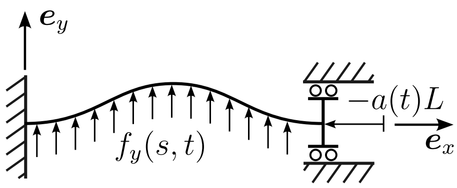
1 Vibration analysis of a beam with midplane stretching

In this assignment, you will study the dynamics of a clamped–clamped Euler–Bernoulli beam with moderate curvature. In Assignment 1.1, you will derive the governing equations for the beam subject to an external lateral force and a prescribed displacement. For the special case of a concentrated force excitation and zero prescribed displacement, you will then apply a single-mode approximation to show how the equations reduce to the Duffing equation. In Assignment 1.2, you will analyze the nonlinear dynamics of the Duffing equation and interpret the results in the physical context of a clamped–clamped beam subjected to a concentrated force.

Assignment 1.1 Assume a horizontal beam of length L that is clamped on both sides. The right hand side of the beam is excited with a prescribed displacement in axial direction $a(t)L$, where $a(t)$ is a known time-dependent function. The beam is subject to a line distributed load $f_y(s,t)$ that acts in $\boldsymbol{e}_y$-direction. The beam has constant line mass density ρA, constant axial stiffness EA and constant bending stiffness EI.



(a) Formulate the kinematic boundary conditions for the displacement fields in axial and transversal direction $u(s,t)$ and $v(s,t)$, respectively.

(b) What is the space of admissible virtual displacements V_{adm}? Specify the internal, external and dynamical virtual work functionals δW^{int}, δW^{ext} and δW^{dyn} for admissible virtual displacements. For the internal virtual work, assume the first- and second-order approximations for curvature $k = v''$ and axial strain $e = u' + \frac{1}{2}v'^2$, respectively. Finally, formulate the principle of virtual work for admissible virtual displacements.

(c) Using integration by parts, manipulate the internal virtual work functional for admissible virtual displacements such that no spatial derivatives remain on the virtual displacements. Then, from the principle of virtual work, derive the governing partial differential equations

$$\rho A\ddot{u} = EA\big(u' + \tfrac{1}{2}v'^2\big)' \tag{1.1}$$

$$\rho A\ddot{u} + EIv^{iv} = EA\big[\big(u' + \tfrac{1}{2}v'^2\big)v'\big]' + f_y\,. \tag{1.2}$$

(d) Assume that the frequencies of the axial oscillations are much higher than those of the transversal oscillations, so that the axial wave speed $c = \sqrt{E/\rho} \to \infty$ tends to infinity. Starting from equation (1.1), show that under this assumption the strain $e(t) = u' + \frac{1}{2}v'^2$ depends only on time t.

(e) Use the kinematic boundary conditions to show that the axial strain is given by

$$e(t) = a(t) + \frac{1}{2L}\int_0^L v'(s,t)^2 \mathrm{d}s\,. \tag{1.3}$$

(f) By substituting the axial strain solution (1.3) into (1.2), derive the governing equations that describe the dynamics of the excited clamped-clamped beam.

The axial strain solution (1.3) can also be substituted into the principle of virtual work. In addition, assume that $a(t) = 0$ and that the line distributed force is replaced by a concentrated force applied at $s = L/2$, which can be expressed using the Dirac delta function δ_{D} as $f_y(s,t) = F(t)\,\delta_{\mathrm{D}}\big(s - \frac{L}{2}\big)$. Under these assumptions, the total virtual work becomes

$$\delta W^{\text{tot}} = -\int_0^L \left\{\delta v'\left(\frac{EA}{2L}\int_0^L v'^2 \mathrm{d}s\right)v' + \delta v'' EI v'' + \delta v \rho A \ddot{v}\right\}\mathrm{d}s + \delta v\big(\tfrac{L}{2}, t\big)F(t)\,. \tag{1.4}$$

This kind of beam is called beam with mid-plane stretching.

