ECS 256 - Problem set 2

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1a)

A coin is flipped k times with p probability of heads. For each head, the coin is flipped one additional time (a bonus flip). The number of bonus flips is referred to as Y and the total number of heads X

Var(X) can be found using the Law of Total Variance, and properties of binomial distributions. We will also need to use part of the derivation of EX:

$$\begin{aligned}
E(X|Y) &= E(X - Y + Y|Y) \\
&= E((X - Y)|Y) + E(Y|Y) && \text{(by 3.13)} \\
&= pY + Y && \text{(by 3.110)} \\
&= (1 + p)Y
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E[Var(X|Y)] + Var[E(X|Y)] && \text{(by 9.8)} \\
&= E[Var(X|Y)] + Var[(1 + p)Y] && \text{(from above)} \\
&= E[Var(X|Y)] + (1 + p)^2 kp(1 - p) && \text{(by 3.34 and 3.109)} \\
&= E[Yp(l - p)] + (1 + p)^2 kp(1 - p) && \text{(by 3.111)} \\
&= kp^2(1 - p) + (1 + p)^2 kp(1 - p) && \text{(by 3.103)} \\
&= kp(1 - p)\left(p + (1 + p)^2\right)
\end{aligned}$$

Using p=0.5

$$\begin{aligned}
&= k(0.25)(0.5 + (1.5^2)) \\
&= 0.6875k
\end{aligned}$$

1b)

In the trapped miner example, a miner chooses between three doors with only one leading to safety after 2 hours. The other two doors lead back to the door room after 3 and 5 hours respectively.

We are interesting the variance of Y, the time it takes to escape the mine. We will build upon Ahmed Ahmedin's solution to EY, where N refers to the total attempts needed to escape and U_i refers to the time spent traveling on the i^{th} attempt.

$$\begin{aligned}
Var(Y) &= E[Var(Y|N)] + Var[E(Y|N)] && \text{(by 9.8)}\\
&= E[Var(Y|N)] + Var[4N-2] && \text{(by 9.16)}\\
&= E[Var(Y|N)] + 16Var[N] && \text{(by 3.34 and 3.41)}\\
&= E[Var(Y|N)] + 16 \cdot \frac{1-1/3}{(1/3)^2} && \text{(by 3.93)}\\
&= E[Var(U_1 + U_2 + ... + U_n|N)] + 96 &&\\
&= E[Var(U_1|N) + ... + Var(U_{N-1}|N + Var(U_N|N)] + 96 && \text{(by 3.51)}\\
&= E[1 + 1 + ...1 + 0] + 96 &&\\
&= E[N-1] + 96 &&\\
&= E[N] - 1 + 96 && \text{(by 3.17)}\\
&= 3 - 1 + 96 && \text{(by 3.92)}\\
&= 98 &&
\end{aligned}$$

We know that Var(U_i—N) is independent because the miner's choice of door does not depend of a previous choice. Since we are conditioning this event on there being N attempts, the values of the first N-1 attempts will either be 3 or 5. So the variance of an individual attempt in this case is 1. The variance of the N^{th} attempt is 0 because that attempt always is the same tunnel.

2a)

For a vector Q of random variables $(Q_1, ..Q_n)$ we have:

$$Cov(Q) = E(QQ') - E(Q)E(Q') \quad \text{(by 13.53)}$$

Let $Q = Y|X$, where Y is vector valued. Then:

$$Cov(Y|X) = E\big((Y|X)(Y|X)'\big) - E(Y|X)E(Y|X)' \quad \text{(by 13.53)}$$

Taking expected value of both sides we have:

$$\begin{aligned}
E\big(Cov(Y|X)\big) &= E\Big(E\big((Y|X)(Y|X)'\big) - E(Y|X)E(Y|X)'\Big)\\
&= E\Big(E\big((Y|X)(Y|X)'\big)\Big) - E\Big(E(Y|X)E(Y|X)'\Big)\\
&= E(YY') - E\Big(E(Y|X)E(Y|X)'\Big) && \text{(by Law of Tot. Expect.)}
\end{aligned}$$

Now let $Q = E(Y|X)$, where Y is vector valued. Then:

$$Cov(E(Y|X)) = E\big(E(Y|X)E(Y|X)'\big) - E\big(E(Y|X)\big)E\big(E(Y|X)\big)' \qquad \text{(by 13.53)}$$
$$= E\big(E(Y|X)E(Y|X)'\big) - E(Y)E(Y)' \qquad \text{(by Law of Tot. Expect.)}$$

Summing up the left sides and the right sides of these 2 equations we get:

$$E\big(Cov(Y|X)\big) + Cov(E(Y|X)) = E(YY') - E\Big(E(Y|X)E(Y|X)'\Big)$$
$$+ E\big(E(Y|X)E(Y|X)'\big) - E(Y)E(Y)'$$
$$E\big(Cov(Y|X)\big) + Cov(E(Y|X)) = E(YY') - E(Y)E(Y)'$$
$$E\big(Cov(Y|X)\big) + Cov(E(Y|X)) = Cov(Y) \qquad \text{(by 13.53)}$$

2.b Yet to be done