Algebraic Topology*

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Contents

*These notes are inspired by Chapters 1 and 2 of [1].

1 Introduction

Algebraic topology is the study of spaces and their shape. More specifically, it aims to determine when spaces have the same shape. For example, we know that the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are the same if and only if $n = m$, but why do we know this? One could argue that they have different dimensions and thus are different, however, for $I \subseteq \mathbb{R}^n$, there exists continuous surjective, so-called space-filling, maps $f : I \to I^2$. However, these maps do not preserve all topological features, for instance, they are not injective, and thus we do not say that I and I^2 are topologically equivalent. To argue that topological spaces X and Y are topologically equivalent we require the existence of a homeomorphism. That is, a map $f : X \to Y$ that is continuous, bijective with f^{-1} being continuous. More generally, algebraic topology aims to understand topological spaces by studying their maps into abstract objects. For example, one usually considers maps of the form $G : X \to G(X)$, where X is a topological space and $G(X)$ is an algebraic object, normally a group. Moreover, for a different topological space Y, algebraic topology tries to understand conditions for the existence of a map $f : X \to Y$ such that $fG(X) = Gf(X)$. That is, an f such that the commutative diagram

$$\begin{tikzcd} X \arrow[r, "G"] \arrow[d, "f"'] & G(X) \arrow[d, "f"] \\ Y \arrow[r, "G"] & G(Y) \end{tikzcd}$$

holds.

1.1 Continuous Functions on Topological Spaces

Throughout, let $I := [0, 1] \subseteq \mathbb{R}$ with the subspace topology. More generally, for any topological space X, a subset $A \subseteq X$ will be endowed with the subspace topology. Moreover, as we will be constantly dealing with maps we will abbreviate the notation for function composition with standard multiplication notation.

Definition 1.1.1. *For topological spaces X and Y, a homotopy is a continuous map $F : X \times I \to Y$ such that for every $t \in I$ the map $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ is continuous.*

One can think of a homotopy as a continuous deformation of a topological space X into a topological space Y.

Definition 1.1.2. *Continuous maps $f, g : X \to Y$ are homotopic if there exists a homotopy $F : X \times I \to Y$ such that*

$$F_0(x) := F(x, 0) = f(x)$$

and

$$F_1(x) := F(x, 1) = g(x)$$

for $x \in X$. In such a case we write $f \simeq g$.

Similarly, one can say that maps are homotopic if there exists a continuous deformation of one into the other.

Exercise 1.1.3. *For topological spaces X and Y, show that $\simeq$ is an equivalence relation on the space of continuous maps from $X \to Y$.*

Definition 1.1.4. *For a subset $A \subseteq X$, a continuous map $r : X \to A$ such that $r(X) = A$ and $r|_A = \mathrm{id}_A$ is referred to as a retraction of X onto A.*

Remark 1.1.5. *Let $r : X \to A$ be a retraction of X onto A. Then as $r(x) \in A$ for all $x \in X$, it follows that $r^2(x) = r(x)$ as $r|_A = \mathrm{id}_A$. Therefore, $r^2 = r$.*

Example 1.1.6. *Suppose $X \neq \emptyset$ and consider $p \in X$. Then $r : X \to \{p\}$, given by $r(x) = p$ for all $x \in X$, retracts X onto p. Clearly, $r|_{\{p\}} = \mathrm{id}_{\{p\}}$, and r is continuous as*

$$r^{-1}(A) = \begin{cases} \emptyset & p \notin A \\ X & p \in A. \end{cases}$$

Definition 1.1.7. *For $A \subseteq X$, a retraction $r : X \to A$ is a deformation retract if $r \simeq \mathrm{id}_X$.*

Remark 1.1.8. *The use of $r \simeq \mathrm{id}_X$ is a slight abuse of notation as strictly speaking we have $r : X \to A$ and $\mathrm{id}_X : X \to X$. What we mean with this notation is that there exists a continuous map $F : X \times I \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1 = ir$ where $i : A \to X$ is the inclusion map.*

Example 1.1.9.

1. *The map $r : \mathrm{D}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\} \to \{0\}$ given by $x \mapsto 0$ is a deformation retract. As $F_t : \mathrm{D}^n \to \mathrm{D}^n$ given by $x \mapsto t \cdot x$ is such that $F_0(x) = r$ and $F_1(x) = \mathrm{id}_{\mathrm{D}^n}$. That is, the n-dimensional disk deformation retracts to a point.*
2. *The map $r : \mathbb{R}^{n+1} \setminus \{0\} \to \mathrm{S}^n := \partial \mathrm{D}^{n+1}$ given by $x \mapsto \frac{x}{\|x\|}$ is a deformation retract. As $F_t : \mathbb{R}^n \setminus \{0\} \to \mathrm{S}^n$ given by $x \mapsto (1-t)x + t\frac{x}{\|x\|}$ is such that $F_0 = \mathrm{id}_{\mathbb{R}^{n+1} \setminus \{0\}}$ and $F_1 = r$. That is, the $(n+1)$-dimensional space excluding the origin deformation retracts to the surface of a $(n+1)$-dimensional disk.*
3. *The map $r : \mathrm{S}^n \times I \to \mathrm{S}^n \times \{0\}$ given by $(x, r) \mapsto (x, 0)$ is a deformation retraction. As $F_t : \mathrm{S}^n \times I \to \mathrm{S}^n \times \{0\}$ given by $F_t(x, r) = (x, (1-t) \cdot r)$ is such that $F_0 = \mathrm{id}_{\mathrm{S}^n \times I}$ and $F_1 = r$.*

Definition 1.1.10. *A continuous map $f : X \to Y$ is a homotopy equivalence if there is a continuous map $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. If there exists a homotopy equivalence between X and Y then the spaces are said to be homotopy equivalent.*

Exercise 1.1.11. *Suppose that $f : X \to Y$ and $g, h : Y \to X$ are such that fg and hf are homotopy equivalences. Show that f is a homotopy equivalence.*

Lemma 1.1.12. *A deformation retraction $r : X \to A$ is a homotopy equivalence.*

Proof. For $A \subseteq X$ with the subspace topology let $r : X \to A$ be a deformation retract, and let $i : A \to X$ be the inclusion map. Then $ri = \mathrm{id}_A$ as $r|_A = \mathrm{id}_A$. Moreover, by the discussions of Remark 1.1.8 we have $ir \simeq \mathrm{id}_X$. ☐

Remark 1.1.13. *Note that if $f : X \to \{p\}$ for $p \in X$ is a homotopy then as $f|_{\{p\}} = \mathrm{id}_{\{p\}}$ it follows that f is also a deformation retraction. Therefore, when $A = \{p\}$ Lemma 1.1.12 has a converse.*

Definition 1.1.14. *A topological space is contractible if it is homotopy equivalent to a point.*

Definition 1.1.15. *A continuous map is null-homotopic if it is homotopic to a constant map.*

Lemma 1.1.16. *If X is contractible then X is path-connected.*

Proof. As X is contractible we have a map $r : X \to \{p\}$ with $r \simeq \mathrm{id}_X$. More specifically, there exists a continuous map $F_t : X \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1 = ir$ where $i : \{p\} \to X$ is the inclusion map. Therefore, for any $x \in X$ let $f_x : I \to X$ be given by $f_x(t) = F_t(x)$. This map is continuous and such that $f_x(0) = F_0(x) = \mathrm{id}_X(x) = x$ and $f_x(1) = F_1(x) = ir(x) = r(x) = p$. Therefore, we have a path between p and any $x \in X$, which implies that X is path-connected. $\square$

Example 1.1.17. *Clearly, every deformation retraction is a retraction, however, a retraction need not be a deformation retraction. Take $X = \{0, 1\}$ with discrete topology, then $x \mapsto 0$ is a retraction. However, it cannot be a deformation retraction as otherwise, X would be path-connected by Lemma 1.1.16 which is not the case.*

Lemma 1.1.18. *A topological space X is contractible if and only if id_X is null-homotopic.*

Proof. $(\Rightarrow)$. For some $p \in X$ there exists continuous maps $f : X \to \{p\}$ and $g : \{p\} \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg = \mathrm{id}_{\{p\}}$. In particular, $(gf)(x) = g(p)$ is a constant map which means id_X is null-homotopic.
$(\Leftarrow)$. We know that $F : X \times I \to X$ exists such that $F(x, 0) = \mathrm{id}_X$ and $F(x, 1) = p$ for some $p \in X$. That is, $\mathrm{id}_X \simeq \mathrm{id}_{\{p\}}$. Let $f : X \to \{p\}$ be given by $f(x) = p$ and $g : \{p\} \to X$ be the inclusion map. Then $fg : \{p\} \to \{p\}$ is such that $fg(x) = p$ for every $x \in X$ meaning $fg = \mathrm{id}_{\{p\}}$. Furthermore, $gf : X \to X$ is such that $gf(x) = p$ for every $x \in X$ meaning $gf = \mathrm{id}_{\{p\}} \simeq \mathrm{id}_X$. Therefore, X is homotopy equivalent to a point and thus contractible. $\square$

Lemma 1.1.19. *Let X be contractible. Then if $r : X \to A$ is a retract for $A \subseteq X$ then A is contractible.*

Proof. Let X be a contractible space, that is there exist continuous maps $f : X \to \{p\}$ and $g : \{p\} \to X$, for some $p \in X$ such that $fg \simeq \mathrm{id}_{\{p\}}$ and $gf \simeq \mathrm{id}_X$ through F. Let $r : X \to A$ be a retraction and suppose $p \in A$. Let $\tilde{f} := f|_A$ and $\tilde{g} = rg$. As $\tilde{f}\tilde{g} : \{p\} \to \{p\}$, it must be the case that $\tilde{f}\tilde{g} = \mathrm{id}_{\{p\}}$. Let $\tilde{F}_t : A \to A$ be given by $\tilde{F}_t(x) = (rF_t)(x)$. Then $\tilde{F}_0 = \tilde{g}\tilde{f}$ and $\tilde{F}_1 = \mathrm{id}_A$. Therefore, $\tilde{g}\tilde{f} \simeq \mathrm{id}_A$. Thus A is homotopy equivalent to a point and is thus contractible. If $p \notin A$, then choose a point $q \in A$. As X is path-connected there exists a path γ from p to q. Then we let $f' : X \to \{q\}$ be given by $f' = \gamma f$ and $g' : \{q\} \to X$ be given by $g' = g\gamma$. It follows that $g'f' \simeq gf$ through the homotopy $g\gamma|_{[0,t]}f\gamma|_{[0,t]}$, and so $g'f' \simeq \mathrm{id}_X$. Thus, we can proceed as before. $\square$

1.2 Constructing Spaces

1.2.1 Quotient Topology

Let X be a topological space and consider an equivalence relation $\sim$ on X. Then $X/\sim$ denotes the set of equivalence classes and we have a map $\pi : X \to X/\sim$ given by $x \mapsto [x]$, where $[x]$ is the equivalence class of x. The most refined topology on $X/\sim$ such that π is continuous is where $U \subseteq X/\sim$ is open if and only if $\pi^{-1}(U) \subseteq X$ is open. This is referred to as the quotient topology.

Proposition 1.2.1. *If $g : X \to Z$ is a continuous map, where Z is a topological space, such that $g(a) = g(b)$ whenever $a \sim b$, then there exists a unique continuous map $f : X/\sim \to Z$ such that $g = f\pi$.*

The quotient topology can be formulated from a partition. Namely, points are equivalent if they are within the same partition of the topological space.

Example 1.2.2.

1. *Consider the torus $T = \mathrm{S}^1 \times \mathrm{S}^1$. Then $\mathrm{S}^1 \subseteq T$ can be thought of as a ring at a particular point along a torus. One can then partition T into rings at different points along the torus. In particular, by identifying*

the points at which rings are defined with S^1 we see that $T/\mathrm{S}^1 = \mathrm{S}^1$. See Figure 1.2.1 for an illustration.*

2. *The Möbius strip M is homeomorphic to $([0,1] \times [0,1])/\sim$, where $(0,y) \sim (1, 1-y)$.*

3. *The Klein bottle K is homeomorphic to $([0,1] \times [0,1])/\sim$, where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$.*

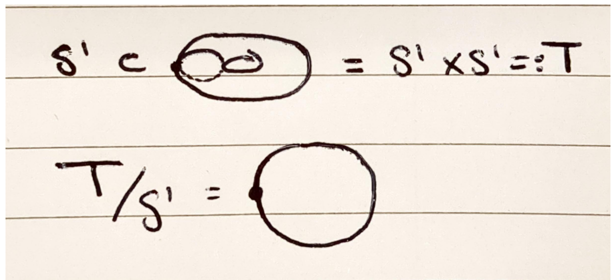


Figure 1.2.1: The quotient of a torus by the circle.

Exercise 1.2.3. *Show that S^1 is a deformation retract of the Möbius strip M.*

1.2.2 Cell Complexes

Cell complexes are also referred to as CW (closure-finite weak-topology) complexes.

Example 1.2.4. *Topological spaces can be constructed using a collection of cells. The torus, $\mathrm{S}^1 \times \mathrm{S}^1$, is the union of a point, two open intervals, and a two-dimensional open disc, see Figure 1.2.2.*

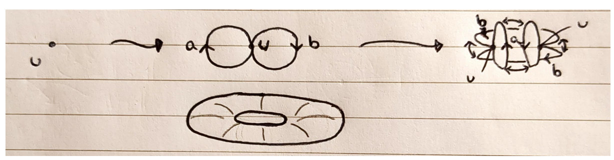


Figure 1.2.2: The torus can be viewed as a collection of cells glued together.

Definition 1.2.5. *A CW complex is a topological space $X = \bigcup_{n \in \mathbb{N}} X^n$ with X^n inductively constructed as follows.*

1. *The set X^0 is a discrete set.*

2. *For $n \geq 1$ let (D^n_α) be a collection of n-dimensional disks, with corresponding continuous maps $\phi_\alpha : \partial \mathrm{D}^n_\alpha \to X^{n-1}$. Then*
$$X^n := X^{n-1} \cup \bigcup_\alpha \mathrm{D}^n_\alpha / \sim,$$
where $\sim$ identifies an $x \in \partial \mathrm{D}^n_\alpha$ with its image $\phi_\alpha(x)$ for every α.

A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for every $n \in \mathbb{N}$.

Remark 1.2.6.

1. *For $n \geq 1$, as sets one can write*
$$X^n = X^{n-1} \cup \bigcup_{\alpha} e_{\alpha}^n$$
where e_{α}^n is homeomorphic to an open n-dimensional disk. The e_{α}^n are referred to as the n-cells of X.

2. *Each cell e_{α}^n has a characteristic map Φ_{α} given by the composition*
$$\mathrm{D}_{\alpha}^n \hookrightarrow X^{n-1} \cup \bigcup_{\alpha} \mathrm{D}_{\alpha}^n \xrightarrow{\phi_{\alpha}} X^n \hookrightarrow X.$$
This map is continuous as in addition to ϕ_{α} being continuous, the first inclusion map is continuous and the second is also continuous by the fact that $X = \bigcup_{n \in \mathbb{N}} X^n$.

3. *If $X = X^m$ for some $m \in \mathbb{N}$, then X is finite-dimensional, with the minimum such m being referred to as the dimension of X.*

4. *In the finite-dimensional case statement 3 of Definition 1.2.5 is not necessary as the quotient already constructs the topology on X. When X is infinite-dimensional then statement 3 is necessary to construct the topology of X.*

Lemma 1.2.7. *Let X be a CW complex. Then $U \subseteq X$ is open if and only if $\Phi_{\alpha}^{-1}(U)$ is open for each characteristic map Φ_{α}.*

Proof. $(\Rightarrow)$. If U is open then $U \cap X^n$ is open for every $n \in \mathbb{N}$. Therefore, $\Phi_{\alpha}^{-1}(U \cap X^n) = \Phi_{\alpha}^{-1}(U) \cap \Phi_{\alpha}^{-1}(X^n)$ is open. Therefore, as $\Phi_{\alpha}^{-1}(X^n)$ is open it must be the case that $\Phi_{\alpha}^{-1}(U)$ is open.
$(\Leftarrow)$. Since $\Phi_{\alpha}^{-1}(U)$ is open in D_{α}^n for all α, it follows that $U \cap X^n$ is open with X^n in the quotient topology. Hence, as
$$U = U \cap X = U \cap \bigcup_{n \in \mathbb{N}} X^n = \bigcup_{n \in \mathbb{N}} U \cap X^n,$$
we have that U is open in X. □

Example 1.2.8.

1. *The topological spaces I, $\mathbb{R}$ and S^1 are CW complexes. Refer to Figure 1.2.3a, Figure 1.2.3b, and Figure 1.2.3c respectively.*

2. *The topological space S^n is a CW complex with a 0-cell and an n-cell. More specifically, the n-cell is attached through the constant map $\mathrm{S}^{n-1} \to e^0$ and can be identified with $\mathrm{D}^n/\partial\mathrm{D}^n$. To gain a geometrical intuition consider S^2, which is the two-dimensional surface of a sphere embedded in three-dimensional space. In this case, D^n is a two-dimensional disk and ∂D^n is its perimeter. Thus $\mathrm{S}^{n-1} \to e^0$ connects the open boundary of the disc to a point to form the surface of a sphere.*

3. *The cell structure of a topological space need not be unique. The space S^n can also be constructed from the space S^{n-1}. More specifically, S^2 can be constructed using a 0-cell, one 1-cell and two 2-cells. Here the 0-cell and the 1-cell are used to construct S^1, and then the two 2-cells form hemispheres that join to form S^2.*

4. *The torus can be constructed as a 0-cell, two 1-cells and a 2-cell.*

5. *The Möbius strip can be constructed as two 0-cells, three 1-cells and one 2-cell.*

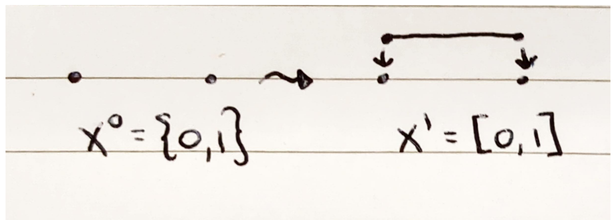
6. *The Klein bottle can be constructed as one 0-cell, two 1-cells and one 2-cell.*

7. *$\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. A non-zero vector determines a line up to scalar multiplication. $\mathbb{RP}^n$ is topologized as the quotient space $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation that*

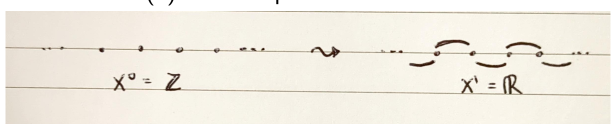
identifies vectors with their non-zero multiples. If we restrict ourselves to vectors of unit norm, then we can equivalently state that $\mathbb{RP}^n$ *is the quotient space* $\mathrm{S}^n/(v \sim -v)$*. Which is equivalent to the quotient of the hemisphere* D^n *with the antipodal points of* $\partial\mathrm{D}^n$ *identified, which is just* $\mathbb{RP}^{n-1}$ *with an* n*-cell attached along the map* $\mathrm{S}^{n-1} \to \mathbb{RP}^{n-1}$*. By induction we conclude that* $\mathbb{RP}^n$ *has the CW structure*

$$e^0 \cup e^1 \cup \cdots \cup e^n,$$

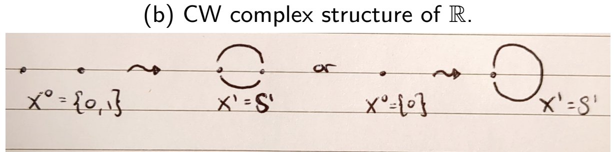
where e^i *is an* i*-cell.*



(a) CW complex structure of I.



(b) CW complex structure of $\mathbb{R}$.



(c) CW complex structure of S^1.

Figure 1.2.3: CW complex structures of topological spaces.

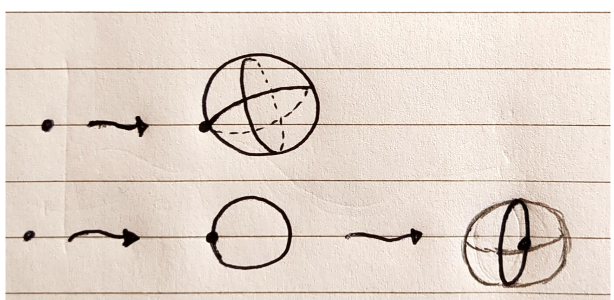


Figure 1.2.4: The different cell structures for S^2. The top illustration corresponds to statement 1 of Example 1.2.8, and the bottom illustration corresponds to statement 2 of Example 1.2.8

Proposition 1.2.9. *A compact subspace of a CW complex is contained in a finite subcomplex.*

Proof. Let X be a CW complex and consider a compact set $C \subseteq X$. Suppose for contradiction that C intersects with infinitely many cells of X. Then there exists a sequence $S := (x_i)_{i\in\mathbb{N}} \subseteq C$ where each x_i lies in a distinct cell. Suppose that $S \cap X^{n-1}$ is closed for some n. Then, for each cell e^n_α of X, the set $\phi_\alpha^{-1}(S)$ is closed in $\partial\mathrm{D}^n_\alpha$, and $\Phi_\alpha^{-1}(S)$ consists of at most one more point in D^n_α which implies that $\Phi_\alpha^{-1}(S)$ is closed in D^n_α. Consequently, $S \cap X^n$ is closed in X^n. Therefore, by induction, it follows that $S \cap X^n$ is closed for every $n \in \mathbb{N}$, which means that S is closed in X. A similar argument shows that any subset of S is closed in X, and thus S has the discrete topology. Therefore, as S is a closed subset of a compact set C it must be finite, which is a contradiction. Thus, we have that C is contained in a finite union of cells. Observe that for a single cell e_α^n the image of the corresponding attracting map ϕ_α is compact. Through induction on the dimension of the image, it follows that the image of ϕ_α lies in a finite subcomplex $A \subseteq X^{n-1}$. Hence, the cell e_α^n is contained in the finite subcomplex $A \cup e_\alpha^n$. As the finite union of subcomplexes is a finite subcomplex it follows that a finite union of cells is contained in a finite subcomplex. Therefore, C is contained in a finite subcomplex. ☐

Let X be a CW complex and $A \subseteq X$. Let $\epsilon = (\epsilon_\alpha)$, where $\epsilon_\alpha > 0$. Then we can inductively construct an open neighbourhood of A, which we denote $N_\epsilon(A)$. More specifically, suppose that $N_\epsilon^n(A)$ is an already constructed open neighbourhood of $A \cap X^n$, starting with $N_\epsilon^0(A) := A \cap X^0$. Then let $\Phi_\alpha^{-1}\left(N_\epsilon^{n+1}(A)\right)$ be the union of an ϵ_α-neighbourhood of $\Phi_\alpha^{-1}(A) \setminus \partial \mathrm{D}^{n+1} \subseteq \mathrm{D}^{n+1} \setminus \partial \mathrm{D}^{n+1}$, and $(1-\epsilon_\alpha] \times \Phi_\alpha^{-1}\left(N_\epsilon^n(A)\right)$ in the spherical coordinates of D^{n+1}. Using these construction let $N_\epsilon(A) := \bigcup_{n \in \mathbb{N}} N_\epsilon^n(A)$.

Proposition 1.2.10. *A CW complex X is Hausdorff.*

Proof. A point in X is pulled back to a point in X^n under an injective map, then for all α the pullback of this point is a closed set as ϕ_α is continuous, and then this closed set is pulled backed to a closed set D_α^n under an injective map. Therefore, the pullback of a point in X under Φ_α^n for any α is a closed set. Thus, using Lemma 1.2.7 it follows that points in a CW complex are closed. Let A and B be disjoint closed sets in X. As A and B are disjoint, the sets $N_\epsilon^0(A) = A \cap X^0$ and $N_\epsilon^0(B) = B \cap X^0$ are disjoint. Assume that $N_\epsilon^n(A)$ and $N_\epsilon^n(B)$ are disjoint. Consider the sets $\Phi_\alpha^{-1}\left(N_\epsilon^n(A)\right)$ and $\Phi_\alpha^{-1}(B)$ in D^{n+1}. These pre-images are disjoint by the inductive assumption but suppose for contradiction that they have zero distance between them. Then by the compactness of D^{n+1} there exists a sequence in $\Phi_\alpha^{-1}(B)$ that converges to a point in $\Phi_\alpha^{-1}(B)$ in $\partial \mathrm{D}^{n+1}$ that is distance zero from $\Phi_\alpha^{-1}\left(N_\epsilon^n(A)\right)$. However, $\Phi_\alpha^{-1}\left(N_\epsilon^n(B)\right)$ is an open-neighbourhood of $\Phi_\alpha^{-1}(B) \cap \partial \mathrm{D}^{n+1}$ disjoint from $\Phi_\alpha^{-1}\left(N_\epsilon^n(A)\right)$. Therefore, $\Phi_\alpha^{-1}\left(N_\epsilon^n(A)\right)$ and $\Phi_\alpha^{-1}(B)$ are at a positive distance apart. Similarly, the sets $\Phi_\alpha^{-1}\left(N_\epsilon^n(B)\right)$ and $\Phi_\alpha^{-1}(A)$ are a positive distance apart. Consequently, $\Phi_\alpha^{-1}(A)$ and $\Phi_\alpha^{-1}(B)$ are a positive distance apart, and so there exists a sufficiently small ϵ_α that will ensure that $\Phi_\alpha^{-1}\left(N_\epsilon^{n+1}(A)\right)$ is disjoint from $\Phi_\alpha^{-1}\left(N_\epsilon^{n+1}(B)\right)$ in D^{n+1}. Hence, through induction, we conclude that there exist disjoint open neighbourhoods $N_\epsilon(A)$ and $N_\epsilon(B)$ for disjoint closed subsets $A, B \subseteq X$. In other words, the CW complex X is Hausdorff. ☐

Corollary 1.2.11. *The topological space $X = \{0,1\}$ with the trivial topology is not a CW complex.*

Proof. The topological space X is not Hausdorff and so cannot be a CW by Proposition 1.2.10. ☐

Definition 1.2.12. *For a CW complex X with finitely many cells, its Euler Characteristic is*

$$\chi(X) := |\{\textit{Even Cells}\}| - |\{\textit{Odd Cells}\}|.$$

Remark 1.2.13. *The Euler characteristic of a CW complex is independent of the cell structure used to construct X.*

Example 1.2.14. *Using the construction of S^n that involves one 0-cell and one n-cell it is clear that*

$$\chi\left(\mathrm{S}^n\right) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even.} \end{cases}$$

1.3 Solution to Exercises

Exercise 1.1.3

Solution.

- For $f : X \to Y$ a continuous map let $F_t : X \to Y$ be given by $x \mapsto f(x)$ for every $t \in I$. Then, $F_0(x) = f(x)$ and $F_1(x) = f(x)$ for every $x \in X$. We note that $F_t : X \to Y$ is continuous for every $t \in I$ as f is continuous, therefore, we conclude that $f \simeq f$ which means that $\simeq$ is reflexive.
- Suppose $f, g : X \to Y$ are continuous maps such that $f \simeq g$. Then there exists a continuous map $F : X \times I \to Y$ such that $F_0 = f$ and $F_1 = g$. Let $G : X \times I \to Y$ be given by $G = F \circ \varphi$ where $\varphi : X \times I \to X \times I$ is given by $(x, t) \mapsto (x, 1-t)$. As F and φ are continuous, G is continuous. Moreover, $G_0 = F_1 = g$ and $G_1 = F_0 = f$. Therefore, $g \simeq f$ meaning $\simeq$ is symmetric.
- Suppose $f, g, h : X \to Y$ are continuous maps with $f \simeq g$ and $g \simeq h$. Let $F : X \times I \to Y$ be a continuous map such that $F_0 = f$ and $F_1 = g$. Let $G : X \times I \to Y$ be a continuous map such that $G_0 = g$ and $G_1 = h$. Let $H : X \times I \to Y$ be given by

$$H_t(x) = \begin{cases} F_{2t}(x) & t \in \left[0, \frac{1}{2}\right] \\ G_{2t-1}(x) & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

  Clearly, H is continuous and such that $H_0 = F_0 = f$ and $H_1 = G_1 = h$. Therefore, $f \simeq h$ meaning $\simeq$ is transitive.

□

Exercise 1.1.11

Solution. As $fg : Y \to Y$ is a homotopy equivalence there exists a $q_1 : Y \to Y$ such that $q_1 fg \simeq fgq_1 \simeq \mathrm{id}_Y$. Similarly, there exists a $q_2 : X \to X$ such that $q_2 hf \simeq hfq_2 \simeq \mathrm{id}_X$. Let $\tilde{g} := q_2 h : Y \to X$. We note that

$$\begin{aligned} \tilde{g} &= q_2 h \\ &= (q_2 h)\mathrm{id}_Y \\ &\simeq (q_2 h)(fgq_1) \\ &= (q_2 hf)(gq_1) \\ &\simeq \mathrm{id}_X gq_1 \\ &= gq_1. \end{aligned}$$

Therefore,

$$f\tilde{g} \simeq fgq_1 \simeq \mathrm{id}_Y$$

and

$$\tilde{g}f = q_2 hf \simeq \mathrm{id}_X.$$

Therefore, f is a homotopy equivalence. □

Exercise 1.2.3

Solution. The Möbius strip is given by $M = ([0,1] \times [0,1]) / \sim$ where $(0, y) \sim (1, 1-y)$. Note that S^1 is homeomorphic to $A := \left([0,1] \times \left\{\frac{1}{2}\right\}\right) / \sim$. Let $F : I \times M \to M$ be given by

$$F_t((x, y)) = \left(x, \frac{t}{2} + (1-t)y\right).$$

Note that

$$\begin{aligned} F_t((0, y)) &= \left(0, \frac{t}{2} + (1-t)y\right) \\ &= \left(1, 1 - \frac{t}{2} - (1-t)y\right) \\ &= \left(1, 1 + \frac{t}{2} - t - (1-t)y\right) \\ &= \left(1, \frac{t}{2} + (1-t)(1-y)\right) \\ &= F_t((1, 1-y)), \end{aligned}$$

and F_t is well-defined. Moreover, it is continuous with $F_0((x,y)) = \mathrm{id}_M$ and $F_1((x,y)) = \left(x, \frac{1}{2}\right) = \mathrm{id}_A$. Hence, S^1 is a deformation retract of the Möbius strip. $\square$

2 The Fundamental Group

2.1 Intuition for the Fundamental Group

Consider two loops A and B which can be linked in different ways. To distinguish between linking mechanisms, suppose A has a front and back. Some examples of ways that A and B could interact include the following.

1. Loops and A and B could be separated.
2. Loop B could pass through A once. Either through the front or through the back of A.
3. Loop B could pass through the front of A twice.
4. Loop B could pass through the front of A and then through the back of A.

Note how in example 4 the loops cancel each other out, meaning A and B remain separated. We observe that there is an additive structure to linking mechanisms. Furthermore, it is clear that for any one of these examples, we can continuously deform the loops whilst maintaining the linking structure. Henceforth, let B_n denote a loop that has n-forward links with A, and let B_{-n} denote a loop with n-backward links with A, with B_0 denoting the loop that is separated from A. Intuitively it makes sense to define the addition of loops B_m and B_n as

$$B_m + B_n = B_{m+n}.$$

With extra work one can see that what we have here is a relationship between the additive group structure of the integers and loops in a topological space. Using this we can give an informal definition of the fundamental group.

Definition 2.1.1 (Informal)**.** *The fundamental group of a space X has elements which are classes of equivalent loops in X that start and end at a fixed base point $x_0 \in X$. Loops are equivalent if one loop can be continuously deformed into the other.*

2.2 Constructions

2.2.1 Path Homotopy

Definition 2.2.1. *Let X be a topological space. Then a path is a continuous map $\gamma : I \to X$, where $I = [0, 1]$.*

Remark 2.2.2. *A loop refers to a path $\gamma : I \to X$ where $\gamma(0) = \gamma(1) = x$, with x being the base point of the loop.*

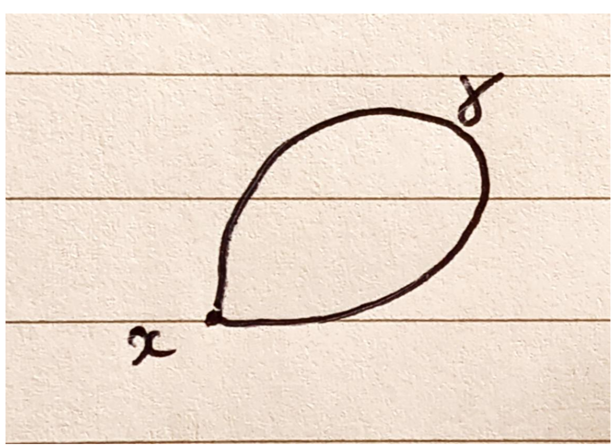


Figure 2.2.1: A loop with base point x.

Definition 2.2.3. *Paths γ_0 and γ_1, with the same endpoints, are homotopic if there exists a homotopy between them that preserves their endpoints. That is, there exists a continuous map $F : I \times I \to X$ such that the following statements hold.*

1. $F_t(0) = \gamma_0(0)$ *and* $F_t(1) = \gamma_1(1)$ *for* $t \in I$.
2. $F_0(s) = \gamma_0(s)$ *and* $F_1(s) = \gamma_1(s)$ *for* $s \in I$.

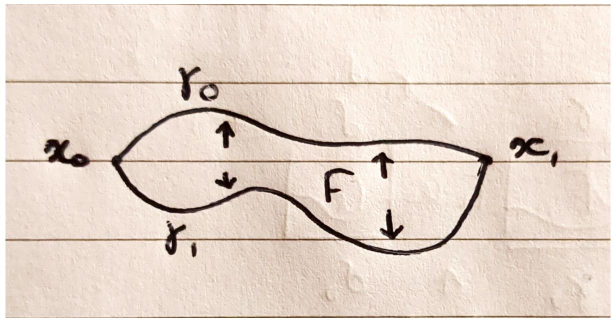


Figure 2.2.2: Homotopy between paths with the same endpoints.

Lemma 2.2.4. *On the set of paths with certain fixed endpoints, homotopy is an equivalence relation.*

Proof. Let $\gamma_0, \gamma_1, \gamma_2 : I \to X$ be paths on a topological space X with the same endpoints.

