Structural Dynamics (4DM90)

Assignments for completion of the part Linear Structural Dynamics

1st Quartile 2025-2026

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Some remarks in advance

- Try first to use the Linear Structural Dynamics slides of the course to solve of the problems.
- You may use the following book as background study material: Kraker, Bram de, 'A Numerical-Experimental Approach in Structural Dynamics', Shaker Publications, Maastricht, 2013. ISBN 978-90-423-0259-4. For sale at Simon Stevin for Euro 15,00.
- Matlab data files, script files and additional pdf files, needed for the solution of the problems, can be found on CANVAS: Assignments_4DM90_LSD_25.zip.
- Work in groups of 3 students. It is not necessary to work out all problems. Choose two problems you like. However, **at least one problem** should be taken from problems **LSD 1-4**, and **at least one problem** should be taken from problems **LSD 5-7**. Every student should be able to answer questions about the two assignments.
- The problems can be solved using MATLAB on a PC/laptop/notebook.
- Try to program in a structured way for fast detection of program errors. Try to program flexible; introduce variables if that is appropriate.
- Pay a lot of attention to the interpretation of calculated data and figures.
- Make a concise report, including (intermediate) results, figures, tables, and their discussions. Please do NOT include MATLAB listings.

LSD1. Large Dynamic Models/Modelling of Damping

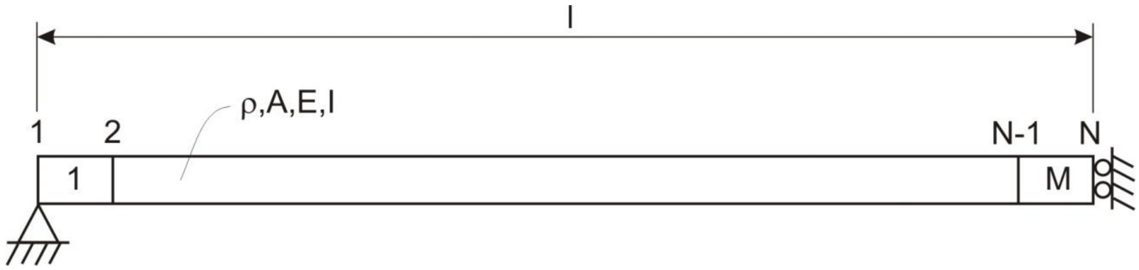


Figure 1.1 Pinned-sliding beam system

Consider the 2D pinned-sliding beam system shown in Figure 1.1. The properties of the steel beam are given in Table 1.1. Only the transversal in plane vibrations of the beam are considered, so axial and torsional vibrations are ignored. The beam is slender. So, when we study transversal vibrations, it suffices to consider pure bending of the beam (shear is neglected). Place the MATLAB command ‘format short e’ in your MATLAB program before you print numerical results.

Length l:	1	m
Young’s Modulus E:	$2.1\ 10^{11}$	N/m^2
Cross section A:	10^{-4}	m^2
Mass density ρ:	7850	kg/m^3
Second moment of area I:	$10^{-8}/12$	m^4

Table 1.1 Beam properties

1. Calculate (see appendix A) the exact analytical solutions of the six lowest eigenfrequencies f_{Ok} (in Hz) and draw the corresponding eigenmodes of the undamped pinned-sliding beam system in Figure 1.1. Although the pinned-sliding boundary conditions are not in the list, the solutions of a beam with other boundary conditions can be used to calculate/draw the required quantities. Determine also the six lowest angular eigenfrequencies $\omega_{Ok} = 2\pi f_{Ok}$ (in rad/s).

2. A Finite Element model of the undamped beam can be obtained based on Euler beam theory because the beam is slender and shear can therefore be neglected (see Appendix B for Euler beam element matrices). In Figure 1.1, M is the number of beam elements and N is the number of nodes.

   a) Assemble the mass and stiffness matrix of the system. Then calculate the six lowest eigenfrequencies (in Hz) of Finite Element models of the beam using the MATLAB command ‘eig’ using respectively (M=) 10, 100, and 1000 Euler beam elements of equal size. Calculate the cpu-time needed for solving the eigenvalue problems by using the MATLAB command ‘cputime’ (place this command just before and just after the ‘eig’ command; the difference gives the used cpu-time). What happens when you try to use 10000 elements?

   b) Again calculate the six lowest eigenfrequencies (in Hz) of Finite Element models of the beam, but now using the MATLAB command ‘eigs’ using respectively (M=) 10, 100, 1000, 10000, and 100000 Euler beam elements of equal size (in the last case, be patient). Use of the command ‘eigs’ requires that the sparse mass and stiffness matrices are defined using the MATLAB command ‘sparse’. Again calculate the cpu-time needed for solving the eigenvalue problems. Type ‘doc eig’ and ‘doc eigs’. What is the essential difference between these two eigenvalue solvers?

c) Compare the all the series of eigenfrequencies calculated above at 2a) and 2b) with the six exact analytical values calculated at 1) and compare the required cpu-times. What are your conclusions?

In the next part of the problem, sometimes it is required to calculate the ‘direct’ (or collocated) Frequency Response Function (FRF) of the transversal displacement of the right end of the beam for the undamped and several damped configurations. ‘Direct’ (or collocated) means that the excitation and the response of the beam is at the same degree of freedom. The FRF must be evaluated in the frequency range [0.2, 500] Hz with equidistant steps of **0.2 Hz**. Both the modulus (on log-log scale) and the phase angle (on lin-log scale) of the FRF must be evaluated. In all questions below, the calculations must be based on a Finite Element model of **100** Euler beam elements. Do **not** use standard MATLAB commands for calculating FRF’s such as bode, nyquist, and freqresp. For all problems given below, interpret the results as much as possible.

