Capsizing a Ship Using the Cannons

Mechanics Final Project 5/10/13
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Abstract:

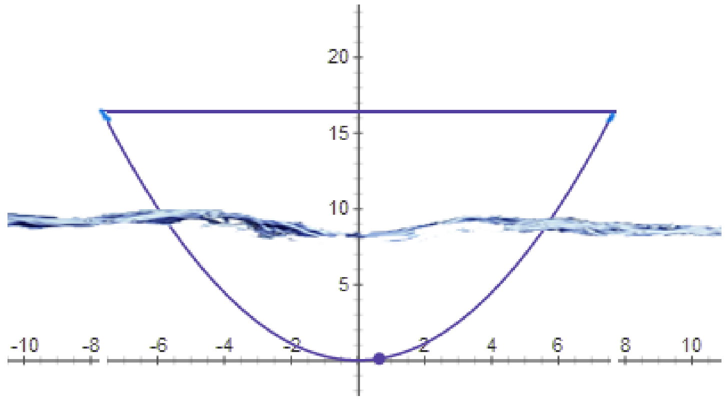
In the earliest days of firepower in naval combat, the ships were extremely large, heavy and slow. Firepower was heavy and was barely an advantage when the less heavily armed enemy was much more nimble. Finally, the Frigate was born and changed Naval combative architecture forever. She was lighter, faster, and--though less armed--nimble enough to take down opponents. However, in order to gain speed, designers reduced the beam of the ship and the amount of weight in the keel. Under normal circumstances, she functioned vallently. However, what if a historic Frigate were overloaded with artillery? Would the recoil from the guns be sufficient to capsize the vessel?

Initial Assumptions:

The inspiration hull for the project is that of the USS Constitution. This hull is a traditional frigate type. These first entered Naval combat in the 17th century. Since then, they have grown and evolved. However, they are still an extremely effective weapon. The hull is designed for speed and maneuverability. In the face of combat, this can be an extremely important fact. To achieve great speed and maneuverability, Frigate hulls are usually very long and relatively narrow. Therefore, they are often more unstable than other hull designs with smaller length to beam ratios.

The USS Constitution firing her cannons. She is a Frigate hull and the oldest commissioned naval hull in existence.

Thus, we were able to abstract the shape of the hull into a parabola. we imagined that this was a solid parabola which had a constant draft along its length. Further, the center of mass for a parabola is below the geometric center, like that of a frigate. To determine an equation for our parabola, we pulled depth and beam data from the US constitution and derived a parabolic curve which fit our research. From this we decided that the draft would be 7 m and the waterline beam would be 10 m. The total height of the boat is 13 m and at the top deck, the beam is 13.6 meters. With these constants, we derived the equation: $y=7\ (X^2)/\ 25$.



This we then considered to be the cross section for the hull. Finally, we collapsed the entire mass of the hull down into one cross section. This is possible because we consider all effects which require the length of the ship to be considered to be negligible. Therefore, accounting for this in our hull design becomes unnecessary.

Capsizing:

In an actual system, capsizing would occur anytime the hull rotated so far that it could no longer right itself. However, this number is heavily related to hydrodynamic forces as well as complex ship design. Therefore, we instead considered capsizing to have occurred anytime the edge of the hull touches the water. We calculated this inside our model, finding the intersection point where the waterline touches the 13-meter tall edge of the hull. This value was 46 degrees.

Calculating Buoyancy:

An object floats when the weight of the volume of fluid displaced by the submerged portion of the object equals that of the object. Therefore, we began to investigate the depth of the parabola submerged at any time. Since the volume is constant, this is a constant over the time period in question. Therefore, since we are examining a parabolic shaped hull, it is easy to find this volume when the hull is not rotated. By solving the curve equation for when the function yields half the desired waterline beam gives one of the limits of integration, henceforth called, “limit1.”. Then consider a rectangle spanning the origin and this point. Its area is clearly and easily defined by A=lw where l is the y axis height and x is the limit of integration. Then, the integral of the curve from zero to limit1 is the area between the curve and the x axis. The result of subtracting this integral from the area of the rectangle is the area of the hull which is underwater. This, we found, to be 46.67 m^2.

