Chapter 2

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1 Section 12: Topological Spaces

1.1 Def: Topology

A topoplogy on a set X is a collection T of subsets of X having the following properties:
(1) $\emptyset$ and X are in T.
(2) T is closed under union.
(3) T is closed under finite intersection.

2 Section 13: Basis for a Topology

2.1 Def: basis

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X such that:
(1) For each $x \in X$, there is at least one basis element B containing x.
(2) If $x \in B_1 \cap B_2$, where B_1 and B_2 are basis elements, then there exists a basis element B_3 such that $B_3 \subset B_1 \cap B_2$.

2.2 Def: Topology Generated by a basis

The topology T generated by $\mathcal{B}$ is the set of all subsets U of X such that for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

2.3 Lemma 13.1

Let X be a set; let $\mathcal{B}$ be a basis for a topology T on X. Then T equals the collection of all unions of elements of $\mathcal{B}$.

2.4 Lemma 13.2

Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element $C \subset \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

2.5 Lemma 13.3

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies T and T', respectively, on X. Then the following are equivalent:
(1) T' is finer than T.
(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

2.6 Def: Subbasis

A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection T of all unions of finite intersections of elements of $\mathcal{S}$.

3 Section 14: The Order Topology

3.1 Def: Order Topology

Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:
(1) All open intervals (a, b) in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.
The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

4 Section 16: The Subspace Topology

4.1 Def: Subspace Topology

Let X be a topological space with topology T. If Y is a subset of X, the collection

$$T_Y = \{Y \cap U : U \in T\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X.

4.2 Lemma 16.1

If $\mathcal{B}$ is a basis for the topology of X then the collection

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

4.3 Lemma 16.2

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

4.4 Theorem 16.3

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$

4.5 Theorem 16.4

Let X be an ordered set in the order topology; let Y be a convex subset of X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

4.6 Theorem 17.1

Let X be a topological space. Then the following conditions hold:
(1) The empty set and X are closed.
(2) Intersections of closed sets are closed.
(3) Finite unions of closed sets are closed.

4.7 Theorem 17.2

Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

4.8 Theorem 17.3

Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

4.9 Def: Closure

Given a subset of A of a topological space X, then closure of A is define as the intersection of all closed sets containing A.

4.10 Theorem 17.4

Let Y be a subspace of X, let A be a subset of Y, let $\overline{A}$ denote their closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

4.11 Def: Continuous

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

4.12 Theorem 17.5

Let A be a subset of the topological space X.
(a) Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.
(b) Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

4.13 Def: Limit Point

If A is a subset of a topological space X and if x is a point of X, we say that x is a limit point of A if every neighborhood of x intersects A in some point other than x itself.

4.14 Theorem 17.6

Let A be a subset of a topological space X, Let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'$$

4.15 Corollary 17.7

A subset of a topological space is close if and only if it contains all of its limit points.

4.16 Def: Hausdorff Space

A topological space X is called a Hausdorff Space if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

4.17 Theorem 17.8

Every finite point set in a Hausdorff space is closed.

4.18 Theorem 17.10

If X is a Hausdorf space, then a sequence of points of X converges to at most one point of X.

4.19 Theorem 17.11

Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

5 Section 18: Continuous Functions

5.1 Def: continuous

A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, then set $f^{-1}(V)$ is an open subset of X.

5.2 Theorem 18.1

Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:
(1) f is continuous.
(2) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.
(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

5.3 Def: homeomorphism

If a function bijective f and its inverse are both continuous, then f is a homeomorphism. Another way to define a homemorphism is to say that it is a bijection f such that $f(U)$ is open if and only if U is open.

5.4 Def: Imbedding

Let $f : X \to Y$ be an injective continuous map, where X and Y are topological spaces. Let Z be the image set f, considered as a subspace of Y. If $f' : X \to Z$ obtained by restricting the range of f is a homemorphism of X with Z, we say that map $f : X \to Y$ is an imbedding of X in Y.

5.5 Theorem 18.2 (Rules for constructing continuous functions)

Let X, Y, Z be topological spaces.

(a) (Constant function) If $f : X \to Y$ maps all of X into the single point y_0 of Y. then f is continuous.

(b) (Inclusion) If A is a subspace of X, then inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspaceo of X, then the restricted function $f|A : A \to Y$ is continuous.

(e) (Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α is continuous for each α.

5.6 Theorem 18.3 (The pasting lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \to A$, and $h(x) = g(x)$ if $x \in B$.

5.7 Theorem 18.4 (Maps into products

Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a))$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \text{ and } f_2 : A \to Y$$

are continuous. The maps f_1 and f_2 are called the coordinate functions of f.