(g) To approximate the nonlinear vibrational behavior in the low-frequency range, we use a single-mode (single-DOF) approximation model for the beam with mid-plane stretching. The transversal displacement field of the beam is approximated as

$$v(s,t) = q(t)\frac{1-\cos(2\alpha s)}{2} \quad \text{with} \quad \alpha = \frac{\pi}{L}\,. \tag{1.5}$$

Sketch the global ansatz function and show that the boundary conditions are satisfied. What is the physical interpretation of q(t)?

(h) Apply the Galerkin projection, i.e., approximate the principle of virtual work (1.4), with the global ansatz function (1.5) and the corresponding virtual displacement. Show that, under the single-mode approximation, the beam dynamics reduces to the single-degree-of-freedom ODE

$$m\ddot{q} + k_1 q + k_3 q^3 = F(t)\,, \tag{1.6}$$

where $m = 3\rho AL/8\,[\mathrm{kg}]$ is the equivalent mass, $k_1 = 2\pi^4 EI/L^3\,[\mathrm{N/m}]$ is the equivalent linear stiffness, $k_3 = \pi^4 EA/8L^3\,[\mathrm{N/m^3}]$ is the equivalent cubic stiffness. *Hint:* $\int_0^L \sin^2(2\alpha s)\mathrm{d}s = \int_0^L \cos^2(2\alpha s)\mathrm{d}s = \frac{L}{2}$, $\int_0^L (1-\cos(2\alpha s))^2 \mathrm{d}s = \frac{3L}{2}$.

Assignment 1.2 In this part, we will study the dynamics of the single-mode approximation (1.6) under the harmonic force excitation $F(t) = F\cos(2\pi f t)$. To account for dissipative effects, we add a linear viscous damping term to the equation. Hence, we study the following Duffing equation

$$m\ddot{q} + c\dot{q} + k_1 q + k_3 q^3 = F\cos(2\pi f t)\,, \tag{1.7}$$

where $m = 3\rho AL/8\,[\mathrm{kg}]$ is the equivalent mass, $k_1 = 2\pi^4 EI/L^3\,[\mathrm{N/m}]$ is the equivalent linear stiffness, $k_3 = \pi^4 EA/8L^3\,[\mathrm{N/m^3}]$ is the equivalent cubic stiffness, and $F\,[\mathrm{N}]$ is the transversal force amplitude. In addition, the cross-sectional area is given by $A = bh\,[\mathrm{m^2}]$ and the second moment of area by $I = bh^3/12\,[\mathrm{m^4}]$.

We take the following parameter values. Mass density $\rho = 2700\,[\mathrm{kg/m^3}]$ and Young's modulus $E = 70\,[\mathrm{GPa}]$, width $b = 0.02\,[\mathrm{m}]$, height $h = 0.02\,[\mathrm{m}]$, length $L = 1.00\,[\mathrm{m}]$, linear viscous damping parameter $c = 2.58\,[\mathrm{Ns/m}]$, transversal force amplitude: $F = 939\,[\mathrm{N}]$.

(a) Compute the magnitude and phase of the frequency response function (FRF) considering the linearized Duffing equation (1.7) (i.e., without the cubic stiffness term) with as input harmonic transversal force excitation $F(t)$ and as output the harmonic transversal displacement $v(\frac{L}{2}, t)$ at the middle of the beam. Use the following parameters: frequency range $f \in [1, 700]$ [Hz] and a frequency step $\Delta f = 1$ [Hz].

Tip: plot the magnitude of the FRF $(\hat{V}(\frac{L}{2}, f)/\hat{F}(f))$ *on a logarithmic scale on the y-axis. Indicate the unit of the FRF on the y-axis.*

Compute the eigenfrequencies in [Hz] of the damped and undamped linearized Duffing equation.

