Lecture Notes on Surface Quasi-Geostrophic

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Table 1: Glossary of Variables and Operators

Variables and Operators			
Symbol	**Variable and Operators**	**Symbol**	**Description**
$\mathbf{v}(x, y, z, t) = (u, v, w)$	Full 3-dimensional Velocity	∇	2D Gradient Operator
$\mathbf{u}(x, y, z, t) = (u, v)$	2-dimensional velocity, in x and y direction	∇_3	3D Gradient
$\widehat{\cdot}$	Dimensionless Variable	$\frac{\mathrm{D}}{\mathrm{D}t}$	Material Derivative
ψ	Geostrophic Streamfunction defined in Eq	∇^2	2D laplacian operator (zonal and meridional)
$(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$	Unit vector in zonal, meridional and vertical direction		

Section 1

Literature Review

Subsection 1.1

Ryan et.al.

Year: 2025

The QG^{+1} model incorporates the first-order corrections that were neglected in the basic QG approximation. It essentially refines the QG equations by accounting for non-geostrophic (ageostrophic) flow components that are dependent on the Rossby number (ϵ).

In **Chapter 2**, the QG^{+1} model is introduced. In this paper,

$$N = f \equiv \text{Constant}^1$$

[1] *page 8*

To facilitate the asymptotic approximation, a potential field is introduced.

$$\mathbf{A} = (-G, -F, \Phi)$$

By Incompressible condition we have

$$\mathbf{v} = \nabla_3 \times \mathbf{A}^2$$

[2] *In this paper ∇_3 is 3D gradient. 2D is just ∇*

Some Physical implications of the model

1. Breaking Symmetry of QG model.
2. It Captures Cyclogeostrophic balance.

   Cyclogeostrophic balance is a fundamental force balance approximation used in meteorology and physical oceanography to describe the motion of fluids (like air and water) in curved paths, where the Coriolis force is balanced by the pressure gradient force and the centrifugal force. It is an essential extension of the simpler geostrophic balance, which only considers straight flow. This balance is particularly important in systems with high curvature and strong winds, such as tropical cyclones (hurricanes/typhoons), mid-latitude low-pressure systems, and strong ocean eddies. The Governing Equation is

$$\underbrace{f\mathbf{v}}_{\text{Coriolis Force}} + \underbrace{\frac{|\mathbf{v}|^2}{R}}_{\text{Centrifugal Force}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial n}}_{\text{Pressure Gradient Force}} \tag{1.1}$$

   Here n is the normal direction pointing toward the center of curvature.

3. Inclusiong of **Frontogenesis** [3].

[3] *Generation of Ocean Fronts*

In **Chapter 3.** A simulation for QG^{+1} is conducted, showing several features:

1. More Vigorous due to captureing ageostrophic frontogenesis.
2. Since the Ageostrophic effects creates stronger surface velocity. Finer structure can be seen on surface using QG^{+1}. [4].

[4] *See Figure 4 in page 24*

In summary, this paper provides a very detailed derivation to the QG^{+1} equation which is introduced more detailed in 4. This paper also demonstrate two simulation to show how QG^{+1} model captures balanced submesoscale dynamics and frontogenesis.

Subsection 1.2

J.Wang et.al. Reconstructing the Ocean's Interior from Surface Data

Year : 2013

In the **Introduction**, the author discussed the current challenge of using SSH and SST [5] measurement to reconstruct subsurface dynamics.

[5] *Surface Sea Height and Surface Sea Temperature*

- Traditional studies assume the signal is dominated by barotropic and first baroclinic modes. However, these modes are typically calculated by **assuming buoyancy anomalies vanish at the surface**.
- SQG theorys works as well. But it normaly assume 0 interior PV.

The author introduced the **Interior plus surface QG** method. It is quasigeostrophic. As introduced in Chapter 2:

1. Surface buoyancy anomoly contributes to the surface part of streamfunction ψ^s :

$$\mathcal{L}\Psi + f_0 + \beta y = Q,$$
$$\mathcal{L} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right), \quad \text{and } -H < z < 0,$$

   Governing equation is

$$\mathcal{L}\psi^s = 0$$

   Essentially, this is same in the SQG theory where we assume 0 interior PV. With boundary condition :

$$\frac{\partial}{\partial z}\psi^s(\mathbf{x}, z, t) = b(\mathbf{x}, z, t)/f_0 \quad \text{at} \quad z = 0, -H,$$

2. The interior part is governed by

$$\frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\hat{\psi}^i - \kappa^2\hat{\psi}^i = \hat{q}^i \quad \text{with} \quad \frac{d\hat{\psi}^i}{dz} = 0 \quad \text{at} \quad z = 0, -H.$$

However, the interior q^i is not clear.

