Barycentric Brownian Bees

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The Particle System

The *barycentric Brownian bees* processes are interacting particle systems defined as follows.

1. **Spatial motion**: N individual particles move in $\mathbb{R}^d$ according to independent Brownian motions.
2. **Branching**: each particle undergoes binary branching at rate one.
3. **Selection**: at each branching event, the particle furthest from the current barycenter of the system is removed.

We denote the BBB process by

$$\mathrm{X} = (X(t), t \geq 0) = ((X_i(t))_{i \in [N]}, t \geq 0),$$

where $[N] := \{1, 2, \ldots, N\}$. We define the associated *barycenter process* $\overline{\mathrm{X}} = (\overline{X}(t), t \geq 0)$ by

$$\overline{X}(t) = N^{-1} \sum_{1 \leq i \leq N} X_i(t).$$

Main Result: An Invariance Principle

Theorem 1: (Theorem 1.1 in [1]) For all $d \geq 1$ and $N \geq 1$, there exists $\sigma = \sigma(d, N) \in (0, \infty)$ such that, as $m \to \infty$,

$$\left(m^{-1/2}\overline{X}(tm), 0 \leq t \leq 1\right) \xrightarrow{\mathrm{d}} (\sigma B(t), 0 \leq t \leq 1),$$

with respect to the Skorohod topology on $\mathcal{D}([0,1], \mathbb{R}^d)$, where $(B(t), 0 \leq t \leq 1)$ is a Brownian motion in $\mathbb{R}^d$ starting at the origin.

Remark: We have $\sigma(d, 1) = 1 = \sigma(d, 2)$ for all $d \geq 1$. For $N \geq 3$, our proof does not yield insight into the value of $\sigma(d, N)$.

A Simulation of the Rescaled Barycenter Process

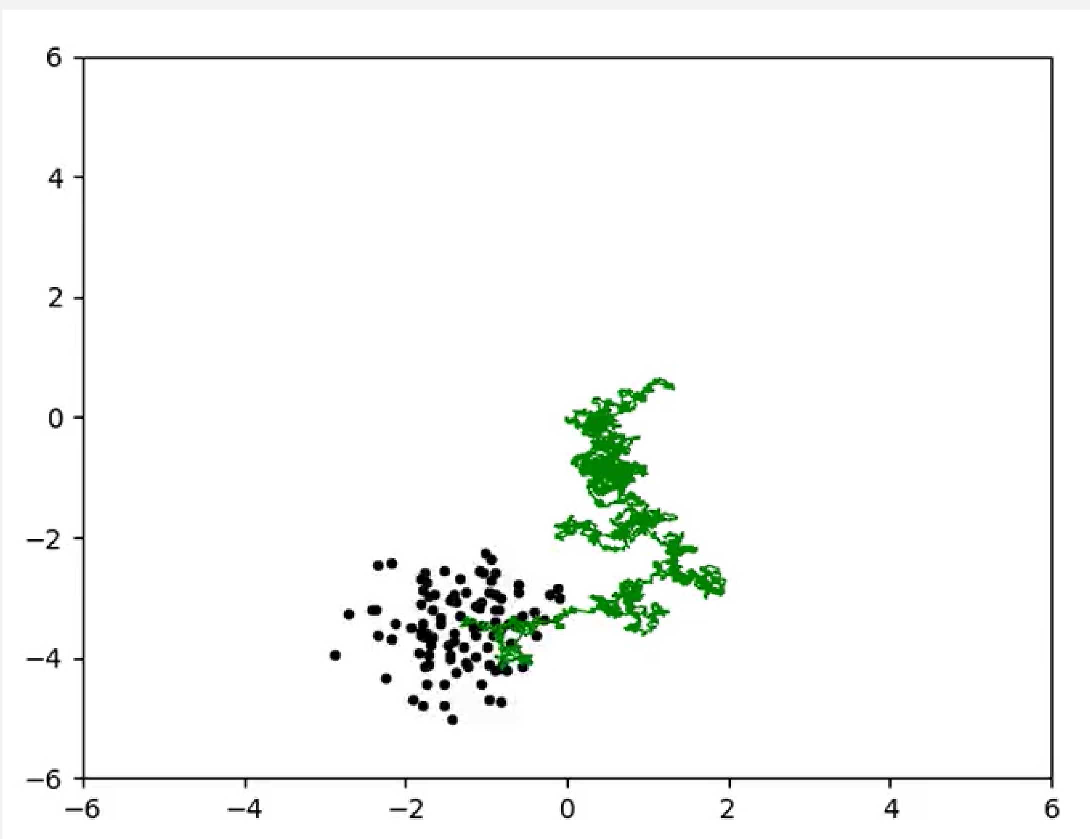

Figure 1: A realization of the BBB process (black particles) and an interpolation of the diffusively rescaled barycenter process (green trajectory) with $d = 2$, $N = 100$, $m = 10^9$, and until 10^4 branching events have occurred.