3. a) Plot the FRF for the undamped system in three ways: first by calculating the last diagonal element (this matrix-element corresponds to the transversal displacement of the right end of the beam) of the inverse of the dynamic stiffness matrix $-4\pi^2 f^2 M + 2\pi j f B + K \;\left(B = O\right)$ for each frequency, secondly by using equation (1.62) of the book using a modal superposition of the lowest six eigenmodes only (but based on a model of 100 elements, i.e. 200 dofs), and thirdly by again using equation (1.62) but now using all modes ($n = 200$). Compare the results and the cpu-times needed for both approaches. What are your conclusions?

   b) Plot the FRF of the proportionally damped system based on a modal superposition of the lowest six modes only, where the dimensionless modal damping coefficient of each mode is equal to $\xi_k = 0.02$ [-] (use equation (1.69) of the book). What are the first six eigenvalues $\lambda_k = \mu_k + j\omega_k$ (in rad/s) corresponding to the lowest six modes (give the six eigenvalues with positive imaginary parts)?

   c) In the Rayleigh damping model, the expression for the damping matrix $B = \alpha M + \beta K$ contains only the two independent parameters α and β. The Rayleigh damping model is a subclass of the class of proportionally damped systems. The dimensionless modal damping coefficient of only two eigenmodes can be tuned independently. Show, that if the (real, undamped) eigenmodes U_O are normalized on the mass matrix, the relation between the dimensionless modal damping coefficients ξ_k and the parameters α and β is: $\xi_k = 0.5\left(\dfrac{\alpha}{\omega_{Ok}} + \beta\omega_{Ok}\right)$. Note that the diagonal elements of the diagonal matrix $U_O^T B U_O$ are written as $2\xi_k\omega_{Ok}$.

   d) Plot the FRF based on a modal superposition of the lowest six modes only using a Rayleigh damping model where the damping matrix is chosen proportionally to the mass matrix: $B = \alpha M$. Choose α so, that the dimensionless modal damping coefficient of the third mode is equal to $\xi_3 = 0.02$. What are the dimensionless modal damping coefficients of the first six eigenmodes for this choice? Compare the FRF with the FRF obtained at 3b).

e) Plot the FRF based on a modal superposition of the lowest six modes only using a Rayleigh damping model where the damping matrix is chosen proportionally to the stiffness matrix: $B = \beta K$. Choose β such, that the dimensionless modal damping coefficient of the third mode is equal to $\xi_3 = 0.02$. What are the dimensionless modal damping coefficients of the first six eigenmodes for this choice? Compare the FRF with the FRF obtained at 3b).

f) Plot the FRF based on a modal superposition of the lowest six modes only using a Rayleigh damping model where the damping matrix is chosen as follows: $B = \alpha M + \beta K$. Choose α and β such, that the dimensionless modal damping coefficients of the 2nd and 4th mode are equal to 0.02. Compare the FRF with the FRF obtained at 3b).

g) At node N at the right end of the beam a linear viscous dashpot with a damping constant of b [Ns/m] is added to the undamped system. Write the equations of motion in first order form using equation (1.21) on page 14 of the book and solve the corresponding eigenvalue problem for the choices $b = 5$ [Ns/m] and $b = 50$ [Ns/m].
(Do not solve the eigenvalue problem in Matlab using the symmetric matrices C and D. This approach leads to erroneous results and illustrates that you should not have blind faith in MATLAB routines. This probably is caused by bad numerical conditioning. Instead, calculate the system matrix $\hat{A}$ as defined in equation (1.21) on page 14 of the book and calculate the eigenvalues and eigenmodes of this matrix using the commands eig or eigs. In principle, this is an unattractive way of solving the eigenvalue problem, because first the inverse of the mass matrix must be calculated and the symmetry of the matrices C and D is lost.)
Compare for both choices of b the six eigenvalues λ_k with the lowest six imaginary parts (take only into consideration those eigenvalues with imaginary parts larger than or equal to zero!) with the lowest six exact undamped angular eigenfrequencies ω_{Ok} obtained at 1).
For the choice $b = 50$ [Ns/m], plot the real displacement parts of the two supercritically damped (or overdamped) eigenmodes (with corresponding real eigenvalues) against the node number (plot only displacements, no rotations). Likewise, plot the real and imaginary displacement parts of the two complex conjugate, subcritically damped (or underdamped) eigenmodes, of which the two corresponding complex conjugate eigenvalues have the smallest non-zero damped eigenfrequency, against the node number. In all eigenmode plots, scale the eigenmodes such, that the displacement element (thus not a rotation element!) with the largest modulus becomes equal to 1.
Plot the FRF (for both choices of b) based on a modal superposition of the lowest twelve eigenmodes using equation (1.54) of the book (the eigenmodes with the twelve lowest absolute values of the imaginary parts of their eigenvalues should be selected).

LSD2. Sensitivity Analysis and Parameter Estimation (Model Updating)

