A RIGHT-INVARIANT SUB-RIEMANNIAN SETTING FOR LARGE DEFORMATION MODELS

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Abstract. The Large Deformation Diffeomorphic Metric Mapping (LDDMM) framework introduced in [9, 38, 14] is a widely recognized method in shape analysis and computational anatomy. It aims at finding a diffeomorphism that deforms optimally a given shape into another one using tools from Riemannian geometry. It was acknowledged in Arguillère's Ph.D. thesis [3, 2, 4, 1] that this approach amounts to endowing the group of deformations $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ with a strong right-invariant sub-Riemannian structure.

In this paper, we extend these ideas to the category of half-Lie groups and we study right-invariant strong sub-Riemannian metrics on them. Several illustrative examples are discussed.

Introduction

The theory of shapes spaces plays a central role in many applications including computational anatomy, medical imaging, biology, etc. A main goal of this field is to define computable metrics between shapes in order to compare them and develop statistical tools. The approach taken in the Large Deformation Diffeomorphic Metric Mapping (LDDMM) framework [9, 38, 14] is to deform a source object into a target object and to study this deformation. The deformation is given by a flow of diffeomorphisms of finite regularity generated by a time-dependent vector field, and the registration problem is induced by the minimization of a functional penalizing the kinetic energy of the deformation. This setting allows to adapt to the group of diffeomorphisms classical results from the theory of right-invariant metrics on Lie groups [5, 15, 25, 40].

The emergence of new datasets, imaging techniques and problems has led to the necessity of extending the classical framework [1] that was focusing on diffeomorphisms to address new situations involving more general infinite dimensional groups. In order to keep a Banach structure, we are forced [32, 33] to deal with topological groups such that the right translations are smooth, namely half-Lie groups [23, 8]. In most examples, the left translations and the inverse mapping may fail to be smooth so that the groups are not Lie groups.

In this paper, we summarize and report on the authors's results in [34] which studies right-invariant sub-Riemannian metrics on more general groups. In particular, we derive a Hamiltonian formulation for the geodesic equations.

1. Half-Lie groups

1.1. **Definitions.** The category of Banach half-Lie groups gives a natural framework to deal with differentiable groups with a Banach manifold structure. Riemannian geometries on such spaces were recently studied by Bauer, Harms, and Michor in [8]:

Definition 1.1 (Half-Lie group)**.** *A Banach (right)* half-Lie group *is a topological group with a smooth Banach manifold structure, such that the left multiplication $R_{g'} : g \mapsto gg'$ is smooth.*

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In the LDDMM framework, the most common example is the group of C^k diffeomorphisms that vanish at infinity, and whose derivatives also vanish at infinity:

$$\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) = \big(\mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)\big) \cap \mathrm{Diff}^1(\mathbb{R}^d).$$

In Arguillère's Ph.D. thesis [2, 3], LDDMM is performed on the group $\mathrm{Diff}_{H^s}(M)$ of Sobolev diffeomorphisms on a finite-dimensional manifold M of bounded geometry. This space is also a half-Lie group [8] and has a Hilbert structure, and was thus highly studied in the context of infinite-dimensional geometry and fluid dynamics [15].

The main issue with dealing with half-Lie groups is that we lose the smoothness of the right multiplication and the inverse, and we therefore cannot define the classical objects from Lie group theory such as the exponential map and the adjoint representation. However, following [8, 23], we can gain regularity on the right by adding an extra assumption on the considered half-Lie groups.

Definition 1.2 (Right-invariant local addition)**.** *Let G be a Banach half-Lie group, and denote $\pi_G : TG \to G$ its tangent bundle. A* right-invariant local addition *on G is a smooth map $\tau : V \subset TG \to G$, where V is an open subset of TG, such that*

- *for every $g \in G$, we have $0_g \in V$ and $\tau(0_g) = g$, where 0_g denotes the zero section of TG*
- *the mapping $(\pi_G, \tau) : V \to G \times G$ is a diffeomorphism onto its range.*
- *for $g \in G$, $TR_g(V) = V$ and $\tau \circ TR_g = R_g \circ \tau$*

Example 1.3. *The mapping $(x, \tau) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto x + \tau$ is a local addition on $\mathbb{R}^d$ (it is the exponential map of the canonical Riemannian metric on $\mathbb{R}^d$) and it induces the following right-invariant map on* $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$

$$\begin{array}{ccc} T\,\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \to & \mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d) \\ (\phi, u \circ \phi) & \mapsto & \phi + u \circ \phi \end{array}$$

Since $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ *is open in* $\mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)$*, for $u \in C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ close enough to 0, the map $\phi + u \circ \phi$ is in* $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$*. Therefore, there exists some subset $V \subset T\,\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ such that the restriction of the previous map is a right-invariant local addition.*

This will allow to gain some regularity for the left translation, as shown in [8]. For $k \in \mathbb{N}$, define G^k as the set of C^k-elements $g \in G$ such that $L_g : G \to G$ and $L_{g^{-1}} : G \to G$ are C^k. Equivalently, denoting $\mathrm{Diff}_{C^k}(G)^G$ the set of C^k right-invariant diffeomorphisms on G, the set G^k can be defined through the identification:

$$\begin{array}{ccc} G^k & \longrightarrow & \mathrm{Diff}_{C^k}(G)^G \\ g & \longmapsto & L_g \end{array} \tag{1}$$

The following result [8] shows the regularity gained with such a construction :

Proposition 1.4 (Differentiable elements)**.** *Let G be a Banach half-Lie group with a right-invariant local addition, and define the family of subgroups of differentiable elements $\{G^k, k \in \mathbb{N}^*\}$ as before. Then, the groups G^k are Banach half-Lie groups, and satisfy the following properties:*

(G.1): *G^{k+1} is a subgroup of G^k with smooth inclusion and, for $l \geq 0$, G^{k+l} is a subset of $(G^k)^l$.*

(G.2): *For $l \geq 0$, the induced multiplication*

$$\begin{array}{ccc} G^{k+l} \times G^k & \longrightarrow & G^k \\ (g', g) & \longmapsto & g'g \end{array}$$

is C^l and C^∞ in the first variable g' for g fixed.

