Parametric Equations at the Circus: Trochoids and Poi Flowers

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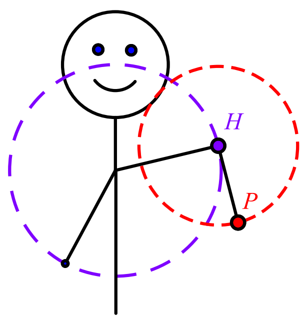
Imagine yourself at the circus. The lights go dim and you catch a faint crackle before whirling flames command your attention with mesmerizing patterns. You are witnessing a performance of poi spinning, see Figure 1(a). First practiced by the Maori in conjunction with music and storytelling, poi spinning is a performance art in the family of juggling in which weights on short chains are swung around to make visually interesting patterns. We are going to concentrate on a class of technical moves for poi, referred to as flowers, and transitions between them. These patterns, naturally described using parametric equations, are familiar to many as the Spirograph [**3**] and are closely related to the cycloid, which was first studied by Galileo [**4**, p. 303] and lead to a wealth of mathematical discovery and controversy [**5**].

(a) Poi flower



(b) Poi spinning

Figure 1. Poi spinners

Building the parametric equations

The basic mechanics of poi flowers are that the arm rotates with the elbow straight and the poi rotates around the hand with its direction of rotation independent from that of

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the arm. Patterns where the hand and poi rotate in the same direction are called *inspin* and patterns where they rotate in opposite directions are called *antispin*.

For simplicity, let the center of the shoulders be the origin, with the positive x-axis extending horizontally to the viewer's right, and let the arm of the spinner be of unit length. The path of the hand rotating in a counterclockwise direction starting from the positive x-axis is, as we learn from the unit circle, given by the parametric equations

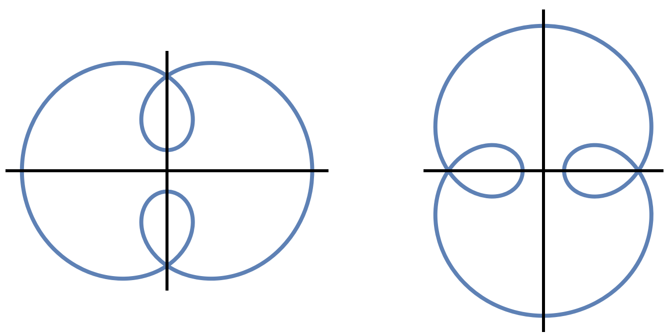
$$x = \cos(t), \quad y = \sin(t).$$

Now for the poi. In Figure 1(b), point H represents the hand and point P the poi head. In practice, the length of the poi is between $1/2$ and one full arm length; we will use a length of $3/4$ of the arm. In the flower patterns we will focus on, the poi head rotates three times for every hand rotation. We give equations for the patterns in which the hands rotate counterclockwise starting at the positive x-axis as examples; in practice the starting point and direction of rotation are freely varied by poi spinners. (Try writing equations for these variations!)

We begin with equations for an inspin flower,

$$x = \cos(t) + \tfrac{3}{4}\cos(3t), \quad y = \sin(t) + \tfrac{3}{4}\sin(3t). \tag{1}$$

Here the first terms represent the hand and the second terms the poi head. Multiplying the second terms by $3/4$ adjusts the radius of the poi's rotation and the $3t$ arguments adjust the frequency of rotation to be three times that of the hand. This pattern is called a vertical two petal inspin flower, shown in Figure 2(a).



(a) Vertical inspin flower (b) Horizontal inspin flower

Figure 2. Inspin flowers

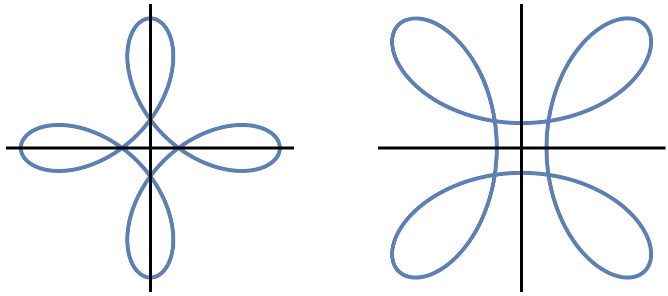
A natural question is how to change the position of the petals. To maintain both vertical and horizontal symmetry, poi spinners often position the petals horizontally by a π radian change in the starting position of the poi,

$$x = \cos(t) + \tfrac{3}{4}\cos(3t + \pi), \quad y = \sin(t) + \tfrac{3}{4}\sin(3t + \pi),$$

giving a horizontal two petal inspin flower, shown in Figure 2(b).