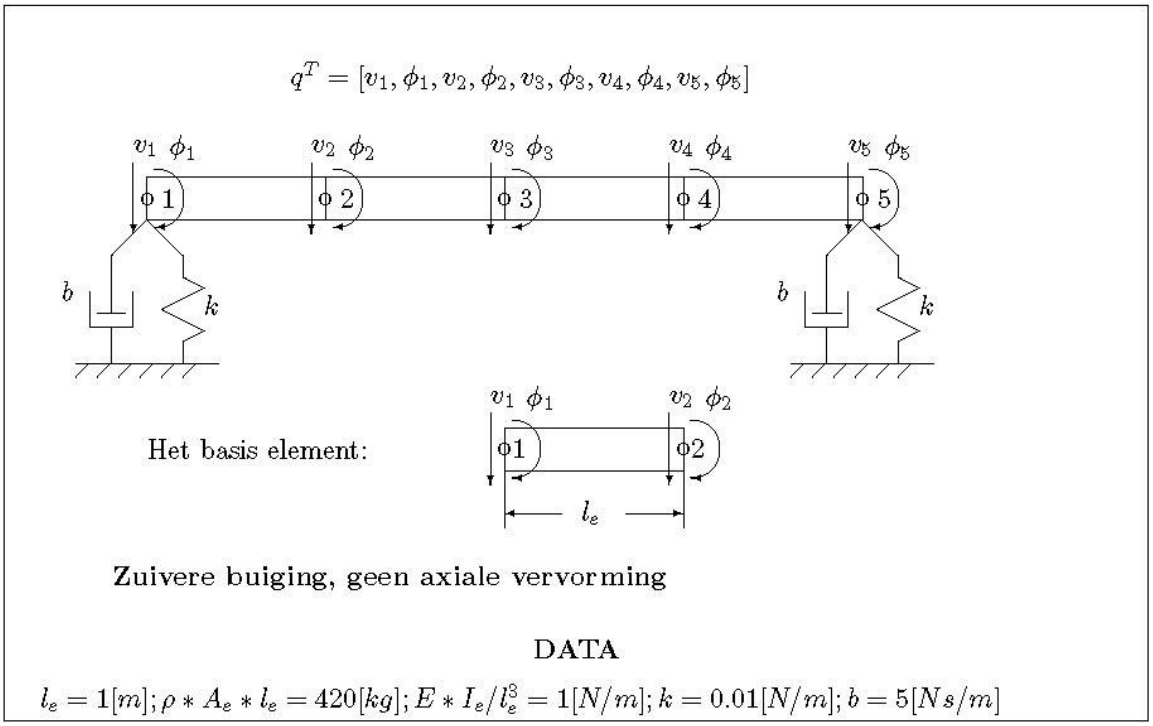


Figure 2.1 Beam system (modelled with 4 beam elements) placed on 2 springs and 2 dampers

Given:
The file 'derivat.mat' containing the stiffness matrix K_T, the damping matrix B_T, and the mass matrix M_T for the beam system presented in Figure 2.1:

$$M_T \ddot{q} + B_T \dot{q} + K_T q = 0$$

Check the correctness of the matrices (see Appendix B for element matrices). Note that only pure bending is considered and that axial and torsional vibrations are ignored.

Three of the eigenvalues of the system given in Figure 2.1 are:

$$\lambda_6 = -0.0120057 + 0.066698j$$
$$\lambda_8 = -0.0121405 + 0.188009j$$
$$\lambda_{10} = -0.0115956 + 0.370725j$$

These eigenvalues are solutions of: $\left(\lambda^2 M_T + \lambda B_T + K_T\right)u = 0$
Check this and interpret the corresponding eigenmodes.

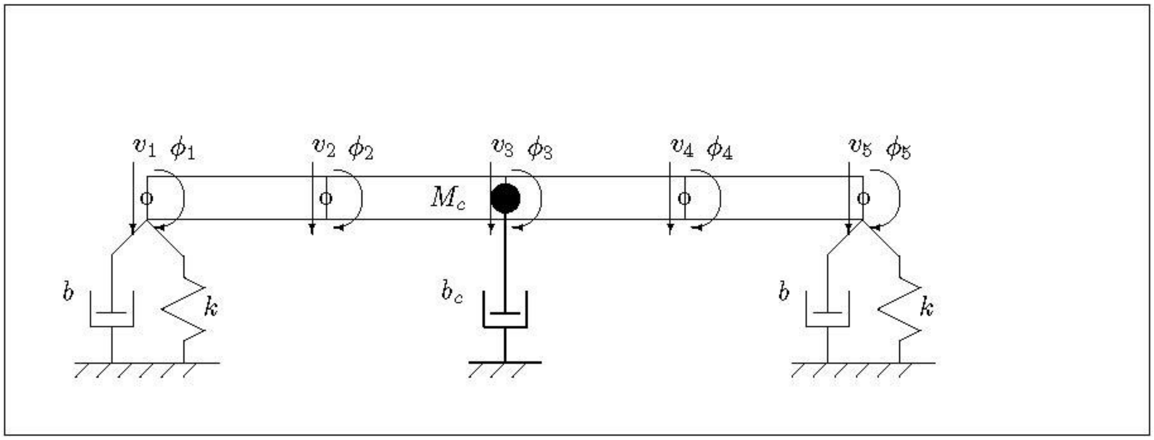


Figure 2.2 Beam system with additional point mass M_c and damper b_c

Investigate how the eigenvalues $\lambda_6, \lambda_8, \lambda_{10}$ change, if an extra point mass M_c or an extra viscous damper b_c are added to the system in the way shown by Figure 2.2. In other words, determine (using the adjoint method or direct method, see section 1.5 in the book):

$\frac{\partial \lambda_i}{\partial M_c}$ and $\frac{\partial \lambda_i}{\partial b_c}$ for $i = 6,8,10$

Interpret your results!

Suppose experiments show that:

$$\lambda_6^{\exp} = -0.013 + 0.058j$$
$$\lambda_{10}^{\exp} = -0.012 + 0.310j$$

Determine those values for M_c and b_c which provide for an optimal tuning between numerical and experimental results using an iterative model updating procedure, see section 4.4 in the book. Give information about the iteration process.

Discuss the meaning of weighting matrices by comparing tuning results obtained using two different weighting matrices.

