



# Probability Theory

An update made by Thomas Walker on notes scribed by Ivan Kirev and Samuel Lam ([4](#))  
from a lecture series on Probability Theory given by Professor Igor Krasovsky.

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# Part I. Measure Theory and Random Variables

## 1 Events, Probability and Random Variables

In developing an abstract mathematical framework to describe the likelihood of events happening in a random experiment we can formalise large-sample results, including the Law of Large Numbers and the Central Limit Theorem. An experiment can be described in the following way.

1. Let the possible outcomes  $\omega$  of the experiment be the sample space,  $\Omega$ .
2. Let events  $A$  be subsets of the sample space  $\Omega$  that we may observe. The collection of such subsets is denoted as  $\mathcal{F}$ .
3. Assign a value  $\mathbb{P}(A) \in [0, 1]$  to each of the subsets  $A \in \mathcal{F}$  to quantify how likely the event is to occur.

With this construction, we have a few problems to resolve.

- What should the collection of events  $\mathcal{F}$  include?
- How should we assign values to the events in  $\mathcal{F}$ ?

It should be the case that we can either observe nothing or something, that is, we should let  $\mathcal{F}_* := \{\emptyset, \Omega\} \subseteq \mathcal{F}$ . However, any useful experiment should be able to differentiate different observed outcomes, and so  $\mathcal{F}$  should include more than just the whole set  $\Omega$  as a potential result.

One could suggest letting  $\mathcal{F}$  contain all possible subsets of  $\Omega$ . We denote this  $\mathcal{F} = \mathcal{F}^* := 2^\Omega$ , and refer to it as the power set of  $\Omega$ . For  $\Omega$  a countable set, this is fine, and it will often be the case that  $\mathcal{F} = 2^\Omega$ . However, issues arise if  $\Omega$  is uncountable, for example, if  $\Omega = \mathbb{R}$ .

Next, there are certain properties that  $\mathbb{P}$  ought to satisfy to make sense. For example, it should be finitely additive. That is, if  $A, B \in \mathcal{F}$  are disjoint outcomes, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

When one extends this property to countably many disjoint events, potential contradictions may arise if  $\mathcal{F}$  doesn't have a certain structure. In the uncountable case,  $2^\Omega$  will often not possess such a structure. Therefore, we want to choose  $\mathcal{F}$  such that it contains more information than  $\mathcal{F}_*$  but is strictly smaller than  $2^\Omega$ .

At this point, one may be wondering why we need to extend the finite additivity property to the countable additivity property. Well, often one is interested in the long-term behaviour of a random experiment, such as the expected value of a dynamical system as it continues to run forward in time. Hence, questions regarding limits arise naturally, for which one needs to reason about countably many events rather than just a finite number.

Mathematicians, therefore, attempted to find a suitable criterion for  $\mathcal{F}$  and  $\mathbb{P}$  so that they would not give rise to contradictions, but still allow  $\mathcal{F}$  to be large enough to be useful. One of the most successful attempts was made by Andrey Kolmogorov in 1933 when he devised the axioms of probability in his Foundations of the Theory of Probability. His work led to the development of the measure theory, which forms the foundations of modern probability theory.

### 1.1 Algebras and $\sigma$ -algebras

Let  $\Omega$  be a set of points  $\omega$ .

**Definition 1.1.1** A nonempty system of subsets of  $\Omega$  is called an algebra  $\mathcal{A}$  if

- $\Omega \in \mathcal{A}$ ,
- $A, B \in \mathcal{A}$  implies that  $A \cup B \in \mathcal{A}$ , and
- $A \in \mathcal{A}$  implies that  $A^c \in \mathcal{A}$ .

If in addition, all countable unions  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$  whenever  $A_1, A_2, \dots \in \mathcal{A}$ , then  $\mathcal{A}$  is a  $\sigma$ -algebra.

**Remark 1.1.2** Note that we can consider the complements of events to show that a  $\sigma$ -algebra (algebra) is also closed under countable (finite) intersections.

### Definition 1.1.3

- A set function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is finitely additive if for any disjoint  $A, B \in \mathcal{A}$  we have

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

- Let  $\mathcal{F}$  be a  $\sigma$ -algebra. A set function  $\mu : \mathcal{F} \rightarrow [0, \infty]$  is called  $\sigma$ -additive if for any disjoint  $A_1, A_2, \dots \in \mathcal{F}$ , it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Such a  $\mu$  is called a measure on  $\mathcal{F}$ . A measure  $\mu$  is a probability measure if  $\mu(\Omega) = 1$ .

- A measure is called  $\sigma$ -finite if there exists a partition  $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ , where the  $\Omega_k$  are pairwise disjoint, such  $\mu(\Omega_k) < \infty$  for  $k \in \mathbb{N}$ .

**Definition 1.1.4** A probability space is a triple  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  is a set called the sample space,  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , and  $\mathbb{P}$  is a probability measure on  $\mathcal{F}$ . An element of  $\mathcal{F}$  is called an event.

**Proposition 1.1.5** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. The probability measure satisfies the following.

1.  $\mathbb{P}(\emptyset) = 0$ .

2. If  $A, B \in \mathcal{F}$  then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

3. If  $A, B \in \mathcal{F}$  and  $B \subseteq A$  then

$$\mathbb{P}(B) \leq \mathbb{P}(A).$$

4. If  $A_1, A_2, \dots \in \mathcal{F}$ , then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

*Proof.*

1. Note that  $\emptyset \in \mathcal{F}$  is disjoint from itself. Therefore, by finite additivity it follows that

$$\mathbb{P}(\emptyset) = \mathbb{P}(\emptyset \cup \emptyset) = 2\mathbb{P}(\emptyset)$$

which implies that  $\mathbb{P}(\emptyset) = 0$ .

2. Note that  $A \cap B^c \in \mathcal{F}$  is disjoint from  $B$ , hence,

$$\mathbb{P}(A \cup B) = \mathbb{P}(B \cup (A \cap B^c)) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c).$$

Similarly,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B).$$

Adding these together it follows that

$$2\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(A^c \cap B) + \mathbb{P}(A \cap B^c).$$

Noting that

$$A \cup B = (A \cap B) \cup (A^c \cap B) \cup (A \cap B^c)$$

where all individual intersections of the right-hand side are disjoint, it follows by finite additivity that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

3. As  $A = (A \cap B) \cup (A \cap B^c)$ ,  $A \cap B = B$ , and  $\mathbb{P}$  is non-negative, it follows that

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c) \geq \mathbb{P}(B).$$

4. Let  $\tilde{A}_1 := A_1$ , and  $\tilde{A}_n = A_n \setminus \left( \bigcup_{k=1}^{n-1} A_k \right)$  for  $n \geq 2$ . It is clear that  $(\tilde{A}_n)_{n \in \mathbb{N}}$  are disjoint and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \tilde{A}_n.$$

Moreover, by statement 3. we have  $\mathbb{P}(\tilde{A}_n) \leq \mathbb{P}(A_n)$ . Therefore, using  $\sigma$ -additivity we have

$$\begin{aligned} \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(\tilde{A}_n) \\ &\leq \sum_{n=1}^{\infty} \mathbb{P}(A_n). \end{aligned}$$

■

**Proposition 1.1.6** Let  $\mathbb{P}$  be a finitely additive set function defined over an algebra  $\mathcal{A}$ , with  $\mathbb{P}(\Omega) = 1$ . Then the following statements are equivalent.

1.  $\mathbb{P}$  is  $\sigma$ -additive, and hence a probability measure.
2.  $\mathbb{P}$  is continuous from below. That is, for any sets  $A_1, A_2, \dots \in \mathcal{A}$  such that  $A_1 \subseteq A_2 \subseteq \dots$  and  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ , we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n).$$

3.  $\mathbb{P}$  is continuous from above. That is, for any sets  $B_1, B_2, \dots$  such that  $B_1 \supseteq B_2 \supseteq \dots$  and  $\bigcap_{n=1}^{\infty} B_n \in \mathcal{A}$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right).$$

4.  $\mathbb{P}$  is continuous at  $\emptyset$ . That is, for any sets  $B_1, B_2, \dots \in \mathcal{A}$  such that  $B_1 \supseteq B_2 \supseteq \dots$  and  $\bigcap_{n=1}^{\infty} B_n = \emptyset$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_n) = 0.$$

*Proof.* (1)  $\Rightarrow$  (2). Consider the sets  $\tilde{A}_1 = A_1$ , and  $\tilde{A}_n = A_n \setminus A_{n-1}$  for  $n \geq 2$ . Then the sets  $\tilde{A}_k$  are a collection of disjoint sets. Moreover,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \tilde{A}_n.$$

Therefore,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right)$$

$$\begin{aligned}
&\stackrel{(1)}{=} \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{P}(\tilde{A}_k) \\
&= \lim_{n \rightarrow \infty} (\mathbb{P}(A_1) + (\mathbb{P}(A_2) - \mathbb{P}(A_1)) + \cdots + (\mathbb{P}(A_n) - \mathbb{P}(A_{n-1})) \\
&= \lim_{n \rightarrow \infty} \mathbb{P}(A_n).
\end{aligned}$$

(2)  $\Rightarrow$  (3). Let  $n \geq 1$  and consider the events  $\tilde{B}_k = B_1 \setminus B_k$ . Then

$$\mathbb{P}(\tilde{B}_n) = \mathbb{P}(B_1) - \mathbb{P}(B_n).$$

The sequence  $(\tilde{B}_k)_{k \in \mathbb{N}}$  is an increasing sequence of events with

$$\bigcup_{n=1}^{\infty} \tilde{B}_n = B_1 \setminus \bigcap_{n=1}^{\infty} B_n.$$

By (2) it follows that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{B}_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{B}_n).$$

Therefore,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{P}(B_n) &= \mathbb{P}(B_1) - \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{B}_n) \\
&= \mathbb{P}(B_1) - \mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{B}_n\right) \\
&= \mathbb{P}(B_1) - \mathbb{P}\left(B_1 \setminus \bigcap_{n=1}^{\infty} B_n\right) \\
&= \mathbb{P}(B_1) - \mathbb{P}(B_1) + \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) \\
&= \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right).
\end{aligned}$$

(3)  $\Rightarrow$  (4). This is clear as  $\mathbb{P}(\emptyset) = 0$ .

(4)  $\Rightarrow$  (1). Let  $A_1, A_2, \dots \in \mathcal{A}$  be pairwise disjoint with

$$A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{A},$$

and

$$B_n := \bigcup_{i=n+1}^{\infty} A_i.$$

Note that  $\bigcup_{i=1}^n A_i$  and  $B_n$  are disjoint and such that

$$A = \bigcup_{i=1}^n A_i \cup B_n.$$

Therefore, by finite additivity we have that

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^n A_i\right) + \mathbb{P}(B_n).$$

The sequence of sets  $(B_n)_{n \in \mathbb{N}}$  is decreasing and such that  $\bigcap_{n=1}^{\infty} B_n = \emptyset$ . Therefore, by finite additivity and (4) it follows that

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{P}(A_i)$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \mathbb{P} \left( \bigcup_{i=1}^n A_i \right) \\
&= \lim_{n \rightarrow \infty} (\mathbb{P}(A) - \mathbb{P}(B_n)) \\
&= \mathbb{P}(A) - \lim_{n \rightarrow \infty} \mathbb{P}(B_n) \\
&= \mathbb{P}(A).
\end{aligned}$$

■

**Proposition 1.1.7** Let  $\mu$  be a finitely additive measure on an algebra  $\mathcal{A}$  and let the sets  $A_1, A_2, \dots \in \mathcal{A}$  be pairwise disjoint with  $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ . Then

$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A).$$

*Proof.* Note that  $\bigcup_{i=1}^n A_i \subseteq A$  for any  $n \in \mathbb{N}$ . Therefore, by finite additivity and statement 3. of Proposition 1.1.5 we have

$$\mu(A) \geq \mu \left( \bigcup_{i=1}^n A_i \right) = \sum_{i=1}^n \mu(A_i).$$

Taking the limit as  $n \rightarrow \infty$  preserves the inequality and so

$$\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$$

as required. ■

**Example 1.1.8** Let  $\Omega$  be a sample space. Then,

1.  $\mathcal{F}_* = \{\emptyset, \Omega\}$ , and
2.  $\mathcal{F}^* = \{A : A \subseteq \Omega\} = 2^{\Omega}$

are  $\sigma$ -algebras.

**Lemma 1.1.9** For any collection  $\mathcal{E}$  of subsets of  $\Omega$  there exists a minimal algebra  $a(\mathcal{E})$  and a minimal  $\sigma$ -algebra  $\sigma(\mathcal{E})$  that contains all elements of  $\mathcal{E}$ . Equivalently,  $a(\mathcal{E})$  ( $\sigma(\mathcal{E})$ ) is the intersection of all algebras ( $\sigma$ -algebras) that contain  $\mathcal{E}$ .

*Proof.* Intersection, countable or uncountable, of algebras ( $\sigma$ -algebras) containing  $\mathcal{E}$  is an algebra ( $\sigma$ -algebra) containing  $\mathcal{E}$ . ■

**Remark 1.1.10** In the context of Lemma 1.1.9 we say that  $\sigma(\mathcal{E})$  is generated by  $\mathcal{E}$ .

**Exercise 1.1.11** Let  $\mathcal{D} = \{D_1, D_2, \dots\}$  be a countable partition of  $\Omega$  such that  $D_i \cap D_j = \emptyset$  for  $i \neq j$  and  $\Omega = \bigcup_{j=1}^{\infty} D_j$ . Show that

$$\sigma(\mathcal{D}) = \left\{ \bigcup_{j \in I} D_j : I \subseteq \mathbb{N} \right\}.$$

## 1.2 Measurable Spaces

**Definition 1.2.1** A measurable space is a pair  $(E, \mathcal{E})$ , where  $E$  is a set and  $\mathcal{E}$  is a  $\sigma$ -algebra on  $E$ .

Let  $\mathbb{R} = (-\infty, \infty)$  be the real line and

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\},$$

for  $-\infty \leq a < b < \infty$ . Let  $\mathcal{A}$  be the algebra of subsets of  $\mathbb{R}$  such that  $A \in \mathcal{A}$  if for some  $n < \infty$  we have

$$A = \bigcup_{i=1}^n (a_i, b_i].$$

Let  $\mathcal{B}(\mathbb{R})$  be the smallest  $\sigma$ -algebra containing  $\mathcal{A}$ . Then for  $a < b$  we observe that

- $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$ ,
- $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b]$ ,
- $\{a\} = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a]$ .

Thus in addition to containing intervals of the form  $(a, b]$ , the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  also contains singleton set  $\{a\}$  and intervals of the form

- $(a, b)$ ,
- $[a, b]$ ,
- $[a, b)$ ,
- $(-\infty, b)$ ,
- $(-\infty, b]$ , and
- $(a, \infty)$ .

**Exercise 1.2.2** Show that  $\mathcal{B}(\mathbb{R})$  can be generated by the collection of

1. open intervals of the form  $(a, b)$ ,
2. closed intervals  $[a, b]$ ,
3. half intervals,
4. intervals of the form  $(-\infty, a]$  or  $[a, \infty)$ ,
5. open sets with respect to the Euclidean metric, or
6. closed sets with respect to the Euclidean metric.

Let  $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_n$ . That is, the set of ordered  $n$ -tuples  $x = (x_1, \dots, x_n)$ , where  $x_k \in \mathbb{R}$  for  $k = 1, \dots, n$ . A rectangle then refers to a set of the form

$$I = I_1 \times \cdots \times I_n = \{x \in \mathbb{R}^n : x_k \in I_k, k = 1, \dots, n\}$$

where  $I_k = (a_k, b_k]$  is known as a side of the rectangle. Let  $\mathcal{I}$  be the collection of all rectangles  $I$ . The smallest  $\sigma$ -algebra  $\sigma(\mathcal{I})$  generated by the system  $\mathcal{I}$  is the Borel  $\sigma$ -algebra of subsets of  $\mathbb{R}^n$  denoted  $\mathcal{B}(\mathbb{R}^n)$ .

Instead of the rectangles  $I = I_1 \times \cdots \times I_n$  let us consider the rectangles  $B = B_1 \times B_2 \times \cdots \times B_n$  with Borel sides. That is,  $B_k$  is a Borel subset of the real line that appears in the  $k^{\text{th}}$  place in the direct product  $\mathbb{R} \times \cdots \times \mathbb{R}$ . The smallest  $\sigma$ -algebra containing all rectangles with Borel sides is denoted by

$$\mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R})$$

and called the direct product of the  $\sigma$ -algebras  $\mathcal{B}(\mathbb{R})$ . In fact,

$$\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}).$$

In other words,  $\sigma$ -algebra generated by the rectangles  $I = I_1 \times \cdots \times I_n$  coincides with the  $\sigma$ -algebra generated by rectangles  $B = B_1 \times \cdots \times B_n$  with Borel sides. We will now justify this.

**Lemma 1.2.3** Let  $\mathcal{E}$  be a collection of subsets of  $\Omega$  and let  $B \subseteq \Omega$ , and consider

$$\mathcal{E} \cap B = \{A \cap B : A \in \mathcal{E}\}.$$

Then

$$\sigma(\mathcal{E} \cap B) = \sigma(\mathcal{E}) \cap B,$$

where on the left-hand side the  $\sigma$ -algebra is over  $B$ , whereas on the right-hand side the  $\sigma$ -algebra is over  $\Omega$ .

*Proof.* Step 1: Show that  $\sigma(\mathcal{E} \cap B) \subseteq \sigma(\mathcal{E}) \cap B$ .

Clearly,  $B = \Omega \cap B \in \sigma(\mathcal{E}) \cap B$ . Moreover, for any  $A \in \sigma(\mathcal{E})$  we have

$$B \setminus (A \cap B) = (\Omega \setminus A) \cap B$$

and so  $B \setminus (A \cap B) \in \sigma(\mathcal{E}) \cap B$  as  $\Omega \setminus A \in \mathcal{E}$ . Note that for  $A_1, A_2, \dots \in \sigma(\mathcal{E})$  we have

$$\bigcup_{i=1}^{\infty} (A_i \cap B) = \left( \bigcup_{i=1}^{\infty} A_i \right) \cap B \in \sigma(\mathcal{E}) \cap B,$$

where the membership follows as  $\bigcup_{i=1}^{\infty} A_i \in \sigma(\mathcal{E})$ . Therefore,  $\sigma(\mathcal{E}) \cap B$  is a  $\sigma$ -algebra over  $B$ . Hence, as  $\mathcal{E} \cap B \subseteq \sigma(\mathcal{E}) \cap B$  it follows that

$$\sigma(\mathcal{E} \cap B) \subseteq \sigma(\sigma(\mathcal{E}) \cap B) = \sigma(\mathcal{E}) \cap B.$$

Step 2: Show that  $\sigma(\mathcal{E}) \cap B \subseteq \sigma(\mathcal{E} \cap B)$ .

Let

$$\mathcal{G} := \{A : A \cap B \in \sigma(\mathcal{E} \cap B)\}.$$

Clearly,  $\Omega \cap B \in \sigma(\mathcal{E} \cap B)$  and so  $\Omega \in \mathcal{G}$ . Let  $A \in \mathcal{G}$ , then note that

$$(\Omega \setminus A) \cap B = B \setminus (A \cap B).$$

By assumption  $A \cap B \in \sigma(\mathcal{E} \cap B)$ , which is a  $\sigma$ -algebra over  $B$  and so  $B \setminus (A \cap B) \in \sigma(\mathcal{E} \cap B)$ . Meaning  $\Omega \setminus A \in \mathcal{G}$ . Next let  $A_1, A_2, \dots \in \mathcal{G}$ , then as

$$\left( \bigcup_{i=1}^{\infty} A_i \right) \cap B = \bigcup_{i=1}^{\infty} (A_i \cap B)$$

it follows that  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$ . Therefore,  $\mathcal{G}$  is a  $\sigma$ -algebra over  $\Omega$ . Hence, as  $\mathcal{E} \subseteq \mathcal{G}$  it follows that  $\sigma(\mathcal{E}) \subseteq \mathcal{G}$ . So for  $A \in \sigma(\mathcal{E})$  we have  $A \in \mathcal{G}$  which implies that  $A \cap B \in \sigma(\mathcal{E} \cap B)$ , and so  $\sigma(\mathcal{E}) \cap B \subseteq \sigma(\mathcal{E} \cap B)$ . ■

It is clear that for  $n = 1$ , the  $\sigma$ -algebras  $\mathcal{B}(\mathbb{R}^n)$  and  $\mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R})$  are the same.

**Lemma 1.2.4**

$$\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}).$$

*Proof.* ( $\subseteq$ ). For any open set,  $A$  we can write

$$A \subseteq \bigcup_{x \in A \cap \mathbb{Q}^2} R(x, \tau(x))$$

where  $R(x, \tau(x))$  is the open square centered at  $x$  and of side length  $\tau(x)$ . As  $A \cap \mathbb{Q}^2$  is countable and  $R(x, \tau(x)) \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ , it follows that  $A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ .

( $\supseteq$ ). For this inclusion, it is sufficient to check that  $B_1 \times B_2 \in \mathcal{B}(\mathbb{R}^2)$  for any Borel sets  $B_1, B_2$ . Note that  $B_1 \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^2)$  since

$$B_1 \times \mathbb{R} \in \sigma(\{\text{open subsets of } \mathbb{R}\}) \times \mathbb{R} = \sigma(\{\text{open subsets of } \mathbb{R} \times \mathbb{R}\}).$$

Similarly,  $\mathbb{R} \times B_2 \in \mathcal{B}(\mathbb{R}^2)$ , and so  $B_1 \times B_2 = (B_1 \times \mathbb{R}) \cap (\mathbb{R} \times B_2) \in \mathcal{B}(\mathbb{R}^2)$ . ■

The case for any  $n > 2$  can be discussed similarly to Lemma 1.2.4. The space  $((\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)))$ , on the other hand, requires a different approach. However, it is useful to outline this as  $((\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)))$  is consistently utilised for constructing probabilistic models of experiments with infinitely many steps. Let  $\mathbb{R}^\infty = \{x = (x_1, x_2, \dots), x_k \in \mathbb{R}\}$ .

**Definition 1.2.5** A set  $C \subseteq \mathbb{R}^\infty$  is called cylindrical if it is of the form

$$C = \left\{ x \in \mathbb{R}^\infty : (x_1, x_2, \dots, x_n) \in \tilde{C}_n \right\}$$

for some  $n \geq 1$  and  $\tilde{C}_n \in \mathcal{B}(\mathbb{R}^n)$ .

**Exercise 1.2.6** Show that the cylindrical sets form an algebra.

The  $\sigma$ -algebra generated by cylindrical sets is called the cylindrical  $\sigma$ -algebra and is denoted  $\mathcal{B}(\mathbb{R}^\infty)$ . One can verify that

$$\mathcal{B}(\mathbb{R}^\infty) = \sigma(\{A_1 \times A_2 \times \dots \subseteq \mathbb{R}^\infty, A_k \in \mathcal{B}(\mathbb{R})\}).$$

**Example 1.2.7** For all  $c \in \mathbb{R}$ , let

$$A = \left\{ x \in \mathbb{R}^\infty : \limsup_n x_n = \inf_n \sup_{k > n} x_k > c \right\}.$$

It follows that  $A \in \mathcal{B}(\mathbb{R}^\infty)$  because

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} \{x \in \mathbb{R}^\infty : x_k > c\}.$$

Similarly, letting

$$B = \left\{ x \in \mathbb{R}^\infty : \liminf_n x_n = \sup_n \inf_{k > n} x_k > c \right\},$$

we have that  $B \in \mathcal{B}(\mathbb{R}^\infty)$  because

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} \{x \in \mathbb{R}^\infty : x_k > c\}.$$

**Exercise 1.2.8** For  $c \in \mathbb{R}$ , show that  $D = \{x \in \mathbb{R}^\infty : \lim_{n \rightarrow \infty} x_n = c\} \in \mathcal{B}(\mathbb{R}^\infty)$ .

### 1.3 Probability Distributions

Throughout this section, we will consider the importance of non-decreasing functions for describing probability measures on measurable spaces.

**Lemma 1.3.1** A non-decreasing function  $g(x)$  on  $\mathbb{R}$  is continuous up to possibly countably many discontinuities of the first kind. That is, for  $\epsilon \searrow 0$  the limits  $g(x + \epsilon)$  and  $g(x - \epsilon)$  exists but distinct.

*Proof.* Note that it must be the case that  $\lim_{\epsilon \searrow 0} (g(x + \epsilon) - g(x - \epsilon)) > 0$  due to the non-decreasing property of  $g$ . Thus we can form open balls at the right limit of the function at each jump. These open sets are distinct and contain at least one rational number. As the rational numbers are countable on  $\mathbb{R}$ , it follows that there are at most countably many such open balls, and hence countably many such jumps. ■

Lemma 1.3.1 is a positive result, as by construction our probability measures behave well in the countable domain. Moreover, we deduce that the derivative of a non-decreasing function,  $g(x)$ , denoted  $g'(x)$  exists Lebesgue almost everywhere.

**Exercise 1.3.2** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$  be a probability space and let  $F(x) := \mathbb{P}((-\infty, x])$  for  $x \in \mathbb{R}$ . Show that,

- $F(x)$  is non-decreasing,
- $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow \infty} F(x) = 1$ , and
- $F(x)$  is right-continuous for all  $x \in \mathbb{R}$ .

**Definition 1.3.3** Every function  $F : \mathbb{R} \rightarrow [0, 1]$  satisfying the properties of Exercise 1.3.2 is called a distribution function on  $\mathbb{R}$ .

Thus to every probability measure  $\mathbb{P}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , there corresponds a distribution function. The opposite is also true and there exists a one-to-one correspondence between distribution functions and probability measures.

**Theorem 1.3.4** Let  $F = F(x)$  be a distribution function on  $\mathbb{R}$ . Then there exists a unique probability measure  $\mathbb{P}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that

$$\mathbb{P}(a, b] = F(b) - F(a),$$

for all  $a, b$  with  $-\infty \leq a < b < \infty$ .

This relies on the following fundamental result in measure theory.

**Theorem 1.3.5 — Caratheodory Theorem.** Let  $\mu_0$  be a  $\sigma$ -additive (pre-)measure on  $(\Omega, \mathcal{A})$ , where  $\mathcal{A}$  is an algebra of subsets of  $\Omega$ . Then there exists a measure  $\mu$  on  $(\Omega, \sigma(\mathcal{A}))$ , such that

$$\mu(A) = \mu_0(A)$$

for all  $A \in \mathcal{A}$ . If  $\mu_0$  is also  $\sigma$ -finite, then the measure  $\mu$  is unique.

**Definition 1.3.6** A measure  $\mu$  on a  $\sigma$ -algebra  $\Sigma$  on  $\Omega$  is complete if any subset of a set of measure zero (null sets) is measurable. That is if  $A \in \Sigma$  is such that  $\mu(A) = 0$ . Then for any  $B \subseteq A$  we have that  $B \in \Sigma$  and  $\mu(B) = 0$ .

Requiring completeness helps to avoid any caveats in proving results relating to measures. The space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$  with  $\mathbb{P}$  constructed from Theorem 1.3.4 is not complete as there are subsets of Borel sets that are not themselves Borel sets. Fortunately, one can enlarge the  $\sigma$ -algebra to include null sets. A measure  $\mu$  on  $\Sigma$  can be completed by extending  $\Sigma$  to

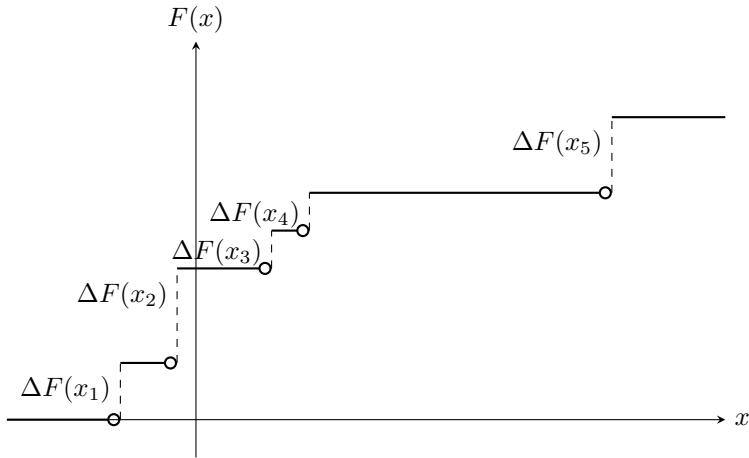
$$\bar{\Sigma} = \sigma(\Sigma \cup \{B \in \Omega : B \subseteq A \in \Sigma, \mu(A) = 0\}),$$

with  $\mu(B) = 0$  for any  $B$  a subset of a null set. The completion of the measure obtained in Theorem 1.3.4 is called the Lebesgue-Stiltjes measure. In particular, the distribution function  $F(x) = x$  corresponds to the Lebesgue measure on  $\mathbb{R}$ .

## 1.4 Measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

### 1.4.1 Discrete Measures

A discrete measure  $\mathbb{P}_{\text{disc}}$  has a piece-wise constant distribution  $F_{\text{disc}} = F(x)$ . The function  $F$  has jumps at the points  $x_1, x_2, \dots$ , that is  $\Delta F(x_i) > 0$ , where  $\Delta F(x) = F(x) - F(x^-)$ .

**Figure 1:** CDF of a discrete measure.

A discrete measure is concentrated at the points  $x_1, x_2, \dots$ , known as atoms. Letting

$$p_k := \mathbb{P}(\{x_k\}) = \Delta F(x_k) > 0,$$

it follows by properties of a distribution function that

$$\sum_{k=1}^{\infty} p_k = 1.$$

In particular we have

$$F_{\text{disc}}(x) = \sum_{x_k \leq x} p_k,$$

and for  $A \subseteq \mathbb{N}$  we have

$$\mathbb{P}(A) = \sum_{k \in A} p_k.$$

We refer to  $(p_1, p_2, \dots)$  as the discrete probability distribution.

### Example 1.4.1

- The Discrete Uniform distribution has  $p_k = \frac{1}{N}$  for  $k = 1, \dots, N$  with  $N$  fixed.
- The Bernoulli  $B(1, p)$  has  $p_1 = p$  and  $p_2 = 1 - p$  where  $0 \leq p \leq 1$ .
- The Binomial  $B(n, p)$  has  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$  for  $k = 0, 1, \dots, n$  and  $0 \leq p \leq 1$ .
- The Poisson  $\text{Po}(\lambda)$  has  $p_k = e^{-\lambda} \frac{\lambda^k}{k!}$  for  $\lambda > 0$  and  $k = 0, 1, \dots$ .

### 1.4.2 Absolutely Continuous Measures

**Proposition 1.4.2** Let  $f$  be an integrable function  $f(x) \geq 0$  such that

$$F(x) = F_{\text{ac}}(x) = \int_{-\infty}^x f(t) dt$$

with respect to the Lebesgue measure. Then the set function  $\mathbb{P}_{\text{ac}}(A) = \int_A f(t) dt$  for  $A \in \mathcal{F}$  is a measure. In particular,  $f$  is a density of  $\mathbb{P}_{\text{ac}}$ .

*Proof.* First define the measure on half-open intervals as  $\mathbb{P}_{\text{ac}}((a, b]) = \int_a^b f(t) dt$ , then use Theorem 1.3.5 to extend the measure to the  $\sigma$ -algebra. ■

The measure  $\mathbb{P}_{\text{ac}}$  is absolutely continuous with respect to the Lebesgue measure  $\mu$ . In the sense that if  $\mu(A) = 0$  then  $\mathbb{P}_{\text{ac}}(A) = 0$ .

**Theorem 1.4.3 — Radon-Nikodym.** If  $\mathbb{P}$  is a measure such that  $\mu(A) = 0$  implies  $\mathbb{P}(A) = 0$ , then  $\mathbb{P}$  has a density.

Note that there is a connection between the absolute continuity of measures and the absolute continuity of functions. If  $\mathbb{P}$  is an absolutely continuous measure then  $F_{ac}(x)$  is an absolutely continuous function, and  $F'_{ac}(x) = f(x)$  almost everywhere.

#### Example 1.4.4

- The Uniform distribution on  $[a, b]$  has density

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

- The Normal or Gaussian distribution has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)$$

for  $x, m \in \mathbb{R}$  and  $\sigma > 0$ .

- The Gamma distribution has

$$f(x) = \frac{x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)}{\Gamma(\alpha)\beta^\alpha}$$

for  $x \geq 0$  and  $\alpha, \beta > 0$ .

### 1.4.3 Singular Continuous Measures

**Definition 1.4.5** A measure  $\nu$  is said to be concentrated on a measurable set  $A$  if  $\nu(E) = 0$  for any  $E \subseteq \mathbb{R} \setminus A$ .

Singular continuous measures are those whose distribution functions are continuous but have all their points of increase on sets of zero Lebesgue measure. We have that  $F(x) = F_{sc}(x)$  is continuous at any  $x$  and  $\mathbb{P}_{sc}$  is concentrated on a set of Lebesgue measure zero. In particular, the distribution has no atoms. For  $x$  in this set,  $F'_{sc}(x) \neq 0$  or does not exist. Thus  $F'_{sc}(x) = 0$  almost everywhere and by continuity we have that  $\mathbb{P}_{sc}\{x\} = 0$  for each point  $x \in \mathbb{R}$ .

**Example 1.4.6** Consider the interval  $[0, 1]$  and construct  $F(x)$  by the following procedure originated by Cantor.

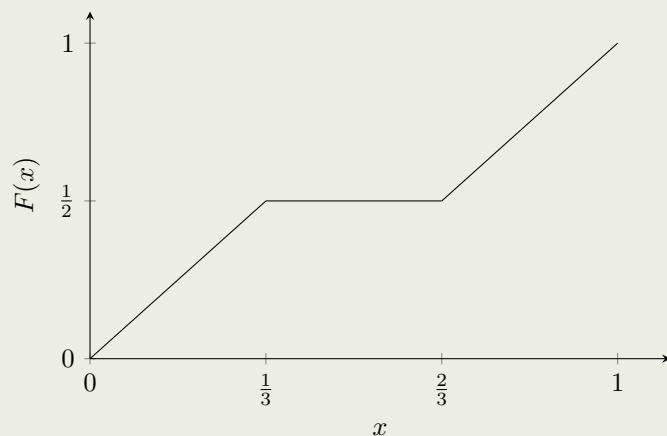
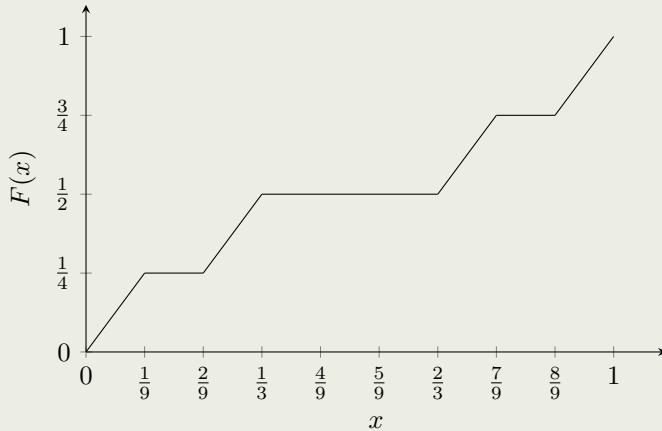


Figure 2: First step of constructing the devil staircase.

Divide  $[0, 1]$  into thirds and let

$$F_2(x) = \begin{cases} \frac{1}{2} & x \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \frac{1}{4} & x \in \left(\frac{1}{9}, \frac{2}{9}\right) \\ \frac{3}{4} & x \in \left(\frac{7}{9}, \frac{8}{9}\right) \\ 0 & x = 0 \\ 1 & x = 1, \end{cases}$$

defining it in the intermediate intervals by linear interpolation. Then divide each of the intervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$  into three parts and construct the function shown below.



**Figure 3:** Second step of constructing the devil staircase.

Continuing with this process, we construct a sequence of functions  $F_n(x)$  for  $n = 1, 2, \dots$  which converges to a non-decreasing continuous function  $F(x)$  (the Cantor function), whose points of increase form a set of Lebesgue measure zero. From the construction of  $F(x)$  we see that the total length of the intervals  $\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right), \dots$  on which the function is constant is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots = 1.$$

Let  $N$  be the set of points of increase of the Cantor function  $F(x)$ . It follows from the sum above that  $\text{Leb}(N) = 0$ . At the same time, if  $\mu$  is the measure corresponding to the Cantor function  $F(x)$ , we have  $\mu(N) = 1$ . This measure is therefore singular with respect to the Lebesgue measure  $\text{Leb}$ .

**Theorem 1.4.7 — Hahn decomposition.** Any probability distribution has a representation of the form

$$F(x) = a_1 F_{\text{disc}}(x) + a_2 F_{\text{ac}}(x) + a_3 F_{\text{sc}}(x)$$

for  $a_1 + a_2 + a_3 = 1$ .

## 1.5 Measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

Distribution functions on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  are defined similarly. For example, when  $n = 2$  we have

$$F(x, y) = \mathbb{P}((-\infty, x] \times (-\infty, y]).$$

For probability spaces  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ ,  $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$  the product measure on  $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$  is defined as follows.

1. Set

$$\mathbb{P}_0(A_1 \times A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2)$$

for  $A_1 \in \mathcal{F}_1$  and  $A_2 \in \mathcal{F}_2$ .

2. Then extend  $\mathbb{P}_0$  to the algebra generated by  $A_1 \times A_2$ , and show that  $\mathbb{P}_0$  is a  $\sigma$ -additive measure on this algebra.

3. Apply Theorem 1.3.5 to obtain the extension.

This extension is the product measure and is denoted  $\mathbb{P}_1 \otimes \mathbb{P}_2$ .

## 1.6 Measures on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$

On  $\mathbb{R}^n$  for  $n \geq 1$ , probability measures were constructed in the following way.

1. It was first defined for elementary sets of the form  $(a, b]$ .
2. The definition was then extended to sets of the form  $A = \sum_{i=1}^n (a_i, b_i]$ .
3. The extension to sets in  $\mathcal{B}(\mathbb{R}^n)$  was provided by Theorem 1.3.5.

A similar procedure of constructing probability measures also works for the space  $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ . Let

$$\mathcal{I}_n(B) = \{x \in \mathbb{R}^\infty : (x_1, \dots, x_n) \in B\}$$

denote the cylinder set in  $\mathbb{R}^\infty$  with base  $B \in \mathcal{B}(\mathbb{R}^n)$ . It is natural to take the cylinder sets as the elementary sets in  $\mathbb{R}^\infty$  whose probabilities enable us to determine the probability measure on the sets of  $\mathcal{B}(\mathbb{R}^\infty)$ .

**Definition 1.6.1** A sequence  $(\mathbb{P}_n)_{n \in \mathbb{N}}$  of probability measures, where  $\mathbb{P}_n$  is a probability measure on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ , is consistent if for all  $n = 1, 2, \dots$  and  $B \in \mathcal{B}(\mathbb{R}^n)$  we have

$$\mathbb{P}_{n+1}(B \times \mathbb{R}) = \mathbb{P}_n(B).$$

**Theorem 1.6.2 — Kolmogorov Extension Theorem.** For any consistent sequence  $(\mathbb{P}_n)_{n \in \mathbb{N}}$ , there exists a unique probability measure  $\mathbb{P}$  on  $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$  such that

$$\mathbb{P}(\mathcal{I}_n(B)) = \mathbb{P}_n(B),$$

for  $B \in \mathcal{B}(\mathbb{R}^n)$  and  $n = 1, 2, \dots$

## 1.7 Random Variables

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.

**Definition 1.7.1** A real function  $\xi : \Omega \rightarrow \mathbb{R}$  is an  $\mathcal{F}$ -measurable function, or a random variable if

$$\{\omega : \xi(\omega) \in B\} \in \mathcal{F}$$

for every  $B \in \mathcal{B}(\mathbb{R})$ . Equivalently,

$$\xi^{-1}(B) = \{\omega : \xi(\omega) \in B\}$$

is a measurable set in  $\Omega$ . When  $(\Omega, \mathcal{F}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ , the  $\mathcal{B}(\mathbb{R}^n)$ -measurable functions are called Borel functions.

Random variables are used to summarise the abstract outcomes  $\omega \in \Omega$  with a real number or vector.

**Example 1.7.2** Consider the experiment of throwing two independent fair six-faced dice. This can be represented by the probability space  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \otimes (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ , where for  $i = 1, 2$ ,  $\Omega_i = \{1, 2, 3, 4, 5, 6\}$  is the outcome from dice  $i$ ,  $\mathcal{F}_i = 2^{\Omega_i}$  and  $\mathbb{P}_i(\{j\}) \equiv \frac{1}{6}$  for all  $j \in \{1, 2, 3, 4, 5, 6\}$ . We can then consider the function  $X : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$  given by

$$X(\omega_1, \omega_2) = \omega_1 + \omega_2$$

which summarises the outcome of dice by their sum. One can check that  $X$  is a measurable function by looking at the possible pre-images of  $X$ .

**Exercise 1.7.3** In the context of Example 1.7.2 answer the following.

- What are the possible outcomes of  $X$ ?
- What are the possible pre-images of  $X$ ?

**Lemma 1.7.4** Let  $\mathcal{D}$  be a collection of subsets on  $\mathbb{R}$  such that  $\sigma(\mathcal{D}) = \mathcal{B}(\mathbb{R})$ . A necessary and sufficient condition that a function  $\xi = \xi(\omega)$  is a random variable is that

$$\xi^{-1}(D) = \{\omega : \xi(\omega) \in D\} \in \mathcal{F}$$

for all  $D \in \mathcal{D}$ .

*Proof.* Let

$$\mathcal{G} = \{B : \xi^{-1}(B) \in \mathcal{F}\} \subseteq \mathcal{B}(\mathbb{R}).$$

Clearly,  $\emptyset \in \mathcal{G}$  as  $\xi^{-1}(\emptyset) = \emptyset \in \mathcal{F}$ . For  $B \in \mathcal{G}$  it follows that

$$\xi^{-1}(B^c) = (\xi^{-1}(B))^c \in \mathcal{F}$$

and so  $B^c \in \mathcal{G}$ . Moreover, for  $B_1, B_2, \dots \in \mathcal{G}$  we have

$$\xi^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} \xi^{-1}(B_i) \in \mathcal{F}$$

and so  $\bigcup_{i=1}^{\infty} B_i \in \mathcal{G}$  which implies that  $\mathcal{G}$  is a  $\sigma$ -algebra. Note that  $\mathcal{G} \subseteq \mathcal{B}(\mathbb{R})$  with  $\sigma(\mathcal{D}) = \mathcal{B}(\mathbb{R})$  so that

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}) \subseteq \sigma(\mathcal{G}) = \mathcal{G} \subseteq \mathcal{B}(\mathbb{R}),$$

therefore,  $\mathcal{G} = \mathcal{B}(\mathbb{R})$  meaning  $\xi$  is a random variable. Conversely, if  $\xi$  is a random variable then  $\mathcal{G} = \mathcal{B}(\mathbb{R})$ , which implies that  $\mathcal{D} \subseteq \mathcal{G}$ . Hence,  $\xi^{-1}(D) \in \mathcal{F}$  for all  $D \in \mathcal{D}$ . ■

**Corollary 1.7.5** A necessary and sufficient condition for  $\xi = \xi(\omega)$  to be a random variable is that

$$\{\omega : \xi(\omega) < x\} \in \mathcal{F}$$

for every  $x \in \mathbb{R}$ , or that

$$\{\omega : \xi(\omega) \leq x\} \in \mathcal{F}$$

for every  $x \in \mathbb{R}$ .

*Proof.* These follow from Lemma 1.7.4 as the collections

$$\mathcal{D}_1 := \{(-\infty, x) : x \in \mathbb{R}\}$$

and

$$\mathcal{D}_2 := \{(-\infty, x] : x \in \mathbb{R}\}$$

are such that  $\sigma(\mathcal{D}_1) = \mathcal{B}(\mathbb{R})$  and  $\sigma(\mathcal{D}_2) = \mathcal{B}(\mathbb{R})$ . ■

**Exercise 1.7.6** Let  $\xi$  be a random variable. Show that

$$\mathcal{F}_{\xi} := \{\xi^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\} \subseteq \mathcal{F}$$

is a  $\sigma$ -algebra.

**Lemma 1.7.7** Let  $\varphi = \varphi(x)$  be a Borel function and  $\xi = \xi(\omega)$  a random variable. Then the composition  $\eta = \varphi \circ \xi$  is also a random variable. In particular,  $\eta$  is  $\mathcal{F}_\xi$ -measurable.