In each of the following questions, you have to perform a slow-stepped sine frequency sweep analysis using the function `sweep` with as input the duffing equation in first order form depending only on time t, state $\boldsymbol{x}$ and excitation frequency f, initial condition $\boldsymbol{x}_0 = [v(\frac{L}{2}, 0), \dot{v}(\frac{L}{2}, 0)]$, start frequency, end frequency, frequency step Δf, number of transient periods $N_\mathrm{t} = 200$, and number of steady-state periods $N_\mathrm{s} = 200$. The function will return a vector of excitation frequencies f [Hz] as well as the displacement amplitude $(v_{\max} - v_{\min})/2$ [m] for each frequency. In each case, plot the displacement amplitude versus the excitation frequency. *Note: the expected computational time to complete the frequency sweep analyses can take up to several minutes.*

(b) Multiply the FRF from *(a)* with $F = 939$ [N] to get for this force excitation amplitude the linear frequency response in [m]. Carry out a slow-stepped sweep-up analysis for the linearized Duffing equation for the frequency range $f \in [1, 700]$ [Hz]. Take the following parameters: start frequency 1 [Hz], frequency step $\Delta f = 1$ [Hz], zero initial condition $\boldsymbol{x}_0 = [v(\frac{L}{2}, 0), \dot{v}(\frac{L}{2}, 0)] = [0, 0]$. Plot the FRF together with the displacement amplitude of the sweep-up analysis in the same plot. *Hint: You should get the same graphs.*

(c) Carry out a slow-stepped sine frequency sweep-up analysis for the frequency range $f \in [1, 50]$ [Hz] with the following parameters: start frequency 1 [Hz], frequency step $\Delta f = 0.25$ [Hz], and zero initial condition $\boldsymbol{x}_0 = [0, 0]$. Let the frequency sweep-up analysis directly be followed by a sweep-down analysis from 50 [Hz] to 1 [Hz].

(d) Carry out a slow-stepped sine frequency sweep-up analysis for the frequency range $f \in [50, 600]$ [Hz] with the following parameter: start frequency 50 [Hz], frequency step $\Delta f = 1$ [Hz], zero initial condition $\boldsymbol{x}_0 = [0, 0]$. Again, let the frequency sweep-up analysis directly be followed by a frequency sweep-down analysis from 600 [Hz] to 50 [Hz].

(e) Carry out a slow-stepped sine frequency sweep-up analysis for the frequency range $f \in [400, 450]$ [Hz] with the following parameters: start frequency 400 [Hz], frequency step $\Delta f = 0.25$ [Hz], initial condition $\boldsymbol{x}_0 = [0.0176, 8.00]$. Repeat the procedure for a frequency sweep-down analysis for the frequency interval $f \in [350, 400]$ [Hz] with start frequency 400 [Hz]. Use the same initial condition $\boldsymbol{x}_0$ as for the sweep-up.

(f) Combine the results computed in questions *(c)*-*(e)* in a single plot. Describe qualitatively the difference between the linear frequency response *(a)* and the nonlinear frequency response obtained here.

In the following carry out several single frequency analyses. Use the provided function `simulate_and_plot`, which plots the time histories of displacement and velocity over the complete time interval. In addition it plots the zoomed time histories of displacement and velocity near the end of the time interval that show the steady-state response over the last 10 excitation periods. Finally, plot the phase portrait together with the Poincaré section using the function `poincare_section`, and the frequency components of the steady-state part of the displacement response on a linear(x)-logarithmic(y) scale for a given frequency range using `frequency_spectrum`. Use in each case $N_\mathrm{o} = 200$ number of points per period and $N_\mathrm{s} = 200$ number of steady-state periods.

(g) Parameters: frequency $f = 34$ [Hz], number of transient periods $N_\mathrm{t} = 200$, initial condition $\boldsymbol{x}_0 = [0, 0]$, frequency range for frequency components $f \in [0, 500]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(h) Parameters: frequency $f = 37$ [Hz], number of transient periods $N_\mathrm{t} = 200$, initial condition $\boldsymbol{x}_0 = [0, 0]$, frequency range for frequency components $f \in [0, 500]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures. Compare the results for $f = 34$ [Hz] from *(g)* and the results for $f = 37$ [Hz]. What is the main difference and how would you call the resonance at $f = 37$ [Hz]?