(G.3): *For $l \geq 0$, the induced left infinitesimal action*

$$\begin{array}{ccc} T_eG^{k+l} \times G^k & \longrightarrow & TG^k \\ (u, g) & \longmapsto & u \cdot g = \partial_{g'}(g'g)_{|g'=e}(u) = T_eR_g(u) \in T_gG^k \end{array}$$

is a C^l mapping, and C^∞ with regards to the first variable.

Proof. The proof is mostly included in the work of Bauer, Harms, and Michor [8, Theorem 3.4] and uses the identification (1). □

We finish this part by introducing equivalents of the adjoint representation and of the Lie bracket on half-Lie groups. For $g \in G^1$, the interior automorphism $\mathrm{int}_g : G \to G, h \mapsto ghg^{-1}$ is C^1. We can thus define the adjoint representation:

$$\mathrm{Ad}_g : \left\{ \begin{array}{ccc} T_eG & \longrightarrow & T_eG \\ v & \longmapsto & \mathrm{Ad}_g(v) = T_e \, \mathrm{int}_g(v) = (T_eR_g)^{-1} \circ T_eL_g(v) \end{array} \right.$$

Following [8], we can also define an analogous of the Lie bracket $[\cdot, \cdot] : T_eG^1 \times T_eG^1 \to T_eG$ by

$$[u, v] = T_e\tilde{u}(v) - T_e\tilde{v}(u)$$

where, for $w \in T_eG^1$, the mapping $\tilde{w} : G \to TG$ is the C^1-right-invariant vector field induced by w, defined by $\tilde{w}(g) = T_eR_g(w) = wg$. The equality has to be understood in charts. This bracket product satisfies many properties of usual Lie brackets:

Proposition 1.5 (Lie bracket)**.** *The Lie bracket $[\cdot, \cdot] : T_eG^1 \times T_eG^1 \to T_eG$ satisfies*

- $[T_eG^{k+1}, T_eG^{k+1}] \subset T_eG^k$ *and coincides with the bracket product $[\cdot, \cdot]_k$ on G^k*
- $[\cdot, \cdot]$ *is a skew-symmetric continuous bilinear mapping and satisfies a Jacobi identity:*

$$\forall u, v, w \in T_eG^2, \ [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \tag{2}$$

- *The inverse of the adjoint representation $\mathrm{Ad}_g^{-1}(v) = \mathrm{Ad}_{g^{-1}} v$ induces a C^1 mapping $\mathrm{Ad}^{-1} : G^2 \times T_eG^1 \to T_eG$ and*

$$\forall u \in T_eG^2, \forall v \in T_eG^1, \ T_e\left(\mathrm{Ad}^{-1}(v)\right)(u) = -[u, v]. \tag{3}$$

1.2. **Regularity and evolution map.** In this part, we deal with regularity properties of half-Lie groups, in the sense that under some regularity conditions, there will exist a unique global flow associated with a right-invariant vector field. Suppose G is a Banach half-Lie group equipped with a right-invariant local addition so that we can define the groups G^k as previously. Let $I \subset \mathbb{R}$ be a compact interval, and suppose G^k is modeled on the Banach space $\mathbb{B}^k$. Let $AC_{L^p}(I, G^k)$ be the set of continuous curves in G^k that are absolutely curves in local charts (see appendix A for some results on the space of absolutely continuous curves, and [16] for a more complete exposition of absolutely continuous curves in infinite dimensional spaces)

Proposition 1.6 (Regularity of Banach half-Lie groups)**.** *Let $t_0 \in I$, and $u \in L^p(I, T_eG^k)$. Then, the ordinary differentiable equation:*

$$\left\{ \begin{array}{l} \dot{g}_t = u_t \cdot g_t = T_eR_{g_t}(u_t) \\ g_{t_0} = e \end{array} \right. \tag{4}$$

has a unique global (i.e. defined on I) solution $g \in AC_{L^p}(I, G^k)$.

Sketch of Proof. The proof is contained in [34], and in [8] for the special case where $u \in C^\infty(I, T_eG^k)$. The local existence and uniqueness follow from an application of the Picard-Lindelöf theorem, and the solution can be extended on the whole interval I using invariance properties. □

Proposition 1.6 allows to define the evolution map for any time-dependent vector field $u \in L^p(I, T_eG^{k+1})$.

Definition 1.7 (Evolution map). *We denote by* $\mathrm{Evol}_{G^k} : L^p(I, T_eG^k) \to AC_{L^p}(I, G^k)$ *the* evolution map *associating to any time-dependent vector field* $u \in L^p(I, T_eG^k)$ *the solution* $g \in AC_{L^p}(I, G^k)$ *of the ODE* (4)*.*

We can prove that the evolution map has some regularity, depending on the space where u lives:

Proposition 1.8 (Derivative of the evolution map). *Suppose I is a compact interval.*

(1) The mapping $\mathrm{Evol}_{G^k} : L^p(I, T_eG^k) \to AC_{L^p}(I, G^k)$ *is locally bounded and the restriction* $\mathrm{Evol}_{G^k} : L^p(I, T_eG^{k+1+l}) \to AC_{L^p}(I, G^k)$ *with* $l \geq 0$ *is* C^l*.*

(2) For $u, \delta u \in L^p(I, T_eG^{k+1+l})$*, its derivative* $\delta g = T_u \mathrm{Evol}_{G^k}(\delta u) \in AC_{L^p}(I \leftarrow g_{opt}^* TG^k)$ *is the unique solution of the linear Cauchy problem:*

$$\delta \dot{g}(t) = \partial_g (T_e R_g u_t)_{|g=g(t)} \delta g(t) + \partial_u (T_e R_{g_t} u)_{|u=u(t)} \delta u(t), \;\; \delta g(0) = 0 \tag{5}$$

where $g(t) = \mathrm{Evol}_{G^k}(u)(t)$*.*

Sketch of Proof. In local charts, the mapping $C : (g, u) \mapsto (g_0, \dot{g}_t - T_e R_g u)$ is differentiable and its derivative $\partial_g C : T_g AC_{L^p}(I, G^k) \to T_eG \times L^p(I, T_eG^k)$ is a Banach isomorphism. The result follows from the implicit function theorem. A complete proof of this result can be found in [34]; it uses ideas from [3, 16]. □

2. Strong right-invariant sub-Riemannian metrics

2.1. **Generalities.** The LDDMM framework uses a space of velocity fields V that is a Reproducing Kernel Hilbert Space (RKHS), continuously embedded in $C_0^{k+2}(\mathbb{R}^d, \mathbb{R}^d) = T_{\mathrm{id}} \mathrm{Diff}_{C_0^{k+2}}(\mathbb{R}^d)$. This induces a right-invariant sub-Riemannian metric on the group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$, as described in [3, 4, 1]. We use this example to define strong right-invariant sub-Riemannian metrics on G.