- Clearly, $\gamma_0 \simeq \gamma_0$ through the homotopy $F_t : I \to X$ given by $F_t(s) = \gamma_0(s)$.
- If $\gamma_0 \simeq \gamma_1$ through $F_t : I \to X$, then $\gamma_1 \simeq \gamma_0$ through $G_t : I \to X$ given by $G_t(s) = F_{1-t}(s)$.
- Suppose $\gamma_0 \simeq \gamma_1$ and $\gamma_1 \simeq \gamma_2$ through $F_t : I \to X$ and $G_t : I \to X$ respectively. Let $H_t : I \to X$ be given by
$$H_t(s) = \begin{cases} F_{2t}(s) & t \in \left[0, \frac{1}{2}\right] \\ G_{2t-1}(s) & t \in \left[\frac{1}{2}, 0\right]. \end{cases}$$
Then $H : I \times I \to X$ is continuous as its restriction to the closed subsets $I \times \left[0, \frac{1}{2}\right]$ and $I \times \left[\frac{1}{2}, 1\right]$ is continuous. Moreover, $H_0 = \gamma_0$ and $H_1 = \gamma_2$ so that $\gamma_0 \simeq \gamma_2$.

Thus, we conclude that $\simeq$ is reflexive, symmetric and transitive on the set of paths with certain fixed endpoints and thus defines an equivalence relation. ☐

Remark 2.2.5. *For a path $\gamma : I \to X$ with certain endpoints, we denote the set of homotopic paths with the same endpoints $[\gamma]$.*

Definition 2.2.6. *Let $\gamma_0, \gamma_1 : I \to X$ be paths such that $\gamma_0(1) = \gamma_1(0)$. Then the product path $\gamma_0 \cdot \gamma_1 : I \to X$ is*
$$(\gamma_0 \cdot \gamma_1)(s) = \begin{cases} \gamma_0(2s) & s \in \left[0, \frac{1}{2}\right] \\ \gamma_1(2s-1) & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Remark 2.2.7.

1. *The product path transverses the path γ_0 and then traverses the path γ_1. However, it does so twice as fast, such that it conforms to Definition 2.2.1.*

2. *If $\gamma_0, \gamma_1 : I \to X$ are paths such that $\gamma_0(1) = \gamma_1(0)$ we will say that γ_1 extends γ_0.*

Proposition 2.2.8. *Let $\gamma_0, \gamma_1, \delta_0, \delta_1 : I \to X$ be paths such that γ_1 extends γ_0, δ_1 extends δ_0, $\gamma_0(0) = \delta_0(0)$, $\gamma_0(1) = \delta_0(1)$ and $\gamma_1(1) = \delta_1(1)$. If $\gamma_0 \simeq \delta_0$ and $\gamma_1 \simeq \delta_1$, then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$.*

Proof. Suppose $\gamma_0 \simeq \delta_0$ through $F_t : I \to X$ and $\gamma_1 \simeq \delta_1$ through $G_t : I \to X$. Let $H_t : I \to X$ be given by

$$H_t(s) = (F_t \cdot G_t)(s).$$

Then $H : I \times I \to X$ is a homotopy between $\gamma_0 \cdot \gamma_1$ and $\delta_0 \cdot \delta_1$. $\square$

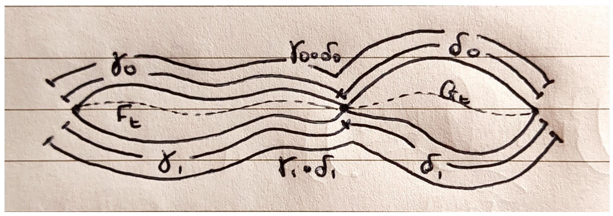


Figure 2.2.3: An intuition of why the homotopy in the proof of Proposition 2.2.8.

A continuous function $\phi : I \to I$ with $\phi(0) = 0$ and $\phi(1) = 1$ can re-parameterises a path $\gamma : I \to X$ into a $\gamma_\phi := \gamma\phi$. In particular, $\gamma \simeq \gamma_\phi$ through the homotopy

$$H(s,t) := \gamma((1-t)\phi(s) + ts).$$

Definition 2.2.9.

1. *For $x \in X$ let $c_x : I \to X$ be the constant path $t \mapsto x$.*
2. *For a path $\gamma : I \to X$ let $\gamma^{-1} : I \to X$ be the path given by $t \mapsto \gamma(1-t)$.*

Lemma 2.2.10. *Let $\gamma_0, \gamma_1, \gamma_2 : I \to X$ be paths. Then the following statements hold.*

1. $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$.
2. $\gamma_0 \cdot c_{\gamma_0(1)} \simeq \gamma_0$ *and* $c_{\gamma_0(0)} \cdot \gamma_0 \simeq \gamma_0$.
3. $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{\gamma_0(0)}$ *and* $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{\gamma_0(1)}$.

Proof.

1. Let $\phi : I \to I$ be given by

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\ s - \frac{1}{4} & s \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 2s - 1 & s \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

   Then $((\gamma_0 \cdot \gamma_1) \cdot \gamma_2)\phi = \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$, and so $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$.

2. Let $\phi : I \to I$ be given by

$$\psi(s) = \begin{cases} 2s & s \in \left[0, \frac{1}{2}\right] \\ 1 & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then $(\gamma_0 \cdot c_{\gamma_0(1)})\phi = \gamma_0$, and so $\gamma_0 \cdot c_{\gamma_0(1)} \simeq \gamma_0$. Similarly, using $\psi : I \to I$ given by

$$\psi(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

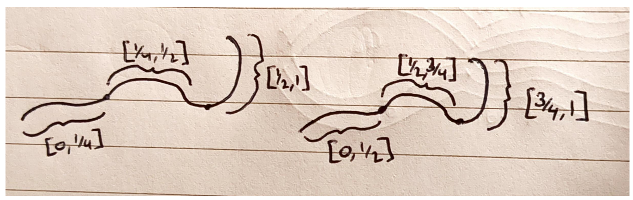
it follows that $c_{\gamma_0(0)} \cdot \gamma_0 \simeq \gamma_0$.

3. Let $H_t : I \times I$ be given by

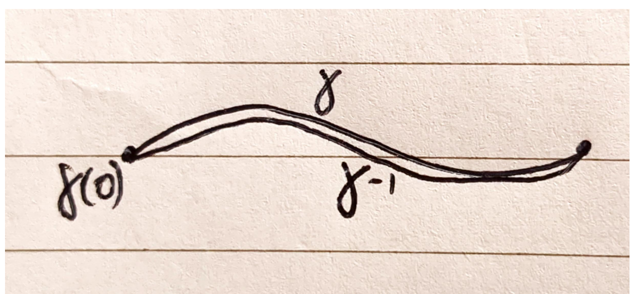
$$H(s,t) = \begin{cases} \gamma_0\left(\max(1-2s, t)\right) & s \in \left[0, \frac{1}{2}\right] \\ \gamma_0\left(\max(2s-1, t)\right) & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then H is continuous with $H(s,0) = \gamma_0^{-1} \cdot \gamma_0$ and $H(s,1) = c_{\gamma_0(1)}$. Thus, $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{\gamma_0(1)}$. Similarly, one deduces that $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{\gamma_0(0)}$.

$\square$



(a) An illustration accompanying the proof of statement 1 of Lemma 2.2.10



(b) An illustration accompanying the proof of statement 3 of Lemma 2.2.10.

Figure 2.2.4

Observe how from Lemma 2.2.10 a group structure on the set of homotopy classes emerges.

Definition 2.2.11. *For $x \in X$ let $\pi_1(X,x)$ denote the set of homotopy classes $[f]$ of loops $f : I \to X$ with base point x.*

Proposition 2.2.12. *For $x \in X$ the set $\pi_1(X,x)$ with binary operation $[f][g] = [f \cdot g]$ is a group with identity element $[c_x]$.*

Proof. By statement 1 of Lemma 2.2.10 the binary operation on $\pi_1(X,x)$ is associative. Using statement 2 of Lemma 2.2.10 we see the loop c_x satisfies the conditions to be the identity element with respect to this binary operation. Moreover using statement 3 of Lemma 2.2.10 a loop $\gamma \in \pi_1(X,x)$ has inverse γ^{-1}, as given by Definition 2.2.9, which is also a loop with base point x, that is $\gamma^{-1} \in \pi_1(X,x)$. $\square$

Remark 2.2.13. *The group of Proposition 2.2.12 is referred to as the fundamental group of X at x.*

The dependence of $\pi_1(X, x)$ on $x \in X$ can be illustrative of the topological properties of X.

Example 2.2.14. *Let $X \subseteq \mathbb{R}^n$ be a convex set. For $x \in X$ let $\gamma_0, \gamma_1 : I \to X$ be loops with base point x. Let $F_t : I \to X$ be given by*

$$F_t(s) = (1-t)\gamma_0(s) + t\gamma_1(s).$$

As X is convex it is indeed the case that $F_t(I) \subseteq X$ and so the map is well-defined. Moreover, it is continuous as γ_0 and γ_1. Thus, as $F_0 = \gamma_0$ and $F_1 = \gamma_1$ we deduce that $\gamma_0 \simeq \gamma_1$. Therefore, $\pi_1(X, x) = \{[c_x]\}$, that is the fundamental group of a convex set in $\mathbb{R}^n$ is trivial. In particular, the fundamental group of X is independent of the base point.

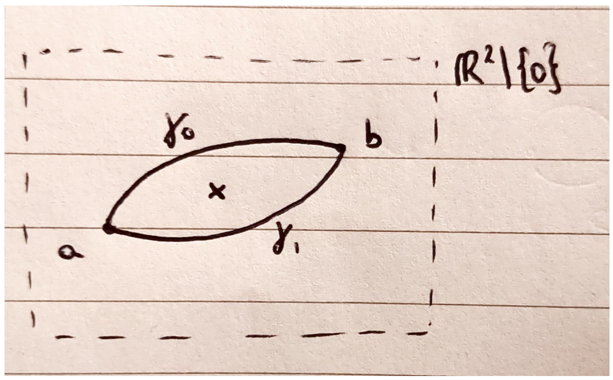


Figure 2.2.5: An example of why the convexity assumption of Example 2.2.14 is required.

Exercise 2.2.15. *Let X be a topological space. Suppose $x_0, x_1 \in X$ are in the same path-connected component of X. That is, there is a path by $h : I \to X$ with $h(0) = x_0$ and $h(1) = x_1$. Show that the map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by*

$$\beta_h([\gamma]) = \left[h \cdot \gamma \cdot h^{-1}\right]$$

is well-defined.

Remark 2.2.16.

1. *The map β_h of Exercise 2.2.15 is referred to as the change-of-base point map.*
2. *Note how $h \cdot f \cdot h^{-1}$ is read from the left, unlike function composition which is read from the right.*

Proposition 2.2.17. *In the context of Exercise 2.2.15, the map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is a group isomorphism.*

Proof. Note that

$$\beta_h([c_{x_1}]) = \left[h \cdot c_{x_1} \cdot h^{-1}\right] = \left[h \cdot h^{-1}\right] = [c_{x_0}].$$

Moreover, for $\gamma_0, \gamma_1 \in \pi_1(X, x_1)$ we have

$$\begin{aligned}
\beta_h([\gamma_0 \cdot \gamma_1]) &= \left[h \cdot \gamma_0 \cdot \gamma_1 \cdot h^{-1}\right] \\
&= \left[h \cdot \gamma_0 \cdot h^{-1} \cdot h \cdot \gamma_1 \cdot h^{-1}\right] \\
&= \left[h \cdot \gamma_0 \cdot h^{-1}\right] \cdot \left[h \cdot \gamma_0 \cdot h^{-1}\right] \\
&= \beta_h([\gamma_0]) \cdot \beta_h([\gamma_1]).
\end{aligned}$$

Thus, β_h is a group homomorphism. As

$$\begin{aligned}\beta_{h^{-1}}\left(\beta_h\left([\gamma]\right)\right) &= \beta_{h^{-1}}\left(\left[h\cdot\gamma\cdot h^{-1}\right]\right)\\ &= \left[h^{-1}\cdot h\cdot\gamma\cdot h^{-1}\cdot h\right]\\ &= [\gamma]\end{aligned}$$

it follows that β_h has inverse $\beta_{h^{-1}}$. Therefore, β_h is bijective and hence an isomorphism. □

Remark 2.2.18. *In light of Proposition 2.2.17, if X is path-connected then we write $\pi_1(X,x) = \pi_1(X)$ for all $x \in X$.*

Definition 2.2.19. *A space X is simply connected if it is path-connected and $\pi_1(X)$ is trivial.*

Proposition 2.2.20. *A space X is simply connected if and only if there exists a unique homotopy class of paths between any points of X.*

Proof. ($\Rightarrow$). Let $\gamma_0, \gamma_1 : I \to X$ be paths from a point $x_0 \in X$ to $x_1 \in X$. Note that $\gamma_0 \cdot \gamma_1^{-1}$ and $\gamma_1 \cdot \gamma_1^{-1}$ are loops with base point x_0. Therefore, as $\pi_1(X)$ is trivial it follows that $\gamma_0 \cdot \gamma_1^{-1} \simeq \gamma_1 \cdot \gamma_1^{-1}$. Therefore,

$$\gamma_0 \simeq \gamma_0 \cdot \gamma_1^{-1} \cdot \gamma_1 \simeq \gamma_1 \cdot \gamma_1^{-1} \cdot \gamma_1 \simeq \gamma_1.$$

($\Leftarrow$). Clearly, X is path-connected. In particular, for any $x_0 \in X$, a path to $x_1 \in X$ must be homotopic to the constant map at x_0, meaning $\pi_1(X)$ is trivial. □

Remark 2.2.21.

1. *The fundamental group is not necessarily abelian.*
2. *For any group G, there exists a two-dimensional CW complex such that $\pi_1(X) \cong G$.*

2.2.2 The Fundamental Group of a Circle

Intuitively, the fundamental group of S^1 should be $\mathbb{Z}$. Consider a non-trivial loop in S^1. Then there is no clear homotopy to point. Moreover, there is no clear homotopy to another loop that wraps around S^1 a different number of times. Likewise, there is no clear homotopy between loops that traverse S^1 in different directions. Therefore, one can imagine identifying loops by how many times they traverse S^1, and in what direction those traversals are made. On the other hand, one expects to be able to continuously deform a non-trivial loop in S^2 to a point. Therefore, one would expect the fundamental group of S^2 to be trivial.

Definition 2.2.22. *For a space X, a covering space is a set $\widetilde{X}$ with a continuous map $p : \widetilde{X} \to X$ such that for any $x \in X$ there is an open neighbourhood $U \subseteq X$ of x such that the following hold.*

- $p^{-1}(U) = \bigcup_{j\in J} \widetilde{U}_j$ *where* $\widetilde{U}_j \subseteq \widetilde{X}$ *is open.*
- $\widetilde{U}_i \cap \widetilde{U}_j = \emptyset$ *for* $i \neq j$.
- $p|_{\widetilde{U}_j} : \widetilde{U}_j \to U$ *is a homeomorphism for all* $j \in J$.

In such a case U is said to be evenly covered with the sheets $\widetilde{U}_j$.

The set $\widetilde{X}$ can be thought of as an embedding of the topological space X. The conditions of Definition 2.2.22 ensure that necessary topological properties are maintained in this embedding.

Proposition 2.2.23. *Let $p : \tilde{X} \to X$ be a covering space. Let $A \subseteq X$ be a subspace. Then $p : p^{-1}(A) \to A$ is a covering space.*

Proof. For $x \in X$ there exists an open set $U \subseteq X$ such that $p^{-1}(U) = \bigcup_j \tilde{U}_j$ for $\tilde{U}_j$ disjoint open sets and $p|_{\tilde{U}_j}$ a homeomorphism. Let $V := U \cap A$ so that $V \subseteq A$ is open. Let $\tilde{V}_j := \tilde{U}_j \cap p^{-1}(A)$ so that $\tilde{V}_j \subseteq p^{-1}(A)$ is open. Then,

$$\begin{aligned} p^{-1}(V) &= p^{-1}(U \cap A) \\ &= p^{-1}(U) \cap p^{-1}(A) \\ &= \bigcup_j \tilde{U}_j \cap p^{-1}(A) \\ &= \bigcup_j \tilde{V}_j. \end{aligned}$$

Note that each $\tilde{V}_j$ is disjoint as the $\tilde{U}_j$ are disjoint. Moreover, it is clear $p|_{\tilde{V}_j} : \tilde{V}_j \to V$ is a homeomorphism and so V is evenly covered by the sheets $\tilde{V}_j$. In particular, this means that $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering space. ☐

Proposition 2.2.24. *Let $p : \tilde{X} \to X$ be a covering space. Then the following statements hold.*

1. *p is an open map.*
2. *$\tilde{X}$ is Hausdorff if X is Hausdorff.*
3. *If $p^{-1}(x)$ is finite for all $x \in X$, then $\tilde{X}$ is compact if and only if X is compact.*

Proof.

1. Let $\tilde{U} \subseteq \tilde{X}$ be open and consider $x \in p\left(\tilde{U}\right) \subseteq X$. Then there exists an open neighbourhood $U \subseteq X$ of x that is evenly covered, $p^{-1}(U) = \bigcup_j \tilde{U}_j$. Now since $x \in p\left(\tilde{U}\right) \cap U$ it follows for $\tilde{x} \in \tilde{U}$ with $p\left(\tilde{x}\right) = x$ that $\tilde{x} \in \tilde{U} \cap p^{-1}(U)$. In particular, there exists a $\tilde{U}_j$ such that $\tilde{x} \in \tilde{U}_j$. Let $\tilde{V} := \tilde{U} \cap \tilde{U}_j$, which we note is open. Then as $p|_{\tilde{U}_j}$ is a homeomorphism, we have that

$$V := p|_{\tilde{U}_j}\left(\tilde{V}\right) \subseteq p\left(\tilde{U}\right) \cap U$$

is open. In particular, $x \in V$ and $V \subseteq p\left(\tilde{U}\right)$, which implies that $p\left(\tilde{U}\right)$ is open. Thus, p is an open map.

2. Let $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ be distinct. Consider $x_1 := p\left(\tilde{x}_1\right) \in X$ and $x_2 := p\left(\tilde{x}_2\right) \in X$.

   - If $x_1 \neq x_2$, then there exists disjoint open neighbourhoods $U_1, U_2 \subseteq X$ of x_1 and x_2 respectively. Then as p is continuous $\tilde{V}_1 := p^{-1}\left(U_1\right) \subseteq \tilde{X}$ and $\tilde{V}_2 := p^{-1}\left(U_2\right) \subseteq \tilde{X}$ are disjoint open neighbourhood of $\tilde{x}_1$ and $\tilde{x}_2$ respectively.
   - If $x_1 = x_2 =: x$ then there exists an open neighbourhood $U \subseteq X$ of x that it is evenly covered, $p^{-1} = \bigcup_{j \in J} \tilde{U}_j$. As $p|_{\tilde{U}_j}$ is injective, it must be the case that $\tilde{x}_1 \in \tilde{U}_{j_1}$ and $\tilde{x}_2 \in \tilde{U}_{j_2}$ for $j_1 \neq j_2$. The sets $\tilde{U}_{j_1}$ and $\tilde{U}_{j_2}$ are open and disjoint by construction, and thus separate $\tilde{x}_1$ and $\tilde{x}_2$.

   Therefore, $\tilde{X}$ is Hausdorff.

3. ($\Rightarrow$). Let (U_α) be an open cover for X. Then as p is surjective, the collection of open sets (V_α), where $V_\alpha = p^{-1}(U_\alpha)$ is an open cover for $\tilde{X}$. Hence, there exists a finite subcover, say $(V_i)_{i=1}^n$. It follows that $(U_i)_{i=1}^n$ is a finite subcover for X, meaning X is compact.
   ($\Leftarrow$). Let $\left(\tilde{U}_\alpha\right)$ be an open cover of $\tilde{X}$. Then for each $x \in X$, as $p^{-1}(x)$ is finite it can be covered by finitely many $\tilde{U}_\alpha$. Let $\left(\tilde{U}_{x,i}\right)_{i=1}^{n_x} \subseteq \left(\tilde{U}_\alpha\right)$ cover $p^{-1}(x)$. Let $\tilde{V}_x := \bigcup_{i=1}^{n_x} \tilde{U}_{x,i}$ and consider $U_x \subseteq X$ an

evenly covered neighbourhood of x. Note the covering of U_x is finite as $p^{-1}(x)$ is finite, so we can write $p^{-1}(U_x) = \bigcup_{j=1}^{m} \tilde{W}_{x,j}$. Let $W_x := \bigcap_{j=1}^{m} p\left(\tilde{V}_x \cap \tilde{W}_{x,j}\right) \subseteq X$, which is open as p is an open map. Doing this for every $x \in X$ yields an open cover $(W_x)_{x\in X}$ of X. Therefore, as X is compact there exists a finite subcover $(W_{x_k})_{k=1}^{n}$ of X. As p is surjective $\left(p^{-1}(W_{x_k})\right)_{k=1}^{n}$ is an open cover of $\tilde{X}$. As

$$p^{-1}(W_{x_k}) \subseteq \tilde{V}_{x_k} = \bigcup_{i=1}^{n_{x_k}} \tilde{U}_{x_k,i},$$

it follows that $\left(\tilde{U}_{x_k,i}\right)_{k=1,i=1}^{n,n_{x_k}}$ is an open cover for $\tilde{X}$. Which is a finite subcover of $\left(\tilde{U}_\alpha\right)$. Therefore, $\tilde{X}$ is compact.

□

Definition 2.2.25. *For $p : \widetilde{X} \to X$ a covering space, the lift of a continuous map $f : Y \to X$ is a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that $p\tilde{f} = f$.*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Example 2.2.26. *Let $p : \mathbb{R} \to \mathrm{S}^1 \subseteq \mathbb{R}^2$ be given by*

$$s \mapsto (\cos(2\pi s), \sin(2\pi s)).$$

Then p is a covering space of S^1 on $\mathbb{R}$. Let $\omega : I \to \mathrm{S}^1$ be the loop with base point $(1,0)$ given by

$$s \mapsto (\cos(2\pi s), \sin(2\pi s))$$

and let $\omega_n : I \to \mathrm{S}^1$ be another loop with base point $(1,0)$ given by

$$s \mapsto (\cos(2n\pi s), \sin(2\pi n s)).$$

Observe the following.

1. $[\omega_n] = [\omega]^n = [\omega] \cdot \ldots \cdot [\omega]$.
2. *The path $\widetilde{\omega_n} : I \to \mathbb{R}$ given by $s \mapsto ns$ is a lift for ω_n with $\widetilde{\omega_n}(0) = 0$ and $\widetilde{\omega_n}(1) = n$. Indeed,*

$$p\widetilde{\omega_n}(s) = p(ns) = (\cos(2\pi ns), \sin(2\pi ns)) = \omega_n(s).$$

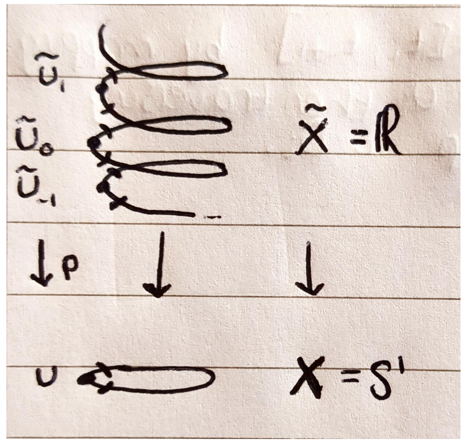


Figure 2.2.6: The covering space $\mathbb{R}$ of S^1 as detailed in Example 2.2.26.

Proposition 2.2.27. *Let $p : \tilde{X} \to X$ be a covering space and $f : Y \to X$ a continuous map. Let $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ be lifts of f such that $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$. Then if Y is connected it follows that $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Let $y \in Y$ and let $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \tilde{U}_j,$$

and suppose that $\tilde{U}_1$ is the sheet such that $\tilde{f}_1(y) \in \tilde{U}_1$ and $\tilde{U}_2$ is the sheet such that $\tilde{f}_2(y) \in \tilde{U}_2$. As $\tilde{f}_1$ and $\tilde{f}_2$ are continuous there exists $N \subseteq Y$ an open neighbourhood of y such that $\tilde{f}_1(N) \subseteq \tilde{U}_1$ and $\tilde{f}_2(N) \subseteq \tilde{U}_2$. As $p\tilde{f}_1 = p\tilde{f}_2$ it follows that $\tilde{f}_1(y) = \tilde{f}_2(y)$ if and only if $\tilde{U}_1 = \tilde{U}_2$ which happens if and only if $\tilde{f}_1|_N = \tilde{f}_2|_N$ as $p|_{\tilde{U}_j}$ is a homeomorphism. Consider

$$A := \left\{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\right\}.$$

By the above arguments we have that A is open. By similar arguments, we deduce that $Y \setminus A$ is also open. Therefore, as $A \neq \emptyset$ by assumption it follows that $A = Y$ which implies that $\tilde{f}_1 = \tilde{f}_2$. □

Proposition 2.2.28. *Let $p : \tilde{X} \to X$ be a covering space and $F : Y \times I \to X$ a continuous map with $\tilde{f}_0 : Y \times \{0\} \to \tilde{X}$ a lift of F_0. Then there exists a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F such that $\tilde{F}_0 = \tilde{f}_0$.*

Proof. For any $y \in Y$ and $t \in I$, there are open neighbourhoods $N_t \subseteq Y$ of y and $(a_t, b_t) \subseteq I$ of t such that

$$F\left(N_t \times (a_t, b_t)\right) \subseteq U,$$

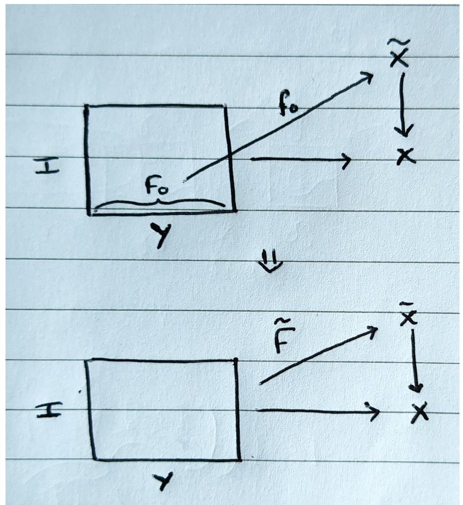
for $U \subseteq X$ open and evenly covered. For fixed $y \in Y$, as I is compact, there exists a finite partition

$$0 < t_0 < \cdots < t_m = 1$$

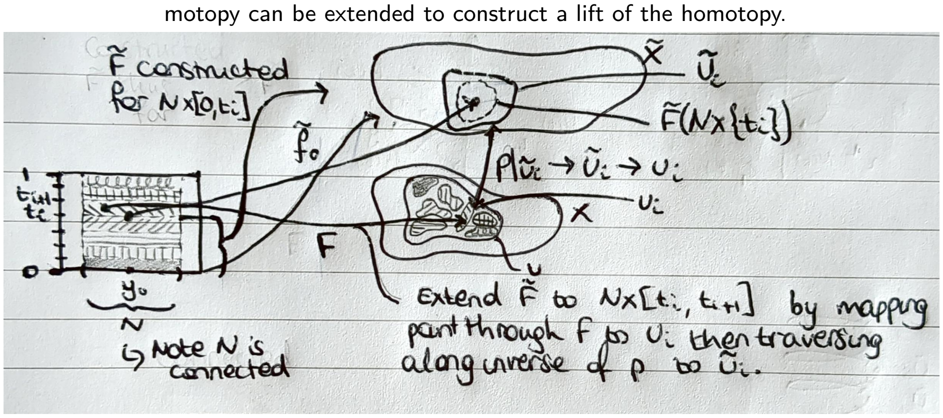
of I, such that for an open neighbourhood $N = \bigcap_{n=0}^m N_{t_n} \subseteq Y$ of y_0 we have $F(N \times [t_i, t_{i+1}]) \subseteq U_i$ where $U_i \subseteq X$ is open and evenly covered. Consequently, we can inductively construct a lift $\tilde{F}|_{N\times I}$ of $F|_{N\times I}$.

- Let $\tilde{F}|_{N\times[0,0]} = \tilde{f}_0|_{N\times[0,0]}$.
- Assume $\tilde{F}|_{N\times[0,t_i]}$ has been constructed. Let $\tilde{U}_i \subseteq \tilde{X}$ be such that $p|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a homeomorphism and $\tilde{F}(y_0, t_i) \in \tilde{U}_i$. One can assume that $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$ by shrinking N if necessary. Then let $\tilde{F}|_{N\times[t_i,t_{i+1}]} := p^{-1} \circ F|_{N\times[t_i,t_{i+1}]}$.

After finitely many steps, for $y \in Y$, we have $\tilde{F} : N \times I \to \tilde{X}$ such that where $N \subseteq Y$ is an open neighbourhood y. So for each $y \in Y$ there exists a neighbourhood $N_y \subseteq Y$ for which a lift $\tilde{F}|_{N_y \times I} : N_y \times I \to \tilde{X}$ of $F : |_{N_y \times I} : N_y \times I \to X$ exists. In particular, as $\{y\} \times I$ is connected, it follows by Proposition 2.2.27 that this lift is unique. Therefore, for $F : Y \times I \to X$ there exists a unique lift $\tilde{F} : Y \times I \to \tilde{X}$. $\square$



(a) Proposition 2.2.28 says that the lift of a strand of a homotopy can be extended to construct a lift of the homotopy.



(b) A visual argument for the proof of Proposition 2.2.28.

Figure 2.2.7

Example 2.2.29. *Let X be a topological space and let A be a discrete set. The map $p : X \times A \to X$ given by $p(x, a) = x$ is continuous. Moreover, for $U \subseteq X$ an open set let $\tilde{U}_j := U \times \{a_j\}$ for $j \in \mathbb{N}$. Note that $\tilde{U}_j \subseteq \tilde{X}$ is open, $\tilde{U}_i \cap \tilde{U}_j$ for $i \neq j$ and*

$$p^{-1}(U) = U \times A = \bigcup_{j \in \mathbb{N}} U \times \{a_j\} = \bigcup_{j \in \mathbb{N}} \tilde{U}_j.$$

Moreover, $p|_{\tilde{U}_j} \tilde{U}_j \to U$ is the map $(x, a_j) \mapsto x$, for fixed a_j, and thus is a homeomorphism. Therefore, $p : X \times A \to X$ is a covering map with $\tilde{X} := X \times A$ a covering space of X. This covering space is referred to as a trivial covering space of X.

Corollary 2.2.30. *Let $f : I \to X$ be a path with $f(0) = x_0$ and suppose $p : \widetilde{X} \to X$ is a covering space. Then for each $\widetilde{x_0} \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \widetilde{X}$ such that $\tilde{f}(0) = \widetilde{x_0}$.*

Proof. Note that f can be viewed as a homotopy $f : Y \times I \to X$ where Y consists of a single point. Thus, as Y is connected, by Proposition 2.2.28 there exists a unique lift $\tilde{f} : Y \times I \to \tilde{X}$. In particular, $\tilde{f} : I \to \tilde{X}$ is just a path with $p\tilde{f} = f$ so that $\tilde{f}(0) = p^{-1}f(0) \in p^{-1}(x_0)$. ☐

Theorem 2.2.31. *Let $x_0 = (1, 0) \in \mathrm{S}^1$. Then $\pi_1\left(\mathrm{S}^1, x_0\right)$ is the infinite cycle group generated by the homotopy class of the loop $\omega : I \to \mathrm{S}^1$ where*

$$\omega(s) = (\cos(2\pi s), \sin(2\pi s)).$$

Proof. Let $f : I \to \mathrm{S}^1$ be a loop at x_0. Then Proposition 2.2.28 implies that there exists a lift $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$. Since by the construction of the lift $p\tilde{f}(1) = f(1) = x_0$, where p is as given by Example 2.2.26. Thus it follows that $\tilde{f}(1) = p^{-1}(x_0) = n$, for some $n \in \mathbb{Z}$. Observe that $\widetilde{\omega_n} : I \to \mathbb{R}$, as given in Example 2.2.26 is another path such that $\widetilde{\omega_n}(0) = 0$ and $\widetilde{\omega_n}(1) = n$. As $\mathbb{R}$ is connected it follows by Proposition 2.2.27 that $\tilde{f} \simeq \widetilde{\omega_n}$. In particular, let $\tilde{f} \simeq \widetilde{\omega_n}$ through $F : I \times I \to \mathbb{R}$, then $pF : I \times I \to \mathrm{S}^1$ is a homotopy between $p\tilde{f} = f$ and $p\widetilde{\omega_n} = \omega_n$, thus $f \simeq \omega_n$. Now let $m, n \in \mathbb{Z}$ and suppose $\omega_m \simeq \omega_n$ through $F : I \times I \to \mathrm{S}^1$. By Proposition 2.2.27 we know that $\widetilde{\omega_n}, \widetilde{\omega_m} : I \to \mathbb{R}$ are the unique lifts of ω_n and ω_m respectively such that $\widetilde{\omega_n}(0) = \widetilde{\omega_m}(0) = 0$. Recall, from Example 2.2.26 that $\widetilde{\omega_n}(1) = n$ and $\widetilde{\omega_n}(1) = m$. Moreover, by Proposition 2.2.28 we know F lifts uniquely to a homotopy $\tilde{F} : I \times I \to \mathbb{R}$ between $\widetilde{\omega_n}$ and $\widetilde{\omega_m}$. As $\tilde{F}(0, 1) = n \in \mathbb{Z}$ and $\tilde{F}$ is continuous it follows that $\tilde{F}(s, 1)$ is a constant function. Therefore,

$$n = \widetilde{\omega_n}(1) = \tilde{F}(0, 1) = \tilde{F}(1, 1) = \widetilde{\omega_m}(1) = m.$$

Thus, $f \simeq \omega_n$ for a unique $n \in \mathbb{Z}$. In particular, $[f] = [\omega_n] = [\omega]^n$ by observation 1. of Example 2.2.26, and so we conclude that ω generates $\pi_1\left(\mathrm{S}^1, x_0\right)$. ☐

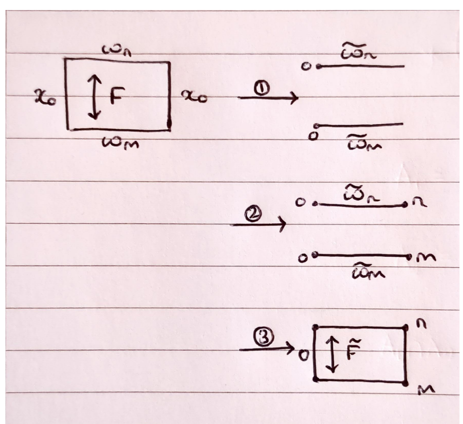


Figure 2.2.8: An illustration of how $\widetilde{\omega_n} \simeq \widetilde{\omega_m}$ implies that $m = n$. Where (1) is an application of Proposition 2.2.27, (2) is using Example 2.2.26 and (3) is an application of Proposition 2.2.28.

