Learning by Belief Merge

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Abstract

In [1] a method for truth-tracking via identification in the limit was introduced for singular learning agents by using belief revision. In this work we study the options for broadening this concept to multiple agents performing truth-tracking as a collective. The intention of the joint approach is to transfer the information requirements on the individual agent to a shared responsibility. Thereby, the information an individual agent has might not be sufficient to perform truth-tracking on their own and their conjectures will be individually inaccurate but their collective conjecture will be precise. The two main approaches we investigate are (1) merging on the agents conjectures and (2) merging on the agents entire belief ordering as a whole. Merging on conjectures (1) turns out to be prone to information loss with relations to Arrow's Impossibility Result [2, 3], while merging on the entirety of the belief orderings (2) has ties to the epistemic notion of distributed knowledge.

1 Introduction

Intelligent agents are those with the capacity to learn form new information by integrating it into their beliefs and knowledge. No matter how efficient an agent is at learning there must be some set of observations required for the agent to draw the necessary conclusions. This means that for an agent to learn something it is required to collect enough data to make the required inference. One of the pillars of human advancement has been the ability to divide this requirement of knowledge acquirement on intelligent learners such that instead of individual agents only performing inference on their own experience they construct their conjectures on the knowledge of the group as a whole. Through this the probability of a learner to have the necessary knowledge goes from the probability that this specific learner experiences all the necessary encounters to the probability that each event occurs for any member in the group. For this reason collective learning is a hugely important notion in human advancement.

While there has definitely been interesting approaches made to model learning agents, such as [**ILAO**, 1], they all neglect this very important aspect of collective learning allowing agents to share the burden of required observations. In this work we will examine methods to allow agents to perform collective learning by combining their knowledge using merging methods. The setting that this merging will be done in is that of plausibility spaces [1], where every possible world is considered and the most believed world, the agents conjecture, is the minimum in a partial ordering. The main objective of the work is to see what method of merging is best suited for this goal. The two approaches that we will investigate is on one hand merging on the conjectures of the agents and on the other merging on the entire plausibility spaces.

2 Learning Agents

Definition 1. An *epistemic space* is $\mathbb{S} = (S, \mathcal{O})$, where:

- S is a set of possible worlds. In this work S will be assumed to be finite (or at most countable).
- $\mathcal{O}$ is a set of subsets of S and represents observations that can be (accurately) made about particular possible worlds. We take $\mathcal{O}$ to be finite (or at most countable). We consider $\mathcal{O}$ to be arbitrary, note however that this set can be closed on various operations, e.g., Boolean operators (forming, together with S, a Boolean algebra). It will be later useful to refer to observations holding at a given world: for any $w \in S, \mathcal{O}_w = \{O \in \mathcal{O} \mid w \in O\}$ is the set of observations that hold for w.

Definition 2. Take an epistemic space $\mathbb{S} = (S, \mathcal{O})$. We will use the following notation:

A *data sequence* $\sigma \in \mathcal{O}^*$ is a finite sequence of observations.

Given a data sequence σ, $\mathrm{set}(\sigma)$ is the set of elements appearing in σ.

A data sequence σ is sound with respect to a possible world $s \in S$ if $\mathrm{set}(\sigma) \subseteq \mathcal{O}_s$.

A data sequence σ is complete with respect to a possible world $s \in S$ if $\mathcal{O}_s \subseteq \mathrm{set}(\sigma)$.

σ_n stands for the $n+1$-th element of σ; if $\sigma = (\sigma_0, \ldots, \sigma_n)$ then $\mathrm{length}(\sigma) = n+1$, and $\sigma[k]$ is the initial segment of σ of length k, where $k \leq \mathrm{length}(\sigma)$.

A *data stream* $\varepsilon \in \mathcal{O}^\infty$ is an infinite sequence of observations; $\mathrm{set}(\varepsilon)$, soundness and completeness of ε, ε_n, $\varepsilon[n]$ are defined analogously as for σ.

Infinite data sequences (data streams) are useful in a setting where both S and $\mathcal{O}$ are infinite, or in the finite cases when repetitions are allowed.

Definition 3. Let $\mathbb{S} = (S, \mathcal{O})$ be an epistemic space. A learner is a function L, that on an input of an epistemic space and a finite sequence of observations outputs a conjecture, i.e., a set of possible worlds, or $\uparrow$. Formally, $L(\mathbb{S}, \sigma^*) = X$, where $X \subseteq S$.

Definition 4. A learner L is *once-defined* on ε if $L(S, \varepsilon[n]) \neq \uparrow$ for exactly one $n \in \mathbb{N}$.

Definition 5. A learner L *finitely identifies* $\mathbb{S} = (S, \mathcal{O})$ if for any stream of data ε for any possible world $s \in S$, L is once-defined for ε and there is an $n \in \mathbb{N}$, such that $L(\varepsilon[n]) = \{s\}$.

An epistemic space $\mathbb{S}$ is finitely identifiable if there exists a learner L which finitely identifies it.

Definition 6. A learner L *identifies* $\mathbb{S} = (S, \mathcal{O})$ *in the limit* if for any stream of data ε for any possible world $s \in S$, there is a $k \in \mathbb{N}$, such that for all $n \leq k$, $L(\mathbb{S}, \varepsilon[n]) = \{s\}$.

An epistemic space $\mathbb{S}$ is identifiable in the limit if there exists a learner L which identifies it in the limit.

Observations can be used to distinguish possible worlds from each other. In some situations a world $s \in S$ can be distinguished from all other worlds in S by a finite set of observations. In such a case we will say that s has a *definite finite tell-tale*.

