Assignment1

October 8, 2024

Assignment 1
Advanced Signal Processing

Thomas Seeberg Christiansen

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importing Jupyter notebook from FunctionLibrary.ipynb
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1 Problem 1

1.1 1)

The system function is defined as:

$$H(z) = 1 + 2.5z^{-1} + z^{-2}$$

which we need to factorize into the two components of a minimum phase filter $H_{min}(z)$ and an all-pass filter $H_{ap}(z)$. The first step is determining the Zeros of $H(z)$.

$$H(z) = (z^{-1} + 2)(z^{-1} + 0.5)$$

$$z_1 = -1/2 \quad \text{and} \quad z_2 = -2$$

Here we have z_1 being inside the unit circle and z_2 being outside and z_2 would need to be reflected back inside the unit circle with its reciprocal $1/z_2 = -0.5$. The decomposition can be constructed with

$$H(z) = \underbrace{H_1(z)(1 - az^{-1})}_{\text{minimum phase}} \underbrace{\frac{(z^{-1} - a^*)}{(1 - az^{-1})}}_{\text{All-pass}}$$

A minimum phase system is defined as having its poles and zeros inside the unit circle and since z_1 being inside, we have the factor

$$H_min(z) = H_1(z)(1 + 2z^{-1}) = (1 + 0.5z^{-1})(1 + 2z^{-1}) = \frac{1}{2}z^{-2} + 2z^{-1} + 2$$

as the minimum phase. The all-pass can be constructed, using the given form, as

$$H_{ap}(z) = \frac{(z^{-1} - a^*)}{(1 - az^{-1})} = \frac{z^{-1} - (-2)}{1 - (-0.5)z^{-1}} = \frac{z^{-1} + 2}{1 + 0.5z^{-1}}$$

Combining everything, we get the decomposition as:

$$H(z) = H_{min}(z)H_{ap}(z) = \left(\frac{1}{2}z^{-2} + 2z^{-1} + 2\right)\left(\frac{z^{-1} + 2}{1 + 0.5z^{-1}}\right)$$

1.2 2)

Filtering with the system function $H(z)$ filter yields a constant group delay due to the filter being a linear phase filter. This means all frequency components of the signal are eqaully delayed without introducing phase distortion. However, filtetering with a minimum phase filter $H_{min}(z)$ does not have a constant group delay because it tries to minimize the group delay at the frequency components, meaning different frequencies experience different delays in turn create phase distortion. In summary, filtetering $H_{min}(z)$ has the least phase delay, but possible distortion, while filtering with $H(z)$ has uniform delay without distortion. Choosing which to use is application dependent if one would want to preserve the phase or minimize the delay.

2 Problem 2

2.1 1) Compute the autocorrelation

Inserting $y[n]$ into $r_y[l]$

$$\begin{aligned} r_y[l] &= E\Bigg(\Big(x[n] - \frac{1}{5}[n-1]\Big) \cdot \Big(x[n-l] - \frac{1}{5}x[n-l-1]\Big)\Bigg) \\ &= E\Big(x[n]x[n-l] - \frac{1}{5}x[n]x[n-l-1] - \frac{1}{5}x[n-1]x[n-l] + \frac{1}{25}x[n-1]x[n-l-1]\Big) \end{aligned}$$

The expected operator is linear meaning the expected value of a sum of variables is equivalent to the sum of the expected value of the individual variables, also scaling linearly with constants. So,

$$r_y[l] = E(x[n]x[n-l]) - \frac{1}{5}E(x[n]x[n-l-1]) - \frac{1}{5}E(x[n-1]x[n-l]) + \frac{1}{25}E(x[n-1]x[n-l-1])$$

Looking at $l = 0$, we get:

$$r_y[0] = E(x[n]^2) - \frac{1}{5}E(x[n]x[n-1]) - \frac{1}{5}E(x[n-1]x[n]) + \frac{1}{25}E(x[n-1]^2)$$

and since $x[n]$ is zero-mean, unit-variance white noise, we have:

$$E(x[n]^2) = 1 \quad \text{and} \quad E(x[n]x[n-1]) = 0.$$

Thus,

$$r_y[0] = 1 - 1/5 \cdot 0 - 1/5 \cdot 0 + 1/25 \cdot 1 = 1 + 1/25 = 26/25$$

Using the same procedure for $l = \pm 1$ and $l = \pm 2$, yields

$$r_y[\pm 1] = 0 - 1/5 \cdot 0 - 1/5 \cdot 1 + 0 = -1/5$$

$$r_y[\pm 2] = 0$$

Finaly, the autocorrelation function can be described with the delta function:

$$r_y[l] = -\frac{1}{5}\delta[l+1] + \frac{26}{25}\delta[l] - \frac{1}{5}\delta[l-1]$$

