Stochastic Optimal Control Matching

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Overview

1. **Setup and Preliminaries**

2. **Stochastic Optimal Control Matching**

3. **Experiments and results**

4. **Conclusion**

Evolution of Generative Models

2020   **DDPM:** Denoising Diffusion Probabilistic Models interpret generation as reversing a discrete noise-adding process, learning to denoise at each step. They produced high-quality samples but required thousands of slow sampling steps.

2021   **Score-based Models:** Score-based generative models extended diffusion to continuous-time SDEs, learning the score function ($\nabla_x \log p_t(x)$) to reverse a stochastic diffusion process. This unified diffusion with stochastic control, allowed probability flow ODEs, and sped up sampling.

2023   **Flow Matching:** Flow matching views generation as learning a deterministic ODE vector field that directly transports a simple distribution (e.g., Gaussian) to data. This removed stochasticity and significantly improved efficiency compared to diffusion/score methods.

What is a Stochastic Control Problem?

A stochastic control problem involves finding an optimal control policy to steer a dynamical system under uncertainty.

Key Components

- **State Process:** $X_t \in \mathbb{R}^d$ (position in state space at time t)
- **Control Process:** $u_t \in \mathbb{R}^d$ (action/decision at time t)
- **Noise Process:** W_t (random disturbances, typically Brownian motion)

Dynamics (SDE)

$$dX_t = f(X_t, u_t, t)dt + g(X_t, t)dW_t \quad (1)$$

Cost Function

$$J(u) = \mathbb{E}\left[\int_0^T L(X_t, u_t, t)dt + \Phi(X_T)\right] \quad (2)$$

The Goal: Finding Optimal Control

Optimal Control u^*

Find the control policy u^* that minimizes the expected cost: $u^* = \arg\min_u J(u)$

Classical Approaches

- **Hamilton-Jacobi-Bellman (HJB) equation:** Partial differential equation approach
- **Pontryagin's Maximum Principle:** Necessary conditions for optimality
- **Dynamic Programming:** Discrete-time recursive approach

Challenge

These classical methods become computationally intractable in high dimensions due to the *curse of dimensionality*.

Introducing Stochastic Optimal Control Matching

SOCM offers a more principled, stable, and accurate way to learn generative dynamics by blending stochastic control theory with modern matching-based generative modeling.

Key Novel Contributions

1. **Controlled Stochastic Process:** Views the generation process as a controlled stochastic process bridging a simple distribution to data.

2. **Least-Squares Matching:** Learning the control via least-squares matching—a stable, convex regression objective.

3. **Joint Optimization:** Optimizing control and variance-reducing reparameterization matrices simultaneously, for efficient learning.

4. **Path-wise Reparameterization:** Introducing a path-wise reparameterization trick, boosting gradient estimation quality.

The SOCM Framework

Key Components

- **Controlled Stochastic Process:** $dX_t = f(X_t, u_t, t)dt + g(X_t, t)dW_t$
- **Cost Function:** $J(u) = \mathbb{E}\left[\int_0^T L(X_t, u_t, t)dt + \Phi(X_T)\right]$
- **Control Policy:** $u^* = \arg\min_u J(u)$

Learning Objective

Minimize the expected cost using a least-squares regression objective:

$$L(u) = \mathbb{E}\left[\|X_T - X_{data}\|^2\right] \tag{3}$$

Advantages of Controlled Stochastic Processes

Why model generation as a controlled stochastic process rather than a fixed deterministic flow?

Key Advantages

1. **Powerful Theoretical Tools:** Access to established methods from stochastic optimal control theory like Hamilton-Jacobi-Bellman equations and dynamic programming.

2. **Learnable Trajectories:** The trajectory of the distribution is no longer fixed (unlike hand-designed noise schedulers in DDPM) and can be learned and optimized to improve efficiency and quality.

3. **Enhanced Interpretability:** The objective becomes finding an optimal control policy, which provides clear insight into how the model moves from its initial distribution to the final complex data distribution.

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Block

Sample text

Alertblock

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Examples

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Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

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Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption

Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$

Figure

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Citation

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References

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