For an antispin flower, we want the poi head rotating in the opposite direction to the hand, thus for a hand rotating counterclockwise, we want the poi rotating clockwise. Changing the sign of the parameter t in the second terms switches the direction, giving

$$x = \cos(t) + \tfrac{3}{4}\cos(-3t), \quad y = \sin(t) + \tfrac{3}{4}\sin(-3t). \tag{2}$$



(a) Diamond mode antispin (b) Box mode antispin

Figure 3. Antispin flowers

These equations create a four petal diamond mode antispin flower, shown in Figure 3(a).

The other common four petal antispin flower, called box mode, is created by changing the starting point of the rotation of the poi by π, as in Figure 3(b). All of these curves are examples of centered trochoids [**3**]. For an alternative explanation of equations similar to ours see [**1, 2**].

Note that three rotations of the poi to one of the hand gave two petals in the inspin case and four in the antispin case. Do you see why? We refer to these patterns as three beat flowers, for the number of rotations of the poi per that of the hand. What other possibilities can you come up with? Think about changing frequencies of rotation or starting positions of the hands and poi. Explore using the Geogebra file available at `http://www.geogebratube.org/student/mHtxkckyQ`, also archived on the supplements webpage of *The College Mathematics Journal*.

Transitions between patterns

We would like to build a performance featuring some of these beautiful three beat flowers. How can we transition between patterns? Transitions that flow naturally look better than having to stop entirely to change patterns. Recall the physical situation, with weights being spun round on the ends of chains; the momentum and pattern of the poi motion cannot easily be changed. This leads us to the following definition.

Definition (Smooth Transition). There is a *smooth transition* between two poi flower patterns if there is a point where the following conditions are met.

1. The hand and poi positions are the same.
2. The velocity vectors of the poi have the same direction.

Poi performers also change direction almost as naturally at moments when the poi head is moving at its slowest.

Definition (Zero Point). A point in a poi flower pattern where the magnitude of the velocity of the poi head is minimum is called a *zero point*.

We will call a transition between poi flower patterns which satisfies Condition 1 but changes the direction of the poi head at a zero point a *semi-smooth transition*. Where are there (semi-) smooth transitions between poi flower patterns? (See [**6**] for an alternative analysis of the transitions.) Begin by thinking through the implications of Condition 1. They boil down to the adage that you can't be in two places at once. What role does timing play in poi transitions? Poi spinners define two flowers as having the same timing if the poi pass through the positive y-axis at the same time. Switching between two patterns which differ only in timing would require changing everything by 180° instantaneously. Thus there are no (semi-) smooth transitions which differ only in timing.

What about changing just the direction? Or direction and timing? Determining when the conditions are met for a smooth transition are easy. (Can you see how?) To consider semi-smooth transitions as well, we will use calculus to identify where the minimum values for the magnitude of velocity occurs. Note that it is sufficient to look at one of each of the inspin and antispin flowers and one direction of rotation.

Refer to (1) for the equations giving the position of the poi in a vertical inspin flower. Recall that velocity is the derivative of position and that parametric functions are differentiated termwise, hence we find the velocity vector

$$\langle v_x, v_y \rangle = \langle -\sin(t) - \tfrac{9}{4}\sin(3t), \cos(t) + \tfrac{9}{4}\cos(3t) \rangle.$$

The squared magnitude is given by $|v|^2 = V = (9/2)\cos(2t) + 97/16$. To find the minimum values, as every good calculus student knows, we find the critical points by setting the derivative equal to zero and classify them using the second derivative test. In our case, we find that the zero points are when $t = (2n+1)\pi/2$ for $n \in \mathbb{Z}$. (Check this. Done? Good.) These are precisely at the ends of the petals of the vertical inspin flowers.

Following the same process in the diamond mode antispin case (2), we find that the flower has squared magnitude of velocity $V = 97/16 - (9/2)\cos(4t)$, which has critical values at $t = n\pi/4$ for $n \in \mathbb{Z}$. This is a relative minimum when n is even, a relative maximum when n is odd. The zero points are again at the ends of the petals. One can show that this is true in general (and could be used as the definition of a petal).

Lemma (Zero Points). *The zero points of three beat poi flowers occur at the ends of the petals.*

Our conditions for smooth and semi-smooth transitions are met when the two graphs of the patterns intersect with parallel tangent lines, see Figure 4. Notice that these types of intersections only occur on the axes, where all but the box mode flowers have zero points. Also, from Figures 4(a) and 4(b) we see that there are no (semi-) smooth transitions between the varieties of inspin flowers or the modes of antispin flowers.