*Proof.* Note that for  $B \in \mathcal{B}(\mathbb{R})$  we have

$$\{w : \eta(\omega) \in B\} = \{w : \varphi(\xi(\omega)) \in B\} = \{w : \xi(\omega) \in \varphi^{-1}(B)\} \in \mathcal{F}_\xi \subseteq \mathcal{F},$$

where the membership to  $\mathcal{F}_\xi$  follows by the fact that  $\varphi^{-1}(B) \in \mathcal{B}(\mathbb{R})$ . Thus we conclude that  $\eta$  is  $\mathcal{F}_\xi$ -measurable, and so also  $\mathcal{F}$ -measurable, meaning it is a random variable. ■

### Example 1.7.8

1. If  $\xi$  is a random variable and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function then  $f(\xi)$  is a random variable.
2. Let  $\xi$  be a random variable.
  - (a) Composing with the continuous function  $x \mapsto x^n$ , for  $n \in \mathbb{N}$ , we deduce that  $\xi^n$  is a random variable.
  - (b) As the function  $x \mapsto \max(x, 0)$  is continuous we deduce that  $\xi^+ := \max(\xi, 0)$  is a random variable.
  - (c) Similarly,  $\xi^- := \max(-\xi, 0)$  is a random variable.
  - (d) As the function  $x \mapsto |x|$  is continuous we deduce that  $|\xi|$  is a random variable.

**Lemma 1.7.9** Let  $\xi$  and  $\eta$  be random variables. Then the following are also random variables.

1.  $\xi + \eta$ .
2.  $\xi - \eta$ .
3.  $\xi\eta$ .
4.  $\frac{\xi}{\eta}$ .
5.  $\max(\xi, \eta)$
6.  $\min(\xi, \eta)$ .

Where we are assuming that all operations are well-defined.

*Proof.*

1. Let  $x \in \mathbb{R}$ . Suppose  $r, s \in \mathbb{Q}$  are such that  $r + s < x$ . If  $\omega \in \Omega$  is such that  $\xi(\omega) < r$  and  $\eta(\omega) < s$  it is clear that  $\xi(\omega) + \eta(\omega) < x$ . On the other hand, if  $\xi(\omega) + \eta(\omega) < x$ , then there exists some  $q \in \mathbb{Q}$  such that  $\xi(\omega) + \eta(\omega) < q < x$ . In particular, there exists  $r, s \in \mathbb{Q}$  such that  $r + s = q$  with the property that  $\xi(\omega) < r$  and  $\eta(\omega) < s$ . Therefore,

$$\{\xi + \eta < x\} = \bigcup_{r+s < x, (r,s) \in \mathbb{Q}^2} \{\xi < r\} \cap \{\eta < s\}.$$

As  $\mathbb{Q}^2$  is countable and  $\xi$  and  $\eta$  are random variables, it follows that  $\{\xi + \eta < x\} \in \mathcal{F}$ . Thus we can conclude that  $\xi + \eta$  is a random variable using Corollary 1.7.5.

2. This follows from 1. by noting that  $-\eta$  is still a random variable.

3. Note that

$$\xi\eta = \frac{1}{2} \left( (\xi - \eta)^2 - \xi^2 - \eta^2 \right).$$

Thus we can conclude that  $\xi\eta$  is a random variable as we have shown previously that all the terms on the right-hand side are random variables.

4. This follows from statement 3. by noting that  $\frac{1}{\eta}$  is a random variable, provided it is well-defined.

5. Note that

$$\max(\xi, \eta) = \frac{1}{2} (|\xi - \eta| + \xi + \eta).$$

We can show that the right-hand side is a random variable, implying that  $\max(\xi, \eta)$  is a random variable.

6. Noting that  $\min(\xi, \eta) = \max(-\xi, -\eta)$  we can deduce that  $\min(\xi, \eta)$  is a random variable by using statement 5. ■

**Example 1.7.10** An alternative way of arriving to statement 2(d) of Example 1.7.8 is by using statement 1. from Lemma 1.7.9 with statements 2(b) and 2(c) from Example 1.7.8.

**Lemma 1.7.11** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of random variables. Provided they exist, the following are also random variables.

1.  $\sup_n(\xi_n)$ .
2.  $\inf_n(\xi_n)$ .
3.  $\lim_n(\xi_n)$ .

*Proof.*

1. Note that for  $x \in \mathbb{R}$  we have

$$\{\sup_n(\xi_n) > x\} = \bigcup_{n=1}^{\infty} \{\xi_n > x\}.$$

The right-hand side is a countable union of sets in  $\mathcal{F}$ , and so  $\{\sup_n(\xi_n) > x\} \in \mathcal{F}$ . Therefore,  $\sup_n(\xi_n)$  is a random variable by Corollary 1.7.5.

2. As  $\inf_n(\xi_n) = -\sup_n(-\xi_n)$  by statement 1. we know that  $\inf_n(\xi_n)$  is also a random variable.
3. If  $\lim_n(\xi_n)$  exists we note that  $\liminf_n(\xi_n) = \lim_n(\xi_n)$ . As  $\liminf_n = \sup_n \inf_{k \geq n}(\xi_n)$  we can use statements 1. and 2. to deduce that  $\lim_n(\xi_n)$  is a random variable. ■

**Definition 1.7.12** A random variable  $\xi$  is called simple if

$$\xi(\omega) = \sum_{j=1}^n x_j \chi_{D_j}(\omega)$$

for some  $n \geq 1$ , with  $D_1, \dots, D_n$  being a partition of  $\Omega$  consisting of measurable sets and

$$\chi_D(\omega) = \begin{cases} 1, & \omega \in D \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 1.7.13**

- For every random variable  $\xi = \xi(\omega)$  there is a sequence of simple random variables  $\xi_1, \xi_2, \dots$ , such that  $|\xi_n| \leq |\xi|$  and  $\xi_n(\omega) \rightarrow \xi(\omega)$  as  $n \rightarrow \infty$ , for all  $\omega \in \Omega$ .
- For any random variable  $\xi(\omega) \geq 0$  there exists a pointwise non-decreasing sequence of simple random variables  $\xi_1(\omega) \leq \xi_2(\omega) \leq \dots \leq \xi(\omega)$  such that

$$\lim_{n \rightarrow \infty} \xi_n(\omega) = \xi(\omega)$$

for all  $\omega \in \Omega$ . Such a sequence is usually denoted  $\xi_n \nearrow \xi$ .

*Proof.* We begin by proving the second statement. For  $n = 1, 2, \dots$ , let

$$\xi_n(\omega) = \sum_{j=0}^{n2^n-1} \frac{j}{2^n} \chi_{\{\omega: \frac{j}{2^n} \leq \xi(\omega) < \frac{j+1}{2^n}\}} + n \chi_{\{\omega: \xi(\omega) \geq n\}}$$

One can verify that the sequence  $\xi_n(\omega)$  is such that  $\xi_n \nearrow \xi$  for all  $\omega \in \Omega$ . The first statement follows from this if we merely observe that  $\xi$  can be represented in the form  $\xi = \xi^+ - \xi^-$ , where  $\xi^+ = \max(\xi, 0)$  and  $\xi^- = \max(-\xi, 0)$ . ■

One can explicitly build a simple function approximation to the function  $f(x) = x^2$  for all  $x \in \mathbb{R}$ .

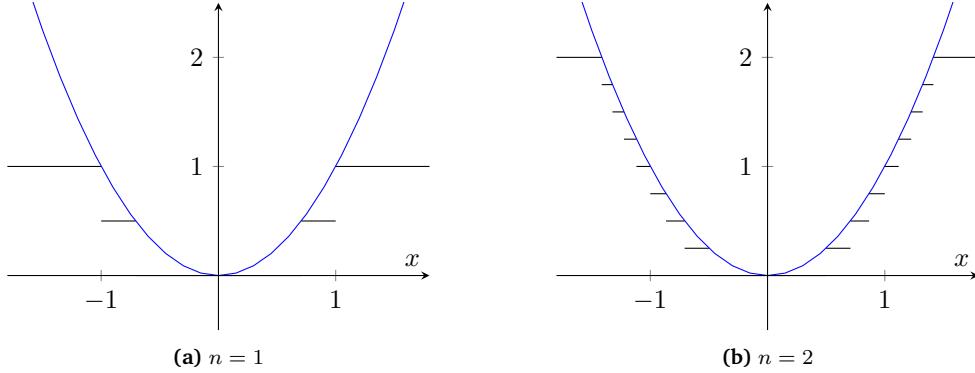


Figure 4: Approximation of  $x^2$ .

More generally, to show results relating to random variables one can follow the four-step proof.

1. First prove the statement to indicator functions.
2. Extend the statement for simple random variables by considering linearity.
3. Extend the statement to non-negative random variables by taking limits.
4. Extend the statement to arbitrary random variables by considering their positive and negative parts.

**Lemma 1.7.14** Consider a measurable space  $(\Omega, \mathcal{F})$  and a disjoint decomposition  $\mathcal{D} = \{D_1, D_2, \dots\}$  of  $\Omega$ . Let  $\xi = \xi(\omega)$  be a  $\sigma(\mathcal{D})$ -measurable random variable. Then  $\xi$  is representable in the form

$$\xi(\omega) = \sum_{k=1}^{\infty} \alpha_k \chi_{D_k}(\omega),$$

where  $\alpha_k \in \mathbb{R}$ . That is,  $\xi(\omega)$  is constant on the elements  $D_k$  of the decomposition.

*Proof.* Suppose that  $\xi = \chi_D$  for  $D \in \sigma(\mathcal{D})$ . Then using Exercise 1.1.11 we know that  $D = \bigcup_{k \in I} D_k$  for some  $I \subseteq \mathbb{N}$ . Therefore,

$$\xi = \sum_{k=1}^{\infty} \alpha_k \chi_{D_k}$$

with

$$\alpha_k = \begin{cases} 1 & k \in I \\ 0 & \text{otherwise.} \end{cases}$$

This is readily extended to  $\xi$  a simple function. When  $\xi$  is a non-negative random variable we know that there exists a sequence of simple functions  $(\xi_n)_{n \in \mathbb{N}}$  such that  $\xi_n \nearrow \xi$ . Note that, as  $\xi$  is a random variable, we know its image does not include  $\infty$ . Letting

$$\xi_n = \sum_{k=1}^{\infty} \alpha_k^{(n)} \chi_{D_k}(\omega)$$

it follows that  $\alpha_k := \lim_{n \rightarrow \infty} \alpha_k^{(n)}$  is well-defined. As each  $\xi_n$  is a non-negative random variable, we can exchange the limit and summation to deduce that

$$\begin{aligned}\xi &= \lim_{n \rightarrow \infty} \xi_n \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{\infty} \alpha_k^{(n)} \chi_{D_k} \right) \\ &= \sum_{k=1}^{\infty} \left( \lim_{n \rightarrow \infty} \alpha_k^{(n)} \right) \chi_{D_k} \\ &= \sum_{k=1}^{\infty} \alpha_k \chi_{D_k}.\end{aligned}$$

For  $\xi$  a general random variable, we can extend the previous observation by using the decomposition  $\xi = \xi^+ - \xi^-$ . ■

## 1.8 Distributions of random variables

**Definition 1.8.1** A probability measure  $\mathbb{P}_\xi$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  with

$$\mathbb{P}_\xi(B) = \mathbb{P}(\{\omega : \xi(\omega) \in B\}),$$

for  $B \in \mathcal{B}(\mathbb{R})$  is called the probability distribution of  $\xi$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

**Definition 1.8.2** The function

$$F_\xi(x) \equiv \mathbb{P}_\xi((-\infty, x]) = \mathbb{P}(\{\omega : \xi(\omega) \leq x\}),$$

for  $x \in \mathbb{R}$  is called the distribution function of  $\xi$ .

**Example 1.8.3** From Example 1.7.2, one can verify that the probability distribution of  $X$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  satisfies

$$\mathbb{P}_X(\{j\}) = \frac{6 - |7 - j|}{36},$$

for  $j \in \{2, 3, \dots, 12\}$ . Using this, we can extend  $\mathbb{P}_X$  to other sets in  $\mathcal{B}(\mathbb{R})$ .

Notice that there are multiple random variables, on potentially different probability spaces, which give the same distribution function. Indeed, we can always construct a probability measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  given a distribution function  $F_\xi$ . Therefore for any random variable  $\xi$  on a probability  $(\Omega, \mathcal{F}, \mathbb{P})$ , the identity random variable  $I(\omega) = \omega$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_\xi)$  has the same distribution as  $\xi$ . For example, consider the space  $(\Omega, 2^\Omega, Q)$ , with  $\Omega = \{2, 3, \dots, 12\}$  and  $Q(\{j\}) = \mathbb{P}_X(\{j\})$ . Then the random variable  $\xi(j) = j$  for all  $j \in \Omega \subseteq \mathbb{R}$  is such that  $Q_\xi(A) \equiv \mathbb{P}_X(A)$  for all  $A \in \mathcal{B}(\mathbb{R})$ .

**Definition 1.8.4** The vector function

$$\xi = (\xi_1, \xi_2, \dots, \xi_n) : \Omega \rightarrow \mathbb{R}^n$$

is a random vector if for any  $B \subseteq \mathcal{B}(\mathbb{R}^n)$ , we have  $\xi^{-1}(B) \in \mathcal{F}$ . As before, we construct  $\mathbb{P}_\xi$  and say that  $\mathbb{P}_\xi = \mathbb{P}_{(\xi_1, \dots, \xi_n)}$  is the joint distribution of  $\xi_1, \dots, \xi_n$ , given by

$$F_\xi(x_1, \dots, x_n) = \mathbb{P}(\omega : \xi_1 \leq x_1, \dots, \xi_n \leq x_n).$$

**Exercise 1.8.5** Show that the vector  $\xi = (\xi_1, \dots, \xi_n)$  is a random variable if and only if  $\xi_1, \dots, \xi_n$  are random variables.

For  $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$  we can similarly define random sequences  $\xi = (\xi_1, \xi_2, \dots)$ .

## 1.9 Solution to Exercises

### Exercise 1.1.11

*Solution.* Let  $\mathcal{C} = \{\bigcup_{i \in I} D_i : I \subseteq \mathbb{N}\}$ .

- As  $\Omega = \bigcup_{i \in \mathbb{N}} D_i$  we have that  $\Omega \in \mathcal{C}$ .
- For  $A \in \mathcal{C}$ , there exists an index set  $I \subseteq \mathbb{N}$  such that  $A = \bigcup_{i \in I} D_i$ . Let  $I' = \mathbb{N} \setminus I$  and define  $B = \bigcup_{i \in I'} D_i \in \mathcal{C}$ . As  $\mathcal{D}$  forms a partition, it is clear that  $B = A^c$  and so  $A^c \in \mathcal{C}$ .
- Let  $A_1, A_2, \dots \in \mathcal{C}$ . Then each  $A_i$  is a countable union of elements from  $\mathcal{D}$ . Therefore,  $A = \bigcup_{i=1}^{\infty} A_i$  is also a countable union of elements from  $\mathcal{D}$  and so  $A \in \mathcal{C}$ .

The above show that  $\mathcal{C}$  is a  $\sigma$ -algebra. ■

### Exercise 1.2.2

*Solution.*

1. As  $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$  it follows that the open intervals also generate  $\mathcal{B}(\mathbb{R})$ .
2. As  $(a, b] = \bigcap_{n=1}^{\infty} [a + \frac{1}{n}, b]$  it follows that the closed intervals also generate  $\mathcal{B}(\mathbb{R})$ .
3. As  $(a, b] = \bigcap_{n=1}^{\infty} [a + \frac{1}{n}, b + \frac{1}{n}]$  it follows that the right-open half intervals also generate  $\mathcal{B}(\mathbb{R})$ .
4. As  $(a, b] = (-\infty, a]^c \cap (-\infty, b]$  it follows that intervals of the form  $(-\infty, a]$  or  $[a, \infty)$  also generate  $\mathcal{B}(\mathbb{R})$ .
5. As  $(a, b)$  is open it follows from (1) that open sets also generate  $\mathcal{B}(\mathbb{R})$ .
6. As  $[a, b]$  are closed sets it follows from (2) that closed sets also generate  $\mathcal{B}(\mathbb{R})$ . ■

### Exercise 1.2.6

*Solution.* The set  $\{x \in \mathbb{R}^{\infty} : (x_1) \in \mathbb{R}\} = \mathbb{R}^{\infty}$ . Similarly, the set  $\{x \in \mathbb{R}^{\infty} : (x_1) \in \emptyset\} = \emptyset$ . Let  $C = \{x \in \mathbb{R}^{\infty} : (x_1, \dots, x_n) \in \tilde{C}_n\}$  with  $\tilde{C}_n \in \mathcal{B}(\mathbb{R}^n)$ . Then as  $(\tilde{C}_n)^c \in \mathcal{B}(\mathbb{R}^n)$  it follows that  $C^c = \{x \in \mathbb{R}^{\infty} : (x_1, \dots, x_n) \in (\tilde{C}_n)^c\}$  is a cylindrical set. Now consider the cylindrical sets  $C_1, \dots, C_m$  where  $C_i = \{x \in \mathbb{R}^{\infty} : (x_1, \dots, x_{n_i}) \in \tilde{C}_{n_i}\}$  for  $\tilde{C}_{n_i} \in \mathcal{B}(\mathbb{R}^{n_i})$ . Let  $n = \max_{i=1, \dots, m}(n_i)$ , then for each  $\tilde{C}_{n_i}$  we can define  $\tilde{D}_n = \tilde{C}_{n_i} \cup \mathbb{R}^{n-n_i} \in \mathcal{B}(\mathbb{R}^n)$ . We note that

$$C_i = \left\{ x \in \mathbb{R}^{\infty} : (x_1, \dots, x_{n_i}, \dots, x_m) \in \tilde{D}_n \right\}.$$

Moreover,

$$\bigcup_{i=1}^m C_i = \left\{ x \in \mathbb{R}^{\infty} : (x_1, \dots, x_m) \in \bigcup_{k=1}^m \tilde{D}_{n_k} \right\}$$

where  $\tilde{D}_{n_k} \in \mathcal{B}(\mathbb{R}^m)$  and so  $\bigcup_{i=1}^m C_i$  is a cylindrical set. Therefore, the cylindrical sets form an algebra. ■

### Exercise 1.2.8

*Solution.* If  $x \in D$  it follows that for all  $\epsilon > 0$  there exists an  $N$  such that for  $n \geq N$  we have that  $|x_n - c| < \epsilon$ . Therefore,

$$D = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ x \in \mathbb{R}^{\infty} : |x_n - c| < \frac{1}{k} \right\}.$$

The set  $\bigcap_{n=N}^{\infty} \{x \in \mathbb{R}^{\infty} : |x_n - c| < \frac{1}{k}\}$  is equivalent to saying that  $x_n \in (c - \frac{1}{k}, c + \frac{1}{k})$  for all  $n \geq N$  and  $x_m$  can be any real number for  $1 \leq m < N$ . Therefore, this is in  $\mathcal{B}(\mathbb{R}^{\infty})$  meaning that  $D \in \mathcal{B}(\mathbb{R}^{\infty})$ . ■

### Exercise 1.3.2

*Solution.*

1. Let  $x \leq y$ . Then  $(-\infty, x] \subseteq (-\infty, y]$  so that by properties of the probability measure it follows that

$$\mathbb{P}((-\infty, x]) \leq \mathbb{P}((-\infty, y])$$

which implies that  $F(x) \leq F(y)$ .

2. Let  $(x_n)$  be a sequence such that  $x_{n+1} \leq x_n$  and  $x_n \rightarrow -\infty$  as  $n \rightarrow \infty$ . Then the sets  $A_n = (-\infty, x_n]$  are a decreasing sequence of events. Therefore,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \lim_{n \rightarrow \infty} F(x_n).$$

Note that  $\bigcap_{n=1}^{\infty} A_n = \emptyset$  so that  $\lim_{n \rightarrow \infty} F(x_n) = 0$ . We can conclude that  $\lim_{x \rightarrow -\infty} F(x) = 0$ . Similarly, let  $(x_n)$  be a sequence such that  $x_n \leq x_{n+1}$  and  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then the sets  $A_n = (x_n, \infty)$  are a sequence of decreasing events. Therefore, as above

$$0 = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n).$$

As  $\mathbb{P}(A_n^c) = F(x_n)$ , it follows that  $1 = \lim_{n \rightarrow \infty} F(x_n)$  from which we conclude that  $\lim_{x \rightarrow \infty} F(x) = 1$ .

3. Let  $x \in \mathbb{R}$  and  $x_n \searrow x$  monotonically. Then the sets  $A_n = (-\infty, x_n]$  are a sequence of decreasing events with the property that  $\bigcap_{n=1}^{\infty} A_n = (-\infty, x]$ . Therefore,

$$F(x) = \mathbb{P}((-\infty, x]) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} F(x_n).$$

More generally we see that  $F(x) = \lim_{y \searrow x} F(y)$  and so  $F(x)$  is right-continuous. ■

### Exercise 1.7.3

*Solution.* The possible outcomes of  $X$  are  $X(\omega_1, \omega_2) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ . The possible pre-images are summarised in Table 1.

Image	Pre-Image
2	$\{\{1, 1\}\}$
3	$\{\{1, 2\}, \{2, 1\}\}$
4	$\{\{1, 3\}, \{2, 2\}, \{3, 1\}\}$
5	$\{\{1, 4\}, \{2, 3\}, \{3, 2\}, \{4, 1\}\}$
6	$\{\{1, 5\}, \{2, 4\}, \{3, 3\}, \{4, 2\}, \{5, 1\}\}$
7	$\{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{4, 3\}, \{5, 2\}, \{6, 1\}\}$
8	$\{\{2, 6\}, \{3, 5\}, \{4, 4\}, \{5, 3\}, \{6, 2\}\}$
9	$\{\{3, 6\}, \{4, 5\}, \{5, 4\}, \{6, 3\}\}$
10	$\{\{4, 6\}, \{5, 5\}, \{6, 4\}\}$
11	$\{\{5, 6\}, \{6, 5\}\}$
12	$\{\{6, 6\}\}$

Table 1: Pre-Images of  $X$

### Exercise 1.7.6

*Solution.* As  $\xi^{-1}(\emptyset) = \emptyset$  it follows that  $\emptyset \in \mathcal{F}_\xi$ . Let  $A \in \mathcal{F}_\xi$  such that  $A = \xi^{-1}(B)$  for some  $B \in \mathcal{B}(\mathbb{R})$ . Then  $A^c = (\xi^{-1}(B))^c = \xi^{-1}(B^c)$ . As  $B^c \in \mathcal{F}$  it follows that  $A^c \in \mathcal{F}_\xi$ . Similarly, let  $A_1, A_2, \dots \in \mathcal{F}_\xi$  with  $A_i = \xi^{-1}(B_i)$  for some  $B_i \in \mathcal{B}(\mathbb{R})$ . Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \xi^{-1}(B_i) = \xi^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right).$$

As  $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}(\mathbb{R})$  it follows that  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\xi$ . We conclude that  $\mathcal{F}_\xi$  is a  $\sigma$ -algebra.  $\blacksquare$

### Exercise 1.8.5

*Solution.* Let  $\xi = (\xi_1, \dots, \xi_n) : \Omega \rightarrow \mathbb{R}^n$  be a random vector, where each  $\xi_i : \Omega \rightarrow \mathbb{R}$ . Without loss of generality consider  $\xi_1$  and  $B \in \mathcal{B}(\mathbb{R})$ . Then,

$$\begin{aligned}\xi_1^{-1}(B) &= \{\omega : \xi_1(\omega) \in B\} \\ &= \{\omega : \xi(\omega) \in B \times \mathbb{R} \times \dots \times \mathbb{R}\}.\end{aligned}$$

As  $B \times \mathbb{R} \times \dots \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^n)$  and  $\xi$  is a random variable we conclude that  $\xi_1^{-1}(B) \in \mathcal{F}$  and hence a random variable. Conversely, if each of the  $\xi_i$  are random variables. Then for  $B \in \mathcal{B}(\mathbb{R}^n)$ , we can write  $B = B_1 \times \dots \times B_n$  for  $B_i \in \mathcal{B}(\mathbb{R})$ . Therefore,

$$\xi^{-1}(B) = \{\omega : \xi_1(\omega) \in B_1, \dots, \xi_n(\omega) \in B_n\} = \bigcap_{i=1}^n \xi_i^{-1}(B_i).$$

Then as each  $\xi_i$  is a random variable we have that  $\xi_i^{-1}(B_i) \in \mathcal{F}$ . Which implies that  $\xi^{-1}(B) \in \mathcal{F}$  and hence a random variable.  $\blacksquare$

## 2 Expectation and Integrals

### 2.1 The Lebesgue Integral

The expectation of a simple random variable  $\xi = \sum_{j=1}^n x_j \chi_{D_j}$  as

$$\mathbb{E}(\xi) = \sum_{j=1}^n x_j \mathbb{P}(D_j),$$

where the sets  $D_j$  form a partition of  $\Omega$ . For an arbitrary non-negative random variable  $\xi = \xi(\omega)$  we can construct a sequence of simple non-negative random variables  $\{\xi_n\}_{n \geq 1}$  such that  $\xi_n(\omega) \nearrow \xi(\omega)$ , as  $n \rightarrow \infty$  for each  $\omega \in \Omega$ . We then set  $\mathbb{E}(\xi) = \lim_{n \rightarrow \infty} \mathbb{E}(\xi_n)$ , which exists since  $\mathbb{E}(\xi_n) \leq \mathbb{E}(\xi_{n+1})$  (possibly taking the value  $+\infty$ ).

**Definition 2.1.1** The expectation  $\mathbb{E}(\xi)$  of a non-negative random variable  $\xi$  is the Lebesgue integral with respect to  $\mathbb{P}$  given by

$$\mathbb{E}(\xi) := \lim_{n \rightarrow \infty} \mathbb{E}(\xi_n) = \int_{\Omega} \xi \, d\mathbb{P} = \int_{\Omega} \xi(\omega) \mathbb{P}(d\omega)$$

which may be infinite.

To see that Definition 2.1.1 is consistent, one has to show it is independent of the choice of  $\xi_n \nearrow \xi$ .

**Definition 2.1.2** An arbitrary random variable  $\xi$  is said to be integrable if  $\mathbb{E}(|\xi|) < \infty$ .

**Remark 2.1.3** An integrable random variable may not be non-negative. However, its expectation is well-defined as  $|\xi|$  is a non-negative random variable.

**Definition 2.1.4** The expectation of an integrable random variable is  $\mathbb{E}(\xi) = \mathbb{E}(\xi^+) - \mathbb{E}(\xi^-)$ .

Definition 2.1.4 is well-defined as  $\mathbb{E}(\xi^-) < \infty$  by assumption.

### 2.2 Properties

**Proposition 2.2.1** Let  $\xi$  and  $\eta$  be integrable random variables and let  $c$  be a constant. Then

1.  $\mathbb{E}(c) = c$ ,
2.  $\mathbb{E}(c\xi) = c\mathbb{E}(\xi)$ ,
3.  $\xi + \eta$  is integrable with  $\mathbb{E}(\xi + \eta) = \mathbb{E}(\xi) + \mathbb{E}(\eta)$ ,
4.  $\xi \leq \eta$  implies that  $\mathbb{E}(\xi) \leq \mathbb{E}(\eta)$ ,
5. if  $\xi = \eta$  almost everywhere with respect to  $\mathbb{P}$ , that is the equality holds up to sets of zero  $\mathbb{P}$ -measure, then  $\mathbb{E}(\xi) = \mathbb{E}(\eta)$ , and
6. if  $\xi \geq 0$  is such that  $\mathbb{E}(\xi) = 0$ , then  $\xi = 0$  almost everywhere.

#### 2.2.1 Exchanging limits and expectations

**Theorem 2.2.2 — Monotone Convergence Theorem.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of random variables such that  $0 \leq \xi_1 \leq \xi_2 \leq \dots$ . Then

$$\lim_{n \rightarrow \infty} \mathbb{E}(\xi_n) = \mathbb{E}\left(\lim_{n \rightarrow \infty} \xi_n\right)$$

exists, potentially being infinite.

**Remark 2.2.3**

- Note that Theorem 2.2.2 also holds for a sequence of random variables  $(\xi_n)_{n \in \mathbb{N}}$  where  $\eta \leq \xi_1 \leq \dots$  for  $\mathbb{E}(\eta) > -\infty$ . We consider  $\xi_n - \eta$  instead of  $\xi_n$  in the original formulation of Theorem 2.2.2.
- Similarly, if  $(\xi_n)_{n \in \mathbb{N}}$  is a sequence of random variables such that  $\dots \leq \xi_2 \leq \xi_1 \leq \eta$ , for  $\mathbb{E}(\eta) < \infty$ . Then we can consider  $\eta - \xi_n$  in the original formulation of Theorem 2.2.2.

**Corollary 2.2.4** Let  $(\xi_n)_{n \geq 1}$  be a sequence of non-negative random variables. Then

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \xi_n\right) = \sum_{n=1}^{\infty} \mathbb{E}(\xi_n).$$

*Proof.* Let  $\xi_n = \sum_{k=1}^n \eta_k$ . Then as each  $\eta_k$  is non-negative, it is clear that  $0 \leq \xi_1 \leq \xi_2 \leq \dots$ . Therefore

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E}(\eta_k) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{E}(\eta_k) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}\left(\sum_{k=1}^n \eta_k\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}(\xi_n) \\ &\stackrel{\text{Thm 2.2.2}}{=} \mathbb{E}\left(\lim_{n \rightarrow \infty} \xi_n\right) \\ &= \mathbb{E}\left(\sum_{k=1}^{\infty} \eta_k\right). \end{aligned}$$

■

**Theorem 2.2.5 — Fatou's Lemma.** Let  $(\xi_n)_{n \geq 1}$  be a sequence of non-negative random variables. Then

$$\mathbb{E}\left(\liminf_n \xi_n\right) \leq \liminf_n \mathbb{E}(\xi_n).$$

*Proof.* Let  $f_n = \inf_{m \geq n} \xi_m$ . Then clearly,  $0 \leq f_n \leq f_{n+1}$  for all  $n$ . Therefore, Theorem 2.2.2 we have that

$$\begin{aligned} \mathbb{E}\left(\liminf_n \xi_n\right) &= \mathbb{E}\left(\lim_{n \rightarrow \infty} f_n\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}(f_n) \\ &= \liminf_n \mathbb{E}(f_n) \\ &\leq \liminf_n \mathbb{E}(\xi_n). \end{aligned}$$

■

**Remark 2.2.6**

- In Theorem 2.2.5,  $\xi_n \geq 0$  can be replaced by  $\xi_n \geq \eta$ , if  $\mathbb{E}(\eta) > -\infty$ .
- In Theorem 2.2.5, if  $\xi_n < \eta$  and  $\mathbb{E}(\eta) < \infty$ , then the statement holds for  $\limsup$  instead.

**Theorem 2.2.7 — Dominated Convergence Theorem.** Let  $(\xi_n)_{n \geq 1}$  be a sequence of random variables such that  $\xi_n \rightarrow \xi$  almost surely. If there exists an integrable random variable  $\eta$  such that  $|\xi_n| \leq \eta$  for all  $n \in \mathbb{N}$ , then  $\xi$  is integrable, with

$$\mathbb{E}(\xi_n) \rightarrow \mathbb{E}(\xi),$$

and

$$\mathbb{E}(|\xi_n - \xi|) \rightarrow 0$$

as  $n \rightarrow \infty$ .

**Corollary 2.2.8** Let  $\eta, \xi, \xi_1, \dots$  be random variables such that  $|\xi_n| \leq \eta$ ,  $\xi_n \rightarrow \xi$  almost everywhere and  $\mathbb{E}(\eta^p) < \infty$  for some  $p > 0$ . Then  $\mathbb{E}(|\xi|^p) < \infty$  and  $\mathbb{E}(|\xi - \xi_n|^p) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* Note that  $|\xi_n|^p \rightarrow |\xi|^p$  almost surely,  $|\xi_n|^p \leq \eta^p$  and  $\mathbb{E}(\eta^p) < \infty$ . Thus, by Theorem 2.2.7 it follows that

$$\mathbb{E}(|\xi_n|^p) \rightarrow \mathbb{E}(|\xi|^p).$$

As the inequality  $\mathbb{E}(|\xi_n|^p) \leq \mathbb{E}(\eta^p)$  is preserved under the limit it follows that  $\mathbb{E}(|\xi|^p) \leq \mathbb{E}(\eta^p) < \infty$ . Next, consider  $\mu_n := \xi - \xi_n$ . It is clear that  $\mu_n \rightarrow 0$  almost surely and  $|\mu_n| \leq \xi + \eta$  with

$$\mathbb{E}(|\xi + \eta|^p) \leq \mathbb{E}(|\xi|^p) + \mathbb{E}(|\eta|^p) \leq 2\mathbb{E}(|\eta|^p) < \infty.$$

So applying our previous observations it follows that

$$\mathbb{E}(|\xi - \xi_n|^p) = \mathbb{E}(|\mu_n|^p) \rightarrow 0$$

as  $n \rightarrow \infty$ . ■

**Remark 2.2.9** In all the above theorems, the integral over  $\Omega$  can be replaced by the integral over any measurable  $A \subseteq \Omega$ .

## 2.2.2 Change of variables

**Theorem 2.2.10** Let  $\xi : \mathcal{F} \rightarrow \mathbb{R}$  be a random variable with probability distribution  $\mathbb{P}_\xi$ . If  $g = g(x)$  is a Borel function, then for all  $A \in \mathcal{B}(\mathbb{R})$ , we have

$$\int_A g(x) d\mathbb{P}_\xi = \int_{\xi^{-1}(A)} g(\xi(\omega)) d\mathbb{P},$$

where the integrals exist or do not exist simultaneously. In particular, for  $A = \mathbb{R}$  we obtain

$$\mathbb{E}(g(\xi(\omega))) = \int_{\Omega} g(\xi(\omega)) d\mathbb{P} = \int_{-\infty}^{\infty} g(x) d\mathbb{P}_\xi \equiv \int_{-\infty}^{\infty} g(x) dF_\xi.$$

*Proof.* We use the four-step proof. The result clearly holds for  $g = \chi_B(x)$  for  $B \in \mathcal{B}(\mathbb{R})$ . Therefore, it also holds for simple  $g(x)$  by the linearity of the integral. For any non-negative measurable function  $g$  we can consider a sequence of simple functions  $g_n \nearrow g$ . The result for  $g$  then follows from the monotone convergence theorem. For arbitrary measurable  $g(x)$  we use  $g(x) = g_+ - g_-$ . ■

**Remark 2.2.11** Theorem 2.2.10 guarantees that the expectation only depends on the probability distribution, and not on the underlying probability space.

1. If  $\xi$  is discrete, so that  $\mathcal{F}_\xi$  is discrete, taking values  $x_1, x_2, \dots$  with probabilities  $p_1, p_2, \dots$  then

$$\mathbb{E}(g(\xi)) = \sum_j g(x_j)p_j.$$

2. If  $\xi$  is absolutely continuous, so that  $F_\xi$  is absolutely continuous, with density  $f(x)$ , then

$$\mathbb{E}(g(\xi)) = \int_{-\infty}^{\infty} g(x)f(x) dx.$$

This provides a way to calculate expectations of  $g(\xi)$  without being conscious of the actual distribution of  $g(\xi)$ . Thus we can make sense of the expectation of probability distributions without specifying its underlying probability space. So is there any point in specifying the underlying probability space of a random variable instead of assuming it to be  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_\xi)$ ? Unfortunately, the answer can be no, since there is a way to develop probability theory, and notions of random variables, without going through the Kolmogorov constructions. Indeed, the underlying probability space is insignificant in most applications; nevertheless, this formulation is still helpful in understanding more complicated random variables like stochastic processes.

### 2.3 Exchanging the Order of Integration

We now consider product measures, and more specifically expressing integration on product measures as repeated integrals. Suppose  $(X_1, \mathcal{M}_1, \mu_1)$  and  $(X_2, \mathcal{M}_2, \mu_2)$  is a pair of measure spaces and consider the product measure  $\mu_1 \times \mu_2$  on the space

$$X = X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}.$$

We assume that the two measure spaces are complete and  $\sigma$ -finite. Given a set  $E$  in  $\mathcal{M}$  we consider the slices

1.  $E_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in E\}$ , and
2.  $E^{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in E\}$ .

**Theorem 2.3.1 — Fubini's Theorem.** In the setting above, suppose that  $f(x_1, x_2)$  is an integrable function on  $(X_1 \times X_2, \mu_1 \times \mu_2)$ . Then the following hold.

- For almost every  $x_2 \in X_2$ , the slice  $f^{x_2}(x_1) = f(x_1, x_2)$  is integrable on  $(X_1, \mu_1)$ .
- $\int_{X_1} f(x_1, x_2) d\mu_1$  is an integrable function on  $X_2$ .
- We can exchange integrals as follows

$$\int_{X_2} \left( \int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1} \left( \int_{X_2} f(x_1, x_2) d\mu_2 \right) d\mu_1 = \int_{X_1 \times X_2} f d\mu_1 \times \mu_2.$$

**Remark 2.3.2** In general, the product space  $(X, \mathcal{M}, \mu)$  is not complete. One can construct the completion of this space  $\bar{\mathcal{M}}$ , as the collection of sets of the form  $E \cup Z$ , where  $E \in \mathcal{M}$  and  $Z \subseteq F$  with  $F \in \mathcal{M}$  and  $\mu(F) = 0$  with  $\bar{\mu}(E \cup Z) := \mu(E)$ . Then,

- $\bar{\mathcal{M}}$  is the smallest  $\sigma$ -algebra containing  $\mathcal{M}$  and all subsets of elements of  $\mathcal{M}$  of measure zero, and
- the function  $\bar{\mu}$  is a measure on  $\bar{\mathcal{M}}$ ,

so that  $\bar{\mathcal{M}}$ . Theorem 2.3.1 continues to hold in this completed space.

In Theorem 2.3.1, we assume that the function  $f$  is integrable over the product space. We can relax this condition, by instead assuming that  $f$  is a non-negative measurable function.

**Theorem 2.3.3 — Tonelli's Theorem.** Suppose that  $f(x_1, x_2) : X_1 \times X_2 \rightarrow [0, \infty]$  is a non-negative

measurable function on  $(X_1 \times X_2, \mu_1 \times \mu_2)$ . Then

$$\int_{X_2} \left( \int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1} \left( \int_{X_2} f(x_1, x_2) d\mu_2 \right) d\mu_1 = \int_{X_1 \times X_2} f d\mu_1 \times \mu_2.$$

**Theorem 2.3.4 — Fubini-Tonelli Theorem.** If  $f$  is a measurable function, then

$$\int_{X_1} \left( \int_{X_2} |f(x_1, x_2)| d\mu_2 \right) d\mu_1 = \int_{X_2} \left( \int_{X_1} |f(x_1, x_2)| d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} |f| d\mu_1 \times \mu_2.$$

Besides, if any one of these integrals is finite, then

$$\int_{X_1} \left( \int_{X_2} f(x_1, x_2) d\mu_2 \right) d\mu_1 = \int_{X_2} \left( \int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f d\mu_1 \times \mu_2.$$

The absolute value of  $f$  in the conditions of Theorem 2.3.4 can be replaced by either the positive or the negative part of  $f$ . Theorem 2.3.4 is a generalisation of Theorem 2.3.3 as one can take the negative part of a non-negative function to be zero whilst maintaining a finite integral. Informally all these conditions say that the double integral of  $f$  is well-defined, though possibly infinite. The advantage of the Theorem 2.3.4 over Theorem 2.3.1 is that the repeated integrals of the absolute value of  $|f|$  may be easier to study than the double integral. As in Theorem 2.3.1, the single integrals may fail to be defined on a zero-measure set.

**Proposition 2.3.5 —  $\mathbb{E}$  and tail probabilities.** Let  $\xi$  be a non-negative integrable random variable. Then

$$\mathbb{E}(\xi) = \int_{[0, \infty)} \mathbb{P}(\xi \geq x) dx. \quad (2.1)$$

*Proof.* We have

$$\begin{aligned} \mathbb{E}(\xi) &= \int_{[0, \infty)} x d\mathbb{P}_\xi \\ &= \int_{[0, \infty)} \left( \int_0^x dt \right) d\mathbb{P}_\xi \\ &= \int_{[0, \infty)} \mathbb{P}(\xi \geq t) dt, \end{aligned}$$

where we applied Theorem 2.3.1 to

$$g(t, x) = \begin{cases} 1, & 0 \leq t \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

■

**Exercise 2.3.6** Generalise the above proof to prove that if  $\xi \geq 0$  and  $p \geq 1$  then

$$\mathbb{E}(\xi^p) = \int_0^\infty py^{p-1}\mathbb{P}(\xi \geq y) dy. \quad (2.2)$$

Verify the formula 2.1 and 2.2 for some simple distributions, such as the exponential  $\text{Exp}(\lambda)$  distribution which has density  $f(x) = \lambda e^{-\lambda x}$ .

## 2.4 Jensen's Inequality and $L^p$ Spaces

### 2.4.1 Convex Functions and Jensen Inequality

**Definition 2.4.1**

- A set  $\Omega \subseteq \mathbb{R}^n$  is convex if for all  $x, y \in \Omega$  and  $\lambda \in [0, 1]$  the point  $(1 - \lambda)x + \lambda y \in \Omega$ .
- Let  $E \subseteq \mathbb{R}^n$  be a convex set. A function  $g : E \rightarrow \mathbb{R}$  is convex if, for all  $x, y \in \Omega$  and  $\lambda \in [0, 1]$ , we have
$$g((1 - \lambda)x + \lambda y) \leq (1 - \lambda)g(x) + \lambda g(y).$$
- Let  $E \subseteq \mathbb{R}^n$  be a convex set. A function  $g : E \rightarrow \mathbb{R}$  is concave if  $-g$  is convex.

For the following, we assume  $E = \mathbb{R}^n$  for simplicity, although removing this assumption won't affect results much.

**Proposition 2.4.2** Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. Then for all  $x_0 \in \mathbb{R}^n$ , there is a vector  $v \in \mathbb{R}^n$ , dependent on  $x_0$ , such that for all  $x \in \mathbb{R}^n$ , we have

$$g(x) \geq g(x_0) + v^\top (x - x_0). \quad (2.3)$$

Any vector  $v$  satisfying (2.3) is called a subgradient of  $g$  at  $x_0$ .

With Proposition 2.4.2 we can prove Jensen's inequality for expectations.

**Theorem 2.4.3 — Jensen's Inequality.** Let  $\xi$  be an integrable random variable and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable, convex function. Then

$$g(\mathbb{E}(\xi)) \leq \mathbb{E}(g(\xi)).$$

*Proof.* If  $g(x)$  is convex then for each  $x_0 \in \mathbb{R}$  there is a  $v \in \mathbb{R}$  such that

$$g(x) \geq g(x_0) + v(x - x_0)$$

for all  $x \in \mathbb{R}$ . Putting  $x = \xi$  and  $x_0 = \mathbb{E}(\xi)$ , we find that

$$g(\xi) \geq g(\mathbb{E}(\xi)) + v(\xi - \mathbb{E}(\xi))$$

and by taking expectation of both sides we get  $g(\mathbb{E}[\xi]) \leq \mathbb{E}[g(\xi)]$ . ■

**Exercise 2.4.4** Prove that the composition of a convex non-decreasing function and a convex function is convex. Hence, conclude that  $f(x) = |x|^r$  is a convex function.

**Corollary 2.4.5 — Lyapunov's Inequality.** If  $0 < p < q < \infty$ , then

$$(\mathbb{E}(|\xi|^p))^{\frac{1}{p}} \leq (\mathbb{E}(|\xi|^q))^{\frac{1}{q}}. \quad (2.4)$$

*Proof.* Let  $r = \frac{q}{p}$ . Then putting  $\eta = |\xi|^p$  and using Exercise 2.4.4 we can apply Jensen's inequality to  $g(x) = |x|^r$  to get that  $|\mathbb{E}(\eta)|^r \leq \mathbb{E}(|\eta|^r)$ . That is,

$$(\mathbb{E}(|\xi|^p))^{\frac{q}{p}} \leq \mathbb{E}(|\xi|^q)$$

from which (2.4) follows. Consequently, if  $\mathbb{E}(|\xi|^q) < \infty$  then  $\mathbb{E}(|\xi|^p) = (\mathbb{E}(|\xi|^q))^{\frac{p}{q}} < \infty$ . ■

The following chain of inequalities among absolute moments

$$\mathbb{E}(|\xi|) \leq (\mathbb{E}(|\xi|^2))^{\frac{1}{2}} \leq \dots \leq (\mathbb{E}(|\xi|^n))^{\frac{1}{n}} \leq \dots$$

is a consequence of Lyapunov's inequality.

**Remark 2.4.6** As a warning, Lyapunov inequality is only true when  $\mathbb{P}$  is a finite measure, which is certainly the case for probability measures.

**Definition 2.4.7** Let  $\xi$  be a random variable with  $\mathbb{E}(|\xi|^p) < \infty$ . For integers  $0 \leq k \leq p$ , the  $k^{\text{th}}$  moment of  $\xi$  is  $\mathbb{E}(\xi^k)$ .