Remark 1 This is the essential modification of this isQG model. They project the interior induced PV equation onto baroclinic modes and **impose additional boundary conditions to deduce the gravest modes.**

This is the origin of QG^{+1} theory.

SECTION 2

Quasi-Geostrophic Equations

In this section, I introduce some basic equations in QG theories. Building a foundation for SQG, eSQG and other variations introduced later. The analysis is already in a **stratified ocean**.

SUBSECTION 2.1

Governing Equations

Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\frac{\nabla p}{\rho} \tag{2.1}$$

Mass Conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2.2}$$

We are in a **stratified ocean**. Breaking the total state variables into a "hydrostatic reference state" (which depends only on z) and a "dynamic perturbation" (which moves the fluid):

$$\rho = \tilde{\rho}(z) + \rho_1(x, y, z, t) \tag{2.3}$$

and

$$p = p_0(z) + p_1(x, y, z, t) \tag{2.4}$$

Then RHS of Eq 2.6 becomes

$$-\frac{1}{\rho}\nabla p_1 \sim -\frac{1}{\rho_0}\nabla p_1$$

Define the **Kinematic Pressure**

$$\phi = \frac{p_1}{\rho_0} \tag{2.5}$$

Momentum Equation becomes

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} = -\nabla \phi} \tag{2.6}$$

Hydrostatic balance is a state of equilibrium in a fluid where the upward force of pressure exactly balances the downward force of gravity.

$$-g\tilde{\rho} = \frac{\mathrm{d}p_0}{\mathrm{d}z} \tag{2.7}$$

2.1.1 Continuity Equation Approximatio

The genearl continuity equation can always be expressed as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot u = 0$$

In Ocean, we assume that the fluid is **Incompressible** and wrote

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \qquad \nabla \cdot \mathbf{u} = 0$$

In Atmosphere, we use the **Anelastic Assumption** Then the mass conservation yeilds

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot ((\tilde{\rho} + \rho_1)\mathbf{v}) = 0 \quad \Rightarrow \quad \boxed{\nabla \cdot (\tilde{\rho}\mathbf{v}) = 0}$$

Remark 2 Here we drop the $\partial_t \rho_1$ term under the **Anelastic assumption**. Essentially by eliminating this partial derivative, we assume the fluid is anelastic, so sound wave is not supported [6] in the system. However, this approximation is normally used for **deep atmospheric or stellar convertion** where density $\tilde{\rho}$ changes significantly with height.

[6] *or less important*

For Oceanography, Next we define the Buoyancy :

$$b = -g\frac{\rho}{\bar{\rho}_0} \tag{2.8}$$

Here $\bar{\rho}_0$ is a constant reference density. The the divergent free condition implies

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = 0} \tag{2.9}$$

And thus the Hydrostatic balance equation Eq 2.7 implies

$$\boxed{\frac{\partial \phi}{\partial z} = b} \tag{2.10}$$

All the Boxed Equation together is the **Hydrostatic Anelastic Equations for Stratified Flow**. If we consider the perturbation of Buoyancy

$$b = \tilde{b}(z) + b_1(x, y, z, t)$$

Expand Eq 2.9 can be writen as

$$\boxed{\frac{\mathrm{D}b_1}{\mathrm{D}t} + w\frac{\mathrm{d}b}{\mathrm{d}z} = 0}^{7} \tag{2.11}$$

[7] *this is the more familiar buoyancy equation we see in lecture*

In a more familiar form we define

$$N^2 = \frac{\mathrm{d}b}{\mathrm{d}z} = -g\frac{\tilde{\rho}_z}{\bar{\rho}_0}$$

Which is the **Brunt Vasala Frequency**.