LSD3. Dynamic Reduction Techniques

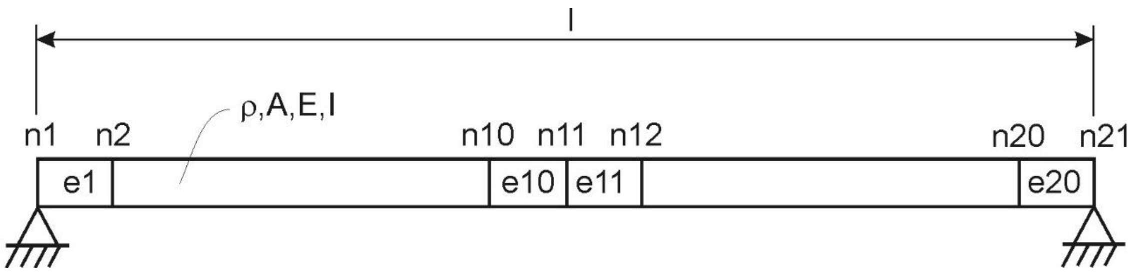


Figure 3.1 Pinned-pinned beam system
($\rho = 7850$ kg/m³, $A = 0.01$ m², $E = 2.1 \cdot 10^{11}$ N/m², $I = 10^{-4}/12$ m⁴, $l = 4$ m)

Consider the pinned-pinned beam shown in Figure 3.1. Only the transversal vibrations of the beam are considered, so axial and torsional vibrations are ignored. The beam is slender. So, when we study transversal vibrations, it suffices to consider pure bending of the beam (shear is neglected). This means that the beam may be modelled using Euler theory (see Appendix B for element matrices).

1. Determine analytically the exact solutions of the six lowest eigenfrequencies (in Hz) and corresponding eigenmodes of the pinned-pinned beam system in Figure 3.1. These analytical exact solutions can be found in Appendix A.
2. Determine the six lowest eigenfrequencies (in Hz) of the pinned-pinned beam system in Figure 3.1 using a Finite Element approach. Use 20 Euler beam elements of equal size to model the system.
3. Apply Guyan reduction to the Finite Element model used in question 2. Take as master degrees of freedom the transversal displacements of nodes 5,9,11,13, and 17 and the rotation of node 11. Determine the eigenfrequencies (in Hz) of the reduced model (consisting of 6 degrees of freedom). Provide a plot of the columns of the obtained Ritz-matrix T^G. Consider only the elements in T^G that correspond to transversal displacements and put the corresponding node numbers on the x-axis. Apply Guyan reduction using 6 other master degrees of freedom and try to choose these dofs so that the six lowest eigenfrequencies are as accurate as possible. Again, provide a plot of the columns of the obtained Ritz-matrix T^G.

The system of Figure 3.1 now is split in two components (also called substructures) of equal size. Component 1 is shown in Figure 3.2.

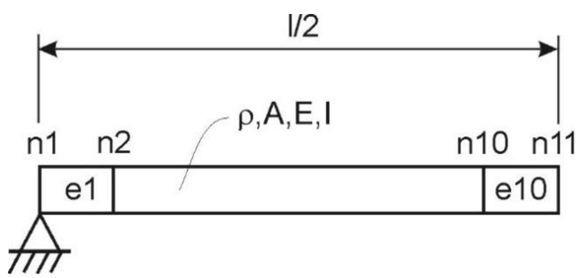


Figure 3.2 Component 1 (pinned-free)

4. Apply Craig-Bampton reduction to each of the two components. Note that for each component we have two boundary dof's at the free end of each component (the transversal displacement and the rotation). For each boundary dof a constraint static mode has to be calculated. Subsequently, keep two fixed-interface normal modes in the reduced component model for each component. Check the eigenfrequencies of these fixed-interface normal modes with the analytical exact values. Each reduced component model thus has four dof's. Determine the reduced component models. Provide a plot of the columns of the obtained Ritz-matrices T^{CB} for components 1 and 2. Again, consider only the elements that correspond to transversal displacements and put the corresponding node numbers on the x-axis.
   Couple these reduced component models to obtain the reduced system model and calculate the six eigenfrequencies (in Hz) of the latter model.
5. (Optional) Apply Rubin reduction to each of the components. Note that for each component we have two boundary dof's (the same ones as above). For each boundary dof a residual flexibility mode must be calculated. Subsequently, keep for each component the rigid body mode and the lowest elastic free-interface normal mode in the reduced component model. Check the eigenfrequency of the lowest elastic free-interface normal mode with the analytical exact value. Each reduced component model thus has four dof's. Plot the columns of the matrices Ritz-matrices T_1^R and T^R in the same way as in steps 3 and 4. Determine the reduced component models.
   Couple these reduced component models to obtain the reduced system model and calculate the six eigenfrequencies (in Hz) of the latter model.
6. Compare the six lowest eigenfrequencies of the system calculated in step 1-5 (calculate the differences with the exact analytical values in terms of percentage) and try to explain the differences. What are the benefits and the drawbacks of each reduction method? Which reduction method do you prefer and why?