Let V be a Hilbert space continuously embedded in T_eG^k. We define, as in [1, 3], the sub-Riemannian structure $(G, G \times V, T_eR, \langle \cdot, \cdot \rangle_V)$ where $\langle \cdot, \cdot \rangle_V$ denotes the scalar product inducing the Hilbert structure on V, and T_eR is the derivative of the right multiplication

$$T_eR : \left\{ \begin{array}{ccc} G \times V & \longrightarrow & TG \\ g, u & \longmapsto & T_eR_g(u) \end{array} \right.$$

By **(G.3)** in Proposition 1.4, the mapping T_eR is C^k. The induced mapping $g \in G \mapsto T_eR_g \in \mathcal{L}(V, T_gG)$ is only C^{k-1} [33, Theorem 5.3], meaning T_eR defines a C^{k-1} bundle morphism (in the sense of [22]), hence the sub-Riemannian structure is only C^{k-1}.

Let us now recall some vocabulary from sub-Riemannian geometry. An absolutely continuous curve $g : I \to G$ is said to be *horizontal* if there exists a continuous lift $t \mapsto u(t) \in V$ such that

$$\dot{g}(t) = T_eR_{g(t)}(u(t)) \text{ for a.e. } t \in I.$$

We call *horizontal system* such a couple (g, u) that belongs to $\mathrm{AC}_{L^p}(I, G) \times L^p(I, V)$. In this right-invariant setting, if $g \in \mathrm{AC}_{L^p}(I, G)$ is a horizontal curve, there can only exist a unique $u \in L^p(I, V)$ such that (g, u) is a horizontal system. The *length* and *energy* of a curve $(g, u) \in \mathrm{AC}_{L^1}(I, G) \times L^1(I, V)$ are given by

$$L(g, u) = \int_I |u(t)|_V dt \qquad \text{and} \qquad E(g, u) = \frac{1}{2} \int_I |u(t)|_V^2 dt.$$

Hence, G becomes a metric space with the following *sub-Riemannian distance*

$$d_V(g_0, g_1) = \inf\{L(g, u),\ (g, u) \text{ is a } L^1 \text{ horizontal system joining } g_0 \text{ to } g_1\}.$$

Properties of the sub-Riemannian distance are summed up in the following proposition.

Proposition 2.1 (Sub-riemannian distance)**.** *The sub-Riemannian distance d_V is a true right-invariant distance and it is also equal to the infimum of the energy on horizontal L^2 systems, i.e. for $g_0, g_1 \in G$:*

$$d_V(g_0, g_1) = \inf\{\sqrt{2E(g, u)},\ (g, u) \text{ is a horizontal system joining } g_0 \text{ to } g_1\}.$$

A horizontal curve between two points of minimal length is called a *minimizing geodesic*. Its length is then equal to the distance between the two endpoints. Since the sub-Riemannian distance is also found by minimizing the energy, any horizontal system that minimizes the energy is also a minimizing geodesic and is parameterized with constant speed. Conversely, any minimizing geodesic (g, u) with constant speed also minimizes the energy, and we have

$$L(g, u) = \sqrt{2E(g, u)}.$$

We end this section with a little discussion on the different types of sub-Riemannian geodesics as well as on the differences with the Riemannian case and the sub-Riemannian finite-dimensional case (as described in [28]). This was described for the infinite-dimensional case in [3, 2]. Let us first define the endpoint mapping

$$\mathrm{End} : \left\{ \begin{array}{ccc} L^2(I, V) & \longrightarrow & G \\ u & \longmapsto & \mathrm{Evol}_G(u)(1) \end{array} \right.$$

The main difficulties will come from the fact that the endpoint mapping End is not a submersion and the space of horizontal curves with fixed endpoints therefore might not be a manifold. More precisely, let $g_1 \in G$, and let $(g_{opt}, u_{opt}) \in AC_{L^2}(I, G) \times L^2(I, V)$ be a minimizing curve for the energy, such that $g_{opt}(1) = g_1$. Then, the mapping $(E, \mathrm{End}) : u \mapsto (E(u), \mathrm{End}(u))$ cannot be a submersion at u_{opt}; otherwise the map (E, End) would be locally surjective from a neighborhood of u_{opt} to a neighborhood of $(E(u_{opt}), \mathrm{End}(u_{opt}))$ [22, Proposition 2.2]. This leads to two different cases:

(i) $(dE(u_{opt}), d\,\mathrm{End}(u_{opt}))$ has closed range.
(ii) $(dE(u_{opt}), d\,\mathrm{End}(u_{opt}))$ has dense range in $\mathbb{R} \times T_{g_1}G$.

By the closed range theorem, case (i) is equivalent to the adjoint mapping $(dE(u_{opt}), d\,\mathrm{End}(u_{opt}))^*$ being not injective, meaning there exist non-zero Lagrange multipliers $(\lambda, p) \in \mathbb{R} \times T_{g_1}G$ such that

$$\lambda dE(u_{opt}) + (d\,\mathrm{End}(u_{opt}))^* p = 0 \tag{6}$$

which is the classical case of finite dimension. It separates into two subcases: horizontal curves that satisfy (6) with $\lambda \neq 0$ (in this case we can simply choose $\lambda = -1$) are called *normal sub-Riemannian geodesics*, or simply *normal geodesics*. In particular, minimizing geodesics that are regular curves for the endpoint map are normal geodesics. We will derive those geodesic equations in the next section. If $\lambda = 0$, the geodesics are *singular curves* for the endpoint map. Such geodesics can also be characterized by an abnormal geodesic equation [28]. In case (ii), nontrivial Lagrange multipliers do not exist. Such geodesics are called *elusive geodesics* [2], and can occur only in infinite dimension. In the next section, we will only focus on the normal geodesics.

