Assignment1

October 6, 2024

Assignment 1
Advanced Signal Processing

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[1]:
```
##### Imports #####
import numpy as np
import matplotlib.pyplot as plt
import scipy
import scipy.signal as sig
import import_ipynb
import FunctionLibrary as FL
```

```
importing Jupyter notebook from FunctionLibrary.ipynb
```

1 Problem 1

1.1 1)

The system function is defined as:

$$H(z) = 1 + 2.5z^{-1} + z^{-2}$$

which we need to factorize into the two components of a minimum phase filter $H_{min}(z)$ and an all-pass filter $H_{ap}(z)$. The first step is determining the Zeros of $H(z)$.

$$H(z) = (z^{-1} + 2)(z^{-1} + 0.5)$$

$$z_1 = -1/2 \quad \text{and} \quad z_2 = -2$$

Here we have z_1 being inside the unit circle and z_2 being outside and z_2 would need to be reflected back inside the unit circle with its reciprocal $1/z_2 = -0.5$. The decomposition can be constructed with

$$H(z) = \underbrace{H_1(z)(1 - az^{-1})}_{\text{minimum phase}} \underbrace{\frac{(z^{-1} - a^*)}{(1 - az^{-1})}}_{\text{All-pass}}$$

A minimum phase system is defined as having its poles and zeros inside the unit circle and since z_1 being inside, we have the factor

$$H_min(z) = H_1(z)(1 + 2z^{-1}) = (1 + 0.5z^{-1})(1 + 2z^{-1}) = \frac{1}{2}z^{-2} + 2z^{-1} + 2$$

as the minimum phase. The all-pass can be constructed, using the given form, as

$$H_{ap}(z) = \frac{(z^{-1} - a^*)}{(1 - az^{-1})} = \frac{z^{-1} - (-2)}{1 - (-0.5)z^{-1}} = \frac{z^{-1} + 2}{1 + 0.5z^{-1}}$$

Combining everything, we get the decomposition as:

$$H(z) = H_{min}(z)H_{ap}(z) = \left(\frac{1}{2}z^{-2} + 2z^{-1} + 2\right)\left(\frac{z^{-1}+2}{1+0.5z^{-1}}\right)$$

1.2 2)

Filtering with the system function $H(z)$ filter yields a constant group delay due to the filter being a linear phase filter. This means all frequency components of the signal are eqaully delayed without introducing phase distortion. However, filtetering with a minimum phase filter $H_{min}(z)$ does not have a constant group delay because it tries to minimize the group delay at the frequency components, meaning different frequencies experience different delays in turn create phase distortion. In summary, filtetering $H_{min}(z)$ has the least phase delay, but possible distortion, while filtering with $H(z)$ has uniform delay without distortion. Choosing which to use is application dependent if one would want to preserve the phase or minimize the delay.

2 Problem 2

2.1 1) Compute the autocorrelation

Inserting $y[n]$ into $r_y[l]$

$$\begin{aligned} r_y[l] &= E\left(\left(x[n] - \frac{1}{5}[n-1]\right)\cdot\left(x[n-l] - \frac{1}{5}x[n-l-1]\right)\right) \\ &= E\left(x[n]x[n-l] - \frac{1}{5}x[n]x[n-l-1] - \frac{1}{5}x[n-1]x[n-l] + \frac{1}{25}x[n-1]x[n-l-1]\right) \end{aligned}$$

The expected operator is linear meaning the expected value of a sum of variables is equivalent to the sum of the expected value of the individual variables, also scaling linearly with constants. So,

$$r_y[l] = E(x[n]x[n-l]) - \frac{1}{5}E(x[n]x[n-l-1]) - \frac{1}{5}E(x[n-1]x[n-l]) + \frac{1}{25}E(x[n-1]x[n-l-1])$$