LSD4. Impedance Coupling

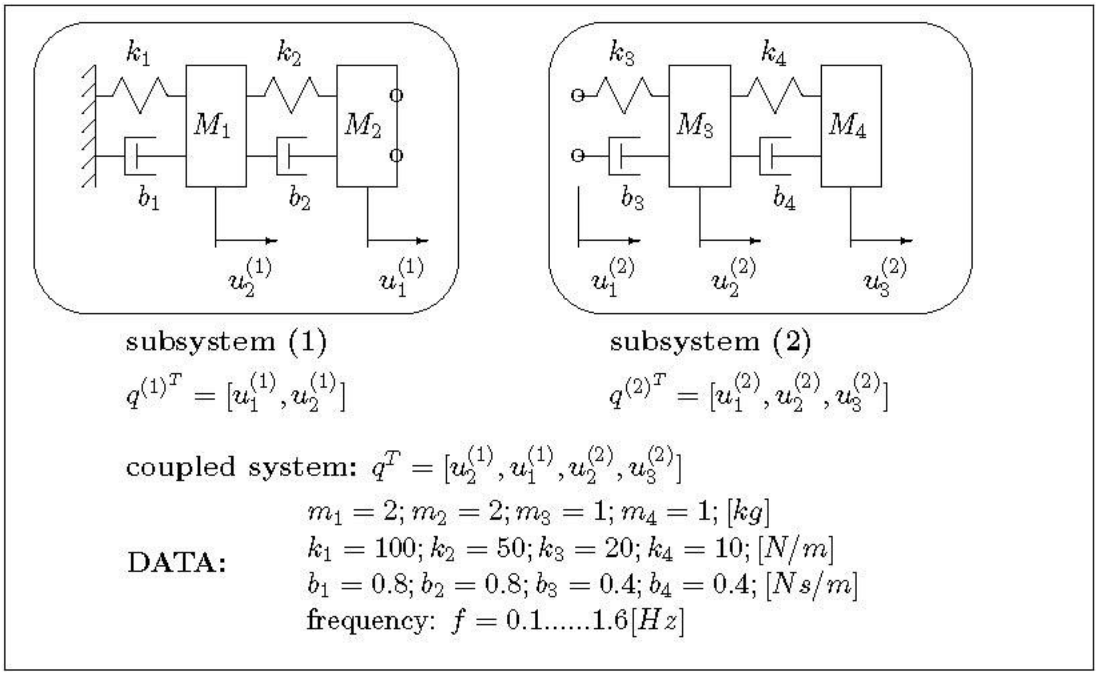


Figure 4.1 Two subsystem models

Consider Figure 4.1 showing two subsystem models.

1. Evaluate the FRF's of the two subsystems.
2. Evaluate the FRF of the total coupled system. This FRF is called the reference solution.
3. Determine the FRF (for example $H_{22}(f)$; excitation and response in degree of freedom number 2 of q) also via Impedance Coupling and compare this FRF with the reference solution. Use one of the two equivalent formulations of Impedance Coupling (in the book only the formulation based on three inversions is presented; on the slides also the formulation based on one inversion is given).
4. Suppose the FRF's of the two subsystems are determined experimentally. This can be simulated by adding some normally distributed (or Gaussian-distributed) input noise and/or normally distributed output noise of a suitable level (mention the standard deviation of the normally distributed noise) to the results of question 1 (should this noise be real or complex?). Investigate what the influence is of these disturbances on the FRF determined by Impedance Coupling. **Compare this FRF with noise (determined by Impedance Coupling) and the reference FRF to which the same level of noise has been added with the reference FRF without noise**. What is your conclusion?
5. Optional: carry out Impedance Coupling with and without noise for the other Impedance Coupling formulation mentioned under 3 and again do the comparison mentioned under 4.

LSD5. DFT/FFT

- Use Matlab's FFT and IFFT algorithm for determining the Fourier transform of the following characteristic time signals:

  1) a harmonic signal,
  2) a signal being the sum of two harmonic functions with close frequencies and different phase,
  3) a square wave signal,
  4) a sawtooth wave signal,
  5) the Dirac delta function,f
  6) a normally distributed (or Gaussian-distributed) white noise signal.

  Describe the time signals properly by mathematical expressions and/or presenting figures. If you use parameters in the mathematical expressions, also mention the parameter values (also mention the parameter values defining the normally distributed white noise signal)! Why are the Dirac delta function and normally distributed white noise very suitable as excitation signals in Experimental Modal Analysis?
  Do not only just present the Fourier transforms. Also mention in a table the parameter values used in FFT algorithm (i.e. the total measurement time $T = 1/\Delta f$, the frequency resolution Δf, the number of discrete time points $N = T/\Delta T$, the time increment ΔT, the sample frequency $f_s = 1/\Delta T$, and the folding/Nyquist frequency $f_{fold} = f_s/2$).
  Suppose that the time signals you consider are measured displacements. Give the unit of the Fourier transform of the time signal on the y-axis (Hint: you have two possibilities: [m] or [m/Hz], for each possibility a different scaling of the Fourier transformed signal has to be applied, see lecture notes).

- For signal 2), investigate the influence of varying the parameters T (while keeping N constant) and N (while keeping T constant) on the Fourier transform.

- For signal 2), investigate the influence of extending a time signal with zeroes (zero-padding) on the Fourier transform.

- Demonstrate the error source 'Aliasing' by means of a well-chosen signal. Show and discuss how 'Aliasing' can be solved.

- Demonstrate the error source 'Signal-leakage' by means of a well-chosen signal. Introduce some windowing functions (standard Matlab routines, such as hanning, hamming, bartlett, blackman, boxcar, chebwin, and kaiser) and discuss the possible advantages and disadvantages of applying the windows on the resulting FFT of the previously mentioned signal.

- Optional: determine zoom-FFT's of properly chosen signals in which the benefit of zoom-FFT becomes clear.