(i) Parameters: frequency $f = 400$ [Hz], number of transient periods $N_\mathrm{t} = 800$, initial condition $\boldsymbol{x}_0 = [0.1, 0]$, frequency range for frequency components $f \in [0, 1000]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(j) Parameters: frequency $f = 400$ [Hz], number of transient periods $N_\mathrm{t} = 400$, initial condition $\boldsymbol{x}_0 = [0.0176, 8.0]$, frequency range for frequency components $f \in [0, 1000]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(k) Parameters: frequency $f = 400$ [Hz], number of transient periods $N_\mathrm{t} = 1400$, initial condition $\boldsymbol{x}_0 = [0.018, 8.0]$, frequency range for frequency components $f \in [0, 1000]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures. Compare these results with the ones obtained in *(i)*.

In the following question, we examine how the force amplitude influences the frequency response curve, that is, the displacement amplitude $(v_\mathrm{max} - v_\mathrm{min})/2$ [m] as a function of the excitation frequency f [Hz]. *Note: the expected computational time to complete the frequency sweep analyses can take up to several minutes.*

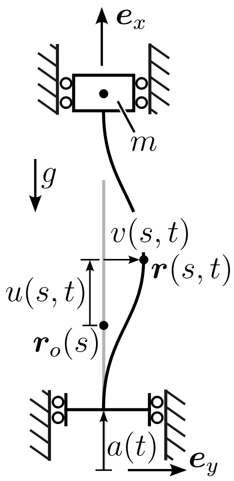
(l) Take the force amplitude: i) twice lower, i.e., $F = 469.5$ [N] and ii) twice higher, i.e., $F = 1878$ [N], and repeat the computations in the questions *(c)* and *(d)* (For the latter, i.e., repeating *(d)* use a frequency range of $f \in [50, 750]$ [Hz]). As in *(f)*, for each force amplitude, combine the results in a single plot. Interpret your results: when does the nonlinear behavior become more pronounced and how do you see that?

(m) Neglecting linear viscous damping and nonlinear stiffness (assuming small vibration amplitudes), calculate the lowest three eigenfrequencies of the undamped, linear beam and plot the corresponding three eigenmodes. *Hint: use the table with exact analytical solutions for the eigenfrequencies and the eigenmodes of a clamped-clamped beam presented in the part with the assignments for Linear Structural Dynamics.* Compare the analytical exact value for the first eigenfrequency with the

approximated first eigenfrequency computed in *(a)*. What is the reason for the possible (small) difference? When you look at the amplitude-frequency diagrams of *(l)* and at the eigenfrequencies and the shapes of eigenmodes 2 and 3, what is your conclusion regarding the accuracy of the amplitude-frequency diagrams of *(l)*?

2 Nonlinear Dynamic Buckling Analysis

In this assignment, you will study the dynamics of a vertical elastic beam with a top mass subjected to harmonic base excitation. In Assignment 2.1, you will derive a single-mode approximation of the system under the assumptions of inextensibility and moderate curvature. In Assignment 2.2, you will investigate the dynamics of the system in the presence of geometric imperfections.



Assignment 2.1 Consider a vertically oriented inextensible beam of length L with constant line mass density ρA and constant bending stiffness EI. The bottom end of the beam is clamped to a slider whose axial displacement in $\boldsymbol{e}_x$-direction is prescribed by a known excitation function $a(t)$. The top end of the beam is clamped to a block of mass m that can move freely in $\boldsymbol{e}_x$-direction. The system is subjected to gravity pointing in negative $\boldsymbol{e}_x$-direction with gravity constant g.