2.2 2) Correlation discussion

To determine whether the two samples are correlated or not, we look at the autocorrelation evaluated at $l = 2$. Which as we saw from above is equal to 0.

$$r_y[2] = -\frac{1}{5}\delta[3] + \frac{26}{25}\delta[2] - \frac{1}{5}\delta[1] = 0$$

Hence, they are uncorrelated. The autocorrelation would need to be nonzero for them to be correlated.

2.3 3) Mean value determination

The mean value of $y[n]$ is equivalent to the expected value.

$$\mu_y = E(y[n]) = E\left(x[n] - \frac{1}{5}x[n-1]\right)$$

Having the same terms as the calculations in 1) and knowning $x[n]$ is a Gaussian random process with zero-mean, the expected values terms are:

$$\mu_y = 0 - \frac{1}{5} \cdot 0 = 0$$

2.4 4) Compute and sketch the PSD

We get the power spectral density (PSD) by taking the Fourier transform of the autocorrelation sequence.

$$S_y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_y[l]e^{j\omega k}$$

The autocorrelation, as seen earlier, is zero for all other lags than $l = -1, 0, 1$, hence we get:

$$\begin{aligned} S_y(e^{j\omega}) &= \sum_{k=-1}^{1} r_y[l]e^{j\omega k} \\ &= r_y[-1]e^{j\omega} + r_y[0] + r_y[1]e^{-j\omega} \\ &= -\frac{1}{5}e^{j\omega} + \frac{26}{25} - \frac{1}{5}e^{-j\omega} \\ &= \frac{26}{25} - \frac{1}{5}(e^{j\omega} + e^{-j\omega}) \end{aligned}$$

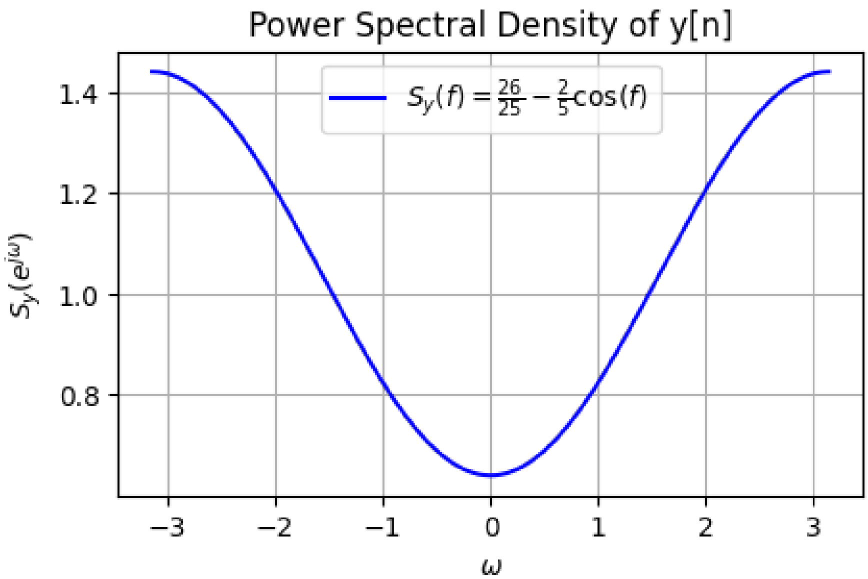
Using Eulers relation, $(2\cos(x) = e^{jx} + e^{-jx})$, we get

$$\frac{1}{5}(e^{j\omega} + e^{-j\omega}) = \frac{2}{5}\cos(\omega)$$

Therefore,

$$S_y(e^{j\omega}) = \frac{26}{25} - \frac{2}{5}\cos(\omega)$$

Plotting gets us the following.