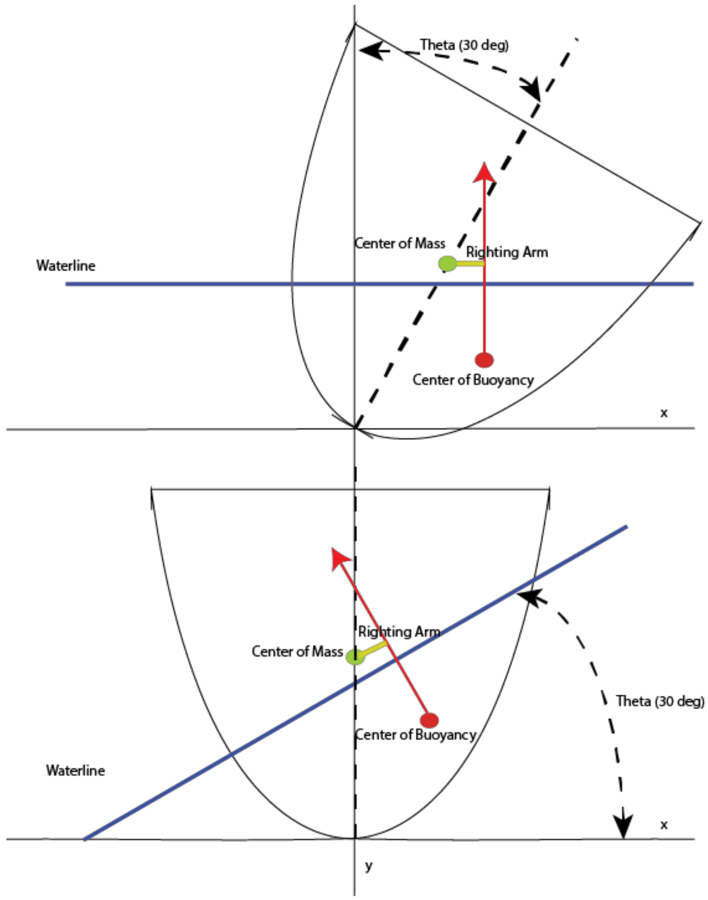
$$10 \times 7 - 2 \int_0^5 \frac{7}{25} x^2 \, dx$$

The mathematical expression evaluated to determine the submerged area of a cross-section. Further, since we have collapsed the volume of the vessel down into a single cross section, we can also consider the weight of the water in a similar fashion. Therefore, the weight of the water displaced equals: $Area_{Underwater}$ * $Depth_{Boat}$ * 1000 kg * m^{-3}. Despite the intuitive proof that this makes sense, we further validated this method by unit analysis: [m^2]*[m] * [kg/m^3] = kg. This yields favorable results and further validates this assumption. With this, we calculate the mass to be 2.333×10^6 kg. This is extremely close to the historic, recorded displacement of the USS Constitution: 2.200×10^6 kg. This final fact also validates this step.

Calculating Restoring Torque:

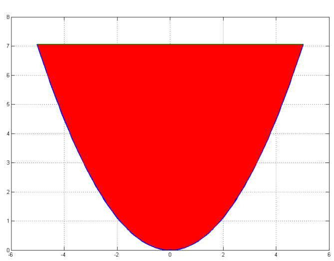
Actual hulls roll in a very complex manner. However, most of our resources abstracted the boat as rolling about an axis through its center of mass, which makes intuitive sense in most situations. Therefore, restoring torque is caused by buoyancy alone. To calculate the magnitude of this restorative force, it is first necessary to calculate the center of buoyancy for a given theta position. At 0 degrees of roll, the center of buoyancy, which is the center of area under the water, is directly under the center of mass. Since buoyancy acts upward, no torque is applied. As the ship rolls, the center of area under the water shifts as the shape of the underwater section changes. Since buoyancy always acts upward, it will exert a restoring torque along the righting arm, which is a measure of the distance between the buoyancy vector and the center of mass.