**Definition 2.4.8** The sequence  $\xi_1, \xi_2, \dots$  of random variables converges in  $L^p$  to the random variable  $\xi$  if

$$(\mathbb{E}(|\xi_n - \xi|^p))^{\frac{1}{p}} \rightarrow 0$$

as  $n \rightarrow \infty$ .

Often one uses the notation  $\|\xi\|_{L^p} = (\mathbb{E}(|\xi|^p))^{\frac{1}{p}}$ . This is done to allude to the fact that  $\|\cdot\|_{L^p}$  defines a norm on the vector space of functions with finite  $p^{\text{th}}$  moments, up to sets of  $\mathbb{P}$ -measure zero. More formally, we say that  $\xi \in \mathcal{L}^p$  if  $\mathbb{E}(|\xi|^p) < \infty$ . Let  $\xi \sim \eta$  if  $\xi = \eta$  almost everywhere. Then  $\|\cdot\|_{L^p}$  defines a norm on  $L^p = \mathcal{L}^p / \sim$ . This result is facilitated by the following inequalities.

**Proposition 2.4.9 — Hölder's Inequality.** Let  $p \in [1, \infty]$  and let  $q \in [1, \infty]$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$  with  $q = \infty$  if  $p = 1$  and vice-versa. If  $\xi \in \mathcal{L}^p$  and  $\eta \in \mathcal{L}^q$ , then

$$\|\xi\eta\|_{L^1(\Omega)} := \mathbb{E}(|\xi\eta|) \leq \|\xi\|_{L^p} \|\eta\|_{L^q}.$$

**Proposition 2.4.10 — Minkowski's Inequality.** If  $\mathbb{E}(|\xi|^p) < \infty$  and  $\mathbb{E}(|\eta|^p) < \infty$  for  $1 \leq p \leq \infty$ , then  $\mathbb{E}(|\xi + \eta|^p) < \infty$  and

$$(\mathbb{E}(|\xi + \eta|^p))^{\frac{1}{p}} \leq (\mathbb{E}(|\xi|^p))^{\frac{1}{p}} + (\mathbb{E}(|\eta|^p))^{\frac{1}{p}}.$$

**Remark 2.4.11** Note that if the sequence  $(\xi_i)_{i \geq 1}$  converges in  $L^p$  to  $\xi$ , then

$$0 \leq |\|\xi_i\|_{L^p} - \|\xi\|_{L^p}| \leq \|\xi_i - \xi\|_{L^p} \rightarrow 0$$

as  $i \rightarrow \infty$ . That is,  $\|\xi_i\|_{L^p} \rightarrow \|\xi\|_{L^p}$ .

## 2.5 Tail Bounds

Most large sample results concern extreme events, for example, whether the value of a random variable deviates from its mean. This section builds the necessary tools to derive upper bounds for the probability of such events. These bounds are usually called "tail bounds" since they correspond to the "tail" of the densities of random variables. In particular, we will see how the tail bounds are related to the integrability of the random variables.

**Example 2.5.1** As a motivation, consider random variables living on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with zero expectation. We would like to bound the probability of the tail event  $\mathbb{P}(X > c)$  for  $c \gg 1$ .

1. Let  $\xi_1$  have a normal distribution  $N(0, 1)$  with density

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

for  $x \in \mathbb{R}$ , such that

$$\begin{aligned} \mathbb{P}(\xi_1 > c) &= \int_c^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &\leq \int_c^\infty \frac{1}{\sqrt{2\pi}} \frac{x}{c} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{1}{c\sqrt{2\pi}} \exp\left(-\frac{c^2}{2}\right) \end{aligned}$$

$$=: r_1(c).$$

2. Let  $\xi_2$  have a double exponential (Laplace) distribution with density

$$f_2(x) = \frac{1}{2}e^{-|x|},$$

such that

$$\begin{aligned}\mathbb{P}(\xi_2 > c) &= \int_c^\infty \frac{1}{2} \exp(-x) dx \\ &= \frac{1}{2} \exp(-c) \\ &=: r_2(c).\end{aligned}$$

3. Let  $\xi_3$  have a standard Cauchy distribution with density

$$f_3(x) = \frac{1}{\pi(1+x^2)}$$

such that

$$\begin{aligned}\mathbb{P}(\xi_3 > c) &= \frac{1}{2} - \frac{1}{\pi} \arctan c \\ &= \frac{1}{\pi} \arctan \frac{1}{c} \\ &\leq \frac{1}{\pi c} \\ &=: r_3(c).\end{aligned}$$

It is clear that  $\mathbb{P}(X > c)$  decays much faster for the normal distribution than the double exponential distribution, and the Cauchy distribution admits the slowest decay. In particular, we have  $r_3(c) \gg r_2(c) \gg r_1(c)$  in the sense that  $\frac{r_1(c)}{r_2(c)} \rightarrow 0$  and  $\frac{r_2(c)}{r_3(c)} \rightarrow 0$  when  $c \rightarrow \infty$ .

**Exercise 2.5.2** Verify the following observations.

1.  $\xi_1$  has zero odd moments, and has  $(2k)^{\text{th}}$  moments  $m_{1,k} = (2k-1)!! := (2k-1) \times \dots \times 3 \times 1 = \frac{(2k)!}{2^k k!}$  for all  $k \in \mathbb{Z}_{\geq 1}$ .
2.  $\xi_2$  also has zero odd moments, and has  $(2k)^{\text{th}}$  moments  $m_{2,k} = (2k)!$ .
3.  $\xi_3$  has  $(2k)^{\text{th}}$  moments  $m_{3,k} = \infty$ .

Therefore, we see that  $\infty = m_{3,k} \gg m_{2,k} \gg m_{1,k}$  as  $k \rightarrow \infty$ , in the sense that as  $k \rightarrow \infty$  we have  $\frac{m_{1,k}}{m_{2,k}} \rightarrow 0$ . We therefore suspect that there is a connection between the tail bounds and the growth of moments.

To standardise the discussion of random variables, one often centralises the moments in the following way.

**Definition 2.5.3** The  $k^{\text{th}}$  central moment, for  $k \geq 1$ , of a random variable  $\xi$  is the expectation  $\mathbb{E}((\xi - \mathbb{E}(\xi))^k)$  whenever  $\mathbb{E}(|\xi|^k) < \infty$ . In particular, the first central moment is zero. The main central moments of interest are the following.

- The 2<sup>nd</sup> central moment  $\mathbb{V}(\xi) := \mathbb{E}((\xi - \mathbb{E}(\xi))^2)$  is called the variance.
- The 3<sup>rd</sup> central moment  $\mathbb{E}((\xi - \mathbb{E}(\xi))^3)$  is called the skewness.
- The 4<sup>th</sup> central moment  $\mathbb{E}((\xi - \mathbb{E}(\xi))^4)$  is called the kurtosis.

**Theorem 2.5.4 — Markov's Inequality.** Let  $\xi$  be a non-negative integrable random variable and  $c > 0$  a constant. Then

$$\mathbb{P}(\xi \geq c) \leq \frac{\mathbb{E}(\xi)}{c}.$$

*Proof.* This follows directly from

$$\mathbb{E}(\xi) \geq \mathbb{E}(\xi \cdot \chi_{\xi \geq c}) \geq c\mathbb{E}(\chi_{\xi \geq c}) = c\mathbb{P}(\xi \geq c).$$

■

**Remark 2.5.5** A generalisation of the Markov inequality considers  $\xi$  a random variable and  $g$  a non-negative Borel function. Let  $c > 0$  be a constant and suppose that  $\mathbb{E}(g(\xi))$  exists, then

$$\mathbb{P}(g(\xi) \geq c) \leq \frac{\mathbb{E}(g(\xi))}{c}. \quad (2.5)$$

Let us interrupt our discussion of tail bound by proving an interesting result regarding  $L^p$  norms. How powerful Markov's inequality (2.5) is for proving tail bounds for specific distributions depends on the integrability of  $\xi$ .

**Corollary 2.5.6** Let  $\xi \in L^p(\Omega)$  for  $p \geq 1$ . Then for all  $\varepsilon > 0$ , we have

$$\mathbb{P}(|\xi - \mathbb{E}(\xi)| \geq \varepsilon) = \mathbb{P}(|\xi - \mathbb{E}(\xi)|^p \geq \varepsilon^p) \leq \frac{\mathbb{E}(|\xi - \mathbb{E}(\xi)|^p)}{\varepsilon^p}. \quad (2.6)$$

When  $p = 2$  we obtain Chebyshev's Inequality, which states that

$$\mathbb{P}(|\xi - \mathbb{E}(\xi)| \geq \varepsilon) \leq \frac{\mathbb{V}(\xi)}{\varepsilon^2}.$$

### 2.5.1 Chernoff Bound and Moment Generating Function

For the case when  $\xi \in L^\infty(\Omega)$  and the  $k^{\text{th}}$  moment does not grow "too quickly", one may choose an optimal  $p$  such that the right-hand side of (2.6) is minimised. This is rarely done in practice. Instead, we consider the moment-generating function.

**Definition 2.5.7 — Moment Generating Function.** The moment generating function of a random variable  $\xi$  is

$$M_\xi(t) = \mathbb{E}(\exp(tX)) = \int_{\mathbb{R}} e^{tx} dF_\xi(x).$$

A moment-generating function does not necessarily exist for all values of  $t \in \mathbb{R}$ . For example, a random variable  $\xi$  with Cauchy distribution has  $M_\xi(t) = \infty$  for all  $t \neq 0$ , and is equal to 1 for  $t = 0$ . However, if we can show that  $M_\xi(t) < \infty$  for a small neighbourhood of zero, say  $t \in (-h, h)$ , then we have the following result.

**Proposition 2.5.8 — Chernoff Inequality.** Let  $\xi$  be a non-negative random variable, then for all  $\varepsilon > 0$  and  $t \in (0, h)$  we have

$$\mathbb{P}(\xi \geq \varepsilon) \leq \frac{M_X(t)}{e^{t\varepsilon}}. \quad (2.7)$$

*Proof.* Noting that  $e^{t\xi}$  is a non-negative random variable, we can use Theorem 2.5.4 to deduce that

$$\mathbb{P}(\xi \geq \varepsilon) = \mathbb{P}(e^{t\xi} \geq e^{t\varepsilon}) \leq \frac{\mathbb{E}(e^{t\xi})}{e^{t\varepsilon}} = \frac{M_X(t)}{e^{t\varepsilon}}.$$

■

**Example 2.5.9** Consider  $\xi$  following a standard normal distribution  $N(0, 1)$ . The  $(2k)^{\text{th}}$  moments of  $\xi$  is given by  $(2k - 1)!!$ , and thus for all  $k \geq 1$  and  $c > 0$ , we have

$$\mathbb{P}(X > c) \leq \frac{(2k - 1)!!}{c^{2k}}. \quad (2.8)$$

Even though using larger  $k$  will lead to a faster rate of decay as  $c \rightarrow \infty$ , the numerator is also larger, so it is harder to use the bound for practical applications. We can obtain a much sharper bound than (2.8) by using the Chernoff bounds. Observe that

$$\begin{aligned} \mathbb{E}(e^{t\xi}) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{tx - x^2/2} dx \\ &= \exp \frac{t^2}{2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-t)^2/2} dx \\ &= \exp \frac{t^2}{2} \end{aligned}$$

so that by the Chernoff bound (2.7) we have

$$\begin{aligned} \mathbb{P}(\xi > c) &\leq \exp \left( \frac{t^2}{2} - ct \right) \\ &= \exp \left( -\frac{c^2}{2} + \frac{1}{2}(t - c)^2 \right). \end{aligned} \quad (2.9)$$

Since (2.9) holds for all  $t > 0$ , we can choose the optimal  $t$  such that the right-hand side is minimised. In our case, we choose  $c > 0$  to obtain

$$\begin{aligned} \mathbb{P}(\xi > c) &\leq \exp \left( \frac{t^2}{2} - ct \right) \\ &= \exp \left( -\frac{c^2}{2} \right). \end{aligned}$$

## 2.6 Solution to Exercises

### Exercise 2.3.6

*Solution.* Note that

$$\mathbb{E}(\xi^p) = \int_0^\infty t^p d\mathbb{P}_\xi(t) = \int_0^\infty \int_0^t py^{p-1} dy d\mathbb{P}_\xi(t).$$

Applying Fubini's theorem to

$$g(y, t) = py^{p-1} \mathbf{1}_{[0 \leq y \leq t]}$$

allows us to interchange the integrals to conclude that

$$\begin{aligned} \mathbb{E}(\xi^p) &= \int_0^\infty \int_y^\infty py^{p-1} d\mathbb{P}_\xi(t) dy \\ &= \int_0^\infty py^{p-1} \mathbb{P}(\xi \geq y). \end{aligned}$$

■

### Exercise 2.4.4

*Solution.* Let  $f = g \circ h$ , where  $g$  is a non-decreasing convex function and  $h$  is a convex function. Then for  $x, y \in \Omega$  and  $\lambda \in [0, 1]$  we have

$$(g \circ h)((1 - \lambda)x + \lambda y) = g(h((1 - \lambda)x + \lambda y))$$

$$\begin{aligned} &\stackrel{(1)}{\leq} g((1-\lambda)h(x) + \lambda h(y)) \\ &\stackrel{(2)}{\leq} (1-\lambda)(g \circ h)(x) + \lambda(g \circ h)(y). \end{aligned}$$

Where in (1) we have used the fact that  $h$  is convex and  $g$  is non-decreasing, and in (2) we have used the fact that  $g$  is convex. Therefore,  $g \circ h$  is a convex function. Hence, with  $g(x) = x^r$  for  $x \geq 0$  and  $h(x) = |x|$ , we deduce that  $f(x) = |x|^r$  is convex. ■

### Exercise 2.5.2

*Solution.*

1. The odd moments are given by

$$\int_{-\infty}^{\infty} \frac{x^{2k+1}}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

which is zero as the integrand is an odd function. For the even moments, we can proceed by induction. For the base case we note that  $\mathbb{E}(\xi^2) = \mathbb{V}(\xi) + \mathbb{E}(\xi) = 1 = (2(1) - 1)!!$ . Therefore, for  $k \geq 1$  we have

$$\begin{aligned} \mathbb{E}(\xi^{2k}) &= \int_{-\infty}^{\infty} \frac{x^{2k}}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= 2 \int_0^{\infty} \frac{x^{2k}}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &\stackrel{(1)}{=} 2 \left( \left[ -x^{2k-1} \exp\left(-\frac{x^2}{2}\right) \right]_0^\infty \int_0^\infty (2k-1)x^{2k-2} \exp\left(-\frac{x^2}{2}\right) dx \right) \\ &= (2k-1) \int_{-\infty}^{\infty} x^{2k-2} \exp\left(-\frac{x^2}{2}\right) dx \\ &\stackrel{\text{Ind Hyp.}}{=} (2k-1)(2k-3)!! \\ &= (2k-1)!. \end{aligned}$$

Where in (1) we have performed integration by parts with  $u = x^{2k-1}$  and  $\frac{dv}{dx} = x \exp\left(-\frac{x^2}{2}\right)$ .

2. The odd moments are given by

$$\int_{-\infty}^{\infty} \frac{x}{2} \exp(-x) dx$$

which is zero as the integrand is an odd function. For the even moments, we can proceed by induction. The case for  $k = 0$  is clear. For  $k \geq 1$  we have

$$\begin{aligned} \mathbb{E}(\xi^{2k}) &= \int_{-\infty}^{\infty} \frac{x^{2k}}{2} \exp(-x) dx \\ &= \int_0^{\infty} \frac{x^{2k}}{2} \exp(-x) dx \\ &= 2 \left( \left[ -x^{2k} \exp(-x) \right]_0^\infty + \int_0^\infty 2kx^{2k-1} \exp(-x) dx \right) \\ &= 2(2k) \int_0^{\infty} x^{2k-1} \exp(-x) dx \\ &= 2(2k) \left( \left[ -x^{2k-1} \exp(-x) \right]_0^\infty + \int_0^\infty (2k-1)x^{2k-2} \exp(-x) dx \right) \\ &= 2(2k)(2k-1) \frac{(2k-2)!}{2} \\ &= (2k)!. \end{aligned}$$

3. Recalling that  $\mathbb{E}(|\xi|) = \infty$ , we can use Corollary 2.4.5 to conclude that all higher-order moments are also infinite. ■

### 3 More on Random Variables

#### 3.1 Transformation of Random Variables

Let us consider the problem of determining the distribution function of a random variable which is the function of other random variables. That is, let  $\xi$  be a random variable with distribution function  $F_\xi(x)$  (and density  $f_\xi(x)$ , if it exists), and let  $\varphi = \varphi(x)$  be a Borel function such that  $\eta = \varphi(\xi)$ . Then we would like to determine the distribution function of  $\eta$ . Proceeding directly we get that

$$F_\eta(y) = \mathbb{P}(\eta \leq y) = \mathbb{P}(\xi \in \varphi^{-1}(-\infty, y]) = \int_{\varphi^{-1}(-\infty, y]} dF_\xi,$$

which expresses the distribution function  $F_\eta(y)$  in terms of  $F_\xi(x)$  and  $\varphi$ .

##### Example 3.1.1

1. Let  $\eta = a\xi + b$  with  $a > 0$ . Then

$$F_\eta(y) = \mathbb{P}(\eta \leq y) = \mathbb{P}\left(\xi \leq \frac{y-b}{a}\right) = F_\xi\left(\frac{y-b}{a}\right).$$

2. Let  $\eta = \xi^2$ . Then it is evident that  $F_\eta(y) = 0$  if  $y < 0$ . While for  $y \geq 0$ , we have

$$\begin{aligned} F_\eta(y) &= \mathbb{P}(\xi^2 \leq y) \\ &= \mathbb{P}(-\sqrt{y} \leq \xi \leq \sqrt{y}) \\ &= \mathbb{P}_\xi(-\infty, \sqrt{y}] - \mathbb{P}_\xi(-\infty, -\sqrt{y}) \\ &= F_\xi(\sqrt{y}) - F_\xi(-\sqrt{y}) + \mathbb{P}(\xi = -\sqrt{y}). \end{aligned}$$

**Proposition 3.1.2 — Probability Integral Transform.** Let  $\xi$  be a random variable on  $(\Omega, \mathcal{F}, \mathbb{P})$  with distribution function  $F_\xi(x)$ . Let  $U$  be a uniformly distributed random variable on  $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$ , where  $\text{Leb}$  denotes the Lebesgue measure. Define the right inverse of  $F_\xi$  on  $[0, 1]$  as

$$F_\xi^{-1}(y) = \sup(\{x : F_\xi(x) < y\}), \quad (3.1)$$

and extend it so that  $F_\xi^{-1}(0) = -\infty$  and  $F_\xi^{-1}(1) = \infty$ . Then  $\xi$  has the same distribution function as  $F_\xi^{-1}(U)$ . In such a case we say that  $F_\xi^{-1}(U)$  is equally distributed as  $\xi$ , or  $\xi \stackrel{d}{=} F_\xi^{-1}(U)$ .

*Proof.* Note that if  $F_\xi$  is invertible we have

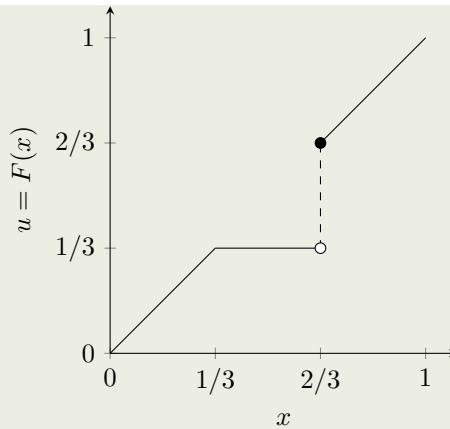
$$F_{F_\xi^{-1}(U)}(y) = \text{Leb}\left(F_\xi^{-1}(U) \leq y\right) \stackrel{(1)}{=} \text{Leb}(U \leq F_\xi(y)) = F_\xi(y),$$

where in (1) we have used the fact that  $F_\xi$  is invertible, and hence bijective, to deduce that  $F_\xi^{-1}(u) \leq y$  if and only if  $u \leq F_\xi(y)$ . Consequently,  $F_\xi^{-1}(U) \stackrel{d}{=} \xi$ . Thus, to complete the proof, it suffices to show that (1) holds for general distribution functions  $F_\xi$ .

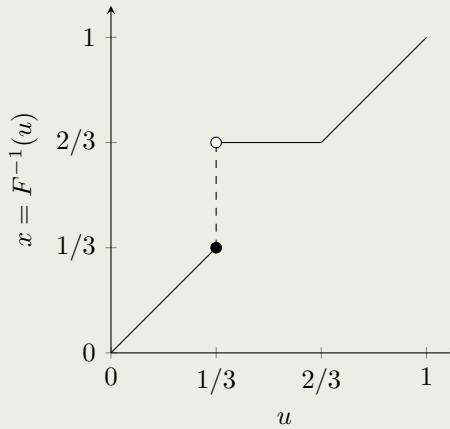
( $\Leftarrow$ ) Assume  $u \leq F_\xi(y)$ . Then clearly whenever  $F_\xi(x) < u$  we have  $F_\xi(x) < F_\xi(y)$  so that  $x \leq y$ . Therefore,  $y$  is an upper bound of the set  $\{x : F_\xi(x) < u\}$ .

( $\Rightarrow$ ) Assume  $F_\xi^{-1}(u) \leq y$  but for contradiction that  $u > F_\xi(y)$ . Note in such case we have  $y \in \{x : F_\xi(x) < u\}$ , so  $y$  is indeed the maximum of  $\{x : F_\xi(x) < u\}$ . In other words, for any  $x > y$  we have  $F_\xi(x) \geq u$ . Consider an arbitrary monotonic decreasing sequence  $(x_n)$  with  $x_n > y$  that converges to  $y$ . We then know that  $\lim_{n \rightarrow \infty} F_\xi(x_n) = F(y)$  by right continuity, and that  $F_\xi(y) \geq u$ . This contradicts our assumption, so we must have  $u \leq F_\xi(y)$ . ■

**Example 3.1.3** Let  $F$  be the distribution function for  $\eta$  which has a uniform distribution on  $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$  with an atom at  $\frac{2}{3}$ .

**Figure 5:** An example of a distribution function.

The inverse  $F^{-1}(u)$  is given in Figure 6.

**Figure 6:** The pseudo-inverse of the previous example.

Now we turn to the problem of determining  $f_\eta(y)$ . Let us suppose that the range of  $\xi$  is a (finite or infinite) open interval  $I = (a, b)$ . Moreover, suppose that  $\varphi = \varphi(x)$ , with domain  $(a, b)$ , is a strictly increasing or decreasing continuously differentiable function. We also suppose that  $\varphi'(x) \neq 0$  for  $x \in I$  so that we can write  $h(y) = \varphi^{-1}(y)$ . For definiteness suppose that  $\varphi$  is strictly increasing so that for  $y \in \varphi(I)$  it follows that,

$$\begin{aligned} F_\eta(y) &= \mathbb{P}(\eta \leq y) \\ &= \mathbb{P}(\varphi(\xi) \leq y) \\ &= \mathbb{P}(\xi \leq \varphi^{-1}(y)) \\ &= \mathbb{P}(\xi \leq h(y)) \\ &= \int_{-\infty}^{h(y)} f_\xi(x) dx \\ &= \int_{-\infty}^y f_\xi(h(z)) h'(z) dz. \end{aligned}$$

Therefore,

$$f_\eta(y) = f_\xi(h(y)) h'(y).$$

On the other hand, if  $\varphi(x)$  is strictly decreasing, then

$$f_\eta(y) = f_\xi(h(y)) (-h'(y)).$$

In either case

$$f_\eta(y) = f_\xi(h(y)) |h'(y)|.$$

**Example 3.1.4** If  $\eta = a\xi + b$  for  $a \neq 0$ , we have

$$h(y) = \frac{y - b}{a}$$

and

$$f_\eta(y) = \frac{1}{|a|} f_\xi\left(\frac{y - b}{a}\right).$$

If  $\varphi = \varphi(x)$  is neither strictly increasing nor strictly decreasing, the above formula is not applicable. However, the following generalisation suffices for many applications.

**Lemma 3.1.5** Let  $\varphi = \varphi(x)$ , defined on the set  $\sum_{k=1}^n [a_k, b_k]$ , be continuously differentiable and either strictly increasing or strictly decreasing on each open interval  $I_k = (a_k, b_k)$ , with  $\varphi'(x) \neq 0$  for  $x \in I_k$ . Let  $h_k = h_k(y)$  be the inverse of  $\varphi(x)$  for  $x \in I_k$ . Then

$$f_\eta(y) = \sum_{k=1}^n f_\xi(h_k(y)) |h'_k(y)| \cdot \chi_{D_k}(y),$$

where  $D_k$  is the domain of  $h_k(y)$ .

**Example 3.1.6** Let  $\eta = \xi^2$  with  $I_1 = (-\infty, 0)$  and  $I_2 = (0, \infty)$ . Observe that  $h_1(y) = -\sqrt{y}$  and  $h_2(y) = \sqrt{y}$ , so that

$$f_\eta(y) = \begin{cases} \frac{1}{2\sqrt{y}} (f_\xi(\sqrt{y}) + f_\xi(-\sqrt{y})) & y > 0 \\ 0 & y \leq 0. \end{cases}$$

If  $\xi \sim N(0, 1)$ , then

$$f_{\xi^2}(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2} & y > 0 \\ 0 & y \leq 0. \end{cases}$$

Similar calculations show that

$$f_{|\xi|}(y) = \begin{cases} f_\xi(y) + f_\xi(-y) & y > 0 \\ 0 & y \leq 0, \end{cases}$$

and

$$f_{\sqrt{|\xi|}}(y) = \begin{cases} 2y (f_\xi(y^2) + f_\xi(-y^2)) & y > 0 \\ 0 & y \leq 0. \end{cases}$$

## 3.2 Independent Random Variables

**Definition 3.2.1** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a measure space.

- A finite collection of events  $\{A_1, \dots, A_n\}$  is independent if  $\mathbb{P}(\cap_{i=1}^n A_i) = \prod_{i=1}^n \mathbb{P}(A_i)$ . An infinite collection  $\{A_1, A_2, \dots\}$  is (mutually) independent if any finite sub-collection of events is independent.
- A finite collection of sub- $\sigma$ -algebras  $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$  of  $\mathcal{F}$  is (mutually) independent if for any  $A_1 \in \mathcal{F}_1, \dots, A_n \in \mathcal{F}_n$ , we have  $\mathbb{P}(\cap_{i=1}^n A_i) = \prod_{i=1}^n \mathbb{P}(A_i)$ . An infinite collection  $\{\mathcal{F}_1, \mathcal{F}_2, \dots\}$  of sub- $\sigma$ -algebras of  $\mathcal{F}$  is (mutually) independent if any finite sub-collection is independent.
- A finite collection  $\{\xi_1, \dots, \xi_n\}$  of random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  is (mutually) independent if the collection of corresponding sub- $\sigma$ -algebras  $\{\sigma(\xi_1), \dots, \sigma(\xi_n)\}$  is independent. In particular

if  $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ , then

$$\mathbb{P}(\xi_1 \in B_1, \dots, \xi_n \in B_n) = \prod_{i=1}^n \mathbb{P}(\xi_i \in B_i) = \prod_{i=1}^n \mathbb{P}_{\xi_i}(B_i).$$

An infinite collection  $\{\xi_1, \xi_2, \dots\}$  of random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  is (mutually) independent if the corresponding collection  $\{\sigma(\xi_1), \sigma(\xi_2), \dots\}$  of sub- $\sigma$ -algebras is (mutually) independent.

**Remark 3.2.2** Another notion of independence says that the collection of events  $\{A_1, \dots, A_n\}$ , is pairwise independent if for all  $i, j$  with  $i \neq j$  we have  $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ . Mutual independence implies pairwise independence but pairwise independence does not imply mutual independence.

**Proposition 3.2.3** A necessary and sufficient condition for the random variables  $\xi_1, \dots, \xi_n$  to be independent is that

$$F_\xi(x_1, \dots, x_n) = F_{\xi_1}(x_1) \dots F_{\xi_n}(x_n)$$

for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ .

**Corollary 3.2.4** If  $\xi = (\xi_1, \dots, \xi_n)$  has a density  $f_\xi$ , then each  $\xi_i$  has a density  $f_{\xi_i}$ . Furthermore,  $\xi_1, \dots, \xi_n$  are independent if and only if

$$f_\xi(x_1, \dots, x_n) = f_{\xi_1}(x_1) \dots f_{\xi_n}(x_n)$$

for all  $(x_1, \dots, x_n)$  except possibly for a Borel subset of  $\mathbb{R}^n$  with Lebesgue measure zero.

**Corollary 3.2.5** If  $\xi_1, \dots, \xi_n$  are independent and  $\xi_i$  has density  $f_{\xi_i}$ , for  $i = 1, \dots, n$ , then  $\xi$  has a density  $f_\xi$  given by

$$f_\xi(x_1, \dots, x_n) = f_{\xi_1}(x_1) \dots f_{\xi_n}(x_n).$$

**Remark 3.2.6** Even if  $\xi_1, \dots, \xi_n$  each has a density, it does not follow that  $(\xi_1, \dots, \xi_n)$  has a density.

The outcome of independent random variables does not depend on the outcome of the other variables. Therefore, it seems as though if we were to combine independent random variables, we should be able to construct a distribution function for this combination, provided we know the distribution functions of the individual random variables.

**Proposition 3.2.7** Let  $\xi$  and  $\eta$  be independent random variables. Then the distribution function  $F_{\xi+\eta}$  of the sum of their sum is given by the convolution of their distribution functions. That is,

$$F_{\xi+\eta}(z) = F_\xi * F_\eta = \int_{-\infty}^{\infty} F_\xi(z-y) dF_\eta(y) = \int_{-\infty}^{\infty} F_\eta(z-x) dF_\xi(x).$$

*Proof.* Using Proposition 3.2.3 we note that  $F_{(\xi, \eta)}(x, y) = F_\xi(x)F_\eta(y)$ . It follows that

$$\begin{aligned} F_{\xi+\eta}(z) &= \int_{\{x, y: x+y \leq z\}} dF_\xi(x) \cdot dF_\eta(y) \\ &= \int_{\mathbb{R}^2} \chi_{x+y \leq z} dF_\xi(x) \cdot dF_\eta(y) \\ &= \int_{-\infty}^{\infty} dF_\xi(x) \left\{ \int_{-\infty}^{\infty} \chi_{x+y \leq z} dF_\eta(y) \right\} \\ &= \int_{-\infty}^{\infty} F_\eta(z-x) dF_\xi(x). \end{aligned}$$

As this argument is symmetric we also get that

$$F_{\xi+\eta}(z) = \int_{-\infty}^{\infty} F_{\xi}(z-y) dF_{\eta}(y).$$

■

**Corollary 3.2.8** If  $\xi$  and  $\eta$  are independent absolutely continuous random variables, then the density  $f_{\xi+\eta}$  is given by the convolution of the densities,

$$f_{\xi+\eta} = f_{\xi} \star f_{\eta} = \int_{-\infty}^{\infty} f_{\xi}(z-y) f_{\eta}(y) dy = \int_{-\infty}^{\infty} f_{\eta}(z-x) f_{\xi}(x) dx.$$

### Example 3.2.9

- Let  $\xi \sim N(m_1, \sigma_1^2)$  and  $\eta \sim N(m_2, \sigma_2^2)$ , so that

$$f_{\xi}(x) = \frac{1}{\sigma_1} \varphi\left(\frac{x-m_1}{\sigma_1}\right)$$

and

$$f_{\eta}(x) = \frac{1}{\sigma_2} \varphi\left(\frac{x-m_2}{\sigma_2}\right),$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Then

$$\begin{aligned} f_{\xi+\eta}(z) &= \int_{-\infty}^{\infty} f_{\eta}(z-x) f_{\xi}(x) dx \\ &= \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \varphi\left(\frac{z-(m_1+m_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right). \end{aligned}$$

Thus the sum of independent normal random variables is the normal random variable  $N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$ .

- Let  $\xi_1, \dots, \xi_n$  be independent  $N(0, 1)$ . Then

$$f_{\xi_1^2 + \dots + \xi_n^2}(x) = \begin{cases} \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

The random variable  $\xi_1^2 + \dots + \xi_n^2$  is usually denoted by  $\chi_n^2$  and its distribution is the  $\chi^2$ -distribution with  $n$  degrees of freedom.

**Proposition 3.2.10** Let  $\xi$  and  $\eta$  be independent random variables with  $E(\xi) < \infty$  and  $E(\eta) < \infty$ . Then  $E(\xi\eta) < \infty$  with  $E(\xi\eta) = E(\xi)E(\eta)$ .

*Proof.* We utilise a four-step proof. First, assume that  $\xi$  and  $\eta$  are non-negative and define

- $\xi_n = \sum_{k=0}^{\infty} \frac{k}{n} \chi_{\{\frac{k}{n} \leq \xi(\omega) < \frac{k+1}{n}\}}$ , and
- $\eta_n = \sum_{k=0}^{\infty} \frac{k}{n} \chi_{\{\frac{k}{n} \leq \eta(\omega) < \frac{k+1}{n}\}}$ .

It follows that  $\xi_n \leq \xi$  and  $\eta_n \leq \eta$  with  $|\xi - \xi_n| \leq \frac{1}{n}$  and  $|\eta - \eta_n| \leq \frac{1}{n}$  for all  $n$ . Since  $\xi$  and  $\eta$  are integrable we can apply the dominated convergence theorem to deduce that

- $\lim_{n \rightarrow \infty} E(\xi_n) = E(\xi)$ , and

- $\lim_{n \rightarrow \infty} \mathbb{E}(\eta_n) = \mathbb{E}(\eta)$ .

Hence,

$$\begin{aligned}\mathbb{E}(\xi_n \eta_n) &\stackrel{(1)}{=} \sum_{i,j \geq 0} \frac{jk}{n^2} \mathbb{E} \left( \chi_{\{\frac{j}{n} \leq \xi < \frac{j+1}{n}\}} \chi_{\{\frac{k}{n} \leq \eta < \frac{k+1}{n}\}} \right) \\ &\stackrel{(2)}{=} \sum_{i,j \geq 0} \frac{jk}{n^2} \mathbb{E} \left( \chi_{\{\frac{j}{n} \leq \xi < \frac{j+1}{n}\}} \right) \mathbb{E} \left( \chi_{\{\frac{k}{n} \leq \eta < \frac{k+1}{n}\}} \right) \\ &= \mathbb{E}(\xi_n) \mathbb{E}(\eta_n),\end{aligned}$$

where (1) is an application of the monotone convergence theorem, and (2) follows from independence. Moreover,

$$\begin{aligned}|\mathbb{E}(\xi \eta) - \mathbb{E}(\xi_n \eta_n)| &\leq \mathbb{E}(|\xi \eta - \xi_n \eta_n|) \\ &= \mathbb{E}(|\xi(\eta - \eta_n) + \eta_n(\xi - \xi_n)|) \\ &\leq \frac{1}{n} \mathbb{E}(|\xi|) + \frac{1}{n} \mathbb{E} \left( |\eta| + \frac{1}{n} \right) \\ &\rightarrow 0\end{aligned}$$

as  $n \rightarrow \infty$ . Therefore,

$$\mathbb{E}(\xi \eta) = \lim_{n \rightarrow \infty} \mathbb{E}(\xi_n \eta_n) = \lim_{n \rightarrow \infty} \mathbb{E}(\xi_n) \lim_{n \rightarrow \infty} \mathbb{E}(\eta_n) = \mathbb{E}(\xi) \mathbb{E}(\eta),$$

and  $\mathbb{E}(\xi \eta) < \infty$ . The general case follows by using the representations

- $\xi = \xi^+ - \xi^-$ , and
- $\eta = \eta^+ - \eta^-$ .

■

**Proposition 3.2.11** Integrable random variables  $\xi$  and  $\eta$  are independent if and only if for all Borel-measurable functions  $f$  and  $g$  we have  $\mathbb{E}(f(\xi)g(\eta)) = \mathbb{E}(f(\xi))\mathbb{E}(g(\eta))$ .

*Proof.* ( $\Leftarrow$ ). For any  $B_1, B_2 \in \mathcal{B}(\mathbb{R})$  let  $f = \chi_{B_1}$  and  $g = \chi_{B_2}$ . Then

$$\begin{aligned}\mathbb{P}(\xi \in B_1, \eta \in B_2) &= \mathbb{E} \left( \chi_{\{\xi \in B_1\}} \cap \{\eta \in B_2\} \right) \\ &= \mathbb{E} \left( \chi_{\{\xi \in B_1\}} \chi_{\{\eta \in B_2\}} \right) \\ &= \mathbb{E}(f(\xi)g(\eta)) \\ &= \mathbb{E}(f(\xi))\mathbb{E}(g(\eta)) \\ &= \mathbb{E} \left( \chi_{\{\xi \in B_1\}} \right) \mathbb{E} \left( \chi_{\{\eta \in B_2\}} \right) \\ &= \mathbb{P}(\xi \in B_1)\mathbb{P}(\eta \in B_2).\end{aligned}$$

Therefore,  $\xi$  and  $\eta$  are independent.

( $\Rightarrow$ ). We proceed in the same way as the proof of Proposition 3.2.10. ■

### 3.3 Correlation

**Definition 3.3.1** Let  $\xi$  and  $\eta$  be random variables defined on the same probability space. Provided that their expectations exist, their covariance is

$$\text{Cov}(\xi, \eta) := \mathbb{E}((\xi - \mathbb{E}(\xi))(\eta - \mathbb{E}(\eta))).$$

**Remark 3.3.2** Note that

$$\mathbb{V}(\xi + \eta) = \mathbb{V}(\xi) + \mathbb{V}(\eta) + 2\text{Cov}(\xi, \eta).$$

Hence,  $\text{Cov}(\xi, \eta) = 0$  implies that  $\mathbb{V}(\xi + \eta) = \mathbb{V}(\xi) + \mathbb{V}(\eta)$ .

**Definition 3.3.3** Random variables  $\xi$  and  $\eta$  are uncorrelated if

$$\text{Cov}(\xi, \eta) = 0.$$

**Corollary 3.3.4** Independent random variables are uncorrelated.

*Proof.* Using Proposition 3.2.10 we deduce that

$$\text{Cov}(\xi, \eta) = \mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = 0.$$

■

**Remark 3.3.5** The converse of Corollary 3.3.4 is not true.

**Example 3.3.6** Consider a random variable  $\alpha$  which takes the values  $\{0, \frac{\pi}{2}, \pi\}$  uniformly. Then  $\xi = \sin \alpha$  and  $\eta = \cos \alpha$  are uncorrelated since

$$\begin{aligned}\text{Cov}(\xi, \eta) &= \mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) \\ &= 0 - \left(\frac{1}{3}\right)(0) \\ &= 0.\end{aligned}$$

However, they are not independent since

$$\mathbb{P}(\xi = 1, \eta = 1) = 0 \neq \frac{1}{9} = \mathbb{P}(\xi = 1)\mathbb{P}(\eta = 1).$$

On the other hand, the random variables  $\xi$  and  $\eta^2$  are correlated since

$$\begin{aligned}\text{Cov}(\xi, \eta^2) &= \mathbb{E}(\xi\eta^2) - \mathbb{E}(\xi)\mathbb{E}(\eta^2) \\ &= 0 - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \\ &= -\frac{2}{9}.\end{aligned}$$

## Part II. Concepts of Convergence

### 4 Convergence in Probability

We now have sufficient tools from measure theory to get into the first serious topic of probability, limiting theorems. Given a random sequence  $(\xi_1, \xi_2, \dots)$  with  $\xi_i$  independently and identically distributed, we would like to study the deviation between the empirical mean  $\frac{S_n}{n}$ , where  $S_n = \xi_1 + \dots + \xi_n$ , and the expectation  $\mathbb{E}(\xi_1)$  as  $n \rightarrow \infty$ .

#### 4.1 Definition and Properties

We have already encountered one form of convergence, namely  $L^p$  convergence. Here we introduce an alternative notion of convergence, known as convergence in probability.

**Definition 4.1.1** A sequence  $\xi_1, \xi_2, \dots$  of random variables from a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  to  $\mathbb{R}$  converges in probability, or in measure  $\mathbb{P}$ , to the random variable  $\xi$ , denoted by  $\xi_n \xrightarrow{p} \xi$ , if for every  $\varepsilon > 0$  we have

$$\mathbb{P}(|\xi_n - \xi| > \varepsilon) \rightarrow 0,$$

as  $n \rightarrow \infty$ .

**Proposition 4.1.2** Let  $\xi, \xi_1, \xi_2, \dots$  be random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$ . If  $\xi_n \xrightarrow{L^p} \xi$ , for  $p \geq 1$  then  $\xi_n \xrightarrow{p} \xi$ .

*Proof.* For the random variable  $\xi_n$  the random variable  $|\xi_n - \xi|^p$  is a non-negative random variable. So for any  $\epsilon > 0$  by Markov's inequality, it follows that

$$\begin{aligned} \mathbb{P}(|\xi_n - \xi| \geq \epsilon) &= \mathbb{P}(|\xi_n - \xi|^p \geq \epsilon^p) \\ &\leq \frac{\mathbb{E}(|\xi_n - \xi|^p)}{\epsilon^p}. \end{aligned}$$

Taking  $n \rightarrow \infty$  the right-hand tends to zero by assumption, so we conclude that  $\xi_n \xrightarrow{p} \xi$ . ■

**Example 4.1.3** The converse of Proposition 4.1.2 is not true. Consider  $\xi_n = n\chi_{(0, \frac{1}{n})}(\omega)$ . Then

$$\mathbb{P}(|\xi_n| > \epsilon) = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore,  $\xi_n \xrightarrow{p} 0$ . However,

$$\mathbb{E}(|\xi_n|^p) = n^{p-1} \not\xrightarrow{n \rightarrow \infty} 0.$$

Therefore,  $\xi_n \not\xrightarrow{L^p} 0$ .

**Exercise 4.1.4** Let  $(\xi_i)_{i \geq 1}$  and  $(\eta_i)_{i \geq 1}$  be sequences of random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$ , and let  $\xi, \eta$  be random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

1. Check that the limit of convergence in probability is almost surely unique. That is, if  $\xi_i$  converges in probability to  $\xi$  and  $\xi'$  then  $\xi = \xi'$  almost surely.
2. Prove that if  $\xi_i \xrightarrow{p} \xi$  and  $\eta_i \xrightarrow{p} \eta$  then for all real numbers  $a, b$  we have  $a\xi_i + b\eta_i \xrightarrow{p} a\xi + b\eta$ .
3. Prove that if  $\xi_i \xrightarrow{p} \xi$  and  $\eta_i \xrightarrow{p} \eta$  then  $\xi_i\eta_i \xrightarrow{p} \xi\eta$ . Notice this is not necessarily true if we have  $L^p$

convergence. What's wrong with the argument, and can we refine the statement?

4. Show that if  $\xi_i \xrightarrow{P} \xi$ ,  $\eta_i \xrightarrow{P} \eta$  and  $\varphi(x, y)$  is a continuous function, then

$$\varphi(\xi_i, \eta_i) \xrightarrow{P} \varphi(\xi, \eta).$$

## 4.2 Coin Flipping Example

We can motivate the notion of convergence in probability by considering flipping  $n$  independent coins. Firstly, consider the flipping of just one coin. Assume the outcomes are zero and one with the probability  $p \in (0, 1)$  of getting a one. We hope to express this as a  $\{0, 1\}$ -valued random variable  $\xi$  on a suitable probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , such that  $\mathbb{P}_\xi(\{0\}) = 1 - p$  and  $\mathbb{P}_\xi(\{1\}) = p$ , or in other words,

$$\mathbb{P}_\xi(\{x\}) = p^x(1 - p)^{1-x}, \quad (4.1)$$

for  $x = 0, 1$ . For the probability space, there are several choices.

- The natural choice would be

- $\Omega = \{0, 1\}$ ,
- $\mathcal{A} = 2^\Omega$ , and
- $\mathbb{P}(\{\omega\}) = p^\omega(1 - p)^{1-\omega}$ .

In this case, our desired random variable would be  $\eta(\omega) = \omega$ .

- A more complicated choice would be

- $\Omega = [0, 1]$ ,
- $\mathcal{A} = \mathcal{B}([0, 1])$ , and
- $\mathbb{P}(E) = \text{Leb}(E)$ .

In this case, our desired random variable would be  $\xi(\omega) = \chi_{(p,1]}(\omega)$ .

**Remark 4.2.1** The more complicated formulation represents how a computer simulates the flipping of a biased coin. First, it generates a random number  $r \in [0, 1]$  from a uniform distribution (for instance, by using the `numpy.random.rand` function in Python), then returns 0 if  $r < 1 - p$  and 1 otherwise.