Subsection 2.2

Scaling Analysis

To simplify our equation, we introduce some scalings.

$$(x, y) \sim L, \quad (u, v) \sim U, \quad t \sim \frac{L}{U}, \quad z \sim H, \quad f \sim f_0$$

Introduce the **Rosbby Number**:

$$\mathrm{Ro} = \frac{U}{f_0 L} \tag{2.12}$$

Now let $\phi = \tilde{\phi}(z) + \phi_1(x, y, z, t)$. Then since the gradient in Eq 2.6 is horizontal, we can replace ϕ by ϕ_1. Now suppose

$$|\mathbf{f} \times \mathbf{u}| \sim |\nabla \phi_1|$$

From Hydrostatic balance we have

$$b \sim \frac{f_0 U L}{H}$$

Then

$$\frac{(\partial b'/\partial z)}{N^2} \sim \frac{f_0 U L}{(HN)^2} \sim Ro\frac{L^2}{L_d^2}$$

Where we have the deformation radius as a function of z.

$$L_d = \frac{NH}{f_0}$$

Introduce dimensionless variables

$$(\widehat{u}, \widehat{v}) = U^{-1}(u, v) \quad \widehat{w} = \frac{L}{UH} w, \quad \widehat{f} = f_0^{-1} f, \quad \widehat{\phi} = \frac{\phi_1}{f_0 UL}, \quad \widehat{b} = \frac{H}{f_0 UL} b_1$$

Remark 3 We then have dimensionless equation of motion for Atmosphere or Ocean. [8]

$$\text{Momentum Equation}: \quad \mathrm{Ro}\frac{\mathrm{D}\widehat{\mathbf{u}}}{\mathrm{D}t} + \widehat{\mathbf{f}} \times \widehat{\mathbf{u}} = -\nabla\widehat{\phi} \tag{2.13}$$

$$\text{Buoyancy Equation}: \quad \mathrm{Ro}\frac{\mathrm{D}\widehat{b}}{\mathrm{D}t} + \left(\frac{L_d}{L}\right)^2 \widehat{w} = 0 \tag{2.14}$$

$$\text{Hydrostatic Balance}: \quad \frac{\partial\widehat{\phi}}{\partial\widehat{z}} = \widehat{b} \tag{2.15}$$

$$\text{Continuity}(\textbf{Atmosphere}): \quad \widehat{\nabla} \cdot \widehat{\mathbf{u}} + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}\widehat{w}}{\partial\widehat{z}} = 0 \tag{2.16}$$

$$\text{Continuity}(\textbf{Oceanography}): \quad \widehat{\nabla} \cdot \widehat{\mathbf{u}} = 0 \tag{2.17}$$

From now on I will drop the hats.

[8] *often people drop the hat for simplicity. However in the first derivation I keep everything with a hat.*

SUBSECTION 2.3

Quasi-Geostrophic Potential Vorticity Equation

We now derive the Quasi-Geostrophic Potential Vorticity Equations. Starting from asymptotic expansions [9]

$$\mathbf{u} = (u, v, w) = \mathbf{u}_g + \mathrm{Ro}\mathbf{u}_1 \quad \phi = \phi_0 + \mathrm{Ro}\phi_1 \quad b = b_0 + \mathrm{Ro}b_1$$

[9] *hat is dropped*

Here we consider the β effect.

$$\mathbf{f} = f_0\boldsymbol{k} + \beta y\boldsymbol{k}$$

Let $\epsilon = \mathrm{Ro}$. **Momentum Equation** :

The $O(1)$ momentum equation gives the Geostrophic balance

$$f_0\boldsymbol{k} \times \mathbf{u}_g = -\nabla\phi_0 \tag{2.18}$$

Immediately this implies

$$\nabla \cdot \mathbf{u}_g = 0$$

And $O(\epsilon)$ is

$$\frac{\mathrm{D}_g\mathbf{u}_g}{\mathrm{D}t} + \beta y\boldsymbol{k} \times \mathbf{u}_g + f_0\boldsymbol{k} \times \mathbf{u}_1 = -\nabla\phi_1 \tag{2.19}$$

Here D_g is the geostrophic material derivative

$$\mathrm{D}_g = \partial_t + \mathbf{u}_g \cdot \nabla$$

Mass Equation :

Since geostrophic velocity is divergent free then $O(1)$ mass equations is

$$\frac{\partial\tilde{\rho}w_0}{\partial z} = 0$$

and $O(\epsilon)$,

$$\nabla \cdot \mathbf{u}_1 + \frac{1}{\tilde{\rho}}\left(\frac{\partial \tilde{\rho} w_1}{\partial z}\right) = 0 \tag{2.20}$$