At all times, the amount of area underwater must remain constant. Were this not true, the ship would rise out of the water or sink. In order to make the math significantly cleaner, we model the water tilting around the boat rather than the boat itself tilting, as explained in the following figure:

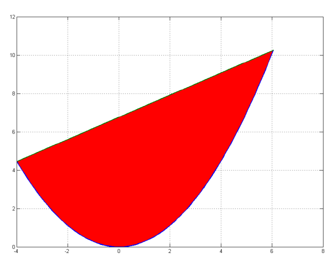


The slope of the waterline to the horizontal is equal to the angle from the boat's centerline to the vertical. We sweep the waterline upward while maintaining the same slope until the area below the water is 46.66 m^3. The point of intersection where the area is 46.66 m^3 is what we use to determine the general y=ax+b equation of the line. Here, a is the slope at the given theta.

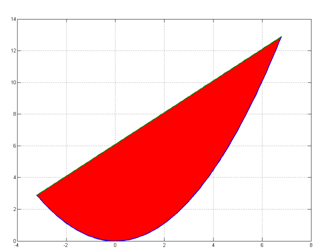
The following three figures illustrate our model's waterline position calculations. All three represent the area underwater at a given angle, and all three have an area of 46.66 degrees.



0 deg



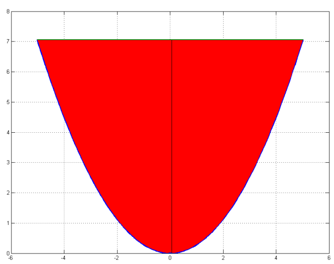
30 deg



45 deg

After this, we need to know the location of a vertical line which divides the submerged area in half--23.3 m^3 on the right and 23.3 m^3 on the left. This is done by changing the limits of integration and once more sweeping through until an equation is found which satisfies equal right-left division of the area. This will be referred to as equation c.

The following Three figures have the vertical half-point line drawn in. The center of buoyancy is somewhere on this line, but we will not know exactly where until we draw one more half-line, which is swept upward.



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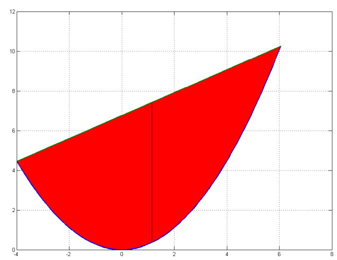

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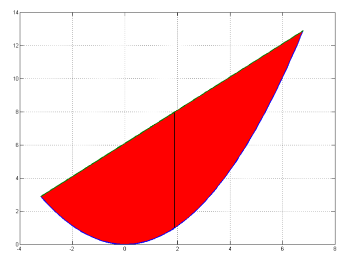

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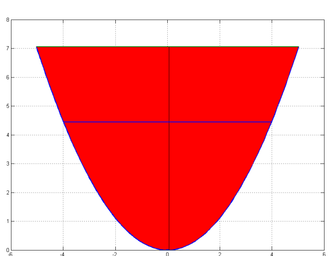
Then, we once more sweep through the general equation until an intercept is found which is half the area underwater. This we will refer to as d. Note that this is half of the submerged area, not half of the intercept.

With these equations, we can now find the intersection of the slope of the waterline and equation d. This is the linear equation where half of the submerged area is below the line.This point will be referred to as point e. Point e is the center of area of the submerged portion; in other words, this is the center of buoyancy.

The following three figures show both half-area lines drawn in. Note how the intersection point between them, the center of buoyancy, shifts as the boat tilts.



