1. The utility function is $u(x_1, x_2) = x_1 x_2$

   (a) Using the definition of indirect utility and duality:

$$\begin{aligned} v(p_1, p_2, m) &= \frac{m^2}{4p_1p_2} \\ v(p_1, p_2, e(p_1, p_2, u)) &= u \\ e(p_1, p_2, u)^2 &= 4p_1p_2u \\ e(p_1, p_2, u) &= \sqrt{4p_1p_2u} \end{aligned}$$

   (b) CV and EV:

      i. Mr Watt's would take a £50 pay cut to achieve the same utility as he currently has.

$$e(p_1, v(p_0, m_0)) - m_0 = \sqrt{4 * \frac{1}{4} * 1 * \frac{100^2}{4 * 1 * 1}} - 100 = -50$$

      ii. A £100 increase pay would be required to achieve the same level of utility as in the new town.

$$m_1 - e(p_0, v(p_1, m_1)) = 100 - \sqrt{4 * 1 * 1 * \frac{100^2}{4 * \frac{1}{4} * 1}} = -100$$

   (c) The consumer surplus will be between -50 and -100.

2. (a) Hotelling's Lemma says that if y^*, x^* are the solution to the PMP for a firm then we have that:

$$\begin{aligned} \frac{\partial \pi}{\partial p} &= y^*(p, \mathbf{w}) \\ \frac{\partial \pi}{\partial w_i} &= -x_i^*(p, \mathbf{w}) \end{aligned}$$

   To see this is true, consider the PMP:

$$\max_{y,x} py(p) - \mathbf{wx}$$

   and apply Shepard's Lemma to the optimal value function $\pi(p, \mathbf{w})$.

   (b) If w_i increases, then we can see that profit must decrease, as $x_i > 0$.

   (c)

$$\begin{aligned} &\max_{L} pL^{\frac{1}{2}} - wL \\ &\frac{1}{2}pL^{-\frac{1}{2}} - w = 0 \\ &L = (\frac{p}{2w})^2 \\ &\pi = \frac{p^2}{2w} - \frac{p^2}{4w} \\ \frac{\partial \pi}{\partial w} = -\frac{p^2}{2w^2} + \frac{p^2}{4w^2} = -&\frac{p^2}{4w^2} = -L \end{aligned}$$

3. (a)

$$40p - 2pz - w = 0$$

The second derivative is -2, therefore this solution is profit maximising.

(b) i. If $z^* = 0$, then $w = 40p$.

ii. If $z^* = 20$, then $w = 0$ and $p \neq 0$.

(c) The input demand function is $z^* = \frac{40p-w}{2p} = 20 - \frac{w}{2p}$. The output supply function is:

$$\begin{aligned}
y^* &= 40z^* - {z^*}^2 \\
&= 40(20 - \frac{w}{2p}) - (20 - \frac{w}{2p})^2 \\
&= 800 - \frac{20w}{p} - 400 + \frac{20w}{p} - \frac{w^2}{4p^2} \\
&= 400 - \frac{w^2}{4p^2} \\
\pi &= py(p) - wz \\
\pi &= 400p - \frac{w^2}{4p} - 20w + \frac{w^2}{2p}
\end{aligned}$$

(d) Hotelling's Lemma:

$$\begin{aligned}
\frac{\partial \pi}{\partial w} &= -(20 - \frac{w}{2p}) \\
&= -z^*(p, w)
\end{aligned}$$