In both cases we see that  $\mathbb{P}_\xi(\{0\}) = 1 - p$  and  $\mathbb{P}_\xi(\{1\}) = p$ . In fact, from Theorem 2.2.10 we know that the distribution functions, and therefore the expectation, will not depend on our choice of probability spaces and random variables, as long as  $\mathbb{P}_\xi$  satisfies (4.1). How can we extend the experiment to the flipping of  $n$  coins? It would be wrong to define  $\xi_1, \dots, \xi_n$  on the same sample space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that  $\xi_1 = \dots = \xi_n$ . This would correspond to the flipping of a single coin once and recording the result  $n$  times. It is clear that in such a construction the random variables would not be independent. It will be hard to write down a large number of independent random variables defined on any of the sample spaces  $(\Omega, \mathcal{A}, \mathbb{P})$  in the above example. A standard way of describing  $n$  independent coin flips (or  $n$  independent trials in general) is to assume that the random variables  $\xi_1, \dots, \xi_n$  lie in different probability spaces. That is,  $\xi_1$  is defined on  $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ ,  $\xi_2$  is defined on  $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$  and so on. Proceeding in this way requires us to operate in these different spaces simultaneously. Thus we need to consider the product space  $(\Omega^{(n)}, \mathcal{A}^{(n)}, \mathbb{P}^{(n)}) = \otimes_{i=1}^n (\Omega_i, \mathcal{A}_i, \mathbb{P}_i)$ . The sample space of this product space is

$$\Omega^{(n)} = \Omega_1 \times \dots \times \Omega_n = \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \Omega_i, \text{ for } i = 1, \dots, n\},$$

the collection of events are

$$\mathcal{A}^{(n)} = \sigma(\{A_1 \times \dots \times A_n : A_i \in \mathcal{A}_i, \text{ for } i = 1, \dots, n\}),$$

and the probability measure  $\mathbb{P}^{(n)}$  satisfies

$$\mathbb{P}^{(n)}(A_1 \times \dots \times A_n) = \prod_{i=1}^n \mathbb{P}_i(A_i).$$

Consequently, we define the family of projection functions onto the  $i^{\text{th}}$  component,  $\text{proj}_i^{(n)} : (\Omega^{(n)}, \mathcal{A}^{(n)}) \rightarrow (\Omega_i, \mathcal{A}_i)$  such that  $\text{proj}_i^{(n)}(\omega_1, \dots, \omega_n) = \omega_i$ . For convenience, we drop the superscript  $(n)$  if there is no ambiguity. Notice that the projection functions are measurable since the pre-image of any set in  $\mathcal{A}_i$  is

$$\text{proj}_i^{-1}(A_i) = \Omega_1 \times \dots \times \Omega_{i-1} \times A_i \times \Omega_{i+1}, \dots, \Omega_n \in \mathcal{A}^{(n)}.$$

We can define the random variables  $\tilde{\xi}_i : (\Omega^{(n)}, \mathcal{A}^{(n)}, \mathbb{P}^{(n)}) \rightarrow \{0, 1\}$  such that  $\tilde{\xi}_i(\omega) = \xi_i(\text{proj}_i(\omega))$ . These random variables are an accurate description of flipping  $n$  coins.

1. The marginal distribution of  $\xi_i$ , defined as the measure  $A \mapsto \mathbb{P}_{\tilde{\xi}_i}(\Omega_1 \times \dots \times A \times \dots \times \Omega_n)$  satisfies (4.1).
2. The family  $(\tilde{\xi}_i)$  of random variables are independent.

**Exercise 4.2.2** Verify the above assertions.

Next, we want to extend the above construction to infinity. More specifically, we want to consider the space  $(\Omega, \mathcal{A}) = \otimes_{i=1}^{\infty} (\Omega_i, \mathcal{A}_i)$  with a suitable probability measure  $\mathbb{P}$ , so that we can discuss large-sample theorems. It should be the case that this probability measure satisfies

$$\mathbb{P}(A_1 \times \dots \times A_n \times \Omega_{n+1} \times \dots) = \prod_{i=1}^n \mathbb{P}_i(A_i).$$

If we use  $\Omega_i \equiv \Omega$  and  $\mathcal{A}_i \equiv \mathcal{A}$  in our above examples and assume the natural choice, then we can safely set

$$\mathbb{P}(\{(\omega_1, \omega_2, \dots)\}) = \prod_{i=1}^{\infty} p^{\omega_i} (1-p)^{1-\omega_i} = p^{\sum \omega_i} (1-p)^{\sum (1-\omega_i)},$$

since the probability measure is well-defined for all singletons  $\{(\omega_1, \omega_2, \dots)\}$ . If we use the example when  $\Omega_i = [0, 1]$ , we can check that our sequence of measures  $(\mathbb{P}^{(n)})$  is consistent and we can apply Theorem 1.6.2 to construct  $\mathbb{P}$ .

The above shows that we need not worry about specifying a single probability space to describe a sequence of independent experiments. If we want to describe an infinite sequence of experiments with an underlying distribution  $\mathbb{P}_{\xi}$ , we can consider the infinite product space  $(\mathbb{R}^{\infty}, \mathcal{B}(\mathbb{R}^{\infty}))$  equipped with the probability measure  $\mathbb{P}$  as determined by the Kolmogorov extension theorem. The projections onto the  $i^{\text{th}}$  component  $\text{proj}_i$  are random variables with distribution  $\mathbb{P}_{\xi}$ . From now on, we abuse notation by not mentioning the underlying probability space, dropping the tilde sign above  $\xi$  and interpreting any operations in the above sense.

### 4.3 Bernoulli's Law of Large numbers

With  $\xi_k$  a  $\{0, 1\}$ -valued random variable, taking value 1 with probability  $p \in (0, 1)$ , let  $S_n = \xi_1 + \dots + \xi_n$ . Then

$$\mathbb{E}(S_n) = \sum_{j=1}^n \mathbb{E}(\xi_j) = \sum_{j=1}^n (1 \cdot \mathbb{P}_{\xi_j}(\xi_j = 1) + 0 \cdot \mathbb{P}_{\xi_j}(\xi_j = 0)) = np.$$

Thus the mean value of  $\frac{S_n}{n}$  is equal to  $p$ . The question now is what does  $|\frac{1}{n} S_n(\omega) - p|$  converge to for large  $n$ ? Moreover, in what sense does this convergence occur? It cannot be that

$$\left| \frac{S_n(\omega)}{n} - p \right| \rightarrow 0$$

uniformly or pointwise in  $\omega$ , because there is always an  $\omega$  such that  $\xi_i(\omega) = 1$  for all  $i$ , so  $S_n(\omega)/n \equiv 1 \not\rightarrow p$ . Therefore, we must consider a weaker notion of convergence, as illustrated in the following exercise.

**Exercise 4.3.1** Verify that  $S_n \sim \text{B}(n, p)$ . Hence, show that

$$\left\| \frac{S_n}{n} - p \right\|_{L^2}^2 \rightarrow 0$$

as  $n \rightarrow \infty$ .

With a more general analysis, we have  $L^p$  convergence for any  $p \in [1, \infty)$ . Moreover, by Chebyshev's inequality, it follows for any fixed  $\epsilon > 0$  that

$$\mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \epsilon\right) \leq \frac{\mathbb{V}\left(\frac{S_n}{n}\right)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2}. \quad (4.2)$$

Hence, we can always make the probability  $\mathbb{P}(|\frac{S_n}{n} - p| > \epsilon)$  arbitrarily small. That is, we have convergence in probability. Stating the above formally gives us the Bernoulli Law of Large numbers.

**Theorem 4.3.2 — Bernoulli Law of Large Numbers.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent and identically distributed Bernoulli random variables, with parameter  $p \in (0, 1)$ . Then  $\frac{S_n}{n}$  converges in probability to  $p$ .

## 4.4 Weak Law of Large Numbers

We can generalise the ideas of the previous section. Note, from Markov's inequality, if  $\xi_n \rightarrow \xi$  in  $L^p$  for  $p \geq 1$ , then we must have  $\xi_n \xrightarrow{p} \xi$ , because

$$\mathbb{P}(|\xi_n - \xi| > \varepsilon) \leq \frac{\|\xi_n - \xi\|_{L^p}^p}{\varepsilon^p} \rightarrow 0.$$

Let  $\xi_1, \dots, \xi_n$  be random variables and let

$$S_n^{(c)} = \sum_{j=1}^n (\xi_j - \mathbb{E}(\xi_j)).$$

Observe that  $\mathbb{E}(S_n^{(c)}) = 0$ . How can we use Chebyshev's inequality to make the weakest assumptions on the  $\xi_1, \dots, \xi_n$  such that  $S_n^{(c)}$  converges? For simplicity, we can first assume that the  $\xi_i \in L^2$  with  $\mathbb{V}(\xi_i) \leq C$  for some constant  $C$  independent of  $i$ . Then, if we assume that the  $\xi_1, \dots, \xi_n$  are pairwise uncorrelated, which is a weaker assumption than independence, we have

$$\mathbb{V}(S_n) = \sum_{i=1}^n \mathbb{V}(\xi_i) \leq Cn.$$

**Theorem 4.4.1 —  $L^2$  Weak Law of Large Numbers.** Let  $\xi_1, \dots, \xi_n$  be uncorrelated  $L^2$  random variables such that  $\mathbb{V}(\xi_j) \leq C$  for some  $C > 0$  and all  $n \geq 1$ . Then

$$\frac{S_n^{(c)}}{n} \xrightarrow{p} 0.$$

*Proof.* Using Chebyshev's inequality and the fact the  $\xi_j$  are uncorrelated, it follows that for all  $\epsilon > 0$  we have

$$\mathbb{P}\left(\left|\frac{S_n^{(c)}}{n}\right| > \epsilon\right) \leq \frac{\mathbb{V}\left(\frac{S_n^{(c)}}{n}\right)}{\epsilon^2} \leq \frac{C}{n\epsilon^2} \xrightarrow{n \rightarrow \infty} 0.$$

■

**Corollary 4.4.2** Let  $\xi_1, \xi_2, \dots$  be integrable, independent and identically distributed random variables, such that  $\mathbb{V}(\xi_j) < \infty$ . Then

$$\frac{S_n}{n} \xrightarrow{p} \mathbb{E}(\xi_1).$$

**Example 4.4.3** Consider  $f$  a measurable function on  $[0, 1]$  with

$$C := \int_0^1 |f(x)|^2 dx < \infty,$$

for example  $f$  could be uniformly bounded by some number  $M$ . The integral

$$\theta = \int_0^1 f(x) dx,$$

is often intractable to compute in practice. However, if  $U$  is a random variable on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with uniform distribution  $\text{Unif}[0, 1]$ , then  $\theta = \mathbb{E}(f(U))$ . For example, one could take  $U : \omega \in ([0, 1], \mathcal{B}[0, 1], \text{Leb}) \mapsto \omega$ . Therefore, an approach to estimating  $\theta$  would be to use an "empirical mean".

1. Define the random variables  $U_1(\omega), U_2(\omega), \dots \stackrel{\text{iid}}{\sim} \text{Unif}[0, 1]$  using a construction as detailed in Section 4.2.
2. Then evaluate the empirical mean

$$\hat{\theta}_n(\omega) := \frac{f(U_1(\omega)) + \dots + f(U_n(\omega))}{n}.$$

It is clear that  $\mathbb{E}(\hat{\theta}_n) = \theta$ . Moreover, since the random variables  $f(U_i)$  are independent and identically distributed with finite variance

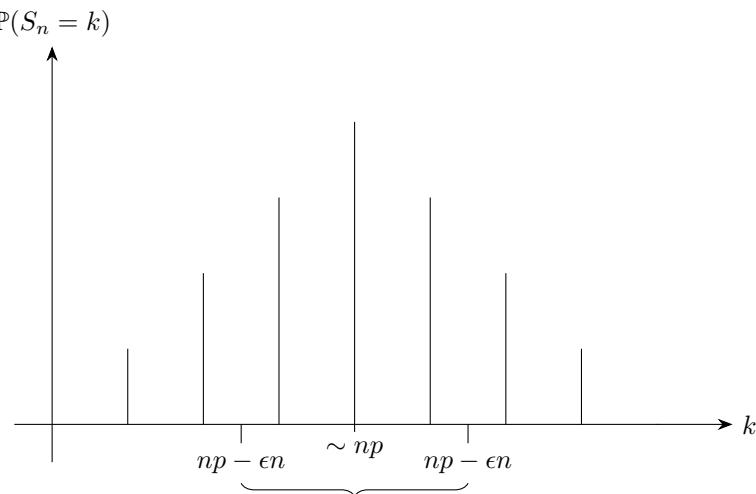
$$\mathbb{V}(f(U_i)) = \mathbb{E}((f(U_i))^2) - \theta^2 = C - \theta^2 < \infty,$$

Theorem 4.4.1 tells us that  $\hat{\theta}_n(\omega) \xrightarrow{n \rightarrow \infty} \theta$  in probability.

**Exercise 4.4.4 —  $L^2$  Weak Law of Large Number for weakly correlated random variables.** Let  $\xi_1, \xi_2, \dots$  be random variables on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\mathbb{E}(\xi_i) = 0$  and  $\mathbb{E}(\xi_i \xi_j) = r_{|i-j|}$ , where  $(r_k)_{k \geq 1}$  is a sequence of real numbers such that  $r_k \xrightarrow{k \rightarrow \infty} 0$ . Let  $S_n = \sum_{i=1}^n \xi_i$ . Show that  $\frac{S_n}{n} \xrightarrow{n \rightarrow \infty} 0$  in probability.

## 4.5 Local and Central Limit Theorem

We return to the coin-flipping scenario. Let  $S_n = \xi_1 + \xi_2 + \dots + \xi_n$ , where  $\xi_k \stackrel{\text{iid}}{\sim} \text{B}(1, p)$ . As discussed in the previous section,  $S_n$  tends to be close to  $np$  for large  $n$ .



Specifically, let us take some interval  $\mathcal{I}_n = (n(p - \varepsilon), n(p + \varepsilon))$ . If we pick some sufficiently large  $\varepsilon$ , say  $\varepsilon = n^\alpha$  where  $\alpha > \frac{1}{2}$ , then by Chebyshev's inequality (4.2) we know that

$$\mathbb{P}(S_n \in \mathcal{I}_n^c) = \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > n^{\alpha-1}\right) \leq \frac{p(1-p)}{n^{1+2\alpha-2}} = \frac{p(1-p)}{n^{2\alpha-1}} \xrightarrow{n \rightarrow \infty} 0.$$

These decaying bounds of  $\mathbb{P}(S_n \in \mathcal{I}_n^c)$  no longer exist when  $\alpha \leq \frac{1}{2}$ . How much do we know about  $\mathbb{P}(S_n \in \mathcal{I}_n^c)$  for this case? Will it tend to some non-trivial constant in  $(0, 1)$ , or will it increase and tend to 1? We will see that the central limit theorem suggests that at the boundary  $\alpha = \frac{1}{2}$ , the quantity  $\mathbb{P}(S_n \in \mathcal{I}_n^c)$  tends to some non-trivial constant in  $(0, 1)$ . This suggests that the rescaled mean  $\frac{S_n}{\sqrt{n}}$  will "converge" in some way to a non-trivial distribution.

**Definition 4.5.1** Consider two sequences  $(f_n)_{n \in \mathbb{N}}$  and  $(g_n)_{n \in \mathbb{N}}$ .

- Big  $O$  notation. We say that  $g_n = O(f_n)$  as  $n \rightarrow \infty$  if  $\left|\frac{g_n}{f_n}\right|$  is bounded for sufficiently large  $n$ . That is, there exists constant  $C > 0$  and  $N$  such that for all  $n \geq N$  we have  $|g_n| \leq C|f_n|$ .
- Small  $o$  notation. We say that  $g_n = o(f_n)$  as  $n \rightarrow \infty$  if  $\left|\frac{g_n}{f_n}\right| \rightarrow 0$  as  $n \rightarrow \infty$ . In other words, for all  $\epsilon > 0$ , there exists  $N := N(\epsilon)$  such that for all  $n \geq N$  we have  $|g_n| \leq \epsilon|f_n|$ . We sometimes write  $g_n \ll f_n$  or  $f_n \gg g_n$ .
- Asymptotic equivalence. We write  $g_n \sim f_n$  if  $\left|\frac{g_n}{f_n}\right| \rightarrow 1$  as  $n \rightarrow \infty$ . Equivalently, we have  $g_n = (1 + o(1))f_n$ .
- Order. We write  $g_n = \Theta(f_n) = \text{ord}(f_n)$  if  $g_n = O(f_n)$  but  $g_n$  is not  $o(f_n)$ .

**Remark 4.5.2**

- The use of equal sign is an abuse of notation.
- The definitions can be extended to any functions  $f(x)$  defined on real or complex numbers, in such case, we can assume  $x$  tends to some point  $x_0$  including  $\infty$ .
- We can also consider order notations for sequences of functions. Let  $g_n = g_n(\alpha)$ ,  $f_n = f_n(\alpha)$ . We say  $g_n(\alpha) = O(|f_n(\alpha)|)$  uniformly if the above definition holds for constants  $C$  and  $N$  independent of  $\alpha$ . We also have analogous definition for  $g_n(\alpha) = o(|f_n(\alpha)|)$ .

**Lemma 4.5.3 — Stirling's Approximation.** As  $n \rightarrow \infty$ , we have

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Using Lemma 4.5.3 we can sketch an asymptotic analysis of the binomial coefficient. Suppose  $n, k, n-k$  all tend to infinity, for example  $k = np$  for  $p \in (0, 1)$ , with Lemma 4.5.3 we obtain

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{k}{e}\right)^k \left(\frac{n-k}{e}\right)^{n-k}} \frac{1 + O\left(\frac{1}{n}\right)}{(1 + O\left(\frac{1}{n}\right))(1 + O\left(\frac{1}{n}\right))} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \exp(n \log n - k \log k - (n-k) \log(n-k)) \frac{1 + O\left(\frac{1}{n}\right)}{(1 + O\left(\frac{1}{n}\right))(1 + O\left(\frac{1}{n}\right))}. \end{aligned}$$

To gain intuition, one can treat  $O\left(\frac{1}{n}\right)$  as being exactly equal to  $\frac{1}{n}$ . That is,

$$\left(1 + O\left(\frac{1}{n}\right)\right)^{-2} = \left(1 + \frac{1}{n}\right)^{-2} = 1 - \frac{2}{n} + \dots,$$

and so,

$$\frac{1 + O\left(\frac{1}{n}\right)}{(1 + O\left(\frac{1}{n}\right))(1 + O\left(\frac{1}{n}\right))} = \left(1 + \frac{1}{n}\right) \left(1 - \frac{2}{n} + \dots\right) = 1 - \frac{1}{n} + \dots = 1 + O\left(\frac{1}{n}\right).$$

Putting this together, we have

$$\binom{n}{k} = \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \exp(n \log n - k \log k - (n-k) \log(n-k)) \left(1 + O\left(\frac{1}{n}\right)\right). \quad (4.3)$$

**Exercise 4.5.4** Show that  $\mathbb{P}(S_n = k)$  is monotone in  $k$  below and above its point of maximum.

With this, we can prove the local limit theorem, which specifies the local asymptotics of a probability mass distribution at the point  $S_n = k$ .

**Theorem 4.5.5 — Local Limit Theorem.** For any  $0 < p < 1$ , we have

$$\max_{0 \leq k \leq n} \left| \mathbb{P}(S_n = k) - \frac{1}{\sqrt{2\pi p(1-p)\sqrt{n}}} e^{-\frac{x^2}{2p(1-p)}} \right| = o\left(\frac{1}{\sqrt{n}}\right),$$

as  $n \rightarrow \infty$  and where  $x = x_{k,n} := \frac{k - np}{\sqrt{n}}$ .

*Proof.* Consider  $k$  such that

$$|x_{k,n}| \leq \frac{A_n}{\sqrt{n}}$$

where  $A_n = n^\epsilon$  for  $\epsilon \in (0, 1)$ . Then as  $k = np + x\sqrt{n}$  we have that

$$k = np \left(1 + O\left(\frac{A_n}{n}\right)\right)$$

which implies that

$$n - k = n(1 - p) \left(1 + O\left(\frac{A_n}{n}\right)\right).$$

Hence,  $k$  and  $n - k$  tend to infinity as  $n \rightarrow \infty$ , which means utilise (4.3) to get that

$$\mathbb{P}(S_n = k) = \underbrace{\frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}}}_{(A)} \underbrace{\exp(n \log n - k(\log k - \log p) - (n-k)(\log(n-k) - \log(1-p)))}_{(B)} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Analysing (A) we notice that

$$\begin{aligned} (A) &= \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \\ &= \frac{1}{\sqrt{2\pi np(1-p)} \left(1 + O\left(\frac{A_n}{n}\right)\right) \left(1 + O\left(\frac{A_n}{n}\right)\right)} \\ &= \frac{1}{\sqrt{2\pi np(1-p)}} \left(1 + O\left(\frac{A_n}{n}\right)\right). \end{aligned}$$

Similarly, analysing (B) we notice that

$$\begin{aligned} (B) &= \exp\left(n \log n - k \left(\log n + \log\left(1 + \frac{x}{p\sqrt{n}}\right)\right) - (n-k) \left(\log n + \log\left(1 - \frac{x}{(1-p)\sqrt{n}}\right)\right)\right) \\ &= \exp\left(-\left((np + x\sqrt{n}) \log\left(1 + \frac{x}{p\sqrt{n}}\right) + (n(1-p) - x\sqrt{n}) \log\left(1 - \frac{x}{(1-p)\sqrt{n}}\right)\right)\right) \end{aligned}$$

$$\begin{aligned}
&= \exp \left( - \left( np \left( \frac{x}{p\sqrt{n}} - \frac{x^2}{2p^2n} + O \left( \frac{x^3}{n^{3/2}} \right) \right) + \frac{x^2}{p} + O \left( \frac{x^3}{n^{1/2}} \right) \right) \right. \\
&\quad \left. + n(1-p) \left( -\frac{x}{(1-p)\sqrt{n}} - \frac{x^2}{2(1-p)^2n} + O \left( \frac{x^3}{n^{3/2}} \right) \right) + \frac{x^2}{(1-p)} + O \left( \frac{x^3}{n^{1/2}} \right) \right) \\
&= \exp \left( -\frac{x^2}{2p(1-p)} + O \left( \frac{x^3}{\sqrt{n}} \right) \right) \\
&= \exp \left( -\frac{x^2}{2p(1-p)} + O \left( \frac{A_n^3}{n^2} \right) \right) \\
&= \exp \left( -\frac{x^2}{2p(1-p)} \right) \left( 1 + O \left( \frac{A_n^3}{n^2} \right) \right).
\end{aligned}$$

Hence,

$$\mathbb{P}(S_n = k) = \frac{1}{\sqrt{2\pi p(1-p)n}} \exp \left( -\frac{x^2}{2p(1-p)} \right) \left( 1 + O \left( \frac{A_n}{n} \right) + O \left( \frac{A_n^3}{n^2} \right) \right).$$

Letting  $\varepsilon = \frac{7}{12}$ , we get that  $\frac{A_n^3}{n^2} = n^{-1/4}$  and  $\frac{A_n}{n} = n^{-5/12}$ , to yield

$$\max_{|x| \leq \frac{A_n}{\sqrt{n}}} \mathbb{P}(S_n = k) = \frac{1}{\sqrt{2\pi p(1-p)n}} \exp \left( -\frac{x^2}{2p(1-p)} \right) \underbrace{\left( 1 + O \left( \frac{1}{n^{5/12}} \right) \right)}_{=1+o\left(\frac{1}{\sqrt{n}}\right)}. \quad (4.4)$$

Now consider  $|x| > \frac{A_n}{\sqrt{n}}$ . Observe that

$$\begin{aligned}
&\max_{|x| > \frac{A_n}{\sqrt{n}}} \left| \mathbb{P}(S_n = k) - \frac{1}{\sqrt{2\pi p(1-p)n}} \exp \left( -\frac{x^2}{2p(1-p)} \right) \right| \\
&\leq \max_{|x| > \frac{A_n}{\sqrt{n}}} |\mathbb{P}(S_n = k)| + \max_{|x| > \frac{A_n}{\sqrt{n}}} \left| \frac{1}{\sqrt{2\pi p(1-p)n}} \exp \left( -\frac{x^2}{2p(1-p)} \right) \right| \\
&\leq \max |\mathbb{P}(S_n = \lfloor np + A_n \rfloor), \mathbb{P}(S_n = \lceil np - A_n \rceil)| + \frac{1}{\sqrt{2\pi p(1-p)n}} \exp \left( -\frac{A_n^2}{2np(1-p)} \right).
\end{aligned}$$

Where the first bound is an application 4.5.4. With  $A_n = n^{\frac{7}{12}}$ , note that the second term is  $o\left(\frac{1}{\sqrt{n}}\right)$ . Furthermore, as

$$n^{\frac{1}{12}} - n^{-\frac{1}{2}} = \frac{np + A_n - np - 1}{\sqrt{n}} \leq \frac{\lfloor np + A_n \rfloor - np}{\sqrt{n}} \leq \frac{np + A_n - np}{\sqrt{n}} = n^{\frac{1}{12}}$$

we note that  $x_{\lfloor np + A_n \rfloor, k} \sim n^{\frac{1}{12}}$ . A similar result holds for  $x_{\lceil np + A_n \rceil, k}$ , and so the first term is also  $o\left(\frac{1}{\sqrt{n}}\right)$ . Combining these results with (4.4) completes the proof. ■

Theorem 4.5.5 says that

$$\mathbb{P} \left( \frac{S_n - np}{\sqrt{n}} = x \right) = \frac{1}{\sqrt{n}} \left( \frac{1}{\sqrt{2\pi p(1-p)}} e^{-\frac{x^2}{2p(1-p)}} + o(1) \right)$$

as  $n \rightarrow \infty$ . At first glance, you may find this result not useful, as it only tells us that the probability decays to zero at a rate of  $O\left(\frac{1}{\sqrt{n}}\right)$ . However, since  $\frac{S_n - np}{np(1-p)}$  seems to converge to a continuous distribution, we really should look at the density by ignoring the  $\frac{1}{\sqrt{n}}$ . The things inside the square bracket suggest that the density function of  $n^{-\frac{1}{2}}(S_n - np)$  converges to a normal distribution  $N(0, p(1-p))$ , which is equivalent to the distribution of  $\frac{S_n - np}{np(1-p)}$  converging to standard normal  $N(0, 1)$ . The above heuristics can be formalised by adding the local probabilities and considering the cumulative distribution function. We therefore arrive at the central limit theorem.

**Theorem 4.5.6 — de Moivre-Laplace CLT.** For any  $0 < p < 1$  and  $x \in \mathbb{R}$  we have,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{S_n - np}{\sqrt{np(1-p)}} \leq x \right) = \Phi(x),$$

where

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy,$$

is the density of  $N(0, 1)$ .

*Proof Sketch.* Note

$$\begin{aligned} \mathbb{P} \left( \frac{S_n - np}{\sqrt{np(1-p)}} \leq x \right) &= \mathbb{P} \left( S_n \leq np + x\sqrt{np(1-p)} \right) \\ &= \sum_{k=0}^{\lfloor np - n^{\frac{7}{12}} \rfloor - 1} \mathbb{P}(S_n = k) + \sum_{k=\lfloor np - n^{\frac{7}{12}} \rfloor}^{\lfloor np + x\sqrt{np(1-p)} \rfloor} \mathbb{P}(S_n = k). \end{aligned}$$

The first term is a sum of a polynomial number of terms of order  $O\left(\exp\left(-n^{\frac{1}{2}}\right)\right)$  and so will vanish as  $n \rightarrow \infty$ . The second term is a Riemann sum. Writing

$$T_n = \left\{ k : \left\lfloor np - n^{\frac{7}{12}} \right\rfloor \leq k \leq \left\lfloor np + x\sqrt{np(1-p)} \right\rfloor \right\}$$

we have

$$\begin{aligned} \sum_{k \in T_n} \mathbb{P}(S_n = k) &= \sum_{k \in T_n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi p(1-p)}} \exp \left( -\frac{1}{2} \left( \frac{k - np}{\sqrt{np(1-p)}} \right)^2 \right) \\ &= \sum_{k \in T_n - np} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi p(1-p)}} \exp \left( -\frac{1}{2} \left( \frac{k/\sqrt{n}}{\sqrt{p(1-p)}} \right)^2 \right). \end{aligned}$$

This is almost a Riemann sum on a partition of  $(-\infty, x]$  with a mesh size of  $\frac{1}{\sqrt{n}}$ , however, it is missing some boundary terms. One can then show that the boundary terms lead to an  $o(1)$  contribution, and conclude that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{S_n - np}{\sqrt{np(1-p)}} \leq x \right) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

We will omit the details here. ■

**Remark 4.5.7** For  $a < b$  it follows that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b \right) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) dy.$$

In particular, we can let  $a = -\epsilon \sqrt{\frac{n}{p(1-p)}}$  and  $b = \epsilon \sqrt{\frac{n}{p(1-p)}}$  to conclude that

$$0 \leq \mathbb{P} \left( \left| \frac{S_n}{n} - p \right| > \epsilon \right) = \lim_{n \rightarrow \infty} \left( \int_{-\infty}^{-\epsilon \sqrt{\frac{n}{p(1-p)}}} + \int_{\epsilon \sqrt{\frac{n}{p(1-p)}}}^{\infty} \right) \exp \left( -\frac{y^2}{2} \right) dy + o(1) = 0.$$

Hence, we obtain Theorem 4.3.2.

Theorem 4.5.6 demonstrates that for a sequence of independent Bernoulli random variables the quantity  $\sqrt{n}(\frac{S_n}{n} - p)$  converges in distribution to a random variable with normal distribution  $N(0, p(1-p))$ .

**Exercise 4.5.8** Using Theorem 4.5.6 prove Theorem 4.3.2.

**Remark 4.5.9** Regarding  $S_n \sim \text{Bin}(n, p)$ , we can consider the behaviour of  $S_n$  if we let  $p$  be a function of  $n$ . Specifically, we let  $p = p(n)$  be such that  $p(n) \rightarrow 0$  and  $p(n) \cdot n \rightarrow \lambda > 0$  as  $n \rightarrow \infty$ . Then for fixed  $k$  we have

$$\begin{aligned}\mathbb{P}(S_n = k) &= \frac{n!}{k!(n-k)!} p(n)^k (1-p(n))^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right)^k \left(1 - \frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right)^{n-k}.\end{aligned}$$

Therefore,

$$\mathbb{P}(S_n = k) \rightarrow \frac{1}{k!} \lambda^k e^{-\lambda},$$

and so  $S_n \sim \text{Bin}(n, p(n))$  converges to a  $\text{Pois}(\lambda)$  distribution.

## 4.6 Solution to Exercises

### Exercise 4.1.4

*Solution.*

- Consider the set  $A_k = \{|\xi - \xi'| > \frac{1}{k}\}$ . It is clear that  $\{\xi \neq \xi'\} = \bigcup_{k=1}^{\infty} A_k$ . Moreover,

$$\begin{aligned}\mathbb{P}(A_k) &\stackrel{(1)}{\leq} \mathbb{P}\left(|\xi - \xi_n| + |\xi_n - \xi'| > \frac{1}{k}\right) \\ &\stackrel{(2)}{\leq} \mathbb{P}\left(|\xi - \xi_n| > \frac{1}{2k}\right) + \mathbb{P}\left(|\xi_n - \xi'| > \frac{1}{2k}\right) \\ &\xrightarrow{n \rightarrow \infty} 0.\end{aligned}$$

Where (1) is an application of the triangle inequality, and in (2) we use the fact that

$$\left\{|\xi - \xi_n| + |\xi_n - \xi'| > \frac{1}{k}\right\} \subseteq \left\{|\xi - \xi_n| > \frac{1}{2k}\right\} \cup \left\{|\xi_n - \xi'| > \frac{1}{2k}\right\}.$$

Therefore,  $\mathbb{P}(A_k) = 0$  which implies that

$$\begin{aligned}\mathbb{P}(\{\xi \neq \xi'\}) &= \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) \\ &\stackrel{(1)}{=} \lim_{n \rightarrow \infty} \mathbb{P}(A_k) \\ &= 0.\end{aligned}$$

Where in (1) we have used the continuity of the measure  $\mathbb{P}$ , as  $A_k$  is a sequence of increasing events. We conclude that  $\xi = \xi'$  almost everywhere.

- Using the triangle inequality we have that

$$|a\xi_n + b\eta_n - a\xi - b\eta| \leq |a||\xi_n - \xi| + |b||\eta_n - \eta|.$$

Therefore,

$$\mathbb{P}(|a\xi_n + b\eta_n - a\xi - b\eta| \geq \epsilon) \leq \mathbb{P}(|a||\xi_n - \xi| + |b||\eta_n - \eta| \geq \epsilon).$$

If  $\omega$  satisfies the inequality on the right-hand side, it must be the case that either

- (a)  $|a||\xi_n - \xi| \geq \frac{\epsilon}{2}$ , or
- (b)  $|b||\eta_n - \eta| \geq \frac{\epsilon}{2}$ .

Hence,

$$\mathbb{P}(|a||\xi_n - \xi| + |b||\eta_n - \eta| \geq \epsilon) \leq \mathbb{P}\left(|\xi_n - \xi| \geq \frac{\epsilon}{2|a|}\right) + \mathbb{P}\left(|\eta_n - \eta| \geq \frac{\epsilon}{2|b|}\right).$$

Both terms on the right-hand side tend to 0 as  $n \rightarrow \infty$  by assumption. Therefore,

$$\mathbb{P}(|a\xi_n + b\eta_n - a\xi - b\eta| \geq \epsilon) \xrightarrow{n \rightarrow \infty} 0$$

which implies that  $a\xi_n + b\eta_n \xrightarrow{P} a\xi + b\eta$ .

3. We proceed in the same way as in the previous parts to arrive at

$$\mathbb{P}(|\xi_i \eta_i - \xi \eta| \geq \epsilon) \leq \mathbb{P}\left(|\xi_i||\eta_i - \eta| \geq \frac{\epsilon}{2}\right) + \mathbb{P}\left(|\eta||\xi_i - \xi| \geq \frac{\epsilon}{2}\right). \quad (4.5)$$

Consider the first term on the right-hand side. We note that given an  $M > 0$  it follows that

$$\left\{|\xi_i||\eta_i - \eta| \geq \frac{\epsilon}{2}\right\} \subseteq \{|\xi_i - \xi| \geq 1\} \cup \{|\xi| \geq M\} \cup \left\{|\eta_i - \eta| \geq \frac{\epsilon}{2(M+1)}\right\}.$$

One can observe this by considering the event than no of the events on the right-hand side hold, then

$$|\xi_i||\eta_i - \eta| \leq (|\xi_i - \xi| + |\xi|)|\eta_i - \eta| < (M+1)\frac{\epsilon}{2(M+1)} = \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(|\xi_i||\eta_i - \eta| \geq \frac{\epsilon}{2}\right) &\leq \mathbb{P}(|\xi_i - \xi| \geq 1) + \mathbb{P}(|\xi| \geq M) + \mathbb{P}\left(|\eta_i - \eta| \geq \frac{\epsilon}{2(M+1)}\right) \\ &\xrightarrow{i \rightarrow \infty} \mathbb{P}(|\xi| \geq M) \\ &\xrightarrow{M \rightarrow \infty} 0. \end{aligned}$$

Similarly, for the second term in the right-hand side of the equation (4.5) we have that

$$\begin{aligned} \mathbb{P}\left(|\eta||\xi_i - \xi| \geq \frac{\epsilon}{2}\right) &\leq \mathbb{P}(|\eta| \geq M) + \mathbb{P}\left(|\eta_i - \eta| \geq \frac{\epsilon}{2M}\right) \\ &\xrightarrow{i \rightarrow \infty} \mathbb{P}(|\eta| \geq M) \\ &\xrightarrow{M \rightarrow \infty} 0. \end{aligned}$$

Therefore, we can conclude that

$$\mathbb{P}(|\xi_i \eta_i - \xi \eta| \geq \epsilon) \xrightarrow{i \rightarrow \infty} 0$$

which implies that  $\xi_i \eta_i \xrightarrow{P} \xi \eta$ .

4. Let  $M \in \mathbb{R}$ . On  $[-M, M]^2$  the function  $\varphi(x, y)$  is uniformly continuous. Therefore, given an  $\epsilon > 0$  there exists a  $\delta > 0$  such that for all  $(x, y), (x', y') \in [-M, M]^2$  with  $|(x, y) - (x', y')| < \delta$  we have

$$|\varphi(x, y) - \varphi(x', y')| < \epsilon.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(|\varphi(\xi_i, \eta_i) - \varphi(\xi, \eta)| \geq \epsilon \mid (\xi_i, \eta_i), (\xi, \eta) \in [-M, M]^2\right) \\ &\leq \mathbb{P}\left(|(\xi_i, \eta_i) - (\xi, \eta)| \geq \delta \mid (\xi_i, \eta_i), (\xi, \eta) \in [-M, M]^2\right) \\ &\leq \mathbb{P}\left(|\xi_i - \xi| \geq \frac{\delta}{2} \mid \xi_i, \xi \in [-M, M]\right) + \mathbb{P}\left(|\eta_i - \eta| \geq \frac{\delta}{2} \mid \eta_i, \eta \in [-M, M]\right) \\ &\xrightarrow{i \rightarrow \infty} 0. \end{aligned}$$

Sending  $M \rightarrow \infty$  completes the proof. ■

**Exercise 4.2.2***Solution.*

1. It is clear that

$$\begin{aligned}\mathbb{P}_{\xi_i}(\{x\}) &= \mathbb{P}_{\tilde{\xi}_i}(\Omega_1 \times \cdots \times \{x\} \times \cdots \times \Omega_n) \\ &= \mathbb{P}^{(n)}(\Omega_1 \times \cdots \times \{x\} \times \cdots \times \Omega_n) \\ &= \mathbb{P}_i(\{x\}) \\ &= p^x(1-p)^{1-x}\end{aligned}$$

for  $x = 0, 1$ .

2. By construction

$$\begin{aligned}\mathbb{P}\left(\tilde{\xi}_1 \in A_1, \dots, \tilde{\xi}_n \in A_n\right) &= \mathbb{P}^{(n)}(A_1 \times \cdots \times A_n) \\ &= \prod_{i=1}^n \mathbb{P}_i(A_i).\end{aligned}$$

Therefore, the  $(\tilde{\xi}_i)$  are independent. ■**Exercise 4.3.1***Solution.* As  $S_n \sim \text{B}(n, p)$  we know that  $\mathbb{E}(S_n) = np$  and  $\mathbb{V} = np(1-p)$ . Therefore,

$$\begin{aligned}\mathbb{E}\left(\left|\frac{S_n}{n} - p\right|^2\right) &= \mathbb{E}\left(\left|\frac{S_n - np}{n}\right|^2\right) \\ &= \frac{1}{n^2} \mathbb{E}((S_n - np)^2) \\ &= \frac{1}{n^2} \mathbb{V}(S_n) \\ &= \frac{p(1-p)}{n} \\ &\xrightarrow{n \rightarrow \infty} 0.\end{aligned}$$
■

**Exercise 4.4.4***Solution.* Note that as  $(r_k)_{k \geq 1}$  converges to 0 it must be bounded. Suppose that  $r_k \leq M$  for all  $k \in \mathbb{N}$ . Moreover, for any  $\delta > 0$  we can find a  $N_\delta \in \mathbb{N}$  such that  $r_k \leq \delta$  for all  $k \geq N_\delta$ . Therefore, for an  $\epsilon > 0$  and  $n > N_\delta$  we have

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) &\leq \frac{\mathbb{V}(S_n)}{n^2 \epsilon^2} \\ &= \frac{\sum_{i,j} \mathbb{E}(\xi_i \xi_j)}{n^2 \epsilon^2} \\ &= \frac{(nr_0 + 2(n-1)r_1 + \cdots + 2(n-N_\delta)r_{N_\delta}) + (2(n-N_\delta-1)r_{N_\delta+1} + \cdots + 2(1)r_n)}{n^2 \epsilon^2} \\ &\leq \frac{2nN_\delta M + 2\delta(1 + \cdots + n - N_\delta - 1)}{n^2 \epsilon^2} \\ &\leq \frac{2nN_\delta M + 2\delta \frac{1}{2}n(n-1)}{n^2 \epsilon^2} \\ &\xrightarrow{n \rightarrow \infty} \frac{\delta}{\epsilon^2}.\end{aligned}$$

As  $\delta > 0$  we arbitrary for a fixed  $\epsilon > 0$  we conclude that

$$\mathbb{P} \left( \left| \frac{S_n}{n} \right| \geq \epsilon \right) \xrightarrow{n \rightarrow \infty} 0.$$

■

### Exercise 4.5.4

*Solution.* Consider the quotient

$$q = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{(n-k)p}{(k+1)(1-p)}.$$

When  $q < 1$  the Binomial probability density function is decreasing and when  $q > 1$  the Binomial probability density function (PDF) is increasing.

- The PDF is increasing for  $k < p(n+1) - 1$ .
- The PDF is decreasing for  $k > p(n+1) - 1$ .

When  $p(n+1)$  is an integer the PDF is maximal for both  $(n+1)p$  and  $(n+1)p - 1$ . ■

### Exercise 4.5.8

*Solution.* Given a  $\delta > 0$  there exists a  $X_\delta > 0$  such that for all  $x > X_\delta$  we have that

$$\Phi(x) - \Phi(-x) = \int_{-x}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \geq 1 - \delta.$$

Observe that from CLT we have that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{S_n - np}{\sqrt{np(1-p)}} \leq x \right) = \Phi(x)$$

which implies that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{S_n}{n} - p \leq \frac{x\sqrt{p(1-p)}}{\sqrt{n}} \right) = \Phi(x).$$

For fixed  $\epsilon > 0$  let  $\delta > 0$  and fix  $x > X_\delta$  and choose  $N \geq \frac{x^2 p(1-p)}{\epsilon^2}$ . It follows that for  $n \geq N$  we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} \left( \left| \frac{S_n}{n} - p \right| < \epsilon \right) &\geq \lim_{n \rightarrow \infty} \mathbb{P} \left( \left| \frac{S_n}{n} - p \right| \leq \frac{x\sqrt{p(1-p)}}{\sqrt{n}} \right) \\ &= \lim_{n \rightarrow \infty} \left( \mathbb{P} \left( \frac{S_n}{n} - p \leq \frac{x\sqrt{p(1-p)}}{\sqrt{n}} \right) - \mathbb{P} \left( \frac{S_n}{n} - p \leq -\frac{x\sqrt{p(1-p)}}{\sqrt{n}} \right) \right) \\ &= \Phi(x) - \Phi(-x) \\ &\geq 1 - \delta. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \left| \frac{S_n}{n} - p \right| \geq \epsilon \right) < \delta.$$

As  $\delta$  was arbitrary we conclude that  $\frac{S_n}{n} \xrightarrow{P} 0$ . ■

## 5 Almost Sure Convergence

### 5.1 Definition

It will be useful now to briefly introduce almost sure convergence, and identify how it relates to convergence in probability and  $L^p$  convergence. The full theory of almost sure convergence will be discussed in Chapter 7.

**Definition 5.1.1** A sequence  $(\xi_n)_{n \geq 1}$  of random variables on probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  converges  $\mathbb{P}$ -almost surely to the random variable  $\xi$ , denoted by  $\xi_n \xrightarrow{\text{a.s.}} \xi$ , if

$$\mathbb{P} \left( \left\{ \omega : \xi_n(\omega) \xrightarrow{n \rightarrow \infty} \xi(\omega) \right\} \right) = 0.$$

Further discussions will follow, for which the aim is to determine the following implications of convergence.

- Almost sure convergence and convergence in  $L^p$  both imply convergence in probability.
- Convergence in probability implies convergence in distribution.

$$\begin{array}{c} \xrightarrow{L^p} \\ \Downarrow \\ \xrightarrow{\text{a.s.}} \implies \xrightarrow{p} \implies \xrightarrow{d} \end{array}$$

### 5.2 Connection to Convergence in Probability

#### 5.2.1 Borel-Cantelli Lemma

**Definition 5.2.1** Let  $A_1, A_2, \dots$  be a sequence of events.

- Let

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k.$$

If  $\omega \in \limsup_{n \rightarrow \infty} A_n$ , we say that  $A_n$  occurs infinitely often.

- Let

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k.$$

If  $\omega \in \liminf_{n \rightarrow \infty} A_n$ , we say that  $A_n$  occurs eventually.

#### Exercise 5.2.2

1. Show that

$$\mathbb{P} \left( \liminf_{n \rightarrow \infty} A_n \right) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n).$$

2. Describe the complement of  $\limsup_{n \rightarrow \infty} A_n$ .

**Theorem 5.2.3 — Borel-Cantelli Lemma.** Let  $A_1, A_2, \dots$  be a sequence of events.

1. If  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$  then  $\mathbb{P}(A_n \text{ i.o.}) = 0$ .
2. If  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$  and  $A_n$  are mutually independent then  $\mathbb{P}(A_n \text{ i.o.}) = 1$ .

