

Explanation 2 Since here N is constant and we assume the Boussinesq Equation, the third term in Eq 1.22 becomes,

$$\frac{f_0}{\cancel{\rho}} \cancel{\rho} \frac{\partial b}{\partial z} \left(\frac{fL}{NH}\right)^2 \sim fb_z \frac{1}{\text{Bu}}$$

The only difference here is the third second order ageostrophic quadratic correction we normally ignore in classical QG theories.