

ECS 256 - Problem set 2

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Problem 1.a

A coin is flipped k times with p probability of heads. For each head, the coin is flipped one additional time (a bonus flip). The number of bonus flips is referred to as Y and the total number of heads X

$\text{Var}(X)$ can be found using the Law of Total Variance, and properties of binomial distributions. We will also need to use part of the derivation of $E(X)$:

$$\begin{aligned} E(X|Y) &= E(X - Y + Y|Y) \\ &= E((X - Y)|Y) + E(Y|Y) && \text{(by 3.13)} \\ &= pY + Y && \text{(by 3.110)} \\ &= (1 + p)Y \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[\text{Var}(X|Y)] + \text{Var}[E(X|Y)] && \text{(by 9.8)} \\ &= E[\text{Var}(X|Y)] + \text{Var}[(1 + p)Y] && \text{(from above)} \\ &= E[\text{Var}(X|Y)] + (1 + p)^2 kp(1 - p) && \text{(by 3.34 and 3.109)} \\ &= E[Yp(1 - p)] + (1 + p)^2 kp(1 - p) && \text{(by 3.111)} \\ &= kp^2(1 - p) + (1 + p)^2 kp(1 - p) && \text{(by 3.103)} \\ &= kp(1 - p) (p + (1 + p)^2) \end{aligned}$$

Using $p=0.5$

$$\begin{aligned} &= k(0.25)(0.5 + (1.5^2)) \\ &= 0.6875k \end{aligned}$$

Problem 1.b

In the trapped miner example, a miner chooses between three doors with only one leading to safety after 2 hours. The other two doors lead back to the door room after 3 and 5 hours respectively.

We are interested in the variance of Y , the time it takes to escape the mine. We will build upon Ahmed Ahmedin's solution to EY , where N refers to the total attempts

needed to escape and U_i refers to the time spent traveling on the i^{th} attempt.

$$\begin{aligned}
Var(Y) &= E[Var(Y|N)] + Var[E(Y|N)] && \text{(by 9.8)} \\
&= E[Var(Y|N)] + Var[4N - 2] && \text{(by 9.16)} \\
&= E[Var(Y|N)] + 16Var[N] && \text{(by 3.34 and 3.41)} \\
&= E[Var(Y|N)] + 16 \cdot \frac{1 - 1/3}{(1/3)^2} && \text{(by 3.93)} \\
&= E[Var(U_1 + U_2 + \dots + U_n|N)] + 96 \\
&= E[Var(U_1|N) + \dots + Var(U_{N-1}|N) + Var(U_N|N)] + 96 && \text{(by 3.51)} \\
&= E[1 + 1 + \dots + 1 + 0] + 96 \\
&= E[N - 1] + 96 \\
&= E[N] - 1 + 96 && \text{(by 3.17)} \\
&= 3 - 1 + 96 && \text{(by 3.92)} \\
&= 98
\end{aligned}$$

We know that $Var(U_i|N)$ is independent because the miner's choice of door does not depend of a previous choice. Since we are conditioning this event on there being N attempts, the values of the first $N-1$ attempts will either be 3 or 5. So the variance of an individual attempt in this case is 1. The variance of the N^{th} attempt is 0 because that attempt always is the same tunnel.

Problem 2.a

For a vector Q of random variables $(Q_1, ..Q_n)$ we have:

$$Cov(Q) = E(QQ') - E(Q)E(Q') \quad (\text{by 13.53})$$

Let $Q = Y|X$, where Y is vector valued. Then:

$$Cov(Y|X) = E((Y|X)(Y|X)') - E(Y|X)E(Y|X)' \quad (\text{by 13.53})$$

Taking expected value of both sides we have:

$$\begin{aligned} E(Cov(Y|X)) &= E\left(E((Y|X)(Y|X)') - E(Y|X)E(Y|X)'\right) \\ &= E\left(E((Y|X)(Y|X)')\right) - E\left(E(Y|X)E(Y|X)'\right) \\ &= E(YY') - E\left(E(Y|X)E(Y|X)'\right) \quad (\text{by Law of Tot. Expect.}) \end{aligned}$$

Now let $Q = E(Y|X)$, where Y is vector valued. Then:

$$\begin{aligned} Cov(E(Y|X)) &= E(E(Y|X)E(Y|X)') - E(E(Y|X))E(E(Y|X))' \quad (\text{by 13.53}) \\ &= E(E(Y|X)E(Y|X)') - E(Y)E(Y)' \quad (\text{by Law of Tot. Expect.}) \end{aligned}$$

Summing up the left sides and the right sides of these 2 equations we get:

$$\begin{aligned} E(Cov(Y|X)) + Cov(E(Y|X)) &= E(YY') - E\left(E(Y|X)E(Y|X)'\right) \\ &\quad + E(E(Y|X)E(Y|X)') - E(Y)E(Y)' \\ E(Cov(Y|X)) + Cov(E(Y|X)) &= E(YY') - E(Y)E(Y)' \\ E(Cov(Y|X)) + Cov(E(Y|X)) &= Cov(Y) \quad (\text{by 13.53}) \end{aligned}$$

Problem 2.b

TODO.

Problem 3

$$\rho(X_{i+j}, X_i) = \frac{\text{Cov}(X_{i+j}, X_i)}{\sqrt{\text{Var}(X_{i+j})} \cdot \sqrt{\text{Var}(X_i)}}$$

A couple things we noticed:

$$E(X_i) = \sum_{l=1}^n l \cdot \pi_l$$

$$E(X_{i+j}|X_i) = \sum_{l=1}^n l \cdot m_{k,l}^j$$

where M^j is the transition matrix $M \cdot M \cdots M$ (j times), and $X_i = k$. First, let's derive $\text{Cov}(X_{i+j}, X_i)$.

$$\text{Cov}(X_{i+j}, X_i) = E(X_{i+j}X_i) - E(X_{i+j})E(X_i)$$

Suppose $Q = X_{i+j}X_i$. By the law of total expectations, $E(Q) = E(E(Q|X_i))$. Thus,

$$\begin{aligned} E(X_{i+j}X_i) &= \sum_{l=1}^n \pi_l l \cdot E(X_{i+j}|X_i) \\ &= \sum_{l=1}^n \left[\pi_l l \cdot \left[\sum_{k=1}^n k \cdot m_{l,k}^j \right] \right]. \end{aligned}$$

Since $X_{i,j}$ and X_i have the same distribution (II), the covariance becomes

$$\text{Cov}(X_{i+j}, X_i) = \sum_{l=1}^n \left[\pi_l l \cdot \left[\sum_{k=1}^n k \cdot m_{l,k}^j \right] \right] - (EX_i)^2.$$

Next, we derive $\text{Var}(X_{i,j})$ and $\text{Var}(X_i)$. **TODO**