(a) The undeformed and straight reference configuration is $\boldsymbol{r}_o(s) = s\boldsymbol{e}_x$. Formulate the kinematic boundary conditions for the displacement fields in axial and transversal direction $u(s,t)$ and $v(s,t)$, respectively.

(b) The beam is assumed to be inextensible, i.e., the axial strain $e(s,t) = 0$ must vanish everywhere along the beam. Show how this constraint relates u' with v'. Keeping terms up to second order, show that this nonlinear relation can be approximated as

$$u' = -\frac{v'^2}{2} + O(\epsilon^3)\,. \tag{2.1}$$

Use the nonlinear relation between u' and v' in the expression for the nonlinear curvature k, and show that in the inextensible case this expression reduces to

$$k = \frac{v''}{\sqrt{1 - v'^2}}\,. \tag{2.2}$$

Assuming v' and v'' to be of order $O(\epsilon)$, show how to expand the curvature (2.2) up to order $O(\epsilon^3)$, i.e.,

$$k = v''\Big(1 + \frac{1}{2}v'^2\Big) + O(\epsilon^4)\,. \tag{2.3}$$

(c) Use (2.1) to derive the displacement u_P, virtual displacement δu_P and acceleration $\ddot{u}_P$ of the center of mass P of the top mass. Neglecting the inertial effects of the beam in axial direction as ($\rho AL \ll m$), formulate the dynamical virtual work of the system δW^{dyn} in terms of $\delta v, \ddot{v}$ and $\delta u_P, \ddot{u}_P$. Moreover, formulate the external virtual work functional δW^{ext} due to gravity in terms of δu_P, neglecting also here the mass of the beam.

(d) Use the expansion (2.3) in the strain energy density of the inextensible nonlinear Euler–Bernoulli beam, retaining terms up to order $O(\epsilon^4)$. Carry out the variation of the strain energy to obtain the internal virtual work functional

$$\delta W^{\text{int}} = -\int_0^L \left\{\delta v' EI v' v''^2 + \delta v'' EI(v'' + v'' v'^2)\right\} \mathrm{d}s\,. \tag{2.4}$$

(e) To approximate the nonlinear vibrational behavior in the low-frequency range, we use a single-mode (single-DOF) approximation model for the beam. The transversal displacement field of the beam is approximated as

$$v(s,t) = q(t)\frac{1-\cos(2\alpha s)}{2} \quad \text{with} \quad \alpha = \frac{\pi}{L}\,. \tag{2.5}$$

Sketch the global ansatz function and show that the boundary conditions are satisfied. What is the physical interpretation of $q(t)$?

(f) Use the ansatz function (2.5) to show that the displacement of point P is approximated as

$$u_P(t) = a(t) - \frac{\pi^2}{4L} q(t)^2\,. \tag{2.6}$$

Use the discrete expression (2.6) as a starting point to obtain the approximations of the virtual displacement δu_P and acceleration $\ddot{u}_P$. *Hint:* $\int_0^L \sin^2(2\alpha s)\mathrm{d}s = \frac{L}{2}$.

(g) Apply the Galerkin projection, i.e., approximate the principle of virtual work

$$\delta W^{\text{tot}} = -\int_0^L \left\{\delta v' EI v' v''^2 + \delta v'' EI(v'' + v'' v'^2) + \delta v \rho A \ddot{v}\right\} \mathrm{d}s - \delta u_P m(g + \ddot{u}_P) \tag{2.7}$$

with the global ansatz function (2.5) and the corresponding virtual displacement. For the virtual displacement of point P, use the previously derived approximations. Show that, under the single-mode approximation, the system dynamics reduces to the single-degree-of-freedom ODE

$$M(q)\ddot{q} + G(q,\dot{q}) + K(q) + \left(\frac{2\pi^4 EI}{L^3} - \frac{\pi^2 mg}{2L}\left(1 + \frac{\ddot{a}(t)}{g}\right)\right) q = 0 \tag{2.8}$$

with

$$M(q) = \frac{3\rho A L}{8} + \frac{\pi^4 m}{4L^2} q^2\,, \quad G(q,\dot{q}) = \frac{\pi^4 m}{4L^2} q \dot{q}^2\,, \quad K(q) = \frac{\pi^6 EI}{L^5} q^3\,. \tag{2.9}$$

