Questions

Foundation

1. The HCF of a and b is 3 and their LCM is 60. Find all possible values of a and b.
Hint: $a \cdot b = HCF(a,b) \cdot LCM(a,b)$

2. Tom has a $\frac{1}{3}$ chance of scoring a basket. How many shots does Tom have to take for their to be more than a $\frac{2}{3}$ chance of Tom having thrown the ball into the net?

Higher

1. Show that the iterative formula: $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ converges to the $\sqrt{2}$ and find x_2 where $x_0 = 1$. You may assume that x_n is always greater than zero.
Hint: Iterative formulas converge if $x_{n+1} - x_n \to 0$ *as* $n \to \infty$

2. Show that $2x^2 + x - 3$ and $x^2 - 1$ have at least one common factor for $x > 2$.

Answers

Foundation
1. The HCF of a and b is 3 and their LCM is 60. Find all possible values of a and b.

Proof. $a = 3m$ and $b = 3n$

$$ab = 9mn = 180 \implies mn = 20$$
$$\implies (m,n) = (1,20),(2,10),(4,5)$$
$$\implies (a,b) = (3,60),(6,15),(12,15)$$

$\square$

2. Tom has a $\frac{1}{3}$ chance of scoring a basket. How many shots does Tom have to take for their to be more than a $\frac{2}{3}$ chance of Tom having thrown the ball into the net?

Proof. The probability of Tom not scoring is $\frac{2}{3}$.
$\therefore$ The probability of Tom not scoring after n attempts is $(\frac{2}{3})^n$.
$\therefore$ The probability of Tom scoring after n attempts is $1 - (\frac{2}{3})^n$.
$n = 3 \implies$ The probability of Tom scoring after 3 attempts is $1 - (\frac{2}{3})^3 = 1 - \frac{8}{27} = \frac{19}{27} > \frac{18}{27} = \frac{2}{3}$. $\square$

Higher
1. Show that the iterative formula: $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ converges to the $\sqrt{2}$ and find x_2 where $x_0 = 1$. You may assume that x_n is always greater than zero.

Proof. The iterative formula converges when $x_{n+1} = x_n$. Let x be the limit of this sequence.

$$x = \frac{x}{2} + \frac{1}{x}$$

Multiplying both sides by $2x$:

$$2x^2 = x^2 + 2$$
$$x^2 = 2 \implies x = \sqrt{2}$$

Calculating x_2:

$$x_0 = 1 \implies x_1 = \frac{1}{2} + 1 = \frac{3}{2}$$
$$x_1 = \frac{3}{2} \implies x_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$$

$\square$

2. Show that $2x^2 + x - 3$ and $x^2 - 1$ have at least one common factor for $x > 2$.

Proof. Factorising:

$$2x^2 + x - 3 = (2x+3)(x-1)$$
$$x^2 - 1 = (x+1)(x-1)$$

Both quadratics share a factor of $(x - 1)$ which is bigger than 1 for $x > 2$ $\square$