Looking at $l = 0$, we get:

$$r_y[0] = E(x[n]^2) - \frac{1}{5}E(x[n]x[n-1]) - \frac{1}{5}E(x[n-1]x[n]) + \frac{1}{25}E(x[n-1]^2)$$

and since $x[n]$ is zero-mean, unit-variance white noise, we have:

$$E(x[n]^2) = 1 \quad \text{and} \quad E(x[n]x[n-1]) = 0.$$

Thus,

$$r_y[0] = 1 - 1/5 \cdot 0 - 1/5 \cdot 0 + 1/25 \cdot 1 = 1 + 1/25 = 26/25$$

Using the same procedure for $l = \pm 1$ and $l = \pm 2$, yields

$$r_y[\pm 1] = 0 - 1/5 \cdot 0 - 1/5 \cdot 1 + 0 = -1/5$$

$$r_y[\pm 2] = 0$$

Finaly, the autocorrelation function can be described with the delta function:

$$r_y[l] = -\frac{1}{5}\delta[l+1] + \frac{26}{25}\delta[l] - \frac{1}{5}\delta[l-1]$$

2.2 2) Correlation discussion

To determine whether the two samples are correlated or not, we look at the autocorrelation evaluated at $l = 2$. Which as we saw from above is equal to 0.

$$r_y[2] = -\frac{1}{5}\delta[3] + \frac{26}{25}\delta[2] - \frac{1}{5}\delta[1] = 0$$

Hence, they are uncorrelated. The autocorrelation would need to be nonzero for them to be correlated.

2.3 3) Mean value determination

The mean value of $y[n]$ is equivalent to the expected value.

$$\mu_y = E(y[n]) = E\left(x[n] - \frac{1}{5}x[n-1]\right)$$

Having the same terms as the calculations in 1) and knowning $x[n]$ is a Gaussian random process with zero-mean, the expected values terms are:

$$\mu_y = 0 - \frac{1}{5} \cdot 0 = 0$$

2.4 4) Compute and sketch the PSD

We get the power spectral density (PSD) by taking the Fourier transform of the autocorrelation sequence.

$$S_y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_y[l]e^{j\omega k}$$

The autocorrelation, as seen earlier, is zero for all other lags than $l = -1, 0, 1$, hence we get:

$$\begin{aligned} S_y(e^{j\omega}) &= \sum_{k=-1}^{1} r_y[l]e^{j\omega k} \\ &= r_y[-1]e^{j\omega} + r_y[0] + r_y[1]e^{-j\omega} \\ &= -\frac{1}{5}e^{j\omega} + \frac{26}{25} - \frac{1}{5}e^{-j\omega} \\ &= \frac{26}{25} - \frac{1}{5}(e^{j\omega} + e^{-j\omega}) \end{aligned}$$