Corollary 2.2.32. *The fundamental group of $\pi_1\left(\mathrm{S}^1\right)$ is isomorphic to $(\mathbb{Z}, +)$.*

Proof. As S^1 is path connected we let $\pi_1\left(\mathrm{S}^1\right) = \pi_1\left(\mathrm{S}^1, x_0\right)$ for $x_0 = (1, 0)$ without any loss of generality. For any $[\gamma] \in \pi_1\left(\mathrm{S}^1\right)$, by Theorem 2.2.31 one can write $[\gamma] = [\omega]^n$ uniquely. Meaning $\Phi : \pi_1\left(\mathrm{S}^1\right) \to \mathbb{Z}$ given by $[\gamma] \mapsto n$

where n is such that $[\gamma] = [\omega]^n$ is well-defined. Moreover, it is surjective and injective as if $\Phi([\gamma]) = \Phi([\phi])$ then $[\gamma] = [\omega]^n = [\phi]$. Furthermore, it is a group homomorphism as

$$\Phi([\gamma][\phi]) = \Phi\left([\omega]^n[\omega]^m\right) = \Phi\left([\omega]^{n+m}\right) = n + m = \Phi([\gamma]) + \Phi([\phi]).$$

Therefore, Φ is a group isomorphism which implies that $\pi_1\left(\mathrm{S}^1\right) \cong \mathbb{Z}$. □

Theorem 2.2.33. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proof. Without loss of generality suppose that $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Assume p has no roots in $\mathbb{C}$, then for $r > 0$ we obtain a loop $f_r : I \to \mathbb{C}$ given by

$$s \mapsto \frac{\left(\frac{p\left(re^{2\pi i s}\right)}{p(r)}\right)}{\left|\frac{p\left(re^{2\pi i s}\right)}{p(r)}\right|},$$

such that $|f_r(s)| = 1$ for every $s \in I$. In particular, as $f_r(0) = f_r(1) = 1$, the curve f_r is a loop with base point 1. Moreover, f_r depends continuously on r and thus $f_r \simeq f_0$ for every $r > 0$ where f_0 is just the constant loop at 1. Therefore, $[f_r] = [f_0] = 0 \in \pi_1\left(\mathrm{S}^1\right)$. Now fix $r > 0$ such that

$$r > \min\left(1, |a_1| + \cdots + |a_n|\right).$$

Observe that for $|z| = r$ we have

$$\begin{aligned} |z^n| &\geq (|a_1| + \cdots + |a_n|)\left|z^{n-1}\right| \\ &\geq \left|a_1 z^{n-1}\right| + \cdots + |a_n| \\ &\geq \left|a_1 z^{n-1} + \cdots + a_n\right|. \end{aligned}$$

Hence, for $0 \leq t \leq 1$, the polynomial

$$p_t(z) = z^n + t\left(a_1 z^{n-1} + \cdots + a_n\right)$$

has no root z with $|z| = r$. Let

$$F_r(t, s) = \frac{\left(\frac{p_t\left(re^{2\pi i s}\right)}{p_t(r)}\right)}{\left|\frac{p_t\left(re^{2\pi i s}\right)}{p_t(r)}\right|}$$

such that $F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$. Therefore, $[\omega_n] = [f_r]$, however, $[f_r] = 0$. Thus, as $[\omega_n] = [\omega]^n$ it follows from Corollary 2.2.32 that $n = 0$ which means that p is constant. □

Proposition 2.2.34. *Let X and Y be path-connected topological spaces. For $x \in X$ and $y \in Y$ we have*

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Proof. Observe that $\gamma : I \to X \times Y$ given by $s \mapsto (\delta(s), \eta(s))$ is continuous if and only if $\delta : I \to X$ and $\eta : I \to Y$ are continuous. Therefore, for a path $\gamma : I \to X \times Y$ with base point (x, y) we obtain paths $\delta : I \to X$ and $\eta : I \to Y$ with base points x and y respectively. Conversely, for paths $\delta : I \to X$ and $\eta : I \to Y$ with base points x and y respectively, we obtain a path $\gamma : I \to X \times Y$, given by $s \mapsto (\delta(s), \eta(s))$, with base point (x, y). In particular, $\gamma_1, \gamma_2 : I \to X \times Y$, where $\gamma_i(s) = (\delta_i(s), \eta_i(s))$ for $i = 1, 2$, are homotopic if and only if $\delta_1 \simeq \delta_2$ and $\eta_1 \simeq \eta_2$. Therefore, $\Phi : \pi(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_2(Y, y)$ given by $[\gamma] \mapsto ([\delta], [\eta])$ is well-defined and in particular bijective. Moreover,

$$\begin{aligned} \Phi([\gamma_1 \cdot \gamma_2]) &= ([\delta_1 \cdot \delta_2], [\eta_1 \cdot \eta_2]) \\ &= ([\delta_1][\delta_2], [\eta_1][\eta_2]) \\ &= ([\delta_1], [\eta_1])([\delta_2], [\eta_2]) \\ &= \Phi([\gamma_1])\Phi([\gamma_2]) \end{aligned}$$

with $\Phi([c_{(x,y)}]) = ([c_x],[c_y])$, and so Φ is a group isomorphism. It follows that

$$\pi_1(X \times Y,(x,y)) \cong \pi_1(X,x) \times \pi_1(Y,y).$$

□

Example 2.2.35. *Using Proposition 2.2.34 it follows that the torus $X = \mathrm{S}^1 \times \mathrm{S}^1$ has fundamental group*

$$\pi_1(X) = \pi_1(\mathrm{S}^1) \times \pi_1(\mathrm{S}^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

2.2.3 Induced Homomorphisms

Throughout, maps will be continuous unless stated otherwise. Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map with the property that $f(x_0) = y_0$. A map $\phi : (X, x_0) \to (Y, y_0)$ induces a map $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where

$$\phi_*([f]) = [\phi f].$$

Exercise 2.2.36. *For a continuous map $\phi : X \to Y$ show that $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is well-defined and a homomorphism.*

Proposition 2.2.37.

1. *For topological spaces X, Y and Z let $\psi : X \to Y$ and $\phi : Y \to Z$. Then $(\phi\psi)_* = \phi_*\psi_*$.*
2. *Let X be a topological space, then $\mathrm{id}_X : X \to X$ induces the identity map on the fundamental group $\pi_1(X, x)$.*

Proof.

1. Let $f : I \to X$ be a loop with base point x, then
$$\begin{aligned}(\phi\psi)_*([f]) &= [(\phi\psi)f] \\ &= [\phi(\psi f)] \\ &= \phi_*([\psi f]) \\ &= \phi_*\psi_*([f]).\end{aligned}$$
2. Let $f : I \to X$ be a loop with base point x, then
$$(\mathrm{id}_X)_*(f) = [\mathrm{id}_X f] = [f],$$
thus, $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$.

□

Corollary 2.2.38. *Let $\phi : X \to Y$ be a homeomorphism. Then $\phi_* : \pi_1(X, x) \to \pi_1(Y, \phi(x))$ is an isomorphism.*

Proof. Let $\psi : Y \to X$ be the inverse of ϕ. Then $\phi\psi = \mathrm{id}_X$, and so using statement 2. of Proposition 2.2.37 it follows that $(\phi\psi)_* = \mathrm{id}_{\pi_1(X,x)}$. Using statement 1. of Proposition 2.2.37 we know that $(\phi\psi)_* = \phi_*\psi_*$. Thus, ψ_* is a group homomorphism that is inverse to ϕ_*, which means that ϕ_*, and ψ_*, is an isomorphism. □

Lemma 2.2.39. *Let $F_t : X \to Y$ be a homotopy and consider $x \in X$. Let $h : I \to Y$ be the path between*

$F_0(x)$ and $F_1(x)$ given by $s \mapsto F_s(x)$. Then $(F_0)_* = \beta_h\,(F_1)_*$, where β_h is as given by Exercise 2.2.15.

$$\begin{tikzcd}
 & \pi_1(Y, F_1(x)) \arrow[dd, "\beta_h"] \\
\pi_1(X, x) \arrow[ur, "(F_1)_*"] \arrow[dr, "(F_0)_*"'] & \\
 & \pi_1(Y, F_0(x))
\end{tikzcd}$$

Proof. For $t \in I$ let $h_t : I \to X$ be the path between $F_0(x)$ and $F_t(x)$ given by $s \mapsto h(ts)$. Then for $f : I \to X$ a loop with base point x let

$$H_t := h_t \cdot (F_t f) \cdot h_t^{-1}.$$

Note that H_t is a loop at $F_0(x)$, and continuous in t. Thus H_t is a homotopy between

$$H_0 = h_0 \cdot (F_0 f) \cdot h_0^{-1} \simeq F_0 f$$

and

$$H_1 = h_1 \cdot (F_1 f) \cdot h_1^{-1} = h \cdot (F_1 f) \cdot h^{-1}.$$

Therefore,

$$\begin{aligned}
(F_0)_*([f]) &= [F_0 f] \\
&= \left[h \cdot (F_1 f) \cdot h^{-1}\right] \\
&= \beta_h([F_1 f]) \\
&= \beta_h\,(F_1)_*([f]).
\end{aligned}$$

□

Corollary 2.2.40. *Let $\phi : X \to Y$ be a homotopy equivalence. Then $\phi_* : \pi_1(X, x) \to \pi_1(Y, \phi(x))$ is an isomorphism for every $x \in X$.*

Proof. Let $\phi : X \to Y$ be a homotopy equivalence with $\psi : Y \to X$ such that $\phi\psi \simeq \mathrm{id}_Y$ and $\psi\phi \simeq \mathrm{id}_X$. From Lemma 2.2.39 there exists a path $h : I \to X$ between x and $(\psi\phi)(x)$ such that $(\psi\phi)_* = \beta_h(\mathrm{id}_X)_* = \beta_h$. Hence, using Proposition 2.2.17 and statement 1 of Proposition 2.2.37 it follows that $(\psi\phi)_* = \psi_*\phi_*$ is a group isomorphism. In particular, $\psi_*\phi_*$ is bijective so ϕ_* is injective and ψ_* is surjective. Similarly, $(\phi\psi)_* = \phi_*\psi_*$ is an so ψ_* is injective and ϕ_* is surjective. Therefore, ϕ_* is bijective and thus a group isomorphism. □

Remark 2.2.41. *Suppose $A \subseteq X$ is a deformation retract of X. Then as a deformation retract is a homotopy equivalence, it follows that $\pi_1(X) \cong \pi_1(A)$.*

Example 2.2.42. *Suppose that $r : \mathrm{D}^2 \to \partial\mathrm{D}^2 = \mathrm{S}^1$ is a retraction and let $i : \mathrm{S}^1 \to \mathrm{D}^2$ be the inclusion map so that $ri = \mathrm{id}_{\mathrm{S}^1}$. Then on the one hand, using Lemma 2.2.39, we have*

$$\pi_1\left(\mathrm{S}^1\right) \xrightarrow{i_*} \pi_1\left(\mathrm{D}^2\right) \xrightarrow{r_*} \pi_1\left(\mathrm{S}^1\right).$$

Therefore, as $\pi_1\left(\mathrm{D}^2\right) = 0$ by Example 2.2.14 it follows that $r_*i_*\left(\pi_1\left(\mathrm{S}^1\right)\right) = 0$. However, as $ri = \mathrm{id}_{\mathrm{S}^1}$ it is also the case that $r_*i_* = (ri)_* = \mathrm{id}_{\pi_1(\mathrm{S}^1)}$ by statement 2 of Proposition 2.2.37. Which implies that $r_*i_*\left(\pi_1\left(\mathrm{S}^1\right)\right) = \mathbb{Z}$ as $\pi_1\left(\mathrm{S}^1\right) = \mathbb{Z}$, and thus we arrive at a contradiction. Therefore, D^2 does not retract onto S^1.

Theorem 2.2.43 (Brouwer Fixed Point Theorem). *Let $h : \mathrm{D}^2 \to \mathrm{D}^2$ be a continuous map. Then there exists an $x \in \mathrm{D}^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in \mathrm{D}^2$. Let $r : \mathrm{D}^2 \to \mathrm{S}^1$ where $r(x)$ is the intersection between S^1 and the ray propagating from $h(x)$ towards x. In particular, let $g : \mathrm{D}^2 \to \left(\mathrm{D}^2 \times \mathrm{D}^2\right) \setminus \left\{(x,x) \in \mathrm{D}^2\right\}$ be given by

$$g(x) = (x, f(x)).$$

Note that g is continuous as f is continuous. For $(x, f) = ((x_1, x_2), (f_1, f_2)) \in \left(\mathrm{D}^2 \times \mathrm{D}^2\right) \setminus \left\{(x,x) \in \mathrm{D}^2\right\}$ consider

$$\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} := \begin{pmatrix} x_1 - f_1 \\ x_2 - f_2 \end{pmatrix} t + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Then observe that $h_1(t)^2 + h_2(t)^2 = 1$ can be written as a quadratic in t of the form $at^2 + bt + c = 0$. More specifically,

$$\begin{cases} a = (x_1 - f_1)^2 + (x_2 - f_2)^2 \\ b = 2\left(f_1(x_1 - f_1) + f_2(x_2 - f_2)\right) \\ c = f_1^2 + f_2^2. \end{cases}$$

In particular, note that $a \neq 0$ as $(x_1, x_2) \neq (f_1, f_2)$. Moreover, one can check that

$$b^2 - 4ac = 4\left(\left(f_1(x_2 - f_2) + f_2(x_1 - f_1)\right)^2 + (x_1 - f_1)^2 + (x_2 - f_2)^2\right) > 0.$$

Therefore, $h_1(t)^2 + h_2(t)^2 = 1$ has distinct solutions, say $t_+ > t_-$. Now consider $h : \left(\mathrm{D}^2 \times \mathrm{D}^2\right) \setminus \left\{(x,x) \in \mathrm{D}^2\right\} \to \partial \mathrm{D}^2$ given by

$$(x, f) = ((x_1, x_2), (f_1, f_2)) \mapsto (h_1(t_+), h_2(t_+)).$$

As $h_1(t_+)$ and $h_2(t_+)$ are polynomial in x_1, x_2, f_1 and f_2 it follows that h is continuous. Observe that $r(x) = h(g(x))$ and so as h and g are continuous it follows that r is continuous. Furthermore, $r(x) = x$ if $x \in \mathrm{S}^1$ which means that r is a retraction. This contradicts Example 2.2.42, thus there exists a $x \in \mathrm{D}^2$ such that $h(x) = x$. $\square$

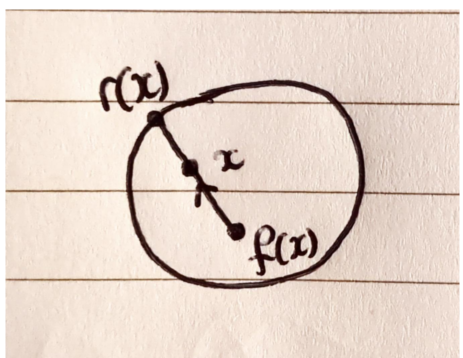


Figure 2.2.9: The construction of the retraction $r : \mathrm{D}^2 \to \mathrm{S}^1$ consider in the proof of Theorem 2.2.43.

Lemma 2.2.44. *For a topological space X and $x_0 \in X$ assume that*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha$$

such that

- A_α *is open and path-connected,*
- $x_0 \in A_\alpha$ *for all* $\alpha \in \Lambda$, *and*

- $A_\alpha \cap A_\beta$ *is path-connected for all* $\alpha, \beta \in \Lambda$.

Then if f *is a loop in* X *at* x_0 *it follows that*

$$[f] = [h_1] \dots [h_m]$$

where the h_i *are loops at* x_0 *and are contained in a single* A_α.

Proof. As f is continuous, for all $s \in I$ there is an open neighbourhood V_s of s such that $f(V_s) \subseteq A_\alpha$ for some $\alpha \in \Lambda$. One can choose V_s such that $V_s = (a_s, b_s)$ and $f([a_s, b_s]) \subseteq A_\alpha$. Therefore, as I is compact, there exists a finite partition

$$0 = s_0 < \dots < s_m = 1$$

with the property that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some $\alpha_i \in \Lambda$. Let $f_i : I \to X$ be the re-scaled path of $f|_{[s_{i-1}, s_i]}$ such that $f \simeq f_1 \cdot \ldots \cdot f_m$. As $A_{\alpha_i} \cap A_{\alpha_{i+1}}$ is path-connected, and $x, f(s_i) \in A_{\alpha_i} \cap A_{\alpha_{i+1}}$, there exists a path g_i from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. With g_0 and g_m being the constant loops at x_0 let $h_i := g_{i-1} \cdot f_i \cdot g_i^{-1}$ such that h_i is a loop with base point x and $h_i(I) \subseteq A_{\alpha_i}$. It follows that

$$f \simeq \left(g_0 \cdot f_1 \cdot g_1^{-1}\right) \cdot \ldots \cdot \left(g_{m-1} \cdot f_m \cdot g_m^{-1}\right)$$

which implies that $[f] = [h_1] \dots [h_m]$.

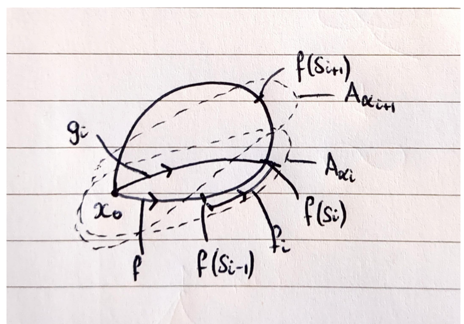


Figure 2.2.10

$\square$

Remark 2.2.45. *Suppose that* $U_1, U_2 \subseteq X$ *are open and path connected such that* $U_1 \cup U_2 = X$ *with* $U_1 \cap U_2$ *path connected. Then for* $x \in U_1 \cap U_2$, *using Lemma 2.2.44, every* $[f] \in \pi_1(X, x)$ *can be written as*

$$[f] = [g_1][h_1] \dots [g_n][h_n]$$

where g_i *is a loop with base point* x *contained in* U_1 *and* h_i *is a loop with base point* x *contained in* U_2. *In particular, with* $i_1 : U_1 \to X$ *and* $i_2 : U_2 \to X$ *the inclusion maps, it follows that* $(i_1)_* (\pi_1(U_1, x)) \cup (i_2)_* (\pi_1(U_2, x))$ *generates* $\pi_1(X, x)$.

Theorem 2.2.46. *For* $n \geq 2$ *we have that* $\pi_1(\mathrm{S}^n) = 0$.

Proof. Let

$$U_1 := \mathrm{S}^n \setminus \{(1, 0, \dots, 0)\}$$

and

$$U_2 := \mathrm{S}^n \setminus \{(-1, 0, \dots, 0)\}.$$

Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$ through stereographic projections. Moreover, $U_1 \cup U_2 = \mathrm{S}^n$ and $U_1 \cap U_2$ is path connected. Let $x \in U_1 \cap U_2$, then as $\pi_1(U_1, x) = \pi_1(U_2, x) = 0$, by Example 2.2.14, it follows by Remark 2.2.45 that $\pi_1(\mathrm{S}^n, x) = 0$ Therefore, as S^n is path-connected we deduce that $\pi_1(\mathrm{S}^n) = 0$. $\square$

2.3 Seifert-van Kampen

The Seifert-van Kampen theorem is a way to compute the fundamental groups of spaces that can be decomposed into simpler spaces.

Example 2.3.1. *Consider the space X consisting of circles A and B that intersect at a single point x. We know that $\pi_1(A)$ and $\pi_1(B)$ are isomorphic to $\mathbb{Z}$.*

- *Let a^n denote n loops of A in one direction.*
- *Let a^{-n} denote n loops of A in the opposite direction.*
- *Let a^0 denote no loops of A.*

We adopt a similar notation for loops of B. Consequently, entries of $\pi_1(X)$ are of the form $a^{n_1}b^{n_2}a^{n_3}$ say, which is the loop that traverses A with n_1 loops, then traverses B with n_2 and then traverses A with n_3 times. We refer to such representations as words and, as expected, words form a group.

- *Words are multiplied by concatenating them and performing any simplifications at the point of joining.*
- *Inverses are formed by changing the sign of each exponent and reversing the ordering of the symbols.*
- *The identity is the empty word.*

The group of words is written $\pi_1(A) * \pi_1(B) = \mathbb{Z} * \mathbb{Z}$ and referred to as the free product. The Seifert-van Kampen theorem will tell us that for a space X its fundamental group is some free product involving the fundamental groups of the space's components.

2.3.1 Free Groups and Free Products

Let S be a set of symbols. Then $S^{-1} := \left\{s^{-1} : s \in S\right\}$ is a corresponding set of symbols where the elements will act as an inverse to a symbol in S. Let $A(S)$ denote the set of all words formed by $S \cup S^{-1}$. That is,

$$A(S) := \left\{s_1^{n_1} \ldots s_k^{n_k} : s_i \in S \cup S^{-1},\, n_i \in \mathbb{Z},\ \text{for some } k \in \mathbb{N}\right\}.$$

Note that included in $A(S)$ is the empty word, e. The empty word has the property that $ss^{-1} = s^{-1}s = e$, and $es = se$ for any $s \in S$. A word $s_1^{n_1} \ldots s_k^{n_k} \in A(S)$ is reduced if $s_{j+1} \neq s_j^{-1}$ and $s_{j+1}^{-1} \neq s_j$ for any $j = 1, \ldots, k$. Let $R(S)$ denote the set of reduced words.

Definition 2.3.2. *For a set of symbols S, the free group generated by F_S is the set $R(S)$ with the group law that concatenates words and performs any cancellations to arrive at a reduced word.*

Exercise 2.3.3. *Verify that F_S, as given by Definition 2.3.2, is a group.*

A free group generated by a set S can equivalently be given with a universal property definition.

Definition 2.3.4. *Let S be a set. The free group F_S generated by S is such that for any group G and function $\varphi : S \to G$ there exists a unique group homomorphism $\tilde{\varphi} : F_S \to G$ which extends φ.*

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, hook] & G \\
F \arrow[ur, "\tilde{\varphi}"] &
\end{tikzcd}$$

Definition 2.3.5. *Let S be a set and $R \subseteq R(S)$. Then let*

$$\langle S|R\rangle = F_S/\langle\langle R\rangle\rangle,$$

where $\langle\langle R\rangle\rangle$ denotes the normal closure of R in F_S. If G is a group and $G \cong \langle S|R\rangle$ then $\langle S|R\rangle$ is referred to as a presentation of G. In such a case, R is referred to as the relation.

Definition 2.3.6. *A group G is finitely presented if there exists a finite set S, with $R \subseteq R(S)$ such that $G \cong \langle S|R\rangle$.*

Theorem 2.3.7. *Let G be a group. Then there exists a set S and $R \subseteq R(S)$ such that $G \cong \langle S|R\rangle$.*

Proof. Let G be a group, and let F_G be the free group generated by G. Then by Definition 2.3.4 there exists a unique homomorphism $\varphi : F_G \to G$ such that $\varphi|_G = \mathrm{id}_G$. In particular, φ is surjective as id_G is surjective. Moreover, $\ker(\varphi) \subseteq F_G$ is a normal subgroup and so by the first isomorphism theorem, it follows that

$$\langle G|\ker(\varphi)\rangle = F_G/\langle\langle\ker(\varphi)\rangle\rangle = F_G/\ker(\varphi) \cong \mathrm{im}(\varphi) = G.$$

Therefore, $\langle G|\ker(\varphi)\rangle$ is a presentation for G. $\square$

Example 2.3.8. *The dihedral group D_n is presented by $\langle S|R\rangle$ where $S = \{r, s\}$ and $R = \left\{r^n, s^2, rsrs^{-1}\right\}$.*

Definition 2.3.9. *Let (G_α) be a collection of disjoint groups. Then the free product of (G_α) is*

$$\underset{\alpha}{*} G_\alpha := \{g_1 \dots g_m : g_i \in G_{\alpha_i} \text{ not the identity, and } \alpha_i \neq \alpha_{i+1}\}.$$

On G_{α_i} multiplication is given by

$$(g_1 \dots g_m)(h_1 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$$

with any cancellations made.

Remark 2.3.10.

1. *The groups (G_α) in Definition 2.3.9 are disjoint in the sense of their symbols, rather than their structure.*
2. *Each group in the collection (G_α) can be identified with a subgroup of the free product. With only the empty word being common to each of these subgroups which are otherwise disjoint.*
3. *A collection of homomorphisms $\phi_\alpha : G_\alpha \to H$ extends uniquely to a homomorphism $\phi : \underset{\alpha}{*} G_\alpha \to H$ by*

$$\phi(g_1 \dots g_m) = \phi_{\alpha_1}(g_1) \dots \phi_{\alpha_m}(g_m).$$

Exercise 2.3.11. *Show that $\underset{\alpha}{*} G_\alpha$, as given by Definition 2.3.9, is a group with the inverse of $g_1 \dots g_m \in \underset{\alpha}{*} G_\alpha$ given by $g_m^{-1} \dots g_1^{-1}$ and the identity being the empty word.*

The amalgamated product is a generalisation of Definition 2.3.9 that deals with collections of groups that have overlapping symbols.

Definition 2.3.12. *Let G_0, G_1 and G_2 be groups, with $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ homomorphisms. A group H with homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ such that $h_1 f_1 = h_2 f_2$ is an amalgamated product of G_1 and G_2 over G_0 if it satisfies the property that for every group G and homomorphisms $h_i' : G_i \to G$, for $i = 1, 2$, with $h_1' f_1 = h_2' f_2$, there exists a unique homomorphism $\alpha : H \to G$ such that $h_1' = \alpha h_2$ and $h_2' = \alpha h_2$.*

$$\begin{tikzcd}
G_0 \arrow[r, "f_1"] \arrow[d, "f_2"] & G_1 \arrow[d, "h_1"] \arrow[ddr, bend left, "h_1'"] & \\
G_2 \arrow[r, "h_2"] \arrow[drr, bend right, "h_2'"] & H \arrow[dr, dashed, "\exists!\alpha"] & \\
 & & G
\end{tikzcd}$$

Theorem 2.3.13. *Given $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$, there exists a unique amalgamated product, up to isomorphism, which we denote $G_1 \underset{G_0}{*} G_2$.*

Remark 2.3.14. *For groups G_1 and G_2, when $G_1 = \{\mathrm{id}\}$ we write $G_1 \underset{G_0}{*} G_2 = G_1 * G_2$.*

Proposition 2.3.15. *For groups G_0, G_1 and G_2 with homomorphisms $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ we have*

$$G_1 \underset{G_0}{*} G_2 \cong (G_1 * G_2)/N,$$

where N is the normal closure of

$$\left\{f_1(g) f_2(g)^{-1}\right\} \subseteq G_1 * G_2.$$

Corollary 2.3.16. *Let $G_1 = \langle S_1 | R_1 \rangle$ and $G_2 = \langle S_2 | R_2 \rangle$, then*

$$G_1 * G_2 = \langle S_1 \cup S_2 | R_1 \cup R_2 \rangle.$$

2.3.2 The Seifert-van Kampen Theorem

Theorem 2.3.17 (Seifert-van Kampen)**.** *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected. With $x \in U_1 \cap U_2$ we have that*

$$\pi_1(X, x) \cong \pi_1(U_1, x) \underset{\pi_1(U_1 \cap U_2, x)}{*} \pi_2(U_2, x) \cong (\pi_1(U_1, x) * \pi_1(U_2, x))/N,$$

where N is the normal closure of

$$\left\{(i_1)_*(\omega)\left((i_2)_*(\omega)\right)^{-1} : \omega \in \pi_1(U_1 \cap U_2, x)\right\}$$

with $i_k : U_1 \cap U_2 \to U_k$ and $j_k : U_k \to X$ the inclusion maps for $k = 1, 2$. That is,

$$\begin{tikzcd}
U_1 \cap U_2 \arrow[r, "i_1"] \arrow[d, "i_2"] & U_1 \arrow[d, "j_1"] \\
U_2 \arrow[r, "j_2"] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_1(U_1 \cap U_2, x) \arrow[rr, "(i_1)_*"] \arrow[d, "(i_2)_*"] & & \pi_1(U_1, x) \arrow[d, "(j_1)_*"] \\
\pi_2(U_2, x) \arrow[r, "(j_2)_*"] & \pi_1(U_1, x) & \underset{\pi_1(U_1 \cap U_2, x)}{*} \pi_2(U_2, x)
\end{tikzcd}$$

Proof. Let $\Phi : \pi_1(U_1, x) * \pi_1(U_2, x) \to \pi_1(X, x)$ be the homomorphism induced by the inclusion map. Then by Remark 2.2.45 the map Φ is surjective. Moreover, if $[\gamma] \in N$, then

$$\begin{aligned}
\Phi([\gamma]) &= \Phi\left((i_1)_*(\omega)\left((i_2)_*(\omega)\right)^{-1}\right) \\
&= [\omega] \cdot [\omega]^{-1} \\
&= [c_x],
\end{aligned}$$

and so $N \subseteq \ker(\Phi)$. A factorisation for $[f] \in \pi_1(X, x)$ is a formal product $[f_1] \dots [f_k]$ such that the following hold.

- Each f_i is a loop at x in one of the U_i, and $[f_i] \in \pi_1(U_j, x)$ is its corresponding homotopy class.
- The loop $f_1 \dots f_k$ is homotopic to f in X.

Note that a factorisation of $[f]$ is a word, possibly not reduced, in $\pi_1(U_1, x) * \pi_1(U_2, x)$ that is mapped to $[f]$ under Φ. Factorisations of $[f]$ are equivalent if they are related by a finite sequence of the following operations.

1. Combine adjacent terms $[f_i][f_{i+1}]$ into a single term $[f_i \cdot f_{i+1}]$ if $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(U_j, x)$.
2. Regard the term $[f_i] \in \pi_1(U_j, x)$ as a term of $\pi_1(U_{j'}, x)$ if f_i is a loop in $U_1 \cap U_2$.

Notice how operation 1 does not change the factorisation as an element of $\pi_1(U_1, x) * \pi_1(U_2, x)$. Notice how operation 2 does not change the factorisation as an element of $(\pi_1(U_1, x) * \pi_1(U_2, x))/N$, by how N is constructed. Therefore, equivalent factorisations correspond to the same element of $(\pi_1(U_1, x) * \pi_1(U_2, x))/N$. Consequently, showing that factorisations of $[f]$ are equivalent implies that the map $(\pi_1(U_1, x) * \pi_1(U_2, x))/N \to \pi_1(X, x)$ is injective, and thus its kernel is exactly N. Hence, let $[f_1] \dots [f_k]$ and $[f'_1] \dots [f'_l]$ be factorisations of $[f]$. Then $f_1 \cdot \dots \cdot f_k$ and $f'_1 \cdot \dots \cdot f'_l$ are homotopic, say through the homotopy $F : I \times I \to X$. By the compactness of $I \times I$ there exists partitions

$$0 < s_0 < \dots < s_m = 1$$

and

$$0 = t_0 < \dots < t_n = 1$$

such that for $R_{ij} := [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ we have $F(R_{ij}) \subseteq U_1$ or $F(R_{ij}) \subseteq U_2$. More specifically, for each $(s, t) \in I \times I$, one can consider its image $F(s, t) \in X$. As U_1 and U_2 cover X, there exists an open neighbourhood of $F(s, t)$ that is contained in either U_1 or U_2. Taking the pre-image of this open neighbourhood generates an open set in $I \times I$. Doing this for all $(s, t) \in I \times I$ one constructs an open cover for $I \times I$. Therefore, using the compactness of $I \times I$ we can reduce this open cover to a finite sub-cover. We can then use the corresponding points of the open sets to construct our partitions. Now with these partitions, we can assume that the partition $\{s_0, \dots, s_m\}$ subdivides the partitions giving the products $f_1 \cdot \dots \cdot f_k$ and $f'_1 \cdot \dots \cdot f'_l$. Now relabel the R_{ij} to $R_1, \dots, R_{mn}$ as in Figure 2.3.1.

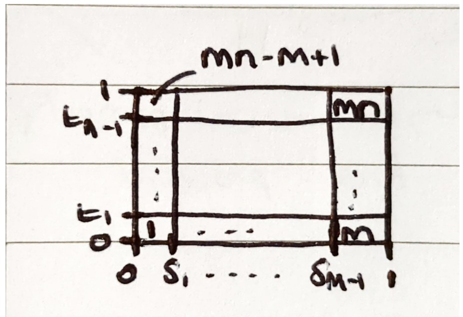


Figure 2.3.1

Then a path γ in $I \times I$ from left to right gives a loop $F|_\gamma$ in X at x. Let γ_r be the path separating the first r rectangles from the others such that

$$F|\gamma_0 \simeq f_1 \cdot \ldots \cdot f_k$$

and

$$F|_{\gamma_{mn}} \simeq f_1' \cdot \ldots \cdot f_l'.$$

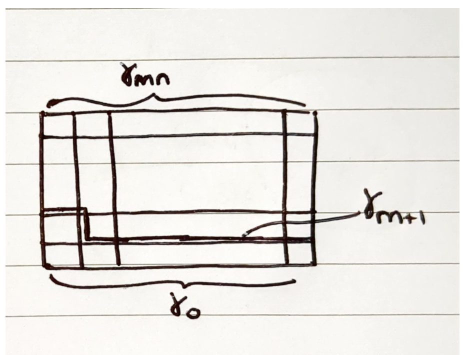


Figure 2.3.2

Let v be a grid point. Let g_v be a path in X from x to $F(v)$ such that g_v is contained in $U_1 \cap U_2$ if $F(v) \in U_1 \cap U_2$, and in a single U_i otherwise. Then this gives a factorisation of $[F|_{\gamma_r}]$ into loops only contained in U_1 or U_2. Thus, the factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F|_{\gamma_r}$ and $F|_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_j. Doing this iteratively from γ_0 it follows that $[F|_{\gamma_0}] = [F|_{\gamma_{mn}}]$ which implies that factorisations of f are equivalent. $\square$

Theorem 2.3.18 (Seifert-van Kampen, strong version)**.** *Let X be a path-connected topological space with the following decomposition.*

- $X = \bigcup_\alpha A_\alpha$.
- A_α, $A_\alpha \cap A_\beta$ *and* $A_\alpha \cap A_\beta \cap A_\gamma$ *open and path-connected for all* α, β, *and* γ.
- $x \in \bigcap_\alpha A_\alpha$.