2.2. **Normal geodesics and Euler-Poincaré equations.** In this section, we study the normal geodesic equation associated with a strong right-invariant sub-Riemannian metric, and we show that this equation is equivalent to a reduced geodesic equation in the dual of the Lie algebra (the Euler-Arnold equations). We recall that $V \hookrightarrow T_eG^k$ is a Hilbert space, and we suppose for the rest of this section that $k \geq 2$. We denote by $K : V^* \to V$ the inverse of the Riesz canonical isometry. For $l \leq k$, let $i_l : V \to T_eG^l$ denote the smooth inclusion, and $i_l^* : T_e^*G^l \to V^*$ be its adjoint. The mapping i_l^* is not necessarily injective unless V is dense in T_eG^l (which is usually the case in the context of LDDMM). We also denote by $K^l : T_e^*G^l \to V$ the composition $K^l = K \circ i_l^*$. This mapping induces a vector bundle morphism, which we also denote by $K^l : T^*G^l \to TG^l$ and which is defined as

$$K^l : \left\{ \begin{array}{ccc} T^*G^l & \longrightarrow & TG^l \\ (g,p) & \longmapsto & K_g^l p, \end{array} \right.$$

where $K_g^l = T_eR_g \circ K^l \circ T_eR_g^*$. This mapping defines the cometric for the sub-Riemannian structure induced by V on G^l, for $l \leq k$. We now define the *normal sub-Riemannian Hamiltonian* as

$$H : \left\{ \begin{array}{ccc} T^*G & \longrightarrow & \mathbb{R} \\ (g,p) & \longmapsto & \frac{1}{2}\big(p \,|\, K_g^0 p\big) = \frac{1}{2}|K^0 T_eR_g^* p|_V^2 \end{array} \right.$$

This Hamiltonian can be derived from the pre-Hamiltonian $\mathcal{H} : T^*G \times V \to \mathbb{R}$ defined as $\mathcal{H}(g,p,u) = (p \,|\, T_eR_g(u)) - \frac{1}{2}|u|_V^2$ by the problem

$$H(g,p) = \sup_{u \in V} \mathcal{H}(g,p,u)$$

whose supremum is attained for $u(g,p) = K^0 T_eR_g^* p$.

This normal Hamiltonian is C^1 whenever $k \geq 2$, i.e. if V is embedded in T_eG^2. Let us compute its derivatives. We work in local coordinates $\boldsymbol{\phi} : \boldsymbol{U} \subset T^*G \to \phi(U) \times \mathbb{B}^*$ where $\boldsymbol{\phi}(g,p) = (\phi(g), d\phi(g)^* p)$ with $\phi : U \to \phi(U) \subset \mathbb{B}$ is a local chart and $U = \pi_{T^*G}(\boldsymbol{U})$ is an open subset of G. In these coordinates, we compute the partial derivative $\partial_p H(g,p) \in T^{**}G$:

$$\partial_p H(g,p)\delta p = \big(\delta p \,|\, K_g^0 p\big),$$

so that we actually have $\partial_p H(g,p) = K_g^0 p \in TG \subset T^{**}G$. Therefore, we can define the symplectic gradient of the normal Hamiltonian with respect to the canonical weak symplectic form (the exterior derivative of the Liouville form):

$$\nabla^\omega H(g,p) = (\partial_p H(g,p), -\partial_g H(g,p)) = \Big(K_g^0 p, -\big(\partial_g K_g^0 p\big)^* p\Big). \tag{7}$$

Proposition 2.2 (Hamiltonian flow)**.** *Let $g \in \mathrm{AC}_{L^2}(I,G)$ be an horizontal curve. Then, g_t is a normal geodesic if and only if it is the projection onto G of a curve $(g,p) \in \mathrm{AC}_{L^2}(I,T^*G)$ satisfying the Hamiltonian equations:*

$$(\dot{g}, \dot{p}) = \nabla^\omega H(g,p). \tag{8}$$

Proof. Let $p \in T_{g_1}G^*$, and let $F : L^2(I,V) \to \mathbb{R}$ be the mapping

$$F(u) = E(u) - (p \,|\, \mathrm{End}(u)).$$

For $u \in L^2(I,V)$, let also $p^u(t) \in AC_{L^2}(I,T^*G)$ be the unique curve such that $p^u(1) = p$ and $\dot{p}_t^u = -\partial_g \mathcal{H}(g(t), p^u(t), u(t))$. A computation then shows that for $\delta u \in L^2(I,V)$,

$$dF(u)\delta u = -\int_I \partial_u \mathcal{H}(g(t), p^u(t), u(t))\delta u dt.$$

In particular, u is a critical point of F if and only if $\partial_u \mathcal{H}(g(t), p^u(t), u(t)) = 0$, in which case the curve $(g(t), p^u(t))$ satisfies by construction the normal equation. This concludes the proof. $\square$

The normal Hamiltonian is right-invariant, that is to say invariant by the cotangent action, $H \circ TR_g^* = H$. This means that, for any $(g,p) \in T^*G$,

$$H(g,p) = h_e \circ m(g,p)$$

where $h_e(\nu) = H(e,\nu) = \frac{1}{2}|\nu|^2_{V^*}$, for $\nu \in T_e^*G$, and $m : T^*G \to T_e^*G$ is the momentum map defined as

$$(m(g,p) \mid u) = (p \mid T_e R_g(u))\,.$$

This momentum map and this new Hamiltonian give rise to another dynamic, which has been particularly studied for Riemannian geometry in infinite dimension [5, 15, 7, 29] and for sub-Riemannian metrics on diffeomorphism groups [3].

Proposition 2.3 (Sub-Riemannian Euler-Arnold equation)**.** *Let $g \in \mathrm{AC}_{L^2}(I,G)$ be a horizontal curve, with $u_t = \dot g_t g_t^{-1} = (T_e R_{g_t})^{-1}\dot g_t$. Then, g_t is a normal geodesic if and only if there exists a momentum map $m_t \in T_e^*G$, that is AC_{L^2} in the space $T_e^*G^1$, such that $K^0 m_t = u_t$ and that satisfies the equation (in $T_e^*G^1$) :*

$$\dot m_t + \mathrm{ad}^*_{u_t} m_t = 0. \tag{9}$$

In such case, the covector $p(t) = (T_e R^*_{g_t})^{-1} m(t)$ defines a curve $(g,p) \in \mathrm{AC}_{L^2}(I, T^*G)$ that satisfies the normal Hamiltonian equation (8)

Remark 2.4. *In this case, since $K^0 m_t = u_t$ a.e., the curve g_t satisfies the Cauchy problem*

$$\dot g_t = T_e R_{g_t}\big(K^0 \,\mathrm{Ad}^*_{g_t^{-1}}(m_0)\big),\; g_0 = e_G. \tag{10}$$

The proof of this result in the context of half-Lie groups (and more general graded group structures) can be found in [34]. We propose in the next section a reformulation of this computation using Lie-Poisson reduction theory.