Hint: $\int_0^L \cos^2(2\alpha s)\mathrm{d}s = \frac{L}{2}$, $\int_0^L \sin^2(2\alpha s)\cos^2(2\alpha s)\mathrm{d}s = \frac{L}{8}$, $\int_0^L (1-\cos(2\alpha s))^2 \mathrm{d}s = \frac{3L}{2}$.

Assignment 2.2 In this part we will investigate the dynamics of an extended version of (2.8) with the physical parameter specified in Table 2.1. In engineering practice, beams will obviously not be perfectly straight, among others due to accuracy limitations in manufacturing processes. Although the geometric imperfections may have a general form, for our analysis it is enough to only take into account the contribution of the

Table 2.1: Physical parameters for the vertical elastic beam with top mass under harmonic base excitation.

physical parameter	**symbol**	**value**	**unit**
mass density of the beam	ρ	7850	[kg/m^3]
Young's modulus of the beam	E	192	[GPa]
width of the beam	b	0.015	[m]
thickness of the beam	h	0.0005	[m]
beam's length	L	0.18	[m]
beam's cross-sectional area	$A = bh$	7.5×10^{-6}	[m^2]
beam's second moment of area	$I = bh^3/12$	1.5625×10^{-13}	[m^4]
beam's shape imperfection	e_1	1.24	[–]
linear viscous damping parameter$^{(a)}$	c_1	0.6/4	[Ns/m]
nonlinear quadratic damping constant$^{(b)}$	$c_{q,1}$	0.2/8	[Ns^2/m^2]
top mass	m	0.51	[kg]
base acceleration excitation factor	r_d	1.55	[–]
gravitational acceleration	g	9.81	[m/s^2]
excitation frequency	f	–	[Hz]
time	t	–	[s]

$^{(a)}$models material damping $^{(b)}$models air damping

considered mode shape (2.5) to the actual geometric imperfection of the beam, i.e., the initial predeformed transversal displacement $v_0(s)$ is assumed to be determined by (2.5) when setting $q(t) = e_1 h$. The variable e_1 [–] is the dimensionless imperfection factor and h [m] is the thickness of the beam; note that halfway the length of the beam (at $s = L/2$ [m]), if $e_1 = 1$ [–], the imperfection equals the beam's thickness.
Adapting the derivation in assignment 2.1 by taking additionally precurvature and dissipative effects into account, the equation of motion (EoM) for this single-mode approximation takes the form

$$M(q)\ddot{q}+G(q,\dot{q})+C(\dot{q})+p_1\left[1-r_0\left(1+\frac{\ddot{a}(t)}{g}\right)-p_2e_1^2\right]q+K(q) = p_3e_1r_0\left(1+\frac{\ddot{a}(t)}{g}\right) \quad (2.10)$$

where

- $P_\mathrm{c} = \dfrac{4\pi^2EI}{L^2}$ [N] is the critical static Euler buckling load for clamped-clamped boundary conditions.

- $r_0 = \dfrac{mg}{P_c}$ [–] is the dimensionless static buckling parameter for the perfect beam; if $0 \leq r_0 < 1$, static buckling does not occur; if $r_0 = 1$, the weight of the top mass equals the critical static buckling load; and if $r_0 > 1$, static buckling occurs.

- $p_1 = \dfrac{2\pi^4EI}{L^3}$ [N/m] is the equivalent linear stiffness.

- $p_2 = \dfrac{\pi^2h^2}{4L^2}$ [–] is the imperfection static buckling load reduction factor.