*Proof.*

1. By continuity of the measure we have

$$\mathbb{P}(A_n \text{ i.o.}) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) = 0.$$

2. Observe that  $\{A_n \text{ i.o.}\}^c = \{A_n^c \text{ ev.}\} = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k^c$ . Hence, we have

$$1 - \mathbb{P}(A_n \text{ i.o.}) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{k \geq n} A_k^c\right).$$

By independence we have that

$$\mathbb{P}\left(\bigcap_{k \geq n} A_k^c\right) = \prod_{k \geq n} \mathbb{P}(A_k^c).$$

Note that  $\log(1-x) \leq -x$  for  $x \in [0, 1)$ , and thus

$$\log \mathbb{P}\left(\bigcap_{k \geq n} A_k^c\right) = \log \prod_{k \geq n} (1 - \mathbb{P}(A_k)) \leq - \sum_{k \geq n} \mathbb{P}(A_k) = -\infty,$$

That is,  $\mathbb{P}\left(\bigcap_{k \geq n} A_k^c\right) = 0$  for all  $n$ . ■

**Remark 5.2.4** Theorem 5.2.3 is an example of a zero-one law.

**Example 5.2.5** For real numbers in  $[0, 1]$ , we consider the event that its binary expansion contains a finite string of  $\{0, 1\}$  infinitely many times. Assume the desired string to be  $(x_1, \dots, x_m)$ . We consider the sequence of events

$$A_n := \{\omega : \xi_{nm+1}(\omega) = x_1, \xi_{nm+2}(\omega) = x_2, \dots, \xi_{nm+m}(\omega) = x_m\}$$

for  $n \geq 0$  on  $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$ . Where  $\xi_k(\omega) = \omega_k$ . The event  $A_n$  occurs when the desired string appears starting from digit  $nm + 1$ . These are mutually independent, given that the  $\xi_k$  are independent. As  $\mathbb{P}(A_n) = \frac{1}{2^m}$  for all  $n \in \mathbb{N}$  we observe that  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ , so by Theorem 5.2.3 we deduce that  $\mathbb{P}(A_n \text{ i.o.}) = 1$ . In other words, our finite string  $\{0, 1\}$  is guaranteed to appear infinitely often in the binary expansion of almost every real number in the interval  $[0, 1]$ .

## 5.2.2 Applications of the Borel-Cantelli Lemma

**Proposition 5.2.6** A necessary and sufficient condition that  $\xi_n \rightarrow \xi$   $\mathbb{P}$ -almost surely is that

$$\mathbb{P}\left(\sup_{k \geq n} |\xi_k - \xi| \geq \varepsilon\right) \xrightarrow{n \rightarrow \infty} 0,$$

for every  $\varepsilon > 0$ .

*Proof.* Note that  $\xi_n(\omega) \not\rightarrow \xi(\omega)$  if and only if there exists an  $\epsilon > 0$  such that

$$|\xi_n(\omega) - \xi(\omega)| \geq \epsilon$$

infinitely often. So let  $A_n^\epsilon = \{\omega : |\xi_n - \xi| \geq \epsilon\}$  and  $A^\epsilon = \limsup A_n^\epsilon$ . It follows that

$$\{\omega : \xi_n(\omega) \not\rightarrow \xi(\omega)\} = \bigcup_{\varepsilon \geq 0} A^\varepsilon.$$

As the sets  $A^\epsilon$  are nested, one can restrict  $\epsilon$  to the form  $\epsilon = \frac{1}{m}$  for some positive integer  $m$ , so that

$$\{\omega : \xi_n(\omega) \not\rightarrow \xi(\omega)\} = \bigcup_{m=1}^{\infty} A^{\frac{1}{m}}.$$

Hence,  $\mathbb{P}(\{\omega : \xi_n \not\rightarrow \xi\}) = 0$  if and only if  $\mathbb{P}\left(\bigcup_{m=1}^{\infty} A^{\frac{1}{m}}\right) = 0$ . If this holds it follows for all  $m \geq 1$  that

$$\mathbb{P}\left(A^{\frac{1}{m}}\right) \leq \mathbb{P}\left(\bigcup_{m=1}^{\infty} A^{\frac{1}{m}}\right) = 0.$$

Conversely, if for  $m \geq 1$  we have  $\mathbb{P}\left(A^{\frac{1}{m}}\right) = 0$  then by the continuity of the measure it follows that  $\mathbb{P}\left(\bigcup_{m=1}^{\infty} A^{\frac{1}{m}}\right) = 0$ . Now  $\mathbb{P}\left(A^{\frac{1}{m}}\right) = 0$  for all  $m \geq 1$  happens if and only if  $\mathbb{P}(A^\epsilon) = 0$  for all  $\epsilon > 0$ . Therefore, as

$$\mathbb{P}(A^\epsilon) = \mathbb{P}\left(\bigcap_{n \geq 1} \bigcup_{k \geq n} A_k^\epsilon\right) = \lim_n \mathbb{P}\left(\bigcup_{k \geq n} A_k^\epsilon\right) = \mathbb{P}\left(\sup_{k \geq n} |\xi_k - \xi| \geq \epsilon\right),$$

we complete the proof. ■

**Corollary 5.2.7** Convergence almost surely implies convergence in probability.

$$\begin{array}{c} \xrightarrow{L^p} \\ \Downarrow \\ \xrightarrow{\text{a.s.}} \implies \xrightarrow{\text{P}} \implies \xrightarrow{d} \end{array}$$

*Proof.* Suppose that  $\xi_n \rightarrow \xi$   $\mathbb{P}$ -almost surely. Let  $\epsilon > 0$ . Then for any  $\delta > 0$  we can use Proposition 5.2.6 to find an  $N \in \mathbb{N}$  such that

$$\mathbb{P}\left(\sup_{k \geq n} |\xi_k - \xi| \geq \epsilon\right) \leq \delta$$

for all  $n \geq N$ . It follows that for all

$$\mathbb{P}(|\xi_n - \xi| < \epsilon) \geq 1 - \delta$$

for all  $n \in \mathbb{N}$ . In particular,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\xi_n - \xi| < \epsilon) = 1$$

which implies that

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\xi_n - \xi| \geq \epsilon) = 0.$$

Meaning  $\xi_n \rightarrow \xi$  in probability. ■

**Exercise 5.2.8** Show that the sequence  $(\xi_n)_{n \geq 1}$  converges almost surely if and only if, for all  $\epsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{k, l \geq n} |\xi_k - \xi_l| \geq \epsilon\right) = 0,$$

or equivalently,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{k \geq 0} |\xi_{n+k} - \xi_n| \geq \epsilon\right) = 0.$$

**Example 5.2.9** Consider the sequence  $f_n := \chi_{A_n}$  for

$$A_n = \left[ \frac{n}{2^k} - 1, \frac{n+1}{2^k} - 1 \right]$$

whenever  $k \geq 0$  and  $2^k \leq n < 2^{k+1}$ . The first few  $A_n$  are

- $[0, 1]$ ,
- $[0, \frac{1}{2}]$ ,
- $[\frac{1}{2}, 1]$ ,
- $[0, \frac{1}{4}]$ ,
- $[\frac{1}{4}, \frac{1}{2}]$ ,
- $\vdots$

Plotting these indicator functions one observes that they move from left to right over  $[0, 1]$ , half their width and repeat. Therefore, it is clear that in probability the function converges to 0. However, given that the indicator function moves from left to right infinitely many times, for all  $\omega \in [0, 1]$ ,  $f_n(\omega) = 1$  (and 0) infinitely often and so  $f_n(\omega)$  does not converge almost surely.

There is a partial converse to the implication that almost sure converge implies convergence in probability. More specifically, we will see that if a sequence converges in probability, then we can extract a subsequence that converges almost surely.

**Lemma 5.2.10** A sufficient condition for  $\xi_n \xrightarrow{\text{a.s.}} \xi$  is that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \varepsilon) < \infty$$

is satisfied for all  $\varepsilon > 0$ .

*Proof.* Denote the event  $A_n^\varepsilon = \{\omega : |\xi_n(\omega) - \xi(\omega)| \geq \varepsilon\}$  as required. Since  $\sum_{n=1}^{\infty} \mathbb{P}(A_n^\varepsilon) < \infty$ , by Theorem 5.2.3 we know that  $\mathbb{P}(A^\varepsilon) := \mathbb{P}(A_n^\varepsilon \text{ i.o.}) = 0$ . Following the arguments in Proposition 5.2.6 we know that  $\mathbb{P}(\{\omega : \xi_n \not\rightarrow \xi\}) = 0$  as desired. ■

**Corollary 5.2.11** Let  $(\varepsilon_n)_{n \geq 1}$  be a sequence of positive numbers such that  $\varepsilon_n \searrow 0$  as  $n \rightarrow \infty$ . Then if  $(\xi_n)_{n \in \mathbb{N}}$  and  $\xi$  are such that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \varepsilon_n) < \infty,$$

then  $\xi_n \xrightarrow{\text{a.s.}} \xi$ .

*Proof.* Fix an arbitrary  $\epsilon > 0$ . Choose  $N$  such that for all  $n \geq N$  we have  $\varepsilon_n < \epsilon$ . Then

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \epsilon) \leq \underbrace{\sum_{n=1}^{N-1} \mathbb{P}(|\xi_n - \xi| \geq \epsilon)}_{\leq N-1 < \infty} + \underbrace{\sum_{n=N}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \epsilon)}_{< \infty},$$

hence by Lemma 5.2.10 we have  $\xi_n \xrightarrow{\text{a.s.}} \xi$  as  $n \rightarrow \infty$ . ■

**Theorem 5.2.12** If  $\xi_n \xrightarrow{p} \xi$ , then there exists a subsequence such that  $\xi_{n_k} \xrightarrow{\text{a.s.}} \xi$ .

*Proof.* Since  $\lim_{n \rightarrow \infty} \mathbb{P}(|\xi_n - \xi| > \frac{1}{k}) = 0$  for all  $k \geq 1$ , we can choose a subsequence such that

$$\mathbb{P}\left(|\xi_n - \xi| > \frac{1}{k}\right) \leq 2^{-k}$$

for all  $k \geq 1$ . Since  $\sum_{k=1}^{\infty} 2^{-k}$  converges, by Corollary 5.2.11 we have  $\xi_{n_k} \xrightarrow{\text{a.s.}} \xi$ . ■

**Corollary 5.2.13** If  $\xi_1 \geq \xi_2 \geq \dots \geq 0$  are random variables and  $\xi_n \xrightarrow{p} 0$ , then  $\xi_n \xrightarrow{\text{a.s.}} 0$ .

*Proof.* Let  $\varepsilon > 0$  and let  $A_n = \{\xi_n > \varepsilon\}$ . Then by continuity,

$$\mathbb{P}(\limsup \xi_n > \varepsilon) \leq \mathbb{P}(\xi_n > \varepsilon \text{ i.o.}) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right).$$

Since  $A_n$  is non-increasing, the right hand side equals  $\lim_{n \rightarrow \infty} \mathbb{P}(A_n)$  which is 0 since  $\xi_n \xrightarrow{p} 0$ . Thus

$$\mathbb{P}(\limsup \xi_n > \varepsilon) = 0$$

for all  $\varepsilon > 0$  and hence

$$\begin{aligned} \mathbb{P}(\xi_n \not\rightarrow \xi) &= \mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} \left\{\xi_n > \frac{1}{m}\right\}\right) \\ &= \mathbb{P}\left(\bigcup_{m=1}^{\infty} \limsup_n \xi_n > \frac{1}{m}\right) \\ &\leq \sum_{m=1}^{\infty} \mathbb{P}\left(\limsup \xi_n > \frac{1}{m}\right) \\ &= 0. \end{aligned}$$

### 5.3 Connection to $L^p$ convergence

Example 4.1.3 shows that almost sure convergence does not guarantee  $L^p$  convergence. However, if we have almost sure convergence and convergence in mean, then we have  $L^1$  convergence.

**Theorem 5.3.1** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of non-negative random variables such that  $\xi_n \xrightarrow{\text{a.s.}} \xi$  and  $\mathbb{E}(\xi_n) \rightarrow \mathbb{E}(\xi) < \infty$ . Then  $\xi_n \xrightarrow{L^1} \xi$ , that is,

$$\mathbb{E}(|\xi_n - \xi|) \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Proof.* We have  $\mathbb{E}(|\xi_n|) < \infty$  for sufficiently large  $n$  and therefore for such  $n$  we have

$$\begin{aligned} \mathbb{E}(|\xi_n - \xi|) &= \mathbb{E}(\xi - \xi_n)\chi_{\xi \geq \xi_n} + \mathbb{E}(\xi_n - \xi)\chi_{\xi_n > \xi} \\ &= 2\mathbb{E}(\xi - \xi_n)\chi_{\xi \geq \xi_n} + \mathbb{E}(\xi_n - \xi). \end{aligned}$$

The second term on the right-hand side tends to zero by assumption. For the first term we note that  $0 \leq (\xi - \xi_n)\chi_{\xi \geq \xi_n} \leq \xi$  so by the dominated convergence theorem it follows that  $\mathbb{E}(\xi - \xi_n)\chi_{\xi \geq \xi_n} \rightarrow 0$ . ■

**Example 5.3.2** Let  $(\xi_n)_{n \in \mathbb{N}}$  be independent  $\{0, 1\}$ -valued random variables on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , with  $\mathbb{P}(\xi_n = 1) = \frac{1}{n}$ . For example,  $(\Omega, \mathcal{F}) = (\{0, 1\}, \mathcal{B}(\{0, 1\}))^{\mathbb{N}}$  and  $\mathbb{P}$  being an appropriate infinite product measure. Then

$$\mathbb{E}(|\xi_n - 0|^p) = \frac{1}{n} \rightarrow 0,$$

so  $\xi_n \xrightarrow{L^p} 0$ . However,

$$\begin{aligned}\{\omega : \xi_n \rightarrow 0\} &= \{\xi_n = 0 \text{ eventually}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} \{\xi_k = 0\},\end{aligned}$$

where the inner intersections are an increasing sequence of sets. Therefore,

$$\begin{aligned}\mathbb{P}(\xi_n \rightarrow 0) &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{k \geq n} \{\xi_k = 0\}\right) \\ &= \lim_{n \rightarrow \infty} \prod_{k \geq n} \mathbb{P}(\xi_k = 0) \\ &= \lim_{n \rightarrow \infty} \prod_{k \geq n} \left(1 - \frac{1}{k}\right) \\ &= 0.\end{aligned}$$

Indeed,

$$\begin{aligned}\prod_{k \geq n} \left(1 - \frac{1}{k}\right) &= \lim_{N \rightarrow \infty} \prod_{k=n}^N \frac{k-1}{k} \\ &= \lim_{N \rightarrow \infty} \frac{n-1}{n} \frac{n}{n+1} \cdots \frac{N-1}{N} \\ &= 0.\end{aligned}$$

Thus  $\xi_n$  does not converge almost surely to 0. Hence, we conclude that  $L^p$  convergence does not imply almost sure convergence.

## 5.4 Strong Law of Large Numbers

**Definition 5.4.1** Let  $\xi_1, \xi_2, \dots$  be a sequence of integrable random variables and let  $S_n = \xi_1 + \dots + \xi_n$ . Then the sequence  $(\xi_n)_{n \geq 1}$  satisfies the strong law of large numbers if

$$\frac{S_n - \mathbb{E}(S_n)}{n} \xrightarrow{\text{a.s.}} 0.$$

Note that the strong law of large numbers properties is stronger than the weak law of large numbers, as now we require convergence almost surely which is stronger than convergence in probability. Recall that for independent and identically distributed random variables  $(\xi_n)_{n \in \mathbb{N}}$  with  $\mathbb{V}(\xi_1) < \infty$  we showed the  $L^2$  weak law of large number. Imposing a stronger moment assumption we arrive at a strong law of large numbers.

**Proposition 5.4.2 — Cantelli's Strong Law of Large Numbers.** Let  $\xi_1, \xi_2, \dots$  be a sequence of independent and identically distributed random variables with  $\mathbb{E}(\xi_1^4) < \infty$ . Then

$$\frac{S_n - \mathbb{E}(S_n)}{n} \xrightarrow{\text{a.s.}} 0.$$

*Proof.* Without loss of generality, we centralise the random variables by subtracting its mean, that is  $\mathbb{E}(\xi_1) = 0$ . Note by Chebyshev's inequality that

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \epsilon\right) < \frac{\mathbb{E}\left(\left|\frac{S_n}{n}\right|^4\right)}{\epsilon^4}.$$

In the expansion of forth moments, the only non-vanishing terms are terms of the form  $\mathbb{E}(\xi_j^4)$  and  $\mathbb{E}(\xi_i^2 \xi_j^2) = \mathbb{E}(\xi_i^2) \mathbb{E}(\xi_j^2)$  with  $i \neq j$ , noticing that the odd moments of  $\xi_i$  vanishes. We therefore have

the expansion

$$\mathbb{E}(S_n^4) = \mathbb{E}\left(\sum_{k=1}^n \xi_k^4 + \binom{4}{2,2} \sum_{j,k=1, j < k} \xi_j^2 \xi_k^2\right)$$

where  $\binom{4}{2,2} = \frac{4!}{(2!)^2} = 6$  comes from the multinomial theorem. Now notice that there are  $\frac{n(n-1)}{2}$  unique ways to choose the indices  $(j, k)$  such that  $j < k$ , therefore we have, from the i.i.d assumption,

$$\mathbb{E}\left(\left|\frac{S_n}{n}\right|^4\right) = \frac{1}{n^4} (n\mathbb{E}(\xi_1^4) + 3n(n-1)(\mathbb{E}(\xi_1^2))^2)$$

From Corollary 2.4.5 we have that  $\mathbb{E}(\xi_1^2)^{1/2} \leq \mathbb{E}(\xi_1^4)^{1/4}$ , and so

$$\mathbb{E}\left(\left|\frac{S_n}{n}\right|^4\right) \leq \frac{3n^2 - 2n}{n^4} \mathbb{E}(\xi_1^4) \lesssim \frac{1}{n^2},$$

and therefore  $\sum_{n \geq 1} \mathbb{P}(|\frac{S_n}{n}| \geq \epsilon) \lesssim \sum_{n \geq 1} \frac{1}{n^2} < \infty$ . Thus we can conclude by applying Lemma 5.2.10.  $\blacksquare$

We will now work towards Kolmogorov's strong law of large numbers.

**Proposition 5.4.3 — Kolmogorov's Maximal Inequality.** Let  $\xi_1, \xi_2, \dots$  be independent random variables with finite variances. Then for all  $n \geq 1$  and  $x > 0$ , we have

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k - \mathbb{E}(S_k)| \geq x\right) \leq \frac{\mathbb{V}(S_n)}{x^2}.$$

*Proof.* Without loss of generality, we can consider  $\mathbb{E}(\xi_1) = 0$ . Consider the event

$$A = \{\omega : \max_{1 \leq k \leq n} |S_k(\omega)| \geq x\}$$

and the events

$$A_k = \{\omega : |S_j| < x \text{ for } j = 1, \dots, k-1 \text{ and } |S_k| \geq x\}$$

for  $k = 1, \dots, n$ . Note that the sets  $A_k$  are mutually disjoint and are such that  $A = \bigcup_{k=1}^n A_k$ . Therefore,

$$\begin{aligned} \mathbb{E}(S_n^2) &\geq \mathbb{E}(S_n^2 \chi_A) \\ &= \sum_{k=1}^n \mathbb{E}(S_n^2 \chi_{A_k}) \\ &= \sum_{k=1}^n \mathbb{E}((S_k + \xi_{k+1} + \dots + \xi_n)^2 \chi_{A_k}) \\ &= \sum_{k=1}^n \underbrace{\mathbb{E}(S_k^2 \chi_{A_k})}_{\geq x^2 \mathbb{P}(A_k)} + \underbrace{2\mathbb{E}((S_k \chi_{A_k})(\xi_{k+1} + \dots + \xi_n))}_{= 2\mathbb{E}(S_k \chi_{A_k})\mathbb{E}(\xi_{k+1} + \dots + \xi_n) = 0} + \underbrace{\mathbb{E}((\xi_{k+1} + \dots + \xi_n)^2)}_{\geq 0} \\ &\geq x^2 \sum_{k=1}^n \mathbb{P}(A_k) \\ &= x^2 \mathbb{P}(A). \end{aligned}$$

Therefore,

$$\mathbb{V}(S_n^2) = \mathbb{E}(S_n^2) \geq x^2 \mathbb{P}(A). \quad \blacksquare$$

**Remark 5.4.4** Note that Proposition 5.4.3 is stronger than Chebyshev's inequality since Chebyshev's inequality only gives

$$\mathbb{P}(|S_k - \mathbb{E}(S_k)| \geq x) \leq \frac{\mathbb{V}(S_n)}{x^2}$$

for  $k = 1, \dots, n$ . Which when combined with a union bound gives

$$\mathbb{P} \left( \max_{1 \leq k \leq n} |S_k - \mathbb{E}(S_k)| \geq x \right) = \mathbb{P} \left( \bigcup_{k=1}^n \{|S_k - \mathbb{E}(S_k)| \geq x\} \right) \leq \frac{n \mathbb{V}(S_n)}{x^2}.$$

Hence, Proposition 5.4.3 removes the factor of  $n$  in the bound.

**Lemma 5.4.5** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of real-valued independent random variables with  $\mathbb{E}(\xi_i) = 0$  for all  $i$ . If  $\sum_{i \geq 1} \mathbb{V}(\xi_i) < \infty$ , then  $\sum_{n=1}^{\infty} \xi_n$  converges almost surely.

*Proof.* Note that

$$\begin{aligned} 0 \leq \sup_{m,n \geq k} |S_n - S_m| &\leq \sup_{m,n \geq k} (|S_n - S_k| + |S_k - S_m|) \\ &= 2 \sup_{n \geq k} |S_n - S_k| =: 2\sigma_k. \end{aligned}$$

By Proposition 5.4.3, we have that

$$\begin{aligned} \mathbb{P}(\sigma_k \geq x) &= \lim_{m \rightarrow \infty} \mathbb{P} \left( \max_{m \geq n \geq k} |S_n - S_k| \geq x \right) \\ &\leq \frac{1}{x^2} \lim_{m \rightarrow \infty} \sum_{n=k+1}^m \mathbb{V}(\xi_n) \\ &= \frac{1}{x^2} \sum_{n=k+1}^{\infty} \mathbb{V}(\xi_n) \\ &\xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Therefore,

$$\lim_{k \rightarrow \infty} \mathbb{P} \left( \sup_{n,m \geq k} |S_n - S_m| \geq \epsilon \right) = 0$$

and so we conclude by using the result of Exercise 5.2.8. ■

**Theorem 5.4.6** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables. If  $\sum_{n=1}^{\infty} \mathbb{E}(\xi_n)$  converges and  $\sum_{n=1}^{\infty} \mathbb{V}(\xi_n)$  converges, then  $\sum_{n=1}^{\infty} \xi_n$  converges almost surely.

*Proof.* Consider

$$\sum_{i=1}^{\infty} \xi_i = \sum_{i=1}^{\infty} (\xi_i - \mathbb{E}(\xi_i)) + \sum_{i=1}^{\infty} \mathbb{E}(\xi_i).$$

By assumption we know that  $\sum_{i=1}^{\infty} \mathbb{E}(\xi_i)$  converges, and  $\sum_{i=1}^{\infty} (\xi_i - \mathbb{E}(\xi_i))$  converges by Lemma 5.4.5. ■

In our aim to prove Kolmogorov's strong law of large numbers, we require some results regarding the convergence of weighted averages.

**Lemma 5.4.7 — Toeplitz.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of non-negative numbers. Let  $b_n = \sum_{i=1}^n a_i$  so that  $b_1 = a_1 > 0$ , and  $b_n \nearrow \infty$  as  $n \rightarrow \infty$ . Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of numbers converging to  $x$ . Then

$$\frac{1}{b_n} \sum_{j=1}^n a_j x_j \rightarrow x.$$

In particular, if  $a_n = 1$ , then

$$\frac{x_1 + \dots + x_n}{n} \rightarrow x.$$

*Proof.* Fix  $\epsilon > 0$ . Choose  $N_0 := N_0(\epsilon)$  such that for all  $n \geq N_0$  we have  $|x_j - x| < \epsilon/2$ . Now choose  $N_1 > N_0$ , which depends on  $N_0$ , such that  $\frac{1}{b_{N_1}} \sum_{j=1}^{N_0} |x_j - x| < \frac{\epsilon}{2}$ , which exists since  $|x_j - x|$  is bounded for  $j = 1, \dots, N_0$ . Then for any  $n > N_1$ , we have

$$\begin{aligned} \left| \frac{1}{b_n} \sum_{j=1}^n a_j x_j - x \right| &\leq \left| \frac{1}{b_n} \sum_{j=1}^{N_0} a_j (x_j - x) \right| + \left| \frac{1}{b_n} \sum_{j=N_0+1}^n a_j (x_j - x) \right| \\ &\leq \frac{1}{b_{N_1}} \left| \sum_{j=1}^N a_j (x_j - x) \right| + \left| \frac{1}{b_n} \sum_{j=N_0+1}^n a_j (x_j - x) \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \underbrace{\left( \frac{1}{b_n} \sum_{j=N_0+1}^n a_j \right)}_{\leq 1} \leq \epsilon. \end{aligned}$$

■

**Exercise 5.4.8** Suppose  $(\xi_i)_{i \in \mathbb{N}}$  is a sequence of independent random variables with common mean  $m$  and variance  $\mathbb{V}(\xi_k) = k\eta(k)$  with the condition that  $\mathbb{V}(\xi_k) \rightarrow \infty$ ,  $\eta(k) > 0$  and  $\eta(k) \searrow 0$  as  $k \rightarrow \infty$ . Using Lemma 5.4.7 prove that the sequence satisfies the weak law of large numbers. That is, show that  $n^{-1} \sum_{i=1}^n \xi_i \rightarrow m$  as  $n \rightarrow \infty$  in  $L^2$  and in probability.

**Lemma 5.4.9** Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  be as in Lemma 5.4.7 and let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of numbers such that  $\sum_{n=1}^{\infty} x_n$  converges. Then

$$\frac{1}{b_n} \sum_{j=1}^n b_j x_j \rightarrow 0$$

as  $n \rightarrow \infty$ . In particular, if  $b_n = n$ ,  $x_n = \frac{y_n}{n}$  and  $\sum_{j=1}^{\infty} \frac{y_j}{j}$  converges, then

$$\frac{y_1 + \dots + y_n}{n} \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Proof.* We let  $b_0 = S_0 = 0$  and  $S_n = \sum_{j=1}^n x_j$ . Then

$$\begin{aligned} \sum_{j=1}^n b_j x_j &= \sum_{j=1}^n b_j (S_j - S_{j-1}) = b_n S_n - b_0 S_0 - \sum_{j=1}^n S_j (b_j - b_{j-1}) \\ &= b_n S_n - b_0 S_0 - \sum_{j=1}^n a_j S_j. \end{aligned}$$

Dividing  $b_n$  gives

$$\frac{1}{b_n} \sum_{j=1}^n b_j x_j = S_n - \underbrace{\frac{b_0 S_0}{b_n}}_{\rightarrow 0} - \frac{1}{b_n} \sum_{j=1}^n a_j S_j.$$

So when  $n \rightarrow \infty$ , we see that  $b_n^{-1} \sum_{j=1}^n b_j x_j \xrightarrow{n \rightarrow \infty} 0$  by Lemma 5.4.7. ■

**Theorem 5.4.10** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables with  $\mathbb{E}(\xi_n^2) < \infty$  for all  $i$ . Let  $(b_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers such that  $b_n \nearrow \infty$  and

$$\sum_{n=1}^{\infty} \frac{\mathbb{V}(\xi_n)}{b_n^2} < \infty.$$

Then

$$\frac{S_n - \mathbb{E}(S_n)}{b_n} \xrightarrow{\text{a.s.}} 0.$$

When  $b_n = n$  we obtain a strong law of large numbers.

*Proof.* Observe that

$$\frac{S_n - \mathbb{E}(S_n)}{b_n} = \frac{1}{b_n} \sum_{i=1}^n b_k \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k}. \quad (5.1)$$

Now

$$\mathbb{V} \left( \sum_{k=1}^n \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k} \right) \sum_{k=1}^n = \mathbb{V} \left( \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k} \right) = \sum_{k=1}^n \frac{\mathbb{V}(\xi_k)}{b_k^2} < \infty.$$

Therefore, by Theorem 5.4.6

$$\sum_{k \geq 1} \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k}$$

converges almost surely. Hence, applying Lemma 5.4.9 to (5.1) completes the proof. ■

**Exercise 5.4.11** For  $\xi$  an integrable non-negative random variable, show that

$$\sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n) \leq \mathbb{E}(\xi) \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n).$$

**Theorem 5.4.12 — Kolmogorov's Strong Law of Large Numbers.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent identically distributed random variables with  $\mathbb{E}(|\xi_1|) < \infty$ . Then

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mathbb{E}(\xi_1)$$

as  $n \rightarrow \infty$ .

*Proof.* Without loss of generality assume that  $\mathbb{E}(\xi_1) = 0$ . By Exercise 5.4.11 it follows that  $\sum_{n \geq 1} \mathbb{P}(|\xi_n| \geq n) \leq \mathbb{E}(|\xi_1|) < \infty$ . By the first Borel-Cantelli lemma, we know that  $\mathbb{P}(|\xi_n| \geq n \text{ i.o.}) = 0$ . That is,  $|\xi_n| < n$  eventually  $\mathbb{P}$ -almost everywhere. So letting  $\tilde{\xi}_n = \xi_n \chi_{|\xi_n| < n}$ , we have

$$\frac{S_n}{n} \xrightarrow{n \rightarrow \infty} 0$$

almost surely if and only if

$$\frac{1}{n} \sum_{i=1}^n \tilde{\xi}_i \xrightarrow{n \rightarrow \infty} 0$$

almost surely. Note that by dominated convergence theorem it follows that  $\mathbb{E}(\tilde{\xi}_n) \xrightarrow{n \rightarrow \infty} \mathbb{E}(\xi_1) = 0$ . Therefore, using Lemma 5.4.7 with  $x_n = \mathbb{E}(\tilde{\xi}_n)$  it follows that  $\frac{1}{n} \sum_{i=1}^n \mathbb{E}(\tilde{\xi}_i) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0$$

if and only if

$$\frac{1}{n} \sum_{i=1}^n (\tilde{\xi}_i - \mathbb{E}(\tilde{\xi}_i)) \xrightarrow{\text{a.s.}} 0. \quad (5.2)$$

By Lemma 5.4.9 we know that (5.2) holds if  $\sum_{n=1}^{\infty} \frac{\tilde{\xi}_n - \mathbb{E}(\tilde{\xi}_n)}{n}$  converges. Using Theorem 5.4.6 this is the case if  $\sum_{n=1}^{\infty} \frac{\mathbb{V}(\tilde{\xi}_n - \mathbb{E}(\tilde{\xi}_n))}{n^2}$  converges. Hence, observe that

$$\mathbb{V} \left( \sum_{n \geq 1} \frac{\tilde{\xi}_n - \mathbb{E}(\tilde{\xi}_n)}{n} \right) \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}(\xi_1^2 \chi_{|\xi_1| < n})}{n^2}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^n \mathbb{E} (\xi_n^2 \chi_{\{|\xi_1| \in [k-1, k]\}}) \\
&= \sum_{k=1}^{\infty} \mathbb{E} (\xi_n^2 \chi_{|\xi_1| \in [k-1, k]}) \underbrace{\left( \sum_{n=k}^{\infty} \frac{1}{n^2} \right)}_{\leq 2/k} \\
&= 2 \sum_{k=1}^{\infty} \mathbb{E} \left( |\xi| \underbrace{\frac{|\xi|}{k}}_{\leq 1} \chi_{\{|\xi_1| \in [k-1, k]\}} \right) \\
&\leq 2\mathbb{E}(|\xi_1|) < \infty.
\end{aligned}$$

Hence, (5.2) holds which completes the proof. ■

**Example 5.4.13** Consider the probability space  $([0, 1), \mathcal{B}([0, 1)), \text{Leb})$ . An  $\omega \in [0, 1)$  has the binary representation

$$\omega = \frac{\omega_1}{2} + \frac{\omega_2}{2^2} + \dots = 0.\omega_1\omega_2\dots$$

where  $\omega_j \in \{0, 1\}$ . Let  $\xi_j(\omega) = \omega_j$ . Then for  $(x_1, \dots, x_n) \in \{0, 1\}^n$  consider

$$\begin{aligned}
A_{(x_1, \dots, x_n)} &= \{\omega : \xi_1 = x_1, \dots, \xi_n = x_n\} \\
&= \left\{ \omega : \frac{x_1}{2} + \dots + \frac{x_n}{2^n} \leq \omega < \frac{x_1}{2} + \dots + \frac{x_n}{2^n} + \frac{1}{2^n} \right\}.
\end{aligned}$$

so  $\mathbb{P}(A_{(x_1, \dots, x_n)}) = \frac{1}{2^n}$ . Therefore, as  $\xi_1, \xi_2, \dots$  are independent and identically distributed Bernoulli random variables with  $\mathbb{P}(\xi_1 = 1) = \frac{1}{2}$  by the strong law of large numbers we conclude that

$$\frac{1}{n} \sum_{k=1}^n \xi_k \xrightarrow{\text{a.s.}} \mathbb{E}(\xi_1) = \frac{1}{2}.$$

That is, for almost every number in  $[0, 1)$  the proportion of 0s and 1s in its binary expansion tends to  $\frac{1}{2}$ . We call such numbers normal.

## 5.5 Kolmogorov's 0-1 Law

**Definition 5.5.1** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.

1. Let  $\mathcal{F}_1, \mathcal{F}_2, \dots$  be sub- $\sigma$ -algebra of  $\mathcal{F}$ . Define the  $\sigma$ -algebra generated by their union as

$$\bigvee_{i=1}^n \mathcal{F}_i = \sigma \left( \bigcup_{i=1}^n \mathcal{F}_i \right),$$

with the natural extension for when  $n = \infty$ .

2. Let  $\xi_1, \xi_2, \dots$  be a sequence of random variables defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Then

$$\sigma(\xi_1, \dots, \xi_n) = \bigvee_{i=1}^n \sigma(\xi_i),$$

with the natural extension for when  $n = \infty$ .

**Definition 5.5.2** Under settings of Definition 5.5.1, let  $\mathcal{F}_n^p = \sigma(\xi_n, \dots, \xi_p)$  for  $p \geq n$  and  $\mathcal{F}_n^\infty =$

$\sigma(\xi_n, \dots)$ . For a sequence of random variables  $\xi_1, \xi_2, \dots$ , let

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{F}_n^{\infty}$$

be referred to as the tail  $\sigma$ -algebra. Events of  $\mathcal{T}$  are called tail events.

### Example 5.5.3

- For some  $B \in \mathcal{B}(\mathbb{R})$ , note that

$$\{\xi_n \in B \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \{\xi_k \in B\} \in \mathcal{T}.$$

- Similarly,  $\{\sum_{n=1}^{\infty} \xi_n \text{ converges}\} \in \mathcal{T}$ .
- The event  $\{\xi_{10} \in B\}$  may not be in  $\mathcal{T}$ , since its occurrence may be affected by changing a finite number of  $\xi_n$ , namely changing  $\xi_{10}$ . Similarly,  $\{\omega : \xi_n \notin \mathcal{B} \text{ for all } n\}$  is not in  $\mathcal{T}$  since its occurrence may be affected by just a single  $\xi_n$ .

**Lemma 5.5.4** If  $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$  are independent, that is for all  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  we have that  $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ , then  $\sigma(\mathcal{A})$  and  $\sigma(\mathcal{B})$  are independent.

*Proof.* Consider  $A \in \mathcal{A}$  and the measures

- $\mathbb{P}_A^{(1)}(B) = \mathbb{P}(AB)$ , and
- $\mathbb{P}_A^{(2)}(B) = \mathbb{P}(A)\mathbb{P}(B)$ .

These measures coincide on  $\mathcal{B}$ , and so by Theorem 1.3.5 they coincide on  $\sigma(\mathcal{B})$ . That is  $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$  for all  $A \in \mathcal{A}$  and  $B \in \sigma(\mathcal{B})$ . Now let  $B \in \sigma(\mathcal{B})$  and consider the measures

- $Q_B^{(1)}(A) = \mathbb{P}(AB)$ , and
- $Q_B^{(2)}(A) = \mathbb{P}(A)\mathbb{P}(B)$ .

Similarly we conclude that using Theorem 1.3.5 that for all  $A \in \sigma(\mathcal{A})$  and  $B \in \sigma(\mathcal{B})$  that  $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$ . ■

**Lemma 5.5.5** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables. Then,  $\mathcal{T}$  is independent of itself.

*Proof.* Note that  $\mathcal{F}_1^n$  is independent of  $\mathcal{F}_{n+1}^{n+k}$  for all  $k$  as the random variables  $\xi_i$  are independent for all  $i$ . Hence,  $\mathcal{F}_1^n$  is independent with  $\bigcup_{k=1}^{\infty} \mathcal{F}_{n+1}^{n+k}$ . So by using Lemma 5.5.4 we deduce that  $\mathcal{F}_1^n$  is independent with  $\mathcal{F}_{n+1}^{\infty}$ . As  $\mathcal{T} \subset \mathcal{F}_{n+1}^{\infty}$  it follows that  $\mathcal{F}_1^n$  is independent of  $\mathcal{T}$ . It follows that  $\bigcup_{n=2}^{\infty} \mathcal{F}_1^n$  is independent with  $\mathcal{T}$ , and so by using Lemma 5.5.4 we have that  $\mathcal{F}_1^{\infty}$  is independent with  $\mathcal{T}$ . However, as  $\mathcal{T} \subset \mathcal{F}_1^{\infty}$  it follows that  $\mathcal{T}$  is independent of  $\mathcal{T}$ . ■

**Theorem 5.5.6 — Kolmogorov's Zero-One Law.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables, and let  $A \in \mathcal{T}$ . Then  $\mathbb{P}(A) \in \{0, 1\}$ .

*Proof.* Using Lemma 5.5.5 we have  $\mathbb{P}(A) = \mathbb{P}(A \cap A) = (\mathbb{P}(A))^2$ . Hence,  $\mathbb{P}(A)$  must have a value of zero or one. ■

**Example 5.5.7** Given independent sequence  $(\xi_n)_{n \in \mathbb{N}}$  of random variables and  $B_1, B_2, \dots \in \mathcal{B}(\mathbb{R})$ , we have

$$\{\xi_n \in B_n \text{ i.o.}\} = \cap_{n=1}^{\infty} \cup_{k \geq n} \{\xi_k \in B_k\} \in \mathcal{T}.$$

Likewise, for the independent events  $A_1, A_2, \dots$ , the sequence of random variables  $\chi_{A_1}, \chi_{A_2}, \dots$  are independent, so that the event  $\limsup_n A_n := \{\chi_{A_i} = 1 \text{ i.o.}\}$  is a tail event. Kolmogorov's Zero-One Law then says that  $\limsup_n A_n$  must have probability zero and one, which coincides with our observation from the Borel-Cantelli lemmas.

## 5.6 Law of Iterated Logarithms

### Definition 5.6.1

1. A function  $\varphi^*(n)$  is called upper for  $S_n$  if  $S_n \leq \varphi^*(n)$  for all  $n \geq n_0$ , for some  $n_0 \in \mathbb{N}$ , with probability 1.
2. A function  $\varphi_*(n)$  is called lower for  $S_n$  if  $S_n > \varphi_*(n)$  for infinitely many  $n$  with probability 1.

**Remark 5.6.2** If a function  $\psi(n)$  is such that for all  $\epsilon > 0$  the function  $(1 + \epsilon)\psi(n)$  is upper for  $S_n$  and the function  $(1 - \epsilon)\psi(n)$  is lower for  $S_n$ , then the function  $\psi(n)$  is an optimal rate of convergence for  $S_n$ .

**Example 5.6.3** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent Bernoulli random variables with  $\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = \frac{1}{2}$  and let  $S_n = \xi_1 + \dots + \xi_n$ . We know that  $\frac{S_n}{n} \rightarrow 0$  almost surely. Moreover, since

$$\sum_{n=1}^{\infty} \frac{1}{(n(\log n)^{2\epsilon+1})} < \infty$$

for all  $\epsilon > 0$ , it follows that  $\frac{S_n}{\sqrt{n(\log n)^{1+2\epsilon}}} \xrightarrow{\text{a.s.}} 0$  for all  $\epsilon > 0$  by Theorem 5.4.10. However, by Theorem 4.5.6 we know that

$$\frac{S_n}{\sqrt{n}} \not\rightarrow 0,$$

instead it converges in distribution to a normal random variable.

Consider some  $\varphi(n)$ .

- Then

$$\left\{ \limsup \frac{S_n}{\varphi(n)} \leq 1 \right\} = \left\{ \lim_{n \rightarrow \infty} \sup_{m \geq n} \frac{S_m}{\varphi(m)} \leq 1 \right\}$$

which means that for all  $\epsilon > 0$  there exists an  $n_1 \in \mathbb{N}$  such that  $\sup_{m \geq n} \frac{S_m}{\varphi(m)} \leq 1 + \epsilon$  for all  $n \geq n_1$ .

Equivalently,  $S_m \leq (1 + \epsilon)\varphi(m)$  for all  $m \geq n_1$ . Therefore, if  $\mathbb{P}\left(\limsup \frac{S_n}{\varphi(n)} \leq 1\right) = 1$ , it follows that  $(1 + \epsilon)\varphi(n)$  is upper for  $S_n$  for all  $\epsilon > 0$ .

- Similarly,

$$\left\{ \limsup \frac{S_n}{\varphi(n)} \geq 1 \right\} = \left\{ \lim_{n \rightarrow \infty} \sup_{m \geq n} \frac{S_m}{\varphi(m)} \geq 1 \right\}$$

means that for all  $\epsilon > 0$  there exists an  $n_1 \in \mathbb{N}$  such that  $\sup_{m \geq n} \frac{S_m}{\varphi(m)} \geq 1 - \epsilon$  for all  $n > n_1$ . Equivalently,  $S_m \geq (1 - \epsilon)\varphi(m)$  for infinitely many  $m$ . So if  $\mathbb{P}\left(\limsup \frac{S_n}{\varphi(n)} \geq 1\right) = 1$  then  $(1 - \epsilon)\varphi(n)$  is lower for  $S_n$  for all  $\epsilon > 0$ .

The following propositions will be useful for proving Theorem 5.6.6. The proof of Proposition 5.6.4 is beyond the scope of these notes, for its proof refer to Theorem 9.4 in [2].

**Proposition 5.6.4** Let  $\xi_1, \xi_2, \dots$  be independent and identically distributed random variables with  $\mathbb{E}(\xi_1) = 0$  and  $\mathbb{E}(\xi_1^2) = 1$ . Let  $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$  be such that  $a_n \rightarrow \infty$  and

$$\frac{a_n}{\sqrt{n}} \rightarrow 0.$$

Furthermore, let  $(b_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$  be a sequence such that  $b_n \rightarrow 0$ . Then

$$\mathbb{P}(S_n \geq a_n \sqrt{n}) = \exp\left(-\frac{a_n^2(1+b_n)}{2}\right).$$

**Proposition 5.6.5** Let  $\xi_1, \xi_2, \dots$  be independent random variables with  $\mathbb{E}(\xi_i) = 0$  and  $\mathbb{E}(\xi_i^2) = 1$ . Then for  $\alpha \geq \sqrt{2}$  we have

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq 2\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right)$$

where  $M_n := \max(S_0, S_1, \dots, S_n)$  for  $S_0 = 0$ .

*Proof.* Let

$$A_j = \{M_{j-1} \leq \alpha\sqrt{n} \leq M_j\}.$$

Then

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \sum_{j=1}^{n-1} \mathbb{P}\left(A_j \cap \left\{\frac{S_n}{\sqrt{n}} \leq \alpha - \sqrt{2}\right\}\right).$$

Note that the variance of  $S_n - S_j$  is  $n - j$ , so using independence and Chebyshev's inequality it follows that

$$\begin{aligned} \mathbb{P}\left(A_j \cap \left\{\frac{|S_n - S_j|}{\sqrt{n}} > \sqrt{2}\right\}\right) &= \mathbb{P}(A_j)\mathbb{P}\left(\frac{|S_n - S_j|}{\sqrt{n}} > \sqrt{2}\right) \\ &\leq \mathbb{P}(A_j)\frac{n-j}{2n} \\ &\leq \frac{1}{2}\mathbb{P}(A_j). \end{aligned}$$

Since  $\bigcup_{j=1}^{n-1} A_j \subseteq \{M_n \geq \alpha\sqrt{n}\}$  we deduce that

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \frac{1}{2}\mathbb{P}\left(\frac{M_n}{\sqrt{n}} + \alpha\right),$$

which upon rearrangement gives the desired inequality. ■

**Theorem 5.6.6 — Law of Iterated Logarithm.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be independent identically distributed random variables with  $\mathbb{E}(\xi_1) = 0$  and  $\mathbb{E}(\xi_1^2) = \sigma^2 > 0$ . Then

$$\mathbb{P}\left(\limsup \frac{S_n}{\psi(n)} = 1\right) = 1, \tag{5.3}$$

where

$$\psi(n) = \sqrt{2\sigma^2 n \log(\log n)}.$$

That is, for all  $\varepsilon > 0$ , the function  $(1 + \varepsilon)\psi$  is upper and the function  $(1 - \varepsilon)\psi$  is lower for  $S_n$ .