Buoyancy Equation :
$O(1)$:

$$\left(\frac{L_d}{L}\right)^2 w_0 = 0$$

and $O(\epsilon)$:

$$\frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \left(\frac{L_d}{L}\right)^2 w_1 = 0 \tag{2.21}$$

Now we take the **Curl** of Eq 2.19, note that

$$\nabla \times (\boldsymbol{k} \times \mathbf{u}_1) = \boldsymbol{k}\nabla \cdot \mathbf{u}_1 - \underbrace{u_1 \nabla \cdot \boldsymbol{k}}_{=0} + \underbrace{(\mathbf{u}_1 \cdot \nabla)\boldsymbol{k}}_{=0} - \underbrace{(\boldsymbol{k} \cdot \nabla)\mathbf{u}_1}_{=0} = \boldsymbol{k}\nabla \cdot \mathbf{u}_1$$

Define the geostrophic vorticity :

$$\xi_g = \nabla \times \mathbf{u}_g$$

Then Eq 2.19 becomes

$$\frac{\mathrm{D}_g \xi_g}{\mathrm{D}t} + \beta v_0 = -f_0 \nabla \cdot \mathbf{u}_1$$ [10]

Plug in Eq 2.20,

$$= \frac{f_0}{\tilde{\rho}} \frac{\partial \tilde{\rho} w_1}{\partial z}$$

Plug in Eq 2.21 to replace w_1

$$= -\frac{f_0}{\tilde{\rho}} \underbrace{\frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 \frac{\mathrm{D}_g b_0}{\mathrm{D}t}\right)}_{\equiv I}$$

Now we examine I, normally in QG theory, we assume L_d is a constant. Thought from its definition, N cound actually depends on z. Since $\nabla \tilde{\rho} = 0$, we can put the first two terms into the material derivative. [11]

$$I = \partial_z \left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2\right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho}\left(\frac{L}{L_d}\right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} \equiv I_1 + I_2$$

Let's go back to the Hydrostatic balance equation, [12] For $O(1)$:

$$\frac{\partial \phi_0}{\partial z} = b_0 \quad + \quad f_0 \boldsymbol{k} \times \mathbf{u}_g = -\nabla \phi_0 \quad \Rightarrow \quad \boldsymbol{k} \times \frac{\partial \mathbf{u}_g}{\partial z} = -\frac{\nabla b_0}{f_0}$$

Then

$$I_2 = \tilde{\rho}\left(\frac{L}{L_d}\right)^2 \partial_z \frac{\mathrm{D}_g b_0}{\mathrm{D}t} = \tilde{\rho}\left(\frac{L}{L_d}\right)^2 \left[\frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} + \underbrace{\partial_z \mathbf{u}_g \cdot \nabla b_0}_{=0}\right]$$

Therefore

$$I = \partial_z \left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2\right) \frac{\mathrm{D}_g b_0}{\mathrm{D}t} + \tilde{\rho}\left(\frac{L}{L_d}\right)^2 \frac{\mathrm{D}_g \partial_z b_0}{\mathrm{D}t} = \frac{\mathrm{D}_g}{\mathrm{D}t}\left[\frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 b_0\right)\right]$$

[10] *this equation is already in $\boldsymbol{k}$ direction so the unit vector is dropped*

[11] *Here we use the fact that N is constant*

[12] *we haven't use it yet.*

Then evantually we have

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\xi_g + f + \frac{f_0}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 b_0\right)\right] = 0 \tag{2.22}$$

We can rewrite this equation using Streamfunction in a more simple form. Recall Eq 2.9, we have

$$b_0 = \frac{\partial \phi_0}{\partial z}$$

From Eq 2.18, the Kinematic Pressure can be expressed in terms of geostrophic streamfunction

$$u_g = -\partial_y \psi_g \quad v_g = \partial_x \psi_g \qquad \text{where } \boxed{\phi_0 = f_0 \psi_g} \quad \Rightarrow \quad \xi_g = \nabla^2 \psi_g$$