Definition 7. Let $\mathbb{S} = (S, \mathcal{O})$ be an epistemic space, and let $s \in S$. We say that $D_s \subseteq \mathcal{O}$ is a *definite finite tell tale* of s if the following conditions hold:

1. D_s is finite;
2. $D_s \subseteq \mathcal{O}_s$;
3. for all $w \in S$, if $s \neq w$ then $D_s \not\subseteq \mathcal{O}_w$.

A weaker type of tell-tale for a world, simply called a *finite tell-tale* is defined as follows.

Definition 8. Let $\mathbb{S} = (S, \mathcal{O})$ be an epistemic space, and let $s \in S$. We say that $F_s \subseteq \mathcal{O}$ is a *finite tell-tale* of s if the following conditions hold:

1. F_s is finite;
2. $F_s \subseteq \mathcal{O}_s$;
3. for all $w \in S$, if $s \neq w$ and $F_s \subseteq \mathcal{O}_w$ then $\mathcal{O}_w \not\subset \mathcal{O}_s$.

Theorem 1. An epistemic space $\mathbb{S} = \{S, \mathcal{O}\}$ is finitely identifiable if and only if all $s \in \mathbb{S}$ have definite finite tell-tales.

Theorem 2. An epistemic space $\mathbb{S} = \{S, \mathcal{O}\}$ is identifiable in the limit if and only if all $s \in \mathbb{S}$ have finite tell-tales.

Proposition 1. Any finite epistemic space (with finite S) is identifiable in the limit.

Proposition 2. There are finite epistemic spaces which are not finitely identifiable.

Definition 9. A *plausibility space*, $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$, equips the epistemic space with a plausibility relation, $\preceq\ \subseteq S \times S$ is a total preorder (called a *plausibility relation*) over S, i.e., it satisfies the following conditions:

1. for any $x, y, z \in S$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity);
2. for any $x, y \in S$, $x \preceq y$ or $y \preceq x$ (strong connectedness).

Alternatively, a total preorder can be defined as a relation that is reflexive, transitive, and connected.

Definition 10. A mapping that takes an epistemic space $\mathbb{S}$ and produces a plausibility space $\mathbb{B}_{\mathbb{S}}$ by appending a plausibility order $\preceq$ is called a *prior plausibility assignment* $Plaus(\mathbb{S}) = \mathbb{B}_{\mathbb{S}} = (\mathbb{S}, \mathcal{O}, \preceq)$.

Definition 11. A plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ is *overgeneralising* if there are $x, y \in S$ such that $x \preceq y$ and $\mathcal{O}_y \subset \mathcal{O}_x$. If a plausibility space is not overgeneralising, we will call it an *Occam plausibility space*.

Example 1. Let $\mathbb{S} = (S, \mathcal{O})$ be an epistemic space, defined as follows: $S = \{x, y\}$ and $\mathcal{O} = \{\{x, y\}, \{y\}\}$, see Figure 1a. A plausibility relation on this space can be introduced in various ways, Figure 1b illustrates the overgeneralising case. The agent prefers the state which makes both observables true. If she starts with the assumption that y is the actual world and in fact x is the one, there is no observable that can change her mind, since both of them are true at y. Figure 1c gives a better order: if the preferred x is the actual world, the agent will be right forever, however if y is the actual world, at some point the agent will receive the observation $\{y\}$ and so she will be able to distinguish x from y, and change her mind.

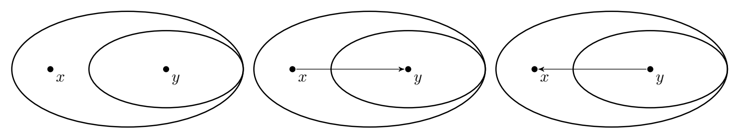

Figure 1: Epistemic space $\mathbb{S}$ from Example 1 with a plausibility relation can result in an overgeneralising plausibility space

Definition 12. Consider the class $\mathfrak{S}$ of epistemic spaces for which every possible world $s \in S$ has a finite tell-tale. We will call the prior plausibility assignment that when provided a $\mathbb{S} \in \mathfrak{S}$ outputs an Occam plausibility space for an *Occam plausibility assignment*.

Proposition 3. For any $\mathbb{S} \in \mathfrak{S}$ there is a plausibility relation $\preceq \subseteq S \times S$, such that $(S, \mathcal{O}, \preceq)$ is an Occam plausibility space. In other words, an Occam plausibility assignment can always provide a plausibility relation to extend $\mathbb{S}$ to an Occam plausibility space.

Proof. Take $\preceq$ to be defined as follows: for any $x, y \in S$, $x \preceq y$ iff $F_x \subseteq \mathcal{O}_y$. This order is also known as a *specialisation order*... $\square$

Definition 13. If a *one-step belief revision* is a function R_1 that for a plausibility space $\mathbb{B}_{\mathbb{S}}$ and an observation $p \in \mathcal{O}$ produces a new plausibility space $R_1(\mathbb{B}_{\mathbb{S}}, p)$

Then a *belief revision method* R is a function that applies a one-step belief revision method sequentially on a plausibility space $\mathbb{B}_{\mathbb{S}}$ and a sequence of observations σ.

$$R(\mathbb{B}_{\mathbb{S}}, \sigma * p) = R_1(R(\mathbb{B}_{\mathbb{S}}, \sigma), p)$$

Belief revision methods $R(\mathbb{B}_{\mathbb{S}}, \sigma)$ are used by learners $L(\mathbb{S}, \sigma)$ to update the current plausibility space $\mathbb{B}_{\mathbb{S}}$ as observations are given from the data sequence. To have a start point for the belief revision method a prior plausibility assignment is used to go from epistemic space to plausibility space.

$$L_R^{Plaus}(\mathbb{S}, \sigma) = \min R(Plaus(\mathbb{S}), \sigma)$$

The two belief revision methods we will look at are *conditioning* and *lexicographic revision*, which we will introduce using their one-step belief revision function.