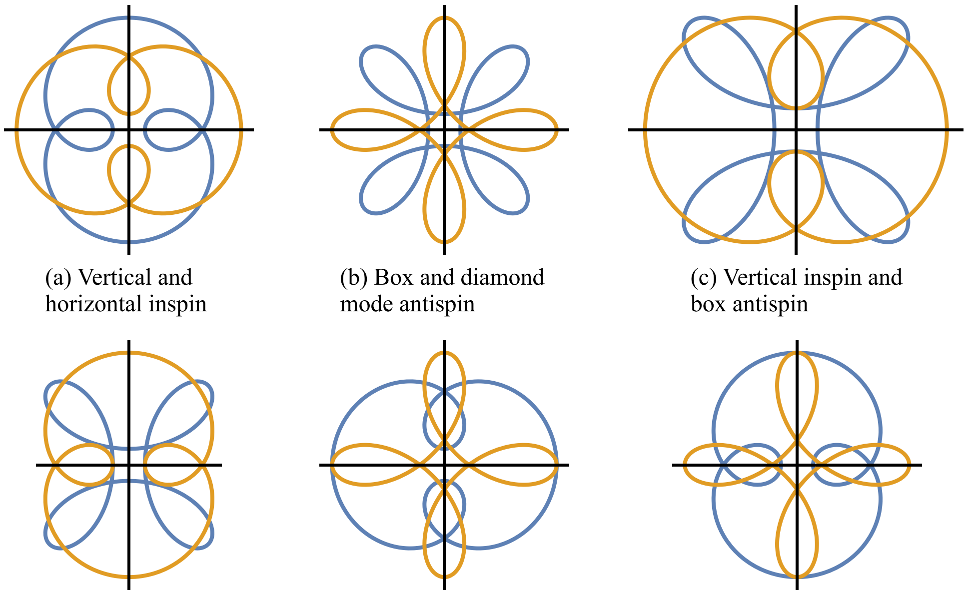
Smooth transitions require that the poi heads be moving in the same direction, so the hands must be moving in opposite directions, forcing the transitions to switch between inspin and antispin. Our definition of timing then implies that these transitions are between patterns with the same timing if the transition is on the y-axis, and different if it is on the x-axis. Also, none of the transition points between inspin and antispin flowers are zero points for both patterns. Together, we have the following result.

Theorem (Transitions Between Patterns). *At points where the hands and poi heads coincide, the possible smooth and semi-smooth transitions are as follows.*

Smooth transitions *only occur between inspin and antispin flowers with different hand directions, and with different timing for x-axis transitions and the same timing for y-axis transitions.*

Semi-smooth transitions *occur anywhere there is a zero point.*

This concludes our introduction to the mathematics of poi spinning, though there is still much to explore. Mathematics really does show up everywhere.



(a) Vertical and horizontal inspin
(b) Box and diamond mode antispin
(c) Vertical inspin and box antispin
(d) Horizontal and box antispin
(e) Vertical inspin and diamond antispin
(f) Horizontal and diamond antispin

Figure 4. Combinations of three beat flower patterns

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Summary. Poi spinning is a performance art, related to juggling, where weights on the ends of short chains are swung to make interesting patterns. We study a certain class of moves for poi where the patterns created are centered trochoids. Like all curves in the cycloid family, they are best expressed using parametric equations. Using the calculus of the curves, we find that there are just a few places where one pattern can be smoothly transformed into another.

References

1. B. Drexler, The Math of Poi: Flowers, Roulettes and Trochoids, *Drex's Tech Poi Blog* 216, 2011, `http://drexfactor.com/weirdscience/2011/12/05/drexs_tech_poi_blog_216_math_poi_flowers_roulettes_and_trochoids`.
2. ———, The Math of Poi part 2—Roulettes for the Unit Circle, *Drex's Tech Poi Blog* 218, 2011, `http://drexfactor.com/weirdscience/2011/12/12/drexs_tech_poi_blog_218_math_poi_part_2_roulettes_unit_circle`.
3. L. M. Hall, Trochoids, roses and thorns—Beyond the spirograph, *College Math. J.* **23** (1992) 20–35, `http://dx.doi.org/10.2307/2686194`.
4. G. Simmons, *Calculus Gems: Brief Lives and Memorable Mathematics*. Mathematical Association of America, Washington DC, 2007.
5. J. Martin, The Helen of geometry, *College Math. J.* **41** (2010) 17–28, `http://dx.doi.org/10.4169/074683410X475083`.
6. N. Yee, Vulcan Tech Gospel, 2011, `http://noelyee.weebly.com/vulcan-tech-gospel.html`.

Additional material for this article is available at `http://www.maa.org/cmj/supplements`.