Using Eulers relation, $(2\cos(x) = e^{jx} + e^{-jx})$, we get

$$\frac{1}{5}(e^{j\omega} + e^{-j\omega}) = \frac{2}{5}\cos(\omega)$$

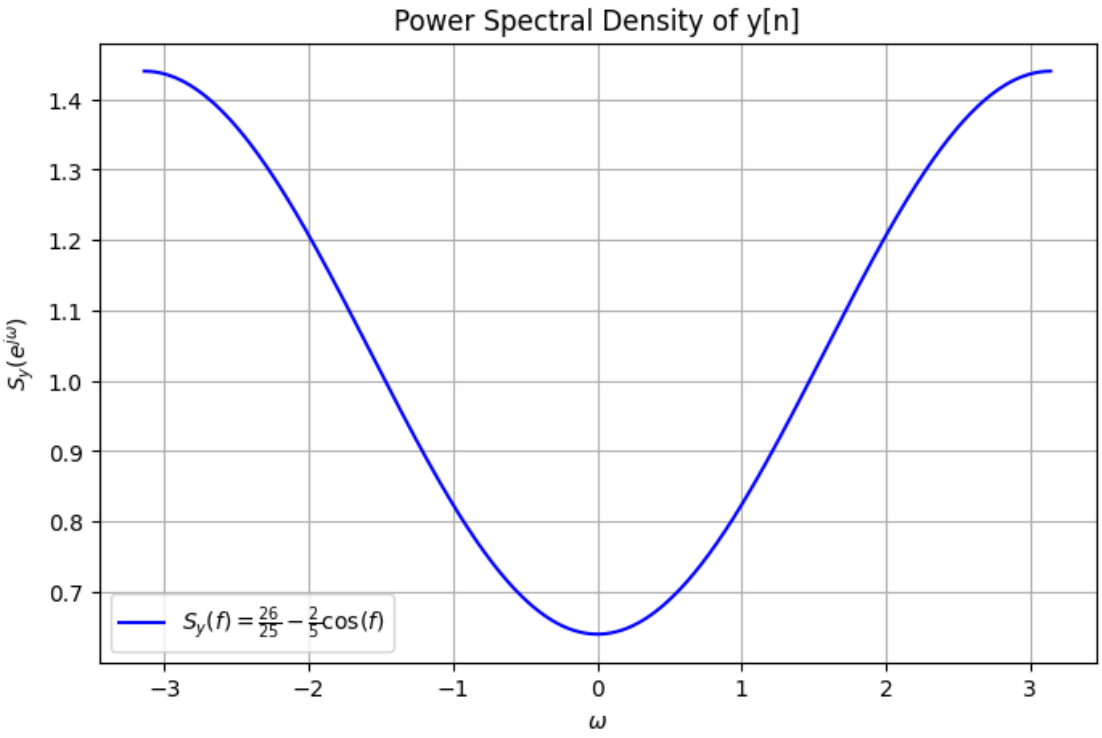
Therefore,

$$S_y(e^{j\omega}) = \frac{26}{25} - \frac{2}{5}\cos(\omega)$$

Plotting gets us the following.

[2]:
```
# Range
omega = np.linspace(-np.pi, np.pi, 500)
# Function
S_y = (26 / 25) - (2 / 5) * np.cos(omega)

# Plot the PSD
plt.figure(figsize=(8, 5))
plt.plot(omega, S_y, label=r'$S_y(f) = \frac{26}{25} - \frac{2}{5} \cos(f)$',␣
↪color='blue')
plt.title('Power Spectral Density of y[n]')
plt.xlabel('$\omega$')
plt.ylabel('$S_y(e^{j\omega})$')
plt.grid(True)
plt.legend()
plt.show()
```



3 Problem 3

Problem 6

The file `dataset.dat` contains 1024 samples of an unknown signal. The signal can be loaded into MATLAB with `x=load('dataset.dat')`.

1. Compute and plot the autocorrelation for an appropriate range of lags. What can be derived about the signal from the plot of autocorrelation values?
2. Estimate the power spectral density of the signal using a Fourier based method of your choice. Explain the pros and cons of your chosen method.

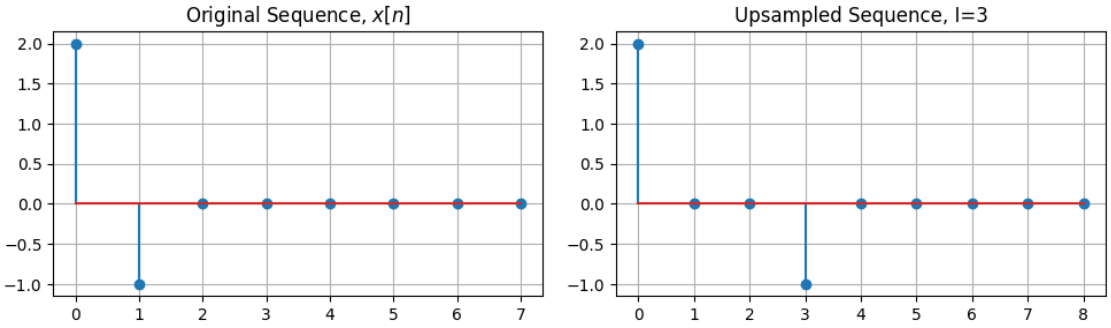
3.1 1) Upsample by 3

Upsampling by a factor of $I = 3$ requires the insertion of $(I - 1) = 2$ samples between samples of $x[n]$. Therefore,

$$x_I[n] = \{2, 0, 0, -1, 0, 0, \dots\}.$$

Can be sketched as below:

[16]:
```
xn = [2, -1, *[0 for i in range(6)]]
I = 3
xn_upsample = FL.upsample(xn,I,3)
fig, ax = plt.subplots(1,2, figsize=(10,3))
fig.tight_layout(pad=1.5)
ax[0].stem(xn)
ax[0].grid()
ax[0].set_title("Original Sequence, $x[n]$")
ax[1].stem(xn_upsample)
ax[1].grid()
ax[1].set_title(f"Upsampled Sequence, I={I}")
plt.show()
```