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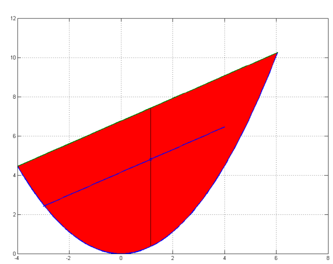
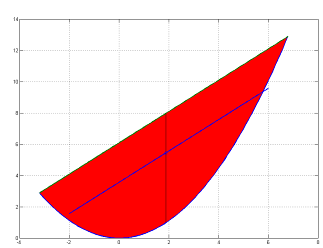
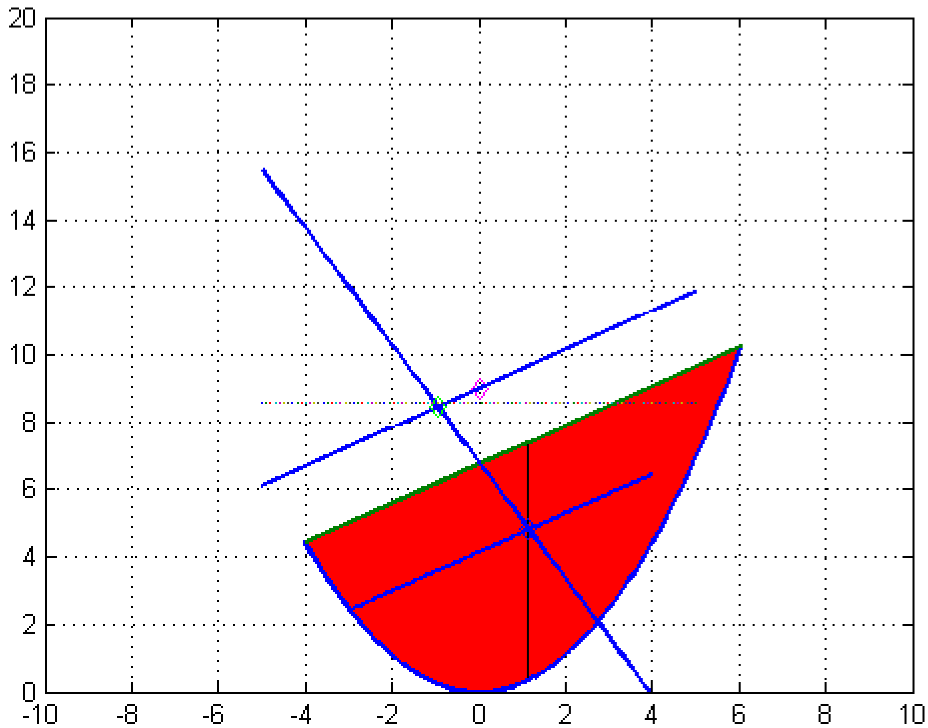


30 deg



45 deg

We now look for the line which is normal to the slope of the waterline and also passes through the intersection point e. This is the intersection point of the vertical division of submerged area and the slope of the waterline. This line perpendicular to the slope of the waterline must be of the form y=mx+b, where m is the slope. The slope is the negative reciprocal of the slope of the waterline, which is also equivalent to 1/tan(theta). We plug in the x and y values for point e and find the intercept of this line. This line is the buoyancy vector. The final step is to geometrically solve for the righting arm, which is perpendicular to the buoyancy vector and connects it to the center of mass. Multiplying this righting arm by the displacement of the boat (the weight) gives us the torque on the hull at any given angle. The following figure is an example Righting Arm calculation for 30 degrees of heel:



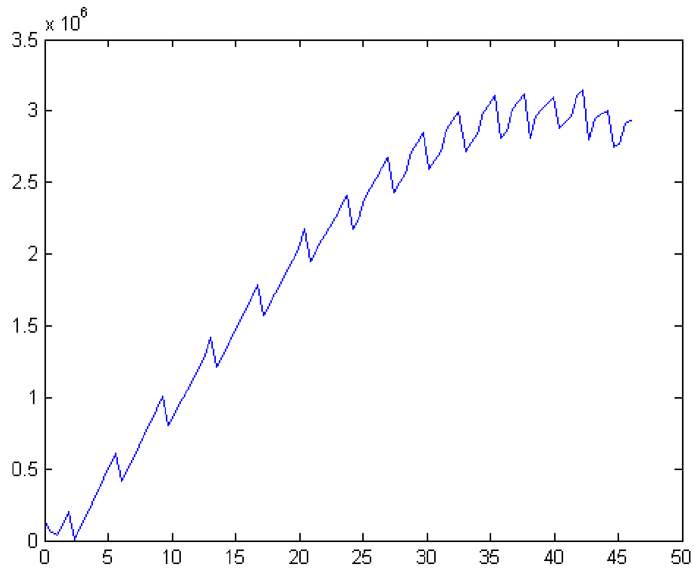
Note the two half-area lines that converge on the center of buoyancy. Through this point is the buoyancy vector, which is normal to the waterline (green). This vector moves upward until it passes the center of mass of the entire hull (the red point). The final blue line, normal to the buoyancy vector, is the line of the righting arm. The actual righting arm is the distance between the green point and the red point. Multiplying this scalar by the displacement yields torque. This seems counterintuitive at first, but we have actually just computed Force*Distance*sin(theta), since the righting arm is just equal to the distance between the center of buoyancy times the angle theta. Though our method required more geometry, we decided to compute the true righting arm because that is how nautical stability calculations are really done.