3 Problem 3

Problem 5

One of the key equations in chapter 12.4 on two-channel filter banks is

$$Y(z) = \frac{1}{2}\Big((T(z)X(z) + A(-z)X(-z)\Big) \qquad (12.99)$$

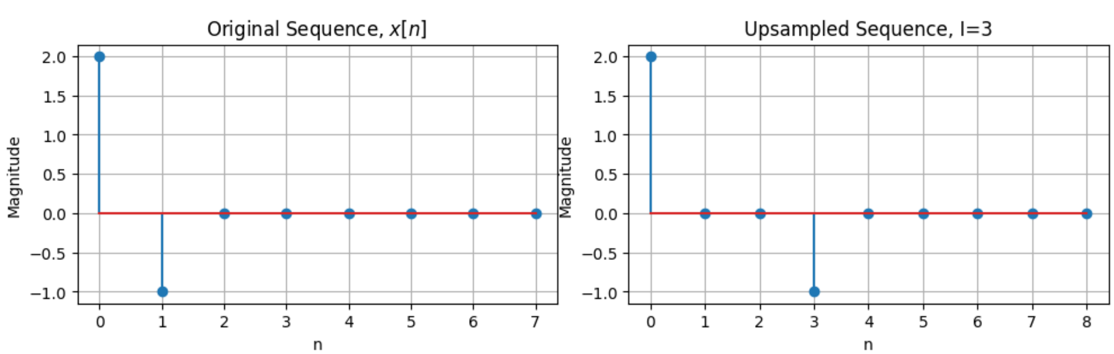
1. Discuss the background and meaning of this equation and explain the different components of the equation.

3.1 1) Upsample by 3

Upsampling by a factor of $I = 3$ requires the insertion of $(I - 1) = 2$ samples between samples of $x[n]$. Therefore,

$$x_I[n] = \{2, 0, 0, -1, 0, 0, \dots\}.$$

Can be sketched as below:



3.2 2) Upsample by 3 and interpolate

Linear interpolation uses the triangular sequence:

$$g_{lin}[n] = \begin{cases} 1 - \frac{|n|}{I}, & -I < n < I \\ 0, & \text{otherwise} \end{cases}$$

Given $I = 3$, we get the following filter:

$$g_{lin}[n] = \begin{cases} \frac{1}{3}, & \text{for } n = -2 \\ \frac{2}{3}, & \text{for } n = -1 \\ 1, & \text{for } n = 0 \\ \frac{2}{3}, & \text{for } n = 1 \\ \frac{1}{3}, & \text{for } n = 2 \end{cases}$$

Then it is just a matter of a convolution between the sequence and the interpolation window.

$$y[n] = x[n] * g_{lin}[n]$$

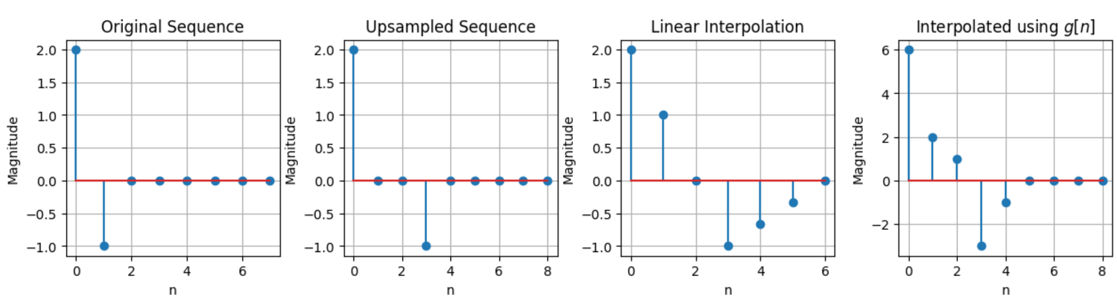
Therefore,

$$\begin{aligned} y[0] &= 1 \cdot x[0] + \frac{2}{3} \cdot x[1] + \frac{1}{3} \cdot x[2] = 2 \\ y[1] &= \frac{2}{3} \cdot x[0] + 1 \cdot x[1] + \frac{2}{3} \cdot x[2] + \frac{1}{3} \cdot x[3] = 1 \\ y[2] &= \frac{1}{3} \cdot x[0] + \frac{2}{3} \cdot x[1] + 1 \cdot x[2] + \frac{2}{3} \cdot x[3] + \frac{1}{3} \cdot x[4] = 0 \\ &\dots \end{aligned}$$

Using the given filter $g[n]$, we get:

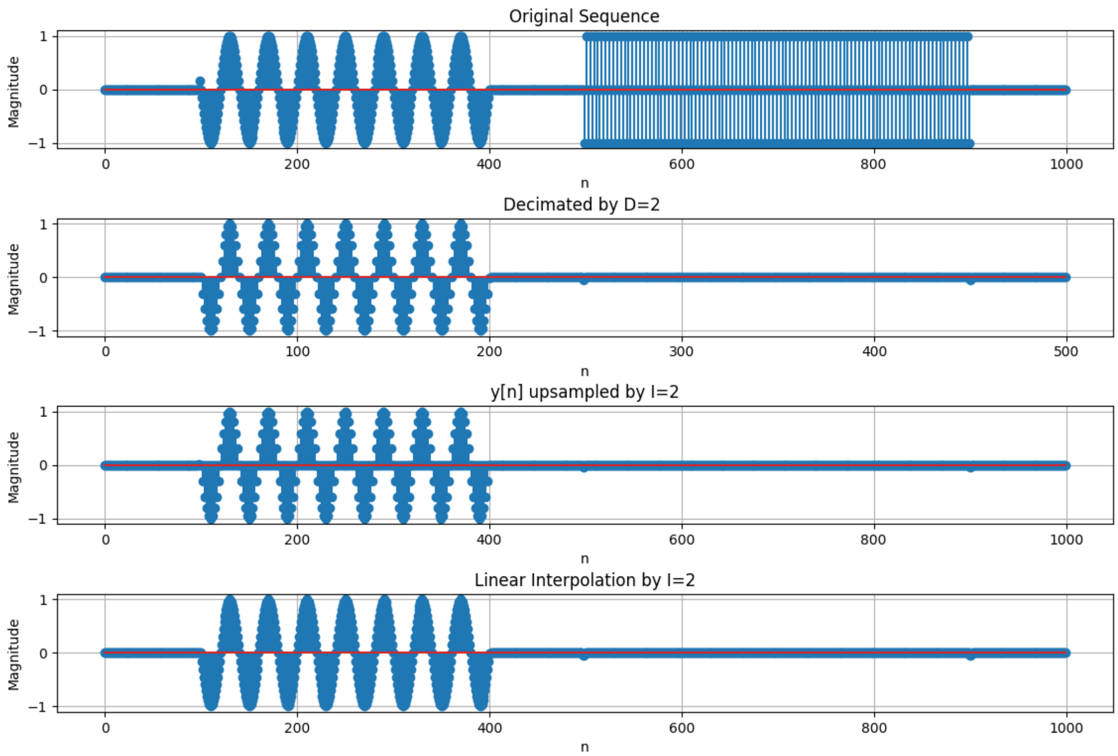
$$\begin{aligned} y[0] &= 3 \cdot x[0] - 1 \cdot x[1] = 6 \\ y[1] &= 1 \cdot x[0] + 3 \cdot x[1] - 1 \cdot x[2] = 2 \\ y[2] &= 1 \cdot x[1] + 3 \cdot x[2] - 1 \cdot x[3] = 1 \\ &\dots \end{aligned}$$

When plotted, we get the following:



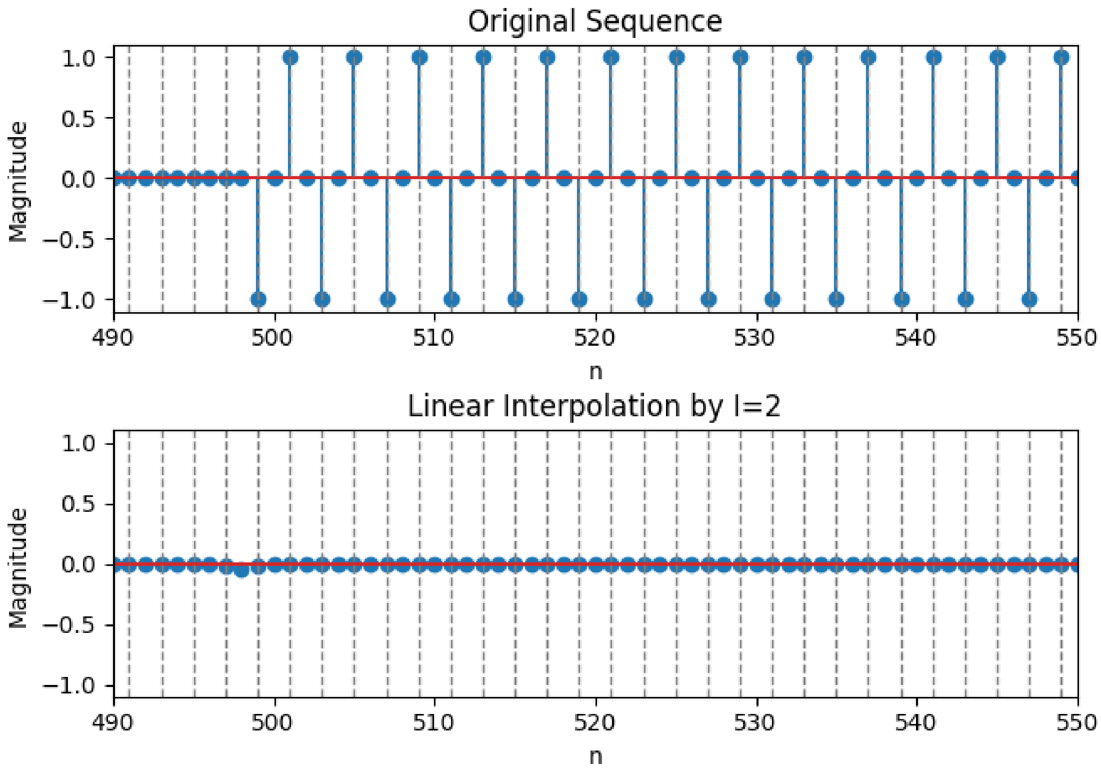
3.3 3) Discuss the performance

4 Problem 4



The original $x[n]$ consists of two waveform parts. The first being a sine/cosine wave and the last somekind of repeating signal with $\{-1, 0, 1, 0\}$. Then we decimate $x[n]$ by a factor of 2, which includes using a lowpass filter before downsampling and removing every $D = 2$ samples and halving the spectrum. Next, we first upsample the signal by $I = 2$, introducing a zero for every second sample, and lastly we linearly interpolate the upsampled signal.

We have 'undone' the decimation with interpolation, but we have lost the last part of our original $x[n]$ sequence because it matched with every other sample being remove. I have tried showing it in the plot below where the vertical dashed lines correspond to every other sample:



5 Problem 5

Problem 5

One of the key equations in chapter 12.4 on two-channel filter banks is

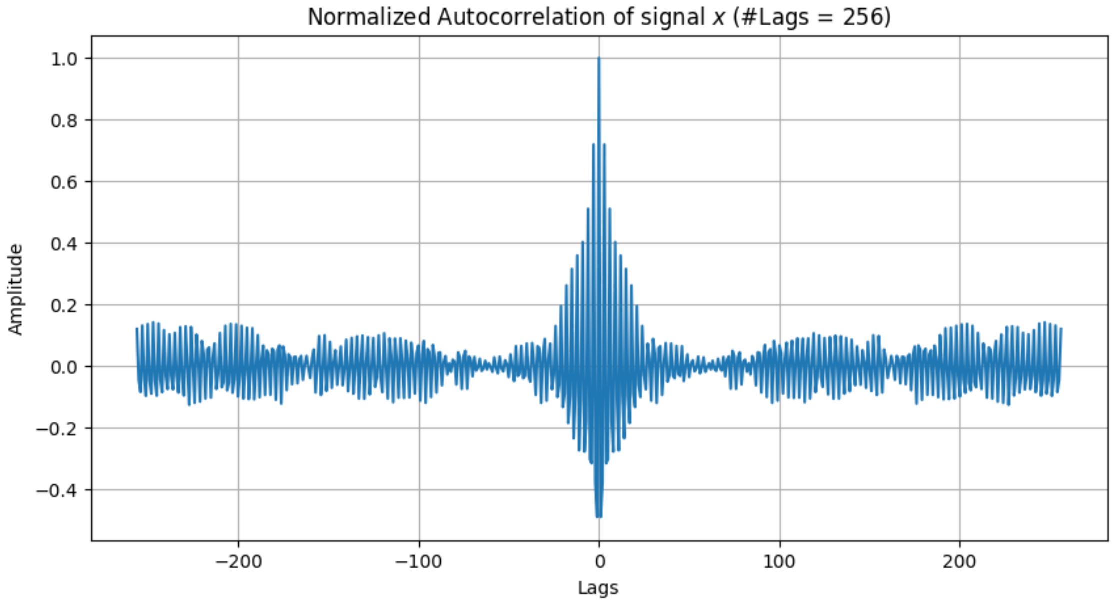
$$Y(z) = \frac{1}{2}\Big((T(z)X(z) + A(-z)X(-z)\Big) \tag{12.99}$$

1. Discuss the background and meaning of this equation and explain the different components of the equation.

6 Problem 6

6.1 1) Autocorrelation

The autocorrelation is plotted below using $l = 256$ lags which captures the key information of our data. The futher from lag 0 we get the more it appears to fluctuate around zero. Hence, we wouldn't want to include too much noise from the distant lags with negligible autocorrelation.