2.2.1. *Lie-Poisson reduction.* Proposition 2.3 can be reformulated using reduction theory in infinite-dimensional manifolds. This was highly studied ([25] for example) for right-invariant Hamiltonians on finite-dimensional Lie groups leading to the standard Lie-Poisson and Euler-Poincaré reductions (see also [24]). In infinite-dimensional geometry, it gets a little bit trickier to define Poisson structures as the spaces we are dealing with are often non-reflexive and the symplectic form and Poisson brackets are therefore only weak [31, 30]. Another issue here is that the tangent space at identity T_eG of G is not a Lie algebra: it is not stable by the Lie brackets defined previously. We can still adapt some of these constructions in this half-Lie group setting and use the language of Poisson geometry. As previously, the cotangent bundle T^*G is naturally endowed with a weak symplectic structure by the canonical symplectic form ω. In local coordinates, we have

$$\omega_{g,p}(\delta g, \delta p, \delta g', \delta p') = \delta p(\delta q') - \delta p'(\delta q).$$

We can then associate to this symplectic form a weak Poisson structure $(\mathcal{A}_{T^*G}, \{\cdot,\cdot\}_\omega)$ as in [30, Proposition 2.18], where

$$\mathcal{A}^\infty_{T^*G} \coloneqq \{H \in C^\infty(T^*G),\ \text{H admits a symplectic gradient}\}$$

is a subalgebra of $C^\infty(T^*G)$. Here, admitting a symplectic gradient means that there exists a (Hamiltonian) vector field X_H such that for any smooth vector field $X \in \Gamma(T^*G)$ on T^*G, the following holds:

$$dH(X) = \omega(X_H, X).$$

This can be rewritten using the interior product $dH = i_{X_H}\omega$. The Poisson bracket is then defined on $\mathcal{A}_{T^*G}$ as

$$\{F, H\}_\omega = \omega(X_F, X_H) = dH(X_F) = -dF(X_H).$$

If $H_V(g, p) = \frac{1}{2}(p \mid K_g^k p)$ is a Hamiltonian coming from a right-invariant metric induced by the Hilbert space V, it belongs to the algebra $\mathcal{A}_{T^*G}$, and its symplectic gradient coincides with the vector field X_{H_V} so that its flow preserves the Poisson bracket $\{\cdot,\cdot\}_\omega$.

Now, we can also endow the dual of the "Lie" algebra T_e^*G with a weak Poisson structure. We define the subalgebra

$$\mathcal{A}_{T_e^*G^\infty} \coloneqq \{h \in C^\infty(T_e^*G^\infty),\ dh \in C^\infty(T_e^*G^\infty, T_eG^\infty)\}.$$

Here, the space T_eG^∞ is the space of vectors $X \in T_eG$ such that the induced right-invariant vector field $\tilde{X} : g \mapsto T_eR_g(X)$ is smooth, so that the space T_eG^∞ is stable by the bracket defined in section 1.1 and then turns into a Lie algebra [8]. Note that $T_eG^\infty \subset T_eG$ is a Fréchet space and we equip its dual $T_e^*G^\infty$ with the Hausdorff locally convex weak-* topology, so that T_eG^∞ identifies with the bidual $T_e^{**}G^\infty$. Here $C^\infty(T_e^*G^\infty)$ is the space of Gâteaux smooth functions of $T_e^*G^\infty$ (as in [18, 30]), and the algebra $\mathcal{A}_{T_e^*G^\infty}$ may be strictly included in $C^\infty(T_e^*G^\infty)$. We can thus define the smooth natural Poisson bracket

$$\{h, f\}_+(m) \coloneqq \big(m \mid [dh(m), df(m)]\big)$$

for any $m \in T_e^*G^\infty$. Each $f \in \mathcal{A}_{T_e^*G^\infty}$ admits a corresponding Hamiltonian vector field $\hat{X}_f \in C^\infty(T_e^*G^\infty, T_e^*G^\infty)$ such that $\hat{X}_f \cdot h = \{h, f\}_+$ for any $h \in \mathcal{A}^\infty_{T_e^*G^\infty}$, and given by:

$$\hat{X}_f(m) = -\operatorname{ad}^*_{df(m)} m.$$

The classical result from reduction theory [25, 30, 31] gives that the momentum map $m : T^*G \to T_e^*G^\infty$, defined by $m(g, p) = T_eR_g^*p$ is a Poisson map between the weak Poisson structures $(T^*G, \mathcal{A}^\infty_{T^*G}, \{\cdot,\cdot\}_\omega)$ and $\big(T_e^*G^\infty, \mathcal{A}_{T_e^*G^\infty}, \{\cdot,\cdot\}_+\big)$. This means that for any right-invariant mappings $H, F \in \mathcal{A}^\infty_{T^*G}$, the following equality holds:

$$\{H, F\}_\omega = \{h, f\}_+ \tag{11}$$

where $h, f \in \mathcal{A}_{T_e^*G^\infty}$ are such that $h \circ m = H$ and $f \circ m = F$. In particular, this implies that, if $(g, p) \in C^\infty(I, T^*G)$ is the flow of a Hamiltonian vector field $X_H = \nabla^\omega H$ where $H = h \circ m \in \mathcal{A}_{T_e^*G^\infty}$ is right-invariant, the momentum $m_t = m(g_t, p_t) \in T_eG^\infty$ is the flow of $\hat{X}_h$, i.e. solves the reduced Lie-Poisson equation:

$$\dot{m}_t + \operatorname{ad}^*_{dh(m)} m = 0.$$

This construction is not yet applicable to Hamiltonian functions coming from the strong right-invariant sub-Riemannian metrics defined previously. Indeed the Hilbert space V is only assumed to be embedded in T_eG^k for a certain $k \in \mathbb{N}$. This means one has to deal with non-smooth Hamiltonians, and momentum maps have to belong to $T_e^*G^k$. However, these Poisson brackets can be extended on the algebras

$$\mathcal{A}^l_{T^*G} \coloneqq \{H \in C^l(T^*G),\ \text{H admits a symplectic gradient}\}$$

and

$$\mathcal{A}_{T_e^*G^l} \coloneqq \{h \in C^\infty(T_e^*G^l),\ dh \in C^\infty(T_e^*G^l, T_eG^l)\}.$$