Then Eq 2.22 becomes

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\nabla^2\psi_g + f + \frac{f_0^2}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\tilde{\rho}\left(\frac{L}{L_d}\right)^2 \frac{\partial \psi_g}{\partial z}\right)\right] = 0 \tag{2.23}$$

Restore the dimensions

$$\frac{\mathrm{D}g}{\mathrm{D}t}\left[\nabla^2\psi_g + f + \frac{f_0^2}{\tilde{\rho}}\frac{\partial}{\partial_z}\left(\frac{\tilde{\rho}}{N^2}\frac{\partial \psi}{\partial z}\right)\right] = 0 \tag{2.24}$$

Subsection 2.4

Ertel PV Conservation

Theorem 2.1 The Ertel PV, denoted usually as q or Q, is defined as:

$$Q = \frac{\boldsymbol{\omega}_a \cdot \nabla \psi}{\rho} \tag{2.25}$$

Where:

1. $\boldsymbol{\omega}_a = \nabla \times \mathbf{u} + 2\boldsymbol{\Omega}$ is the absolute vorticity.
2. ψ is a conserved scalar (like potential temperature θ or density).
3. ρ is the density.

The derivation starts from the Momentum Equation

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p = -2\boldsymbol{\Omega} \times \mathbf{u}$$

Use the vector identity then take Curl

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}^2) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

We have

$$\frac{\partial \boldsymbol{\omega}_a}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}_a) = \nabla \times \left(-\frac{1}{\rho}\nabla p\right)$$

Then apply the vector identity

$$\nabla \times (\frac{1}{\rho}\nabla p) = \frac{1}{\rho^2}\nabla\rho \times \nabla p$$

We get

$$\frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = (\boldsymbol{\omega}_a \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}_a(\nabla \cdot \mathbf{u}) + \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{2.26}$$

Section 3

Surface Quasi-Geostrophic Equations

The surface Quasi-Geostrophic Equation takes the problem to the next step, how could we retrive the interior motion from surface measurements such as SSH and SST. Recall the Buoyancy Equation

$$\frac{\mathrm{D}b_1}{\mathrm{D}t} + wN^2 = 0 \tag{3.1}$$

At the surface, $z = \eta$, the boundary condition yeilds that $w = 0$. We denote the surface buoyancy as b_s and surface velocity $\mathbf{u}_s$ Then

$$\frac{\partial b_s}{\partial t} + \mathbf{u}_s \cdot \nabla b_s = 0$$

and

$$b_s = f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0}$$

Explanation 1 The critical principle of SQG is to view surface buoyancy as a PV sheet. Since

$$\int_0^\epsilon f + \nabla^2 \psi_g + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) \mathrm{d}z = 0$$

Then we impose a boundary condition

$$\frac{\partial \psi}{\partial z}\Big|_{z=\epsilon} = \frac{b_s}{f_0} \quad \Rightarrow \quad \int_0^\epsilon \frac{b_s}{f_0} = \frac{\partial \psi}{\partial z}\Big|_0^\epsilon$$

Compare the latter witht the integral, by defining

$$q_{\mathrm{SQG}} = \nabla^2 \psi_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) + \frac{b_s}{f_0} \delta(z)$$

We have

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \frac{\partial \psi}{\partial z} = 0$$

Then the surface buoyancy appears in the QGPV equation naturally, it is as if adding an additional PV sheet at the surface. This inspires us to seperate the surface induced dynamics and interior dynamics.

Interior Dynamics

$$\begin{cases} q & = \nabla^2 \psi + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial_z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right) \\ f_0 \frac{\partial \psi}{\partial z}\Big|_{z=0} & = 0 \\ \frac{\mathrm{D}q}{\mathrm{D}t} & = 0 \end{cases}$$

Subsection 3.1

Surface Buoyancy Induced Dynamics

Surface Dynamics, this is the Surface Quasi-Geostrophic Dynamics:

$$\begin{cases} q & = \nabla^2\psi + f + \dfrac{f_0^2}{\tilde{\rho}}\dfrac{\partial}{\partial_z}\left(\dfrac{\tilde{\rho}}{N^2}\dfrac{\partial\psi}{\partial z}\right) = 0^{13} \\ f_0\dfrac{\partial\psi}{\partial z}\Big|_{z=0} & = b_s \\ \dfrac{\mathrm{D}b_s}{\mathrm{D}t} & = 0 \end{cases}$$