*Proof.* For simplicity, we will assume that  $\sigma^2 = 1$ . Note that (5.3) is equivalent to

$$\mathbb{P}(S_n \geq (1 + \epsilon)\psi(n) \text{ i.o.}) = 0 \tag{5.4}$$

and

$$\mathbb{P}(S_n \geq (1 - \epsilon)\psi(n) \text{ i.o.}) = 1 \tag{5.5}$$

for all  $\epsilon > 0$ . We first show (5.4). Given an  $\epsilon > 0$ , choose  $\theta > 1$  such that  $\theta^2 < 1 + \epsilon$ . Let  $n_k = \lfloor \theta^k \rfloor$  and  $x_k = \theta \sqrt{2 \log(\log(n_k))}$ . Therefore, using Proposition 5.6.4 and Proposition 5.6.5, it follows that

$$\mathbb{P} \left( \frac{M_{n_k}}{\sqrt{n_k}} \geq x_k \right) \leq 2 \exp \left( -\frac{1}{2} (x_k - \sqrt{2})^2 (1 + b_k) \right)$$

where  $b_k \rightarrow 0$ . Note that  $\frac{1}{2} (x_k - \sqrt{2}) (1 + b_k) = O(\theta^2 \log(k))$ , and so for large  $k$  we have

$$\mathbb{P} \left( \frac{M_{n_k}}{\sqrt{n_k}} \geq x_k \right) \leq \frac{2}{k^\theta}.$$

Since  $\theta > 1$ , it follows by statement 1. of Theorem 5.2.3 that

$$\mathbb{P}(M_{n_k} \geq \theta \psi(n_k) \text{ i.o.}) = 0.$$

Suppose that  $n_{k-1} < n \leq n_k$  and  $S_n > (1 + \epsilon) \psi(n)$ . Then  $\psi(n) \geq \psi(n_{k-1}) \sim \frac{1}{\sqrt{\theta}} \psi(n_k)$ . Hence, by our choice of  $\theta$  it follows that

$$(1 + \epsilon) \psi(n) > \theta \psi(n_k)$$

for large enough  $k$ . Thus for  $k$  sufficiently large we have that  $S_n > (1 + \epsilon) \psi(n)$  implies  $M_{n_k} \geq \theta \psi(n_k)$ , and so we have shown (5.4). We now show (5.5). Given  $\epsilon > 0$ , choose  $\theta \in \mathbb{N}$  such that  $\frac{3}{\sqrt{\theta}} < \epsilon$ . Let  $n_k = \theta^k$  so that  $n_k - n_{k-1} \rightarrow \infty$ . Let  $x_k = (1 + \frac{1}{\theta}) \psi(n_k)$ , and apply Proposition 5.6.4 with  $a_n = \frac{x_k}{\sqrt{n_k - n_{k-1}}}$  to deduce that

$$\mathbb{P}(S_{n_k} - S_{n_{k-1}} \geq x_k) = \mathbb{P}(S_{n_k - n_{k-1}} \geq x_k) = \exp \left( -\frac{x_k^2}{2(n_k - n_{k-1})} (1 + b_k) \right),$$

where  $b_k \rightarrow 0$ . Note that

$$\frac{x_k^2}{2(n_k - n_{k-1})} (1 + b_k) = O \left( \left( 1 + \frac{1}{\theta} \right) \log(k) \right)$$

and so for  $k$  large enough we have that

$$\mathbb{P}(S_{n_k} - S_{n_{k-1}} \geq x_k) \geq \frac{1}{k}.$$

As the events are independent we can apply statement 2. of Theorem 5.2.3 to deduce that

$$\mathbb{P}(S_{n_k} - S_{n_{k-1}} \geq x_k \text{ i.o.}) = 1.$$

On the other hand, one can apply (5.4) to  $(-\xi_n)_{n \in \mathbb{N}}$  to deduce that

$$\mathbb{P}(-S_{n_{k-1}} \leq 2\psi(n_{k-1}) \text{ e.v.}) = 1.$$

As  $2\psi(n_{k-1}) \leq \frac{2}{\sqrt{\theta}} \psi(n_k)$  it follows that

$$\mathbb{P} \left( S_{n_k} \geq x_k - \frac{2}{\sqrt{\theta}} \psi(n_k) \text{ i.o.} \right) = 1.$$

Noting that  $x_k - \frac{2}{\sqrt{\theta}} \psi(n_k) > (1 - \epsilon) \psi(n_k)$ , by our choice of  $\theta$ , the equation (5.5) follows. ■

## 5.7 Solution to Exercises

### Exercise 5.2.2

*Solution.*

- Recall that

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k.$$

Note that

$$\bigcap_{k \geq n} A_k \subseteq \bigcap_{k \geq n+1} A_k$$

so that by the continuity of the measure it follows that

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{k \geq n} A_k\right).$$

Moreover, as  $\bigcap_{k \geq n} A_k \subseteq A_k$  for all  $k \geq n$  it follows that

$$\mathbb{P}\left(\bigcap_{k \geq n} A_k\right) \leq \inf_{k \geq n} \mathbb{P}(A_k)$$

for all  $m \geq n$ . Therefore,

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n).$$

- Using de Morgan's law it follows that

$$(\limsup A_n)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k\right)^c = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k^c = \liminf A_n^c.$$

■

### Exercise 5.2.8

*Solution.* For  $n \in \mathbb{N}$  and  $\epsilon > 0$  consider the set

$$B_n^\epsilon = \left\{ \omega : \sup_{k,l \geq n} |\xi_k(\omega) - \xi_l(\omega)| \geq \epsilon \right\}.$$

It is clear that if  $\xi_n(\omega) \not\rightarrow \xi(\omega)$  then there exists an  $\epsilon > 0$  such that for all  $n \in \mathbb{N}$  there exists  $k, l \geq n$  such that  $|\xi_k(\omega) - \xi_l(\omega)| \geq \epsilon$ . Otherwise,  $(\xi_n(\omega))_{n \in \mathbb{N}}$  would be Cauchy and therefore convergent. Therefore,

$$\{\omega : \xi_n(\omega) \not\rightarrow \xi(\omega)\} \subseteq \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}.$$

On the other hand, if  $\omega \in \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}$  for some  $m \in \mathbb{N}$  this implies that for all  $n \in \mathbb{N}$  there exists  $k, l \geq n$  such that  $|\xi_k(\omega) - \xi_l(\omega)| \geq \frac{1}{2m}$ . Hence,  $\xi_n(\omega) \not\rightarrow \xi(\omega)$ . Therefore,

$$\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}} \subseteq \{\omega : \xi_n(\omega) \not\rightarrow \xi(\omega)\}.$$

Note that for any  $m' \in \mathbb{N}$  we have

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m'}}\right) \leq \mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) \leq \sum_{m=1}^{\infty} \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right).$$

So that  $\mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) = 0$  if and only if  $\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) = 0$  for all  $m \in \mathbb{N}$ . As the sets  $B_n^{\frac{1}{m}}$  are decreasing in  $n$  we know that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(B_n^{\frac{1}{m}}\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{k,l \geq n} |\xi_k - \xi_l| \geq \frac{1}{m}\right)$$

for all  $m \in \mathbb{N}$ . Therefore,  $\xi_n \rightarrow \xi$  almost surely if and only if

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{k,l \geq n} |\xi_k - \xi_l| \geq \epsilon\right) = 0$$

for all  $\epsilon > 0$ .

■

**Exercise 5.4.8**

*Solution.* Let  $S_n = \frac{1}{n} \sum_{k=1}^n \xi_k$ . Then

$$\mathbb{V}(S_n) = \frac{1}{n^2} \sum_{k=1}^n k\eta(k). \quad (5.6)$$

In the notation of Lemma 5.4.7 let  $a_k = k$  and  $x_k = \eta(k)$  so that

$$b_k = \sum_{i=1}^n a_i = \frac{1}{2}n(n+1) \leq \frac{1}{2}n^2.$$

Consequently,

$$\frac{2}{n^2} \sum_{k=1}^n k\eta(k) \leq \frac{1}{b_n} \sum_{k=1}^n k\eta(k) \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, using (5.6) it is clear that  $\mathbb{V}(S_n) \rightarrow 0$  as  $n \rightarrow \infty$ , hence,  $S_n \rightarrow m$  in  $L^2$ . Moreover for  $\epsilon > 0$ , using Chebyshev's inequality we observe that

$$\mathbb{P}(|S_n - m|) \leq \frac{\mathbb{V}(S_n)}{\epsilon^2}.$$

Hence,  $S_n \rightarrow m$  in probability as well. ■

**Exercise 5.4.11**

*Solution.* Proceeding directly from  $\sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n)$  it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n) &= \sum_{n=1}^{\infty} \sum_{k \geq n} \mathbb{P}(k \leq \xi < k+1) \\ &= \sum_{k=1}^{\infty} k \mathbb{P}(k \leq \xi < k+1) \\ &= \sum_{k=0}^{\infty} \mathbb{E}(k \chi_{[k, k+1]}) \\ &\leq \sum_{k=0}^{\infty} \mathbb{E}(\xi \chi_{[k, k+1]}) \\ &= \mathbb{E}(\xi) \\ &\leq \sum_{k=0}^{\infty} \mathbb{E}((k+1) \chi_{[k, k+1]}) \\ &= 1 + \sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n). \end{aligned}$$

■

## 6 Convergence in Distribution

In this section, we focus on the weak convergence of measures defined  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

### 6.1 Weak Convergence

**Definition 6.1.1** Let  $\xi_n : (\Omega_n, \mathcal{F}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  be a random variable for  $n = 1, 2, \dots$ . Then  $\xi_n \rightarrow \xi$  weakly as  $n \rightarrow \infty$  if

$$\lim_{n \rightarrow \infty} (\mathbb{E}_{\mathbb{P}_n}(f(\xi_n))) = \mathbb{E}_{\mathbb{P}}(f(\xi))$$

for all  $f \in \mathcal{C}_b(X)$ .

It is important to note that we do not need to specify the probability space of  $\xi_n$  when establishing convergence in distribution, what matters is the distribution of  $\xi_n$ . Weak convergence is often taken to be the definition of convergence in distribution, however, convergence in distribution has also been defined differently. Specifically, we say that  $\xi_n \rightarrow \xi$  in distribution if the distribution function  $F_{\xi_n}(x) \rightarrow F_\xi(x)$  pointwise for all  $x$  where  $F_\xi$  is continuous. To show that weak convergence is equivalent to our usual definition of convergence in distribution, we have to show that weak convergence can be formulated on a single probability space.

**Theorem 6.1.2** Suppose  $\mu$  and  $(\mu_n)_{n \in \mathbb{N}}$  are probability measures such that the corresponding distribution functions  $F_n$ , converge pointwise to  $F$  at the points of continuity of  $F$ . Then there exists random variables  $\xi$  and  $(\xi_n)_{n \in \mathbb{N}}$  defined on a single probability space  $(\Omega', \mathcal{F}', \mathbb{P}')$  such that  $\xi$  has distribution  $\mu$ ,  $\xi_n$  has distribution  $\mu_n$ , and  $\xi_n \rightarrow \xi$   $\mathbb{P}'$ -almost surely.

*Proof.* Let  $(\Omega', \mathcal{F}', \mathbb{P}') = ([0, 1], \mathcal{B}([0, 1]), \text{Leb})$ . Then  $\mu$  and  $\mu_n$  induce distribution functions  $F_n$  and  $F$ . Let  $F^{-1}$  and  $F_n^{-1}$  be their right inverses as defined in equation (3.1). Then by Proposition 3.1.2 the random variables  $F^{-1}(U)$  and  $F_n^{-1}(U)$ , where  $U$  is the uniformly distributed random variables on  $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$ , have the same distribution as  $\mu$  and  $\mu_n$  respectively. It remains to prove that  $F_n^{-1}(u) \rightarrow F^{-1}(u)$  almost surely as  $n \rightarrow \infty$ . Note that if the preimage of  $\{u\}$  under  $F$  is not finite, then  $u$  is a point of discontinuity of  $F^{-1}$ . Hence, there are only countably many such points as  $F^{-1}$  is a non-decreasing right-continuous function. Hence, it suffices to show that the limit holds for points  $u$  whose preimage under  $F$  is finite. Let  $u$  be such a point and observe the following.

- If  $x < F^{-1}(u)$ , then using the argument in the proof of Proposition 3.1.2 we have  $F(x) < u$ . If  $x$  is a point of continuity of  $F$ , then  $F_n(x) \rightarrow F(x)$  by assumption, which implies that  $F_n(x) < u$  for sufficiently large  $n$ . For such  $n$  we have that  $x \leq F_n^{-1}(u)$  which implies that  $x \leq \liminf F_n^{-1}(u)$ . Therefore, we can choose a sequence  $(x_k)$  of points of continuity of  $F$  such that  $x_k \nearrow F^{-1}(u)$ . Thus we obtain that  $F^{-1}(u) \leq \liminf F_n^{-1}(u)$ .
- If  $x > F^{-1}(u)$ , then  $F(x) \geq u$ . In fact we must have  $F(x) > u$ , for if  $F(x) = u$  then  $F(y) = u$  for any  $y \in [F^{-1}(u), x]$ , contradicting the assumption that the preimage of singleton  $\{u\}$  is finite. Repeating the above arguments yields  $F^{-1}(u) \geq \limsup_{n \rightarrow \infty} F_n^{-1}(u)$ .

Combining the observations we have  $F_n^{-1}(u) \rightarrow F^{-1}(u)$ , completing the proof. ■

**Theorem 6.1.3** Let  $\xi, (\xi_n)_{n \in \mathbb{N}}, \dots : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  be integrable random variables. Then the following are equivalent.

1.  $\xi_n \rightarrow \xi$  weakly.
2.  $\limsup \mathbb{P}(\xi_n \in E) \leq \mathbb{P}(\xi \in E)$  for any closed set  $E \subseteq \mathbb{R}$ .
3.  $\liminf \mathbb{P}(\xi_n \in O) \geq \mathbb{P}(\xi \in O)$  for any open set  $O \subseteq \mathbb{R}$ .
4.  $\lim \mathbb{P}(\xi_n \in C) = \mathbb{P}(\xi \in C)$  for any  $C$  such that  $\mathbb{P}(\xi \in \partial C) = 0$ .
5. Let  $F_{\xi_n}(x)$  be the distribution function of  $\xi_n$  and similarly for  $F_\xi(x)$ , then  $F_{\xi_n}(x) \rightarrow F_\xi(x)$  pointwise at any point of continuity of  $F_\xi(x)$ .

6. \*For all bounded Lipschitz functions  $f$ , it follows that  $\mathbb{E}(f(\xi_n)) \rightarrow \mathbb{E}(f(\mu))$ .

**Remark 6.1.4** Point 6. of Theorem 6.1.3 is not examinable, and consequently we will not include it in our proof but is useful to note for upcoming results.

*Proof.* (1)  $\implies$  (2). Let  $E \subseteq \mathbb{R}$  be a closed set and consider the function  $f(x) = \chi_E(x)$ . Let

$$g(t) = \begin{cases} 1 & t \leq 0 \\ 1-t & 0 \leq t \leq 1 \\ 0 & t \geq 1. \end{cases}$$

Then define

$$f_\epsilon(x) = g\left(\frac{1}{\epsilon}\rho(x, E)\right)$$

where

$$\rho(x, E) = \inf\{|x - y| : y \in E\}.$$

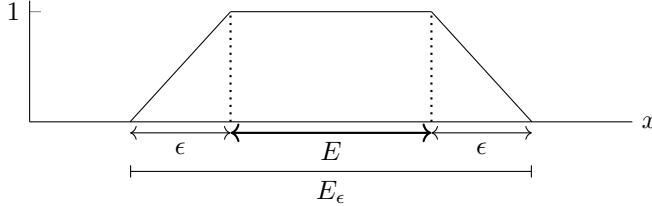
Note that  $E_\epsilon := \{x : \rho(x, E) < \epsilon\}$  forms a decreasing sequence of sets as  $\epsilon \searrow 0$  such that  $E_\epsilon \searrow E$ . Observe that,

$$\mathbb{P}_n(\xi_n \in E) = \int f \, d\mathbb{P}_n \leq \int f_\epsilon \, d\mathbb{P}_n.$$

Consequently,

$$\begin{aligned} \limsup \mathbb{P}_n(\xi_n \in E) &\leq \limsup \int f_\epsilon \, d\mathbb{P}_n \\ &\stackrel{(1)}{=} \int f_\epsilon \, d\mathbb{P} \\ &\leq \mathbb{P}(\xi \in E_\epsilon) \\ &\xrightarrow{\epsilon \searrow 0} \mathbb{P}(\xi \in E). \end{aligned}$$

Where (1) follows from the fact that  $\xi_n \rightarrow \xi$  weakly.



(2)  $\implies$  (3). This is clear after observing that  $O = \mathbb{R} \setminus E$  for  $E$  a closed set. Moreover, it is clear from this that (2) and (3) are equivalent.

(3)  $\implies$  (4). Recall that  $\bar{C} = C \cup \partial C$  and  $\mathring{C} = C \setminus \partial C$ . As  $\mathbb{P}(\xi \in \partial C) = 0$  it follows that

- $\limsup \mathbb{P}_n(\xi_n \in C) \leq \limsup \mathbb{P}_n(\xi_n \in \bar{C}) \leq \mathbb{P}(\xi \in \bar{C}) = \mathbb{P}(\xi \in C)$ , and
- $\liminf \mathbb{P}_n(\xi_n \in C) \geq \liminf \mathbb{P}_n(\xi_n \in \mathring{C}) \geq \mathbb{P}(\xi \in \mathring{C}) = \mathbb{P}(\xi \in C)$ .

Therefore,  $\lim \mathbb{P}_n(\xi_n \in C) = \mathbb{P}(\xi \in C)$ .

(4)  $\implies$  (5). This is clear.

(5)  $\implies$  (1). Let  $f$  be a bounded continuous function. Let  $X, X_1, X_2, \dots$  be the random variables given by Theorem 6.1.2, defined on the probability space  $(\Omega', \mathcal{F}', \mathbb{P}')$ . It follows that  $f(X_n) \rightarrow f(X)$  almost surely with respect to  $\mathbb{P}'$ . Therefore, using the dominated convergence theorem we conclude that

$$\mathbb{E}_n(f(\xi_n)) = \mathbb{E}(f(\xi_n)) \rightarrow \mathbb{E}(f(X)) = \mathbb{E}(f(\xi)).$$

Hence,  $\xi_n \rightarrow \xi$  weakly. ■

**Remark 6.1.5** Consequently, we can establish that a sequence of random variables  $(\xi_n)_{n \in \mathbb{N}}$  converges in distribution to a random  $\xi$  using the notion of weak convergence or (5) of Theorem 6.1.3. In any case, we will denote this convergence as  $\xi_n \xrightarrow{d} \xi$ .

**Corollary 6.1.6** Suppose that the random variables  $\xi_n, \xi$  have densities  $f_n(x), f(x)$ , respectively. If  $f_n(x) \rightarrow f(x)$  for any  $x$  then  $\xi_n \xrightarrow{d} \xi$ .

*Proof.* It is sufficient to show that

$$F_n(x) = \int_{-\infty}^x f_n(y) dy \rightarrow F(x) = \int_{-\infty}^x f(y) dy$$

for any  $x$ . Using

$$|F_n(x) - F(x)| \leq \int_{-\infty}^x |f_n(y) - f(y)| dy,$$

we need to check that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_n(y) - f(y)| dy = 0.$$

Note that  $a = a_+ - a_-$  and  $|a| = a_+ + a_-$  for any real  $a$ . Since  $f_n$  and  $f$  are densities, they integrate to 1, so for each  $n$  we have

$$0 = \int_{-\infty}^{\infty} f(y) - f_n(y) dy = \int_{-\infty}^{\infty} (f(y) - f_n(y))_+ - (f(y) - f_n(y))_- dy.$$

Then

$$\int_{-\infty}^{\infty} |f(y) - f_n(y)| dy = 2 \int_{-\infty}^{\infty} (f(y) - f_n(y))_+ dy,$$

which goes to zero by the dominated convergence theorem. Indeed

$$0 \leq (f(y) - f_n(y))_+ \leq f(y),$$

for any  $n$  and the function  $f(y)$  is integrable. ■

### Exercise 6.1.7

1. Let  $F$  be continuous and suppose  $F_n(x) \rightarrow F(x)$  pointwise. Show that  $F_n$  converges uniformly to  $F$ . That is,

$$\sup_x |F_n(x) - F(x)| \xrightarrow{n \rightarrow \infty} 0.$$

2. Give an example of distribution functions  $F_n(x), F(x)$  such that  $F_n(x) \xrightarrow{d} F(x)$ , but

$$\sup_x |F_n(x) - F(x)| \not\rightarrow 0,$$

as  $n \rightarrow \infty$ .

3. Give an example of probability measures  $\mathbb{P}, \mathbb{P}_n$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that  $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$ , but convergence  $\mathbb{P}_n(B) \rightarrow \mathbb{P}(B)$  does not hold for all  $B \in \mathcal{B}(\mathbb{R})$ .

## 6.2 Connection to Convergence in Probability

We will show convergence in probability is a strictly stronger notion than convergence in distribution. However, there does exist a partial converse that we will explore.

**Theorem 6.2.1** Let  $\xi$  and  $(\xi_n)_{n \in \mathbb{N}}$  be random variables from  $(\Omega, \mathcal{F}, \mathbb{P})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . If  $\xi_n \xrightarrow{p} \xi$  then  $\xi_n \xrightarrow{d} \xi$ .

$$\begin{array}{c} \xrightarrow{L^p} \\ \Downarrow \\ \xrightarrow{\text{a.s.}} \xrightarrow{p} \xrightarrow{d} \end{array}$$

*Proof.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function, such that

$$|f(x)| \leq C$$

for all  $x \in \mathbb{R}$ . Choose  $M$  such that  $\mathbb{P}(|\xi| > M) \leq \frac{\epsilon}{6C}$ . Note that for  $x \in [-M, M]$  the function  $f$  is uniformly continuous. Therefore, there exists a  $\delta > 0$  such that for  $|x - y| < \delta$ , with  $x, y \in [-M, M]$ , we have that

$$|f(x) - f(y)| \leq \frac{\epsilon}{3}.$$

Moreover, there exists an  $N$  such that

$$\mathbb{P}(|\xi_n - \xi| > \delta) < \frac{\epsilon}{6C}$$

for  $n \geq N$ . Hence for  $n \geq N$  it follows that,

$$\begin{aligned} \mathbb{E}(|f(\xi_n) - f(\xi)|) &\leq \mathbb{E}(|f(\xi_n) - f(\xi)| | |\xi_n - \xi| \leq \delta, |\xi| \leq M) \\ &\quad + \mathbb{E}(|f(\xi_n) - f(\xi)| | |\xi_n - \xi| \leq \delta, |\xi| > M) \\ &\quad + \mathbb{E}(|f(\xi_n) - f(\xi)| | |\xi_n - \xi| > \delta) \mathbb{P}(|\xi_n - \xi| > \delta) \\ &\leq \frac{\epsilon}{3} + 2C \frac{\epsilon}{6C} + 2C \frac{\epsilon}{6C} \\ &= \epsilon. \end{aligned}$$

■

**Proposition 6.2.2** Let  $(\xi_n)_{n \in \mathbb{N}}$  be random variables such that  $\xi_n \rightarrow c$  in distribution, where  $c$  is a constant. Then  $\xi_n \rightarrow c$  in probability.

*Proof.* Given an  $\epsilon > 0$ , the set  $E_\epsilon = \mathbb{R} \setminus B_\epsilon(c)$  is closed. Therefore, by Theorem 6.1.3 we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(|\xi_n - c| \geq \epsilon) &= \lim_{n \rightarrow \infty} \mathbb{P}(\xi_n \in E_\epsilon) \\ &= \limsup_n \mathbb{P}(\xi_n \in E_\epsilon) \\ &\leq \mathbb{P}(c \in E_\epsilon) \\ &= 0. \end{aligned}$$

Therefore,  $\xi_n \rightarrow c$  in probability. ■

**Example 6.2.3** Note that convergence in distribution does not imply convergence in probability. Consider a real-valued random variable  $X$  that is symmetric about zero, such as  $\xi \sim N(0, 1)$ . Then the sequence  $\xi_n := (-1)^{n+1}\xi$  converges in distribution, as they are identically distributed, but not in probability.

**Proposition 6.2.4 — Continuous Mapping Theorem.** Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function with points of discontinuity,  $U_\varphi$ . Let  $\xi, (\xi_n)_{n \in \mathbb{N}} : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  be random variables where  $\mathbb{P}(\xi \in U_\varphi) = 0$  and  $\xi_n \xrightarrow{d} \xi$ , then  $\varphi(\xi_n) \xrightarrow{d} \varphi(\xi)$ .

*Proof.* Let  $E \subseteq \mathbb{R}$  be a closed set. Let  $x \in \overline{\varphi^{-1}(E)} \setminus U_\varphi$ . By definition, there exists a sequence  $(x_n) \subseteq \varphi^{-1}(E)$  such that  $x_n \rightarrow x$ . As  $\varphi$  is continuous at  $x$  it follows that  $\varphi(x_n) \rightarrow \varphi(x)$ . As  $(\varphi(x_n)) \subseteq E$  and  $E$  is closed, it follows that  $\varphi(x) \in E$  which implies that  $x \in \varphi^{-1}(E)$ . Therefore,

$$\overline{\varphi^{-1}(E)} \subseteq \varphi^{-1}(E) \cup U_\varphi.$$

Therefore,

$$\begin{aligned} \limsup \mathbb{P}(\varphi(\xi_n) \in E) &= \limsup \mathbb{P}(\xi_n \in \varphi^{-1}(E)) \\ &\stackrel{(1)}{\leq} \limsup \mathbb{P}\left(\xi_n \in \overline{\varphi^{-1}(E)}\right) \\ &\stackrel{\text{Thm 6.1.3 } 2}{\leq} \mathbb{P}\left(\xi \in \overline{\varphi^{-1}(E)}\right) \\ &\leq \mathbb{P}(\xi \in \varphi^{-1}(E)) + \mathbb{P}(U_\varphi) \\ &= \mathbb{P}(\xi \in \varphi^{-1}(E)). \end{aligned}$$

Where (1) is justified as  $\varphi^{-1}(E) \subseteq \overline{\varphi^{-1}(E)}$ . Therefore, by point 5 from Theorem 6.1.3 we conclude that  $\varphi(\xi_n) \rightarrow \varphi(\xi)$  in distribution. ■

**Theorem 6.2.5** Let  $\xi, (\xi_n)_{n \in \mathbb{N}} : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  be random variables such that  $\xi_n \rightarrow \xi$  in distribution. Moreover, let  $\eta_1, \eta_2, \dots : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  be random variables such that  $\mathbb{P}(|\xi_n - \eta_n| > \epsilon) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $\epsilon > 0$ . Then  $\eta_n \rightarrow \xi$  in distribution.

The proof Theorem 6.2.5 relies on point 6. of Theorem 6.1.3, and so we will omit it here.

**Corollary 6.2.6** Consider the random variables  $\xi, (\xi_n)_{n \in \mathbb{N}}$  and  $(\eta_n)_{n \in \mathbb{N}}$  from  $(\Omega, \mathcal{F}, \mathbb{P})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . Assume  $\xi_n \xrightarrow{n \rightarrow \infty} \xi$  in distribution and  $\eta_n \xrightarrow{n \rightarrow \infty} c$  in probability, with  $c$  being a constant. Show that for the random variables  $T_n : \omega \in \Omega \mapsto (\xi_n(\omega), \eta_n(\omega))$  and  $T : \omega \in \Omega \mapsto (\xi_n(\omega), c)$  we have that  $T_n \xrightarrow{n \rightarrow \infty} T$  in distribution.

One can refer to [3] for proof of these results.

**Theorem 6.2.7 — Slutsky's Theorem.** Consider the random variables  $\xi, (\xi_n)_{n \in \mathbb{N}}$  and  $(\eta_n)_{n \in \mathbb{N}}$  from  $(\Omega, \mathcal{F}, \mathbb{P})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . If  $\xi_n \xrightarrow{d} \xi$  and  $\eta_n \xrightarrow{p} c$ , with  $c$  being a constant, then the following hold.

1.  $\xi_n + \eta_n \xrightarrow{d} \xi + c$ .
2.  $\xi_n \eta_n \xrightarrow{d} c\xi$ .
3. If  $c \neq 0$ , then  $\frac{\xi_n}{\eta_n} \xrightarrow{d} \frac{\xi}{c}$ .

*Proof.* Using Corollary 6.2.6 we can apply Proposition 6.2.4 to  $\varphi(x, y) = x+y$ ,  $\varphi(x, y) = xy$  and  $\varphi(x, y) = \frac{x}{y}$  respectively. ■

**Example 6.2.8** Consider a real-valued random variable  $\xi$ , that is symmetric about zero. Let  $\xi_n = \xi$  and  $\eta_n = (-1)^{n+1}\xi$ . Then  $\xi_n + \eta_n$  takes the form  $(2\xi, 0, 2\xi, 0, \dots)$  which does not converge in distribution. Hence, it is not true in general that  $\xi_n \xrightarrow{d} \xi$  and  $\eta_n \xrightarrow{d} \eta$  implies  $\xi_n + \eta_n \xrightarrow{d} \xi + \eta$ .

### 6.3 Relative Compactness and Tightness

Soon we will prove the central limit theorem by looking at characteristic functions. To establish the relationship between characteristic functions and measures, we need some tools relating to the collections of measures and clarify the meaning of relatively compact and tight collections. These concepts are also useful for working with stochastic processes and ergodic theory.

**Definition 6.3.1** A family of probability measures  $\mathcal{P} = (\mathbb{P}_\alpha)_{\alpha \in A}$ , with the corresponding set of distribution functions  $(F_\alpha)_{\alpha \in A}$  is relatively compact if every sequence of measures from  $\mathcal{P}$  contains a subsequence that weakly converges to a probability measure.

**Remark 6.3.2** We emphasise that in this definition the limit measure needs to be a probability measure, although it need not belong to the original class  $\mathcal{P}$ . In fact,  $\mathcal{P}$  is relatively compact if its closure with respect to the Levi-Prokhorov metric is compact.

**Example 6.3.3** The collection consisting of a weakly convergent sequence of measures is relatively compact.

**Lemma 6.3.4** Let  $\mathbb{P}$  be a probability measure and  $(\mathbb{P}_n)_{n \in \mathbb{N}}$  a family of probability measures. Then  $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$  if and only if every subsequence  $(\mathbb{P}_{n'_k})_{k \in \mathbb{N}}$  of  $\{\mathbb{P}_n\}$  contains a subsequence  $(\mathbb{P}_{n''_k})_{k \in \mathbb{N}}$  such that  $\mathbb{P}_{n''_k} \xrightarrow{d} \mathbb{P}$ .

*Proof.* ( $\Rightarrow$ ). For  $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$  it is the case that

$$\int f(x) d\mathbb{P}_n(x) \rightarrow \int f(x) d\mathbb{P}(x)$$

for all bounded and continuous functions  $f$ . Clearly, if this holds then any subsequence  $(\mathbb{P}_{n'_k})_{k \in \mathbb{N}}$  converges in distribution to  $\mathbb{P}$ . Hence, one can simply take the subsequence  $n''_k = n'_k$ . Conversely, let every subsequence  $(\mathbb{P}_{n'_k})_{k \in \mathbb{N}}$  has a subsequence  $(\mathbb{P}_{n''_k})_{k \in \mathbb{N}}$  such that  $\mathbb{P}_{n''_k} \xrightarrow{d} \mathbb{P}$ .

( $\Leftarrow$ ). Suppose for contradiction that  $\mathbb{P}_n \not\xrightarrow{d} \mathbb{P}$ , then there exists a bounded and continuous function  $f$  such that

$$\left| \int f(x) d\mathbb{P}_k(x) - \int f(x) d\mathbb{P} \right| \geq \epsilon$$

for infinitely many  $k \in \mathbb{N}$ . Hence, we can extract a subsequence  $(\mathbb{P}_{n'_k})_{k \in \mathbb{N}}$  such that

$$\left| \int f(x) d\mathbb{P}_{n'_k}(x) - \int f(x) d\mathbb{P} \right| \geq \epsilon$$

for all  $k \in \mathbb{N}$ . However, it is clear that this has no convergent subsequence convergent to  $\mathbb{P}$ , and hence we get a contradiction. ■

A given family of probability measures  $\mathcal{P}$  is not necessarily relatively compact.

**Example 6.3.5** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of real-valued random variables and suppose  $F_{\xi_n}(x) \rightarrow F(x)$  for all  $x \in U_F$ . Then  $F(x)$  is not necessarily a distribution function. Hence, the family  $(\xi_n)_{n \in \mathbb{N}}$  is not relatively compact.

1. The variables could run away to infinity. Let  $\xi_n$  be  $U[n, n+1]$ , then  $F(x) \equiv 0$ .
2. The variables could spread across infinity. Let  $\xi_n$  be  $U[-n, n]$ , then  $F(x) \equiv 1/2$ .

Let us denote the collection of non-decreasing, right-continuous functions with an image in  $[0, 1]$  as  $\mathcal{G} = \{F : \mathbb{R} \rightarrow [0, 1]\}$ , and refer to it as the collection of generalised distribution functions.

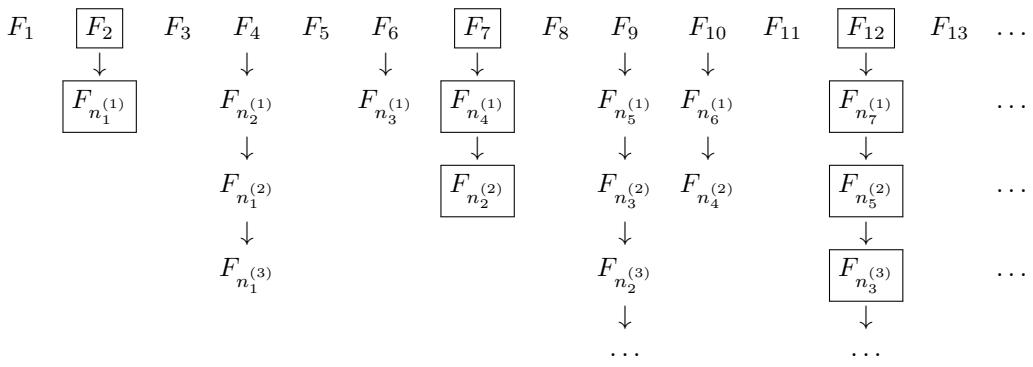
**Remark 6.3.6** Distribution functions form a subset of  $\mathcal{G}$  for which  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$ .

**Theorem 6.3.7 — Helly's Selection Theorem.** The collection  $\mathcal{G}$  of generalised distribution functions is sequentially compact. That is, for any sequence  $(F_n)_{n \in \mathbb{N}} \subseteq \mathcal{G}$ , there exists a function  $F \in \mathcal{G}$  and a subsequence  $F_{n_k} \subseteq \{F_n\}$  such that  $F_{n_k}(x) \rightarrow F(x)$  for every point  $x \in \mathbb{R} \setminus U_F$ , where  $U_F$  is the set of discontinuities of  $F$ .

*Proof.* Let  $(q_k)_{k \geq 1}$  be an enumeration of  $\mathbb{Q}$ , that is a bijection from  $\mathbb{N}$  to  $\mathbb{Q}$ .

- The sequence  $(F_n(q_1))_{n \geq 1}$  is bounded, so by the Bolzano-Weierstrass theorem it has a subsequence  $(F_{n_k^{(1)}}(q_1))_{k \geq 1}$  which converges to some number  $G(q_1) \in [0, 1]$ .
- Now consider the sequence  $(F_{n_k^{(1)}}(q_2))_{k \geq 1}$ . This is also bounded and so by the Bolzano-Weierstrass theorem it has a subsequence  $(F_{n_k^{(2)}}(q_2))_{k \geq 1}$  which converges to some number  $G(q_2) \in [0, 1]$ .
- We repeat to extract further subsequences.

We can illustrate the above procedure with the following diagram.



We now extract the diagonal subsequence, that is we let  $n_k = n_k^{(k)}$  for all  $k \in \mathbb{N}$ . Note that for any  $m \in \mathbb{N}$  we have  $(n_k)_{k \in \mathbb{N}} \subseteq (n_k^{(m)})_{k \in \mathbb{N}}$ . Hence,  $F_{n_k}(q_m) \rightarrow G(q_m)$  as  $(F_{n_k^{(m)}})_{k \in \mathbb{N}}$  is a converging sequence, and any subsequence of a converging sequence converges to the same limit as the main sequence. Therefore,  $F_{n_k}(q) \rightarrow G(q)$  for all  $q \in \mathbb{Q}$ . Moreover, for any  $p, q \in \mathbb{Q}$  we have  $G(p) \leq G(q)$  by the monotonicity of limits. Now define

$$F(x) = \inf\{F(q) : q \in \mathbb{Q}, q > x\} = \lim_{q \rightarrow x^+} G(q).$$

It is clear that  $F \in \mathcal{G}$ . Moreover, for all  $q \in \mathbb{Q}$  we have  $F(q) \geq G(q)$  and for  $x < q$  we have  $F(x) \leq G(q)$ . It remains to show that  $F_{n_k}(x) \rightarrow F(x)$  for all  $x \in \mathbb{R} \setminus U_F$ , so let us fix  $x \in \mathbb{R} \setminus U_F$  and some arbitrary  $\epsilon > 0$ . By continuity one can choose  $y < r < x < q$  with  $r, q \in \mathbb{Q}$ , such that

$$F(x) - \epsilon < F(y) \leq F(r) = G(r) \leq F(x) \leq F(q) = G(q) < F(x) + \epsilon,$$

and so for sufficiently large  $k$ , it follows that  $F_{n_k}(r), F_{n_k}(q) \in (F(x) - \epsilon, F(x) + \epsilon)$  which implies that  $F_{n_k}(x) \in (F(x) - \epsilon, F(x) + \epsilon)$ . Since  $\epsilon > 0$  is arbitrary, the proof is complete. ■

It turns out that the following notion of tightness provides a necessary and sufficient condition for a family of probability measures (or finite measures) to be relatively compact.

**Definition 6.3.8** A family of probability measures  $\mathcal{P} = (\mathbb{P}_\alpha)_{\alpha \in A}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  is tight if for all  $\epsilon > 0$  there exists a compact set  $K \subset \mathbb{R}$  such that

$$\sup_{\alpha \in A} \mathbb{P}_\alpha(\mathbb{R} \setminus K) \leq \epsilon.$$

**Theorem 6.3.9 — Prokhorov's theorem.** A family of probability measures  $\mathcal{P}$  is tight if and only if it is relatively compact.

*Proof.* ( $\Rightarrow$ ). Let us suppose that  $\mathcal{P} := (\mathbb{P}_\alpha)_{\alpha \in A}$  is relatively compact but not tight. Then there exists a  $\epsilon > 0$  such that for any compact  $K \subset \mathbb{R}$  we have  $\sup_\alpha \mathbb{P}_\alpha(\mathbb{R} \setminus K) > \epsilon$ . Hence, for any  $n$  there is a  $\mathbb{P}_{\alpha_n}$  such that

$$\mathbb{P}_{\alpha_n}(\mathbb{R} \setminus (-n, n)) > \epsilon. \quad (6.1)$$

By relative compactness, there is a subsequence  $(\mathbb{P}_{\alpha_{n_k}})_{k \geq 1}$  such that  $\mathbb{P}_{\alpha_{n_k}} \xrightarrow{k \rightarrow \infty} Q$  weakly, where  $Q$  is a probability measure. Therefore, by 6.1.3, it follows that

$$\limsup_{k \rightarrow \infty} \mathbb{P}_{\alpha_{n_k}}(\mathbb{R} \setminus (-n, n)) \leq Q(\mathbb{R} \setminus (-n, n)).$$

But the right-hand side  $\searrow 0$  as  $n \rightarrow \infty$ , which contradicts (6.1). Hence  $\mathcal{P}$  must be tight.

( $\Leftarrow$ ). Now let  $\mathcal{P}$  be tight, and  $(\mathbb{P}_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $\mathcal{P}$  with corresponding distribution functions  $(F_n)_{n \in \mathbb{N}}$ . By Helly's selection theorem, there exists a subsequence  $F_{n_k}(x) \rightarrow F(x)$  at  $x \in \mathbb{R} \setminus U_F$  where  $F$  is some generalised distribution function. We now check that  $F(-\infty) = 0$  and  $F(+\infty) = 1$ . Fix  $\epsilon > 0$ , then from tightness there is an  $I = (a, b)$  such that,

$$\sup_n \mathbb{P}_n(\mathbb{R} \setminus I) < \epsilon.$$

Consequently,  $1 - \inf_n \mathbb{P}_n(I) > \epsilon$  which implies that  $\mathbb{P}_n(I) > 1 - \epsilon$  for all  $n \in \mathbb{N}$ . Now let  $a' < a < b < b'$  where  $a', b' \in \mathbb{R} \setminus U_F$ , then

$$\begin{aligned} 1 - \epsilon &< \mathbb{P}_{n_k}((a, b]) \\ &< \mathbb{P}_{n_k}((a', b']) \\ &= F_{n_k}(b') - F_{n_k}(a') \\ &\xrightarrow{k \rightarrow \infty} F(b') - F(a'). \end{aligned}$$

This implies that  $F(+\infty) - F(-\infty) \geq 1$ . Since,  $F : \mathbb{R} \rightarrow [0, 1]$  we must have  $F(+\infty) = 1$  and  $F(-\infty) = 0$ . Therefore,  $F$  is a distribution function which implies that  $\mathcal{P}$  is relatively compact. ■

### Remark 6.3.10

- From Theorem 6.3.9, one sees that for a family of random variables  $(\xi_n)_{n \in \mathbb{N}}$ , if

$$F_{\xi_n}(x) \rightarrow F(x)$$

for all points of continuity of  $F$ , where  $F$  is a distribution function, then  $(\xi_n)_{n \in \mathbb{N}}$  is tight.

- Theorem 6.3.9 remains true for measures on  $\mathbb{R}^n$ ,  $\mathbb{R}^\infty$  and more generally on any complete separable metric space with a Borel  $\sigma$ -algebra of sets.

**Exercise 6.3.11** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a family of random variables such that  $\mathbb{E}(|\xi_n|) \leq M$  for all  $n \in \mathbb{N}$ . Show that  $(\xi_n)_{n \in \mathbb{N}}$  is tight.

## 6.4 Solution to Exercises

### Exercise 6.1.7

*Solution.*

- Fix  $\epsilon > 0$ . As  $\lim_{x \rightarrow \infty} F(x) = 1$ ,  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $F(x)$  is non-decreasing, there exists an  $M \in \mathbb{R}$  such that for  $x > M$  we have  $F(x) \in (1 - \frac{\epsilon}{2}, 1]$  and for  $x < -M$  we have  $F(x) \in [0, \frac{\epsilon}{2}]$ . Let  $x_\epsilon > M$  then there exists an  $N_1 \in \mathbb{N}$  such that  $|F_n(x_\epsilon) - F(x_\epsilon)| < \frac{\epsilon}{2}$  for  $n \geq N_1$ . Therefore, for all  $x \geq x_\epsilon$  and  $n \geq N_1$  it follows that  $F_n(x) \in (1 - \epsilon, 1]$ , so that  $|F_n(x) - F(x)| < \epsilon$ . Similarly, there exists an  $N_2 \in \mathbb{N}$  such that for  $x \leq -x_\epsilon$  and  $n \geq N_2$  we have  $|F_n(x) - F(x)| < \epsilon$ . Then as  $F$  is continuous it is uniformly continuous on  $[-x_\epsilon, x_\epsilon]$  so that we can choose

$$-x_\epsilon = x_0 < x_1 < \dots < x_k = x_\epsilon$$

such that  $|F(x_{i+1}) - F(x_i)| < \frac{\epsilon}{5}$  for each  $i \in \{0, \dots, k\}$ . For each  $i \in \{0, \dots, k\}$  let  $\tilde{N}_i \in \mathbb{N}$  be such that

$$|F_n(x_i) - F(x_i)| \leq \frac{\epsilon}{5}$$

for all  $n \geq \tilde{N}_i$ . For  $n \geq \tilde{N} = \max_i (\tilde{N}_i)$  it follows that

$$\begin{aligned} |F_n(x_{i+1}) - F_n(x_i)| &\leq |F_n(x_{i+1}) - F(x_{i+1})| + |F(x_{i+1}) - F(x_i)| + |F(x_i) - F_n(x_i)| \\ &\leq \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\ &\leq \frac{3\epsilon}{5}. \end{aligned}$$

Therefore, for any  $x \in [-x_\epsilon, x_\epsilon]$ , it is true that  $x_i \leq x < x_{i+1}$  for some  $i \in \{0, \dots, k-1\}$ . By the non-decreasing property of  $F$  we know that  $F(x_i) \leq F(x) \leq F(x_{i+1})$  and similarly for each  $F_n$ . Therefore,

$$\begin{aligned} |F_n(x) - F(x)| &\leq |F_n(x) - F_n(x_i)| + |F_n(x_i) - F(x_i)| + |F(x_i) - F(x)| \\ &\leq |F_n(x_{i+1}) - F_n(x_i)| + |F_n(x_i) - F(x_i)| + |F(x_i) - F(x_{i+1})| \\ &\leq \frac{3\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\ &< \epsilon. \end{aligned}$$

Therefore, for  $n \geq N = \max(N_1, N_2, \tilde{N})$  we have that

$$|F_n(x) - F(x)| < \epsilon$$

for all  $x \in \mathbb{R}$ . Hence,  $\sup_x |F_n(x) - F(x)| \rightarrow 0$  which is equivalent to uniform convergence.