3.2 2) Upsample by 3 and interpolate

Linear interpolation uses the triangular sequence:

$$g_{lin}[n] = \begin{cases} 1 - \frac{|n|}{I}, & -I < n < I \\ 0, & \text{otherwise} \end{cases}$$

Given $I = 3$, we get the following filter:

$$g_{lin}[n] = \begin{cases} \frac{1}{3}, & \text{for } n = -2 \\ \frac{2}{3}, & \text{for } n = -1 \\ 1, & \text{for } n = 0 \\ \frac{2}{3}, & \text{for } n = 1 \\ \frac{1}{3}, & \text{for } n = 2 \end{cases}$$

Then it is just a matter of a convolution between the sequence and the interpolation window.

$$y[n] = x[n] * g_{lin}[n]$$

Therefore,

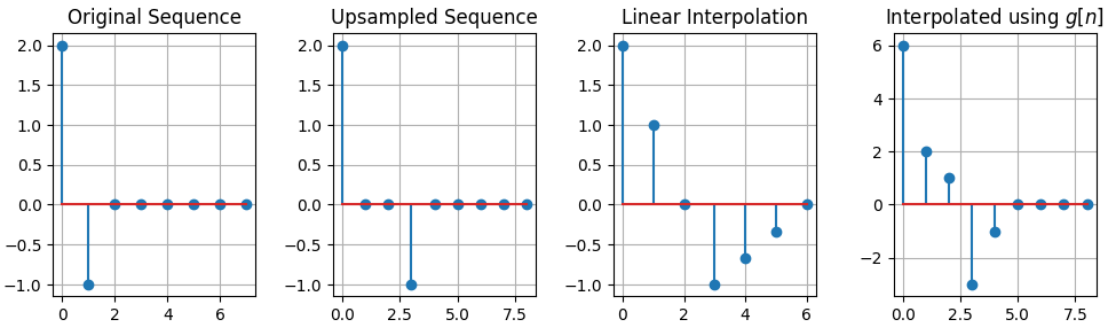
$$\begin{aligned} y[0] &= 1 \cdot x[0] + \frac{2}{3} \cdot x[1] + \frac{1}{3} \cdot x[2] = 2 \\ y[1] &= \frac{2}{3} \cdot x[0] + 1 \cdot x[1] + \frac{2}{3} \cdot x[2] + \frac{1}{3} \cdot x[3] = 1 \\ y[2] &= \frac{1}{3} \cdot x[0] + \frac{2}{3} \cdot x[1] + 1 \cdot x[2] + \frac{2}{3} \cdot x[3] + \frac{1}{3} \cdot x[4] = 0 \\ &\dots \end{aligned}$$

Using the given filter $g[n]$, we get:

$$\begin{aligned} y[0] &= 3 \cdot x[0] - 1 \cdot x[1] = 6 \\ y[1] &= 1 \cdot x[0] + 3 \cdot x[1] - 1 \cdot x[2] = 2 \\ y[2] &= 1 \cdot x[1] + 3 \cdot x[2] - 1 \cdot x[3] = 1 \\ &\dots \end{aligned}$$

[33]:
```
xn = [2, -1, *[0 for i in range(6)]]
gn = [-1, 3, 1]
I = 3
xn_upsample = FL.upsample(xn,I,3)
xn_lin_interp = FL.linInterp(xn_upsample, I)
xn_interp = np.convolve(xn_upsample, gn,'same')
fig, ax = plt.subplots(1,4, figsize=(10,3))
fig.tight_layout(pad=1.5)
ax[0].stem(xn)
ax[0].grid()
ax[0].set_title("Original Sequence")
ax[1].stem(xn_upsample)
ax[1].grid()
```

```
ax[1].set_title("Upsampled Sequence")
ax[2].stem(xn_lin_interp)
ax[2].grid()
ax[2].set_title("Linear Interpolation")
ax[3].stem(xn_interp)
ax[3].grid()
ax[3].set_title("Interpolated using $g[n]$")
plt.show()
```



3.3 3) Discuss the performance

[ ]:

4 Problem 4

Problem 6

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