Then

$$\pi_1(X, x) \cong \underset{\alpha}{*} \pi_1(A_\alpha, x)/N$$

where N is the normal closure of

$$\left\{(i_{\alpha\beta})_*(\omega)\left((i_{\beta\alpha})_*(\omega)\right)^{-1} : \omega \in \pi_1(A_\alpha \cap A_\beta)\right\} \subseteq \mathop{*}_{\alpha} \pi_1(A_\alpha, x_0)$$

with $i_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$ the inclusion map.

Definition 2.3.19. *A pointed topological space is a topological space X, with a designated base point x, written (X, x).*

Definition 2.3.20. *For pointed topological spaces $(X_i, x_i)_{i\in I}$ the wedge sum of the spaces is*

$$\bigvee_{i\in I} X_i = \left(\bigsqcup_{i\in I} X_i\right) / \sim$$

where $x_i \sim x_j$ for every $i, j \in I$.

Example 2.3.21. *Consider the pointed topological spaces $\left(\mathrm{S}^1, x_0\right)$ and $\left(\mathrm{S}^1, x_1\right)$. Then graphically $X = \mathrm{S}^1 \vee \mathrm{S}^1$ is represented by two circles attached at a point, henceforth we will refer to this point as x. Let A_1 be the first circle with a semi-circular region of the second circle which includes x. Similarly, let A_2 be the second circle with a semi-circular region of the first circle which includes x.*

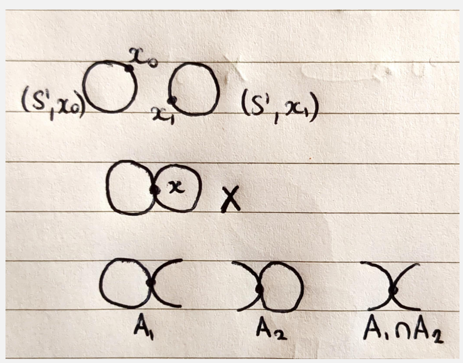


Figure 2.3.3: Graphical representation of the wedge sum of unit circles and the subsets A_1 and A_2 referenced in Example 2.3.21.

If a portion of a loop with base point x enters the semi-circular region, it is intuitive that this segment of the loop is equivalent to the constant map at x. Thus, $\pi_1(A_1) \cong \pi_1\left(\mathrm{S}^1\right) \cong \mathbb{Z}$. Similarly, $\pi_1(A_2) \cong \mathbb{Z}$. Moreover, $\pi_1(A_1 \cap A_2)$ is trivial as any loop is equivalent to the constant map. As A_1, A_2 and $A_1 \cap A_2$ are path connected Theorem 2.3.17 implies that

$$\pi_1\left(\mathrm{S}^1 \vee \mathrm{S}^1\right) \cong \pi_1(A_1) * \pi_1(A_2) \cong \mathbb{Z} * \mathbb{Z} = F_{\{a,b\}}.$$

Through induction it follows that

$$\pi_1\left(\bigvee_{i=1}^{n} \mathrm{S}^1\right) \cong \mathbb{Z} * \cdots * \mathbb{Z} = F_{\{a_1,\ldots,a_n\}}.$$

More generally, one applies Theorem 2.3.18 to deduce that

$$\pi_1\left(\bigvee_{\alpha\in\Lambda} \mathrm{S}^1\right) = *_{\alpha\in\Lambda}\mathbb{Z} = F_{\alpha\in\Lambda}.$$

Example 2.3.22. *Let T be a torus and consider a point $x \in T$. Then let $\mathrm{D} \subseteq T$ be a closed disc containing x. Moreover, let $A_1 := T \setminus D$ and let A_2 be an open set containing D. Note that A_1 and A_2 are open and path connected.*

- *A_1 is homotopy equivalent to $\mathrm{S}^1 \vee \mathrm{S}^1$ and so $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$ by Example 2.3.21.*
- *A_2 is homeomorphic to D^2 and so $\pi_1(A_2)$ is trivial by Example 2.2.14.*
- *$A_1 \cap A_2$ is open and path connected with $x \in A_1 \cap A_2$. Moreover, $A_1 \cap A_2$ is homotopy equivalent to S^1 and so $\pi_1(A_1 \cap A_2) \cong \mathbb{Z}$.*

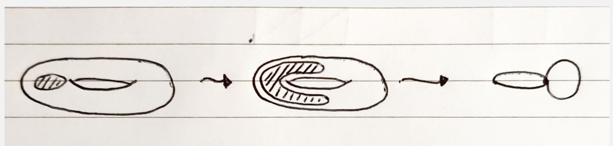


Figure 2.3.4: The torus and the identified components referenced in Example 2.3.22.

From Theorem 2.3.17 it follows that

$$\pi_1(T) \cong \pi_1(A_1) / \langle\langle i_*(\pi_1(A_1 \cap A_2))\rangle\rangle,$$

where $i : A_1 \cap A_2 \to T$ is the inclusion map. In particular, the inclusion map $i : A_1 \cap A_2 \to T$ induces a map $i_* : \pi_1(\mathrm{S}^1) \to \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)$. Viewing a loop in in S^1 as a loop in $\mathrm{S}^1 \vee \mathrm{S}^1$we have

$$i_*(\omega) = aba^{-1}b^{-1},$$

where ω generates $\pi_1(A_1 \cap A_2)$, and $\{a, b\}$ generate $\pi_1(A_1)$.

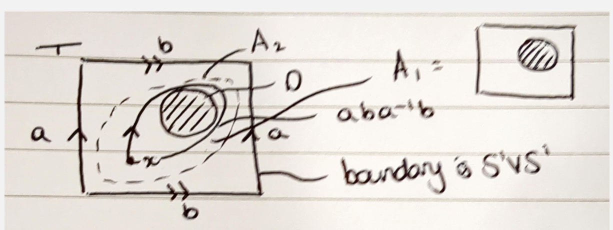


Figure 2.3.5: We can view loops in $A_1 \cap A_2$, which is equivalent to S^1, as loops on the boundary of A_1, which is equivalent to $\mathrm{S}^1 \vee \mathrm{S}^1$.

Hence,

$$\begin{aligned}\pi_1(T) \cong (\mathbb{Z} * \mathbb{Z}) / \langle\langle aba^{-1}b^{-1}\rangle\rangle &= F_{\{a,b\}} / \langle\langle aba^{-1}b^{-1}\rangle\rangle \\ &= \langle a, b | aba^{-1}b^{-1}\rangle \\ &\cong \mathbb{Z}^2.\end{aligned}$$

2.3.3 Application to CW-complexes

Consider a path-connected topological space X. Let Y be the space obtained by attaching 2-cells $\left(e_\alpha^2\right)$ to X along the maps $\phi_\alpha : \partial \mathrm{D}^2 = \mathrm{S}^1 \to X$. For each α let $\phi'_\alpha : I \to X$ be the loop

$$\phi'_\alpha(s) = \phi_\alpha(\cos 2\pi s, \sin 2\pi s),$$

which has base point $\phi'_\alpha(0)$. Fix $x \in X$ and then for each α let γ_α be a path from x to $\phi'_\alpha(0)$. Let $N \subseteq \pi(X, x)$ be the normal closure of the collection of equivalence classes $\left[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}\right]$. Note that N is independent of the path γ_α chosen for each α as suppose instead that η_α is chosen, then

$$\eta_\alpha \cdot \phi_\alpha \cdot \eta_\alpha^{-1} = \left(\eta_\alpha \cdot \gamma_\alpha^{-1}\right) \cdot \gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1} \left(\gamma_\alpha \cdot \eta_\alpha^{-1}\right).$$

Which means that $\eta_\alpha \cdot \phi_\alpha \cdot \eta_\alpha^{-1}$ and $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ are conjugate in $\pi_1(X, x_0)$. Note that $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is not necessarily null homotopic in X, however, it is certainly is null homotopic in Y. Therefore, N is contained in the kernel of the homomorphism $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion map $i : X \to Y$.

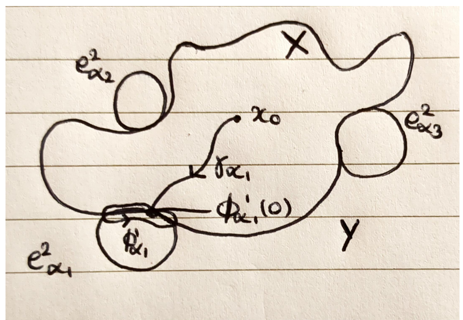


Figure 2.3.6: The attached 2-cells $\left(e_\alpha^2\right)$ to a path connected topological space Y, the illustrated loops that traverse the attached boundary of the 2-cells.

> **Proposition 2.3.23.** *The inclusion $i : X \to Y$ induces the surjection $i_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ with $\ker(i_*) = N$ so that*
>
> $$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N.$$

Proof. Let Z be Y with an attached $I \times I$ strip, S_α, for each α where $(t, 0) \in I \times \{0\}$ is identified with $\gamma_\alpha(t)$, $(1, t) \in \{1\} \times I$ is identified with an arc on e_α^2 and the edges $(0, t) \in \{0\} \times I$ of each of the strips for the α are identified with each other. Note that Z deformation retracts to Y. For each α let $y_\alpha \in e_\alpha^2$ be such that $y_\alpha \notin X$ and $y_\alpha \notin S_\alpha$.

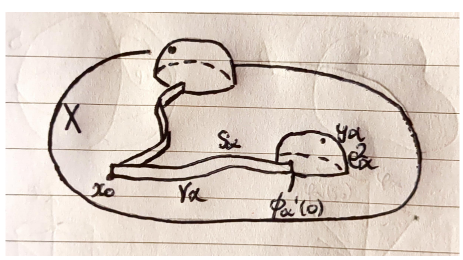


Figure 2.3.7: Illustrations of the constructions used in the proof of Proposition 2.3.23

Let $A_1 := Z \setminus \bigcup_\alpha \{y_\alpha\}$ and $A_2 = Z \setminus X$. Then A_1 deformation retracts to X and A_2 is homotopy equivalent to a point. Thus, $\pi_1(A_2)$ is trivial and Theorem 2.3.17 says that

$$\pi_1(Z, x_0) \cong \pi_1(A_1, x_0)/\left\langle\left\langle j_*\left(\pi_1(A_1 \cap A_2)\right)\right\rangle\right\rangle,$$

where $j : A_1 \cap A_2 \to A_1$ is the inclusion map. Thus, as $\pi_1(Z, x_0) \cong \pi_1(Y, x_0)$ and $\pi_1(A_1, x_0) \cong \pi_1(X, x_0)$ we have that $\langle\langle j_* (\pi_1(A_1 \cap A_2))\rangle\rangle = \ker(i_*)$. For $z_0 \in A_1 \cap A_2$, near x_0 on the segment where the S_α intersect, let δ_α be a loop from z_0 based in $A_1 \cap A_2$ that represents the element of $\pi_1(A_1, z_0)$ corresponding to $\left[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}\right] \in \pi_1(A_1, z_0)$ under the change-of-base point isomorphism β_h for h the line segment connecting z_0 to x_0 along the intersection of the S_α. Observe that $A_\alpha = A_1 \cap A_2 \setminus \bigcup_{\beta \neq \alpha} e_\beta^2$ is an open set that deformation retracts onto a circle in $e_\alpha^2 \setminus \{y_\alpha\}$ and so $\pi_1(A_\alpha) \cong \mathbb{Z}$. In particular, $\pi_1(A_\alpha)$ is generated by δ_α. Applying Theorem 2.3.17 to the cover of $A_1 \cap A_2$ given by the open sets A_α, it follows that $\pi_1(A_1 \cap A_2)$ is generated by the loops δ_a. Thus, $\langle\langle j_* (\pi_1(A_1 \cap A_2))\rangle\rangle = N$. □

Corollary 2.3.24. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

Proof. Let $\langle g_\alpha | (r_\beta)\rangle$ be a presentation of G, that is $G = F_{(g_\alpha)} / \langle\langle (r_\beta)\rangle\rangle$. From Example 2.3.21 we have

$$\pi_1\left(\bigvee_{g_\alpha} \mathrm{S}^1_{g_\alpha}\right) = F_{g_\alpha}.$$

Each word r_β is a loop in $\bigvee_{g_\alpha} \mathrm{S}^1_{g_\alpha}$. Let Y be the space constructed by attaching 2-cells to $\bigvee_{g_\alpha} \mathrm{S}^1_{g_\alpha}$ along the loops r_β. Then from Proposition 2.3.23 it follows that

$$\pi_1(Y) \cong \pi_1(X) / \langle\langle (r_\beta)\rangle\rangle \cong F_{(g_\alpha)} / \langle\langle (r_\beta)\rangle\rangle \cong G.$$

□

Proposition 2.3.25. *Let X be a topological space. Suppose that Y is obtained by attaching n-cells, for fixed $n > 2$, to X. Then the inclusion $i : X \to Y$ induces isomorphism.*

Proof. Consider the same construction of the topological space Z as considered in the proof of Proposition 2.3.23, where now we consider the n-cells e_α^n rather than the 2-cells e_α^2. In this case, A_α deformation retracts onto a sphere S^{n-1}, and so $\pi_1(A_\alpha) = 0$ for $n > 2$. Hence, $\pi_1(A_1 \cap A_2)$ is trivial which implies that $\pi_1(Y, x_0) \cong \pi_1(A_1, x_0) \cong \pi_1(X, x_0)$ and $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism. □

Corollary 2.3.26. *Let $X = \bigcup_{n \in \mathbb{N}} X^n$ be a path-connected CW complex.*

1. *The inclusion map $i : X^1 \to X$ induces a surjective homomorphism $i_* : \pi_1\left(X^1\right) \to \pi_1(X)$.*
2. *The inclusion map $i : X^2 \to X$ induces an isomorphism $i_* : \pi_1\left(X^2\right) \to \pi_1(X)$.*

Proof.

- Suppose $X = \bigcup_{n=1}^m X^n$ is finite-dimensional. Let $i_j : X^j \to X^{j+1}$ for $j = 1, \ldots, m-1$ be the inclusion map. Then applying Proposition 2.3.23 to i_1 and Proposition 2.3.25 to i_j for $j = 2, \ldots, m-1$ yields the surjective homomorphism

$$\pi_1\left(X^1\right) \xrightarrow{(i_1)_*} \pi_1\left(X^2\right) \xrightarrow{(i_2)_*} \ldots \xrightarrow{(i_{m-1})_*} \pi_1\left(X^m\right) = \pi_1\left(X\right).$$

  Moreover, we obtain the isomorphism

$$\pi_1\left(X^2\right) \xrightarrow{(i_2)_*} \ldots \xrightarrow{(i_{m-1})_*} \pi_1\left(X^m\right) = \pi_1\left(X\right).$$

- Suppose X is infinite-dimensional. Here we only argue for statement 2 as the argument for statement 1 is similar. Let $f : I \to X$ be a loop with base point $x_0 \in X^2$. Then the image of f is compact in X and so contained in X^n for some $n \in \mathbb{N}$ by Proposition 1.2.9. Using Proposition 2.3.25 it follows that f is homotopic to a loop in X^2. Therefore, $\pi_1\left(X^2, x_0\right) \to \pi_1(X, x_0)$ is surjective. Now suppose that

$f \in \pi_1\left(X^2, x_0\right)$ is null-homotopic via $F : I \times I \to X$. The image of this homotopy is compact in X and so lies in some X^n, where we can assume that $n > 2$. Therefore, since $\pi_1\left(X^2, x_0\right) \to \pi_1\left(X^n, x_0\right)$ is injective by Proposition 2.3.25 it follows that f is null-homotopic in X^2, meaning $\pi_1\left(X^2, x_0\right) \to \pi_1\left(X, x_0\right)$ is also injective.

□

2.4 Covering Spaces

We would now like a classification of covering spaces for a topological space.

Example 2.4.1.

1. *On the space S^1 we have already encountered a covering space, namely Example 2.2.26. Visually this can viewed as an infinite helical vertical extrapolation of S^1. However, we also have a covering space $p : \mathrm{S}^1 \to \mathrm{S}^1$ given by*

$$p(z) = z^n$$

where z is viewed as a complex number and $n \in \mathbb{N}$. Visually this can be viewed as an n-fold helical extrapolation of S^1 around a torus. It turns out, that these covering spaces exhaust all the possible covering spaces of S^1.

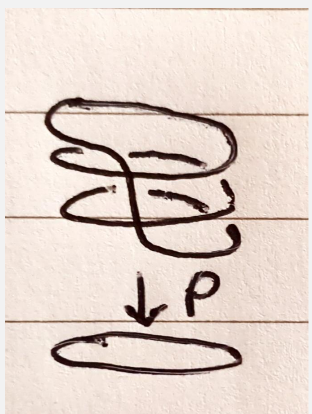


Figure 2.4.1: The covering space $p : \mathrm{S}^1 \to \mathrm{S}^1$ given by $z \mapsto z^3$.

2. *Note that $\mathrm{S}^1 \vee \mathrm{S}^1$ can be viewed as a graph with one vertex and two oriented edges. More specifically, we label the edges a and b. For consistency let $X = \mathrm{S}^1 \vee \mathrm{S}^1$.*

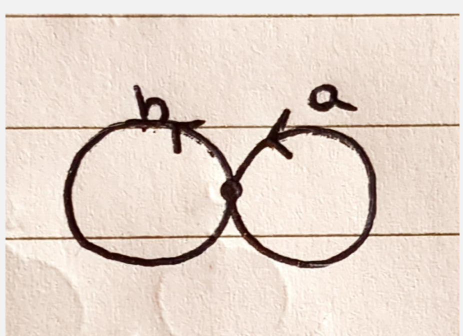


Figure 2.4.2: $X = \mathrm{S}^1 \vee \mathrm{S}^1$.

Now let $\tilde{X}$ be any graph with four edge ends meeting at each vertex. Suppose that the edges of $\tilde{X}$ are labelled a and b and oriented in such a way that the local picture at each vertex of $\tilde{X}$ looks like the vertex of X. A structure constructed in this way is referred to as a 2-oriented graph. Given a 2-oriented graph one can construct a map $p : \tilde{X} \to X$ sending all vertices of $\tilde{X}$ to vertices of X and sending each edge of $\tilde{X}$ to the correspondingly labelled edge of X. As each vertex of $\tilde{X}$ locally resembles the vertex of X it follows that $p : \tilde{X} \to X$ is a covering map. What's more, any covering map of X induces such an oriented structure. Indeed, one can show that any graph with four edge ends at each vertex can be 2-oriented. Thus we see that the possible covering spaces for $X = \mathrm{S}^1 \vee \mathrm{S}^1$ are much more plentiful than those of S^1.

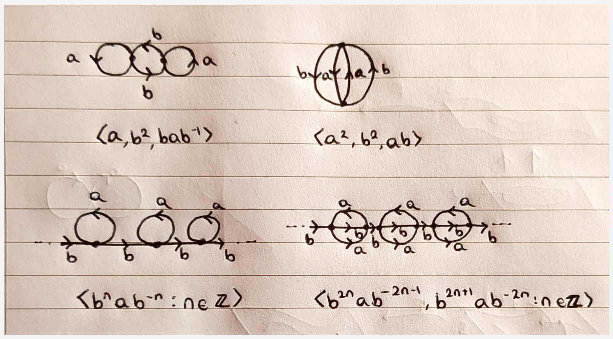


Figure 2.4.3: Covering spaces for $\mathrm{S}^1 \vee \mathrm{S}^1$.

2.4.1 Lifting Properties

Throughout this section, we will let $f : Y \to X$ be a continuous map, with lift $\tilde{f} : Y \to \widetilde{X}$. Meaning $\tilde{f}$ has the property that $p\tilde{f} = f$ where $p : \widetilde{X} \to X$ is a covering space.
For Y connected recall the following results regarding the lift of maps to covering spaces.

- Proposition 2.2.27 says that if lifts $\tilde{f}_1$ and $\tilde{f}_2$ coincide at a point, then they coincide on all of Y.
  - If Y is a single point we can contextualise this result in the following way. Let $f : I \to X$ is a path with $f(0) = x_0$. For $\widetilde{x_0} \in p^{-1}(x_0)$, there exists a unique path $\tilde{f} : I \to \tilde{X}$ that lifts f and starts at $\widetilde{x_0}$.
  - In particular, this means that the lift of a constant path is a constant path.
- Proposition 2.2.28 says that if $F_t : Y \to X$ is a homotopy and $\tilde{F}_0$ is a lift of F_0, then there exists a unique homotopy $\tilde{F}_t : Y \to \widetilde{X}$ from $\tilde{F}_0$ that lifts F_t.
  - Recall that a homotopy of paths must fix the endpoints of the paths, namely if $F : I \times I \to X$ is a homotopy of paths then $F_t(0) : I \to X$ and $F_t(1) : I \to X$ are constant paths. Therefore, as our remark above, their lifts are constant paths, and so $\tilde{F}_t$ is also a homotopy of paths.

Fixing $x_0 \in X$ and $\widetilde{x_0} \in \tilde{X}$ one can consider the induced map $p_* : \pi_1\left(\tilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$.

Proposition 2.4.2. *Fix $x_0 \in X$ and $\widetilde{x_0} \in \widetilde{X}$ so that $p\left(\widetilde{x_0}\right) = x_0$, then the following statements hold.*

1. $p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$ *is injective.*
2. *The set $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops starting at x_0 whose lift to $\tilde{X}$ starting at $\widetilde{x_0}$ are loops.*

Proof.

1. Let $\tilde{f}_0 : I \to \tilde{X}$ be a loop at $\widetilde{x_0}$ such that $\left[\tilde{f}_0\right] \in \ker(p_*)$, meaning $p\tilde{f}_0 = f_0$ is homotopic to a constant loop. More specifically, let f_0 be homotopic to a constant loop through $F_t : I \to X$. By Proposition 2.2.28,

it follows that F_t lifts to a homotopy $\tilde{F}_t$ between $\tilde{f}_0$ and the lift of the constant loop which must be the constant loop by uniqueness, which implies that $\left[\tilde{f}_0\right] = \mathrm{id} \in \pi_1\left(\tilde{X}, \widetilde{x_0}\right)$. Thus p_* is injective.

2. On the one hand, for $f : I \to X$ a loop at x_0 that lifts to a loop $\tilde{f}$ at $\widetilde{x_0}$. Then $p\tilde{f} = f$ and so $p_*\left(\left[\tilde{f}\right]\right) = [f]$. On the other hand, if $f : I \to X$ is a loop at x_0 such that there exists a loop $\tilde{f} : I \to \tilde{X}$ at $\widetilde{x_0}$ with $p_*\left(\left[\tilde{f}\right]\right) = [f]$, then f is homotopic to $p\tilde{f}$. By Proposition 2.2.28 there exists a loop $\tilde{f}' : I \to \tilde{X}$ at $\widetilde{x_0}$ such that $p\tilde{f}' = f$.

□

Definition 2.4.3. *A topological space X has a property P locally if for each $x \in X$ and neighbourhood U of x there is an open neighbourhood $V \subseteq U$ of x that has property P.*

Example 2.4.4.

1. *Let $X = (0,1) \cup (2,3) \subseteq \mathbb{R}$ be endowed with the subspace topology. Then X is locally path connected but it is not path connected.*

2. *Let*

$$X = \left(\bigcup_{n\in\mathbb{N}} \left\{\frac{1}{n}\right\} \times I\right) \cup (\{0\} \times I) \cup (I \times \{0\}) \subseteq \mathbb{R}^2$$

be endowed with the subspace topology. Then X is path connected but it is not locally path connected. Therefore, path-connected is not a stronger property than locally path-connected, and similarly, locally path-connected is not a stronger property than path-connected.

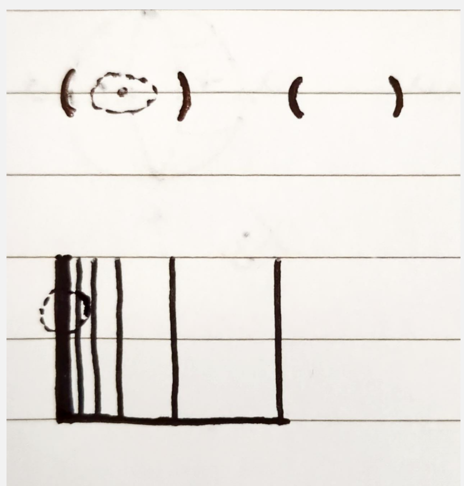


Figure 2.4.4: The distinction between path-connected and locally path-connected

For a covering space $p : \tilde{X} \to X$, suppose $U \subseteq X$ is an evenly covered neighbourhood of $x \in X$ with

$$p^{-1}(U) = \bigcup_{\alpha\in\Lambda} \tilde{U}_\alpha.$$

Then, note that $\left|p^{-1}(x)\right|$ is exactly equal to $|\Lambda|$. In other words, the sheets of U are in bijection to $p^{-1}(x)$. Consequently, the cardinality of $p^{-1}(x)$ is locally constant. Hence, if X is connected then the cardinality of $p^{-1}(x)$ is constant.

Proposition 2.4.5. *Let X and $\widetilde{X}$ be path-connected and let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a covering space.*

> *Then the number of sheets of p equals the index of $H := p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ in $\pi_1(X, x_0)$.*

Proof. Let g be a loop in X at x_0 with $\tilde{g}$ its lift to a path in $\tilde{X}$ starting at $\widetilde{x_0}$. Then for $[h] \in H$, the path $h \cdot g$ lifts to a path $\tilde{h} \cdot \tilde{g}$ in $\tilde{X}$ that starts at $\widetilde{x_0}$ and ends at the endpoint of $\tilde{g}$ as Proposition 2.4.2 says that $\tilde{h}$ is a loop. Therefore, the map $\Phi : \pi_1(X, x_0)/H \to p^{-1}(x_0)$ be given by

$$H[g] \mapsto \tilde{g}(1)$$

is well-defined.

- For $\widetilde{x_0}' \in p^{-1}(x_0)$, as $\tilde{X}$ is path connected, there is a path $\tilde{g}$ in $\tilde{X}$ from $\widetilde{x_0}$ to $\widetilde{x_0}'$. Then $g = p \cdot \tilde{g}$ is such that $\Phi(H[g]) = \widetilde{x_0}'$, and so Φ is surjective.
- Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$, then the lift $\tilde{g}_1 \cdot \tilde{g}_2^{-1}$ of $g_1 \cdot g_2^{-1}$ is a loop in $\tilde{X}$ at $\widetilde{x_0}$. Hence, Proposition 2.4.2 implies that $[g_1][g_2]^{-1} \in H$ and so $H[g_1] = H[g_2]$, and so Φ is injective.

Therefore, Φ is a bijective, meaning the index of H in $\pi_1(X, x_0)$ equals to number of sheets of p which is given by $\left|p^{-1}(x_0)\right|$. ☐

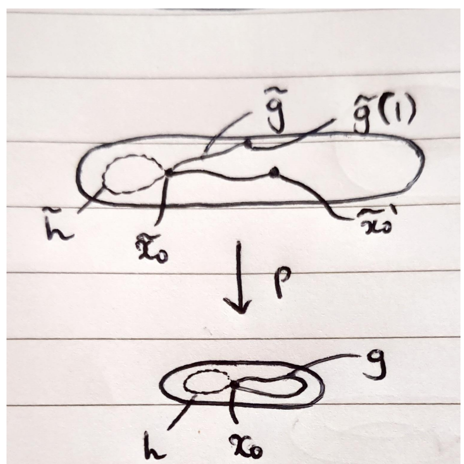


Figure 2.4.5

> **Proposition 2.4.6.** *Let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a covering space and $f : (Y, y_0) \to (X, x_0)$ a continuous map, where Y is path-connected and locally path-connected. Then there is a lift $\tilde{f} : (Y, y_0) \to \left(\tilde{X}, \widetilde{x_0}\right)$ of f if and only if*
>
> $$f_*\left(\pi_1(Y, y_0)\right) \subseteq p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_0}\right)\right).$$
>
> $$\begin{tikzcd} & (\tilde{X}, \widetilde{x_0}) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Proof. $(\Rightarrow)$. As $f = p\tilde{f}$ it follows that $f_* = p_*\tilde{f}_*$. Thus

$$f_*\left(\pi_1(Y, y_0)\right) = p_*\left(\tilde{f}_*\left(\pi_1(Y_1, y_0)\right)\right) \subseteq p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_0}\right)\right)$$

$(\Leftarrow)$. For $y \in Y$ choose a path γ from y_0 to y such that $f\gamma : I \to X$ is a path from x_0 to $f(y)$. By Corollary 2.2.30 the path $f\gamma$ can be lifted to a path $\widetilde{f\gamma} : I \to \tilde{X}$ that starts at $\widetilde{x_0}$. Thus, consider the map $\tilde{f} : (Y, y_0) \to \left(\tilde{X}, \widetilde{x_0}\right)$ given by $y \mapsto \widetilde{f\gamma}(1)$.

- Suppose γ' is path from y_0 to y, then $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$ is a loop with base point x_0 with

$$[h_0] \in f_* (\pi_1(Y, y_0)) \subseteq p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_0}\right)\right).$$

  Using statement 2. of Proposition 2.4.2, this means that there is a homotopy h_t of h_0 to a loop h_1 that lifts to a loop $\tilde{h}_1$ in $\tilde{X}$ with base point $\widetilde{x_0}$. Using Proposition 2.2.28 we obtains a lift $\tilde{h}_t$ of h_t. As $\tilde{h}_1$ is a loop with base point $\widetilde{x_0}$, it follows that $\tilde{h}_0$ is also a loop with base point $\widetilde{x_0}$. By the uniqueness of lifted paths, it follows that the first half of $\tilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half of $\tilde{h}_0$ is $\widetilde{f\gamma}^{-1}$. Hence, $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$, which means that $\tilde{f}$ is well-defined.

- Let $y \in Y$ and U an evenly covered neighbourhood of $f(y)$. Let $\tilde{U}$ be the sheet such that $\tilde{f}(y) \in \tilde{U}$, meaning that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. As Y is locally path connected, there exists $V \subseteq Y$ a path-connected neighbourhood of y such that $f(V) \subseteq U$. Consider a path γ from y_0 to y and for $y' \in V$ consider a path η from y to y' such that $\gamma \cdot \eta$ is a path from y_0 to y'. Then $(f\gamma) \cdot (f\eta) : I \to U$ is a path from x_0 to $f(y')$. In particular, $\widetilde{f\eta} = (p|_{\tilde{U}})^{-1} f\eta$. Which means that

$$\tilde{f}|_V (y') = \widetilde{f\eta}(1) = (p|_{\tilde{U}})^{-1} f\eta(1) = (p|_{\tilde{U}})^{-1} f (y')$$

  and so $\tilde{f}|_V = (p|_{\tilde{U}})^{-1} f$. Hence, $\tilde{f}|_{\tilde{U}} : V \to \tilde{U}$ is continuous, which implies that $\tilde{f}$ is continuous at y.

$\square$

2.4.2 Classification of Covering Spaces

For a fixed space X there exists a classification of its different covering spaces.

Definition 2.4.7. *A covering space $p : \widetilde{X} \to X$ is a universal cover if $\widetilde{X}$ is simply-connected.*

As universal covers have advantageous properties, it will be useful to determine when spaces have universal covers.

Definition 2.4.8. *A topological space X is semi-locally simply connected if each $x \in X$ has a neighbourhood U such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial for $i : U \hookrightarrow X$ being the inclusion map.*

Example 2.4.9. *Consider $X = \bigcup_{n\in\mathbb{N}} C_n \subseteq \mathbb{R}^2$ where $C_n \subseteq \mathbb{R}^2$ is the circle of radius $\frac{1}{n}$ centred at $\left(\frac{1}{n}, 0\right)$. For $0 \in X$, any neighbourhood U of zero contains a circle C_n for some $n \in \mathbb{N}$. Hence, $i_* : \pi_1(U, 0) \to \pi_1(X, 0)$ is not trivial for any neighbourhood U of zero and so X is not semi-locally simply connected.*

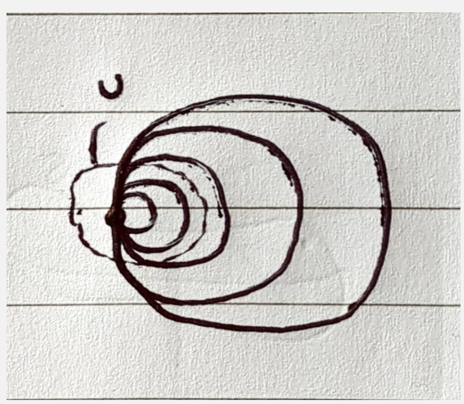


Figure 2.4.6: A non-example of a semi-locally simply connected topological space.

Proposition 2.4.10. *If $p : \widetilde{X} \to X$ is a universal cover, then X is semi-locally simply connected.*

Proof. For $x \in X$ let U be an evenly covered neighbourhood with a sheet $\tilde{U}$. For $\gamma \in \pi_1(U, x)$ the path $\tilde{\gamma} := p^{-1}|_{\tilde{U}}\gamma$ is a loop at $\tilde{x} = p^{-1}|_{\tilde{U}}(x) \in \tilde{U}$. In particular, $\tilde{\gamma}$ is homotopic to the constant loop at $\tilde{x}$ as $\tilde{X}$ is simply connected. Composing this homotopy with p implies that $\gamma : I \to X$ is homotopic to the constant loop at x, and so $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. □

Definition 2.4.11. *A basis on a set Y is a collection $\mathcal{B} \subseteq \mathcal{P}(Y)$ such that the following hold.*

1. *$Y = \bigcup_{U \in \mathcal{B}} U$.*
2. *If $U_1, U_2 \in \mathcal{B}$, and $y \in U_1 \cap U_2$, then there exists a $V \in \mathcal{B}$ such that $y \in V$ and $V \subseteq U_1 \cap U_2$.*

Remark 2.4.12. *A basis on a set Y defines a topology. Namely, $A \subseteq Y$ is open if and only if A is the union of elements of $\mathcal{B}$. Consequently, $f : Z \to Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathcal{B}$.*

Suppose that $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ is a universal cover. Then, we have an equivalence between

1. points in $\widetilde{X}$,
2. $[\gamma]$ where γ is a path in $\widetilde{X}$ starting at $\widetilde{x_0}$, and
3. $[\gamma]$ where γ is a path in X starting at x_0.