LSD6. System Estimation

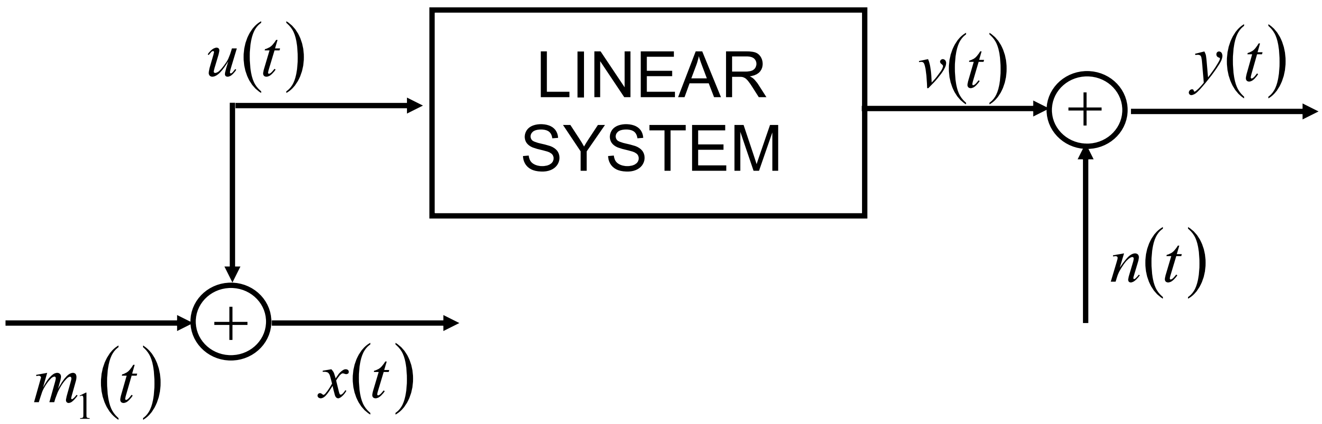


Figure 6.1: SISO System

Consider the Single-Input-Single-Output (SISO) System in Figure 6.1.

Given are the following three files corresponding to 3 different SISO systems:

- A file 'sch1dof.mat' containing the noisy time series 'excit' ($=u(t)=x(t), m_1(t)=0$) and 'output' ($=v(t)=y(t), n(t)=0$), both having 4000 time points, sampled with a time increment of $\Delta t = 0.1$ [s], so with a sample frequency of $f_s = 10$ [Hz]. The output signal (a displacement in mm) has been generated by numerical integration of a single-dof model using the excitation signal (a force in N) as input.
- A file 'sch2dof.mat' analogous to 'sch1dof.mat'. However, now a two-dof model has been used.
- A file 'schsweep.mat', containing the time series 'excit' and 'respons', however, now containing 20 data blocks of 128 time points each. Each block in 'excit' contains a frequency-modulated sweep as excitation signal. Each block in 'respons' contains the corresponding calculated system response. In this situation again $\Delta t = 0.1$ [s].

For each of the three files given:

Determine the FRF estimators $\widehat{H}_1$ and $\widehat{H}_2$ and the coherence function estimator $\hat{\gamma}^2_{xy}$ for the case $m_1(t) = n(t) = 0$ using the estimators $\hat{S}_{xx}, \hat{S}_{xy}, \hat{S}_{yx}, \hat{S}_{yy}$ (use the definitions given on the slides). Do not use Matlab-procedures such as tfestimate.m, spectrum.m, etc. Use two different block sizes. Use for the first two systems 31 blocks with 128 discretization points and 15 blocks with 256 discretization points (i.e. no overlap); use for the third system 20 blocks of 128 points and 10 blocks of 256 points (again no overlap). In all cases, apply the Hanning window to each block.

Then, investigate the influence of adding some normally distributed (or Gaussian-distributed) input noise $m_1(t)$ (choose an appropriate standard deviation and mention the value used) on the FRF estimators; the measured input signal now becomes $x(t) = u(t) + m_1(t)$. In addition, investigate the influence of adding some normally distributed output noise $n(t)$ (choose an appropriate standard deviation and mention the value used) on the FRF estimators; the measured output signal now becomes $y(t) = v(t) + n(t)$.

In all 3 cases (without disturbance noise, with input noise, and with output noise) and for the three systems, discuss the differences in the results of the $\widehat{H}_1$ and the $\widehat{H}_2$ estimator and evaluate the role of the coherence function. Discuss also the influence of using two different block sizes.

You may also investigate the effect of using 50% overlap in the blocks which results in more blocks available for averaging and may result in using the experimental data more effectively.

LSD7. Experimental Modal Analysis: A Frequency-Domain Modal Parameter Fit Procedure

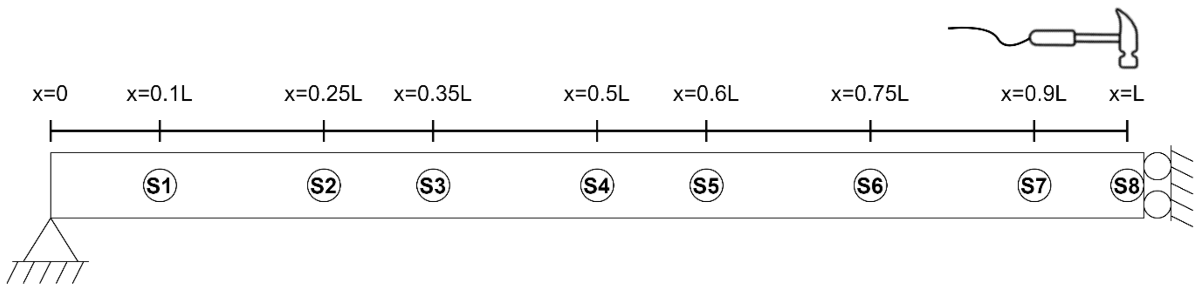


Figure 7.1: Experimental setup of the pinned-sliding beam, where the location of the 8 displacement sensors and the point of excitation are indicated relative to the length of the beam.

You are an engineer performing experimental modal analysis in order to identify a modal model of the pinned-sliding beam structure in Figure 7.1. As such, you place 8 displacement sensors along the length L of the beam and excite the right end of the beam with an impact hammer including a force sensor. From this, you obtain the 8 measured complex frequency response functions (FRFs) in [m/N], for the 1000 logarithmically spaced data points within the frequency range 10 to 2000 [rad/s], as given in *Measurement1.mat*. In this assignment, these were obtained from a model, so in fact we have a simulated experiment. Your plan is to perform a multi-mode modal parameter fit in the frequency domain.