Proof Strategy

Overview: We approximate $\bar{\mathrm{X}}$ by a sum of Independent and Identically Distributed (IID) random variables constructed from X, and which satisfy the assumptions of Donsker's Theorem. Specifically, we construct an increasing sequence of almost surely finite stopping times $(\tau_i; i \geq 1)$ such that the following conditions hold:

C1 (IID-ness)

i. $(\tau_{i+1} - \tau_i; i \geq 1)$ are IID with positive finite mean, and
ii. $(X_1(\tau_{i+1}) - X_1(\tau_i); i \geq 1)$ are IID.

C2 (Control of the jumps)

i. $\left(\sup_{\tau_i \leq t \leq \tau_{i+1}} |X_1(t) - X_1(\tau_i)|; i \geq 1\right)$ are identically distributed, and
ii. $\mathbf{E}_x \left[\sup_{\tau_1 \leq t \leq \tau_2} |X_1(t) - X_1(\tau_1)|^2\right] < \infty$.

C3 (Approximation of $\bar{\mathrm{X}}$) We have

$$m^{-1/2} \sup_{0 \leq t \leq 1} |\bar{X}(tm) - X_1(tm)| \to 0,$$

in probability as $m \to \infty$.

Conditions C1 and C2 allow us to apply Donsker's Theorem to the random variables

$$(X_1(\tau_{i+1}) - X_1(\tau_i); i \geq 1),$$

and to conclude that X_1 satisfies an invariance principle. Condition C3 then implies Theorem 1.

Construction of the $(\tau_i; i \geq 1)$: A key aspect of the argument is to choose the $(\tau_i; i \geq 1)$ in such a way that the BBB process *regenerates*, i.e. starts over from a single *queen particle*, at each time τ_i. We therefore define our prototypical stopping time as the first time that the following three events occur successively. Given a constant $L > 0$ and for some $r_N > 0$ small enough and appropriate constant γ close to the origin,

1. the extent $E(X(t)) := \max_{i \neq j \in [N]} |X_i(t) - X_j(t)|$ of the BBB process satisfies $E(X(t)) \leq L$,
2. in the next unit of time, the configuration $X - \bar{X}$ arranges itself as:

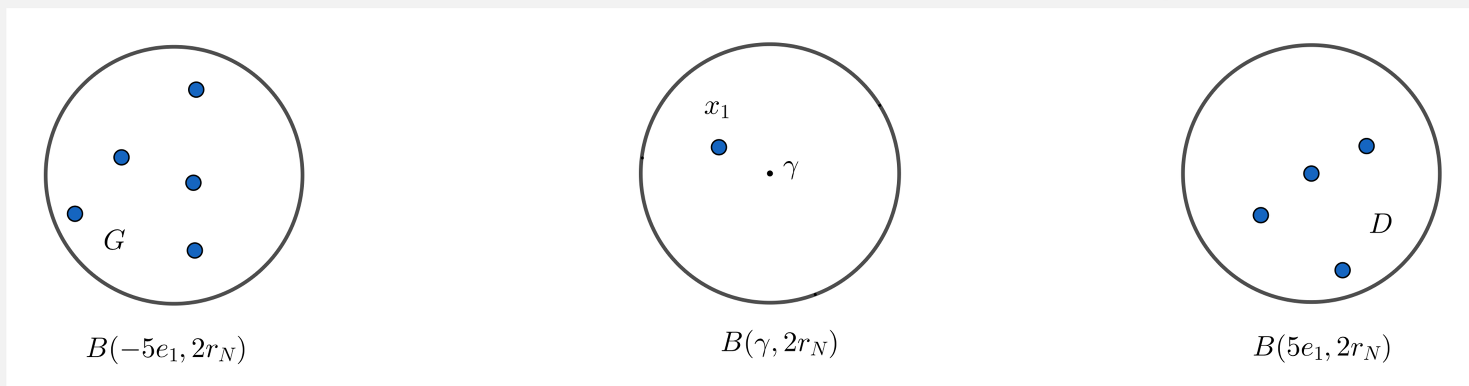

Figure 2: A particle configuration $x = (x_i)_{i \in [N]}$ arranged in two groups G and D of approximately equal size around particle 1.