- $p_3 = hp_1$ [N] is the imperfection direct excitation load.
- $M(q) = \dfrac{3\rho AL}{8} + \dfrac{\pi^4 m}{4L^2}\left(h^2e_1^2 + 2he_1q + q^2\right)$ [kg] is the equivalent mass.
- $G(q,\dot{q}) = \dfrac{\pi^4 m}{4L^2}\dot{q}^2(he_1 + q)$ [N] contains the centrifugal and Coriolis forces.
- $C(q,\dot{q}) = c_1\dot{q} + c_{q,1}|\dot{q}|\dot{q}$ [N] contains the damping forces.
- $K(q) = \dfrac{\pi^6 EI}{4L^5}\left(4q^3 + 9he_1q^2\right)$ [N] contains the nonlinear elastic forces.
- $\ddot{a}(t) = r_{\mathrm{d}} g \sin(2\pi f t)$ [m/s^2] is the prescribed base acceleration excitation. This excitation appears as a direct excitation on the right-hand side of (2.10) (only for the imperfect beam), and as a parametric excitation in the linear stiffness term.

(a) For the static case, calculate $r_0 = mg/P_{\mathrm{c}}$ and conclude if the beam will statically buckle or not, i.e., due to the weight of the top mass only and without dynamic base excitation. Calculate the static buckling diagram for the beam: plot r_0 on the vertical axis (imagine varying the value for the top mass m as a way to vary the dimensionless static axial load r_0) and plot the dimensionless transversal displacement of the beam half way its length $v(L/2,t)/h = q(t)/h$ on the horizontal axis. Indicate the static buckling point P_{c} on the vertical axis (at this point the linear stiffness becomes 0) and indicate the location in the static buckling diagram for the value of m specified in the table above. Describe the possible solutions for axial loads $mg > P_{\mathrm{c}}$. Plot the diagram first for the perfect beam ($e_1 = 0$) and then for an imperfect beam ($e_1 = 1.24$). *Hint: first rewrite the EoM of the system for the static situation by removing all time dependent terms. Vary the value of the top mass so that the interval* $0 < r_0 < 2$ *is covered, take a step of* $\Delta r_0 = 1 \cdot 10^{-4}$. *Solve the nonlinear algebraic equations that results in a polynomial equation in the static condition with Matlab's* `roots` *function. To obtain the physically relevant solutions, only consider purely real solutions (i.e., discard the ones with complex values).*

(b) Neglecting damping, neglecting nonlinear terms, neglecting base excitation ($r_{\mathrm{d}} = 0$ [–]), assuming a perfect beam ($e_1 = 0$ [–]), and assuming small vibration amplitudes, at which frequency does the first natural resonance of the unbuckled beam occur, when the beam is perturbed in the transversal (or lateral) direction (without base excitation, so $r_{\mathrm{d}} = 0$ [–]), as a function of the dimensionless axial load for the range $0 \leq r_0 \leq 1$ [–], i.e., plot f_n [Hz] against r_0 [–]. What is the value for the first natural resonance frequency for our case $m = 0.51$ [kg]?

(c) For the imperfect beam, carry out a slow stepped sine frequency sweep-up and sweep-down analysis for the frequency interval $f \in [1, 160]$ [Hz] with the following parameters: start frequency 1 [Hz], frequency step $\Delta f = 0.25$ [Hz], and zero initial condition $\boldsymbol{x}_0 = [v(\frac{L}{2},0), \dot{v}(\frac{L}{2},0)] = [0,0]$. Let the frequency sweep-up analysis directly be followed by a sweep-down analysis from 160 [Hz] to 1 [Hz].