Therefore, for an arbitrary space X we can consider constructing a universal cover $\widetilde{X}$ as the space of homotopy classes of paths starting at x_0. Indeed the collection $\mathcal{U}$ of path-connected open sets $U \subseteq X$ such that i_* is trivial forms a basis for the topological space X. In particular, if for a given set $U \in \mathcal{U}$ and path γ in X from x_0 to a point in U we let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ a path in } U \text{ with } \eta(0) = \gamma(1)\},$$

it follows that the collection $\left(U_{[\gamma]}\right)_{U \in \mathcal{U}, \gamma}$ forms a basis which we can use to define a topology on $\widetilde{X}$. One can then show that the map $p : \widetilde{X} \to X$ with $U_{[\gamma]} \mapsto U$ gives our desired covering space.

Theorem 2.4.13. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then there exists a universal cover $p : \widetilde{X} \to X$.*

Proof. For $x_0 \in X$, let

$$\tilde{X} := \{[\gamma] : \gamma : I \to X \text{ a path with } \gamma(0) = x_0\}$$

and let $p : \tilde{X} \to X$ be given by

$$[\gamma] \to \gamma(1).$$

- Let $\mathcal{U}$ be the collection of all path connected open subsets $U \subseteq X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial.
  - For $x \in X$, as X is semi-locally simply connected, there exists $V \subseteq X$ open such that $\pi_1(V, x) \to \pi_1(X, x)$ is trivial. Then as X is locally path connected there exists a $U \subseteq V \subseteq X$ such that U is path connected. In particular, the map $\pi_1(U) \to \pi_1(V) \to \pi_1(X)$ is trivial. Therefore, $U \in \mathcal{U}$. Hence, $X = \bigcup_{U \in \mathcal{U}} U$.
  - Let $U_1, U_2 \in \mathcal{U}$ and $y \in U_1 \cap U_2$. Then there exists a path-connected neighbourhood $V \subseteq U_1 \cap U_2$ of y. In particular, from $V \hookrightarrow U_1 \hookrightarrow X$ we have $\pi_1(V) \to \pi_1(U_1) \to \pi_1(X)$, and so $\pi_1(V) \to \pi_1(X)$ must be trivial as $\pi_1(U_1) \to \pi_1(X)$ is trivial. Hence, $V \in \mathcal{U}$.

Thus, we deduce that $\mathcal{U}$ is a basis for the topology on X. Now for $U \in \mathcal{U}$ and a path $\gamma : I \to X$ from x_0 to a point in U, let

$$U_{[\gamma]} := \{[\gamma \cdot \eta] : \eta : I \to U \text{ a path with } \eta(0) = \gamma(1) \in U\} \subseteq \tilde{X}.$$

The set $U_{[\gamma]}$ depends only on the class $[\gamma]$. Indeed, let $[\gamma] = [\gamma']$ then $[\gamma \cdot \eta] = [\gamma' \cdot \eta]$ and so $U_{[\gamma]} = U_{[\gamma']}$.

- As $X = \bigcup_{U \in \mathcal{U}} U$ we have $\bigcup_{U \in \mathcal{U}, \gamma} U_{[\gamma]} = \tilde{X}$.
- If $[\gamma'] \in U_{[\gamma]}$ then $\gamma' = \gamma \cdot \eta$ for $\eta : I \to U$ a path. Hence, elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$ and so $U_{[\gamma']} \subseteq U_{[\gamma]}$. Similarly, elements of $U_{[\gamma]}$ have the form

$$[\gamma \cdot \mu] = \left[\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu\right] = \left[\gamma' \cdot \eta^{-1} \cdot \mu\right]$$

  and so $U_{[\gamma]} \subseteq U_{[\gamma']}$, meaning $U_{[\gamma']} = U_{[\gamma]}$. Now consider $U_{[\gamma]}$ and $V_{[\gamma']}$, and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ so that $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. As $\mathcal{U}$ is a basis, let $W \in \mathcal{U}$ be such that $W \subseteq U \cap V$, and such that $\gamma''(1) \in W$. Then $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']} = U_{[\gamma']} \cap V_{[\gamma']}$ with $[\gamma''] \in W_{[\gamma'']}$.

Therefore, $\left(U_{[\gamma]}\right)_{U \in \mathcal{U}, \gamma}$ forms a basis for $\tilde{X}$.

- Note that U is path connected and so $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is surjective. Moreover, suppose $\gamma \cdot \eta_1$ and $\gamma \cdot \eta_2$ have the same endpoint. Then $\left[\eta_1 \cdot \eta_2^{-1}\right]$ is a loop at $\gamma(1)$ in U. Therefore, as X is semi-locally simply connected we have that $\left[\eta_1 \cdot \eta_2^{-1}\right] = \mathrm{id}$ and so $[\eta_1] = [\eta_2]$. In particular, $[\gamma \cdot \eta_1] = [\gamma \cdot \eta_2]$ and so $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is injective. Hence, $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is bijective. Let $V_{[\gamma']} \subseteq U_{[\gamma]}$ be an element of the basis. Then by construction we have that $p\left(V_{[\gamma']}\right) = V \in \mathcal{U}$ which is open, meaning $p^{-1}|_{U_{[\gamma]}}$ is continuous. Moreover, for $V \subseteq U$ with $V \in \mathcal{U}$, let $[\gamma'] \in p^{-1}(V) \cap U_{[\gamma]}$. Then $V_{[\gamma']} \subseteq U_{[\gamma']} = U_{[\gamma]}$ and so $p|_{U[\gamma]}$ is bijective as

$$p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']} \cap U_{[\gamma]} = V_{[\gamma']}.$$

  Thus, as $V_{[\gamma']}$ is open it follows that $p|_{U_{[\gamma]}}$ is continuous. In particular, this also means that p is continuous. Now note that if $U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset$ then for $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ we have

$$U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']},$$

  and so

$$p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$$

  is indeed an even covering of U. Therefore, $p : \tilde{X} \to X$ is a covering space.

- Let $\widetilde{x_0} \in \widetilde{X}$ be the class of the constant path. Let $[\gamma] \in \widetilde{X}$ be arbitrary. Then $\gamma : I \to X$ with $\gamma(0) = x_0$. Let $\gamma_t : I \to X$ be given by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0, t] \\ \gamma(t) & s \in [t, 1]. \end{cases}$$

  Then $\tilde{\gamma} : I \to \tilde{X}$ given by $t \mapsto [\gamma_t]$ is a path from $\widetilde{x_0}$ to $[\gamma]$, which means that $\tilde{X}$ is path connected. Now let $[\gamma] \in p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_0}\right)\right)$. Then γ lifts to a loop at $\widetilde{x_0}$ by statement 2 of Proposition 2.4.2. Since $t \mapsto [\gamma_t]$ provides a homotopy to such a loop, by Proposition 2.2.28 it follows that the lift must be given by this homotopy. In particular, $\widetilde{x_0} = [\gamma_0] = [\gamma_1] = [\gamma]$ and so γ is homotopic to the constant loop. Therefore, $p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_0}\right)\right) = \{\mathrm{id}\}$. Hence, as p_* is injective, by statement 1 of Proposition 2.4.2, it follows by the first isomorphism theorem that $\pi_1\left(\tilde{X}, \widetilde{x_0}\right)$ is trivial and so we conclude that $\tilde{X}$ is simply connected.

$\square$

Proposition 2.4.14. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x_0})) = H$.*

Proof. Let $\tilde{X}$ be as constructed in Theorem 2.4.13 and let $X_H := \tilde{X}/\sim$ where $[\gamma] \sim [\gamma']$ if and only if $\gamma(1) = \gamma'(1)$ and $\left[\gamma \cdot (\gamma')^{-1}\right] \in H$. Note the following.

- $[\gamma] \sim [\gamma]$ as $\mathrm{id} \in H$.
- $[\gamma] \sim [\gamma']$ implies that $[\gamma'] \sim [\gamma]$ as H contains all inverses.
- If $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ then $[\gamma] \sim [\gamma'']$ since H is closed under products.

Therefore, $\sim$ is an equivalence relation and X_H is well-defined. Let $p : X_H \to X$ be the natural projection given by $[\gamma] \mapsto \gamma(1)$. Observe that if $\gamma(1) = \gamma'(1)$ then $[\gamma] \sim [\gamma']$ which implies that $[\gamma \cdot \eta] = [\gamma' \cdot \eta]$, for η a path with $\eta(0) = \gamma(1) = \gamma'(1)$. Therefore, $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified to each other. As $U_{[\gamma]}$ form a basis it follows that p is a covering space, and more specifically,

$$p^{-1}(U) = \bigcup_{\gamma} U_{[\gamma']}.$$

For the base point $\widetilde{x_0} \in X_H$ we choose the equivalence class of the constant path c_{x_0}. Note that a loop γ in X with base point x_0, lifts to $\tilde{X}$ starting at $[c_{x_0}]$ and ending at $[\gamma]$. So the image of this loop in X_H is a loop if and only if $[\gamma] \sim [c_{x_0}]$, or equivalently $[\gamma] \in H$. Therefore, the image of $p_* : \pi_1(X_H, \widetilde{x_0}) \to \pi_1(X, x_0)$ is exactly H. □

Remark 2.4.15. *Note how Theorem 2.4.13 is Proposition 2.4.14 with H being the trivial subgroup.*

Definition 2.4.16. *Covering spaces*

$$p_1 : \widetilde{X}_1 \to X$$

and

$$p_2 : \widetilde{X}_2 \to X$$

are isomorphic if there exists a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 f = p_1$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Remark 2.4.17. *Note that the inverse of an isomorphism between covering spaces, f^{-1}, is also an isomorphism between covering spaces. Moreover, the composition of isomorphisms is an isomorphism. Thus, isomorphism defines an equivalence relation between covering spaces.*

Proposition 2.4.18. *Let X be a path-connected, locally path-connected and let $x_0 \in X$. Path-connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic through an $f : \tilde{X}_1 \to \tilde{X}_2$ that maps $\widetilde{x_1} \in p^{-1}(x_0)$ to $\widetilde{x_2} \in p_2^{-1}(x_0)$ if and only if*

$$(p_1)_* \left(\pi_1 \left(\tilde{X}_1, \widetilde{x_1}\right)\right) = (p_2)_* \left(\pi_1 \left(\tilde{X}_2, \widetilde{x_2}\right)\right).$$

Proof. $(\Rightarrow)$. If $f : \left(\tilde{X}_1, \widetilde{x_1}\right) \to \left(\tilde{X}_2, \widetilde{x_2}\right)$ is an isomorphism then $p_1 = p_2 f$ and so

$$(p_1)_* \left(\pi_1 \left(\tilde{X}_1, \widetilde{x_1}\right)\right) \subseteq (p_2)_* \left(\pi_1 \left(\tilde{X}_2, \widetilde{x_2}\right)\right).$$

Furthermore, $p_2 = p_1 f^{-1}$ and so

$$(p_2)_* \left(\pi_1 \left(\tilde{X}_2, \widetilde{x_2}\right)\right) \subseteq (p_1)_* \left(\pi_1 \left(\tilde{X}_1, \widetilde{x_1}\right)\right).$$

($\Leftarrow$). Using Proposition 2.4.6 we can lift p_1 to a continuous map

$$\tilde{p}_1 : \left(\tilde{X}_1, \widetilde{x_1}\right) \to \left(\tilde{X}_2, \widetilde{x_2}\right).$$

Similarly, we can lift p_2 to a continuous map

$$\tilde{p}_2 : \left(\tilde{X}_2, \widetilde{x_2}\right) \to \left(\tilde{X}_1, \widetilde{x_1}\right).$$

In particular, $p_1\tilde{p}_2 = p_2$ and $p_2\tilde{p}_1 = p_1$. Note that $\tilde{p}_1\tilde{p}_2$ fixes $\widetilde{x_2} \in \tilde{X}_2$ and so by Proposition 2.2.27 we have $\tilde{p}_1\tilde{p}_2 = \mathrm{id}_{\widetilde{x_2}}$. Similarly, $\tilde{p}_2\tilde{p}_1 = \mathrm{id}_{\widetilde{x_1}}$ and so $f = \tilde{p}_1$ is an isomorphism. ☐

Remark 2.4.19. *Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be covering spaces. Let $x_0 \in X$ be fixed. Then for $\widetilde{x_1} \in p_1^{-1}(x_0)$ and $\widetilde{x_2} \in p_2^{-1}(x_0)$, an isomorphism $f : \left(\tilde{X}_1, \widetilde{x_1}\right) \to \left(\tilde{X}_2, \widetilde{x_2}\right)$ is referred to as a base point preserving isomorphism.*

Theorem 2.4.20 (Galois Correspondence)**.** *Let X be a path-connected, locally path-connected, and semi-locally simply-connected, and let $x_0 \in X$. Then the following statements hold.*

1. *Path-connected covering spaces up to base point preserving isomorphisms $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ are in bijection with subgroups $H \subseteq \pi_1(X, x_0)$.*

2. *Path connected covering spaces $p : \widetilde{X} \to X$ up to isomorphisms are in bijection with conjugacy classes of subgroups $H \subseteq \pi_1(X, x_0)$.*

Proof.

1. For a covering space $p : \left(\tilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ we associate the subgroup

$$p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0).$$

Using Proposition 2.4.14 and Proposition 2.4.18 it follows that this is well-defined on the isomorphism classes, and is in particular bijective.

2. Let $p : \tilde{X} \to X$ be a covering space and consider $\widetilde{x_1}, \widetilde{x_2} \in p^{-1}(x_0)$. Let

$$H_i := p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_i}\right)\right) \subseteq \pi_1(X, x_0)$$

for $i = 1, 2$. Let $\tilde{\gamma}$ be a path from $\widetilde{x_1}$ to $\widetilde{x_2}$ then $\gamma = p\tilde{\gamma}$ is a loop at x_0. Let $[f] \in \pi_1(X, x_0)$. Then $[f] \in H_1$ if and only if the lift $\tilde{f}$ is a loop at $\widetilde{x_1}$. In which case $\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma}$ is a loop at $\widetilde{x_2}$, so as

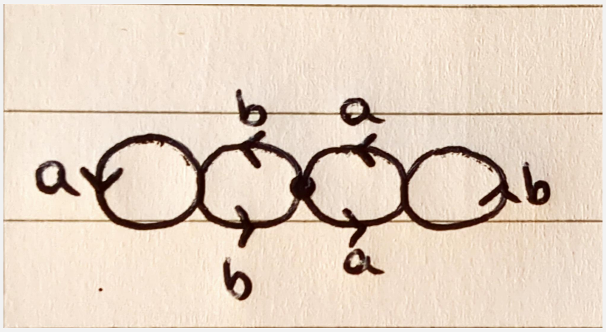
$$p_* \left(\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma}\right) = \gamma^{-1} \cdot f \cdot \gamma$$

we have $[\gamma]^{-1}[f][\gamma] \in H_2$. Hence, $[\gamma]^{-1}H_1[\gamma] \subseteq H_2$. Similarly, $[\gamma]H_2[\gamma]^{-1} \subseteq H_1$. Conversely, let $H \subseteq \pi_1(X, x_0)$, and let $[\delta] \in \pi_1(X, x_0)$ be arbitrary. Let $\tilde{\delta}$ be a lift of δ such that $\tilde{\delta}(0) = \widetilde{x_0}$ and consider $\widetilde{x_3} = \tilde{\delta}(1)$. Then through the same construction

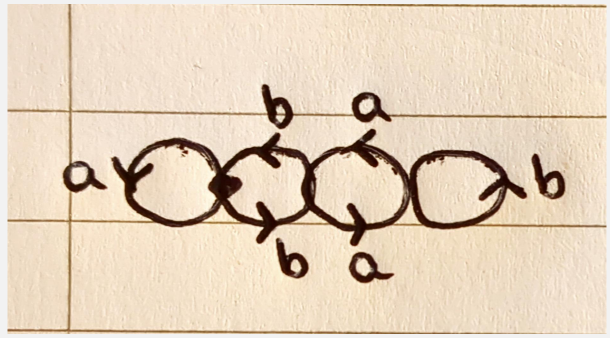
$$p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_3}\right)\right) = [\delta]^{-1}H[\delta].$$

☐

Example 2.4.21. *Recall the construction of covering spaces for $\mathrm{S}^1 \vee \mathrm{S}^1$ given in statement 2 of Example 2.4.1. Consider the covering spaces of $\mathrm{S}^1 \vee \mathrm{S}^1$ as depicted in Figure 2.4.7.*



(a) Covering space $\mathrm{S}^1 \vee \mathrm{S}^1$.



(b) Covering space $\mathrm{S}^1 \vee \mathrm{S}^1$.

Figure 2.4.7

These covering spaces are isomorphic, but not under a base point preserving isomorphism. The subgroup of $\pi_1\left(\mathrm{S}^1 \vee \mathrm{S}^1, x_0\right)$ induced by Figure 2.4.7a is

$$H_1 := \left\langle a^2, b^2, aba^{-1}, bab^{-1} \right\rangle,$$

and the subgroup of $\pi_1\left(\mathrm{S}^1 \vee \mathrm{S}^1, x_0\right)$ induced by Figure 2.4.7b is

$$H_2 := \left\langle a^2, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \right\rangle.$$

Indeed, H_1 and H_2 are distinct but they are conjugate, as expected by Theorem 2.4.20.

Proposition 2.4.22. *Let $p : \tilde{X} \to X$ be a covering space for X which is path-connected, locally path-connected, and semi-locally simply connected. Then the following hold.*

1. *The path connected components of $\tilde{X}$ are in bijection with the orbits of the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.*
2. *For fixed $\widetilde{x_0} \in p^{-1}(x_0)$, let $Y \subseteq \tilde{X}$ be a path connected component containing $\widetilde{x_0}$. Then $(p|_Y)_* \left(\pi_1\left(Y, \widetilde{x_0}\right)\right)$ is the stabiliser subgroup of $\widetilde{x_0}$.*

Proof.

1. Let $x, x' \in p^{-1}(x_0)$ be in the same orbit under the action of $\pi_1(X, x_0)$. Then there exists a $\gamma : I \to X$ such that its lift $\tilde{\gamma}$ is a path from x to x'. In particular, this means that x and x' are in the same components of $\tilde{X}$. On the other hand, let $\tilde{A} \subseteq \tilde{X}$ be a connected component of $\tilde{X}$. Let $x, x' \in \tilde{A} \cap p^{-1}(x_0)$. Then since $\tilde{A}$ is connected, there exists a path $\tilde{\gamma}$ from x to x'. The path $p\tilde{\gamma}$ is a loop with base point x_0, hence, x and x' lie in the same orbit.
2. Let $\gamma \in \pi_1(X, x_0)$ be a stabiliser of $\widetilde{x_0}$. Then γ lifts to a loop $\tilde{\gamma}$ with base point $\widetilde{x_0}$. Thus, $\tilde{\gamma} \in \pi_1\left(\tilde{X}, \widetilde{x_0}\right)$ which implies that $\gamma \in (p|_Y)_* (\pi_1(Y, \widetilde{x_0}))$. On the other hand, let $\gamma \in (p|_Y)_* (\pi_1(Y, \widetilde{x_0}))$. Then γ lifts to $\tilde{\gamma}$ which is a loop at $\widetilde{x_0}$. That is γ acts on $\widetilde{x_0}$ by sending it to itself and thus is in the stabiliser of $\widetilde{x_0}$.

$\square$

2.4.3 Deck Transformations

Definition 2.4.23. *Let $p : \widetilde{X} \to X$ be a covering space. A deck transformation is an isomorphism between p*

and itself. In particular, $f : \tilde{X} \to \tilde{X}$ is a deck transformation if f is a homeomorphism and $pf = p$. The group of deck transformations is denoted by $G\left(\widetilde{X}\right)$.

Example 2.4.24.

1. *Consider the covering space $p : \mathbb{R} \to \mathrm{S}^1 \subseteq \mathbb{C}$. Then $f : \mathbb{R} \to \mathbb{R}$ is a deck transformation if and only if $p(f(t)) = p(t)$, which happens if and only if $e^{2\pi i f(t)} = e^{2\pi i t}$ which happens if and only if $f(t) = t + n$ for some $n \in \mathbb{Z}$. Therefore, $G\left(\tilde{X}\right) \cong \mathbb{Z}$.*

2. *Recall the covering space $p : \mathrm{S}^1 \to \mathrm{S}^1$ where $p(z) = z^n$, for some integer n. Then $f : \mathrm{S}^1 \to \mathrm{S}^1$ is a deck transformation if and only if $p(f(t)) = p(t)$, which happens if and only if $f(t)^n = f(t)$, which happens if and only if $f(t) = e^{\frac{2k\pi}{n}} t$ for $k \in \{0, \ldots, n-1\}$. Therefore, $G\left(\widetilde{X}\right) \cong \mathbb{Z}/n\mathbb{Z}$.*

Note that a deck transformation can be viewed as a lift of the covering space p. Hence, from Proposition 2.2.27 we see that if $\widetilde{X}$ is path-connected then $f \in G\left(\widetilde{X}\right)$ is determined by where it sends a single point. Consequently, the identity is the only deck transformation with a fixed point.

Definition 2.4.25. *A covering space $p : \widetilde{X} \to X$ is normal if for each $x \in X$ and every pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is an $f \in G\left(\widetilde{X}\right)$ such that $f(\tilde{x}) = \tilde{x}'$.*

Example 2.4.26. *The covering spaces of Example 2.4.24 are normal.*

1. *For $x \in \mathrm{S}^1$, we have $p^{-1}(x) = \{\alpha + n : n \in \mathbb{Z}\}$ for some $\alpha \in \mathbb{R}$. Therefore, for $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ we have $\tilde{x} = \alpha + n_1$ and $\tilde{x}' = \alpha + n_2$ for some $n_1, n_2 \in \mathbb{Z}$. Note that $f(t) = t + (n_2 - n_1)$ is a deck transformation, and in particular*

$$f\left(\tilde{x}\right) = \alpha + n_1 + n_2 - n_1 = \alpha + n_2 = \tilde{x}'.$$

2. *For $e^{i\theta} \in \mathrm{S}^1$, we have $p^{-1} = \left\{e^{i\frac{\theta}{n} + \frac{2k\pi}{n}} : k \in \{0, \ldots, n-1\}\right\}$. Therefore, for $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ we have $\tilde{x} = e^{i\frac{\theta}{n} + \frac{2k_1\pi}{n}}$ and $\tilde{x}' = e^{i\frac{\theta}{n} + \frac{2k_2\pi}{n}}$ for $k_1, k_2 \in \{0, \ldots, n-1\}$. Note that $f(t) = e^{\frac{2(k_2 - k_1)\pi}{n}} t$, where $k_2 - k_1$ is taken modular n, is a deck transformation, and in particular*

$$f\left(\tilde{x}\right) = e^{\frac{2(k_2-k_1)\pi}{n}} e^{i\frac{\theta}{n} + \frac{2k_1\pi}{n}} = e^{i\frac{\theta}{n} + \frac{2k_2\pi}{n}} = \tilde{x}'.$$

Proposition 2.4.27. *Let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a path-connected covering space, and X locally path-connected. Then $p : \widetilde{X} \to X$ is normal if and only if $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ is a normal subgroup.*

Proof. Let $\widetilde{x_1} \in p^{-1}(x_0)$, and let $\tilde{\gamma}$ be a path from $\widetilde{x_0}$ to $\widetilde{x_1}$ then $\gamma = p\tilde{\gamma}$ is a loop at x_0. Then

$$[\gamma] p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_1}\right)\right)[\gamma]^{-1} = H.$$

Hence, $[\gamma] H [\gamma]^{-1} = H$ if and only if $H = p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_1}\right)\right)$ which by Proposition 2.4.18 happens if and only if there exists a deck transformation $f : \left(\tilde{X}, \widetilde{x_0}\right) \to \left(\tilde{X}, \widetilde{x_1}\right)$. In particular, as $f\left(\widetilde{x_0}\right) = \widetilde{x_1}$ we have that $G\left(\tilde{X}\right)$ acts transitively on $p^{-1}(x_0)$ if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup. Now let $x_0' \in X$ and h a path from x_0 to x_0'. Let $\tilde{h}$ be a lift of h such that $\tilde{h}(0) = \widetilde{x_0}$. Set $\widetilde{x_0}' = \tilde{h}(1)$ so that $p\left(\widetilde{x_0}'\right) = x_0'$. Then the following diagram commutes.

$$
\begin{tikzcd}
\pi_1\left(\tilde{X}, \widetilde{x_0}\right) \arrow[d, "p_*"] & \pi_1\left(\tilde{X}, \widetilde{x_0}'\right) \arrow[l, "\beta_{\tilde{h}}"'] \arrow[d, "p_*"] \\
\pi_1(X, x_0) & \pi_1\left(X, x_0'\right) \arrow[l, "\beta_h"']
\end{tikzcd}
$$

Thus as β_h and $\beta_{\tilde{h}}$ are isomorphisms we have that $H \subseteq \pi_1(X, x_0)$ is normal if and only if

$$p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_0}'\right)\right) \subseteq \pi_1\left(X, x_0'\right)$$

is normal. This, as argued before, happens if and only if $G\left(\tilde{X}\right)$ acts transitively on $p^{-1}\left(x_0'\right)$. Therefore, we have that H is normal if and only if $G\left(\tilde{X}\right)$ acts transitively on X, which is equivalent to saying that p is normal. $\square$

Proposition 2.4.28. *Let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a path connected covering space, and X locally path-connected. Let $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normalizer of H. Then $G\left(\widetilde{X}\right)$ is isomorphic to $N(H)/H$. In particular, the following hold.*

- *If $\tilde{X}$ is normal, then*
$$G\left(\widetilde{X}\right) \cong \pi_1(X, x_0)/H.$$
- *If $\widetilde{X}$ is the universal cover, then*
$$G\left(\widetilde{X}\right) \cong \pi_1(X, x_0).$$

Proof. Let $\varphi : N(H) \to G\left(\tilde{X}\right)$ be the map given by sending $[\gamma]$ to the deck transformation τ taking $\widetilde{x_0}$ to $\widetilde{x_1}$, as constructed in the proof of Proposition 2.4.27. Let γ' be a another loop corresponding to the deck transformation τ', which takes $\widetilde{x_0}$ to $\widetilde{x_1}'$. Then $\gamma \cdot \gamma'$ lifts to $\tilde{\gamma} \cdot (\tau(\tilde{\gamma}'))$, which is a path from $\widetilde{x_0}$ to $\tau\left(\widetilde{x_1}'\right) = \tau\tau'\left(\widetilde{x_0}\right)$. Thus, as $\tau\tau'$ is deck transformation, and thus determined at a single point, it must be the case that $\tau\tau'$ is the deck transformation corresponding to $[\gamma][\gamma']$, which shows that φ is a homomorphism.

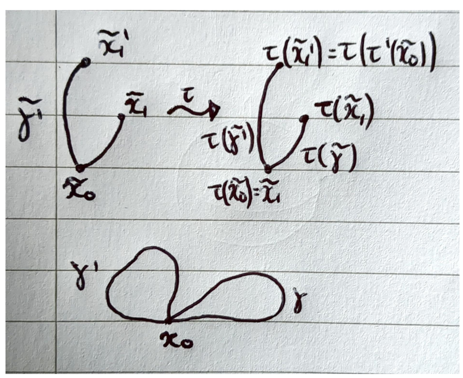


Figure 2.4.8

Moreover, φ is surjective as for any $f \in G\left(\tilde{X}\right)$ we can let $f\left(\widetilde{x_0}\right) = \widetilde{x_1}$ and consider γ a path from $\widetilde{x_0}$ to $\widetilde{x_1}$. By the arguments made in the proof of Proposition 2.4.27 we have that $[\gamma] \in N(H)$. Moreover, the kernel of φ consists of the classes $[\gamma]$ that lift to loops in $\widetilde{X}$, which are exactly the elements of $p_*\left(\pi_1\left(\tilde{X}, \widetilde{x_0}\right)\right) = H$, by statement 2 of Proposition 2.4.2. $\square$

Remark 2.4.29. *A group G acts on a set X by defining a homeomorphism $X \to X$. More specifically, each $g \in G$ corresponds to a homeomorphism $g : X \to X$ such that $(g_1 g_2)(x) = g_1(g_2(x))$ for any $g_1, g_2 \in G$ and*

$x \in X$. The action of G on X is free if for each $x \in X$ we have

$$\{g \in G : g(x) = x\} = \{e\}.$$

We denote by X/G the set of orbits of G in X, where the orbit of $x \in X$ is

$$G(x) = \{g(x) : g \in G\}.$$

The space X/G is turned into a topological space by endowing it with the quotient topology.

Example 2.4.30. *Recall the real projective n-space $\mathbb{RP}^n$ from statement 7 of Example 1.2.8. The antipodal map of S^n, given by $x \mapsto -x$, generates an action of $\mathbb{Z}/2\mathbb{Z}$ on S^n with orbit space $\mathbb{RP}^n$. Moreover, the action is a covering space $\mathrm{S}^n \to \mathbb{RP}^n$ since each open hemisphere in S^n is disjoint from its antipodal image. Since, S^n is simply-connected for $n \geq 2$ it follows that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ by Proposition 2.4.28. One can explicitly see this, as a generator γ of $\pi_1(\mathbb{RP}^n)$ is any loop obtained by projecting a path in S^n connecting antipodal points. The path must be between antipodal points to be non-trivial and thus a generator. Such a loop has order two in $\pi_1(\mathbb{RP}^n)$, if $n \geq 2$, since $\gamma \cdot \gamma$ lifts to a loop in S^n which is homotopic to the trivial loop since $\pi_1(\mathrm{S}^n) = 0$. So the projection of this homotopy into $\mathbb{RP}^n$ gives a null homotopy.*

Definition 2.4.31. *For X a path-connected, locally path-connected and semi-locally simply connected space, a path-connected covering space $p : \tilde{X} \to X$ is abelian if it is normal and has an abelian deck transformation group.*

Proposition 2.4.32. *Let X be path-connected, locally path-connected and semi-locally simply connected. Then X has a universal abelian cover $p : \tilde{X} \to X$, in the sense that $\tilde{X}$ is a covering space of any abelian covering space of X up to isomorphism.*

Proof. Let $H \leq \pi_1(X)$ be the commutator subgroup. Then, by Proposition 2.4.14, there exists a covering space $p_H : X_H \to X$ such that $(p_H)_*(\pi_1(X_H)) = H$. As H is normal, by Proposition 2.4.27, the covering space p_H is normal. Hence, by Proposition 2.4.28, we have that $G(X_H) \cong N(H)/H = \pi_1(X)/H$. Hence, $G(X_H)$ is abelian and so p_H is an abelian covering. Suppose that $q : Y \to X$ is an abelian cover. Then, by Proposition 2.4.27, the subgroup $q_*(\pi_1(Y)) := H' \subseteq \pi_1(X)$ is normal as q is normal. Thus, by Proposition 2.4.28, the quotient $\pi_1(X)/H' \cong G(Y)$ is abelian. Hence, $H \leq H'$, meaning there exists a continuous lift $\tilde{p} : X_H \to Y$ of $p : X_H \to X$. Therefore, $\tilde{p}$ is a covering space of Y. $\square$

2.5 Solution to Exercises

Exercise 2.2.15

Solution. Let $\gamma_0, \gamma_1 \in \pi_1(X, x_1)$ be homotopic. Then an application of Proposition 2.2.8 shows that

$$h \cdot \gamma_0 \simeq h \cdot \gamma_1,$$

and then another application of Proposition 2.2.8 shows that

$$h \cdot \gamma_0 \cdot h^{-1} \simeq h \cdot \gamma_1 \cdot h^{-1}.$$

Therefore,

$$\beta_h([\gamma_0]) = \left[h \cdot \gamma_0 \cdot h^{-1}\right] = \left[h \cdot \gamma_1 \cdot h^{-1}\right] = \beta_h([\gamma_1]).$$

$\square$

Exercise 2.2.36

Solution. Suppose $f_0 \simeq f_1$ through $F_t : I \to X$. Then $\phi F_t : I \to Y$ is a homotopy between ϕf_0 and ϕf_1. Hence,

$$\phi_*([f_0]) = [\phi f_0] = [\phi f_1] = \phi_*([f_1]),$$

which means that ϕ_* is well-defined. Moreover,

$$\phi_*([f \cdot g]) = [\phi(f \cdot g)] = [\phi f \cdot \phi g] = [\phi f][\phi g] = \phi_*([f])\phi_*([g]),$$

and

$$\phi_* \left([c_{x_0}]\right) = [\phi c_{x_0}] = \left[c_{\phi(x_0)}\right].$$

Therefore, ϕ_* is a group homomorphism. $\square$

3 Homology

3.1 Motivation

Just as we considered base point preserving homotopies of the form $\phi : I \to X$ with the fundamental group $\pi_1(X, x_0)$. We can also consider higher homotopies groups $\pi_n(X, x_0)$ which are groups of base point preserving homotopies of the form $\phi : I^n \to X$ where $\phi(\partial I^n) = x_0$. This will be useful to probe higher dimensional spaces, as currently with the fundamental group we cannot distinguish between the S^n's for $n \geq 2$. More specifically, we see by Theorem 2.2.46 that the fundamental group is dependent only on the 2-skeleton of X. More generally, $\pi_n(X)$ depends only on the $(n+1)$-skeleton of X.

Theorem 3.1.1. $\pi_i(\mathrm{S}^n) = 0$ *for* $i < n$ *and* $\mathbb{Z}$ *for* $i = n$.

However, high-order homotopy groups are difficult to compute. Fortunately, homology groups share many of the same dependencies on structure as homotopy groups and are more computable. However, their definition is less transparent than the definition of homotopy and thus requires the introduction of different ideas before they can be implemented.