[13] *no interior Potential vorticity*

The key assumption for SQG theories is that all the Potential vorticity is injected into the system by surface buoyancy. [14]

[14] *surface buoyancy is actually first order, so it is also called surface buoyancy anomoly in some context.*

Subsection 3.2

Retriving Vertical Velocity

Section 4

QG^{+1} Model

The derivation starts from the same set of equations above. We set

$$\epsilon = \mathrm{Ro} = \frac{U}{fL} \qquad \mathrm{Bu} = \left(\frac{NH}{fL}\right)^2$$

The Derivation starts from Eq 2.14, 2.17, 2.15 and 2.13. The Ertel Potential Vorticity is conserved. In this case, the conserved quantity is the **total Buoyancy** since Eq 2.14 we define it as

$$b_{\mathrm{tot}} = N^2 z + b$$

and

$$\boldsymbol{\omega}_a = \nabla_3 \times (u, v, 0) + f\hat{z} = (-v_z, u_z, f + \xi)$$

Then

$$Q = \underbrace{fN^2}_{\text{Background}} + \underbrace{(N^2\xi + fb_z)}_{\text{Linear terms, QGPV}} + \underbrace{(\xi b_z - v_z b_x + u_z b_y)}_{\text{Nonlinear terms}} \tag{4.1}$$

and

$$q_{\mathrm{QG}} = N^2\xi + fb_z$$

Use the vector identity

$$\nabla_3 \cdot (\boldsymbol{\omega} b) = \boldsymbol{\omega} \cdot \nabla_3 b + \underbrace{b(\nabla \cdot \boldsymbol{\omega})}_{=0, \boldsymbol{\omega} = \nabla \times \cdot}$$

We note that

$$\xi b_z - v_z bx + u_z b_y = \boldsymbol{\omega} \cdot \nabla_3 b$$

where

$$\boldsymbol{\omega} = (-v_z, u_z, \xi) = (-v_z, u_z, v_x - u_y)$$

Then after scaling anlysis we have

$$Q = N^2 f + \epsilon q$$

where

$$q = N^2\xi + \frac{1}{\mathrm{Bu}} fb_z + \frac{\epsilon}{\mathrm{Bu}} \nabla_3 \cdot (\boldsymbol{\omega} b) \tag{4.2}$$

The first two term is the calssical Quasi Geostrophic Potential Vorticity.

Explanation 2 Since here N is constant and we assume the Boussinesq Equation, the third term in Eq 2.22 becomes,

$$\frac{f_0}{\cancel{\rho}} \cancel{\rho} \frac{\partial b}{\partial z} \left(\frac{fL}{NH}\right)^2 \sim fb_z \frac{1}{\mathrm{Bu}}$$

The only difference here is the third second order ageostrophic quadratic correction we normally ignore in classical QG theories.

Subsection 4.1

QG^{+1} Vector Field

In order to facilitate asymptotic approximation, the velocity field $\mathbf{u}$ are writen as the Curl of a vector field $\boldsymbol{A}$. [15]. One convention is writing

$$\boldsymbol{A} = (-G, F, \Phi) \tag{4.3}$$

[15] *Changing three variables in velocity to* $\boldsymbol{A}$

and

$$\mathbf{u} = \nabla \times \boldsymbol{A}$$
$$u = -\Phi_y - F_z$$
$$v = \Phi_x - G_z$$
$$w = F_x + G_y$$

However, this $\boldsymbol{A}$ isn't uniquely correspond to a velocity field. Since Gradient is Curl free, $\mathbf{u}$ admits a gauge freedom in that the transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla_3 \Gamma$$

Left $\mathbf{u}$ unchange. Now instead of assuming $\boldsymbol{A}$ is divergent free. [16] We fix the buoyancy to be roughly scaled version of $\nabla_3 \cdot \boldsymbol{A}$.