Definition 14. Conditioning, $cond_1$, updates the plausibility space $\mathbb{B}_{\mathbb{S}}$ by removing inconsistent worlds from S.

$$cond_1(\mathbb{B}_{\mathbb{S}}, p) = (S^p, \mathcal{O}, \preceq^p),\ \text{where } S^p = S \cap p, \text{ and } \preceq^p = \preceq \cap (S^p \times S^p)$$

Lexicographic revision, lex_1, updates the plausibility relation $\preceq$ by making consistent worlds the most plausible and inconsistent worlds the least plausible, while keeping internal relations.

$$lex_1(\mathbb{B}_{\mathbb{S}}, p) = (S, \mathcal{O}, \preceq')$$

where for all $t, w \in S,\ \ t \preceq' w$ iff ($t \preceq_p w$ or $t \preceq_{\overline{p}} w$ or $(t \in p \wedge w \notin p)$), and $\preceq_p = \preceq \cap (p \times p)$, $\preceq_{\overline{p}} = \preceq \cap (\overline{p} \times \overline{p}$.

Proposition 4. The two learners that uses either conditioning or lexicographic revision as belief revision method together with an Occam plausibility assignment can identify in the limit any epistemic space $\mathbb{S}$ when given a data sequence σ that is sound and complete w.r.t. the true world s.

This is the major finding in [1] and as such wont be proved here, however it would be amiss to not give some insight.

For conditioning this is easy to argue because only the set of worlds satisfying F_s will remain in $\mathbb{B}_{\mathbb{S}}$ after some sufficient amount of updates and knowing that s is guaranteed to be the world in $\cap F_s$ with the fewest observations, due to the Occam plausibility assignment, it will also be the most prioritised world in $\cap F_s$.

The argument for lexicographic revision builds largely on the Occam property and the data sequence being sound and complete w.r.t. s.

Observation 1. If a plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ satisfies the *non-overgeneralising property*, then an $s \in S$ will always (after any number of belief revision updates using observations from $\mathcal{O}_s$) be prioritised over world w, if $\mathcal{O}_s \subset \mathcal{O}_w$.[1]

First note that if $\mathcal{O}_s \subset \mathcal{O}_w$ then $w \in \cap F_s$, meaning w satisfy the finite tell tale of s. Given that σ is sound w.r.t. s it is enforced that any observation from σ will satisfy both w and s, thereby any lexicographic revision operation using σ will keep the relation between w and s intact. This together with the fact that s is the world with the fewest observations in $\cap F_s$ (otherwise it would not be the finite tell tale for s) means that in $\preceq$ the world s will always be the most prioritised world from the $\cap F_s$ set. Making s the most prioritised world in $\preceq$ from the set S simply requires σ to contain every observation from $\mathcal{O}_s$, which is why σ is assumed to be complete w.r.t. s.

Definition 15. To discuss the differences and similarities between learning methods, we introduce some relevant properties. Learning method L is:

- *weakly data-retentive* iff $L(\mathbb{S}, \sigma) \neq \emptyset$ implies $L(\mathbb{S}, \sigma) \subseteq \sigma_n$ for any $\mathbb{S}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$;
- *strongly data-retentive* iff $L(\mathbb{S}, \sigma) \neq \emptyset$ implies $L(\mathbb{S}, \sigma) \subseteq \bigcap_{i \in \{1,\ldots,n\}} \sigma_i$, for any $\mathbb{S}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$;
- *conservative* iff $\emptyset \neq L(\mathbb{S}, \sigma) \subseteq p$ implies $L(\mathbb{S}, \sigma) = L(\mathbb{S}, \sigma * p)$, for any $\mathbb{S}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$;
- *data-driven* if it is both conservative and weakly data-retentive;
- *memory-free* iff $L(\mathbb{S}, \sigma) = L(\mathbb{S}, \sigma')$ implies $L(\mathbb{S}, \sigma * p) = L(\mathbb{S}, \sigma' * p$, for any $\mathbb{S}$ and every two data sequences σ, σ' and every $p \in \mathcal{O}$.

3 Multi-agent Learning

We will extend the above setting to the multi-agent case of n agents by combining their individual data streams.

Definition 16. A *data stream profile* is a tuple of data streams $E = \{\varepsilon^1, \ldots, \varepsilon^n\}$. Analogously, a *data sequence profile* is a tuple $\Sigma = (\sigma^1, \ldots, \sigma^n)$. The notation for initial segments of streams is extended to profiles by $E[m] = (\varepsilon^1[m], \ldots, \varepsilon^n[m])$ where $m \in \mathbb{Z}^+$ is called the current *learning step*.

To allow modelling of agents receiving information asynchronously we add the *empty observation* $\top$. It can be thought of as an observation that holds in every world, so receiving it provides no information.