3.1.1 Intuition

Consider the graph X_1 as depicted in Figure 3.1.1. In the case of the fundamental group, a base point is fixed and the loops from that base point are investigated. For example, ab^{-1} is a loop with base point x. In the context of the fundamental group, the loop $b^{-1}a$ would be distinguished from ab^{-1} as they have different base points. However, in some sense, it may be useful to treat $b^{-1}a$ and ab^{-1} as the same object. In effect, we are attempting to simplify matters by abelianizing. Hence, we adopt the additive notation, namely ab^{-1} is represented with $a - b$. Moreover, loops are now referred to as cycles and can be linearly combined as chains of edges. Note that chains can have different decompositions into cycles. For example, the chain $a - b + c - d$ can be decomposed into the the cycles $(a - c) + (d - b)$ or $(a - d) + (b - c)$. We can algebraically formalise this with the following.

- Let C_0 be the free abelian group with basis $\{x, y\}$. Such that elements of C_0 are 0-dimensional chains.
- Let C_1 be the free abelian group with basis $\{a, b, c, d\}$. Such that elements of C_1 are 1-dimensional chains.
- Let $\partial : C_1 \to C_0$ be the map sending each basis element of C_1 to $y - x$, that is the vertex at the head of the edge minus the vertex at its tail. Then

  $$\partial(ka + lb + mc + nd) = (k + l + m + n)y - (k + l + m + n)x,$$

  and so the kernel of ∂ is precisely the cycles on X_1.

With this algebraic formulation, it becomes straightforward to generalise these ideas. Consider the graph X_2, which is X_1 with an attached 2-cell. One can think of this 2-cell as being oriented clockwise, by letting its boundary be $a - b$. Note that by adding A, the loop $a - b$ is homotopic to a point as it can be slid across A. Suggesting that we can quotient subgroup generated by $a - b$ out of C_1, for example, we can now identify $a - c$ with $b - c$.

- Let C_2 be the infinite cyclic group generated by A.
- Let $\partial_2 : C_2 \to C_1$ be the homomorphism where $\partial_2(A) = a - b$.

With this notation, the quotient group we are seeking is $\ker(\partial_1)/\mathrm{im}(\partial_2)$. This quotient group is precisely the homology group $H_1(X_2)$ that we are interested in investigating. We can take these ideas further by now considering the graph X_3, which attaches another 2-cell, B, to X_2 along the same cycle $a - b$.

- Let C_2 be the infinite cyclic group generated by A and B.
- Let $\partial_2 : C_2 \to C_1$ be the homomorphism where $\partial_2(A) = a - b$ and $\partial_2(B) = a - b$.

As before, the homology group is $H_2(X_3) = \ker(\partial_1)/\mathrm{im}(\partial_2)$. However, as the kernel of ∂_2 is non-trivial, it is generated by $A - B$, we can view $A - B$ as a 2-dimensional cycle and obtain the homology group $H_2(X_3) = \ker(\partial_2) \cong \mathbb{Z}$.

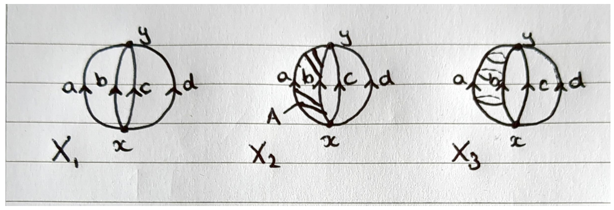


Figure 3.1.1: From left to right, the graphs X_1, X_2 and X_3.

3.2 $\triangle$-Complexes

Definition 3.2.1.

- *An n-simplex in $\mathbb{R}^m$ is the convex hull of a set V of $n+1$ points in $\mathbb{R}^m$ that are not all contained in an affine $(n-1)$-dimensional subspace of $\mathbb{R}^m$. The standard n-simplex is given by*

$$\left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, x_0 + \dots + x_n = 1\right\}.$$

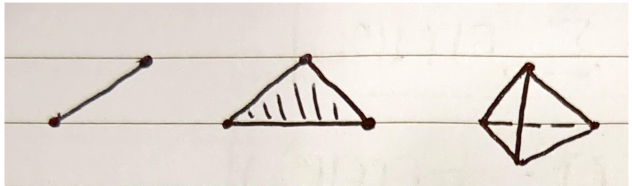
- *An ordered n-simplex is an n-simplex where the vertices have some order defined on them. We denote this by $[v_0, \dots, v_n]$. The standard ordered n-simplex is $[e_1, \dots, e_{n+1}]$, which is the convex hull of the standard basis of $\mathbb{R}^{n+1}$ with the natural ordering imposed on the vertices. We denote this simplex by Δ^n.*

- *For the n-simplex $[v_0, \dots, v_n]$ in $\mathbb{R}^m$ let $L = \mathrm{Sp}(v_0, \dots, v_n)$. Then there is a unique affine morphism $L \to \mathbb{R}^{n+1}$ defined by $v_i \mapsto e_{i+1}$ for $i = 0, \dots, n$. More specifically, we have a homeomorphism from $[v_0, \dots, v_n]$ to Δ^n which preserves the ordering.*

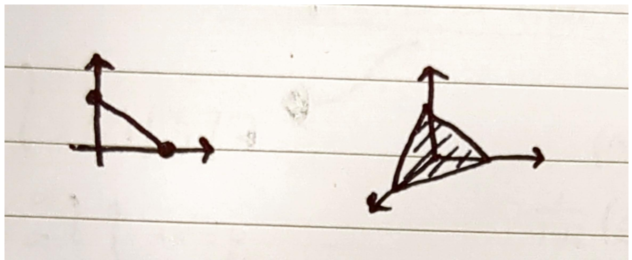
- *The faces of $[v_0, \dots, v_n]$ are defined to be $[v_0, \dots, \hat{v}_i, \dots, v_n]$ for $i = 0, \dots, n$, where $\hat{v}_i$ means this vertex is omitted.*

- *The boundary of a simplex Δ is the union of all the faces.*

- *The (relative) interior of a simplex Δ is $\mathring{\Delta} = \Delta \setminus \partial\Delta$.*



(a) The 1, 2, and 3 simplices



(b) The standard 1 and 2 simplices.

Figure 3.2.1: Δ-complexes.

Example 3.2.2. *For $\Delta^2 = [e_1, e_2, e_3]$ we have*

$$\partial\Delta^2 = [e_2, e_3] \cup [e_1, e_3] \cup [e_1, e_2].$$

Definition 3.2.3. *A Δ-complex structure on a topological space X is a collection of maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ for $\alpha \in A$ and $n(\alpha) \in \mathbb{N}$ with the following properties.*

1. *$\sigma_\alpha|_{\mathring{\Delta}^{n(\alpha)}}$ is injective for all $\alpha \in A$, and for $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha\left(\mathring{\Delta}^{n(\alpha)}\right)$.*
2. *The restriction of σ_α to faces is equal to σ_β for some $\beta \in A$ with $n(\beta) = n(\alpha) - 1$.*
3. *$U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{n(\alpha)}$ for all $\alpha \in A$.*

Example 3.2.4.

1. *The torus is two Δ^2, three Δ^1 and one Δ^0.*
2. *The dunce hat is constructed by identifying the faces of a Δ^2 with each other. As a Δ complex this can be seen as one Δ^2, one Δ^1 and one Δ^0.*

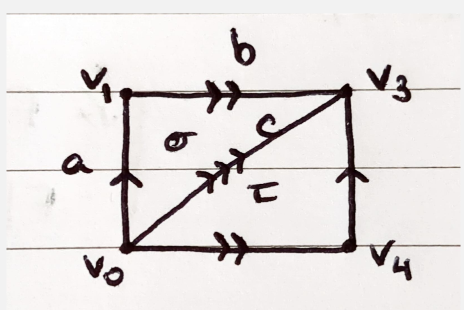


Figure 3.2.2: The torus as a Δ-complex.

3.3 Homology Groups

3.3.1 Simplicial Homology

For X a Δ-complex let the free abelian group of the $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ for $n(\alpha) = n$ be referred to as n-chains, denoted $\Delta_n(X)$. Hence, we denote elements of $\Delta_n(X)$ as

$$\sum_{\alpha \in A, n(\alpha)=n} c_\alpha \sigma_\alpha$$

for $c_\alpha \in \mathbb{Z}$ with finitely many of the c_α non-zero.

Example 3.3.1. *For T the torus as given by Figure 3.2.2 the n-chains are given by the following.*

- $\Delta_0(T) \cong \mathbb{Z}v.$
- $\Delta_1(T) \cong a\mathbb{Z} \oplus b\mathbb{Z} \oplus c\mathbb{Z}.$
- $\Delta_2(T) \cong \sigma\mathbb{Z} \oplus \tau\mathbb{Z}.$
- $\Delta_n(T) \cong 0$ *for* $n \geq 3.$

Definition 3.3.2. *The boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is given by*

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}.$$

In particular, we let $\partial_0 = 0$.

Lemma 3.3.3. *With ∂_n and ∂_{n-1} as given by Definition 3.3.2, we have $\partial_{n-1} \cdot \partial_n = 0$.*

Proof. Observe that

$$\begin{aligned}\partial_{n-1}\partial_n(\sigma) =& \sum_{j<i} (-1)^i (-1)^j \sigma|_{v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n} \\ &+ \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} \\ =& 0\end{aligned}$$

□

Remark 3.3.4. *Geometrically, Lemma 3.3.3 holds as the composed map gives us the n-simplices restricted to the $(n-2)$-ordered faces by removing two vertices from the n-simplices. Removing these vertices in different orders results in each restriction appearing twice in the sum. Due to the changing signs, these cancel each other and thus we are left with the zero map.*

Definition 3.3.5. *A chain complex of abelian groups $(C_\bullet, \partial)$ is the chain*

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0, \tag{3.3.1}$$

where each C_i is an abelian group and the ∂_n are homomorphisms such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}$.

Let $(C_\bullet, \partial)$ be a chain complex. Note that

$$Z_n := \ker(\partial_n) \subseteq C_n$$

and

$$B_n := \mathrm{im}(\partial_{n+1}) \subseteq C_n.$$

In particular, elements of Z_n are referred to as cycles and elements of B_n are referred to as boundaries. For $b \in B_n$ we have that $b = \partial_{n+1}(a)$ for some $a \in C_{n+1}$, so that $\partial_n(b) = (\partial_n \circ \partial_{n+1})(a) = 0$ and so $B_n \subseteq Z_n$.

Lemma 3.3.6. *The sequence $(C_\bullet, \partial)$ is a chain complex if and only if $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$.*

Proof. ($\Rightarrow$). Let $x \in \mathrm{im}(\partial_{n+1})$, then there exists a $y \in C_{n+1}$ such that $\partial_{n+1}(y) = x$. Therefore, as $\partial_n \circ \partial_{n+1} = 0$ it follows that

$$\partial_n(x) = \partial_n \circ \partial_{n+1}(y) = 0,$$

which implies that $x \in \ker(\partial_n)$.
($\Leftarrow$). Let $y \in C_{n+1}$, then $\partial_{n+1}(y) \in \mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$, which implies that

$$(\partial_n \circ \partial_{n+1})(y) = 0.$$

Therefore, $\partial_n \circ \partial_{n+1} \equiv 0$, making $(C_\bullet, \partial)$ a chain complex. $\square$

Definition 3.3.7. *The n^{th} homology group of a chain complex, as given by* (3.3.1), *is*

$$H_n(C_\bullet, \partial) = Z_n / B_n.$$

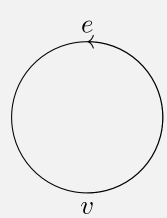
Definition 3.3.8. *The n^{th} simplicial homology group of a chain complex, as given by* (3.3.1), *is*

$$H_n^\Delta(X) = H_n(\Delta_\bullet(X), \partial) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

Remark 3.3.9. *Elements of a homology group H_n are cosets referred to as homology classes.*

Example 3.3.10.

1. *Consider $X = \mathrm{S}^1$, with a vertex v, and an edge e.*

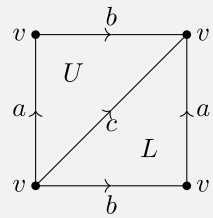


- *The groups $\Delta_0\left(\mathrm{S}^1\right)$ and $\Delta_1\left(\mathrm{S}^1\right)$ are both $\mathbb{Z}$ as they are generated by $\{v\}$ and $\{e\}$ respectively.*
- *For $n \geq 2$, the groups $\Delta_n\left(\mathrm{S}^1\right)$ are trivial as S^1 has no n-simplices.*
- *The boundary map $\partial_1 = v - v = 0$.*

Therefore,

$$H_n^\Delta\left(\mathrm{S}^1\right) \cong \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n \geq 2. \end{cases}$$

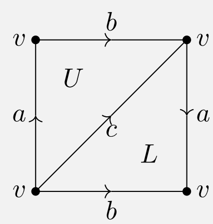
2. Consider $X = T$ the torus, with a vertex v, three edges (a, b, c) and two 2-simplices (U, L).



   - $\Delta_0(T) \cong \mathbb{Z}$ as it is generated by $\{v\}$. $\Delta_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as it is generated by $\{a, b, a+b-c\}$. Similarly, $\Delta_2(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.
   - The boundary $\partial_1 = v - v = 0$.
   - The boundary map ∂_2 is given by $\partial_2(U) = \partial_2(L) = a + b - c$. In particular, $\partial_2(pU + qL) = 0$ only if $p = -q$. So $\ker(\partial_2)$ is generated by $U - L$.

   Therefore,
$$H_n^{\Delta}(T) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0, 2 \\ 0 & n \geq 3. \end{cases}$$

3. Consider $X = K$ the Klein bottle, with a vertex v, three edges and two 2-simplices (U, L).



   Then
$$\Delta_n(K) = \begin{cases} \langle v \rangle & n = 0 \\ \langle a, b, c \rangle & n = 1 \\ \langle U, L \rangle & n = 2. \end{cases}$$

   Moreover,
$$\begin{cases} \partial_0 = 0 \\ \partial_1(a) = \partial_1(b) = \partial(c) = v - v = 0 \\ \partial_2(U) = a + b - c,\ \partial_2(L) = a - b + c. \end{cases}$$

   Note that
$$0 = \partial_2(pU + qL) = (p+q)a + (p-q)b + (q-p)c$$
   implies that $p + q = 0$ and $p - q = 0$, which happens if and only if $p = q = 0$ and thus $\ker(\partial_2) = 0$. Therefore,
$$H_0(K) = \ker(\partial_0)/\mathrm{im}(\partial_1) = \langle v \rangle \cong \mathbb{Z},$$

and

$$
\begin{aligned}
H_1(K) &= \ker(\partial_1)/\mathrm{im}(\partial_1) \\
&= \langle a, b, c\rangle / \langle a+b-c, a-b+c\rangle \\
&= \langle a+b-c, b, c\rangle / \langle a+b-c, 2b-2c\rangle \\
&= \langle b, c\rangle / \langle 2b-2c\rangle \\
&= \langle b-c, c\rangle / \langle 2b-2c\rangle \\
&= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.
\end{aligned}
$$

Similarly,

$$H_2(K) = \ker(\partial_2)/\mathrm{im}(\partial_1) = 0,$$

therefore,

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 2. \end{cases}$$

3.3.2 Singular Homology

Definition 3.3.11. *A singular n-simplex in a topological space X is a continuous map $\sigma : \Delta^n \to X$.*

The σ does not have to be a well-behaved map, in particular, it can have singularities. Thus, a singular n-simplex is more general than a Δ-complex as the image of σ does not necessarily have to be a simplex. The free abelian group of the singular n-simplices in X is denoted $C_n(X)$. Elements of $C_n(X)$ are finite singular n-chains of the form $\sum_i n_i \sigma_i$ for $n_i \in \mathbb{Z}$ and σ_i a singular n-simplex. As the space of singular n-simplicies is larger, when working with $C_n(X)$ in practice we are less likely to encounter finitely generated groups as we did in the case of simplicial homology.

Definition 3.3.12. *The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_1,\ldots,\hat{v}_i,\ldots,v_n]},$$

where σ is a n-simplex.

Lemma 3.3.13. *With ∂_n and ∂_{n-1} as given by Definition 3.3.12, we have $\partial_{n-1} \circ \partial_n = 0$.*

Proof. Proceeds in the same way as Lemma 3.3.3. □

From Lemma 3.3.13 it follows that

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \tag{3.3.2}$$

is a chain complex.

Definition 3.3.14. *The n^{th} singular homology group of the chain complex* (3.3.2) *is*

$$H_n(X) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

Proposition 3.3.15. *If X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$.*

A simplicial homology can be constructed from a singular homology. Let X be an arbitrary space, and define the simplicial complex $S(X)$ as the following Δ-complex.

1. One n-simplex Δ^n_σ for each n-simplex $\sigma : \Delta^n \to X$.
2. Attach Δ^n_σ to the restrictions of σ to the $(n-1)$-simplices of $\partial\Delta^n$.

From this construction we have that $H^\Delta_n(S(X)) = H_n(X)$ for all $n \in \mathbb{N}$.

Proposition 3.3.16. *For a topological space $X = \bigcup_\alpha X_\alpha$, where the X_α are path-connected, we have that*

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

Proof. Since Δ^n is path-connected and σ is continuous, it follows that a singular n-complex $\sigma : \Delta^n \to X$ has a path-connected image. In other words, $\sigma : \Delta^n \to X_\alpha$ for some α. Thus,

$$C_n(X) = \bigoplus_\alpha C_n(X_\alpha).$$

The boundary maps ∂_n preserves this decomposition, so $\partial_n(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$ implies that $\ker(\partial_n)$ and $\mathrm{im}(\partial_{n+1})$ split into direct sums. Therefore,

$$H_n = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}) \cong \bigoplus_\alpha H_n(X_\alpha).$$

□

Proposition 3.3.17. *If X is a non-empty, path-connected topological space, then*

$$H_0(X) \cong \mathbb{Z}.$$

Proof. Since $\partial_0 = 0$ we have $H_0(X) = C_0(X)/\mathrm{im}(\partial_1)$. Let $\epsilon : C_0(X) \to \mathbb{Z}$ be given by

$$\sum_i n_i\sigma_i \mapsto \sum_i n_i.$$

Clearly, ϵ is surjective. Let $\sigma : \Delta^n \to X$ be a 1-simplex. Then

$$\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]},$$

and so $\epsilon(\partial_1(\sigma)) = 1 - 1 = 0$, which implies that $\mathrm{im}(\partial_1) \subseteq \ker(\epsilon)$. On the other hand, $\epsilon\left(\sum_i n_i\sigma_i\right) = 0$ implies that $\sum_i n_i = 0$. The σ_i are singular 0-simplicies, which are just points of X. Let $x_0 \in X$ and let σ_0 be the corresponding 0-simplex. That is, $\sigma_0 : \Delta^0 \to X$ is given by $\Delta^0 = [v] \mapsto x_0$. Let $\tau_i : I \to X$ be a path from a base point x_0 to $\sigma_i([v])$. We can view τ_i as a singular 1-simplex since $[v_0, v_1]$ is homeomorphic to Δ^1. In particular, $\tau_i : [v_0, v_1] \to X$ is such that $\partial_1(\tau_i) = \sigma_i - \sigma_0$. Hence,

$$\partial_1\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i,$$

using the fact that $\sum_i n_i = 0$. This means that $\sum_i n_i\sigma_i \in \mathrm{im}(\partial_1)$ and so $\ker(\epsilon) \subseteq \mathrm{im}(\partial_1)$. Consequently, we have $\ker(\epsilon) = \mathrm{im}(\partial_1)$. Therefore, from the first isomorphism theorem, it follows that

$$H_0(X) = \ker(\partial_0)/\mathrm{im}(\partial_1) = C_0(X)/\ker(\epsilon) \cong \mathbb{Z}.$$

□

Remark 3.3.18. *From Proposition 3.3.16 and Proposition 3.3.17 we deduce that for an arbitrary space X the 0^{th} singular homology group $H_0(X)$, is a direct sum of $\mathbb{Z}$. In particular, the rank of this partial sum indicates the number of path-connected components of X.*

Proposition 3.3.19. *If X is a point, then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}$$

Proof. Since X is just a point, for each $n \in \mathbb{N}$ there exists a unique singular n-simplex, $\sigma_n : \Delta^n \to X$, so $C_n(X) \cong \mathbb{Z}$ for all $n \in \mathbb{N}$. Observe,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd}, \\ \sigma_{n-1} & n \text{ even}, \end{cases}$$

which shows that $\partial_n \equiv 0$ if n is odd, and an isomorphism if n is even. Therefore,

$$\begin{tikzcd} \dots \arrow[r, "\partial_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[d, "\sim"] & C_0(X) \arrow[r, "\partial_0"] \arrow[d, "\sim"] & \dots \\ \dots \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z} \arrow[r, "0"] & \dots \end{tikzcd}$$

So $H_n(X) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}) = 0$ for $n \geq 1$. Using Proposition 3.3.17 we know that $H_0(X) = \mathbb{Z}$ as X is path-connected. $\square$

3.3.3 Reduced Homology Group

Definition 3.3.20. *The reduced homology group $\widetilde{H_n}(X)$ is the homology group of the augmented chain*

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\partial_0} 0,$$

where $\epsilon : C_0(X) \to \mathbb{Z}$ is given by

$$\epsilon\left(\sum_i n_i \sigma_i\right) = \sum_i n_i.$$

Remark 3.3.21. *Recall from Proposition 3.3.17 that ϵ is surjective and $\epsilon \circ \partial_1 = 0$. Thus,*

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\partial_0} 0,$$

is a chain complex so that $\widetilde{H_n}(X)$ is well-defined.

Proposition 3.3.22. *For a space X we have $H_0(X) \cong \widetilde{H_0}(X) \oplus \mathbb{Z}$.*

Proof. Since $\epsilon : C_0(X) \to \mathbb{Z}$ is surjective it induces a surjective homomorphism $\phi_\epsilon : H_0(X) = C_0(X)/\mathrm{im}(\partial_1) \to \mathbb{Z}$. In particular, $\ker(\phi_\epsilon) = \ker(\epsilon)/\mathrm{im}(\partial_1) = \widetilde{H_0}(X)$. Therefore, by the first isomorphism theorem we have

$$H_0(X)/\widetilde{H_0}(X) \cong \mathbb{Z}$$

which implies that

$$H_0(X) \cong \widetilde{H_0}(X) \oplus \mathbb{Z}.$$

$\square$

Corollary 3.3.23. *If X is a point, then*

$$\widetilde{H_n}(X) = 0$$

for all $n \in \mathbb{N}$.

Proof. Follows immediately from Proposition 3.3.19 and Proposition 3.3.22. □

Note that for $n \geq 1$ we have $H_n(X) \cong \widetilde{H_n}(X)$ as the augmented chain is the same as the original chain complex at these points. For the augment chain complex we can also consider

$$\ker(\partial_0)/\mathrm{im}(\epsilon) = \mathbb{Z}/\mathbb{Z} = \{0\}.$$

3.4 Homotopy Invariance

Definition 3.4.1. *For chain complexes $(A_\bullet, \partial^A)$ and $(B_\bullet, \partial^B)$ a chain map $f : (A_\bullet, \partial^A) \to (B_\bullet, \partial^B)$ is collection of homomorphisms $f_n : A_n \to B_n$ such that $\partial_n^B \circ f_n = f_{n-1} \circ \partial_n^A$.*

$$\begin{tikzcd} \dots \arrow[r, "\partial^A_{n+2}"] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "\partial^A_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r, "\partial^B_{n+2}"] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r, "\partial^B_{n-1}"] & \dots \end{tikzcd}$$

Remark 3.4.2.

1. *For our purposes, we have $\partial_n^A = \partial_n^B$ being the boundary map.*
2. *A chain map gives a mechanism to transfer between chain complexes. For topological spaces X and Y let $f : X \to Y$ be a map and let $f_\# : C_n(X) \to C_n(Y)$ by*

$$f_\#(\sigma) = f \circ \sigma$$

for $\sigma : \Delta^n \to X$ and extend it linearly to elements of $C_n(X)$. Observe that

$$\begin{aligned} (f_\# \circ \partial)(\sigma) &= f_\# \left(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \right) \\ &= \sum_{i=0}^{n} (-1)^i (f \circ \sigma)|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \\ &= (\partial \circ f_\#)(\sigma) \end{aligned}$$

so $f_\# \circ \partial = \partial \circ f_\#$ meaning $f_\#$ is a chain map between $(C_\bullet(X), \partial)$ and $(C_\bullet(Y), \partial)$. Note that for $\alpha \in C_n(X)$ with $\partial(\alpha) = 0$ we have

$$(\partial \circ f_\#)(\alpha) = (f_\# \circ \partial)(\alpha) = 0$$

and so $f_\#$ maps cycles to cycles. Similarly, as

$$f_\# \circ (\partial(\beta)) = \partial \circ (f_\#(\beta))$$

we deduce that boundaries are mapped to boundaries. In particular,

$$f_\#(\ker(\partial_n)) \subseteq \ker(\partial_n)$$

and

$$f_\#(\mathrm{im}(\partial_{n+1}) \subseteq \mathrm{im}(\partial_{n+1})$$

so that f induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

- *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_* = g_* \circ f_*$.*
- $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

Despite Δ^n being a simplex, the space $\Delta^n \times I$ can be divided into $(n+1)$-simplices. Let

$$\Delta^n \times \{0\} = [v_0, \dots, v_n]$$

and

$$\Delta^n \times \{1\} = [w_0, \dots, w_n],$$

where v_i and w_i have the same projection under $\Delta^n \times I \to \Delta^n$. We can interpolate from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ through a sequence of n-simplices. More specifically, starting from $[v_0, \dots, v_n]$ we move to

$$[v_0, \dots, v_{n-1}, w_n],$$

then to

$$[v_0, \dots, v_{n-2}, w_{n-1}, w_n]$$

until arriving at $[w_0, \dots, w_n]$. The region between $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ and $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$ is the $(n+1)$ simplex

$$[v_0, \dots, v_i, w_i, \dots, w_n].$$

Altogether, we see that $\Delta^n \times I$ is the union of the $(n+1)$-simplices given by

$$[v_0, \dots, v_i, w_i, \dots, w_n],$$

that intersect along the face of the next n-simplex in the sequence.

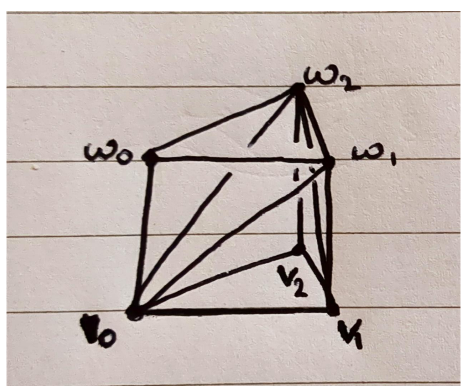


Figure 3.4.1: The decomposition of $\Delta^3 \times I$.

Theorem 3.4.3. *If $f, g : X \to Y$ are homotopic then $f_* = g_*$.*

Proof. Let $F : X \times I \to Y$ be a homotopy from f to g. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Consider

$$\Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{F} Y.$$

Using the decomposition of $\Delta^n \times I$ into $(n+1)$-simplices, consider $P : C_n(X) \to C_{n+1}(Y)$ given by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \dots, v_i, w_i, \dots, w_n]}.$$

Let $\sigma : \Delta^n \to X$ be a singular n-simplex, then

$$\begin{aligned}(\partial \circ P)(\sigma) &= \partial \left(\sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]} \right) \\ &= \sum_{i=0}^{n} \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]} \\ &\quad + \sum_{i=0}^{n} \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]}.\end{aligned}$$

If $i = j$, then the terms cancel except for

$$F \circ (\sigma \times \mathrm{id})|_{[\hat{v}_0,w_0,\ldots,w_n]} = g \circ \sigma = g_{\#}(\sigma)$$

and

$$F \circ (\sigma \times \mathrm{id})|_{[v_0,\ldots,v_n,\hat{w}_n]} = -f \circ \sigma = -f_{\#}(\sigma).$$

The terms $i \neq j$ sum up to $-(P \circ \partial)(\sigma)$ since

$$\begin{aligned}(P \circ \partial)(\sigma) &= \sum_{i=0}^{n} \sum_{j<i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]} \\ &\quad + \sum_{i=0}^{n} \sum_{j>i} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]}.\end{aligned}$$

Hence,

$$\partial \circ P = g_{\#} - f_{\#} - P \circ \sigma.$$

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, bend right, "f_{\#}"'] \arrow[d, shift left, bend left, "g_{\#}"] & C_{n-1}(X) \arrow[dl, "P"'] \\ C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \end{tikzcd}$$

Now if $\alpha \in C_n(X)$ is a cycle, that is $\partial(\alpha) = 0$, then

$$g_{\#}(\alpha) - f_{\#}(\alpha) = (\partial \circ P)(\alpha) + (P \circ \partial)(\alpha) = (\partial \circ P)(\alpha),$$

meaning $g_{\#}(\alpha) - f_{\#}(\alpha)$ is a boundary. Thus, $g_{\#}(\alpha) - f_{\#}(\alpha)$ is in the identity homology class since the homology group quotients out the boundary. This means that $g_{\#}(\alpha)$ and $f_{\#}(\alpha)$ are in the same homology class and so $g_*([\alpha]) = f_*([\alpha])$. □

Remark 3.4.4. *The relation*

$$\partial \circ P + P \circ \partial = g_{\#} - f_{\#} \tag{3.4.1}$$

is expressed by saying that P is a chain homotopy between the chain maps $f_{\#}$ and $g_{\#}$. The proof of Theorem 3.4.3 shows that if (3.4.1) *is satisfied then the chain maps induce the same homomorphism on homology.*

Corollary 3.4.5. *If $f : X \to Y$ is a homotopy equivalence, then f_* is an isomorphism.*

Proof. Let $g : Y \to X$ be a continuous map such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. Then

$$f_* g_* = (fg)_* = (\mathrm{id}_Y)_* = \mathrm{id}.$$

Similarly, $g_* f_* = \mathrm{id}$ which implies that f_* is an isomorphism. □

Similar results hold for reduced homology groups, as given in Definition 3.3.20. Let $f : X \to Y$ be a map, and then construct the chain map $f_\# : C_n(X) \to C_n(Y)$ for the chains $(C_\bullet(X), \partial)$ and $(C_\bullet(Y), \partial)$ as in statement 2 of Remark 3.4.2. Then for the corresponding augmented chain complexes, we let $f_\# : \mathbb{Z} \to \mathbb{Z}$ be the identity map. Doing so we observe that

$$
\begin{aligned}
(f_\# \circ \epsilon)(\sigma) &= \epsilon\left(\sum_i n_i \sigma_i\right) \\
&= \sum_i n_i \\
&= \sum_i f_\#(n_i) \\
&= \epsilon\left(\sum_i f_\#(n_i)\sigma_i\right) \\
&= \left(\epsilon \circ f_\#\right)(\sigma).
\end{aligned}
$$

Therefore, we have

$$
\begin{tikzcd}
\dots \arrow[r] & C_1(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_0(X) \arrow[r, "\epsilon"] \arrow[d, "f_\#"] & \mathbb{Z} \arrow[r] \arrow[d, "f_\#"] & 0 \\
\dots \arrow[r] & C_1(Y) \arrow[r, "\partial"] & C_0(Y) \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

in turn constructing a chain map for the augmented chains of $(C_\bullet(X), \partial)$ and $(C_\bullet(Y), \partial)$. Let the prism operator $P : \mathbb{Z} \to C_0(Y)$ be the zero map such that for $v \in C_0(X)$ we have

$$
\begin{aligned}
\partial(P(v)) - P(\partial(v)) &= \partial\left(F \circ (v \times \mathrm{id})\right) - 0 \\
&= F \circ (v \times \mathrm{id})|_{[w]} - F \circ (v \times \mathrm{id})|_{[v]} \\
&= g_\#(v) - f_\#(v).
\end{aligned}
$$

Therefore, P is a chain homotopy between the extended chain maps $f_\#$ and $g_\#$ for the augmented chains of $(C_\bullet(X), \partial)$ and $(C_\bullet(Y), \partial)$.

$$
\begin{tikzcd}
C_1(X) \arrow[r, "\partial"] & C_0(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, bend right, "f_\#"'] \arrow[d, bend left, "g_\#"] & \mathbb{Z} \arrow[r] \arrow[dl, "P"'] & 0 \\
C_1(Y) \arrow[r, "\partial"] & C_0(Y) \arrow[r, "\partial_n"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

Hence, the proof of Theorem 3.4.3 holds for the augmented chain complexes, allowing us to conclude that the induced homomorphism between reduced homology groups is invariant under homotopy. Thus, we obtain a result similar to Corollary 3.4.5, which says that if $f : X \to Y$ is a homotopy equivalence then f_* is an isomorphism between the reduced homology groups.

Example 3.4.6. *Let X be a nonempty set.*

1. *Endow X with the trivial topology $\{\emptyset, X\}$, then X is homotopic to a point x_0 through the homotopy $H : X \times I \to X$ given by*

$$
H_t(x) = \begin{cases} x & t \in [0,1) \\ x_0 & t = 1. \end{cases}
$$

   Therefore, with Corollary 3.4.5 it follows by Proposition 3.3.19 that

$$
H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}
$$

2. *Endow X with the discrete topology, then the path-connected components of X are the individual points*

of X. Therefore, by Proposition 3.3.16 we have that

$$H_n(X) \cong \bigoplus_{x\in X} H_n(\{x\}) = \begin{cases} \oplus_{x\in X}\mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}$$

Exercise 3.4.7. *For $k \in \mathbb{N}$ find $\widetilde{H_n}\left(\mathbb{R}^k\right)$ for every $n \in \mathbb{N}$.*

3.5 Exact Sequences and Excision

3.5.1 Exact Sequences

Relationships between $H_n(X), H_n(A)$ and $H_n(X/A)$ for $A \subseteq X$ are useful to establish as CW-complexes are built inductively from subspaces.

Definition 3.5.1. *A sequence of group homomorphisms of abelian groups,*

$$\ldots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} \ldots,$$

is exact at A_n if $\ker(\alpha_n) = \operatorname{im}(\alpha_{n+1})$. A sequence

$$\ldots \xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \ldots,$$

is exact if it is exact at A_n for all $n \in \mathbb{N}$.

Remark 3.5.2. *Let $(A_\bullet, \alpha)$ be an exact sequence.*

1. *Using Lemma 3.3.6 it follows that $(A_\bullet, \alpha)$ is a chain complex.*
2. *The homology groups*

$$H_n(A_\bullet, \alpha) = \ker(\alpha_n)/\operatorname{im}(\alpha_{n+1}) = \ker(\alpha_n)/\ker(\alpha_n) = \{0\}$$

   are trivial.