Movement of the Boat:

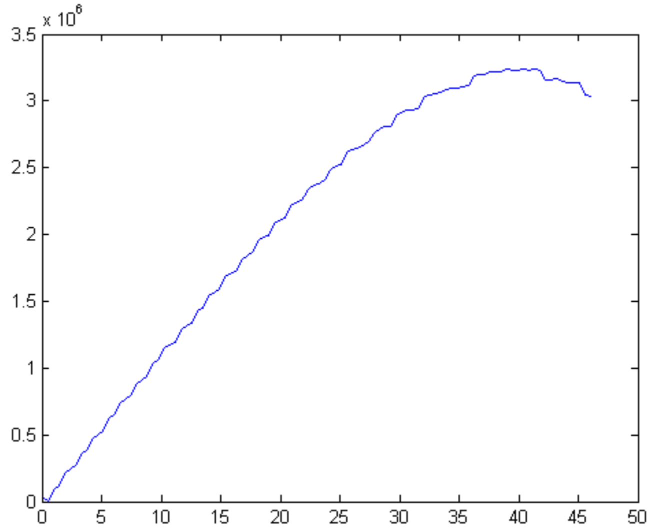
From the previous work, we now have an equation to find the restorative torque from the keel at a given theta. Therefore, we now need the moment of inertia. Mass distribution is arbitrary, so we essentially estimated a moment based on keel weight and then tweaked it as necessary. Since angular acceleration is torque/moment, we have the capability to calculate angular acceleration (or deceleration, in this case) for any angle. We then ran a computational analysis of the motion of the boat, using ever-smaller time steps (on the order of 0.01 seconds in the final model). Beginning with an angle of zero, an alpha (acceleration) of zero, and an initial velocity provided by the cannons (which we swept until finding the key value), we calculated a new angle at each time step from the previous velocity and a new velocity from the angular acceleration, which itself was calculated from the current angle. This is analogous to using a right rectangular method for calculating an integral. By sweeping the velocity and running the simulation, we found a velocity that causes the ship to capsize.

Results:

The following is an early graph our model produced of torque as a function of angle. This is important to include because it demonstrates the validity of our mathematical process:



Note the jagged behavior; torque dips and spikes when it should be smooth. Also, there is some very unexpected behavior near 0 angle. This was a result of having time steps that were too large when we were drawing the two lines to calculate center of buoyancy. Increasing the time steps yields: (again, torque as a function of angle)

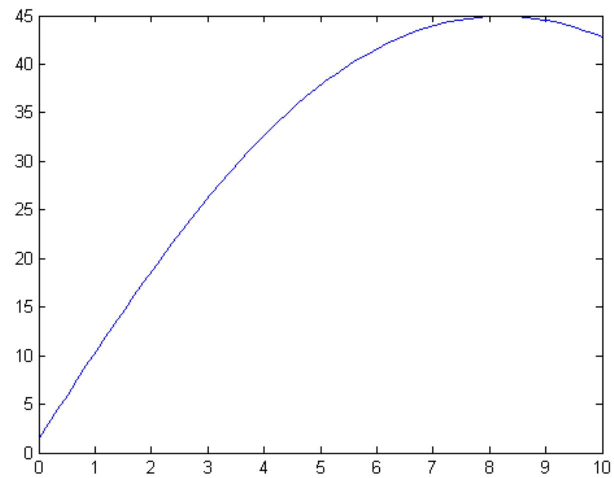


Note how the torques are still somewhat jagged. This artifact is likely impossible to eliminate, as shorter time steps require far more computational time. The built-in MatLab solvers that could replace our crude sweeps are challenged by our torque calculation function because it itself is so computationally expensive. As a result, the accuracy pictured above is what we used for the rest of our calculations.