These algebras are not stable but satisfy $\{\mathcal{A}^{l+1}_{T^*G}, \mathcal{A}^{l+1}_{T^*G}\}_\omega \subset \mathcal{A}^l_{T^*G}$ and $\{\mathcal{A}_{T_e^*G^{l+1}}, \mathcal{A}_{T_e^*G^{l+1}}\}_+ \subset \mathcal{A}_{T_e^*G^l}$. The C^{l-1} co-restriction of the momentum map $m : T^*G \to T_e^*G^l$ sends $\mathcal{A}_{T_e^*G^l}$ to $\mathcal{A}^{l-1}_{T^*G}$ and satisfies again:

$$\forall h, f \in \mathcal{A}_{T_e^* G^l},\ \{h, f\}_+ = \{h \circ m, f \circ m\}.$$

In the particular case $h(m) = \frac{1}{2}|K^1 m|_V^2$ for $m \in T_e^* G^1$, we get the previous subriemannian Euler-Arnold equation.

2.3. **Completeness.** In infinite-dimensional Riemannian geometry, the Hopf-Rinow theorem does not hold anymore [6, 22], even with strong metrics. For right-invariant strong Riemannian metrics defined on Banach half-Lie groups, we can still recover most of the completeness properties [8]. We prove similar results for the strong sub-Riemannian metrics we have defined in previous sections.

Theorem 2.5 (Completeness properties)**.** *Let G be a right-invariant half-Lie group carrying a right-invariant local addition. Let $V \hookrightarrow T_eG$ be a Hilbert space, and $d_V : G \times G \to \mathbb{R} \cup \{\infty\}$ the induced sub-Riemannian distance. Then, the following properties hold:*

(1) The space (G, d_V) is metrically complete

(2) Assume that the endpoint mapping $\mathrm{End} : L^2(I, V) \to G$ *is weakly-continuous with regards to some Hausdorff topology on G. Then, for any $g, g' \in G$ such that $d_V(g, g') < \infty$, there exists a minimizing geodesic connecting g and g'*

(3) Assume V is continuously embedded in T_eG^2. Then, for every $p_0 \in T_e^*G$, there exists a unique global curve $(g, p) \in \mathrm{AC}_{L^2}(I, T^*G)$ satisfying the normal geodesic equation. In other words, the normal Hamiltonian geodesic flow is well-defined and global.

(4) Assume V is continuously embedded in T_eG^2. Then, the Euler-Arnold equation (9) *is globally well-posed, i.e. for any $m_0 \in T_e^*G$, there exists a unique solution of the equation*

$$\begin{cases} \dot{m}_t + \mathrm{ad}_{u_t}^* m_t = 0 \\ K^0 m_t = u_t \end{cases}$$

3. Examples and applications to LDDMM models

3.1. **LDDMM, inexact matching, and induced quotient metrics.** In this section, we come back to the LDDMM framework. We introduce a Banach manifold of shapes $\mathcal{Q}$, a Banach half-Lie group G endowed with a right-invariant local addition, and a Hilbert space $V \hookrightarrow T_eG^2$. We assume that G acts continuously on the left on $\mathcal{Q}$. Let $A(g, q) = g \cdot q$ denote this left action, where $g \in G$ and $g \in \mathcal{Q}$. Given a source object q_S and a target q_T, we want to find the best deformation $g \in G$ matching q_S and q_T, and we consider the following inexact problem

$$\inf_{g \in G} d_V(e, g) + b(g \cdot q_S) \tag{12}$$

with $b : \mathcal{Q} \to \mathbb{R}^+$ a data attachment term that measures the discrepancy between the deformed object $g \cdot q_S$ and the target object q_T. Using the definition of the distance d_V, we can reformulate the matching problem (12) and optimize the functional $J : L^2(I, V) \to \mathbb{R}^+$ defined as

$$J(u) = E(u) + b(\mathrm{End}(u) \cdot q_S). \tag{13}$$

If we assume, as in Theorem 2.5, that the endpoint mapping $\mathrm{End} : L^2(I, V) \to G$ is weakly continuous with regards to some Hausdorff topology on G, and that b is continuous for the same Hausdorff topology, then there exists $u \in L^2(I, V)$ such that $J(u)$ is minimal [3, 34].

In the following, we assume some regularity conditions on the action of G on $\mathcal{Q}$ and endow $\mathcal{Q}$ with a sub-Riemannian metric induced by G. We also assume that for all $q \in \mathcal{Q}$, the mapping

$A_q = A(\cdot, q) : g \mapsto g \cdot q$ is C^1, and we denote by $\xi_q = \xi(\cdot, q) = \partial_g A(g, q)_{|g=e}$ its continuous differential in e. Furthermore, we assume that for every $k > 0$, the mappings

$$A : \begin{cases} G^k \times \mathcal{Q} & \longrightarrow & \mathcal{Q} \\ (g,\ q) & \longmapsto & g \cdot q \end{cases} \quad \text{and} \quad \xi : \begin{cases} T_e G^k \times \mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (u,\ q) & \longmapsto & \xi_q(u) = u \cdot q \end{cases}$$

are C^k. The triple $(\mathcal{Q} \times V, \xi, \langle \cdot, \cdot \rangle_V)$ now defines a sub-Riemannian structure on $\mathcal{Q}$: it defines a horizontal distribution $\Delta \subset T\mathcal{Q}$ and each horizontal space $\Delta_q = \xi_q(V)$ can be endowed with a Hilbert structure by defining a scalar product $\langle \cdot, \cdot \rangle_q$:

$$\langle X, X \rangle_q = \inf_{u \in V,\ \xi_q u = X} \langle u, u \rangle_V = \langle \xi_q^{-1} X, \xi_q^{-1} X \rangle_V$$

where ξ_q^{-1} is the inverse of the restriction $\xi_q : (\ker \xi_q)^\perp \to \Delta_q$, and $(\ker \xi_q)^\perp$ is the orthogonal of $\ker \xi_q$ in V. As in section 2.2, this sub-Riemannian structure defines a cometric