Exercise 3.5.3.

1. *Show that $0 \longrightarrow A \xrightarrow{\alpha} B$ is exact if and only if $\ker(\alpha) = \{0\}$.*
2. *Show that $A \xrightarrow{\alpha} B \longrightarrow 0$ is exact if and only if $\operatorname{im}(\alpha) = B$.*
3. *Show that $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ is exact if and only if α is an isomorphism.*
4. *Show that $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is exact if and only if the following hold.*
   (a) $\ker(\alpha) = \{0\}$.
   (b) $\ker(\beta) = \operatorname{im}(\alpha)$.
   (c) $\operatorname{im}(\beta) = C$.

   In particular, we have that $C \cong B/A$.

Remark 3.5.4. *An exact sequence of the form of statement 4 in Exercise 3.5.3 is referred to as a short exact sequence.*

Lemma 3.5.5. *Consider the following commutative diagram of abelian group homomorphisms, where the rows are exact.*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\epsilon"] & B \arrow[r, "\beta"] \arrow[d, "\zeta"] & C \arrow[r, "\gamma"] \arrow[d, "\eta"] & D \arrow[r, "\delta"] \arrow[d, "\theta"] & E \arrow[d, "\iota"] \\ A' \arrow[r, "\kappa"] & B' \arrow[r, "\lambda"] & C' \arrow[r, "\mu"] & D' \arrow[r, "\nu"] & E' \end{tikzcd}$$

Then if ϵ, ζ, θ and ι are isomorphism then η is an isomorphism.

Proof.

1. Let $c' \in C'$. Then $\mu(c') \in D'$ so that there exists a $d \in D$ such that

$$\theta(d) = \mu(c')$$

as θ is an isomorphism. In particular,

$$\nu(\theta(d)) = \iota(\delta(d))$$

due to commutativity. By the exactness of D' we have $\nu(\theta(d)) = 0$ and so $\iota(\delta(d)) = 0$. As ι is an isomorphism it follows that $\delta(d) = 0$. Thus, as D is exact

$$d = \gamma(c)$$

for some $c \in C$. In particular,

$$\theta(\gamma(c)) = \theta(d) = \mu(c').$$

As $\theta(\gamma(c)) = \mu(\eta(c))$ by commutativity it follows that

$$\mu(\eta(c) - c') = 0$$

as μ is a homomorphism. Therefore, by the exactness at C' we have

$$\eta(c) - c' = \lambda(b')$$

for some $b' \in B'$. As ζ is an isomorphism $b' = \zeta(b)$ for some $b \in B$, and thus

$$\lambda(b') = \lambda(\zeta(b)) = \eta(\beta(b))$$

where the second equality is by commutativity. Hence,

$$\eta(c) - c' = \eta(\beta(b))$$

and so

$$\eta(c - \beta(b)) = c'$$

which shows that η is surjective.

2. Let $\eta(c) = 0$. Then $\mu(\eta(c)) = 0$. Therefore, by commutativity, we have that $\theta(\gamma(c)) = 0$. By the injectivity of θ it follows that $\gamma(c) = 0$. Using the exactness at C it follows that $c = \beta(b)$ for some $b \in B$. Thus,

$$0 = \eta(c) = \eta(\beta(b)) = \lambda(\zeta(b))$$

where the last equality follow by commutativity. Using the exactness of B' we have

$$\zeta(b) = \kappa(a')$$

for some $a' \in A'$. In particular, as ϵ is an isomorphism there exists an $a \in A$ such that

$$\kappa(a') = \kappa(\epsilon(a)) = \zeta(\alpha(a))$$

where the second equality follows by commutativity. Therefore,

$$\zeta(b) = \zeta(\alpha(a))$$

and so by the injectivity of ζ we have that

$$b - \alpha(a) = 0.$$

Applying β, it follows that

$$c = \beta(\alpha(a)) = 0,$$

where the second equality follows by exactness at B. Thus, η is injective.

By combining 1 and 2 it follows that η is bijective and thus an isomorphism. □

Definition 3.5.6. *Let $A \subseteq X$, for X a topological space. Then A is a strong deformation retract of X if there is a deformation retraction $r : X \to A$, and a map $F : I \times X \to X$ such that for $x \in X, a \in A$ and $t \in I$ we have*

1. $F(0, x) = x$,
2. $F(1, x) = r(x)$, *and*
3. $F(t, a) = a$.

Remark 3.5.7. *Statement 3 of Definition 3.5.6 is the strong aspect of a strong deformation retraction since the other conditions hold under a regular retraction. A strong deformation retraction requires a homotopy which fixes the set A.*

Definition 3.5.8. *Let $\emptyset \neq A \subseteq X$ be closed, for X a topological space. Then (X, A) is a good pair if A has a neighbourhood in X that is a strong deformation retract to A.*

Example 3.5.9.

1. *The pair $\left(\mathrm{D}^n, \mathrm{S}^{n-1}\right)$ is a good pair, since S^{n-1} is a deformation retraction of $\mathrm{D}^n \setminus \{0\}$.*

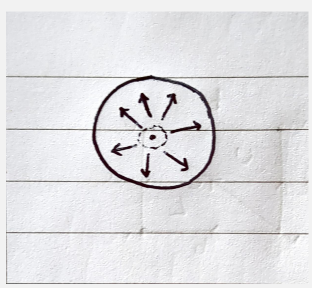


Figure 3.5.1

2. *Let*

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \subseteq [0, 1].$$

Then $([0, 1], A)$ is not a good pair. To see this, note that any neighbourhood $U \subseteq X$ of A necessarily contains a path connected component $\tilde{U} \subseteq U$ such that $\left|\tilde{U} \cap A\right| \geq 2$. A deformation retract $r : U \to A$ will be such that $r\left(\tilde{U}\right)$ is not path connected, which contradicts the connectedness of $\tilde{U}$.

3. *Let X be a CW complex, and let A be a non-empty subcomplex of X. Then (X, A) is a good pair.*

Theorem 3.5.10. *Let (X, A) be a good pair, then there is an exact sequence*

$$\cdots \longrightarrow \widetilde{H_1}(A) \xrightarrow{i_*} \widetilde{H_1}(X) \xrightarrow{j_*} \widetilde{H_1}(X/A) \xrightarrow{\partial} \widetilde{H_0}(A) \xrightarrow{i_*} \widetilde{H_0}(X) \xrightarrow{j_*} \widetilde{H_0}(X/A) \longrightarrow 0, \tag{3.5.1}$$

where $i : A \hookrightarrow X$ is the inclusion map and $j : X \to X/A$ is the quotient map.

Corollary 3.5.11. *In the setting of Theorem 3.5.10 we have that*

$$\widetilde{H_i}\left(\mathrm{S}^n\right) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Proof. Recall from statement 1 of Example 3.5.9 that $\left(\mathrm{D}^n, \mathrm{S}^{n-1}\right)$ is a good pair. Moreover, for $n > 0$ recall that $\mathrm{D}^n/\mathrm{S}^{n-1} \cong \mathrm{S}^n$. Therefore,

$$\cdots \to \widetilde{H_i}\left(\mathrm{S}^{n-1}\right) \xrightarrow{i_*} \underbrace{\widetilde{H_i}\left(\mathrm{D}^n\right)}_{0} \xrightarrow{j_*} \widetilde{H_i}\left(\mathrm{S}^n\right) \xrightarrow{\partial} \widetilde{H_{i-1}}\left(\mathrm{S}^{n-1}\right) \xrightarrow{i_*} \underbrace{\widetilde{H_{i-1}}\left(\mathrm{D}^n\right)}_{0} \xrightarrow{j_*} \widetilde{H_{i-1}}\left(\mathrm{S}^n\right) \to \cdots.$$

Thus, using statement 4 of Exercise 3.5.3, for $n > 0$ we have $\widetilde{H_i}\left(\mathrm{S}^n\right) \cong \widetilde{H_{i-1}}\left(\mathrm{S}^{n-1}\right)$ for $i > 0$ and $\widetilde{H_0}\left(\mathrm{S}^n\right) = 0$.

- Since S^0 is just two points, from Proposition 3.3.16 and Proposition 3.3.19 it follows that that $\widetilde{H_0}\left(\mathrm{S}^0\right) = \mathbb{Z}$ and $\widetilde{H_i}\left(\mathrm{S}^0\right) = 0$ for $i > 0$.
- Suppose that

$$\widetilde{H_i}\left(\mathrm{S}^{n-1}\right) = \begin{cases} \mathbb{Z} & i = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

  Then

$$\widetilde{H_i}\left(\mathrm{S}^n\right) = \widetilde{H_{i-1}}\left(\mathrm{S}^{n-1}\right) = \begin{cases} \mathbb{Z} & i-1 = n-1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we conclude by induction that

$$\widetilde{H_i}\left(\mathrm{S}^n\right) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 3.5.12. *There exists no retraction $r : \mathrm{D}^n \to \partial\mathrm{D}^n$.*

Proof. Assume that $r : \mathrm{D}^n \to \partial\mathrm{D}^n$ is a retraction. Let $i : \partial\mathrm{D}^n \to \mathrm{D}^n$ be the inclusion map such that $ri = \mathrm{id}_{\partial\mathrm{D}^n}$, then $r_* i_* = (ri)_* = \mathrm{id}$. However,

$$\underbrace{\widetilde{H_{n-1}}\left(\partial\mathrm{D}^n\right)}_{\mathbb{Z}} \xrightarrow{i_*} \underbrace{\widetilde{H_{n-1}}\left(\mathrm{D}^n\right)}_{0} \xrightarrow{r_*} \underbrace{\widetilde{H_{n-1}}\left(\partial\mathrm{D}^n\right)}_{\mathbb{Z}},$$

implies that $i_* = 0$ and $r_* = 0$ which gives a contradiction. □

Theorem 3.5.13 (Brouwer Fixed Point Theorem)**.** *Every continuous map $f : \mathrm{D}^n \to \mathrm{D}^n$ has a fixed point.*

Proof. Assuming no such fixed point exists one can construct a retraction $r : \mathrm{D}^n \to \partial\mathrm{D}^n$ in the same way as done in the proof of Theorem 2.2.43. This contradicts Corollary 3.5.12 and so f must have a fixed point. □

3.5.2 Relative Homology Group

We introduce relative homology groups to prove Theorem 3.5.10. Intuitively relative homology groups allow one to ignore certain data and structures. More specifically, for a topological space X and $A \subseteq X$ let

$$C_n(X, A) = C_n(X)/C_n(A).$$

Effectively, $C_n(X, A)$ ignores the chains in the subspace A. Let $\partial : C_n(X) \to C_{n-1}(X)$ be the boundary map so that

$$\partial(\sigma : \Delta^n \to A) \in \partial(C_n(A)) \subseteq C_{n-1}(A).$$

This induces the homomorphism $\partial : C_n(X, A) \to C_{n-1}(X, A)$ with $\partial \circ \partial = 0$ so that we have the chain complex

$$\cdots \longrightarrow C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \longrightarrow \dots$$

for which we make the following remarks.

- The homology groups are the relative homology groups, $H_n(X, A)$.
- The relative n-chains are elements of $C_n(X, A)$.
- The relative n-cycles are elements $[\alpha]$ of $\ker(\partial) \subseteq C_n(X, A)$ such that $\partial(\alpha) \in C_{n-1}(A)$.
- The relative n-boundaries are elements $[\alpha]$ of $\mathrm{im}(\partial) \subseteq C_n(X, A)$ such that $\alpha = \partial(\beta) + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Definition 3.5.14. *A short exact sequence of chain complexes is*

$$0 \to (A_\bullet, \delta) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \longrightarrow 0$$

where i and j are chain maps such that

$$0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \longrightarrow 0$$

is a short exact sequence for every $n \in \mathbb{N}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "\partial"] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i"] & \dots \\
\dots \arrow[r, "\partial"] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "j"] & \dots \\
\dots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It will be interesting to understand whether one could zig-zag along a short exact sequence of chain complexes, to encounter the homology groups of the different groups. In particular, we would like to consider

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to \dots.$$

However, to do so requires an extension of a connecting map $\partial : H_n(C) \to H_{n-1}(A)$.

Definition 3.5.15. *Let*
$$0 \to (A_\bullet, \delta) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \longrightarrow 0$$
be a short exact sequence of chain complexes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \dots \\
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $c \in C_n$ be a cycle. Then $c = j(b)$ for some $b \in B_n$, as j is surjective by exactness. Then
$$j(\partial(b)) = \partial(j(b)) = \partial c = 0,$$
and so $\partial b \in \ker(j) \subseteq B_{n-1}$. Thus, $\partial(b) = i(a)$ for some $a \in A_{n-1}$. In particular,
$$i(\partial(a)) = \partial(i(a)) = \partial(\partial(b)) = 0,$$
which implies that $\partial(a) = 0$ as i is injective by exactness.

$$\begin{tikzcd}
 & a \in A_{n-1} \arrow[d, mapsto, "i"] \\
b \in B_n \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "j"] & \partial b \in B_{n-1} \\
c \in C_n &
\end{tikzcd}$$

Let $\partial : H_n(C) \to H_{n-1}(A)$ be the map given by $[c] \mapsto [a]$.

Exercise 3.5.16. *Show that the map $\partial : H_n(C) \to H_{n-1}(A)$ of Definition 3.5.15 is well-defined and a homomorphism.*

Theorem 3.5.17 (Zig-Zag)**.** *Suppose*
$$0 \to (A_\bullet, \delta) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \longrightarrow 0$$
is a short exact sequence of chain complexes. Then the sequence
$$\dots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to \dots,$$
is exact.

Proof.

- Since $ji = 0$, we have $j_* i_* = 0$ which implies that $\mathrm{im}(i_*) \subseteq \ker(j_*)$.

- We have $\partial j_* = 0$ since in this case $\partial b = 0$ from Definition 3.5.15, which implies that $\operatorname{im}(j_*) \subseteq \ker(\partial)$.
- Since $i_*\partial$ takes $[c]$ to $[\partial b] = 0$, we have that $i_*\partial = 0$ which implies that $\operatorname{im}(\partial) \subseteq \ker(i_*)$.
- A homology class in $\ker(j_*)$ is represented by a cycle $b \in B_n$ with $j(b)$ a boundary, such that $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjective, we have $c' = j(b')$ for some $b' \in B_{n+1}$. In particular,

$$j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial(j(b')) = 0$$

since $\partial j(b') = \partial(c') = j(b)$. Therefore, $b - \partial(b') = i(a)$ for some $a \in A_n$. This a is a cycle since

$$i(\partial(a)) = \partial(i(a)) = \partial(b - \partial(b')) = \partial(b) = 0,$$

and i is injective. Hence,

$$i_*([a]) = [b - \partial b'] = [b],$$

which shows that $\ker(j_*) \subseteq \operatorname{im}(i_*)$.

- With the notation as in Definition 3.5.15, if c represents a homology class in $\ker(\partial)$, then $a = \partial(a')$ for some $a' \in A_n$. Then $b - i(a')$ is a cycle since

$$\partial(b - i(a')) = \partial(b) - \partial(i(a')) = \partial(b) - i(\partial(a')) = \partial(b) - i(a) = 0.$$

Moreover,

$$j(b - i(a')) = j(b) - j(i(a')) = j(b) = c,$$

and so j_* maps $[b - i(a')]$ to $[c]$, hence $\ker(\partial) \subseteq \operatorname{im}(j_*)$.

□

With this, for any pair (X, A), consider the sequence of short exact sequences

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0,$$

for i the inclusion map and j the quotient map. Using Theorem 3.5.17 we obtain a long exact sequence of homology groups,

$$\cdots \to H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \to 0.$$

More specifically, if $[a] \in H_n(X, A)$ is represented by a cycle $\alpha \in C_n(X)$, then $\partial([\alpha])$ is the class of the cycle $\partial(a)$. Hence, $\partial : H_n(X, A) \to H_{n-1}(A)$ is given by $\partial([\alpha]) = [\partial(\alpha)]$. In particular, this long exact sequence of homology group formalises the intuition that $H_n(X, A)$ measures the difference between the groups $H_n(X)$ and $H_n(A)$. Indeed, exactness implies that if $H_n(X, A) = 0$ then $A \hookrightarrow X$ induces an isomorphism $H_n(A) \cong H_n(X)$. We can also consider the sequence of short exact sequences provided by the augment chains.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "\partial"] & C_1(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_0(A) \arrow[r, "\epsilon"] \arrow[d, "i"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r, "\partial"] & C_1(X) \arrow[r, "\partial"] \arrow[d, "j"] & C_0(X) \arrow[r, "\epsilon"] \arrow[d, "j"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r, "\partial"] & C_1(X, A) \arrow[r, "\partial"] & C_0(X, A) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

In particular, note that when $A \neq \emptyset$ we have that $\widetilde{H_n}(X, A) \cong H_n(X, A)$. Thus, applying Theorem 3.5.17 to this sequence of short exact sequences yields the long exact sequence

$$\cdots \to \widetilde{H_n}(A) \to \widetilde{H_n}(X) \to \widetilde{H_n}(X, A) \to \widetilde{H_{n-1}}(A) \to \widetilde{H_{n-1}}(X) \to \widetilde{H_{n-1}}(X, A) \to \dots. \tag{3.5.2}$$

Observe the following.

1. If $A = \{x\} \subseteq X$, then since $\widetilde{H_n}(\{x\}) = 0$ for all $n \in \mathbb{N}$, it follows from (3.5.2) that $H_n(X, x) \cong \widetilde{H_n}(X)$.
2. A map $f : X \to Y$ such that $f(A) \subseteq B$ induces the chain map $f_\# : C_n(X, A) \to C_n(Y, B)$. Since $f_\#\partial = \partial f_\#$ holds for absolute chains it also holds for relative chains. Therefore, by statement 2 of Remark 3.4.2 we get an induced homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$. Such induced maps have the property that $(f \circ g)_* = f_* \circ g_*$.

Example 3.5.18. *For the pair $(\mathrm{D}^n, \partial\mathrm{D}^n)$, the maps $\partial : H_i(\mathrm{D}^n, \partial\mathrm{D}^n) \to \widetilde{H_{i-1}}(\mathrm{S}^{n-1})$ are isomorphisms for $i > 0$ since $\widetilde{H_i}(\mathrm{D}^n) = 0$ for all i. Thus,*

$$H_i(\mathrm{D}^n, \partial\mathrm{D}^n) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 3.5.19. *A homotopy between maps $f, g : (X, A) \to (Y, B)$ is a map $F : I \times X \to Y$ such that*

1. *$F(0, x) = f(x)$,*
2. *$F(1, x) = g(x)$, and*
3. *$F(s, a) \in B$,*

for all $x \in X$, $s \in I$ and $a \in A$.

Proposition 3.5.20. *If $f, g : (X, A) \to (Y, B)$ are homotopic then $f_* = g_*$.*

Proof. Recall the prism operator $P : C_n(X) \to C_{n+1}(Y)$ constructed in the proof of Theorem 3.4.3. The operator P maps $C_n(A)$ to $C_{n+1}(B)$ and so induces a map $P' : C_n(X)/C_n(A) \to C_{n+1}(Y)/C_{n+1}(B)$. As (3.4.1) remains valid after passing through quotients it follows by Remark 3.4.4 that $f_* = g_*$. ☐

For a topological space X consider the triple (X, A, B), where $B \subseteq A \subseteq X$ such that

$$(A, B) \to (X, B) \to (X, A).$$

This induces the short exact sequence of chain complexes

$$0 \to \underbrace{C_n(A, B)}_{C_n(A)/C_n(B)} \to \underbrace{C_n(X, B)}_{C_n(X)/C_n(B)} \to \underbrace{C_n(X, A)}_{C_n(X)/C_n(A)} \to 0$$

which in turn yields the long exact sequence

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to H_{n-1}(X, B) \to H_{n-1}(X, A) \to \dots$$

3.5.3 Excision

Let X be a topological space, and consider a collection of subspaces $\mathcal{U} = (U_j) \subseteq X$ whose interiors form an open cover of X. Then let $C_n^{\mathcal{U}}(X) \subseteq C_n(X)$ be the subgroup of chains of the form $\sum_i n_i\sigma_i$ where the image of each σ_i is contained in some $U_j \in \mathcal{U}$. Observe that the map $\partial : C_n(X) \to C_{n-1}(X)$ satisfies $\partial\left(C_n^{\mathcal{U}}(X)\right) \subseteq C_{n-1}^{\mathcal{U}}(X)$, and so the groups $C_n^{\mathcal{U}}(X)$ form a chain complex. Let $H_n^{\mathcal{U}}(X)$ be the homology groups of this chain complex. In particular, the inclusion map $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain map, so we get an induced homomorphism $i_* : H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$.

Remark 3.5.21. *If $\mathcal{U} = \{A, B\}$, then we denote $C_n^{\mathcal{U}}(X)$ by $C_n(A + B)$.*

Proposition 3.5.22. *The inclusion $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence, in the sense of Remark 3.4.4. Hence, i induces isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.*

Proof (Sketch). Step 1: Barycentric subdivision of simplices.
The barycentre of the simplex $[v_0, \dots, v_n]$ is the point $b = \frac{1}{n+1}\sum_i v_i$. Inductively, the barycentric subdivision of $[v_0, \dots, v_n]$ is the decomposition of $[v_0, \dots, v_n]$ into the n-simplices $[b, w_0, \dots, w_{n-1}]$ where $[w_0, \dots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \dots, \hat{v}_i, \dots, v_n]$. The n-simplices of the barycentric subdivision of Δ^n together with their faces form a Δ-complex structure on Δ^n. In particular, the diameter of each simplex of the barycentric subdivision of $[v_0, \dots, v_n]$ is at most $\frac{n}{n+1}$ times the diameter of $[v_0, \dots, v_n]$.

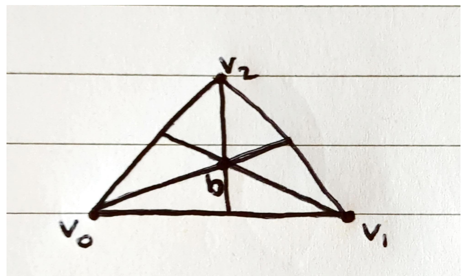


Figure 3.5.2: Barycentric subdivision of Δ^2.

Step 2: Barycentric subdivision of linear chains.
For a convex set Y in Euclidean space, the linear maps $\Delta^n \to Y$ generate a subgroup of $C_n(Y)$ referred to as the linear chains, $\mathrm{L}C_n(Y)$. The boundary map ∂ takes $\mathrm{L}C_n(Y)$ to $\mathrm{L}C_{n-1}(Y)$, so linear chains form a subcomplex of the singular chain complex of Y. By convention, $\mathrm{L}C_{-1}(Y) = \mathbb{Z}$. Each $b \in Y$ determines a homomorphism $b : \mathrm{L}C_n(Y) \to \mathrm{L}C_{n+1}(Y)$ by

$$b([w_0, \dots, w_n]) = [b, w_0, \dots, w_n].$$

One can see that $\partial b + b\partial = \mathrm{id}$, so that b is a chain homotopy between the identity map and the zero map of the augmented chain complex $\mathrm{L}C(Y)$. Let $\lambda : \Delta^n \to Y$ be a generator of $\mathrm{L}C_n(Y)$ with b_λ the image of the barycentre of Δ^n under λ. Inductively, $S(\lambda) = b_\lambda(S\partial\lambda)$ where $b_\lambda : \mathrm{L}C_{n-1}(Y) \to \mathrm{L}C_n(Y)$, where S is the identity on $\mathrm{L}C_{-1}(Y)$. One can check that $\partial S = S\partial$ so that S is a chain map from the chain complex $\mathrm{L}C(Y)$ to itself. Inductively let $T : \mathrm{L}C_n(Y) \to \mathrm{L}C_{n+1}(Y)$ be given by $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ for $n \geq 0$ and setting $T = 0$ for $n = -1$. One can show that $\partial T + T\partial = \mathrm{id} - S$ so that T is chain homotopy between S and the identity.

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{L}C_2(Y) \arrow[r] \arrow[d, "S"] & \mathrm{L}C_1(Y) \arrow[r] \arrow[d, "S"] \arrow[dl, "T"] & \mathrm{L}C_0(Y) \arrow[r] \arrow[d, "S=\mathrm{id}"] \arrow[dl, "T"] & \mathrm{L}C_{-1}(Y) \arrow[r] \arrow[d, "S=\mathrm{id}"] \arrow[dl, "T=0"] & 0 \\
\dots \arrow[r] & \mathrm{L}C_2(Y) \arrow[r] & \mathrm{L}C_1(Y) \arrow[r] & \mathrm{L}C_0(Y) \arrow[r] & \mathrm{L}C_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

Step 3: Barycentric subdivision of general chains.
Let $S : C_n(X) \to C_n(X)$ be given by $S\sigma = \sigma_\# S\Delta^n$ for a singular n-simplex $\sigma : \Delta^n \to X$. One can show that $\partial S\sigma = S\partial\sigma$ so that S is a chain map. Let $T : C_n(X) \to C_{n+1}(X)$ be given by $T\sigma = \sigma_\# T\Delta^n$. One can show that $\partial T + T\partial = \mathrm{id} - S$ so that T is a chain homotopy between S and the identity.
Step 4: Iterated barycentric subdivision.
A chain homotopy between id and the iterative S^m is given by $D_m = \sum_{0 \leq i < m} TS^i$ as it satisfies $\partial D_m + D_m\partial = \mathrm{id} - S^m$. For each singular n-simplex $\sigma : \Delta^n \to X$, there exists an $m \in \mathbb{N}$ such that $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$ since the diameter of $S^m(\Delta^n)$ will be less than the Lebesgue number of the cover of Δ^n by the open sets $\sigma^{-1}\left(\mathring{U}_j\right)$. Let $m(\sigma)$ be the smallest such m. Let $D : C_n(X) \to C_{n+1}(X)$ be given by $D\sigma = D_{m(\sigma)}\sigma$. Let $\rho = \mathrm{id} - \partial D - D\partial$. One can show that $\partial\rho = \rho\partial$ such that $\rho : C_n(X) \to C_n(X)$ is a chain map. In particular, we have that $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$, thus $\partial D + D\partial = \mathrm{id} - i\rho$. Furthermore, $\rho i = \mathrm{id}$ since D is identically zero on $C_n^{\mathcal{U}}(X)$ as $m(\sigma) = 0$ if $\sigma \in C_n^{\mathcal{U}}(X)$, meaning i is a chain homotopy equivalence with inverse ρ. $\square$

Remark 3.5.23.

1. *The Lebesgue number for an open cover of a compact metric space is a number $\epsilon > 0$ such that every set of diameter less than ϵ lies in some set of the cover.*
2. *Barycentric constructions allow for the computation of homology groups using singular simplices.*

Theorem 3.5.24. *Let X be a topological space with $Z \subseteq A \subseteq X$ subspaces such that the closure of Z is contained in the interior of A, $\bar{Z} \subseteq \mathring{A}$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces the isomorphism*

$$H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$$

for all $n \in \mathbb{N}$. Equivalently, if $A, B \subseteq X$ are such that $\mathring{A} \cup \mathring{B} = X$, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces the isomorphism

$$H_n(B, A \cap B) \cong H_n(X, A)$$

for all $n \in \mathbb{N}$.

Proof. Consider the case where $X = A \cup B$ and $\mathcal{U} = \{A, B\}$ is a cover. From the proof of Proposition 3.5.22 we deduce that $\partial D + D\partial = \mathrm{id} - i\rho$, and $\rho i = \mathrm{id}$. In particular, all these maps take chains of A to chains of A, so they induce quotient maps when we factor out chains in A. These quotient maps also satisfy the previous formula, thus the inclusion $C_n(A+B)/C_n(A) \to C_n(X)/C_n(A)$ induces an isomorphism on homology. The map $C_n(B)/C_n(A \cap B) \to C_n(A+B)/C_n(A)$ induced by the inclusion is an isomorphism since both quotient groups are free with a basis consisting of the singular n-simplices in B that do not lie in A. Hence, $H_n(B, A \cap B) \cong H_n(X, A)$. □

Remark 3.5.25.

1. *Theorem 3.5.24 gives conditions for which the impact of relative groups $H_n(X \setminus A)$ may be ignored.*
2. *The equivalent statement of Theorem 3.5.24 follows as one can set $B = X \setminus Z$ and $Z = X \setminus B$. Then $A \cap B = A \setminus Z$ and $\bar{Z} = X \setminus \mathring{B}$ such that $\bar{Z} \subseteq \mathring{A}$ if and only if $X = \mathring{A} \cup \mathring{B}$.*

Proposition 3.5.26. *Let (X, A) be a good pair. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphism $q_* : H_n(X, A) \to H_n(X/A, A/A)$ for all $n \in \mathbb{N}$.*

Proof. Let $V \subseteq X$ be a neighbourhood of A that strongly deformation retracts to A. Note that

$$\begin{tikzcd}
H_n(X, A) \arrow[r] \arrow[d, "q_*"] & H_n(X, V) \arrow[r] \arrow[d] & H_n(X \setminus A, V \setminus A) \arrow[d] \\
H_n(X/A, A/A) \arrow[r] & H_n(X/A, V/A) \arrow[r] & H_n((X/A) \setminus (A/A), (V/A) \setminus (A/A))
\end{tikzcd}$$

commutes.

- Note (V, A) is homotopy equivalent to A such that $H_n(V, A) \cong H_n(A, A) = 0$. As $A \subseteq V \subseteq X$, the triple (X, V, A) induces a long exact sequence

  $$\cdots \to \underbrace{H_n(V, A)}_{0} \to H_n(X, A) \to H_n(X, V) \to \underbrace{H_{n-1}(V, A)}_{0} \to \dots.$$

  Thus,

  $$H_n(X, A) \cong H_n(X, V).$$

  Therefore, the upper-left map of the diagram is an isomorphism.

- Similarly, with the triple $(X/A, V/A, A/A)$ and using $H_n(V/A, A/A) \cong H_n(A/A, A/A) = 0$ we get that

  $$H_n(X/A, A/A) \cong H_n(X/A, V/A).$$

  Therefore, the lower-left map of the diagram is an isomorphism. Hence, with statement 1 and the commutativity of the diagram, we get an induced isomorphism $H_n(X, V) \cong H_n(X/A, V/A)$.

- The upper-right and lower-right maps of the diagram are isomorphism by Theorem 3.5.24.

- The right-most vertical map of the diagram is an isomorphism since the quotient map $X \to X/A$ induces a homeomorphism $X/A \to (X/A) \setminus (A/A)$.

Therefore, by the commutativity of the diagram, it follows that the left-most vertical map q_* is an isomorphism. □

Remark 3.5.27. *Note that A/A is just a point, so that $H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$. Hence, using Proposition 3.5.26 we see that for a good pair (X, A) we have $H_n(X, A) \cong \widetilde{H}_n(X/A)$ for all $n \in \mathbb{N}$.*

Proof. (Theorem 3.5.10). Using Proposition 3.5.26 we deduce that

$$\widetilde{H}_n(X, A) \cong H_n(X, A) \cong \widetilde{H}_n(X/A).$$

Thus, using (3.5.2) we deduce the long exact sequence (3.5.1) as required. □

Corollary 3.5.28. *Let $(X_\alpha)_{\alpha \in A}$ be a collection of topological spaces with $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair for all $\alpha \in A$. Let $\bigvee_{\alpha \in A} X_\alpha$ denote the wedge sum with respect to the x_α. Then there is an isomorphism*

$$\widetilde{H}_n\left(\bigsqcup_\alpha X_\alpha\right) = \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \cong \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right).$$

Proof. Since, $(X, A) = (\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\})$ is a good pair, it follows by Proposition 3.5.26 that

$$H_n(X, A) \cong H_n\left(\bigvee_\alpha X_\alpha, \left(\bigsqcup_\alpha \{x_\alpha\}\right) / \left(\bigsqcup_\alpha \{x_\alpha\}\right)\right).$$

Since,

$$H_n\left(\bigvee_\alpha X_\alpha, \left(\bigsqcup_\alpha \{x_\alpha\}\right) / \left(\bigsqcup_\alpha \{x_\alpha\}\right)\right) \cong \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

and

$$H_n(X, A) \cong \bigoplus_\alpha H_n\left(X_\alpha, x_\alpha\right) \cong \bigoplus_\alpha \widetilde{H}_n\left(X_\alpha\right),$$

it follows that

$$\bigoplus_\alpha \widetilde{H}_n\left(X_\alpha\right) \cong \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

□

Example 3.5.29. *Using Corollary 3.5.28 it follows that*

$$\widetilde{H}_n\left(\mathrm{S}^1 \vee \mathrm{S}^1\right) \cong \widetilde{H}_n\left(\mathrm{S}^1\right) \oplus \widetilde{H}_n\left(\mathrm{S}^1\right) \cong \begin{cases} 0 & n = 0,\, n \geq 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1. \end{cases}$$

Similarly,

$$\widetilde{H_n}\left(\mathrm{S}^1 \vee \mathrm{S}^1 \vee \mathrm{S}^2\right) \cong \widetilde{H_n}\left(\mathrm{S}^1\right) \oplus \widetilde{H_n}\left(\mathrm{S}^1\right) \oplus \widetilde{H_n}\left(\mathrm{S}^2\right) \cong \begin{cases} 0 & n=0,\, n \geq 3 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2. \end{cases}$$

Recall from statement 2 of Example 3.3.10 that

$$H_n^{\Delta}\left(\mathrm{S}^1 \times \mathrm{S}^1\right) \cong \begin{cases} 0 & n=0,\, n \geq 3 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2. \end{cases}$$

In particular, using Theorem 4.1.2 we can deduce that

$$\widetilde{H_n}\left(\mathrm{S}^1 \times \mathrm{S}^1\right) \cong \begin{cases} 0 & n=0,\, n \geq 3 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2. \end{cases}$$

Hence, $\mathrm{S}^1 \vee \mathrm{S}^1 \vee \mathrm{S}^2$ *and* $\mathrm{S}^1 \times \mathrm{S}^1$ *have isomorphic homology groups. However,* $\pi_1\left(\mathrm{S}^1 \vee \mathrm{S}^1 \vee \mathrm{S}^2\right) \cong \mathbb{Z} * \mathbb{Z}$*, whereas* $\pi_1\left(\mathrm{S}^1 \times \mathrm{S}^1\right) \cong \mathbb{Z} \times \mathbb{Z}$*. Therefore, these spaces are not homotopy equivalent despite having isomorphic homology groups.*

Theorem 3.5.30. *Let* $U \subseteq \mathbb{R}^m$*,* $V \subseteq \mathbb{R}^n$ *be open and non-empty. Then, if* U *and* V *are homeomorphic then* $m = n$*.*

Proof. For $x \in U$, let $A := \mathbb{R}^m \setminus \{x\}$ and $B := U$. Then by Theorem 3.5.24 we have that

$$H_k(U, U \setminus \{x\}) \cong H_k\left(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}\right).$$

From the long exact sequence

$$\cdots \to \underbrace{\widetilde{H_k}\left(\mathbb{R}^m\right)}_{0} \to \widetilde{H_k}\left(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}\right) \to \widetilde{H_{k-1}}\left(\mathbb{R}^m \setminus \{x\}\right) \to \underbrace{\widetilde{H_{k-1}}\left(\mathbb{R}^m\right)}_{0} \to \dots,$$

it follows that

$$H_k\left(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}\right) \cong \widetilde{H_{k-1}}\left(\mathbb{R}^m \setminus \{x\}\right).$$

Since $\mathbb{R}^m \setminus \{x\}$ deformation retracts to S^{n-1} we have

$$H_k(U, U \setminus \{x\}) \begin{cases} \mathbb{Z} & k = m \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$H_k(V, V \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Let $h : U \to V$ be a homeomorphism, as $h_* : H_k(U, U \setminus \{x\}) \to H_k(V, V \setminus \{h(x)\})$ is an isomorphism for all k, Proposition 3.3.15 it must be the case that $m = n$. ☐

3.5.4 Naturality

Theorem 3.5.31. *Let $(A_\bullet, \partial)$, $(B_\bullet, \partial)$, $(C_\bullet, \partial)$, $(A'_\bullet, \partial)$, $(B'_\bullet, \partial)$, and $(C'_\bullet, \partial)$ be chain complexes that satisfy*

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & B_\bullet \arrow[r, "j"] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r, "i'"] & B'_\bullet \arrow[r, "j'"] & C'_\bullet \arrow[r] & 0 \end{tikzcd}$$

where each row is a short exact sequence. Then the induced diagram

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & \mathrm{H}_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_{n-1}(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_{n-1}(C) \arrow[r] \arrow[d, "\gamma_*"] & \dots \\ \dots \arrow[r] & \mathrm{H}_n(A') \arrow[r, "(i')_*"] & \mathrm{H}_n(B') \arrow[r, "(j')_*"] & \mathrm{H}_n(C') \arrow[r, "\partial"] & \mathrm{H}_{n-1}(A') \arrow[r, "(i')_*"] & \mathrm{H}_{n-1}(B') \arrow[r, "(j')_*"] & \mathrm{H}_n(C') \arrow[r] & \dots \end{tikzcd}$$

is commutative.