[16] *if $\boldsymbol{A}$ is divergent free, then this is adding an additioal information to the system and $\boldsymbol{A}$ can be uniquely determined.*

$$\nabla_3 \cdot \boldsymbol{A} = -G_x + F_y - \Phi_z \quad \Rightarrow \quad b = f\Phi_z + \mathrm{Bu}\frac{N^2}{f}(G_x - F_y)$$

The advantage can be seen by calculating the QGPV Eq 4.2 (**Dimensionless form**) :

$$q_{\mathrm{QG}} = N^2\xi + \frac{f}{\mathrm{Bu}} b_z = N^2(\Phi_{xx} + \Phi_{yy}) + \frac{f^2}{\mathrm{Bu}}\Phi_{zz} = \mathcal{L}\Phi \tag{4.4}$$

Where

$$\mathcal{L} = N^2\nabla^2 + \frac{f^2}{\mathrm{Bu}}\partial_{zz} \tag{4.5}$$

Then Φ is the only component related to the QGPV compare to dependence on all three component of velocity in the classical QGPV. When is the Buoyancy unchanged?

Explanation 3 Suppose we add a gradient to the original Vector Potential

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla_3 \Gamma$$

Then

$$G \to G - \Gamma_x \quad F \to F + \Gamma_y \to \Phi \to \Phi - \Gamma_z$$

And

$$b_{new} = \underbrace{\left[f\Phi_z + \{Bu\}\frac{N^2}{f}(G_x - F_y)\right]}_{b_{origin}} - \underbrace{\left[f\Gamma_{zz} + \{Bu\}\frac{N^2}{f}(\Gamma_{xx} + \Gamma_{yy})\right]}_{b_{change}}$$

So in order to have $b_{new} = b_{origin}$. We have

$$b_{change} = f\Gamma_{zz} + \{Bu\}\frac{N^2}{f}(\Gamma_{xx} + \Gamma_{yy}) = \boxed{\mathcal{L}(\Gamma) = 0} \tag{4.6}$$

Some Discussions

1. **In a triply-periodic domain** :

   Eq 4.6 implies that Γ is constant in the domain.

2. **Rigid Lid and flat bottom** :

   since $w = 0$ at upper and lower boundary, we have

$$F_x + G_y = 0$$

Subsection 4.2

Evolution and Inversion Equations of QG^{+1}

Assume

$$G^0 = F^0 = 0$$

and

$$\epsilon \ll 1 \qquad \mathrm{Bu} \sim O(1)$$

Then expand $\boldsymbol{A}$ asymptotically we have

$$u = -\Phi^0_y - \epsilon(\Phi^1_y + F^1_z) \tag{4.7a}$$

$$v = \Phi^0_x + \epsilon(\Phi^1_x - G^1_z) \tag{4.7b}$$

$$w = 0 + \epsilon(F^1_x + G^1_y) \tag{4.7c}$$

$$b = f\Phi^0_z + \epsilon f\left(\Phi^1_z + \mathrm{Bu}\frac{N^2}{f^2}(G^1_x - F^1_y)\right) \tag{4.7d}$$

Evaluate Eq 4.2 we have up to $O(\epsilon)$.

$$\begin{aligned} q &= N^2(\nabla^2\Phi^0 + \nabla^2\Phi^1 - G^1_{zx} + F^1_{zy}) + \frac{f^2}{\mathrm{Bu}}\Phi^0_z + \epsilon\frac{f^2}{\mathrm{Bu}}\left(\Phi^1_z + \mathrm{Bu}\frac{N^2}{f^2}(G^1_x - F^1_y)\right) \\ &\quad + \underbrace{\left[-\Phi^0_{xz}, -\Phi^0_{yz}, \nabla^2\Phi^0\right]}_{\text{This is } \boldsymbol{\omega}=(-v_z,u_z,\xi)} \cdot \underbrace{\left[f\Phi^0_{zx}, f\Phi^0_{zy}, f\Phi^0_{zz}\right]}_{\text{this is } \nabla_3 b} \\ &= \boxed{\mathcal{L}(\Phi^0) + \epsilon\mathcal{L}(\Phi^1) + \epsilon\frac{f}{\mathrm{Bu}}\Big(-|\nabla\Phi^0_z|^2 + \Phi^0_{zz}\nabla^2\Phi^0\Big) + O(\epsilon^2)} \end{aligned} \tag{4.8}$$

The surface Buoyancy from

$$b^t = f\Phi^0_z\Big|_{z=0} + f\Phi^1_z\Big|_{z=0} + O(\epsilon^2) \tag{4.9a}$$

$$b^b = f\Phi^0_z\Big|_{z=-H} + f\Phi^1_z\Big|_{z=-H} + O(\epsilon^2) \tag{4.9b}$$