2. Consider the random variable  $\xi : \Omega \rightarrow \mathbb{R}$  where  $\mathbb{P}(\xi = 1) = 1$ . The distribution function of  $\xi$  is given by

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1. \end{cases}$$

For  $n \in \mathbb{N}$ , let  $\xi_n : \Omega \rightarrow \mathbb{R}$  be the random variable where  $\mathbb{P}(\xi_n = 1 - \frac{1}{n}) = 1$ . Similarly, the distribution function of  $\xi_n$  is given by

$$F_n(x) = \begin{cases} 0 & x < 1 - \frac{1}{n} \\ 1 & x \geq 1 - \frac{1}{n}. \end{cases}$$

Then for  $x > 1$  it is clear that  $F_n(x) = F(x) = 1$  for all  $n \in \mathbb{N}$  which implies that  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ . For  $x < 1$ , there exists a  $N \in \mathbb{N}$  such that  $\frac{1}{N} < 1 - x$ . Therefore, for  $n \geq N$  we have that  $F_n(x) = F(x) = 0$  and so  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ . As  $F$  is only continuous for  $\mathbb{R} \setminus \{0\}$  we conclude that  $F_n \xrightarrow{d} F$ . However,  $\sup_x |F_n(x) - F(x)| = 1$  for all  $n \in \mathbb{N}$ .

3. Consider the setting of the previous part and let  $B = (-\infty, 1) \in \mathcal{B}(\mathbb{R})$ . Then  $\mathbb{P}(\xi_n \in B) = 1$  for all  $n \in \mathbb{N}$ , however,  $\mathbb{P}(\xi \in (0, 1)) = 0$ . ■

### Exercise 6.3.11

*Solution.* For  $\epsilon > 0$  let  $K = [-\frac{2M}{\epsilon}, \frac{2M}{\epsilon}]$  then using Markov's inequality it follows that

$$\begin{aligned} \mathbb{P}(\xi_n \in \mathbb{R} \setminus K) &= \mathbb{P}\left(|\xi_n| > \frac{2M}{\epsilon}\right) \\ &\leq \frac{\mathbb{E}(|\xi_n|)}{\frac{2M}{\epsilon}} \\ &\leq \frac{\epsilon}{2M} M \\ &< \epsilon. \end{aligned}$$

Therefore, the family  $(\xi_n)_{n \in \mathbb{N}}$  is tight. ■

## 7 Convergence of Characteristic Functions

In this chapter, we look at the characteristic function of measures of  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ .

### 7.1 Characteristic Function

**Definition 7.1.1** The characteristic function of a random variable  $\xi : \Omega \rightarrow \mathbb{R}$  is

$$\varphi(t) := \varphi_\xi(t) \equiv \mathbb{E}(e^{it\xi}) := \int_{\Omega} e^{it\xi(\omega)} \mathbb{P}(\mathrm{d}\omega)$$

for  $t \in \mathbb{R}$ .

**Remark 7.1.2** We may generalise Definition 7.1.1 to random variables on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ . The characteristic function of a random vector  $\xi := (\xi_1, \dots, \xi_n)$  is

$$\varphi_\xi(t_1, \dots, t_n) := \mathbb{E} \left( \exp \left( i \sum_{k=1}^n t_k \xi_k \right) \right).$$

The characteristic function of a random variable only depends on its distribution. If  $F(x)$  has density  $f(x)$ , with respect to the Lebesgue measure, then

$$\varphi(t) = \int_{\mathbb{R}^n} e^{i(t^\top x)} f(x) \mathrm{d}x.$$

#### Proposition 7.1.3

1. If  $\xi$  is a random variable and  $\eta = a\xi + b$  for constants  $a, b$ , then  $\varphi_\eta(t) = e^{itb} \mathbb{E}(e^{iat\xi})$ .
2. For a characteristic function  $\varphi$  we have  $|\varphi(t)| \leq \varphi(0) = 1$ .
3. Let  $\xi$  be a random variable. Then  $\varphi_\xi(t)$  is uniformly continuous on  $\mathbb{R}$ .
4. If  $\xi_1, \dots, \xi_n$  are independent random variables and  $S = \xi_1 + \dots + \xi_n$ , then

$$\varphi_S(t) = \prod_{j=1}^n \varphi_{\xi_j}(t).$$

*Proof.*

1. Clear from properties of the expectation.
2. Clear from the definition.
3. Note that

$$|\varphi(t+h) - \varphi(t)| = |\mathbb{E}(e^{it\xi}(e^{ih\xi} - 1))| \leq \mathbb{E}(|e^{ih\xi} - 1|).$$

By dominated convergence theorem we know that  $\mathbb{E}(|e^{ih\xi} - 1|) \rightarrow 0$  as  $h \rightarrow 0$ . Therefore, we have the uniform continuity of  $\varphi$ .

4. Clear from properties of the expectation. ■

The moment-generating function also shares properties 1. and 4., but the lack of properties 2. and 3. means it is preferable to use characteristic functions to establish weak convergence.

#### Exercise 7.1.4

1. Let  $\xi \sim \text{B}(n, p)$ . Then

$$\varphi_\xi(t) = (pe^{it} + (1-p))^n.$$

2. Let  $\xi \sim N(m, \sigma^2)$ . Then

$$\varphi_\xi(t) = \exp\left(itm - \frac{t^2\sigma^2}{2}\right).$$

3. Let  $\xi \sim Po(\lambda)$ . Then

$$\varphi_\xi(t) = e^{-\lambda + \lambda e^{it}}.$$

## 7.2 Obtaining Moments

The existence of moments for a real-valued random variable is determined by the smoothness of its characteristic function at zero.

**Proposition 7.2.1** Let  $\xi$  be a random variable with a characteristic function  $\varphi$  and distribution function  $F$ .

1. If  $\mathbb{E}(|\xi|^n) < \infty$  for some  $n \geq 1$  then  $\varphi^{(\tau)}(t)$  exists for any  $0 \leq \tau \leq n$  and the following hold.

(a)  $\varphi^{(\tau)}(t) = \int_{\mathbb{R}} (ix)^\tau e^{itx} dF(x)$ .

(b)  $\mathbb{E}(\xi^\tau) = \frac{\varphi^{(\tau)}(0)}{i^\tau}$ .

(c)  $\varphi(t) = \sum_{\tau=0}^{n-1} \frac{(it)^\tau}{\tau!} \mathbb{E}(\xi^\tau) + \frac{(it)^n}{n!} \varepsilon_n(t)$ , where  $|\varepsilon_n(t)| \leq 3\mathbb{E}(|\xi|^n)$  and  $\varepsilon_n(t) \rightarrow 0$  as  $t \rightarrow 0$ .

2. If  $\mathbb{E}(|\xi|^n) < \infty$  for all  $n \geq 1$  and

$$\limsup_{n \rightarrow \infty} \left( \frac{(\mathbb{E}(|\xi|^n))^{\frac{1}{n}}}{n} \right) = \frac{1}{e \cdot R} < \infty,$$

for  $R > 0$ , then

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathbb{E}(\xi^n)$$

converges for all  $|t| < R$ .

*Proof.*

1. Since  $\mathbb{E}(|\xi|^n) < \infty$ , we have  $\mathbb{E}(|\xi|^r) < \infty$  for any  $r \leq n$  by Lyapunov's inequality. Consider the difference quotient

$$\frac{\varphi(t+h) - \varphi(t)}{h} = \mathbb{E}\left(e^{it\xi} \left(\frac{e^{ih\xi} - 1}{h}\right)\right).$$

Note that

$$\begin{aligned} \left| e^{it\xi} \frac{e^{ihx} - 1}{h} \right| &= \left| \frac{e^{ihx} - 1}{h} \right| \\ &= \left| \frac{1 + ihx + O(h^2) - 1}{h} \right| \\ &\leq |x|. \end{aligned}$$

So if  $\mathbb{E}(|\xi|) < \infty$  it follows from the dominated convergence theorem that

$$\begin{aligned} \varphi'(t) &= \lim_{h \rightarrow 0} \frac{\varphi(t+h) - \varphi(t)}{h} \\ &= \lim_{h \rightarrow 0} \mathbb{E}\left(e^{it\xi} \left(\frac{e^{ih\xi} - 1}{h}\right)\right) \\ &= \mathbb{E}\left(e^{it\xi} \lim_{h \rightarrow 0} \left(\frac{e^{ih\xi} - 1}{h}\right)\right) \\ &= \mathbb{E}(i\xi e^{it\xi}) \end{aligned}$$

$$= i \int_{-\infty}^{\infty} xe^{itx} dF(x).$$

The existence of the derivatives  $\varphi^{(r)}(t)$  for  $1 < r \leq n$  follows by induction. Note that (b) follows immediately from (a). To establish the (c), consider

$$e^{iy} = \cos y + i \sin y = \sum_{k=0}^{n-1} \frac{(iy)^k}{k!} + \frac{(iy)^n}{n!} (\cos \theta_1 y + i \sin \theta_2 y)$$

for real  $y$  with  $|\theta_1| \leq 1$ ,  $|\theta_2| \leq 1$ . Therefore,

$$e^{it\xi} = \sum_{k=0}^{n-1} \frac{(i\xi)^k}{k!} + \frac{(i\xi)^n}{n!} (\cos \theta_1 \xi + i \sin \theta_2 \xi)$$

and so

$$\mathbb{E}(e^{it\xi}) = \sum_{k=0}^{n-1} \frac{(it)^k}{k!} \mathbb{E}(\xi^k) + \frac{(it)^n}{n!} (\mathbb{E}(\xi^n) + \varepsilon_n(t)),$$

where

$$\varepsilon_n(t) = \mathbb{E}(\xi^n (\cos \theta_1(\omega)t\xi + i \sin \theta_2(\omega)t\xi - 1)).$$

It is clear that  $|\varepsilon_n(t)| \leq 3\mathbb{E}(|\xi^n|)$ . Then dominated convergence shows that  $\varepsilon_n(t) \rightarrow 0$  as  $t \rightarrow 0$ .

2. Let  $0 < t_0 < T$ . Then,

$$\limsup \frac{(\mathbb{E}(|\xi|^n))^{1/n}}{n} \leq \frac{1}{et_0}$$

implies that

$$\limsup \frac{(\mathbb{E}(|\xi|^n e^n t_0^n))^{1/n}}{n} \leq 1.$$

Thus

$$\limsup \left( \frac{\mathbb{E}(|\xi|^n e^n t_0^n)}{n^n} \right)^{1/n} < 1.$$

By Stirling's formula we have that

$$\limsup \left( \frac{\mathbb{E}(|\xi|^n t_0^n)}{n!} \right)^{1/n} < \limsup \left( \frac{\mathbb{E}(|\xi|^n e^n t_0^n)}{n^n} \right)^{1/n} < 1.$$

Consequently, the series  $\sum \mathbb{E}\left(\frac{|\xi|^n t_0^n}{n!}\right)$  converges by Cauchy's test and so the series  $\sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mathbb{E}(\xi^r)$  converges for  $|t| \leq t_0$ . By the previous statement for  $n \geq 1$  we know that

$$\varphi(t) = \sum_{r=0}^n \frac{(it)^r}{r!} \mathbb{E}(\xi^r) + R_n(t),$$

where  $|R_n(t)| \leq \frac{3|t|^n}{n!} \mathbb{E}(|\xi|^n)$ . Therefore

$$\varphi(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mathbb{E}(\xi^r)$$

for all  $|t| < T$ . ■

**Remark 7.2.2** The second part of Proposition 7.2.1 gives a sufficient condition for the moments  $\mathbb{E}(\xi^n)$  to determine  $\varphi(t)$  uniquely. Indeed, they already determine  $\varphi(t)$  for  $-R < t < R$ . Take  $s$  such that  $|s| < \frac{R}{2}$  and follow the proof to get

$$\varphi(t) = \sum_{k=0}^{\infty} i^k \frac{(t-s)^k}{k!} \varphi^{(k)}(s),$$

where

$$\varphi^{(k)}(s) = \mathbb{E}(\xi^k e^{is\xi}),$$

for  $-\frac{R}{2} < s < \frac{R}{2}$ . Therefore, the moments uniquely determine  $\varphi(t)$  for  $|t| < \frac{3}{2}R$ . Proceed analogously to increase the domain in which  $\varphi(t)$  is defined.

**Theorem 7.2.3 — Carleman's test.** A sufficient condition for the unique determination of the characteristic function  $\varphi(t)$  is

$$\sum_{n=0}^{\infty} \frac{1}{(\mathbb{E}(\xi^{2n}))^{\frac{1}{2n}}} = \infty.$$

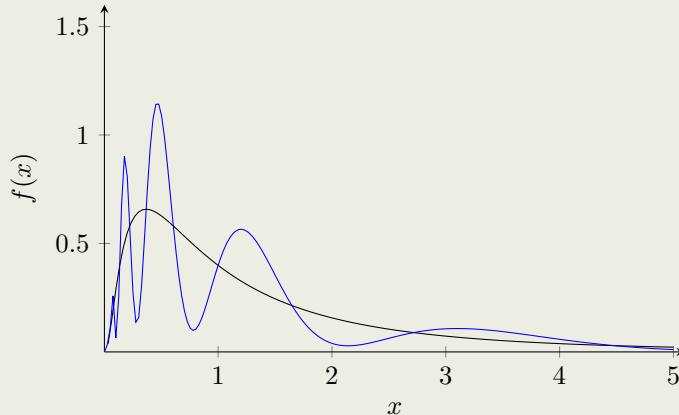
**Example 7.2.4** If  $\mathbb{E}(\xi^n)$  grows too fast, there may be multiple characteristic functions  $\varphi(t)$  with the same moments. Consider a random variable distributed as a standard log-normal distribution,

$$f_0(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log(x))^2}{2}\right),$$

for  $x \geq 0$ . Consider another random variable with density

$$f_a(x) = f_0(x) \times (1 + a \sin(2\pi \log(x))),$$

for  $x \geq 0$  and  $a \in [-1, 1]$ .



These seemingly different random variables have the same  $r^{\text{th}}$  moment. To see this, it suffices to evaluate the integral

$$\int_0^\infty x^{r-1} \exp\left(-\frac{(\log(x))^2}{2}\right) \sin(2\pi \log(x)) dx,$$

for  $r = 0, 1, \dots$ . With a variable substitution of  $s = \log(x)$ , the integral becomes

$$\int_{-\infty}^\infty \exp((r-1)s) \exp\left(-\frac{s^2}{2}\right) \sin(2\pi s) ds.$$

The integrand is an  $L^1$  function multiplied by  $\sin(2\pi s)$ . Therefore, by the Riemann-Lebesgue lemma, the integral is zero, and the two random variables have the same moments.

### Exercise 7.2.5

1. Verify that if  $\xi$  has a standard normal distribution, then  $\exp(\xi)$  has a standard log-normal distribution.
2. Show that the  $r^{\text{th}}$  moment of  $\exp(\xi)$  is equal to  $\exp\left(\frac{r^2}{2}\right)$ .

Notice that the moments grow too fast for the characteristic function to be analytic.

### 7.3 Inversion Formula

**Theorem 7.3.1 — Inversion formula I.** Let  $\xi$  be a random variable on  $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  with distribution  $F(x)$  and characteristic function  $\varphi(t)$ . If  $a < b$  are points of continuity of  $F$  then

$$F(b) - F(a) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt.$$

To prove Theorem 7.3.1 let us define

$$S(T) = \int_0^T \frac{\sin x}{x} dx.$$

Note  $S(T)$  is a differentiable function with  $S(T) > 0$  whenever  $T > 0$ . Moreover,

$$S(+\infty) = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

This can be proven using standard calculus techniques<sup>1</sup>. We therefore know that  $S(T)$  is a bounded function and  $\sup_{T>0} S(T)$  exists. We also note that

$$\int_0^T \frac{\sin(kx)}{x} dx = \int_0^T \frac{\sin(kx)}{kx} d(kx) = S(kT)$$

for  $k > 0$  and when  $k < 0$ , we have

$$\int_0^T \frac{\sin(kx)}{x} dx = - \int_0^T \frac{\sin(|k|x)}{x} dx = -S(|k|T).$$

Equivalently, we can write

$$\int_0^T \frac{\sin(kx)}{x} dx = \text{sgn}(k)S(|k|T).$$

Moreover, as the integrand is even we know that

$$\int_{-T}^T \frac{\sin(kx)}{x} dx = 2 \text{sgn}(k)S(|k|T).$$

*Proof.* (Theorem 7.3.1). For fixed  $T$  let

$$I_T = \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt = \int_{-T}^T \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} e^{itx} dF(x) dt$$

We note that the integrand is bounded uniformly in  $(t, x)$  by

$$\left| \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \right| \leq \left| \int_a^b e^{-its} ds \right| \leq |b - a|, \quad (7.1)$$

and

$$\int_T^T \int_{\mathbb{R}} |b - a| dF(x) dt = 2T|b - a| < \infty.$$

Therefore, by Fubini's theorem we may exchange the order of integration so that

$$I_T = \int_{\mathbb{R}} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} e^{itx} dt dF(x) = \int_{\mathbb{R}} \int_{-T}^T \frac{e^{it(x-a)} - e^{it(x-b)}}{it} dt dF(x).$$

Since the domain of integration of the inner integral is symmetric, we can ignore the odd parts of the integrand, so that

$$I_T = \int_{\mathbb{R}} \int_{-T}^T \frac{\sin(t(x-a)) - \sin(t(x-b))}{t} dt dF(x).$$

<sup>1</sup><https://www.wikihow.com/Integrate-the-Sinc-Function>

Now let

$$J_{T,x} := \int_{-T}^T \frac{\sin(t(x-a)) - \sin(t(x-b))}{t}$$

so that

$$I_T = \int_{\mathbb{R}} J_{T,x} dF(x).$$

Note that

$$|J_{T,x}| \leq \left| \int_{-T}^T \frac{\sin(t(x-a))}{t} \right| + \left| \int_{-T}^T \frac{\sin(t(x-b))}{t} \right| \leq 4 \sup_{T>0} S(T) < \infty,$$

which is integrable with respect to  $F$ . Therefore by the dominated convergence theorem we have that

$$I_\infty := \lim_{T \rightarrow \infty} I_T = \int_{\mathbb{R}} \lim_{T \rightarrow \infty} J_{T,x}.$$

Note that

$$\lim_{T \rightarrow \infty} J_{T,x} = \begin{cases} 0 & x \notin [a, b] \\ \pi & x \in \{a, b\} \\ 2\pi & x \in (a, b). \end{cases}$$

Therefore,

$$I_\infty = 2\pi(F(b-) - F(a)) - \pi(F(a) - F(a-) - (F(b-) - F(b))),$$

but as  $a$  and  $b$  are points of continuity we conclude that

$$F(b) - F(a) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt.$$

■

**Corollary 7.3.2** There is a one-to-one correspondence between probability distributions on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and characteristic functions.

*Proof.* Let  $F$  and  $G$  be probability distribution functions on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  with the same characteristic function. By Theorem 7.3.1 we note that  $F(b) - F(a) = G(b) - G(a)$  for any  $a < b$  that are points of continuity of  $F$  and  $G$ , which is dense in  $\mathbb{R}$ . Since the collection of open intervals  $\{(a, b) : a < b\}$  generates  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , we must have  $F = G$ . ■

**Proposition 7.3.3 — Inversion formula II.** Let  $\xi$  be a random variable on  $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  with distribution  $F(x)$  and characteristic function  $\varphi(t)$ . If  $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$ , then  $F(x)$  is absolutely continuous with density  $f(x)$ , and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt. \quad (7.2)$$

*Proof.* Let  $f(x)$  be as defined in (7.2). Then for  $|h| > 0$  consider

$$f(x+h) - f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{-it(x+h)} - e^{-itx}) \varphi(t) dt.$$

Note that

$$\left| e^{-it(x+h)} - e^{-itx} \right| |\varphi(t)| \leq 2|\varphi(t)|$$

and

$$(e^{-it(x+h)} - e^{-itx}) \varphi(t) \xrightarrow{|h| \rightarrow 0} 0.$$

Therefore, as

$$\int_{-\infty}^{\infty} |2\varphi(t)| dt = 2 \int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$$

we can deduce from the dominated convergence theorem that

$$|f(x+h) - f(x)| \xrightarrow{|h| \rightarrow 0} 0$$

which implies that  $f$  is continuous. Similarly, one can show that  $f$  is differentiable. Hence,  $f$  is integrable on  $[a, b]$  for  $a < b$ . Therefore, we can deduce that

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt dx \\ &\stackrel{\text{Fubini.}}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt \\ &\stackrel{\text{Thm 7.3.1}}{=} F(b) - F(a) \end{aligned}$$

where  $F(x) = \int_{-\infty}^x f(y) dy$  for all  $x \in \mathbb{R}$  and so  $F$  is absolutely continuous. Suppose  $F$  has density  $g$ , then

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} g(x) dx.$$

As  $\varphi(t)$  is integrable we can apply the inversion formula for Fourier transforms to deduce that

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt = f(x).$$

Therefore,  $F$  the density of  $F$  is  $f$ . ■

## 7.4 Central Limit Theorems

The following lemma will be useful for proving Theorem 7.4.3.

**Lemma 7.4.1** If  $\mathbb{P}$  is a probability measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  with characteristic function  $\varphi(t)$ , then

$$\mathbb{P}\left(\left\{x : |x| \geq \frac{2}{\epsilon}\right\}\right) \leq \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} 1 - \varphi(t) dt$$

for all  $\epsilon > 0$ .

*Proof.* Note that for  $x \neq 0$ , we have

$$\int_{-\epsilon}^{\epsilon} 1 - e^{itx} dt = 2\epsilon - \frac{e^{it\epsilon} - e^{-it\epsilon}}{ix} = 2\epsilon \left(1 - \frac{\sin \epsilon x}{\epsilon x}\right).$$

Therefore,

$$\begin{aligned} \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} 1 - \varphi(t) dt &= \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \left(1 - \int_{\mathbb{R}} e^{itx} \mu(dx)\right) dt \\ &= \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\mathbb{R}} 1 - e^{itx} \mu(dx) dt \\ &\stackrel{\text{Fubini.}}{=} \int_{\mathbb{R}} \left(\frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} 1 - e^{itx} dt\right) \mathbb{P}(dx) \\ &= \int_{\mathbb{R}} 2 \underbrace{\left(1 - \frac{\sin \epsilon x}{\epsilon x}\right)}_{\geq 0} \mathbb{P}(dx) \\ &\geq 2 \int_{-\infty}^{-\frac{2}{\epsilon}} \left(1 - \frac{\sin \epsilon x}{\epsilon x}\right) \mathbb{P}(dx) + 2 \int_{\frac{2}{\epsilon}}^{\infty} \left(1 - \frac{\sin \epsilon x}{\epsilon x}\right) \mathbb{P}(dx). \end{aligned}$$

Note that

$$1 - \frac{\sin \epsilon x}{\epsilon x} \geq \frac{1}{2}$$

for  $|x| \geq \frac{2}{\epsilon}$ . Therefore,

$$\frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} 1 - \varphi(t) dt \geq \int_{-\infty}^{-\frac{2}{\epsilon}} \mathbb{P}(dx) + \int_{\frac{2}{\epsilon}}^{\infty} \mathbb{P}(dx) = \mathbb{P}\left(\left\{x : |x| \geq \frac{2}{\epsilon}\right\}\right).$$

■

**Remark 7.4.2** Lemma 7.4.1 shows that the tail of the measure  $\mathbb{P}$ , hence the existence of moments, is determined by the smoothness of  $\varphi$  at zero.

**Theorem 7.4.3 — Levi's Continuity theorem.** Let  $(\varphi_n)_{n \in \mathbb{N}}$  be the characteristic functions corresponding to the sequence of distribution functions  $(F_n)_{n \in \mathbb{N}}$ .

1. If  $F_n \rightarrow F$  weakly, where  $F$  is a distribution function, then  $\varphi_n(t) \rightarrow \varphi(t)$  pointwise for all  $t \in \mathbb{R}$ , where  $\varphi$  is the characteristic function of  $F$ .
2. If  $\varphi(t) := \lim_{n \rightarrow \infty} \varphi_n(t)$  exists for all  $t \in \mathbb{R}$ , and is continuous at  $t = 0$ , then  $\varphi(t)$  is a characteristic function of some distribution function  $F$ . Moreover,  $F_n \rightarrow F$  weakly.
3. If  $\varphi_n(t)$  is a characteristic function corresponding to a distribution function  $F_n$  and  $\varphi(t)$  is a characteristic function corresponding to some distribution function  $F$ . Then  $\varphi_n(t) \rightarrow \varphi(t)$  for all  $t \in \mathbb{R}$  if and only if  $F_n \rightarrow F$  weakly.

*Proof.*

1. Recall that  $F_n \rightarrow F$  weakly means that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) dF_n(x) = \int_{\mathbb{R}} f(x) dF(x)$$

for every bounded and real continuous functions  $f$ . Therefore, as  $\text{Re}(e^{itx})$  and  $\text{Im}(e^{itx})$  are bounded and real continuous functions, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi_n(t) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} e^{itx} dF_n(x) \\ &= \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}} \text{Re}(e^{itx}) dF_n(x) + \int_{\mathbb{R}} \text{Im}(e^{itx}) dF_n(x) \right) \\ &= \int_{\mathbb{R}} \text{Re}(e^{itx}) dF(x) + \int_{\mathbb{R}} \text{Im}(e^{itx}) dF(x) \\ &= \int_{\mathbb{R}} e^{itx} dF(x) \\ &= \varphi(t). \end{aligned}$$

2. As  $\varphi$  is continuous at 0 we know that  $\varphi(0) = 1$  as  $\varphi_n(0) = 1$  for all  $n \in \mathbb{N}$ . Therefore, for all  $\epsilon > 0$  there is  $u > 0$  small enough such that for all  $t \in [-u, u]$  we have

$$1 - \varphi(t) \leq \frac{\epsilon}{4}.$$

Hence, using the dominated convergence theorem it follows that

$$\frac{\epsilon}{2} \geq \frac{1}{u} \int_{-u}^u 1 - \varphi(t) dt = \lim_{n \rightarrow \infty} \frac{1}{u} \int_{-u}^u 1 - \varphi_n(t) dt.$$

Using Lemma 7.4.1, there is an  $n_0$  such that

$$\mathbb{P}_n\left(\left\{x : |x| \geq \frac{2}{u}\right\}\right) \leq \frac{1}{u} \int_{-u}^u 1 - \varphi_n(t) dt \leq \epsilon \quad (7.3)$$

for all  $n \geq n_0$ . In particular, one can choose  $u$  small enough such that (7.3) holds for all  $n \geq 1$ . Thus,  $(\mathbb{P}_n)_{n \geq 1}$  is tight. Consequently, using Theorem 6.3.9, we have that for any subsequence

$(F_{n_k})_{k \geq 1} \subseteq (F_n)_{n \geq 1}$ , there is a further subsequence that converges weakly to some  $F$ . Using statement 1. it follows that  $\varphi(t)$  is the characteristic function of this  $F$ . In particular, this shows that the limiting distribution function is the same regardless of the subsequence we choose. Suppose that  $F_n \not\rightarrow F$  weakly. Then there is a point  $y \in \mathbb{R} \setminus U_F$ , such that a subsequence  $(F'_{n_k})_{k \geq 1}$  exists with the property that

$$|F'_{n_k}(y) - F(y)| \geq \epsilon$$

for all  $k$ . However, by the above arguments, there is a further subsequence  $(F_{n_{k_j}})_{j \geq 1}$  which converges to  $F$ , giving rise to a contradiction. Therefore, we conclude that  $F_n \rightarrow F$  weakly.

3. ( $\Leftarrow$ ). This is precisely statement 1.

( $\Rightarrow$ ). As  $\varphi(t)$  is a characteristic function, it is continuous at 0 so we can use statement 2. to deduce that  $F_n \rightarrow F$  weakly.  $\blacksquare$

**Remark 7.4.4** The utility of statement 2. in Theorem 7.4.3 is that it does not assume a priori that  $\varphi(t)$  is a characteristic function. Instead, it shows that with the additional requirement that  $\varphi(t)$  is continuous at zero, then  $\varphi(t)$  must be a characteristic function.

**Theorem 7.4.5 — Central Limit Theorem.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent identically distributed non-degenerate random variables with  $\mathbb{E}(\xi_1^2) < \infty$  and  $S_n = \xi_1 + \dots + \xi_n$ . Then

$$\mathbb{P}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} \leq x\right) \rightarrow \Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$$

as  $n \rightarrow \infty$  for all  $x \in \mathbb{R}$ .

**Exercise 7.4.6** Show that Theorem 7.4.5 can be written as

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} \xrightarrow{d} N(0, 1).$$

**Theorem 7.4.7 — Lindeberg's Central Limit Theorem.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables on the  $(\Omega, \mathcal{F}, \mathbb{P})$  with finite second moments  $\mathbb{E}(\xi_j^2) < \infty$ . Let

1.  $m_j = \mathbb{E}(\xi_j)$ ,
2.  $\sigma_j^2 = \mathbb{V}(\xi_j) > 0$ ,
3.  $S_n = \xi_1 + \dots + \xi_n$ , and
4.  $D_n^2 = \sum_{j=1}^n \sigma_j^2$ .

Moreover, suppose that

$$\frac{1}{D_n^2} \sum_{k=1}^n \mathbb{E}(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \varepsilon D_n\}}) \xrightarrow{n \rightarrow \infty} 0 \quad (7.4)$$

for every  $\varepsilon > 0$ . Then

$$\frac{S_n - \mathbb{E}(S_n)}{D_n} \xrightarrow{d} N(0, 1).$$

**Remark 7.4.8** The condition given in equation (7.4) is known as the Lindeberg condition.

We focus on some special cases in which the Lindeberg condition is satisfied and consequently, the central limit theorem is valid. One of the most prominent is the Lyapunov condition.

**Corollary 7.4.9 — Lyapunov's Central Limit Theorem.** Assume the conditions of Theorem 7.4.7 and in addition assume that the sequence  $(\xi_k)_{k \in \mathbb{N}}$  is such that

$$\frac{1}{D_n^{2+\delta}} \sum_{k=1}^n \mathbb{E}(|\xi_k - m_k|^{2+\delta}) \rightarrow 0 \quad (7.5)$$

for some  $\delta > 0$  as  $n \rightarrow \infty$ . Then the sequence  $(\xi_k)_{k \in \mathbb{N}}$  satisfies the Lindeberg condition and so the conclusions of Theorem 7.4.5 hold.

*Proof.* Let  $\varepsilon > 0$ . Then

$$\begin{aligned} \mathbb{E}(|\xi_k - m_k|^{2+\delta}) &\geq \mathbb{E}(|\xi_k - m_k|^{2+\delta} \chi_{\{|\xi_k - m_k| \geq \varepsilon D_n\}}) \\ &\geq \varepsilon^\delta D_n^\delta \mathbb{E}(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \varepsilon D_n\}}). \end{aligned}$$

Which implies that

$$\frac{1}{D_n^2} \sum_{k=1}^n \int_{\{x: |x - m_k| \geq \varepsilon D_n\}} (x - m_k)^2 dF_k(x) \leq \frac{1}{\varepsilon^\delta} \frac{1}{D_n^{2+\delta}} \sum_{k=1}^n \mathbb{E}(|\xi_k - m_k|^{2+\delta}).$$

Consequently,  $(\xi_k)_{k \in \mathbb{N}}$  satisfies the Lindeberg condition. ■

**Remark 7.4.10** The condition of equation (7.5) is known as the Lyapunov condition.

**Exercise 7.4.11** Under the settings of Corollary 7.4.9, suppose that there exists  $K$  such that

$$|\xi_k| \leq K < \infty$$

for all  $k$ , and that  $D_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Show that the sequence  $(\xi_k)_{k \in \mathbb{N}}$  satisfies the Lindeberg condition.

Theorem 7.4.5 does not hold when  $\mathbb{E}(\xi_1^2) = \infty$ . Let  $(\xi_n)_{n \in \mathbb{N}}$  be independent and identically distributed with the Cauchy distribution, that is they have the density

$$f(x) = \frac{\theta}{\pi(x^2 + \theta^2)}$$

for  $\theta > 0$ . One can see that

$$\varphi_{\xi_1}(t) = e^{-\theta|t|}$$

for  $t \in \mathbb{R}$ , which implies that

$$\varphi_{\frac{\xi_n}{n}}(t) = \left( \exp\left(\frac{-\theta|t|}{n}\right) \right)^n = e^{-\theta|t|},$$

Therefore,  $\frac{\xi_n}{n}$  also has a Cauchy distribution.

## 7.5 Berry-Esseen Inequality

Theorem 7.4.5 implies that if  $F_n(x)$  is the distribution function of the random variable  $(S_n - \mathbb{E}(S_n)) / \sqrt{\mathbb{V}(S_n)}$ , then

$$\sup_x |F_n(x) - \Phi(x)| \xrightarrow{n \rightarrow \infty} 0.$$

At what rate does the left-hand side decay? If  $(\xi_k)_{k \in \mathbb{N}}$  are independent and identically distributed with  $\xi_1 \in L^3$ , then we get bounds on the rate of decay.

**Theorem 7.5.1 — Berry-Esseen Inequality.** Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of independent and identically distributed random variables with  $\mathbb{E}(|\xi_1|^3) < \infty$ . Then

$$\sup_x \left| \mathbb{P}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} \leq x\right) - \Phi(x) \right| \leq C \frac{\mathbb{E}(|\xi_1 - \mathbb{E}(\xi_1)|^3)}{\sigma^3 \sqrt{n}},$$

where the constant  $C$  satisfies

$$\frac{1}{\sqrt{2\pi}} \leq C \leq \frac{1}{2}.$$

Although we do not provide a proof of Theorem 7.5.1, we note that the rate  $O\left(\frac{1}{\sqrt{n}}\right)$  is optimal.

**Remark 7.5.2** Let  $(\xi_n)_{n \in \mathbb{N}}$  be independent and identically distributed Bernoulli random variables with  $\mathbb{P}(\xi_k = 1) = p$  and  $\mathbb{P}(\xi_k = -1) = 1-p$ . By symmetry, we know that,

$$2\mathbb{P}\left(\sum_{k=1}^{2n} \xi_k < 0\right) + \mathbb{P}\left(\sum_{k=1}^{2n} \xi_k = 0\right) = 1$$

and therefore,

$$\begin{aligned} \left| \mathbb{P}\left(\sum_{k=1}^{2n} \xi_k < 0\right) - \frac{1}{2} \right| &= \frac{1}{2} \mathbb{P}\left(\sum_{k=1}^{2n} \xi_k = 0\right) \\ &= \frac{1}{2} \binom{2n}{n} \frac{1}{2^{2n}} \\ &\sim \frac{1}{2\sqrt{\pi n}} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{2n}}. \end{aligned}$$

Then  $\mathbb{E}(|\xi_1|^3) = 1 = \sigma$ , and Theorem 7.5.1 cannot be improved in terms of  $O\left(\frac{1}{\sqrt{n}}\right)$  and  $C \geq \frac{1}{\sqrt{2\pi}}$ .

## 7.6 Constructing Characteristic Functions

The following theorems determine whether a function  $\varphi$  is a characteristic function of some measure on  $\mathbb{R}$ , and if so, whether we can easily construct the underlying measure. The constructions are usually difficult and therefore are not usually covered in great detail. Nevertheless, the proofs in this section serve as great examples of using tools developed in the previous chapter. For further discussions refer to [1].

**Theorem 7.6.1 — Bochner-Khinchin.** Let  $\varphi(t)$  be continuous,  $t \in \mathbb{R}$ , with  $\varphi(0) = 1$ . A necessary and sufficient condition that  $\varphi(t)$  is a characteristic function is that it is positive semi-definite. That is, for all  $t_1, \dots, t_n \in \mathbb{R}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , and  $n = 1, 2, \dots$  we have

$$\sum_{j,k=1}^n \varphi(t_j - t_k) \lambda_j \bar{\lambda}_k \geq 0.$$

To show necessity, we note that if  $\varphi$  is a characteristic function of a real-valued random variable  $\xi$ , then

$$\sum_{j,k=1}^n \varphi(t_j - t_k) \lambda_j \bar{\lambda}_k = \mathbb{E}(\eta \bar{\eta}) = \mathbb{E}(|\eta|^2) \geq 0,$$

where  $\eta = \sum_{j=1}^n \lambda_j e^{t_j \xi}$ .

### 7.6.1 Polya's Criterion

**Theorem 7.6.2 — Polya's criterion.** Let a continuous even real-valued function  $\varphi(t)$  satisfy  $\varphi(t) \geq 0$ ,  $\varphi(0) = 1$ ,  $\varphi(t) \rightarrow 0$  as  $t \rightarrow \infty$  and let  $\varphi(t)$  be convex on  $0 \leq t < \infty$ . Then  $\varphi(t)$  is a characteristic function.

As an observation, we note that the function  $\varphi(t)$  must be strictly decreasing over  $[0, \infty)$ . To see this, we

let  $0 < r < s$ , and consider  $t_0 > s$  such that  $0 < f(t_0) < \frac{f(r)}{2}$ . By convexity, we have

$$f(s) \leq \frac{f(t_0) - f(r)}{t_0 - r}(s - r) + f(r) < f(r).$$

**Exercise 7.6.3** For  $k \in \{1, \dots, n\}$  let  $\varphi_k(t)$  be a characteristic function and let the non-negative numbers  $\lambda_k$  satisfy  $\sum_{k=1}^n \lambda_k = 1$ . Show that

$$\sum_{k=1}^n \lambda_k \varphi_k(t)$$

is a characteristic function. Extend the above result for  $n = \infty$ .

## 7.6.2 Marcinkiewicz Theorem

**Theorem 7.6.4 — Marcinkiewicz's Theorem.** If a characteristic function  $\varphi(t)$  is of the form  $e^{p(t)}$ , where  $p(t)$  is a polynomial, then the degree of  $p(t)$  is at most 2.

**Example 7.6.5** As a simple example,  $e^{-t^4}$  is not a characteristic function of any real-valued random variables.

## 7.6.3 Cumulants

**Definition 7.6.6** If an expansion

$$\log \varphi_\xi(t) = \sum_{k=0}^n \frac{(it)^k}{k!} s_k + o(|t|^n),$$

exists as  $t \rightarrow 0$ , then the coefficients  $s_k$  are called cumulants of  $\xi$ .

**Exercise 7.6.7** Show that

1.  $\mathbb{E}(\xi) = s_1$ , and
2.  $\mathbb{V}(\xi) = s_2$ .

**Remark 7.6.8** If  $\xi \sim N(m, \sigma^2)$  then

- $s_1 = m$ ,
- $s_2 = \sigma^2$ , and
- $s_k = 0$  for  $k \geq 3$ .

In general, by Marcinkiewicz's Theorem if for a random variable  $\xi$  there exists  $n$  such that  $s_k = 0$ , for all  $k \geq n$ , then  $s_k = 0$  for all  $k \geq 3$  and  $\xi \sim N(s_1, s_2)$ .

## 7.6.4 Degenerate distributions

The following theorem shows that a property of the characteristic function of a random variable can lead to a non-trivial conclusion about the nature of the random variable.

**Theorem 7.6.9** Let  $\varphi(t)$  be a characteristic function of a random variable  $\xi$ . If  $|\varphi(t_0)| = 1$  for some  $t_0 \neq 0$ , then  $\xi$  is concentrated at the points  $a + nh$  for some  $a$  and where  $h = \frac{2\pi}{t_0}$ . That is,

$$\sum_{n=-\infty}^{\infty} \mathbb{P}(\xi = a + nh) = 1,$$

where  $a$  is a constant.

*Proof.* If  $|\varphi(t_0)| = 1$  for some  $t_0 \neq 0$  then there is a number  $a$  such that  $\varphi(t_0) = e^{it_0a}$ . Therefore,

$$e^{it_0a} = \int_{-\infty}^{\infty} e^{it_0x} dF(x)$$

which implies that

$$1 = \int_{-\infty}^{\infty} e^{it_0(x-a)} dF(x).$$

Equating real parts we see that

$$1 = \int_{-\infty}^{\infty} \cos t_0(x-a) dF(x)$$

which then implies that

$$\int_{-\infty}^{\infty} 1 - \cos t_0(x-a) dF(x) = 0.$$

Since  $1 - \cos t_0(x-a) \geq 0$ , it follows that

$$1 = \cos t_0(x-a)$$

$\mathbb{P}$ -almost surely. That is to say that  $\mathbb{P}$  is concentrated at the points  $x = a + n\left(\frac{2\pi}{t_0}\right)$  for  $n \in \mathbb{Z}$ . ■

### Exercise 7.6.10

1. Let  $\varphi(t)$  be a characteristic function. Show that the following are also characteristic functions.
  - (a)  $|\varphi(t)|^2$ .
  - (b)  $e^{\lambda(\varphi(t)-1)}$  for  $\lambda \geq 0$ .
  - (c)  $\int_0^1 \varphi(ut) du$ .
  - (d)  $\int_0^\infty e^{-u} \varphi(ut) du$ .
2. Let  $X$  and  $Y$  be independent identically distributed random variables with zero mean and unit variance. Prove using characteristic functions that if the distribution  $F$  of  $(X+Y)/\sqrt{2}$  is the same as that of  $X$  and  $Y$ , then  $F$  is the standard normal distribution.
3. Let  $\xi$  be an integer-valued random variable with characteristic function  $\varphi_\xi(t)$ . Show that

$$\mathbb{P}(\xi = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi_\xi(t) dt,$$

for  $k \in \mathbb{Z}$ .

4. Show that if  $\varphi(t)$  is a characteristic function, then  $\operatorname{Re}(\varphi(t))$  is also a characteristic function, but  $\operatorname{Im}(\varphi(t))$  is not.

## 7.7 Solution to Exercises

### Exercise 7.1.4

*Solution.*

1. Note that if  $\xi_1 = \text{B}(1, p)$  the

$$\varphi_{\xi_1} = \mathbb{E}(e^{it\xi_1}) = e^0(1-p) + e^{it}p.$$

Therefore, by Proposition 7.1.3 4. we deduce that

$$\varphi_\xi(t) = (pe^{it} + (1-p))^n.$$

2. Let  $\eta = \frac{\xi - m}{\sigma}$ . Then  $\eta \sim \mathcal{N}(0, 1)$  has density

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

It is sufficient to show that  $\varphi_\eta(t) = e^{-\frac{t^2}{2}}$ . We have

$$\begin{aligned} \varphi_\eta(t) &= \mathbb{E}(e^{it\eta}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx - \frac{x^2}{2}} dx \\ &= e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x-it)^2} dx \\ &= e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty-it}^{\infty-it} e^{-\frac{z^2}{2}} dz \\ &\stackrel{(1)}{=} e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\ &= e^{-\frac{t^2}{2}}. \end{aligned}$$

We justify this equality using contour integration. Let

$$I_R = \oint_{\gamma} e^{-\frac{z^2}{2}} dz$$

where  $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$  for

- $\gamma_1 := \{z = -u : -R \leq u \leq R\}$ ,
- $\gamma_2 := \{z = -R - itu : 0 \leq u \leq 1\}$ ,
- $\gamma_3 := \{z = u - it : -R \leq u \leq R\}$ , and
- $\gamma_4 = \{z = R - it(1-u) : 0 \leq u \leq 1\}$ .