Hint: Use the function `sweep` *with as input the EoM in first order form, i.e.,* `beam_w_top_mass`*, initial condition* $\boldsymbol{x}_0$*, start frequency, end frequency, frequency step* Δf*, number of transient periods* $N_{\mathrm{t}} = 350$*, and number of steady-state periods*

$N_s = 100$. *The function will return a vector of excitation frequencies f [Hz] as well as the displacement amplitude $(v_{\max} - v_{\min})/2$ [m] for each frequency.*

Plot the displacement amplitude versus the excitation frequency of the sweep-up and sweep-down in the same graph. Use a logarithmic scale on the vertical axis (between $3 \cdot 10^{-5}$ and $3 \cdot 10^{-3}$) [m].

Note: the expected computational time to complete the frequency sweep analyses can take up to several minutes.

In the sweep-up branch, you will see three major resonance peaks. How are those approximately related. Can you already guess, how those solutions could be classified? *Hint: If you are unsure at this point, complete the single-frequency analysis below first, then revisit this question.* In the sweep-down branch, you will see a resonance peak at $f \approx 145$ [Hz] that bends over to lower frequencies. This effect is called softening. Can you explain this effect? *Hint: look at the static buckling diagram derived in (a) for the imperfect beam. What changes in the axial beam stiffness and the effective mass of the system $M(q)$ for increasing transversal displacement?*

In the following carry out several single frequency analyses with initial condition $\boldsymbol{x}_0 = [v(\frac{L}{2}, 0), \dot{v}(\frac{L}{2}, 0)]$. Use the provided function `simulate_and_plot`, which plots the time histories of displacement and velocity over the complete time interval. In addition, it plots the zoomed time histories of displacement and velocity near the end of the time interval that show the steady-state response over the last 10 excitation periods. Finally, plot the phase portrait together with the Poincaré section using the function `poincare_section`, and the frequency components of the steady-state part of the displacement response on a linear(x)-logarithmic(y) scale for a given frequency range using `frequency_spectrum`. Use in each case $N_o = 200$ number of points per period and $N_s = 200$ number of steady-state periods. To improve numerical accuracy, use tighter solver tolerances for the time integration: set the relative tolerance to 10^{-9} and the absolute tolerance to 10^{-12}. This can be done in `simulate_and_plot` by defining `odeopts = odeset('RelTol',1e-9,'AbsTol',1e-12)`.

(d) Parameters: frequency $f = 65.5 \approx f_n$ [Hz], number of transient periods $N_t = 200$, initial condition $\boldsymbol{x}_0 = [0.01, 4.8]$, frequency range for frequency components $f \in [0, 300]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(e) Parameters: frequency $f = 65.5 \approx f_n$ [Hz], number of transient periods $N_t = 200$, initial condition $\boldsymbol{x}_0 = [0.01, 4.9]$, frequency range for frequency components $f \in [0, 300]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(f) Parameters: frequency $f = 35.5 \approx f_n/2$ [Hz], where in *(c)* we found a resonance, and at frequency $f = 43$ [Hz] (just after the resonance) with number of transient periods $N_t = 100$, zero initial condition $\boldsymbol{x}_0 = [0, 0]$, frequency range for frequency components $f \in [0, 300]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(g) Parameters: frequency $f = 145 \approx 2f_n$ [Hz], number of transient periods $N_t = 100$, zero initial condition $\boldsymbol{x}_0 = [0.0, 0.0]$, frequency range for frequency components $f \in$

$[0, 300]$ [Hz]. Interpret the obtained steady-state behavior based on the presented figures.

(h) Use the function `bruteforce` to plot the bifurcation diagrams for the frequency sweep-up $f \in [1,\ 160]$ Hz and the frequency sweep-down (use two separate figures). Adopt the following parameter values: $\Delta f = 0.25$ Hz, $N_t = 300$, $N_s = 10$, and zero initial conditions $\boldsymbol{x}_0 = [0,\ 0]$. Plot the displacement amplitude $v(L/2,\ 0 \bmod (1/f))$ versus the excitation frequency f [Hz]. Do the obtained results agree with your expectations? Discuss why or why not.

(i) What type of bifurcations occur at frequencies 139, 151.5 and 11.5 [Hz]?