6 Section 19: The Product Topology

6.1 Def: J-tuple

Let J be an index set. Given a set X, we define a J-tuple of elements of X to be a function $x : J \to X$. If α is an element of J, we often denote the value of x at α by x_α rather than $x(\alpha)$; we call it the αth coordinate of x. And we often denote the function x itself by the symbol

$$(x_\alpha)_{\alpha \in J}$$

which is as close as we can come to a "tuple notation" for an arbitrary index set J. We denote the set of all J-tuples of element of X by X^J.

6.2 Def: Cartesian Product

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup\limits_{\alpha \in J} A_\alpha$. The Cartesian product of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

6.3 Def: Box Topology

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} U_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

6.4 Def: Product Topology

Let S_β denote the collection

$$S_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\}$$

and let S denote the union of these collections

$$S = \bigcup_{\beta \in J} S_\beta$$

The topology generated by the subbasis S is called the product topology.

6.5 Theorem 19.1 (Box vs Product Topologies)

The box topology on $\prod X_\alpha$ hass as bais all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α, and U_α equals X_α expect for finitely many values of α.

6.6 Theorem 19.2

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod\limits_{\alpha \in J} X_\alpha$.

6.7 Theorem 19.3

Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

6.8 Theorem 19.4

If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

6.9 Theorem 19.5

Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$$

6.10 Theorem 19.6

Let $f : A \to \prod\limits_{\alpha \in J} X_\alpha$ be given the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

7 Section 20 and 21: The Metric Topology

7.1 Def: Metric Topology

If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

7.2 Def: Metrizable

If X is a topological space, X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.

7.3 Def: Bounded and Diameter

Let X be a metric space with metric d. A subset A of X is said to be bouned if there is some number M such that

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the diameter of A is defined to be the number

$$diam A = sup\{d(a_1, a_2) : a_1, a_2 \in A\}$$

7.4 Theorem 20.1

Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by the equation

$$\overline{d}(x, y) = min\{d(x, y), 1\}$$

Then $\overline{d}$ is a metric that induces the same topology as d.

7.5 Lemma 20.2

Let d and d be two metrics on the set X; let T and T' be the topologies they induct, respectively. Then T' is finer than T if and only if for each x in X and each ε, there exists a $\delta > 0$ such that

$$B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$$

7.6 Theorem 20.3

The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric p are the same as the product topology on $\mathbb{R}^n$.

7.7 Def: Uniform Topology

Given an index set J_m and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric on $\overline{p}$ on $\mathbb{R}^J$ by the equation

$$\overline{p}(x, y) = sup\{\overline{d}(x_\alpha, y_\alpha) : \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. $\overline{p}$ is called the uniform metric on $\mathbb{R}^J$, and the topology it induces is called the uniform topology.

7.8 Theorem 20.4

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.

7.9 Theorem 20.5

Let $\overline{d}(a, b) = min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^w$, define

$$D(x, y) = sup\{\frac{\overline{d}(x_i, y_i)}{i}\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^w$.

7.10 Theorem 21.1

Let $f : X \to Y$; let X and Y be metrizable with metrics d_x and d_y, respectively. THen continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

7.11 Lemma 21.2 (The sequence lemma)

Let X be a topological space; $A \subset X$. If there is a sequence of point of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.

7.12 Theorem 21.3

Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges $f(x)$. The converse holds if X is metrizable.

7.13 Def: Countable Basis at Point x

A space X is said to have a countable basis at the point x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains at lease one of the sets U_n. A space X that has a countable basis at ech of its points is said to satisfy the first countability axiom.

7.14 Lemma 21.4

The addition, subtraction, and multplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.

7.15 Theorem 21.5

If X is topological space, and if $f, g : X \to \mathbb{R}$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

7.16 Def: Converges Uniformly

Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) converges uniformly to the function $f : X \to Y$ if given $\varepsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \varepsilon$$

for all $n < N$ and all x in X.

7.17 Theorem 21.6 (Uniform limit theorem)

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

8 Section 22: The Quotient Topology

8.1 Def: Quotient Map

Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

8.2 Def: Quotient Topology

If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology T on A relative to which p is a quotient map; it is called the quotient topology induced by p. This is defined by letting T consist of all subsets U of A such that $p^{-1}(U)$ is open in X.

8.3 Def: Quotient Space

Let X be a topological space, and let X^* be a parition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space of X.

8.4 Theorem 22.1

Let $p : X \to Y$ be a quotient map let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.
(1) If A is either open or closed in X, then q is a quotient map.
(2) If p is either an open map or a closed map, then q is a quotient map.

8.5 Theorem 22.2

Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if an only if g is continuous; f is a quotient map if and only is g is a quotient map.

8.6 Corollary 22.3

Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) : z \in Z\}$$

Give X^* the quotient topology.
(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.
(b) If Z is a Hausdorff, so is X^*.