Note that

$$\begin{aligned} \int_{\gamma_2} \exp\left(-\frac{z^2}{2}\right) dz &= \int_0^1 \exp\left(-\frac{(-R - itu)^2}{2}\right) dz \\ &= \int_0^1 \exp\left(-\frac{R^2}{2}\right) \exp(-Rtui) \exp\left(\frac{t^2u^2}{2}\right) du. \end{aligned}$$

Hence,  $\int_{\gamma_2} \exp\left(-\frac{z^2}{2}\right) dz \xrightarrow{R \rightarrow \infty} 0$ . Similarly,  $\int_{\gamma_4} \exp\left(-\frac{z^2}{2}\right) dz \xrightarrow{R \rightarrow \infty} 0$ . On the other hand,

$$\int_{\gamma_1} \exp\left(-\frac{z^2}{2}\right) dz = \int_{-R}^R \exp\left(\frac{u^2}{2}\right) du$$

and

$$\begin{aligned} \int_{\gamma_3} \exp\left(-\frac{z^2}{2}\right) dz &= \int_{-R}^R \exp\left(-\frac{(u - it)^2}{2}\right) du \\ &\stackrel{v=u-it}{=} \int_{-R-it}^{R-it} \exp\left(-\frac{v^2}{2}\right) dv. \end{aligned}$$

Therefore, as  $e^{-\frac{z^2}{2}}$  is analytic in the region defined by  $\gamma$  we deduce that

$$0 = \int_{\gamma} \exp\left(-\frac{z^2}{2}\right) dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}\right) \exp\left(-\frac{z^2}{2}\right) dz.$$

Consequently, as  $R \rightarrow \infty$  we deduce that

$$0 = \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + \int_{-\infty-it}^{\infty-it} \exp\left(-\frac{v^2}{2}\right) dv$$

which implies that

$$\int_{-\infty}^{\infty} \exp\left(\frac{u^2}{2}\right) du = \int_{-\infty-it}^{\infty-it} \exp\left(-\frac{v^2}{2}\right) dv$$

by using the evenness of the integrand.

3. Proceeding directly we see that

$$\begin{aligned} \varphi_{\xi}(t) &= \mathbb{E}(e^{it\xi}) \\ &= e^{-\lambda} \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{it}\lambda)^k}{k!} \\ &= e^{-\lambda} e^{\lambda e^{it}} \\ &= e^{-\lambda + \lambda e^{it}}. \end{aligned}$$

■

### Exercise 7.2.5

*Solution.* Let  $\eta = \exp(\xi)$ . Then as  $\exp(\cdot)$  is a strictly increasing function we note that

$$f_{\eta}(x) = f_{\xi}(\ln(x)) \frac{1}{x} = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\ln(x))^2}{2}\right)$$

for  $x > 0$ . Therefore,  $\eta$  has a standard log-normal distribution. Moreover,

$$\begin{aligned} \mathbb{E}(\eta^r) &= \mathbb{E}(\exp(\xi))^r \\ &= \int_{-\infty}^{\infty} \exp(rx) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-r)^2}{2} + \frac{r^2}{2}\right) dx \\ &= \exp\left(\frac{r^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-r)^2}{2}\right) dx \\ &= \exp\left(\frac{r^2}{2}\right). \end{aligned}$$

■

### Exercise 7.4.6

*Solution.* Set  $m = \mathbb{E}(\xi_1)$ ,  $\sigma^2 = \mathbb{V}(\xi_1)$ , and  $\varphi(t) = \mathbb{E}(e^{it(\xi_1 - m)})$ . If we put

$$\varphi_n(t) = \mathbb{E}\left(\exp\left(it\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}}\right)\right),$$

by independence

$$\varphi_n(t) = \left(\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n.$$

Since  $\mathbb{E}(\xi_1^2) < \infty$ , we have by properties of characteristic functions that

$$\varphi(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$$

as  $t \rightarrow 0$ . So

$$\varphi_n(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \rightarrow e^{-\frac{t^2}{2}}$$

for all  $t \in \mathbb{R}$ . This is the characteristic function of  $N(0, 1)$  and so the result follows by Theorem 7.4.3. ■

### Exercise 7.4.11

*Solution.* Note that

$$|m_k| = |\mathbb{E}(\xi_k)| \leq \mathbb{E}(|\xi_k|) \leq K$$

for all  $k$ . Therefore,  $|\xi_k - m_k| \leq 2K$  for all  $k$ . As  $D_n \rightarrow \infty$ , and  $D_n$  is monotonically increasing, it follows that for some  $N \in \mathbb{N}$  we have that

$$\chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}} = 0$$

for  $n \geq N$ . Therefore, for  $n \geq N$  we have

$$\begin{aligned} \frac{1}{D_n^2} \sum_{k=1}^n \mathbb{E}(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}) &= \frac{1}{D_n^2} \sum_{k=1}^{N-1} \mathbb{E}(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}) \\ &\quad + \frac{1}{D_n^2} \sum_{k=N}^n \mathbb{E}(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}) \\ &= \frac{1}{D_n^2} \sum_{k=1}^{N-1} \mathbb{E}(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}) \\ &\leq \frac{4K^2(N-1)}{D_n^2} \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Hence, Linderberg's condition is satisfied. ■

### Exercise 7.6.3

*Solution.* For  $k = 1, \dots, n$  let  $\zeta_k$  be a random variable with characteristic function  $\phi_k(t)$ . Let  $\eta$  be the random variable where  $\mathbb{P}(\eta = k) = \lambda_k$  for  $k = 1, \dots, n$ . As

$$\zeta_\eta(\omega) = \sum_{k=1}^n \zeta_k \mathbf{1}_{\{\eta=k\}}(\omega)$$

we note that  $\zeta_\eta$  is the sum of the product of random variables as each  $\zeta_k$  is a random variable and  $\mathbf{1}_{\{\eta=k\}}(\omega)$  is measurable as  $\eta$  is a random variable. Therefore, we can consider the characteristic function of  $\zeta_\eta$ . Namely,

$$\begin{aligned} \phi_{\zeta_\eta}(t) &= \mathbb{E}(e^{it\zeta_\eta}) \\ &= \sum_{k=1}^n \mathbb{E}(e^{it\zeta_k}) \mathbb{P}(\eta = k) \\ &= \sum_{k=1}^n \lambda_k \phi_k(t). \end{aligned}$$

One can readily extend to  $n = \infty$  by letting  $\eta$  be a discrete random variable with a countable image. ■

### Exercise 7.6.7

*Solution.* On the one hand,

$$\frac{d}{dt} \log(\varphi_\xi(t)) = \frac{\varphi'_\xi(t)}{\varphi_\xi(t)}$$

and on the other hand,

$$\frac{d}{dt} \log(\varphi_\xi(t)) = \sum_{k=1}^n \frac{i(it)^{k-1}}{(k-1)!} s_k + o(|t|^{n-1}).$$

Substituting  $t = 0$  into both expressions we deduce that

$$is_1 = \frac{i\mathbb{E}(\xi)}{(1)} = i\mathbb{E}(\xi)$$

and so  $\mathbb{E}(\xi) = s_1$ . Similarly,

$$\frac{d^2}{dt^2} \log(\varphi_\xi(t)) = \frac{\varphi_\xi(t)\varphi''_\xi(t) - (\varphi'_\xi(t))^2}{(\varphi_\xi(t))^2}$$

and

$$\frac{d^2}{dt^2} \log(\varphi_\xi(t)) = \sum_{k=2}^n \frac{i^2(it)^{k-2}}{(k-2)!} s_k + o(|t|^{n-2}).$$

Substituting  $t = 0$  into both expressions we deduce that

$$i^2 s_2 = \frac{(1)(i^2\mathbb{E}(\xi^2)) - (i\mathbb{E}(\xi))^2}{1^2} = i^2 \mathbb{V}(\xi).$$

Hence,  $\mathbb{V}(\xi) = s_2$ . ■

### Exercise 7.6.10

*Solution.*

1. (a) Note that  $\overline{\varphi(t)} = \varphi(-t)$  and  $\varphi(-t)$  is also a characteristic function. Therefore,  $|\varphi(t)|^2 = \varphi(t)\varphi(\bar{t})$  is also a characteristic function.
- (b) Let  $(\xi_k)_{k \in \mathbb{N}}$  be a sequence of independent and identically distributed random variables with the characteristic function  $\varphi(t)$ , and let  $\eta$  be a random variable with  $\text{Pois}(\lambda)$  distribution, for  $\lambda \geq 0$ . Moreover,  $\eta$  is independent of the  $(\xi_k)_{k \in \mathbb{N}}$ . Let  $X = \sum_{k=0}^\eta \xi_k$ , then

$$\begin{aligned} \varphi_X(t) &= \mathbb{E}(\exp(itX)) \\ &= \mathbb{E}\left(\exp\left(\sum_{k=0}^\eta it\xi_k\right)\right) \\ &= \sum_{n=0}^\infty \mathbb{E}\left(\prod_{k=0}^n \exp(it\xi_k)\right) \mathbb{P}(\eta = n) \\ &\stackrel{(1)}{=} \sum_{n=0}^\infty \prod_{k=0}^n \mathbb{E}(\exp(it\xi_k)) \frac{e^{-\lambda}\lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^\infty \frac{\varphi(t)^n \lambda^n}{n!} \\ &= e^{-\lambda} e^{\lambda\varphi(t)} \\ &= e^{\lambda(\varphi(t)-1)} \end{aligned}$$

where in (1) we have used the independence assumption of the sequence  $(\xi_k)_{k \in \mathbb{N}}$ . Therefore,  $e^{\lambda(\varphi(t)-1)}$  defines a characteristic function.

- (c) Let  $\psi(t) = \int_0^1 \varphi(ut) du$ . Then  $\psi(0) = \int_0^1 1 du = 1$ . Moreover for all  $t_1, \dots, t_n \in \mathbb{R}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  and  $n = 1, 2, \dots$  we have that

$$\sum_{j,k=1}^n \psi(t_j - t_k) \lambda_j \bar{\lambda}_k = \sum_{j,k=1}^n \left( \int_0^1 \varphi(u(t_j - t_k)) du \right) \lambda_j \bar{\lambda}_k$$

$$= \int_0^1 \sum_{j,k=1}^n \varphi(t'_j - t'_k) \lambda_j \bar{\lambda}_k \, du$$

where  $t'_i = ut_i \in \mathbb{R}$ . Therefore, as  $\varphi(t)$  is a characteristic function we know that

$$\sum_{j,k=1}^n \varphi(t'_j - t'_k) \lambda_j \bar{\lambda}_k \geq 0$$

by Theorem 7.6.1. This implies that

$$\sum_{j,k=1}^n \psi(t_j - t_k) \lambda_j \bar{\lambda}_k \geq 0$$

and so by Theorem 7.6.1 we know that  $\psi(t) = \int_0^t \varphi(ut) \, du$  is a characteristic function.

- (d) Let  $\psi(t) = \int_0^\infty e^{-u} \varphi(ut) \, dt$ . Then  $\psi(0) = \int_0^\infty e^{-u} \, du = 1$ . A similar argument to the above works to conclude that  $\psi(t)$  is a characteristic function.

2. Let  $\varphi(t)$  be the characteristic function  $F$ . Then we have that

$$\varphi(t) = \varphi_{\frac{X+Y}{\sqrt{2}}}(t) = \varphi_X\left(\frac{t}{\sqrt{2}}\right) \varphi_Y\left(\frac{t}{\sqrt{2}}\right) = \varphi\left(\frac{t}{\sqrt{2}}\right)^2.$$

Let  $\psi(t) = \log(\varphi(t))$  so that  $\psi(t) = 2\psi\left(\frac{t}{\sqrt{2}}\right)$ . Consequently, we have that

- $\psi'(t) = \sqrt{2}\psi'\left(\frac{t}{2}\right)$ , and
- $\psi'' = \psi''\left(\frac{t}{2}\right)$ .

As  $\psi''(t)$  is continuous at 0 we must have that  $\psi''(t)$  is a constant. In particular, it follows that

$$\psi''(t) = \psi''(0) = \frac{\varphi(0)\varphi''(0) - (\varphi'(0))^2}{\varphi(0)^2} = \frac{i^2 \mathbb{V}(X)}{1} = -1.$$

Therefore,  $\psi'(t) = -t + c$ . As  $\psi'(0) = 0$  we know that  $c = 0$  and  $\psi(t) = -\frac{t^2}{2}$ . Hence,  $\varphi(t) = \exp\left(-\frac{t^2}{2}\right)$  and so  $F$  is a standard normal distribution.

3. Note that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi_\xi(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \sum_{m \in \mathbb{Z}} e^{ipt} \mathbb{P}(\xi = m) \, dt.$$

As  $\sum_{m \in \mathbb{Z}} e^{imt} \mathbb{P}(\xi = m)$  is absolutely convergent we can interchange the integral and the sum to deduce that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi_\xi(t) \, dt = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{i(m-k)t} \mathbb{P}(\xi = m) \, dt.$$

Then as

$$\int_{-\pi}^{\pi} e^{i(m-k)t} \, dt = \begin{cases} 2\pi & m = k \\ 0 & \text{otherwise} \end{cases}$$

we conclude that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi_\xi(t) \, dt = \mathbb{P}(\xi = k).$$

4. As  $\varphi(0) = 0$  it follows that  $\text{Im}(\varphi(0)) \neq 0$  and so cannot itself be a characteristic function. On the other hand, let  $\xi$  be a random variable for which  $\varphi(t)$  is the characteristic function. Then it is clear that  $-\xi$  is also a random variable and has characteristic function  $\varphi(-t)$ . In particular, this means that  $\varphi(-t)$  is also a random variable and so  $\text{Re}(\phi) = \frac{1}{2}(\varphi(t) + \varphi(-t))$  is a characteristic function by Exercise 7.6.3. ■

## Part III. Introduction to Stochastic Analysis

### 8 Conditional Expectation

When studying stochastic processes  $(\xi_\alpha)_{\alpha \in A}$ , it is natural to determine how different random variables are related. In particular, we want to know if observing one random variable will give more information on the other random variables in the process. For this, we need the notion of conditional probability and conditional expectation.

#### 8.1 Preliminary Measure Theory

To ensure that our notions of conditional probability and conditional expectation are well-defined it will be useful to make note of the following result in measure theory.

**Theorem 8.1.1 — Radon-Nikodym Theorem.** Let  $\mu$  be a finite measure on the measure space  $(\Omega, \mathcal{F})$ . Let  $\lambda$  be a measure on  $\mathcal{F}$  that is absolutely continuous with respect to  $\mu$ . That is,  $\lambda(A) = 0$  whenever  $\mu(A) = 0$ . Then there exists an  $\mathcal{F}$ -measurable function  $f$  such that

$$\lambda(A) = \int_A f d\mu$$

for all  $A \in \mathcal{F}$ . Moreover,  $f$  is determined uniquely up to sets of measure zero. Consequently,  $f$  is called the derivative of  $\lambda$  with respect to  $\mu$  and is often denoted  $f = \frac{d\lambda}{d\mu}$ .

If  $\xi$  is a non-negative random variable on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $\mathcal{G} \subseteq \mathcal{F}$  is a  $\sigma$ -algebra. Then the measure defined by  $Q(G) = \int_G \xi d\mathbb{P}$  for all  $G \in \mathcal{G}$  is an absolutely continuous measure with respect to  $\mathbb{P}$ . Hence, by Theorem 8.1.1 there exists a  $\mathcal{G}$ -measurable function,  $f$ , such that  $Q(G) = \int_G f d\mathbb{P}$  for all  $G \in \mathcal{G}$ . We can extend this naturally to general random variables  $\xi$  that are such that  $\min(\mathbb{E}(\xi^+|\mathcal{G}), \mathbb{E}(\xi^-|\mathcal{G})) < \infty$  almost surely.

#### 8.2 Conditional Expectation and Probability

**Definition 8.2.1** Let  $\xi$  be an integrable random variable on  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -algebra. Then there exists a random variable  $\mathbb{E}(\xi|\mathcal{G})$ , referred to as the conditional expectation of  $\xi$  given  $\mathcal{G}$ , that satisfies the following.

- $\mathbb{E}(\xi|\mathcal{G})$  is  $\mathcal{G}$ -measurable and integrable.
- For every  $G \in \mathcal{G}$  we have

$$\mathbb{E}(\chi_G \mathbb{E}(\xi|\mathcal{G})) = \int_G \mathbb{E}(\xi|\mathcal{G}) d\mathbb{P} = \int_G \xi d\mathbb{P} = \mathbb{E}(\chi_G \xi).$$

**Remark 8.2.2** Theorem 8.1.1 ensures that the conditional expectation of an integrable random variable is unique up to sets of measure zero.

**Definition 8.2.3** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Then the conditional probability of  $B \in \mathcal{F}$  with respect to a  $\sigma$ -algebra  $\mathcal{G} \subseteq \mathcal{F}$  is

$$\mathbb{P}(B|\mathcal{G}) = \mathbb{E}(\chi_B|\mathcal{G}).$$

Note that for a fixed  $B \in \mathcal{F}$ , the conditional probability  $\mathbb{P}(B|\mathcal{G})$  is a  $\mathcal{G}$ -measurable random variable such that

$$\int_G \mathbb{P}(B|\mathcal{G}) d\mathbb{P} = \int_G \chi_B d\mathbb{P} = \mathbb{P}(G \cap B)$$

for all  $G \in \mathcal{G}$  as we would expect from traditional notions of conditional probabilities. Moreover, let  $\xi$  be a random variable on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\mathcal{G} = \sigma(\{D_1, D_2, \dots\})$  where  $\{D_1, D_2, \dots\}$  forms a disjoint partition of  $\Omega$ . That is,

$$\Omega = \bigcup_{i=1}^{\infty} D_i$$

with each  $D_i$  disjoint. Moreover, suppose that  $\mathbb{P}(D_i) > 0$  for  $i = 1, 2, \dots$ . Then all  $\mathcal{G}$ -measurable functions have the form

$$f(\omega) = \sum_{i=1}^{\infty} c_i \chi_{D_i}(\omega)$$

and thus are constant on each  $D_i$ . As  $\mathbb{E}(\xi | \mathcal{G})$  is  $\mathcal{G}$  measurable it must be of this form. Observe that

$$\begin{aligned}\mathbb{E}(\xi \chi_{D_i}) &= \int_{D_i} \xi d\mathbb{P} \\ &= \int_{D_i} \mathbb{E}(\xi | \mathcal{G}) d\mathbb{P} \\ &= \mathbb{E}(\xi | D_i) \mathbb{P}(D_i),\end{aligned}$$

that is

$$\mathbb{E}(\xi | D_i) = \frac{\mathbb{E}(\xi \chi_{D_i})}{\mathbb{P}(D_i)}.$$

Consequently, we also note that

$$\begin{aligned}\mathbb{P}(B | D_i) &= \mathbb{E}(\chi_B | D_i) \\ &= \frac{\mathbb{E}(\chi_B \chi_{D_i})}{\mathbb{P}(D_i)} \\ &= \frac{\mathbb{P}(B \cap D_i)}{\mathbb{P}(D_i)}.\end{aligned}$$

Hence,

$$\mathbb{P}(B | \mathcal{G}) = \sum_{i=1}^{\infty} \mathbb{P}(B | D_i) \chi_{D_i}$$

and

$$\mathbb{P}(B | \{\emptyset, \Omega\}) = \mathbb{P}(B).$$

### 8.3 Properties of Conditional Expectation

**Proposition 8.3.1** Let  $\xi$  and  $\eta$  be random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -algebra. Then,

$$\mathbb{E}(a\xi + b\eta + c | \mathcal{G}) = a\mathbb{E}(\xi | \mathcal{G}) + b\mathbb{E}(\eta | \mathcal{G}) + c$$

almost surely.

*Proof.* For all  $G \in \mathcal{G}$ , we have

$$\begin{aligned}\mathbb{E}(\chi_G \mathbb{E}(a\xi + b\eta + c | \mathcal{G})) &= \mathbb{E}(\chi_G (a\xi + b\eta + c)) \\ &= a\mathbb{E}(\chi_G \xi) + b\mathbb{E}(\chi_G \eta) + c\mathbb{E}(\chi_G) \\ &= a\mathbb{E}(\chi_G \mathbb{E}(\xi | \mathcal{G})) + b\mathbb{E}(\chi_G \mathbb{E}(\eta | \mathcal{G})) + c\mathbb{E}(\chi_G) \\ &= \mathbb{E}(\chi_G (a\mathbb{E}(\xi | \mathcal{G}) + b\mathbb{E}(\eta | \mathcal{G}) + c)).\end{aligned}$$

As  $a\mathbb{E}(\xi | \mathcal{G}) + b\mathbb{E}(\eta | \mathcal{G}) + c$  is  $\mathcal{G}$ -measurable, the result follows by the uniqueness of the conditional expectation. ■

**Proposition 8.3.2** Let  $\xi$  and  $\eta$  be random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $\xi \leq \eta$  almost surely and let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -algebra. Then,

$$\mathbb{E}(\xi | \mathcal{G}) \leq \mathbb{E}(\eta | \mathcal{G})$$

almost surely.

*Proof.* Consider the event  $G = \{\mathbb{E}(\xi|\mathcal{G}) > \mathbb{E}(\eta|\mathcal{G})\}$ . We know  $G \in \mathcal{G}$  as  $\mathbb{E}(\xi|\mathcal{G})$  and  $\mathbb{E}(\eta|\mathcal{G})$  are  $\mathcal{G}$ -measurable. Thus,

$$\begin{aligned}\int_G \xi \, d\mathbb{P} &= \int_G \mathbb{E}(\xi|\mathcal{G}) \, d\mathbb{P} \\ &> \int_G \mathbb{E}(\eta|\mathcal{G}) \, d\mathbb{P} \\ &= \int_G \eta \, d\mathbb{P}.\end{aligned}$$

Hence,  $\mathbb{P}(G) = 0$  as to not contradict  $\xi \leq \mu$  almost surely. Therefore,  $\mathbb{E}(\xi|\mathcal{G}) \leq \mathbb{E}(\eta|\mathcal{G})$  almost surely. ■

**Corollary 8.3.3** Let  $\eta$  be a random variable on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\eta \geq 0$  almost surely. Then for  $\mathcal{G} \subseteq \mathcal{F}$  a  $\sigma$ -algebra we have

$$\mathbb{E}(\eta|\mathcal{G}) \geq 0$$

almost surely.

*Proof.* Letting  $\xi = 0$  in the context of Proposition 8.3.2, it follows that

$$\mathbb{E}(\eta|\mathcal{G}) \geq \mathbb{E}(\xi|\mathcal{G}) = 0$$

almost surely. ■

**Exercise 8.3.4** Throughout, let  $\xi$  and  $\eta$  be random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -algebra.

1. Show that  $\mathbb{E}(\xi|\{\emptyset, \Omega\}) = \mathbb{E}(\xi)$ .
2. Show that  $\mathbb{E}(\xi|\mathcal{F}) = \xi$  almost everywhere.
3. Suppose that  $\xi$  is independent of  $\mathcal{G}$ , which means that  $\xi$  and  $\chi_B$  are independent for all  $B \in \mathcal{G}$ , then  $\mathbb{E}(\xi|\mathcal{G}) = \mathbb{E}(\xi)$ .

**Theorem 8.3.5** Suppose that  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space and  $\xi$  is an integrable random variable taking values in an open interval  $I \subseteq \mathbb{R}$ . Let  $g : I \rightarrow \mathbb{R}$  be convex and let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -algebra. If  $\mathbb{E}(|g(\xi)|) < \infty$ , then

$$\mathbb{E}(g(\xi)|\mathcal{G}) \geq g(\mathbb{E}(\xi|\mathcal{G}))$$

almost surely.

**Corollary 8.3.6** Suppose that  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space and  $\xi$  is an integrable random variable. Then

$$\mathbb{E}(|\xi||\mathcal{G}) \geq |\mathbb{E}(\xi|\mathcal{G})|$$

almost surely.

*Proof.* Clearly, the result follows from Theorem 8.3.5 by letting  $\varphi(x) = |x|$ . However, we can provide a more elementary proof. Specifically, note that for any  $G \in \mathcal{G}$  we have

$$\begin{aligned}\int_G |\mathbb{E}(\xi|\mathcal{G})| \, d\mathbb{P} &\leq \int_G \mathbb{E}(\xi^+|\mathcal{G}) \, d\mathbb{P} + \int_G \mathbb{E}(\xi^-|\mathcal{G}) \, d\mathbb{P} \\ &= \int_G \xi^+ + \xi^- \, d\mathbb{P}\end{aligned}$$

$$\begin{aligned}
&= \int_G |\xi| d\mathbb{P} \\
&= \int_G \mathbb{E}(|\xi| | \mathcal{G}) d\mathbb{P}.
\end{aligned}$$

As this holds for all  $G \in \mathcal{G}$  it follows that  $\mathbb{E}(|\xi| | \mathcal{G}) \geq |\mathbb{E}(\xi | \mathcal{G})|$  almost surely. ■

**Proposition 8.3.7** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $\xi$  an integrable random variable and  $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$   $\sigma$ -algebras with  $\mathcal{F}_1 \subseteq \mathcal{F}_2$ . Then,

$$\mathbb{E}(\mathbb{E}(\xi | \mathcal{F}_2) | \mathcal{F}_1) = \mathbb{E}(\xi | \mathcal{F}_1) = \mathbb{E}(\mathbb{E}(\xi | \mathcal{F}_1) | \mathcal{F}_2) \quad (8.1)$$

almost surely.

*Proof.* Let  $G \in \mathcal{F}_1$ , then  $G \in \mathcal{F}_2$  and hence

$$\int_G \mathbb{E}(\xi | \mathcal{F}_1) d\mathbb{P} = \int_G \xi d\mathbb{P}$$

by definition of  $\mathbb{E}(\xi | \mathcal{F}_1)$ . Similarly,

$$\int_G \mathbb{E}(\mathbb{E}(\xi | \mathcal{F}_2) | \mathcal{F}_1) d\mathbb{P} = \int_G \mathbb{E}(\xi | \mathcal{F}_2) d\mathbb{P} = \int_G \xi d\mathbb{P}.$$

Hence, the first equality of (8.1) follows. Now let  $G \in \mathcal{F}_2$ . As  $\mathbb{E}(\xi | \mathcal{F}_1)$  is  $\mathcal{F}_1$ -measurable it follows that it is also  $\mathcal{F}_2$  measurable. Hence,

$$\int_G \mathbb{E}(\xi | \mathcal{F}_1) d\mathbb{P} = \int_G \mathbb{E}(\mathbb{E}(\xi | \mathcal{F}_1) | \mathcal{F}_2) d\mathbb{P}.$$

Thus we get the second equality of (8.1). ■

**Corollary 8.3.8** For  $\xi$  an integrable random variable on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  we have

$$\mathbb{E}(\mathbb{E}(\xi | \mathcal{G})) = \mathbb{E}(\xi).$$

*Proof.* Take  $\mathcal{F}_1 = \{\emptyset, \Omega\}$  and  $\mathcal{F}_2 = \mathcal{G}$  in Proposition 8.3.7 and then use Exercise 8.3.4 to conclude. ■

**Proposition 8.3.9** Let  $(\xi_n)_{n \geq 1}$  be a sequence of random variables.

- Suppose  $|\xi_n| \leq \eta$  where  $\mathbb{E}(\eta) < \infty$ , and  $\xi_n \rightarrow \xi$  almost surely. Then,

$$\mathbb{E}(\xi_n | \mathcal{G}) \xrightarrow{\text{a.s.}} \mathbb{E}(\xi | \mathcal{G})$$

and

$$\mathbb{E}(|\xi_n - \xi| | \mathcal{G}) \xrightarrow{\text{a.s.}} 0.$$

- Suppose  $\xi_n \geq \eta$  where  $\mathbb{E}(\eta) > -\infty$ ,  $\xi_n \nearrow \xi$  almost surely and  $\mathbb{E}(|\xi|) < \infty$ . Then

$$\mathbb{E}(\xi_n | \mathcal{G}) \nearrow \mathbb{E}(\xi | \mathcal{G})$$

almost surely.

- Suppose  $\xi_n \leq \eta$  where  $\mathbb{E}(\eta) < \infty$ , and  $\xi_n \searrow \xi$  almost surely. Then

$$\mathbb{E}(\xi_n | \mathcal{G}) \searrow \mathbb{E}(\xi | \mathcal{G})$$

almost surely.

4. Suppose  $\xi_n \geq \eta$  where  $E(\eta) > -\infty$ . Then

$$E(\liminf \xi_n | \mathcal{G}) \leq \liminf E(\xi_n | \mathcal{G})$$

almost surely.

5. Suppose  $\xi_n \leq \eta$  where  $E(\eta) < \infty$ . Then

$$E(\limsup \xi_n | \mathcal{G}) \leq \limsup E(\xi_n | \mathcal{G})$$

almost surely.

6. If  $\xi_n \geq 0$  then

$$E\left(\sum_{n=1}^{\infty} \xi_n \middle| \mathcal{G}\right) = \sum_{n=1}^{\infty} E(\xi_n | \mathcal{G})$$

almost surely.

*Proof.*

1. Let  $\zeta_n = \sup_{m \geq n} |\xi_m - \xi|$ . Then  $0 \leq |\zeta_n| \leq 2\eta$  and  $\zeta_n \rightarrow 0$  almost surely, so by the dominated convergence theorem we have  $E(\zeta_n) \xrightarrow{n \rightarrow \infty} 0$ . Now by the triangle inequality it follows that

$$0 \leq |E(\xi_n | \mathcal{G}) - E(\xi | \mathcal{G})| \leq E(|\xi_n - \xi| | \mathcal{G}) \leq E(\zeta_n | \mathcal{G}).$$

Since the sequence  $E(\zeta_n | \mathcal{G})(\omega)$  is decreasing in  $n$  for fixed  $\omega$ , its limit exists  $\omega$ -almost surely. In particular, note that

$$0 \leq E\left(\lim_{n \rightarrow \infty} E(\zeta_n | \mathcal{G})\right) \leq E(E(\zeta_n | \mathcal{G})) = E(\zeta_n) \xrightarrow{n \rightarrow \infty} 0.$$

Hence,  $\lim_{n \rightarrow \infty} E(\zeta_n | \mathcal{G}) = 0$  almost surely which completes the proof.

2. Suppose that  $\eta = 0$ . Then  $E(\xi_n | \mathcal{G}) \geq 0$  almost surely by Corollary 8.3.3. By the assumption of monotonic convergence almost surely, it follows that for  $A_n := \{E(\xi_n | \mathcal{G}) < E(\xi_{n-1} | \mathcal{G})\} \in \mathcal{G}$  we have  $P(A_n) = 0$ . Let  $\tilde{\xi} := \limsup_{n \rightarrow \infty} (E(\xi_n | \mathcal{G}))$  and  $A = \bigcup_{n \geq 2} A_n$ . Then  $A \in \mathcal{G}$  and  $P(A) = 0$ , furthermore, for all  $\omega \in A^c$  it follows that  $E(\xi_n | \mathcal{G})(\omega) \nearrow \tilde{\xi}(\omega)$ . Therefore, for all  $G \in \mathcal{G}$  we have

$$\begin{aligned} E(\tilde{\xi} \chi_G) &= E(\tilde{\xi} \chi_{G \cap A^c}) \\ &\stackrel{\text{MCT}}{=} \lim_{n \rightarrow \infty} (E(E(\xi_n | \mathcal{G}) \chi_{G \cap A^c})) \\ &\stackrel{(1)}{=} \lim_{n \rightarrow \infty} (\xi \chi_{G \cap A^c}) \\ &\stackrel{\text{MCT}}{=} E(\xi \chi_{G \cap A^c}) \\ &= E(\xi \chi_G), \end{aligned}$$

where in (1) we have used the fact that  $G \cap A^c \in \mathcal{G}$  and the definition of the conditional expectation. Consequently, we see that  $\limsup_{n \rightarrow \infty} (E(\xi_n | \mathcal{G}))$  is integrable, and equal to  $E(\xi | \mathcal{G})$  almost surely as desired. When  $\eta \neq 0$  we can just consider the random variables  $\xi_n + \eta$  instead, which is well-defined as  $E(\eta) > -\infty$ .

3. This is equivalent to 2. by considering the sequence  $\xi - \xi_n$ .
4. This is a direct application of 2. to the sequence  $\eta_n = \inf_{k \geq n} \xi_k$ , which is increasing.
5. This is equivalent to 4.
6. This is a direct application of 2. ■

**Corollary 8.3.10** Let  $\xi$  and  $\eta$  be random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\xi$ ,  $\eta$  and  $\xi\eta$  integrable. Let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -algebra and suppose that  $\eta$  is  $\mathcal{G}$ -measurable. Then

$$\mathbb{E}(\xi\eta|\mathcal{G}) = \eta\mathbb{E}(\xi|\mathcal{G})$$

almost everywhere.

*Proof.* Note that  $\mathbb{E}(\xi|\mathcal{G})$  and  $\eta$  are both  $\mathcal{G}$  measurable.

Step 1. Consider  $\eta = \chi_A$  for  $A \in \mathcal{G}$ .

Let  $B \in \mathcal{G}$ . On the one hand,

$$\begin{aligned}\mathbb{E}(\chi_B\mathbb{E}(\xi\eta|\mathcal{G})) &= \mathbb{E}(\chi_B\xi\eta) \\ &= \mathbb{E}(\chi_{A \cap B}\xi),\end{aligned}$$

where for the first equality we have used the definition of conditional expectation. On the other hand,

$$\begin{aligned}\mathbb{E}(\chi_B\eta\mathbb{E}(\xi|\mathcal{G})) &= \mathbb{E}(\chi_{A \cap B}\mathbb{E}(\xi|\mathcal{G})) \\ &= \mathbb{E}(\chi_{A \cap B}\xi),\end{aligned}$$

where for the second equality we have used the fact that  $A \cap B \in \mathcal{G}$  and the definition of conditional expectation. Thus, we have proved the result for  $\eta = \chi_A$  where  $A \in \mathcal{G}$ .

Step 2. Consider  $\eta$  to be a simple random variable.

We can extend the result of Step 1 to simple random variables by using Proposition 8.3.1.

Step 3. Consider  $\eta$  a general integrable random variable.

Recall that any integrable random variable  $\eta$  can be approximated by simple functions  $(\eta_n)_{n \in \mathbb{N}}$  such that  $|\eta_n| \leq \eta$ . Moreover,  $\eta_n \xi \rightarrow \eta \xi$  almost surely, with  $|\eta_n \xi| \leq |\eta \xi|$ . Therefore, using as  $\mathbb{E}(\eta \xi) < \infty$  we can apply Proposition 8.3.9 1. to deduce that  $\mathbb{E}(\eta_n \xi|\mathcal{G}) \rightarrow \mathbb{E}(\eta \xi|\mathcal{G})$ . Using our previous steps we know that  $\mathbb{E}(\eta_n \xi|\mathcal{G}) = \eta_n \mathbb{E}(\xi|\mathcal{G})$ . Therefore as  $\eta_n \mathbb{E}(\xi|\mathcal{G}) \rightarrow \eta \mathbb{E}(\xi|\mathcal{G})$  we deduce that  $\mathbb{E}(\eta \xi|\mathcal{G}) = \eta \mathbb{E}(\xi|\mathcal{G})$ . ■

## 8.4 Conditioning on a Random Variable

**Definition 8.4.1** The conditional expectation of a random variable  $\xi$  with respect to a random variable  $\eta$  is

$$\mathbb{E}(\xi|\eta) := \mathbb{E}(\xi|\sigma(\eta)),$$

where  $\sigma(\eta)$  is the  $\sigma$ -algebra generated by  $\eta$ .

**Theorem 8.4.2** Let  $\mu$  and  $\eta$  be random variables such that  $\mu$  is  $\sigma(\eta)$ -measurable. Then there exists a Borel-measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\mu = f(\eta).$$

In particular, there exists a Borel-measurable function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\mathbb{E}(\mu|\eta) = g(\eta).$$

*Proof.* Step 1. Consider  $\mu = \sum_{i=1}^n c_j \chi_{A_j}$ , with  $\{A_j\}_{j=1}^n$  partitions  $\Omega$ .

As  $\mu$  is  $\sigma(\eta)$ -measurable it must be the case that  $A_j \in \sigma(\eta)$  for all  $j$ . Hence, for all  $j$  there exists  $B \in \mathcal{B}(\mathbb{R})$  such that  $\eta^{-1}(B_j) = A_j$ . It is clear that  $\{B_j\}_{j=1}^n$  partitions  $\eta(\Omega)$ . Hence, set

$$f(x) = \begin{cases} \sum_{i=1}^n c_j \chi_{B_j}(x) & x \in \bigcup_{j=1}^n B_j \\ 0 & \text{otherwise,} \end{cases}$$

so that  $f(\eta(\omega)) = \mu(\omega)$  as required.

Step 2. Consider  $\mu$  a general random variable.

We can approximate  $\mu$  with a sequence of simple random variables,  $(\mu_n)_{n \in \mathbb{N}}$  such that  $\mu_n(\omega) \rightarrow \mu(\omega)$  for all  $\omega \in \Omega$ . By Step 1 we can define Borel-measurable functions  $f_n$  such that  $\mu_n = f_n(\eta)$ . Now set

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} f_n(x) & \text{if it exists on } \eta(\Omega) \\ 0 & \text{otherwise.} \end{cases}$$

Then  $f(x)$  is Borel measurable and

$$\mu(\omega) = \lim_n \mu_n(\omega) = \lim_n f_n(\eta(\omega)) = f(\eta(\omega))$$

as required. ■

**Example 8.4.3** Consider real-valued random variables  $X$  and  $Y$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Assume the random vector  $(X, Y)$  has a continuous joint density  $f_{X,Y}(x, y) > 0$ . Recall that  $X$  has density  $f_X(x) = \int_{\Omega} f_{X,Y}(x, y) dy$  and  $Y$  has density  $f_Y(y) = \int_{\Omega} f_{X,Y}(x, y) dx$ . Assume  $f_X(x), f_Y(y) > 0$  almost everywhere in  $\mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a Borel measurable function with  $\mathbb{E}(|h(X)|) < \infty$ . By Theorem 8.4.2, we know that  $\mathbb{E}(h(X) | Y) = \phi(Y)$  for some unique Borel-measurable  $\phi$ , almost everywhere. That is,

$$\mathbb{E}(\chi_A \phi(Y)) = \mathbb{E}(\chi_A h(X))$$

for all  $A \in \sigma(Y)$ . Since  $A \in \sigma(Y) \subseteq \mathcal{F}$ , we know that  $A = Y^{-1}(B)$  for some  $B \in \mathcal{B}(\mathbb{R})$ . Then,

$$\begin{aligned} \mathbb{E}(\chi_A h(X)) &= \mathbb{E}(\chi_B(Y) h(X)) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) \chi_B(y) f_{X,Y}(x, y) dy dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) \chi_B(y) \frac{f_{X,Y}(x, y)}{f_Y(y)} f_Y(y) dy dx \\ &\stackrel{\text{Fubini}}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) \chi_B(y) \frac{f_{X,Y}(x, y)}{f_Y(y)} f_Y(y) dx dy \\ &= \int_{\mathbb{R}} \chi_B(y) \left( \int_{\mathbb{R}} h(x) \frac{f_{X,Y}(x, y)}{f_Y(y)} dx \right) f_Y(y) dy. \end{aligned}$$

So by the uniqueness of  $\phi$  we deduce that

$$\phi(y) = \int_{\mathbb{R}} h(x) \frac{f_{X,Y}(x, y)}{f_Y(y)} dx.$$

#### Exercise 8.4.4

1. Let  $X$  and  $Y$  be random variables taking value in  $\mathbb{N}$ , with joint mass  $p_{X,Y}(x, y)$  for  $x, y \in \mathbb{N}$ . Assume  $h : \mathbb{N} \rightarrow \mathbb{R}$  is such that  $\mathbb{E}(|h(X)|) < \infty$ . Verify that  $\mathbb{E}(h(X) | Y) = \phi(Y)$  where

$$\phi(y) = \sum_{x \in \mathbb{N}} h(x) \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

for  $p_Y(y) \neq 0$ .

2. Consider random variables  $Z_1, Z_2$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $Z_1 \sim \text{Po}(\lambda_1)$  and  $Z_2 \sim \text{Po}(\lambda_2)$ . Assuming  $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$ , show that

$$\mathbb{P}(Z_1 = k | Z_1 + Z_2 = n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

**Proposition 8.4.5** If  $\mathbb{E}(\xi^2) < \infty$ , then

$$\min_f \mathbb{E}((\xi - f(\eta))^2) = \mathbb{E}((\xi - \mathbb{E}(\xi|\eta))^2),$$

where the minimum is taken over all  $\sigma(\eta)$ -measurable functions such that  $\mathbb{E}(f^2(\eta)) < \infty$ .

## 8.5 Solution to Exercises

### Exercise 8.3.4

*Solution.*

- As  $\mathbb{E}(\xi)$  is just a constant it is  $\{\emptyset, \Omega\}$ -measurable. Moreover,

$$\int_{\emptyset} \xi d\mathbb{P} = 0 = \int_{\emptyset} \mathbb{E}(\xi) d\mathbb{P}$$

and

$$\int_{\Omega} \xi d\mathbb{P} = \mathbb{E}(\xi) = \mathbb{E}(\xi) \int_{\Omega} d\mathbb{P} = \int_{\Omega} \mathbb{E}(\xi) d\mathbb{P}.$$

It follows that  $\mathbb{E}(\xi|\{\emptyset, \Omega\}) = \mathbb{E}(\xi)$ .

- This is clear as  $\xi$  is  $\mathcal{F}$ -measurable and the integrals of both coincide for any  $A \in \mathcal{F}$ .

- Let  $B \in \mathcal{G}$ , then

$$\begin{aligned} \int_B \xi d\mathbb{P} &= \mathbb{E}(\xi \chi_B) \\ &= \mathbb{E}(\xi) \mathbb{E}(\chi_B) \\ &= \mathbb{E}(\xi) \int_B d\mathbb{P} \\ &= \int_B \mathbb{E}(\xi) d\mathbb{P}. \end{aligned}$$

■

### Exercise 8.4.4

*Solution.*

- It suffices to consider singleton  $\{Y = y\}$  sets as  $Y$  takes values in  $\mathbb{N}$ . Proceeding as in Example 8.4.3 we see that

$$\begin{aligned} \mathbb{E}(h(X)\chi_{\{Y=y\}}) &= \sum_{x \in \mathbb{N}} h(x)p_{X,Y}(x,y) \\ &= \sum_{x \in \mathbb{N}} h(x) \frac{p_{X,Y}(x,y)}{p_Y(y)} p_Y(y) \\ &= \phi(y) \mathbb{E}(\chi_{\{Y=y\}}) \\ &= \mathbb{E}(\phi(y)\chi_{\{Y=y\}}). \end{aligned}$$

- Recall that  $Z_1 + Z_2 \sim \text{Po}(\lambda_1 + \lambda_2)$ . Let  $X = Z_1$ ,  $Y = Z_1 + Z_2$  and  $h = \chi_{\{Z_1=k\}}$ . Then

$$\mathbb{P}(Z_1 = k | Z_1 + Z_2 = n) = \mathbb{E}(h(X)|Y)(n).$$

Hence,

$$\begin{aligned} \mathbb{P}(Z_1 = k | Z_1 + Z_2 = n) &= \sum_{x \in \mathbb{N}} \chi_{\{Z_1=x\}} \frac{\mathbb{P}(Z_1 = x, Z_1 + Z_2 = n)}{\mathbb{P}(Z_1 + Z_2 = n)} \\ &= \frac{\mathbb{P}(Z_1 = k)\mathbb{P}(Z_2 = n - k)}{\mathbb{P}(Z_1 + Z_2 = n)} \\ &= \frac{\left(\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right) (\lambda_2^{n-k} e^{-\lambda_2})}{\left(\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}\right)} \\ &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}. \end{aligned}$$

■

## References

- [1] Eugene Lukacs. “A Survey of the Theory of Characteristic Functions”. In: **Advances in Applied Probability** 4.1 (1972), pp. 1–38. ISSN: 00018678. URL: <http://www.jstor.org/stable/1425805> (visited on 08/19/2022).
- [2] Patrick Billingsley. **Probability and Measure**. 3rd ed. Wiley Series in Probability and Mathematical Statistics. New York: Wiley, 1995. 593 pp. ISBN: 978-0-471-00710-4.
- [3] Wikipedia contributors. **Proofs of convergence of random variables — Wikipedia, The Free Encyclopedia**. 2023. URL: [https://en.wikipedia.org/w/index.php?title=Proofs\\_of\\_convergence\\_of\\_random\\_variables&oldid=1180638140](https://en.wikipedia.org/w/index.php?title=Proofs_of_convergence_of_random_variables&oldid=1180638140).
- [4] Samuel Lam Ivan Kirev. **Imperial Probability Theory**. 2023. URL: <https://github.com/Samuel-CHLam/Imperial-Probability-Theory>.