1. Analyze the complete dataset of 8 FRFs. For each of the 4 resonances, estimate the eigenvalues $\lambda_k = \mu_k + j\nu_k$ [rad/s], with $k \in \{1, \ldots, 4\}$. Since the eigenvalues are identical for all 8 FRFs, it is enough to select one FRF for this estimation (choose one for which the resonances appear clearly, e.g. the FRF at position S8). For this selected FRF, estimate ν_k by means of peak-peaking and estimate μ_k by assuming proportional damping with dimensionless modal damping coefficient $\xi_k = 0.02$ [-] $\forall\, k$.

2. With these initial estimates for the eigenvalues, obtain, for each measured FRF, initial estimates for the remaining parameters $\alpha_s = [a_0, a_{1R}, a_{2R}, a_{3R}, a_{4R}, a_{1I}, a_{2I}, a_{3I}, a_{4I}]_s^\top$ with $s \in \{1, \ldots, 8\}$. As such, we will minimize the relative error between the i^{th} measured FRF H_i and i^{th} fitted FRF $\hat{H}_i$. Note that, by excluding a_1 from α_s, we assume the residual mass to be zero. Why is this a valid assumption?

   Firstly, find a matrix F to write the fitted FRF in the form $\hat{H}_i = F\alpha_s$, such that equation (2.106) from the book (or SD2 slide 29) holds.

   Secondly, solve the linear least-squares problem:

   $$\min_{\alpha_s} [e_{re}(\alpha_s)^\top \quad e_{im}(\alpha_s)^\top][e_{re}(\alpha_s)^\top \quad e_{im}(\alpha_s)^\top]^\top,$$

   where $e_{re}(\alpha) = Re(H_i - F\alpha) \oslash Re(H_i)$ and $e_{im}(\alpha) = Im(H_i - F\alpha) \oslash Im(H_i)$.
   Here, $\oslash$ indicates the element-wise division operator. Use the Matlab backslash operator (\) to solve the linear least-squares problem. Show the fitted FRF at sensor S3 and compare it to the respective measured FRF. Plot the magnitude of FRFs on a log-

log scale and the phase on a lin-log scale, both for the complete frequency range in [rad/s]. Also, plot the magnitude of the complex difference between the FRFs on a log-log scale for the complete frequency range.

3. Now you have obtained the initial estimates for all parameters $\beta_s = [a_0, a_{1R}, a_{2R}, a_{3R}, a_{4R}, a_{1I}, a_{2I}, a_{3I}, a_{4I}, \mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2, \nu_3, \nu_4]_s^\top$ with $s \in \{1, \ldots, 8\}$, improve this estimate by solving the non-linear least-squares problem:

$$\min_{\beta_s} [e_{re}(\beta_s)^\top \quad e_{im}(\beta_s)^\top][e_{re}(\beta_s)^\top \quad e_{im}(\beta_s)^\top]^\top,$$

where $e_{re}(\beta_s) = Re\big(H_i - \hat{H}_i(\beta_s)\big) \oslash Re(H_i)$ and $e_{im}(\beta_s) = Im\big(H_i - \hat{H}_i(\beta_s)\big) \oslash Im(H_i)$.

Use the Matlab function lsqnonlin with default settings to iteratively solve the non-linear least-squares problem. Show the obtained estimates for the eigenvalues of each FRF in a table and compare them to the initially estimated eigenvalues from subquestion 1.

4. Finally, obtain the final estimates of all parameters. For the eigenvalues, compute the global estimates by averaging the estimated eigenvalues over all 8 FRFs. For the other parameters, again solve the linear least squares problem for each FRF, but now using the global estimate of the eigenvalues. Show the final fit of the FRF at sensor S3 and compare it to the measured FRF. Again, plot the FRFs and magnitude of the complex difference between them. What is your conclusion?

5. Using the model that is used to produce the simulated experimental data, the first 6 true eigenmodes are produced as well, as given in *Numericalmodel.mat*. You want to compare your experimentally identified model with the underlying finite-element model. Determine the eigenmodes of the system from your estimated residues. Note that, since the rank of the residue matrix is 1 by definition, knowing just a single row or column is sufficient to determine the eigenmodes, apart from a constant (possibly complex) scaling factor.

   Firstly, compare the identified eigenmodes and the numerical eigenmodes by means of a MAC-matrix. Note that scaling of an eigenmode does not influence the values of the MAC-matrix. Which identified eigenmode corresponds to which numerical eigenmode?

   Secondly, plot the identified eigenmodes with their corresponding (as determined by the MAC-matrix) numerical eigenmode. To plot this, find the sensor location for each of the 4 identified eigenmodes, where the (complex) value of the mode shape has maximum magnitude and scale the measured eigenmode by this (complex) number. This ensures that the eigenmode becomes 1 at that location and gets a magnitude smaller than 1 at all other locations. Then plot the eigenmodes, making sure to account for the location x of the sensors along the beam, as given in Figure 1. Is it enough to plot only the real part of the mode, or is the eigenmode intrinsically complex? What is your conclusion: is the system proportionally damped or generally viscously damped?

6. Optional: We will now look at a more realistic situation, where the measured FRFs contain measurement noise, as given in *Measurement2.mat*. Again, perform the parameter fit procedure on the noisy data. How does the noise affect the results?