6.2 2) PSD

The signal seems to be very noisy and would benefit from the smoothing by averaging overlapping segments when using the Welch method. This provides a clear picture of the frequency components while reducing the leakage effects. I have used a segment size of 256 and an overlap of 50%, 128 samples, with each segment being windowed by a Hanning window. The segment size is rather large to capture a higher degree of frequency resolution, while trading off some of the variance.

Comparing the Welch method to the periodogram, we clearly see the differences with the periodogram being a noisy and hard to read, while the Welch method is improved.

In both cases, the spectrum seems to be dominated about the frequency $\omega = 0.33$ where a significant amount of energy is located.

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0.328125
0.328125
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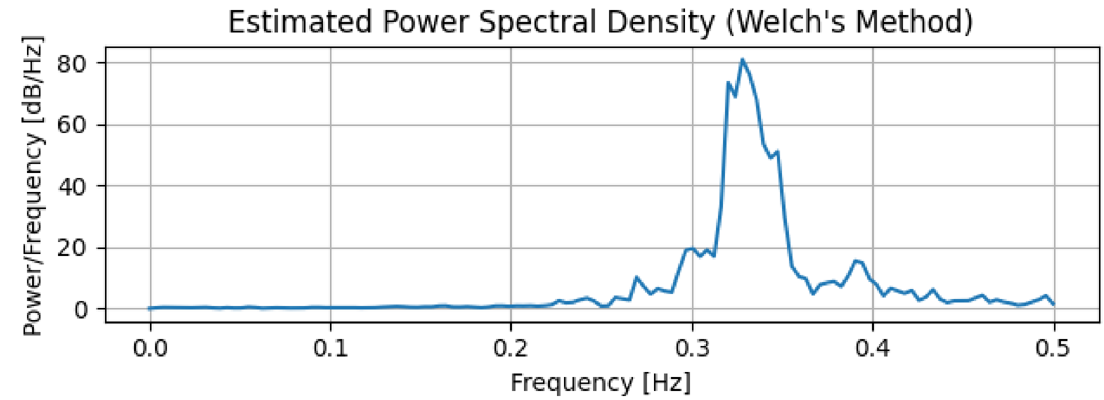
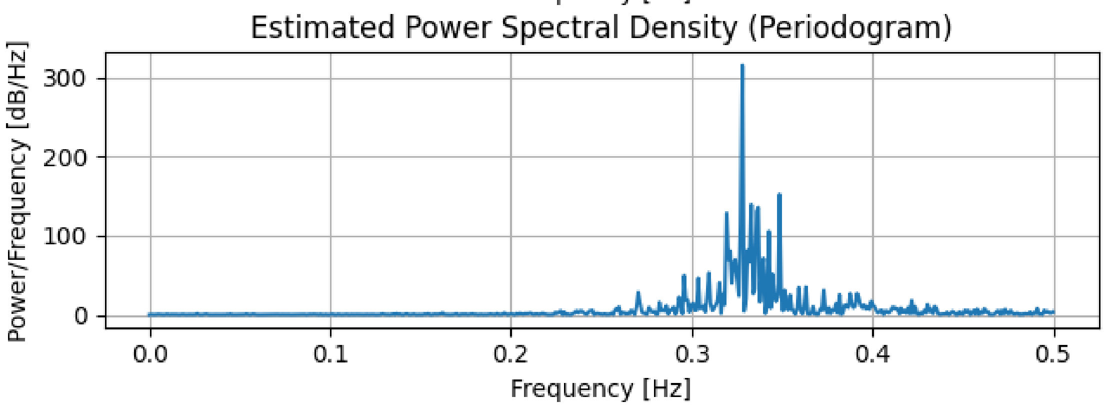

Estimated Power Spectral Density (Welch's Method)
Power/Frequency [dB/Hz]
Frequency [Hz]
0
20
40
60
80
0.0
0.1
0.2
0.3
0.4
0.5


Estimated Power Spectral Density (Periodogram)
Power/Frequency [dB/Hz]
Frequency [Hz]
0
100
200
300
0.0
0.1
0.2
0.3
0.4
0.5