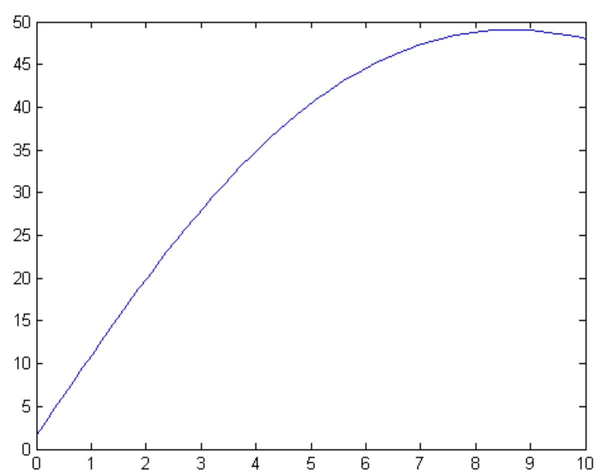
It is also interesting that the torque actually begins to fall off and decrease at 40 degrees of heel. This is inherent to how parabola area distribution works, and is not representative of a problem with our model.

With this information, we were able to run a step-by-step simulation of the rolling ship for a given starting velocity (imparted by the guns). Graphs of the simulations follow, with descriptions:

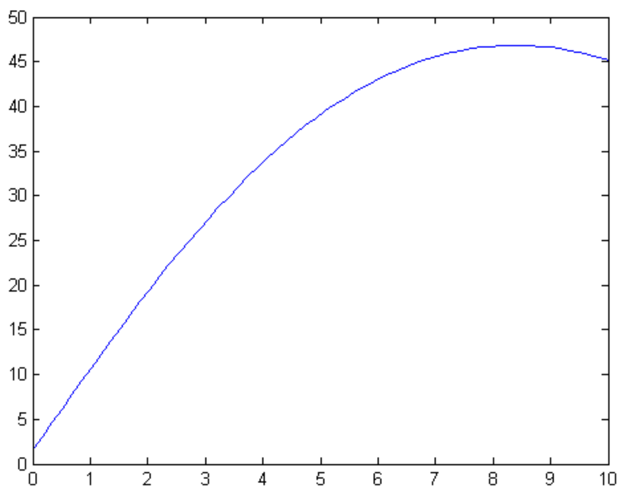
15 degrees/second initial speed, plot of angle of displacement over time (angle of displacement is 0 when the ship is upright, and the ship sinks at 45 degrees, as this is when the edge of the deck touches the water). Note how the ship does not quite capsize.



16 degrees/second initial speed, again a plot of angle of displacement over time. The ship reaches beyond 45 degrees, so this speed is excessive.



The proper initial velocity appears to be 15.5 degrees/second. This causes the waterline to reach the deck and no further, as can be seen in the following graph:



Our simulation indicates that if the hull is moving at 15.5 degrees/second then the ship will not be able to right itself and will capsize. Rotational energy is equal to (1/2)*Moment of Inertia*(angular velocity)^2 This gives us an initial rotational energy of 95,139,000 joules. In other words, 95,139,000 joules imparted to the ship from any outside force will cause it to capsize.We can now determine how many cannons are needed. The U.S.S. constitution carried 20 32-pounder cannons and 30 24-pounder cannons. Together, these all provide around 60,000,000 joules-more than half of the required energy, if all were pointed the same direction and fired at once. Doubling the cannon complement should allow the ship to capsize itself. This seems like an unlikely result, because ships-of-the-line did not capsize under their own firepower. However, recall that only half of the guns on board are fired at once, so a real broadside from our modeled U.S.S. Constitution would only impart ¼ of the required energy. Also, our ship is very unstable, the reasons for which are discussed below.

Since we can consider any propulsion system, we can also consider somewhat more modern options. For example, the U.S. navy Railgun fires at 33,000,000 joules. Three of these fired at once would capsize our ship.

However, there are several issues with the model. The three largest are:

1: Hull shape. The parabolic hull is not used because it is very unstable. True hulls are designed to be much much harder to capsize.

2: Mass distribution in the hull is fairly arbitrary, essentially designed to help our model work more effectively. Real ships tailor mass distribution to maintain stability.

3: We simply discounted friction, which would have a major damping effect on the motion of the boat.

These three together explain why our ship is so easy to capsize-real boats take much more force to tip over, so broadsides never endangered sailing vessels.

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