3. in the next unit of time, the process regenerates from particle 1:

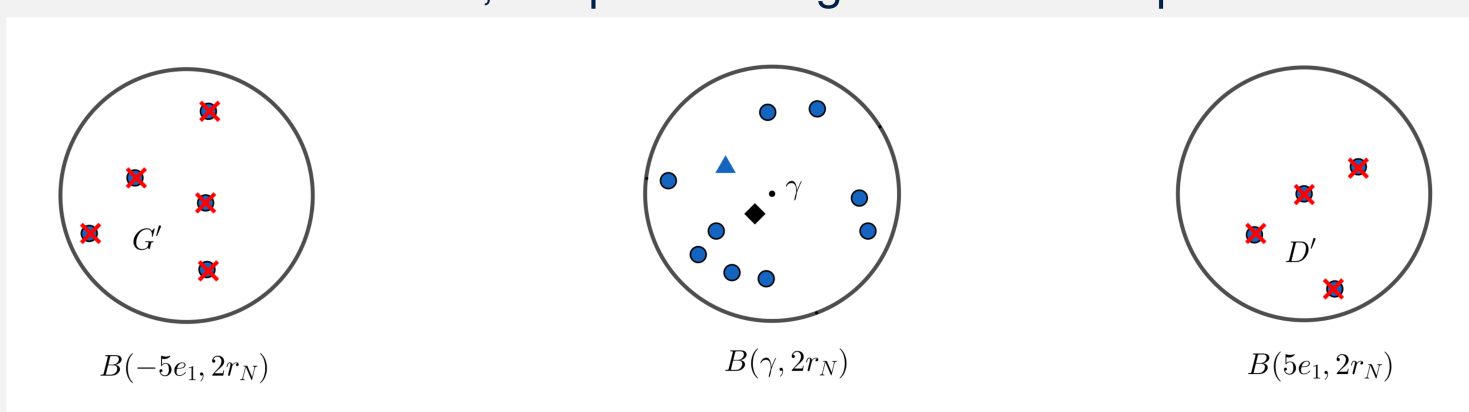

Figure 3: A particle configuration after regeneration of the process. The blue triangle is the location of particle 1, and the black square is the location of the recentered barycenter.

A By-Product of the Proof: Harris Recurrence

Definition: We say that a time-homogeneous càdlàg Markov process $\Phi = (\Phi_t, t \geq 0)$ on the state space $\mathbb{R}^{d \times N}$ is Harris recurrent if there exists a σ-finite and nonzero Borel measure φ on $\mathbb{R}^{d \times N}$ such that for any Borel set $A \subset \mathbb{R}^{d \times N}$ with $\varphi(A) > 0$, for all $x \in \mathbb{R}^{d \times N}$,

$$\mathbf{P}_x(\eta_A = \infty) = 1$$

where $\eta_A := \int_0^\infty \mathbb{1}_{\{\Phi_t \in A\}} dt$ is the total time spent in A by Φ.

Theorem 2: (Theorem 1.3 in [1]) The recentered BBB process $\mathrm{X} - \overline{\mathrm{X}} := (X(t) - \overline{X}(t), t \geq 0)$ is Harris recurrent.

Overview of the proof: In step 1 of the construction of the $(\tau_i; i \geq 1)$, we prove the existence of an increasing sequence of a.s. finite stopping times $(T_i; i \geq 1)$ at which the extent process becomes smaller than L. Moreover, if φ denotes the N-fold product measure of a d-dimensional standard Gaussian, and $\mu_{t,x}$ is the law of $\Phi_t := X(t) - \bar{X}(t)$ with $X(0) = x$, then there exists $C = C_{L,N} > 0$ such that

$$\inf_{x \in \mathbb{R}^{d \times N}: E(x) \leq L} \inf_{1 \leq t \leq 2} \mu_{t,x}(A) \geq C\varphi(A), \text{ for all Borel } A \subset \mathbb{R}^{d \times N}.$$

We use this bound in a stochastic domination argument to obtain

$$\eta_A \geq \sum_{i \geq 1} \int_{T_i+1}^{T_i+2} \mathbb{1}_{\{\Phi_t \in A\}} dt \overset{a.s.}{=} \infty.$$

Open Questions

1. We expect that
$$\lim_{N \to \infty} \sigma(d, N) = 0,$$
for all dimensions $d \geq 1$.
2. For Borel $A \subset \mathbb{R}^d$, write
$$\pi_t^N(A) := \frac{1}{N} \#\{\{X_1(t) - \overline{X}(t), \cdots, X_N(t) - \overline{X}(t)\} \cap A\}$$
for the empirical measure of the BBB process viewed from its barycenter. We expect that π_t^N converges weakly as first $t \to \infty$, then $N \to \infty$, to a continuous Borel measure with compact support. ([2])

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References

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[2] Julien Berestycki, Éric Brunet, James Nolen, and Sarah Penington. Brownian bees in the infinite swarm limit. *Annals of Probability*, to appear.