Proof.

- The first two squares from the left commute since $\beta i = i'\alpha$ implies that $\beta_* i_* = (i')_* \alpha_*$ and $\gamma j = j'\beta$ implies that $\gamma_* j_* = (j')_* \beta_*$.
- Recall, from Definition 3.5.15, that $\partial : H_n(C) \to H_{n-1}(A)$ is given by $\partial([c]) = [a]$, where $c = j(b)$ and $i(a) = \partial(b)$. As $\gamma(c) = \gamma(j(b)) = j'(\beta(b))$ and $i'(\alpha(a)) = \beta(i(a)) = \beta(\partial(b)) = \partial(\beta(b))$ we have that $\partial([\gamma(c)]) = [\alpha(a)]$. Hence, $\partial(\gamma_*[c]) = \alpha_*([a]) = \alpha_*(\partial([c]))$, which means that the third square from the left commutes.

□

3.6 Mayer-Vietoris Sequences

3.6.1 The Sequence

Theorem 3.6.1. *Let X be a topological space with $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$. Let*

- $i_1 : A \cap B \hookrightarrow A$,
- $i_2 : A \cap B \hookrightarrow B$,
- $j_1 : A \hookrightarrow X$, *and*
- $j_2 : B \hookrightarrow X$

be inclusion maps. Then we have the exact sequence

$$\cdots \to H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \xrightarrow{\Psi} H_1(X) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} H_0(X) \to 0,$$

where

1. $\Phi(x) = ((i_1)_*(x), -(i_2)_*(x))$,
2. $\Psi(x, y) = (j_1)_*(x) + (j_2)_*(y)$,
3. *and ∂ is the connecting homomorphism.*

Proof. Consider the sequence of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0,$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$. Note the following.

1. $\ker(\phi) = \{0\}$ since a chain in $A \cap B$ that is a zero chain in A, or in B, must be the zero chain. Hence, the sequence is exact at $C_n(A \cap B)$ by statement 1 of Exercise 3.5.3.

2. $\operatorname{im}(\phi) \subseteq \ker(\psi)$ as $\psi\varphi = 0$. Moreover, $\ker(\psi) \subseteq \operatorname{im}(\phi)$ since for any $(x, y) \in C_n(A) \oplus C_n(B)$, as $x + y = 0$ it follows that $x = -y$ and so x is a chain in both A and B. That is, $x \in C_n(A \cap B)$ and $(x, y) = (x, -x) \in \operatorname{im}(\phi)$. Hence, the sequence is exact at $C_n(A) \oplus C_n(B)$.

3. Exactness at $C_n(A + B)$ follows by definition.

Therefore, we have a short exact sequence of chain complexes, and so by Theorem 3.5.17 we have an induced long exact sequence

$$\cdots \to H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \xrightarrow{\Psi} H_1(X) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} H_0(X) \to 0.$$

□

Remark 3.6.2.

1. *Theorem 3.6.1 utility lies in inductive arguments. If one has results for A, B and $A \cap B$ one can argue by induction using Theorem 3.6.1 that the result is holds for $A \cup B$.*

2. *Note that in the proof of Theorem 3.6.1 we have used notation as given by Remark 3.5.21.*

If $A \cap B \neq \emptyset$ then we have an analogous sequence for the augmented chain complex. More specifically, we have the short exact sequence between the augmented chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_0(A \cap B) \arrow[r, "\phi"] \arrow[d, "\epsilon"] & C_0(A) \oplus C_0(B) \arrow[r, "\psi"] \arrow[d, "\epsilon"] & C_0(A + B) \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

which induces the long exact sequence of homology groups

$$\cdots \to \widetilde{H_1}(A \cap B) \xrightarrow{\Phi} \widetilde{H_1}(A) \oplus \widetilde{H_1}(B) \xrightarrow{\Psi} \widetilde{H_1}(X) \xrightarrow{\partial} \widetilde{H_0}(A \cap B) \xrightarrow{\Phi} \widetilde{H_0}(A) \oplus \widetilde{H_0}(B) \xrightarrow{\Psi} \widetilde{H_0(X)} \to 0,$$

referred to as the Mayer-Vietoris sequence for reduced homology groups. If $A \cap B$ is additionally path connected then $\widetilde{H_0}(A \cap B) = 0$ and so the exact sequence

$$\cdots \to \widetilde{H_1}(A \cap B) \xrightarrow{\Phi} \widetilde{H_1}(A) \oplus \widetilde{H_1}(B) \xrightarrow{\Psi} \widetilde{H_1}(X) \xrightarrow{\partial} \underbrace{\widetilde{H_0}(A \cap B)}_{0} \to \ldots$$

implies that

$$H_1(X) \cong H_1(A) \oplus H_1(B)/\Phi(H_1(A \cap B)).$$

Example 3.6.3. *Let $X = \mathrm{S}^n \subseteq \mathbb{R}^{n+1}$ with $x \in \mathrm{S}^n$. Let $A := \mathrm{S}^n \setminus \{x\}$ and $B = \mathrm{S}^n \setminus \{-x\}$. Then A and B are contractible so that $\widetilde{H_n}(A) = \widetilde{H_n}(B) = 0$ for all n, and $A \cap B$ deformation retracts to S^{n-1}. In particular, by Theorem 3.6.1 we have*

$$\cdots \to \underbrace{\widetilde{H_i}(A) \oplus \widetilde{H_i}(B)}_{0} \to \widetilde{H_i}(X) \to \underbrace{\widetilde{H_{i-1}}(A \cap B)}_{\widetilde{H_{i-1}}} \to \underbrace{\widetilde{H_{i-1}(\mathrm{S}^{n-1})}(A) \oplus \widetilde{H_{i-1}}(B)}_{0} \to \ldots$$

which implies that $\widetilde{H_i}(\mathrm{S}^n) \cong \widetilde{H_{i-1}}(\mathrm{S}^{n-1})$ *for* $n \geq 1$. *As*

$$\widetilde{H_0}(\mathrm{S}^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & \textit{otherwise,} \end{cases}$$

it follows that

$$\widetilde{H_k}(\mathrm{S}^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & \textit{otherwise.} \end{cases}$$

3.6.2 Application

Definition 3.6.4. *A continuous map* $\phi : X \to Y$ *between topological spaces is an embedding if it is homeomorphic to its image.*

Proposition 3.6.5. *For* $h : \mathrm{D}^k \to \mathrm{S}^n$ *an embedding. Then*

$$\widetilde{H_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{D}^k\right)\right) = 0$$

for all i.

Proof.

- For $k = 0$, we have $\mathrm{S}^n \setminus h\left(\mathrm{D}^0\right) \cong \mathbb{R}^n$ and so

$$\widetilde{H_i}(\mathrm{S}^n \setminus h(\mathrm{D}^n)) = 0$$

  for all n.

- Suppose $\widetilde{H_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{D}^{k-1}\right)\right) = 0$ for all n. Now let $h : I^k \to \mathrm{S}^n$, where we replace D^k with I^k for convenience. For contradiction assume there is a cycle α in $\mathrm{S}^n \setminus h\left(I^k\right)$ that is not a boundary in $\mathrm{S}^n \setminus h\left(I^k\right)$. Let $A := \mathrm{S}^n \setminus h\left(I^{k-1} \times \left[0, \frac{1}{2}\right]\right)$ and $B := \mathrm{S}^n \setminus h\left(I^{k-1} \times \left[\frac{1}{2}, 1\right]\right)$ such that $A \cap B = \mathrm{S}^n \setminus h\left(I^k\right)$ and $A \cup B = \mathrm{S}^n \setminus h\left(I^{k-1} \times \left\{\frac{1}{2}\right\}\right)$. Thus by assumption $\widetilde{H_j}(A \cup B) = 0$ for all j, and so from the long exact sequence provided by Theorem 3.6.1 it follows that

$$\widetilde{H_k}\left(\mathrm{S}^n \setminus h\left(I^k\right)\right) \cong \widetilde{H_j}\left(\mathrm{S}^n \setminus h\left(I^{k-1} \times \left[0, \frac{1}{2}\right]\right)\right) \oplus \widetilde{H_j}\left(\mathrm{S}^n \setminus h\left(I^{k-1} \times \left[\frac{1}{2}, 1\right]\right)\right).$$

  Hence, α is a cycle but not a boundary in $\mathrm{S}^n \setminus h\left(I^{k-1} \times \left[0, \frac{1}{2}\right]\right)$ or $\mathrm{S}^n \setminus h\left(I^{k-1} \times \left[\frac{1}{2}, 1\right]\right)$. Let $I_1 \subseteq I$ be the interval such that α is a cycle but not a boundary in $\mathrm{S}^n \setminus h\left(I^{k-1} \times I_1\right)$. Repeating this argument yields a sequence of nested intervals

$$[0, 1] = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$$

  where each I_i is of length $\frac{1}{2^i}$. In particular, this means that α is a boundary of some cycle β in $\mathrm{S}^n \setminus h\left(I^{k-1} \times \{x\}\right)$ for $\{x\} = \bigcap_i I_i$. Note that $\beta = \sum_i n_i \sigma_i$ is a sum of finitely many simplifies, with the image of each σ_i compact. As the $\mathrm{S}^n \setminus h\left(I^{k-1} \times \{x\}\right)$ for an open cover of $\mathrm{S}^n \setminus h\left(I^{k-1} \times \{x\}\right)$ it follows that β is a chain in $\mathrm{S}^n \setminus h\left(I^{k-1} \times \{x\}\right)$ for some i. Thus α is a boundary in $\mathrm{S}^n \setminus h\left(I^{k-1} \times \{x\}\right)$ which is a contradiction.

Therefore, we conclude by induction. □

Proposition 3.6.6. *For $h : \mathrm{S}^k \to \mathrm{S}^n$ an embedding for $k < n$ it follows that*

$$\widetilde{H_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^k\right)\right) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.

- For $k = 0$ we have $\mathrm{S}^n \setminus h\left(\mathrm{S}^0\right) \cong \mathrm{S}^{n-1} \times \mathbb{R}$ and so

$$\widetilde{H_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^0\right)\right) \cong \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose the result holds for $k-1$. Let $h : \mathrm{S}^{k-1} \to \mathrm{S}^n$ be an embedding. Let $A := \mathrm{S}^n \setminus h\left(\mathrm{D}_+^k\right)$ and $B := \mathrm{S}^n \setminus h\left(\mathrm{D}_-^k\right)$ where D_+^k and D_-^k are the positive and negative hemispheres respectively. Using Proposition 3.6.5 it follows that $\widetilde{H_i}(A) = 0$ and $\widetilde{H_i}(B) = 0$ for all i. With the observation $A \cap B = \mathrm{S}^n \setminus \mathrm{S}^n \setminus h\left(\mathrm{S}^k\right)$ and $A \cup B = \mathrm{S}^n \setminus h\left(\mathrm{S}^{k-1}\right)$, from the long exact sequence given by Theorem 3.6.1, it follows that

$$\widetilde{H_{i+1}}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^{k-1}\right)\right) \cong \widetilde{H_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^k\right)\right).$$

  Thus, using the inductive hypothesis it follows that

$$\widetilde{H_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^k\right)\right) \cong \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we conclude by induction. □

Corollary 3.6.7. *Let $h : \mathrm{S}^1 \to \mathrm{S}^2$ be an embedding. Then $S^2 \setminus h\left(S^1\right)$ consists of exactly two path-connected components.*

Proof. From Proposition 3.6.6 we have $\widetilde{H_0}\left(\mathrm{S}^2 \setminus h\left(\mathrm{S}^1\right)\right) \cong \mathbb{Z}$ which implies that $H_0(\mathrm{S}^2 \setminus h\left(\mathrm{S}^1\right)) \cong \mathbb{Z} \oplus \mathbb{Z}$. In other words, $\mathrm{S}^2 \setminus h\left(\mathrm{S}^1\right)$ has two path-connected components. □

Remark 3.6.8.

1. *One could just as well replace S^2 in Corollary 3.6.7 with $\mathbb{R}^2$ as $\mathrm{S}^2 \setminus \{x\}$ is still connected and $\mathbb{R}^2$ is homeomorphic to $\mathrm{S}^2 \setminus \{x\}$ through a stereographic projection. This particular case is referred to as the Jordan curve theorem.*
2. *Corollary 3.6.7 can be generalised to say that a subspace of S^n, or $\mathbb{R}^n$, that is homeomorphic to S^{n-1} separates S^n into two path connected components that have the same homology groups as a point.*

3.7 Degree

For the continuous map $f : \mathrm{S}^n \to \mathrm{S}^n$ let $f_* : H_n\left(\mathrm{S}^n\right) \to H_n\left(\mathrm{S}^n\right)$ be the induced homomorphism. Since $H_n \cong \mathbb{Z}\left(\mathrm{S}^n\right) \cong \mathbb{Z}$, it follows that

$$f_*(\alpha) = d\alpha$$

for some $d \in \mathbb{Z}$. The integer d is known as the degree of f and is denoted $\deg(f)$.

Proposition 3.7.1.

1. $\deg\left(\mathrm{id}_{\mathrm{S}^n}\right) = 1$.
2. *If f is not surjective, then $\deg(f) = 0$.*

3. *If $f \simeq g$ then $\deg(f) = \deg(g)$.*
4. $\deg(fg) = \deg(f)\deg(g)$. *In particular, if f is a homotopy equivalence then $\deg(f) = \pm 1$.*
5. *For $R_i : \mathrm{S}^n \to \mathrm{S}^n$ given by*

$$R_i(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$$

, referred to as the reflection map, we have $\deg(R_i) = -1$.

6. *For $-\mathrm{id}_{S^n}(x) = -x$, referred to as the antipodal map, we have $\deg(-\mathrm{id}_{\mathrm{S}^n}) = (-1)^{n+1}$.*
7. *If $f : \mathrm{S}^n \to \mathrm{S}^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.*

Proof.

1. This is clear since $(\mathrm{id}_{\mathrm{S}^n})_* (\alpha) = \alpha$.
2. Choose a point $x_0 \in \mathrm{S}^n \setminus f(\mathrm{S}^n)$, then f can be factored through

$$\mathrm{S}^n \hookrightarrow \mathrm{S}^n \setminus \{x_0\} \hookrightarrow \mathrm{S}^n.$$

Since $\mathrm{S}^n \setminus \{x_0\}$ is contractible we have that $H_n(\mathrm{S}^n \setminus \{x_0\}) = 0$ and so $f_* = 0$ which means that $\deg(f) = 0$.

3. This is clear as $f \simeq g$ implies that $f_* = g_*$.
4. Since $(fg)_* = f_* g_*$ it is clear that $\deg(fg) = \deg(f)\deg(g)$. In particular, if f is a homotopy equivalence then there exists a $g : \mathrm{S}^n \to \mathrm{S}^n$ such that $fg \simeq \mathrm{id}_{S_n}$. Using statement 1 and our previous discussion it follows that

$$\deg(f)\deg(g) = \deg(fg) = \deg(\mathrm{id}_{\mathrm{S}^n}) = 1.$$

Therefore, as $\deg(f), \deg(g) \in \mathbb{Z}$ we must have that $\deg(f) = \pm 1$.

5. It is sufficient to show the result for $i = 1$.
   - For $n = 1$, let $\omega : [0, 2\pi] \to \mathrm{S}^1$ be given by

$$\omega(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Then $R_1([\omega]) = -[\omega]$ and so $\deg(R_1) = -1$.

   - Assume that $\deg(R_1) = -1$, for $R_i : \mathrm{S}^{n-1} \to \mathrm{S}^{n-1}$. Let $U = \mathrm{S}^n \setminus \{N\}$ and $V = \mathrm{S}^n \setminus \{S\}$, where $N = (0, \ldots, 1)$ and $S = (0, \ldots, 1)$. Note that $R_1(U) = U$ and $R_1(V) = V$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(U \cap V) \arrow[r] \arrow[d, "(R_1)_\#"] & C_\bullet(U) \oplus C_\bullet(V) \arrow[r] \arrow[d, "(R_1)_\# \oplus (R_1)_\#"] & C_\bullet(U + V) \arrow[r] \arrow[d, "(R_1)_\#"] & 0 \\
0 \arrow[r] & C_\bullet(U \cap V) \arrow[r] & C_\bullet(U) \oplus C_\bullet(V) \arrow[r] & C_\bullet(U + V) \arrow[r] & 0
\end{tikzcd}$$

commutes. In particular, this induces the diagram

$$\begin{tikzcd}
H_n(\mathrm{S}^n) \arrow[r, "\partial"] \arrow[d, "(R_1)_*"] & H_{n-1}(U \cap V) \arrow[d, "(R_1)_*"] & H_{n-1}(\mathrm{S}^{n-1}) \arrow[l, "i_*"'] \arrow[d, "(R_1)_*"] \\
H_n(\mathrm{S}^n) \arrow[r, "\partial"] & H_{n-1}(U \cap V) & H_{n-1}(\mathrm{S}^{n-1}) \arrow[l, "i_*"']
\end{tikzcd}$$

where $i : \mathrm{S}^{n-1} \to U \cap V$ is given by

$$i(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$$

which is a homotopy equivalence and thus i_* is an isomorphism. Moreover, ∂ is an isomorphism as shown in Example 3.6.3. The left-hand square of the diagram commutes by Theorem 3.5.31, and the right-hand square commutes by the functoriality of the induced homomorphism, namely $(R_1 i)_* = (R_1) * i_*$. Therefore, there exists an isomorphism $H_n(\mathrm{S}^n) \cong H_{n-1}(\mathrm{S}^{n-1})$. Thus we can use the inductive hypothesis to conclude.

Therefore, using induction we conclude the proof.

6. Since $-\mathrm{id}_{\mathrm{S}^n} = R_1 \dots R_{n+1}$, it follows by statement 5 that
$$\deg(-\mathrm{id}_{\mathrm{S}^n}) = \deg(R_1)\dots\deg(R_{n+1}) = (-1)^{n+1}.$$

7. If $f(x) \neq x$ for all $x \in \mathrm{S}^n$, then $t \mapsto (1-t)f(x) - tx$ is a line segment from $f(x)$ to $-x$ that does not pass through the origin. Let
$$f_t(x) := \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|},$$
such that f_t is a homotopy from f to $-\mathrm{id}_{\mathrm{S}^n}$. Thus, using statement 3 and statement 6 we have
$$\deg(f) = \deg(-\mathrm{id}_{\mathrm{S}^n}) = (-1)^{n+1}.$$

□

Remark 3.7.2. *It is also the case that if $\deg(f) = \deg(g)$, then $f \simeq g$, however, we do not show this here.*

Recall that the action of a group G on a space X is a homomorphism from G to the group of homeomorphisms $X \to X$, denoted $\mathrm{Homeo}(X)$. In particular, the action is free if the homeomorphism corresponding to each non-identity element of G has no fixed points.

Proposition 3.7.3. *If n is even, then $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely by homeomorphisms on S^n.*

Proof. Let G be a free group action on S^n, such that $G \subseteq \mathrm{Homeo}(\mathrm{S}^n)$. For $f \in G$, by statement 4 of Proposition 3.7.1 we have $\deg(f) = \pm 1$. Moreover, for $f, g \in G$ we have $\deg(fg) = \deg(f)\deg(g)$ by statement 3 of Proposition 3.7.1. Hence, the degree defines a homomorphism $d : G \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. As the action is free it follows that if $g \in G \setminus \{id\}$ then g has no fixed points and so by statement 7 of Proposition 3.7.1 we have that $\deg(g) = (-1)^{n+1} = -1$, as n is even. Hence, $\ker(d) = \{\mathrm{id}\}$ which implies that either $G = \{\mathrm{id}\}$ or $G \cong \mathbb{Z}/2\mathbb{Z}$. □

Definition 3.7.4. *A vector field on S^n is a continuous map $v : \mathrm{S}^n \to \mathbb{R}^{n+1}$ such that for each $x \in \mathrm{S}^n$ the vector $v(x)$ is tangent to S^n at x.*

Theorem 3.7.5. *The space S^n admits a continuous vector field $v : \mathrm{S}^n \to \mathbb{R}^{n+1}$ that is nowhere zero if and only if n is odd.*

Proof. ($\Rightarrow$). If $v(x) \neq 0$ for all $x \in \mathrm{S}^n$ let $v' : \mathrm{S}^n \to \mathbb{R}^{n-1}$ be given by
$$v'(x) = \frac{v(x)}{|v(x)|}.$$
Consider
$$f_t(x) := \cos(t\pi)x + \sin(t\pi)v'(x).$$
Then $f_t \in \mathrm{S}^n$ for all $x \in \mathrm{S}^n$ and for all $t \in I$. Thus, f_t is a homotopy from $\mathrm{id}_{\mathrm{S}^n}$ $-\mathrm{id}_{\mathrm{S}^n}$, so
$$1 = \deg(\mathrm{id}_{\mathrm{S}^n}) = \deg(-\mathrm{id}_{\mathrm{S}^n}) = (-1)^{n+1}.$$

Thus n is odd.
($\Leftarrow$). Let $n = 2k - 1$, then

$$v\left(x_1, \ldots, x_{2k}\right) = \left(-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}\right)$$

is a vector field on S^n. ☐

3.8 Solution to Exercises

Exercise 3.4.7

Solution. The space $\mathbb{R}^k$ is homotopy equivalent to a point. Therefore, using Corollary 3.4.5 and Corollary 3.3.23 it follows that $\widetilde{H}_n\left(\mathbb{R}^k\right) = 0$ for every $n \in \mathbb{N}$. ☐

Exercise 3.5.3

Solution.

1. The image of the map into A has image $\{0\}$, hence the sequence is exact if and only if $\ker(\alpha) = \{0\}$.
2. The kernel of the map out of B is equal to B, hence the sequence is exact if and only if $\operatorname{im}(\alpha) = B$.
3. Using statement 1 the sequence is exact at A if and only if $\ker(\alpha) = \{0\}$. Using statement 2 the sequence is exact at B if and only if $\operatorname{im}(\alpha) = B$. Hence, the sequence is exact if α is a bijection, and thus an isomorphism.
4. To be exact at A requires $\ker(\alpha) = \{0\}$ by statement 1. To be exact at B requires $\ker(\beta) = \operatorname{im}(\alpha)$. To be exact at C requires $\operatorname{im}(\beta) = C$ by statement 2.

☐

Exercise 3.5.16

Solution.

- To show ∂ is well-defined it suffices to understand what happens if at each stage a different element is chosen.
  - The choice of a is uniquely determined by $\partial(b)$ since i injective.
  - Suppose b' is chosen instead, then $j\left(b'\right) = j(b)$ and so $b' - b \in \ker(j) = \operatorname{im}(i)$. Hence, $b' - b = i\left(a'\right)$ for some a' which implies that $b' = b + i\left(a'\right)$. Hence replacing b with b' has the effect of changing a to $a + \partial\left(a'\right)$ as

    $$\begin{aligned} i\left(a + \partial\left(a'\right)\right) &= i(a) + i\left(\partial\left(a'\right)\right) \\ &= \partial(b) + \partial\left(i\left(a'\right)\right) \\ &= \partial\left(b + i\left(a'\right)\right). \end{aligned}$$

    As $\partial\left(a + \partial\left(a'\right)\right) = \partial(a)$, it follows that a and $a + \partial\left(a'\right)$ are in the same homology class. Thus, the construction is independent of the representative chosen from the homology class of b.
  - A different representative from the homology class of c has the form $c + \partial\left(c'\right)$. Since, $c' = j\left(b'\right)$ for some b' it follows that

    $$c + \partial\left(c'\right) = c + \partial\left(j\left(b'\right)\right) = c + j\left(\partial\left(b'\right)\right) = j\left(b + \partial\left(b'\right)\right).$$

    Thus, b is replaced by $b + \partial\left(b'\right)$, which leaves $\partial(b)$ and thus a unchanged.

- Suppose $\partial([c_1]) = [a_1]$ and $\partial([c_2]) = [a_2]$ through elements b_1 and b_2 respectively. Then

  $$j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$$

  and

  $$i(a_1 + a_2) = i(a_1) + i(a_2) = \partial(b_1) + \partial(b_2) = \partial(b_1 + b_2).$$

  Therefore,

  $$\partial([c_1] + [c_2]) = [a_1] + [a_2],$$

  which means that ∂ is a homomorphism.

$\square$

4 Appendix

4.1 Equivalence of Simplicial and Singular Homology

For $n \geq 1$ note that

$$H_n(\Delta^n, \partial\Delta^n) \overset{\text{Prop. 3.5.26}}{\cong} \widetilde{H_n}(\Delta^n/\partial\Delta^n) \overset{\text{Cor. 3.5.11}}{\cong} \widetilde{H_n}(\mathrm{S}^n) = \mathbb{Z}.$$

Similarly, $H_0(\Delta^0, \partial\Delta^0) = \mathbb{Z}$.

Lemma 4.1.1. *The homology group $H_n(\Delta^n, \partial\Delta^n)$ is generated by the class of the cycle $i_n : \Delta^n \to \Delta^n$.*

Proof.

- For $n = 0$, we have that $H_0(\Delta^0, \emptyset)$ is generated by $[i_0]$.
- Assume that $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$ is generated by $i_{n-1} : \Delta^{n-1} \to \Delta^{n-1}$. Let $\Lambda \subseteq \partial\Delta^n$ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n. Then Δ^n strongly deformation retracts to Λ thus

  $$H_i(\Delta^n, \Lambda) = H_i(\Lambda, \Lambda) = 0.$$

  Consequently, the long exact sequence for $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ is

  $$\cdots \to \underbrace{H_n(\Delta^n, \Lambda)}_{0} \to H_n(\Delta^n, \partial\Delta^n) \to H_{n-1}(\partial\Delta^n, \Lambda) \to \underbrace{H_{n-1}(\Delta^n, \Lambda)}_{0} \to \dots$$

  which implies that $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda)$. Note that $\partial\Delta^n/\Lambda$ is homeomorphic to $\Delta^{n-1}/\partial\Delta^{n-1}$ which are good pairs, thus,

  $$\begin{aligned} H_n(\Delta^n, \partial\Delta^n) &\cong H_{n-1}(\partial\Delta^n, \Lambda) \\ &\cong \widetilde{H_{n-1}}(\partial\Delta^n/\Lambda) \\ &\cong \widetilde{H_{n-1}}(\Delta^{n-1}/\partial\Delta^{n-1}) \\ &\cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}). \end{aligned}$$

  As $[i_n]$ maps to $[\pm i_{n-1}]$ along these isomorphisms, it follows by the inductive assumption that $H_n(\Delta^n, \partial\Delta^n)$ is generated by $[i_n]$.

□

For X a topological space with a Δ complex structure, we have the simplicial chain complex

$$\cdots \to \Delta_{n+1}(X) \to \Delta_n(X) \to \Delta_{n-1}(X) \to \dots.$$

In particular, every simplicial chain complex can be viewed as a singular n-chain, thus we obtain an inclusion of chain complexes $\Delta_\bullet(X) \to C_\bullet(X)$.

Theorem 4.1.2. *The inclusion of chain complexes $\Delta_\bullet(X) \to C_\bullet(X)$ induces an isomorphism $H_n^\Delta(X) \to H_n(X)$.*

Proof. We consider only the case where the Δ-complex structure on X is finite-dimensional, with dimension k.

- For $k = 0$, X is a collection of points hence $H_n^\Delta(X) \cong H_n(X)$.
- Suppose that $H_n^\Delta(X) \cong H_n(X)$ for all n when the dimension of Δ is $k-1$. Let X have a k dimensional Δ-complex structure. Let X^l be the l-skeleton of X consisting of all simplices of dimension at most l. Then the chain complex

  $$\cdots \to \Delta_{n+1}(X^k)/\Delta_{n+1}(X^{k-1}) \to \Delta_n(X^k)/\Delta_n(X^{k-1}) \to \Delta_{n-1}(X^k)/\Delta_{n-1}(X^{k-1}) \to \dots$$

has homology groups $H_n^{\Delta}\left(X^k, X^{k-1}\right)$. Since

$$\Delta_n\left(X^k\right)/\Delta_n\left(X^{k-1}\right) = \begin{cases} \Delta_n\left(X^k\right) & n = k \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$H_n^{\Delta}\left(X^k, X^{k-1}\right) = \begin{cases} F_{\Lambda_k} & n = k \\ 0 & \text{otherwise,} \end{cases}$$

where Λ_k contains the k-simplices of X. Furthermore,

$$\begin{tikzcd}
0 \arrow[r] & \Delta_n\left(X^{k-1}\right) \arrow[r] \arrow[d, hook] & \Delta_n\left(X^k\right) \arrow[r] \arrow[d, hook] & \Delta_n\left(X^k\right)/\Delta_n\left(X^{k-1}\right) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C_n\left(X^{k-1}\right) \arrow[r] & C_n\left(X^k\right) \arrow[r] & C_n\left(X^k\right)/C_n\left(X^{k-1}\right) \arrow[r] & 0
\end{tikzcd}$$

commutes, with each row being a short exact sequence. Therefore,

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}^{\Delta}\left(X^k, X^{k-1}\right) \arrow[r] \arrow[d, "\alpha"] & H_n^{\Delta}\left(X^{k-1}\right) \arrow[r] \arrow[d, "\beta"] & H_n^{\Delta}\left(X^k\right) \arrow[r] \arrow[d, "\gamma"] & H_n^{\Delta}\left(X^k, X^{k-1}\right) \arrow[r] \arrow[d, "\delta"] & H_{n-1}^{\Delta}\left(X^{k-1}\right) \arrow[r] \arrow[d, "\epsilon"] & \dots \\
\dots \arrow[r] & H_{n+1}\left(X^k, X^{k-1}\right) \arrow[r] & H_n\left(X^{k-1}\right) \arrow[r] & H_n\left(X^k\right) \arrow[r] & H_n\left(X^k, X^{k-1}\right) \arrow[r] & H_{n-1}\left(X^{k-1}\right) \arrow[r] & \dots
\end{tikzcd}$$

commutes by Theorem 3.5.31. Moreover, by the inductive hypothesis, we have that β and ϵ are isomorphisms. Consider the continuous map $\Phi : \bigsqcup_\alpha \left(\Delta_\alpha^k, \partial\Delta_\alpha^k\right) \to \left(X^k, X^{k-1}\right)$ formed by the characteristic maps $\Delta^k \to X$ for all k-simplices of X. Since Φ induces a homeomorphism between $\bigsqcup_\alpha \Delta_\alpha^k / \bigsqcup_\alpha \partial\Delta_\alpha^k$ and X^k/X^{k-1} it follows by Proposition 3.3.15 that $H_n\left(\bigsqcup_\alpha \Delta_\alpha^k / \bigsqcup_\alpha \partial\Delta_\alpha^k\right) \cong H_n\left(X^k/X^{k-1}\right)$. Therefore,

$$H_n\left(X^k, X^{k-1}\right) \cong H_n\left(\bigsqcup_\alpha \Delta_\alpha^k, \bigsqcup_\alpha \partial\Delta^k\right) = \bigoplus_\alpha H_n\left(\Delta_\alpha^k, \partial\Delta_\alpha^k\right),$$

which is the free abelian group on $i_{n\alpha} : \Delta_\alpha^n \to \Delta_\alpha^n$ by Lemma 4.1.1, and so α and δ are isomorphisms. Therefore, using Lemma 3.5.5 it follows that γ is an isomorphism.

$\square$

Remark 4.1.3. *The proof of Theorem 4.1.2 requires more work.*

References

[1] Allen Hatcher. *Algebraic Topology*. Cambridge: University Press, 2001. ISBN: 978-0-521-79540-1.