Appendix A. Exact analytical eigenfrequencies and eigenmodes for some single span beam structures

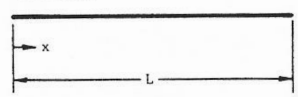
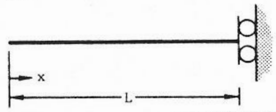
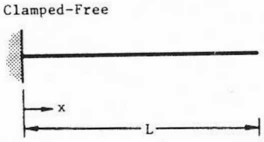
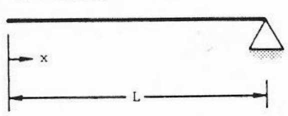
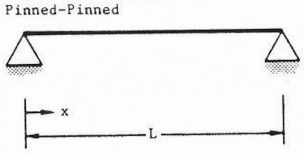
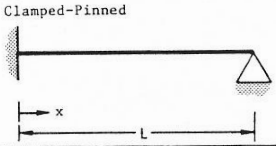
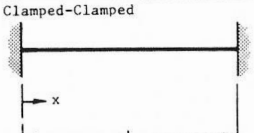
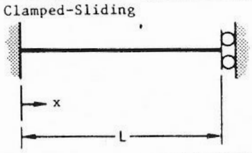
The table on the following page is taken from the book ‘Formulas for natural frequency and mode shape’ written by Robert D. Blevins, Van Nostrand Reinhold Company, 1979, ISBN 0-442-20710-7. Because the printed version of the table may be difficult to read because of the small font, it may be necessary to zoom in on the formulas on your computer screen….

The table shows exact expressions for the eigenfrequencies and corresponding eigenmodes for single span beams based on Euler beam theory (i.e. shear is not modelled).

Table 8-1. Single-Span Beams.

Notation: x = distance along span of beam; m = mass per unit length of beam;
E = modulus of elasticity;
I = area moment of inertia of beam about neutral axis (Table 5-1); L = span of beam;
see Table 3-1 for consistent sets of units

Natural Frequency (hertz); $f_i = \frac{\lambda_i^2}{2\pi L^2}\left(\frac{EI}{m}\right)^{1/2}$; i=1,2,3...

Description[(a)]	λ_i; i=1,2,3...	Mode Shape, $\tilde{y}_i\left(\frac{x}{L}\right)$	σ_i; i=1,2,3...
1. Free-Free 	4.73004074 7.85320462 10.9956078 14.1371655 17.2787597 $(2i+1)\frac{\pi}{2}$; i>5	$\cosh\frac{\lambda_i x}{L} + \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} + \sin\frac{\lambda_i x}{L}\right)$	0.982502215 1.000777312 0.999966450 1.000001450 0.999999937 ≈1.0 for i>5 See Ref. 8-2
2. Free-Sliding 	2.36502037 5.49780392 8.63937983 11.78097245 14.92256510 $(4i-1)\frac{\pi}{4}$; i>5	$\cosh\frac{\lambda_i x}{L} + \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} + \sin\frac{\lambda_i x}{L}\right)$	0.982502207 0.999966450 0.999999933 0.999999993 0.999999993 1.0; i>5
3. Clamped-Free 	1.87510407 4.69409113 7.85475744 10.99554073 14.13716839 $(2i-1)\frac{\pi}{2}$; i>5	$\cosh\frac{\lambda_i x}{L} - \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} - \sin\frac{\lambda_i x}{L}\right)$	0.734095514 1.018467319 0.999224497 1.000033553 0.999998550 ≈1.0; i>5 See Ref. 8-2
4. Free-Pinned 	3.92660231 7.06858275 10.21017612 13.35176878 16.49336143 $(4i+1)\frac{\pi}{4}$; i>5	$\cosh\frac{\lambda_i x}{L} + \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} + \sin\frac{\lambda_i x}{L}\right)$	1.000777304 1.000001445 1.000000000 1.000000000 1.000000000 1.0; i>5
5. Pinned-Pinned 	$i\pi$	$\sin\frac{i\pi x}{L}$	--
6. Clamped-Pinned 	3.92660231 7.06858275 10.21017612 13.35176878 16.49336143 $(4i+1)\frac{\pi}{4}$; i>5	$\cosh\frac{\lambda_i x}{L} - \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} - \sin\frac{\lambda_i x}{L}\right)$	1.000777304 1.000001445 1.000000000 1.000000000 1.000000000 1.0; i>5
7. Clamped-Clamped 	4.73004074 7.85320462 10.9956079 14.1371655 17.2787597 $(2i+1)\frac{\pi}{2}$; i>5	$\cosh\frac{\lambda_i x}{L} - \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} - \sin\frac{\lambda_i x}{L}\right)$	0.982502215 1.000777312 0.999966450 1.000001450 0.999999937 1.0; i>5 See Ref. 8-2
8. Clamped-Sliding 	2.36502037 5.49780392 8.63937983 11.78097245 14.92256510 $(4i-1)\frac{\pi}{4}$; i>5	$\cosh\frac{\lambda_i x}{L} - \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} - \sin\frac{\lambda_i x}{L}\right)$	0.982502207 0.999966450 0.999999933 0.999999993 0.999999993 1.0; i>5

Appendix B. Euler beam element matrices

The 2D 2-node Euler beam element matrices presented below have two degrees of freedom per node: a transversal displacement in y -direction and a rotation around the z -axis.

Euler beam element stiffness matrix:

$$f^e = K^e u^e; \quad f^e = \begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{bmatrix}; \quad u^e = \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{bmatrix}; \quad K^e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ sym & & & 4l^2 \end{bmatrix}$$

Consistent Euler beam element mass matrix:

$$f^e = M^e \ddot{u}^e; \quad f^e = \begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{bmatrix}; \quad \ddot{u}^e = \begin{bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{bmatrix}; \quad M^e = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ sym & & & 4l^2 \end{bmatrix}$$

Sign conventions: