Master Économiste d'entreprise

Auction theory

The Revenue Equivalence Theorem

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1 Types of standard auctions

Auctions are transactions with a specific set of rules detailing resource allocation according to participants' bids (amounts of money they are willing to pay). In game theory auctions are categorized as games with incomplete information because in the vast majority, one player will possess information that other players don't.
Standard auctions require that the winner of the auction be the participant with the highest bid. There are traditionally four types of auction that are used for the allocation of a single item:

1. *Ascending-bid auction* where the price is successively raised until only one bidder remains and that bidder wins the item at the final price.

2. *Descending-bid auction* works in exactly the opposite way. The auctioneer starts at a very high price, and then lowers the price continuously. The first bidder who agrees with the current price wins the item at that price.

3. *First-price sealed-bid auction* where each bidder independently submits a single bid, without seeing others' bids, and the object is sold to the bidder who makes the highest bid. The winner pays its bid.

4. *Second-price sealed-bid auction* works exactly the same way as the first-price sealed-bid auction except that the price the winner pays is the second-highest bidder's bid. This type of auction is also called *Vickrey's auction*.

This four types can be shortened to two types. Descending-bid auction and First-price sealed-bid auctions are based on the same principles. Each bidder must choose a price to call out, conditional on no other bidder having yet called out; and the bidder who chooses the highest price wins the item at the price called out. In both case, the winner pays the bid he/she called out. The Ascending-bid auction and the Second-price sealed-bid rest on the same principles as well but a little more reflection is needed. In an Ascending-bid auction, it is clearly a dominant strategy to stay in the bidding until the price reaches your maximum valuation of the good, that is, until you are just indifferent between winning and not winning. The next-to-last person will drop out when his/her maximum valuation of the good is reached, so the person with the highest bid will win at a price equal to the bid of the second-highest bidder. In a Second-price sealed-bid auction, a Nash equilibrium strategy is to bid the maximal valuation of the good you have, because if the worst comes to the worst you would pay your maximal valuation of the good minus ε so in every case you will have a positive payoff so there is no incentive to deviate from this strategy. Here again, the person with the highest bid will win at a price equal to the bid of the second-highest bidder. Now that we can distinguish two types of standard auction from the bidder's point of view, we should ask which is the most profitable type? The answer is in the *Revenue Equivalence Theorem* (Vickrey, 1961) which states that for certain economic environments, the expected revenue and bidder profits for a broad class of auctions will be the same provided that bidders use equilibrium strategies.

2 Revenue Equivalence Theorem analysis

Theorem 1 *For any two Bayesian-Nash incentive compatible mechanisms, if the surplus function is the same in both mechanisms, the valuation of each player is drawn from the same continuous distribution and each player bid their optimal strategy then the expected payments of all types are the same in both mechanisms, and hence the expected revenue (sum of payments) is the same in both mechanisms.*

Let's clarify some important terms. The maximal valuation that a consumer has for a good is the maximal price he is willing to pay for this good. The consumer's surplus is the difference between the valuation that a consumer have of a good and the price he pays for this good. Namely, it is the gain of a consumer after a purchase. Finally the optimal strategy is the bid that a player should call in order to maximize his surplus. We call it as a Nash equilibrium when no deviation is worth it.

This part will bring a proof of the revenue equivalence theorem. To do so we will consider two standard auction mechanisms: second-price and first-price.
Let's assume an auction with two risk-neutral bidders and one seller. The seller sells a single item. Each bidder i has a valuation v_i of the good, v_i is drawn independantly in a uniform distribution $[0, 1]$. We study standard auction so the bidder with the highest bid wins. Bidder i calls out a bid $b_i(v_i)$ that is increasing in v_i so that the bidder with the highest valuation of the good wins. We recall that the revenue equivalence theorem states that if there are 2 bidders with values drawn from $U[0, 1]$ then for any standard auction the winner bidder with valuation v will pay in average $\frac{1}{3}$ and will have an expected surplus $\frac{1}{2}v^2$. More generally, if there are N bidders with valuation from a continuous distribution, then any standard auction leads to the same expected highest bid and the same expected bidder surplus.

2.1 Second-price auction

To evolve the proof of the revenue equivalence theorem let's study the case in the second-price auction. We will first clarify the payment rule for this auction, then we will show that the optimal strategy for each player is to bid their maximal valuation of the good noticed v_i for player i. Next step will be to find the expected v_i assuming that $v_i > v_j$ is $\frac{2}{3}$ in our framework. Finally we will deduce that the expected bid of player i is $b_i^* = \frac{1}{2}v_i = \frac{1}{3}$ and the expected surplus is given by $S(v_i)^* = \frac{1}{2}v_i^2 = \frac{2}{9}$.

The payment rule of this auction is the following:

- if $b_i < b_j$, bidder i pays 0
- if $b_i > b_j$, bidder i pays b_j

The Nash equilibrium for a Vickrey auction is that all players bid their maximal valuation for the good namely: $b_i = v_i$. It is a dominant strategy because if $v_i > v_j$ then player i has a payoff equal to $v_i - v_j > 0$ because i would win the auction and pays the second price namely v_j. Player j's payoff is 0. There is no incentive to deviate for each player, indeed, if player i deviates, he loses surplus. If player j wants to change is payoff of 0, then he has to bid more than v_i in which case the payoff would be $v_j - v_i < 0$ so neither player i nor player j have an incentive to deviate.

The equilibrium strategies for both players are:
$b_i^* = v_i$
$b_j^* = v_j$
Now let's assume that $v_i > v_j$ so that player i wins. The next step is to find to what v_i and v_j are equal. To do so we have to find the expected value of the highest draw in a uniform law $U[0, 1]$.

Consider N independent draws from a uniform distribution over $[0, 1]$. On average, what is the highest draw?

Let X_i be a single draw from the uniform distribution. Then it follows that it has a cumulative density function of

$$F(x) = \begin{cases} x, & 0 \leq x < 1, \\ 1, & x = 1, \\ 0, & \text{Otherwise.} \end{cases}$$

For N draws, then, what we are looking for is $Y = \max(X_1, X_2, \ldots X_N)$. The cumulative density function of Y is equal to $\mathbb{P}(Y \leq y)$.
Since Y is the max, no independant draw X_i can be greater than y thus we can write:

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \mathbb{P}(X_1 \leq y, X_2 \leq y, \ldots, X_N \leq y) \\ &= \mathbb{P}(X_1 \leq y)\mathbb{P}(X_2 \leq y) \ldots \mathbb{P}(X_N \leq y) \\ &= F(y)F(y) \ldots F(y) \\ &= [F(y)]^N \end{aligned}$$

The above cumulative density function is continuous, so we can find the probability density function of Y by taking its derivative: $NF(y)^{N-1} \times f(y)$ and since we are over $[0, 1]$, $f(y) = 1$.
Then reminding that $\mathbb{E}(X) = \int_a^b xf(x)\, dx$, we can calculate:

$$\begin{aligned} \mathbb{E}(Y) &= \int_0^1 y(Ny^{N-1})\, dy \\ &= \int_0^1 Ny^N\, dy \\ &= \left[\frac{N}{N+1} y^{N+1}\right]_0^1 \end{aligned}$$

So we obtain

$$\boxed{\mathbb{E}(Y) = \frac{N}{N+1}}$$

Symmetrically we can obtain that on average the lowest draw is given by:

$$\boxed{\mathbb{E}(Y) = 1 - \frac{N}{N+1}}$$

Let's go back to our model. As we have 2 players, $N = 2$ so the expected highest valuation of the good is $v_i = \dfrac{2}{3}$ and the expected lowest valuation is $v_j = \dfrac{1}{3}$ which is more generally $v_j = \dfrac{1}{2}v_i$.
We can see that the expected payment by the winner is as we have seen in the theorem:

$$\boxed{b_i^* = v_j = \frac{1}{2}v_i = \frac{1}{3}}$$

Now let's calculate the expected surplus of the winner. Notice that if a bidder has value v_i, he expects to win whenever the other bidder has a value less than v_i; which happens with probability equal to v_i. We can deduce the expected surplus: $S(v_i) = v_i(v_i - \dfrac{1}{2}v_i)$.

$$\boxed{S(v_i)^* = \frac{1}{2}v_i^2 = \frac{2}{9}}$$

2.2 First-price auction

Now we need to find the expected bid and revenu of this type of auction in order to compare with the second-price auction results and so prove the revenue equivalence theorem. We will first clarify the payment rule for this auction, then we will show that the optimal strategy for each player is to bid half of their maximal valuation of the good noticed v_i for player i. Finally we will deduce that the expected bid of player i is $b_i^* = \frac{1}{2}v_i = \frac{1}{3}$ and the expected surplus is given by $S(v_i)^* = \frac{1}{2}v_i^2 = \frac{2}{9}$ exactly the same as the second price auction.

The payment rule of this auction is the following:

- if $b_i < b_j$, bidder i pays 0
- if $b_i > b_j$, bidder i pays b_i

A first price auction with two risk-neutral bidders whose valuations are independantly drawn from a uniform distribution $U[0, 1]$ has Nash equilibrium strategies: $(\frac{1}{2}v_i, \ \frac{1}{2}v_j)$. Let's proove that this equilibrium exists.

Assume that bidder j bids $b_j = \dfrac{1}{2}v_j$, we need to find the best response b_i of bidder i.
Player i wins when $b_i > \dfrac{1}{2}v_j$, namely $v_j < 2b_i$. If this inequality is true, then player i has a surplus equals to $v_i - b_i$.
Inversely, player i loses when $v_j > 2b_i$ and so has a surplus equals to 0.