$$K^{\mathcal{Q}} : \begin{cases} T^*\mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (q, p) & \mapsto & K_q^{\mathcal{Q}} p = \xi_q K_V \xi_q^* \end{cases}$$

and a normal Hamiltonian $H_{\mathcal{Q}}(q, p) = \langle K_q^{\mathcal{Q}} p, K_q^{\mathcal{Q}} p \rangle_q$. Reduction theory can be applied again and we introduce a momentum map $m_{\mathcal{Q}}(q, p) = \xi_q^* p \in V^*$. Since the metric on $\mathcal{Q}$ comes from the right-invariant metric on G, normal geodesics on $\mathcal{Q}$ can be lifted as normal geodesics on G. Horizontal curves in $\mathcal{Q}$ are absolutely continuous curves $q : I \to \mathcal{Q}$ such that $\dot{q} \in \Delta_q$ (a.e.). For such curves, we can define the corresponding *minimal lift* $g \in AC(I, G)$ as the only curve satisfying

$$\begin{cases} g_0 = e \\ \dot{g}_t = T_e R_{g_t} \left(\xi_{q_t}^{-1} \dot{q}_t \right) \end{cases}$$

Note that there can exist other horizontal lifts in G of the same curve in $\mathcal{Q}$, and the minimal lift is the lift that minimizes the energy:

Proposition 3.1 (Normal geodesics in $\mathcal{Q}$)**.** *Let $q \in AC_{L^2}(I, T\mathcal{Q})$ be a normal geodesic in $\mathcal{Q}$, and let $g \in AC_{L^2}(I, TG)$ be its minimal lift. Then, q_t is a normal geodesic if and only if g_t is a normal geodesic in G.*

Moreover, if $t \mapsto (g_t, p_t^G)$ and $t \mapsto (q_t, p_t^{\mathcal{Q}})$ satisfy the normal Hamiltonian equations, with starting condition

$$m_G(g_0, p_0^G) = m_{\mathcal{Q}}(q_0, p_0^{\mathcal{Q}}) \tag{14}$$

then g_t is the minimal lift of q and the momentum maps are equal:

$$\forall t \in I,\ m_{\mathcal{Q}}(q_t, p_t^{\mathcal{Q}}) = m_G(g_t, p_t^G).$$

This tells us that studying the sub-Riemannian normal geodesics in $\mathcal{Q}$ is in a way equivalent to studying the sub-Riemannian normal geodesics in G. Furthermore, if we come back to the inexact problem (12), and we assume the data attachment term $b : \mathcal{Q} \to \mathbb{R}^+$ to be C^1, we see that the critical points $u \in L^2(I, V)$ of the functional J satisfy

$$dJ(u) = dE(u) + db(q_1) d\,\mathrm{End}_{q_0}(u) = 0. \tag{15}$$

This means that the critical points of J are normal geodesics of the sub-Riemannian structures on $\mathcal{Q}$, with covector $p(1) = dg\,(q(1))$.

3.2. **Combining rotations and diffeomorphisms.** In classical LDDMM, a first step is often performed to align source and target objects through rigid motions. In the multi-scale setting [27, 34], one can include the rigid motion alignment as a coarse first layer. In [13], authors develop an equivalent formulation of multi-scale registration through an iterated semi-direct product. We adapt their idea to include the rigid motion. We introduce the group of affine isometries of $\mathbb{R}^d$

$$\mathrm{Isom}(\mathbb{R}^d) = \mathbb{R}^d \ltimes SO(d)$$

with the product law given by $(\tau, R)(\tau', R') = (\tau + R\tau', RR')$. This allows us to associate any element (R, τ) of the Lie group $\mathrm{Isom}(\mathbb{R}^d)$ with the rigid deformation $(R, \tau) : x \in \mathbb{R}^d \mapsto Rx + \tau$. We consider the following right action of $\mathrm{Isom}(\mathbb{R}^d)$ on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ by conjugation,

$$S \star \varphi := S^{-1}\varphi S,$$

where $S = (\tau, R) \in \mathrm{Isom}(\mathbb{R}^d)$. This action is well defined, continuous, and we can thus consider the group $H^k := \mathrm{Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ with the following composition law:

$$(S, \varphi) \cdot (S', \varphi') := (SS', S'^{-1}\varphi S' \circ \varphi'). \tag{16}$$

This group H^k is a half-Lie group and can be equipped with a right-invariant local addition, so that we simply have $(H^k)^l = H^{k+l}$ for $l \geq 0$.

We now want to endow the group H^k with a right-invariant sub-Riemannian metric. The group of affine isometries is equipped with a bi-invariant Riemannian metric induced by the metric on $\mathfrak{isom}(\mathbb{R}^d) = \mathbb{R}^d \oplus \mathfrak{so}(d)$:

$$\langle (\sigma, A), (\sigma', A') \rangle_{\mathfrak{isom}} = \langle \sigma, \sigma' \rangle_{\mathbb{R}^d} - \mathrm{tr}(AA')$$

where $\sigma, \sigma' \in \mathbb{R}^d$, and $A, A' \in \mathfrak{so}(d)$ are skew-symmetric matrices. Furthermore, if $V \subset C_0^{k+2}(\mathbb{R}^d, \mathbb{R}^d)$ is a Hilbert space, the metric on $T_e H^k = \mathfrak{isom}(\mathbb{R}^d) \oplus T_{\mathrm{id}} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ is defined as

$$\langle (s, u), (s', u') \rangle = \langle s, s' \rangle_{\mathfrak{isom}} + \langle u, u' \rangle_V.$$

Suppose we want to perform shape registration in a Banach manifold $\mathcal{Q}$ and that both groups $\mathrm{Isom}(\mathbb{R}^d)$ and $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ act continuously on the left on $\mathcal{Q}$. This defines a left action of H^k on $\mathcal{Q}$ by

$$(S, \varphi) \cdot q = S(\varphi q)$$

where $q \in \mathcal{Q}$ and $(S, \varphi) \in H^k$. For $q_S, q_T \in \mathcal{Q}$ source and target objects, we can therefore define the following inexact matching problem

$$\inf_{(s,u) \in \mathfrak{isom}(\mathbb{R}^d) \times V} \frac{1}{2} \int_I \Big(|s|^2_{\mathfrak{isom}(\mathbb{R}^d)} + |u|_V^2 \Big)\, dt + b_{\mathrm{rigid}}\big(S_1 q_0\big) + b\big((S_1, \varphi_1) \cdot q_0\big). \tag{17}$$

3.3. **Extensions of diffeomorphism groups.** In this part, we give another example of a half-Lie group equipped with a right-invariant sub-Riemannian metric. We follow ideas from [27, 34] that study actions on varifolds, and that keep track of the change of weights induced by the action of diffeomorphisms during the dynamic. We consider here the split extension:

$$L^k = \mathrm{Diff}_{C_0^k} \ltimes C_0^{k-1}(\mathbb{R}^d, \mathbb{R}_{>0})$$

where the group operation is given for all $(\varphi, \omega), (\varphi', \omega') \in L^k$ by

$$(\varphi, \omega) \cdot (\varphi', \omega') = (\varphi \circ \varphi', \omega \circ \varphi' \omega').$$

The group L^k is also equipped with a right-invariant local addition, and $(L^k)^l = L^{k+l}$ The diffeomorphism group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ is a subgroup of H^k through the smooth inclusion

$$i_{L^k} : \left\{ \begin{array}{ccc} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & L^k \\ \varphi & \longmapsto & (\varphi, |J\varphi|) \end{array} \right.$$

where $|J\varphi|$ is the Jacobian determinant of φ. Let $V \hookrightarrow C_0^{k+2}(\mathbb{R}^d, \mathbb{R}^d)$ be a Hilbert space, endowing $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ with the previous right-invariant sub-Riemannian metric. Since V is also embedded in $T_e L^{k+2}$, the group L^k is also endowed with a right-invariant metric, which is equivalent to the metric induced by the left action of $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ on L^k. In particular, according to the discussion of section 3.1, any curve $t \mapsto (\varphi_t, P_t)$ satisfying the normal Hamiltonian equations with starting point $(\mathrm{id}, P_0) \in C_0^k(\mathbb{R}^d, \mathbb{R}^d) \times C_0^k(\mathbb{R}^d, \mathbb{R}^d)^*$ defines a normal geodesic $i_{L^k}(\varphi_t) = (\varphi_t, |J\varphi_t|)$ in L^k. To get a solution $(i_{L^k}(\varphi_t), p_t^{L^k})$ of the normal Hamiltonian equations in L^k, with $p_t^{L^k} = (p_t^\varphi, p_t^\omega) \in C_0^k(\mathbb{R}^d, \mathbb{R}^d)^* \times C_0^{k-1}(\mathbb{R}^d, \mathbb{R}_{>0})^*$, one needs to find an initial covector satisfying $P_0 = di_{L^k}^* p_0^{L^k}$, i.e.

$$p_0^\varphi - \nabla p_0^\omega = P_0. \tag{18}$$

In this case, the covectors will satisfy, for all $t \in I$,

$$p^\varphi - |J\varphi| \left(d\varphi^{-1}\right)^* \nabla p^\omega = P_t. \tag{19}$$

One can immediately take $p_0^\varphi = P_0$ to get $p_t^\varphi = P_t$ for all $t \in I$. However, the idea of this construction is to add variables in the optimization problem in order to gain regularity. Indeed, since the covectors are in the dual, one can hope that the relation (18) may hide some singularities of P_0 in the (weak) gradient of p_0^ω, making the couple $(p_0^\varphi, p_0^\omega)$ a bit more regular. An example is given in particular in [27, 34] in the context of matching of images.

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Appendix A. The set of absolutely continuous curves on a half-Lie group

In this part, we define the set of absolutely continuous curves and expose its differentiable structure. A more complete exposition of absolutely continuous curves and their properties in infinite-dimensional manifolds and in Lie groups can be found in [16]. The differentiable structure of this set is given in [20, 17, 34].

Let $I \subset \mathbb{R}$ be a compact interval, and $t_0 \in I$, and $p \geq 1$. Let $\mathbb{B}$ be a Banach space. We define the vector space $AC_{L^p}(I, \mathbb{B})$ of continuous curves $\eta : I \to \mathbb{B}$ such that there exists $\gamma \in L^p(I, \mathbb{B})$ verifying for any $t \in I$

$$\eta(t) = \eta(a) + \int_a^t \gamma(t) dt.$$

This is equivalent to saying that η is almost everywhere differentiable with $\eta' \in L^p(I, \mathbb{B})$. The space $AC_{L^p}(I, \mathbb{B})$ is equipped with a Banach structure with the norm $|\eta|_{AC_{L^p}} = |\eta(t_0)|_{\mathbb{B}} + |\eta'|_{L^p}$ and is continuously included in $C(I, \mathbb{B})$. Assume now G is a Banach half-Lie group modeled on the Banach space $\mathbb{B}$. The space of absolutely curves $AC_{L^p}(I, G)$ in the group G is defined as the set of continuous curves $\eta : I \to G$, such that for any local charts (U, φ) and any $a < b$ such that $\eta([a, b]) \subset U$, the curve $\varphi \circ \eta : [a, b] \to \mathbb{B}$ is in $AC_{L^p}([a, b], \mathbb{B})$. The manifold structure of this space is given in the following proposition.

Proposition A.1 (Banach manifold structure of $AC_{L^p}(I, G)$)**.** *Assume $I = [a, b]$ with $a < b \in \mathbb{R}$.*

(1) The space $AC_{L^p}(I,G)$ is a Banach manifold.

(2) For $t \in I$, the evaluation

$$\mathrm{ev}_t : \left\{ \begin{array}{ccc} AC_{L^p}(I,G) & \longrightarrow & G \\ \eta & \longmapsto & \eta(t) \end{array} \right.$$

is smooth.

(3) The tangent bundle $TAC_{L^p}(I,G)$ is identified with the vector bundle $AC_{L^p}(I,TG) \to AC_{L^p}(I,G)$. For $g \in AC_{L^p}(I,G)$, we therefore have

$$T_g AC_{L^p}(I,G) = AC_{L^p}(I \leftarrow g^*TG)$$

where $AC_{L^p}(I \leftarrow g^*TG) = \{\gamma \in AC_{L^p}(I,TG),\ \gamma(t) \in T_{g(t)}G,\ \forall t \in I\}$.

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