Definition 17. A *plausibility profile* $B_{\mathbb{S}} = \{\mathbb{B}^1_{\mathbb{S}}, \ldots, \mathbb{B}^n_{\mathbb{S}}\}$ is a multi-set of n plausibility spaces that all relate to the same epistemic space $\mathbb{S}$, but are allowed to differ on $\preceq$. Plausibility profiles statically represents the n agents current beliefs on $\mathbb{S}$.

$R(B_{\mathbb{S}}, E[m])$ will represent the static plausibility profile of $B_{\mathbb{S}}$ after having revised each plausibility space using revision method R with its related data sequence $\varepsilon[m]$.

$$R(B_{\mathbb{S}}, E[m]) = (R(\mathbb{B}^1_{\mathbb{S}}, \varepsilon^1[m]), \ldots, R(\mathbb{B}^n_{\mathbb{S}}, \varepsilon^n[m]))$$

Proposition 5. Given a triplet of plausibility profile $B_{\mathbb{S}}$, stream profile E and revision method R there is a deterministic B_S for each learning step $m \in \mathbb{Z}^+$.

Note how any plausibility profile will always be based on a data sequence profile. This is the case because plausibility spaces are a static representation of the agent's belief at learning step m, so even if an agent is operating on a stream any plausibility space that the agent produces will always be based on a finite amount of observations $\varepsilon[m]$.

3.1 Plausibility Aggregation Functions

Let the class of all plausibility spaces $\mathbb{B}_{\mathbb{S}}$ be denoted $\mathcal{B}$, and the class of all plausibility profiles B be $\mathfrak{B}$.

[1]This is due to s being prioritised over w in the initial plausibility relation, and any update that is sound w.r.t. s cannot change that relation.

Definition 18. A *plausibility aggregation function* is a function that takes a plausibility profile of size n and returns a plausibility space.

$$f : \mathfrak{B} \to \mathcal{B}$$

Definition 19. A *selection function* γ takes a plausibility space $\mathbb{B}_{\mathbb{S}}$ and selects a set of worlds called the *conjecture* of $\mathbb{B}_{\mathbb{S}}$. With a proper choice of selection function the conjecture should represent the set of worlds most believed to be the actual world.

$$\gamma : \mathbb{B}_{\mathbb{S}} \to \mathcal{P}(\bigcup_{i=1}^{n} S_i)$$

Definition 20. Given a plausibility aggregation function f and a selection function γ, the *collective learning method* Λ_f^γ assigns a set of worlds to each plausibility profile by first aggregating to a single plausibility space and applying the selection function after.

$$\Lambda_f^\gamma : \mathfrak{B} \to \mathcal{P}(\bigcup_{i=1}^{n} S_i)$$

An agent applying a collective learning method is a *collective learning agent*.

For this work we take the selection function to be the $\min$ of the plausibility relation $\preceq$ of $f(E)$, and so $\Lambda_f^{\min}(E) = \min(f(E))$. The result of the $\min$ selection function on an individual plausibility space is called the *conjecture* and can consist of multiple worlds. Whenever specifically referring to the multi-agent case we will call the output of Λ_f^γ a *collective conjecture*.

Endeavouring to study collective learning require us to make the above setting dynamic. There are many ways to achieve that. The path we want to pursue in this paper is characterised by the following principles:

We make the following assumptions:

1. the agents are homogeneous, they all revise their belief according to the same belief revision method (either of conditioning, lexicographic, or minimal revision);
2. the agents start with the same plausibility space, $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$, with one world $s \in S$ (unknown to the agents) is designated to be the actual one;
3. each agent i is inductively given a stream ρ_i that is sound with respect to the real world, which is an $s \in S$;
4. collectively their data streams sum up to a sound and complete data stream for s, i.e., $set(\Sigma) = \mathcal{O}_s$;

Assumption 4 is a relaxation of the requirements for a singular agent to identify in the limit that is possible because the completeness burden can now be shared between all of the agents involved.

Definition 21. A plausibility profile $B_{\mathbb{S}}$ is sound and complete w.r.t. s if for the data sequence profile Σ it is the case that $set(\Sigma)$ is sound and complete w.r.t. s.

The following are the two interesting cases of their group communication:

1. at each stage their individual plausibility spaces are aggregated, corresponding to them submitting their full individual plausibility spaces;
2. at each stage their individual conjectures are combined, corresponding to them individually submitting their sets of most plausible worlds, or equivalently, communicating the conjunction of all believed propositions.

4 Merging Perspectives

There exists suggestions of frameworks for convergence of multiple sources of belief bases into a single belief base, commonly called *belief merge*. One notable framework is the *integrity constraint belief mergers* by Konieczny and Pino Pérez, [4] [5]. The standard objective of belief mergers is to combine a profile of belief bases into a consistent belief base that best represents the whole profile. This concept somewhat deviates from the goal of truth tracking since if the profile agrees on some incorrect observation then the incorrect observation should be in the output. With the assumption that the information given is sound w.r.t. the true world this difference does not matter as such an incorrect observation can never occur, however we would like to let go of that assumption if possible, at least to some degree.

Synthetic View: From the synthetic perspective the goal is to find a collective belief base that best represents the information of the profile, even if some of it is incorrect.

Epistemic View: From the epistemic perspective the goal is to identify the true world, even if the given information is incorrect.

Therefore a belief merger of synthetic view is not satisfactory and frameworks of such a perspective are not adequate, including the integrity constraint belief mergers. This can be argued by a simple example.

Example 2. Consider three agents working on a plausibility space $\mathbb{B}_{\mathbb{S}}$ containing worlds s and t where s is the true world, along with four observations $\{z, q, p_1, p_2\}$.

$$S = \{\underline{s}, t\}$$
$$\mathcal{O} = \{z, q, p_1, p_2\} = \{\{s, t\}, \{t\}, \{s\}, \{s\}\}$$
$$\preceq_1 = \preceq_2 = \preceq_3 = \{t < s\}$$

$$\sigma_1 = \sigma_2 = (z)$$
$$\sigma_3 = (z, p_1)$$

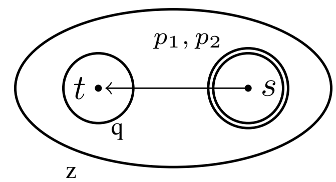

Figure 2: Plausibility Space, $\mathbb{B}_{\mathbb{S}}$

The Occam plausibility assignment will prioritise t. Three agents a_1, a_2, a_3 start with this initial plausibility space and they each receive their respective data streams σ_1, σ_2 and σ_3. Applying lexicographic revision will result in $\preceq_1 = \preceq_2 = \{t < s\}$ and $\preceq_2 = \{s < t\}$. The synthetic perspective would want us to find the plausibility space that best represents the profile as a whole, which in this case would be $\{t < s\}$ even though agent a_3 was provided with information that strictly shows us that it is incorrect.

Most current belief mergers, such as Konieczny and Pino Pérez's integrity constraint mergers, are of the synthetic perspective which is enough argumentation to consider alternatives, however there is another problem.

The integrity constraint mergers work on profiles of belief bases instead of profiles of plausibility spaces. To make this work regardless you could go from plausibility spaces to belief bases by using the learning agent's conjectures as its belief base, however this has a major problem in that the conjectures are by design allowed to contain observations that don't hold in the true world since we don't require the streams to be complete w.r.t. the true world. If an integrity constraint merger satisfying this form of majority voting scheme is used for merging the profile, then it is easy to imagine a situation where the majority believes an incorrect observation and the majority scheme will drown out the potentially correct observation only present in the minority.

Consider the extreme case with a set of agents $\{a_1, \ldots, a_n\}$ where a_1 receives every observation $O_s \in \mathcal{O}_s$ and the remaining agents $\{a_2, \ldots, a_n\}$ receive none. Since we assume all of the agents start with the same plausibility space we run into a problem where every agent except a_1 has the same conjecture and all of the information that a_1 received is drowned out by the majority.

This occurs because going from a plausibility space to its conjecture with insufficient information can allow the conjecture to contain observations that have yet to be shown inconsistent with the true world. Applying traditional aggregation functions from the synthetic view, such as majority, on the conjectures allows for the possibility of incorrect observations to outweigh the correct observations.

For this reason having a belief merger of the epistemic view is not enough on its own, it must also not fall victim to the problem of assuming the conjecture of the agents to be correct. For this reason we will consider ways of merging on profiles of plausibility spaces that contain the whole range of preference relations between all of the possible worlds.

4.1 Preaggregation

A well known result from social choice theory is *Arrow's impossibility theorem*, [2], for which one interpretation is that when working with a profile of preference relations represented by total preorders there is no aggregation function that successfully merges the profile without being a dictatorship (or oligarchy), [6]. Since the plausibility spaces are total preorders Arrow's result tells us that there is no aggregation that combines the spaces without being a dictatorship. Unfortunately having a dictator results in the same problem that the majority example had where the less prioritised agents can possibly have information that is crucial for the truth tracking but will be discarded due to being less prioritised.

However, aggregation functions are not an atomic operation and looking into where dictatorship is introduced could help us avoid it.

In a paper by Philippe Vincke, [3], they divide aggregation functions into two steps. The first is regarded as the preaggregation function, for which the point is to combine the information present in the profile to more precisely find the best possible candidates for the final result of the aggregation function. The second part is then to select the result of the aggregation function between the best candidates from the preaggregation function. What is notable here is that the problem of Arrow's impossibility theorem occurs during the selection after the preaggregation has concluded. Consider an example of four agents merging their total preorders on three possible worlds a, b and c.

Example 3.
$A = \{a, b, c\}, \quad n = 4$

$$(\preceq_1, \preceq_2, \preceq_3, \preceq_4) = \begin{cases} a \preceq_1 b \preceq_1 c, \\ a \preceq_2 b \preceq_2 c, \\ b \preceq_3 a \preceq_3 c, \\ b \preceq_4 a \preceq_4 c. \end{cases}$$

For $\preceq$ to be the result of an aggregation on $(\preceq_1, \preceq_2, \preceq_3, \preceq_4)$ we must have $a \preceq c$ and $b \preceq c$ since every agent agrees, but there is no way to fairly decide between $(a \prec b)$ or $(b \prec a)$ or $(a \preceq b$ and $b \preceq a)$.

$$\text{preaggregation}(\preceq_1, \preceq_2, \preceq_3, \preceq_4) = \begin{cases} a \preceq b \preceq c, \\ b \preceq a \preceq c, \\ a = b \preceq c. \end{cases}$$

It is in selecting between the candidates from the preaggregation function that a dictatorship arises.

From the synthetic perspective all three candidates perform equally as a final option, so selecting either of them as the main result satisfies the requirements. From the epistemic perspective there is only one correct answer, the true world, so just selecting one isn't good enough. This makes it more accurate to have the merger announce that not enough information is present and give the set of all three candidates rather than selecting one for which there is no proof. In fact, the learning agents of [1] work in a similar fashion. They output a conjecture consisting of the best candidates for a result which is guaranteed to be of size one when the conditions of identifying in the limit are met. In the same fashion when performing a merging on the plausibility profile it is not necessary that the output is one specific possible world that must be the correct world but rather a plausibility space with a preference order that takes all of the information in consideration and then that plausibility space will have a conjecture the same fashion as the learners.

As long as the selection function used does not arbitrarily decide between equally likely candidates then we avoid the dictatorship.

5 Collective learning

In pursuit of a method for combining plausibility profiles that follows the epistemic perspective and does not make arbitrary selection choices we will be looking into the details of the revision methods used by the learners and how they impact the methods for merging the learners.

For sound and complete data streams using conditioning for learning agents is a very elegant solution that fits simulating learning. Removing the worlds that don't comply with the observations is a very simple solution to truth tracking as long as the information never goes against the true world. For singular learning agents working on sound and complete sequences it will eventually remove any world that disagrees with the true world, and then the non-overgeneralising property will ensure that the correct world is selected from the set of worlds that satisfy the finite tell tale. To have this same concept work in a multi agent setting means the information that is split over all of the agents would have to be combined in such a way that the set of worlds satisfying the finite tell tale can be found, from which the non-overgeneralising property will again ensure selecting the correct world. This also implies that the non-overgeneralising property has to persist through the merging.

Definition 22. An aggregation function f is *collectively rational* w.r.t. graph property p if $f(E)$ satisfy p when all $\preceq \in E$ do. [6].

Combining the information from each agent in the plausibility profile turns out to be very simple due to the simplicity of conditioning.

Observation 2. Learning agents using conditioning as revision method comply with intersection modelling distributed knowledge. Intersection on a set A of learning agents remaining worlds S_i will output the set S_A of worlds that every agent in A still considers plausible.

$$\bigcap_{i=1}^{n} S_i = S_A$$

After intersection on a plausibility profile that is sound and complete w.r.t. the true world s only worlds that satisfy every observation in $\mathcal{O}_s$ will remain. In other words, the result of the intersection will be exactly the worlds that satisfy the finite tell tale F_s, which is $\cap F_s$. The method that the singular learning agents of [1] use to identify the true world in the limit is by finding $\cap F_s$ and then use the non-overgeneralising property to select the true world from $\cap F_s$. In the following section we will define a collective learning method $\Lambda_\cap$ that finds $\cap F_s$ by intersection and uses it to identify in the limit the same way the individual learning agents do.

5.1 Collective learning method with intersection

A collective learning method requires an aggregation function that provides a single plausibility space which contains the entire plausibility profiles information and then some selection function that selects the correct world from the collective plausibility space.

Aggregation Function

$$f_\cap(E) = (\bigcap_{i=1}^{n} S_i, \mathcal{O}, \bigcap_{i=1}^{n} \preceq_i)$$

Since the initial plausibility space of each agent is the same and no conditioning operation can alter the individual relation directions finding the correct relations for the aggregated plausibility space is as simple as figuring out what worlds remain after the aggregation and then keep the relations where both worlds are kept. A faster way of doing this is to think of the relations as simple elements, by enumeration for example, and apply intersection.

Proposition 6. Conditioning can be defined in terms of intersection.

$$cond(\mathbb{B}_\mathbb{S}, p) = (S \cap p, \mathcal{O}, \preceq^{S \cap p})$$

Where $\preceq^{S \cap p}$ are the relations between the worlds that remain in $S \cap p$.

Proof. Conditioning removes any world that is inconsistent with p from S. This means any of the remaining worlds must be the worlds in S that are consistent with p, which is exactly $S \cap p$. □

Proposition 7. The aggregation function $f_\cap$ is a generalisation of conditioning.

$$cond(\mathbb{B}_\mathbb{S}, p) = f_\cap(\{\mathbb{B}_\mathbb{S}, \mathbb{B}_p\})$$

Where $\mathbb{B}_p = (p, \mathcal{O}, \preceq^p)$ is the plausibility space that can be created around the observation p. Since both plausibility relations $\preceq^p$ and $\preceq^S$ are built following the Occam's razor property they will agree on any relation between two worlds that are in both plausibility spaces.

Proof. Both conditioning and the intersection aggregation function removes worlds from S and then removes relations in $\preceq$ that are no longer necessary. This means that we can solely look at S as everything else follows from what is done to S. Since both functions in the formula can be expressed through intersection on the set of plausible worlds, we can see if they equate when transforming both of them.

As per proposition 6 we know that the conditioning can be written as $S \cap p$ and as for the aggregation function all that is done to the set plausible worlds is an intersection between the two.

$$\begin{aligned} cond(\mathbb{B}_\mathbb{S}, p) &= f_\cap(\{\mathbb{B}_\mathbb{S}, \mathbb{B}_p\}) \\ (S \cap p) &= f_\cap(\{\mathbb{B}_\mathbb{S}, \mathbb{B}_p\}) \\ &= (S \cap p) \end{aligned}$$

□

Proposition 8. If for data sequence σ and plausibility profile E their data sets are the same, $set(\sigma) = set(\Sigma)$, and their starting points are equal, $S = \bigcap_i^n S_i$, then applying conditioning on the whole σ provides the same plausibility space as the aggregation function on E.

$$cond(\mathbb{B}_{\mathbb{S}}, \sigma) = f_{\cap}(\{cond(\mathbb{B}^1_{\mathbb{S}}, \sigma_1), \dots, cond(\mathbb{B}^n_{\mathbb{S}}, \sigma_n)\})$$

$$\text{when} \quad set(\sigma) = set(\Sigma) \text{ and } S = \bigcap_i^n S_i,\ S \in \mathbb{B}_{\mathbb{S}}, S_i \in \mathbb{B}^i_{\mathbb{S}}$$

Proof. Both conditioning and the intersection aggregation function are done by intersection and thus follow the distributive law saying that serial operations can be expressed by applying the operations individually and then on the results, [7]. Shown abstractly with intersection in our system this can be expressed as

$$((S \cap p) \cap q) \dots = (S \cap p) \cap (S \cap q) \cap \dots$$

Together with the requirement that $S = \bigcap_i^n S_i$ we get

$$((S \cap p) \cap q) \dots = (S_1 \cap p) \cap (S_2 \cap q) \cap \dots$$

And then converted back to the operations

$$\begin{aligned} &cond(\dots cond(cond(\mathbb{B}_{\mathbb{S}}, p), q), \dots) \\ &\qquad = f_{\cap}(\{cond(\mathbb{B}^1_{\mathbb{S}}, p), cond(\mathbb{B}^2_{\mathbb{S}}, q), \dots\}) \end{aligned}$$

Writing the sequences of observations as data sequences

$$cond(\mathbb{B}_{\mathbb{S}}, \sigma) = f_{\cap}(\{cond(\mathbb{B}_{\mathbb{S}}, \sigma_1), \dots, cond(\mathbb{B}_{\mathbb{S}}, \sigma_n)\})$$

□

Lemma 1. If E is sound and complete w.r.t. s then it is the case that $\bigcap_{i=1}^n S_i = \bigcap F_s$ when $S_i \in \mathbb{B}^i_{\mathbb{S}}$.

Proof. Since the plausibility profile E is sound w.r.t. s the possible world sets S_i will all contain every world w that agrees with s on every observation in $\mathcal{O}_s$ including s.

That E is collectively complete w.r.t. s necessarily implies that for any world v that is not in $\bigcap \mathcal{O}_s$ there has to be at least one S_i that doesn't contain v, and thus the intersection $\bigcap_{i=1}^n S_i$ will not contain v either.

From these two arguments we know that the intersection $\bigcap_{i=1}^n S_i$ will only contain worlds that agree with s on every observation in $\mathcal{O}_s$, therefore $\bigcap_{i=1}^n S_i = \bigcap \mathcal{O}_s$. By definition the set of worlds that satisfy the finite tell tale of s is exactly the worlds that satisfy every observation in $\mathcal{O}_s$, formally written $\bigcap \mathcal{O}_s = \bigcap F_s$.

$$\bigcap_{i=1}^n S_i = \bigcap \mathcal{O}_s = \bigcap F_s$$

□

Selection Function

$$\gamma_{\cap}(\mathbb{B}_{\mathbb{S}}) = \min(\preceq)$$

From Proposition 8 we know that the resulting plausibility space of the aggregation function will be the same as if it was done by a single learning agent given identical information, so the same selection function that the learning agent uses can be used. Said selection function utilises the non-overgeneralising property, so we need to ensure that the property follows through the merging, i.e. it is required to be collectively rational with respect to this property.

Proposition 9. $f_{\cap}$ is collectively rational w.r.t. the non-overgeneralising property.

Proof. For a plausibility relation to follow the non-overgeneralising property the observations used for belief revision operations must be sound and complete w.r.t. the true world s and the relations must follow the Occam's razor property, stating that the worlds with the least amount of observations are prioritised. Since soundness and completeness are assumed all that is needed to show that the non-overgeneralising property persists through the merging is that Occam's razor property does.

Intersection will by definition only ever remove relations. Since the plausibility relation is total and transitive removing a world and all the relations tied to it will not have an impact on the relations between the other worlds. As such the Occam's razor property will be upheld and therefore the non-overgeneralising property will as well. □

Intersection learning method

$$\Lambda_{\cap}(E) = \gamma_{\cap}(f_{\cap}(E))$$

The intersection learning method applies the selection function on the output of the aggregation function.

Proposition 10. Given a plausibility space E that is sound and complete w.r.t. s, the intersection learning method $\Lambda_{\cap}(E)$ will produce the singleton set $\{s\}$.

Proof. Lemma 1 tells us that an intersection on the set of possible worlds from a plausibility profile E that is sound and complete w.r.t. s will provide the set of worlds that satisfy the finite tell tale of s, namely F_s. So $f_{\cap}(E) = (\bigcap F_s, \mathcal{O}, \preceq_{\cap F_s})$. Proposition 9 says that $f_{\cap}$ is collectively rational w.r.t. the non-overgeneralising property, and since the plausibility spaces are built such that they are non-overgeneralising (from assumption) then so is $f_{\cap}(E)$. Given that $f_{\cap}(E)$ is both the plausibility space for the finite tell tale of s and non-overgeneralising we know that the most prioritised world must be s, and therefore $\Lambda_{\cap}(E) = \{s\}$ □

Proposition 11. A learning agent using conditioning on σ equates to applying the intersection learning method on plausibility profile E with data profile Σ when $set(\sigma) = set(\Sigma)$ and $S = \bigcap_i^n S_i$.

$$L_{cond}(\mathbb{S}, \sigma) = \Lambda_{\cap}(\{cond(\mathbb{B}^1_{\mathbb{S}}, \sigma_1), \dots, cond(\mathbb{B}^n_{\mathbb{S}}, \sigma_n)\})$$

$$\text{when} \quad set(\sigma) = set(\Sigma) \text{ and } S = \bigcap_i^n S_i,\ S \in \mathbb{S}, S_i \in \mathbb{B}^i_{\mathbb{S}}$$

Proof. From Proposition 8 we know that the plausibility space from $f_{\cap}(E)$ and $cond(\mathbb{B}_{\mathbb{S}}, \sigma_A)$ is the same when $set(\sigma) = set(\Sigma)$ and $S = \bigcap_i^n S_i$.

$$\begin{aligned} \Lambda_{\cap}(E) &= \gamma_{\cap}(f_{\cap}(E)) \\ &= \gamma_{\cap}(cond(\mathbb{B}_{\mathbb{S}}, \sigma_A)) \end{aligned}$$

The selection function $\gamma_{\cap} = \min(\preceq)$ is exactly the method used by the learning agents to produce the conjecture, and since the two methods produce the same plausibility space the selection function will also select the same conjecture.

$$\gamma_\cap(f_\cap(E)) = \min(cond(\mathbb{B}_\mathbb{S}, \sigma_A)) = L_{cond}(\mathbb{S}, \sigma_A)$$

□

The output of an agent doing the intersection conjecture function on a profile E cannot be distinguished from a singular learning agent performing conditioning on σ when $set(\sigma) = set(\Sigma)$ and $S = \bigcap_i^n S_i$.

Consider a set of n singular learning agents whose data streams together make up a data stream profile $P = \{\rho^1, \dots, \rho^n\}$. Each ρ^i is sound w.r.t. the world s and collectively the streams are complete w.r.t. s, meaning the profile is sound and complete w.r.t. s. For each step in the streams there is a plausibility profile E that represents the agents static belief at that stage in the learning. The different iterations of E will be denoted E_0 for the initial profile before any conditioning is applied, E_j for learning step j and E_m for the last iteration.

For each step every stream in the profile must include another observation, however since the empty observation $\emptyset$ is allowed we can still model the case of an agent not receiving any new observation.

Step	Profile		Conjecture
0	$E_0 = \{\mathbb{B}^1_\mathbb{S}, \dots, \mathbb{B}^n_\mathbb{S}\}$	$\rightarrow$	$\Lambda_\cap(E_0)$
1	$E_1 = \{\mathbb{B}^1_\mathbb{S}, \dots, \mathbb{B}^n_\mathbb{S}\}$	$\rightarrow$	$\Lambda_\cap(E_1)$
$\vdots$	$\vdots$		$\vdots$
m	$E_m = \{\mathbb{B}^1_\mathbb{S}, \dots, \mathbb{B}^n_\mathbb{S}\}$	$\rightarrow$	$\Lambda_\cap(E_m)$

Definition 23. A collective learner is *monotonic* if $\Lambda(E_j) \subseteq \Lambda(E_i)$ when $i < j$.

Proposition 12. An agent using the intersection learning method $\Lambda_\cap$ is a monotonic learner.

Proof. Both functions, conditioning and intersection learning method, operate by removing worlds that have been shown to be inconsistent with s. Therefore any application of the two functions will either produce the same plausibility space as previous, if the information is already known, or reduce the state space in some fashion, so $\Lambda(E_j) \subseteq \Lambda(E_i)$ when $i < j$.

□

Since conditioning based learning agents and intersection merging agents behave interchangeably when provided the same information and starting point we can consider merging agents as learning agents and thereby also apply the same concept of singular learning agents identifying in the limit from [1] to collective learners.

Definition 24. Given an epistemic space $\mathbb{S} = (S, \mathcal{O})$, a world $s \in S$ is *collectively identified in the limit by learning pair* (R, Λ) if, for every sound and complete data stream profile for s, there exists a learning step j after which $\Lambda(R(\mathbb{B}_\mathbb{S}, \rho^1[j]), \dots, R(\mathbb{B}_\mathbb{S}, \rho^n[j]))$ outputs the singleton $\{s\}$.

The epistemic space $\mathbb{S}$ is *identified in the limit by* (R, Λ) iff all its worlds are identified in the limit by (R, Λ).

An epistemic space $\mathbb{S}$ is *identifiable in the limit* (learnable) if there exists a learning pair that can identify it in the limit.

Lemma 2. Given a data stream profile P that is sound and complete w.r.t. s the collective learning pair $(L_{cond}, \Lambda_\cap)$ will collectively identify in the limit the world s.

Proof. If the reader has read [1] this is easily to argue with proposition 11 considering that the collective learner $\Lambda_\cap$ is indistinguishable from a singular learning agent L_{cond}, which has been showed in [1] to identify s in the limit for such a situation. However let us still provide proof without relying on this relation.

Since the data stream profile P is sound and complete w.r.t. s there must be some learning step j where the resulting plausibility profile will be sound and complete w.r.t. s.

In proposition 10 it is shown that the intersection learning method on a plausibility profile that is sound and complete w.r.t. s produces the singleton set $\{s\}$. Once the intersection learning method has produced the singleton $\{s\}$ it will stay on $\{s\}$ forever since any observation O that any agent could update its individual plausibility space with would never remove s from its S_i due to $O \in \mathcal{O}_s$. So we know that the learning pair will by some learning step j return the singleton $\{s\}$ and continually do so forever, thereby having collectively identified in the limit the world s. □

Definition 25. A learning method pair are *universal on a class C of epistemic spaces* if they can identify in the limit every epistemic space in C that is identifiable in the limit. A *universal learning method pair* is one that is universal on the class of all epistemic spaces.

Proposition 13. The learning method pair $(L_{cond}, \Lambda_\cap)$ are universal.

Proof. □

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