We can then deduce the expected surplus of bidder i given the strategy of bidder j.

$$\begin{aligned}\mathbb{E}(S_i) &= \int_0^{2b_i} (v_i - b_i)\ dv_j + \int_{2b_i}^{1} 0\ dv_j \\ &= \Big[(v_i - b_i)v_j\Big]_0^{2b_i} \\ &= 2v_ib_i - 2b_i^2\end{aligned}$$

To find the best b_i we have to maximize the expected surplus by taking its derivative and by equalizing it to 0.

$\dfrac{\partial \mathbb{E}(S_i)}{\partial b_i} = 0 \Leftrightarrow 2v_i - 4b_i = 0$, so we have: $b_i = \dfrac{1}{2}v_i$.
The best strategy of player i is to bid half of his valuation, and as the game is symmetric player j's best response is $\dfrac{1}{2}v_j$.
As shown in the second-auction case, assuming that valuations are drawn independantly from a uniform distribution and that $v_i > v_j$, the expected valuation of bidder i is $v_i = \dfrac{2}{3}$.
As $v_i > v_j$, bidder i wins and pay its bid, so we have:

$$\boxed{b_i^* = \frac{1}{2}v_i = \frac{1}{3}}$$

The optimal bid of the winner is exactly the same under second-price and first-price auction. Here again, notice that if a bidder has value v_i, he expects to win whenever the other bidder has a value less than v_i; which happens with probability equal to v_i. The expected surplus will be the same as the second-price auction one: $S(v_i) = v_i(v_i - \dfrac{1}{2}v_i)$.

$$\boxed{S(v_i)^* = \frac{1}{2}v_i^2 = \frac{2}{9}}$$

More generally, for any given standard auction, denoting $S(v)$ the expected utility of player with a valuation v of the good and b the bid he called out, his surplus is the following:
$S(v) = v\mathbb{P}(v) - b$, so
$S(v)' = \mathbb{P}(v) = v$ in our context. $S(v)'$ is the probability to win the auction. Now for any standard auction, we denote $S(0) = 0$, namely a bidder with the lowest possible valuation of the good make 0 expected surplus.
Given a uniform distribution $U[0,1]$, the *Fundamental Theorem of Calculus* gives us that: $S(v) = S(0) + \int_0^v S(v)'\ dv = \int_0^v v\ dv$ which gives us the expected surplus of a bidder with valuation v for any standard action.

$$\boxed{S(v) = \frac{1}{2}v^2}$$

3 Entry cost in Vickrey's auction

The objective of this part is to ease the assumption that there is no entry cost in order to find the optimal bidding strategy as part of the second-price auction. The main difference with the previous analysis is that the game is now in two steps: the first step is to decide whether to participate to the auction and the second step is to define the optimal bidding strategy.

Let's focus on the simple case to understand the mechanism of the first step. We will see that the objective is to find a cut-off, where we decide to enter the auction. The idea is that this cut-off will be called $\tilde{v}$ and if our valuation is higher than $\tilde{v}$ then we enter, is our valuation is lower then we do not participate.

To find $\tilde{v}$ let's take a simple model in the case of a Vickrey's auction (second-price auction). There are two bidders whose valuations are drawn from a continuous distribution $U[0,1]$ and there is a cost equal to F if we enter the auction.

Firstly, the objective is to find the threshold $\tilde{v}$. As $\tilde{v}$ is the minimal valuation that allows a participation, a bidder with a valuation equals to $\tilde{v}$ wins only if he is the only one to participate. He is the only one to participate only if other players have a valuation inferior to $\tilde{v}$ which happens with the probability: $\mathbb{P}(v_1 < \tilde{v}) \times ... \times \mathbb{P}(v_{N-1} < \tilde{v})$ which is equal to $\tilde{v}^{N-1}$. We assume that the reserve price is 0. Then the bidder with valuation $\tilde{v}$ would have a payoff equals to: $\tilde{v} \times \tilde{v}^{N-1} = \tilde{v}^N$. The last step in order to find the cut-off $\tilde{v}$ is to consider the case when the bidder which has a valuation $\tilde{v}$ is indifferent between participating or not to the auction. This player is indifferent when his payoff is equal to the cost of entry, namely: $\tilde{v}^N = F$ which gives us: $\tilde{v} = F^{\frac{1}{N}}$.

Now that we have determine the first stage of the game, we can deduce the optimal bidding strategy in the second stage. If a player has a valuation higher than $\tilde{v}$ then the optimal strategy doesn't change with the previous case. The Nash equilibrium strategy is to bid his own valuation of the good. The solution with entry cost in a Vickrey's auction is the following:

$$b_i = \begin{cases} v_i, & v_i > \tilde{v} = F^{\frac{1}{N}}, \\ \text{No entry}, & v_i < \tilde{v} = F^{\frac{1}{N}} \end{cases}$$

The very interesting thing about the cost of entry is the idea of sunk cost fallacy. The basic previous model predicts that once you have paid the entry cost you should bid the same way as you would if there were no entry cost. We can imagine that intuitively players are incitated to deduce the entry cost to the bid they wanted to call before the auction. However, the previous model show that when a bidder has paid to enter the auction, any entry fee that he might have paid should have no bearing on his bidding strategy.

Sources

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