Coursework 2

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1 Problem Sheet 7 Problem 3

Step 1: T is well-defined.
Let $f \in L^2(0,1)$, then

$$\begin{aligned}
\|Tf\|_{L^2(0,1)}^2 &= \int_0^1 \left|\int_0^1 e^{-st} f(t)\,dt\right|^2 ds \\
&\overset{\text{T.I}}{\leq} \int_0^1 \left(\int_0^1 \left|e^{-st}\right| |f(t)|\,dt\right)^2 ds \\
&\overset{\text{Höder's}}{\leq} \int_0^1 \left(\left(\int_0^1 e^{-2st}\,dt\right)^{\frac{1}{2}} \|f\|_{L^2(0,1)}\right)^2 ds \\
&= \|f\|_{L^2(0,1)}^2 \int_0^1 \int_0^1 e^{-2st}\,dt\,ds \\
&= \|f\|_{L^2(0,1)}^2 \int_0^1 \frac{1-e^{-2s}}{2s}\,ds \\
&\overset{(1)}{\leq} \|f\|_{L^2(0,1)}^2 \\
&< \infty
\end{aligned}$$

where (1) follows as $\frac{1-e^{-2s}}{2s} \leq 1$ for $s \in (0,1)$. Therefore, $Tf \in L^2(0,1)$ and so the map $T : L^2(0,1) \to L^2(0,1)$ is well-defined.
Step 2: $T \in \mathcal{L}\left(L^2(0,1), L^2(0,1)\right)$.
Note that for $f_1, f_2 \in L^2(0,1)$ and $\lambda \in \mathbb{R}$ we have that

$$\begin{aligned}
T(f_1 + \lambda f_2)(s) &= \int_0^1 e^{-st}(f_1 + \lambda f_2)(t)\,dt \\
&= \int_0^1 e^{-st} f_1(t)\,dt + \lambda \int_0^1 e^{-st} f_2(t)\,dt \\
&= Tf_1(s) + \lambda T f_2(s).
\end{aligned}$$

Therefore, T is a linear map. Recall from Step 1 that

$$\|Tf\|_{L^2(0,1)} \leq \|f\|_{L^2(0,1)}$$

and so

$$\|T\|_{L^2(0,1)\to L^2(0,1)} = \sup_{0 \neq f \in L^2(0,1)} \frac{\|Tf\|_{L^2(0,1)}}{\|f\|_{L^2(0,1)}} \leq 1.$$

Hence, T is a bounded linear map which implies that $T \in \mathcal{L}\left(L^2(0,1), L^2(0,1)\right)$.
Step 3: $T \in \mathcal{K}\left(L^2(0,1), L^2(0,1)\right)$.

For $f \in L^2(0,1)$ such that $\|f\|_{L^2(0,1)} \le 1$ let $x, y \in (0,1)$. Then observe that

$$\begin{aligned}
|Tf(x) - Tf(y)| &= \left| \int_0^1 \left(e^{-xt} - e^{-yt}\right) f(t)\, dt \right| \\
&\overset{\text{T.I}}{\le} \int_0^1 \left|e^{-xt} - e^{-yt}\right| |f(t)|\, dt \\
&\overset{\text{Höder's}}{\le} \left(\int_0^1 \left|e^{-xt} - e^{-yt}\right|^2 \, dt \right)^{\frac{1}{2}} \|f\|_{L^2(0,1)} \\
&\le \left(\int_0^1 \left|e^{-xt} - e^{-yt}\right|^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the the mean value theorem we note that for $t \in (0,1)$ we have that

$$\begin{aligned}
\left|e^{-xt} - e^{-yt}\right| &\le |x-y| \sup_{z \in (0,1)} \left| \frac{d}{dt} e^{-zt} \right| \\
&= |x-y| \sup_{z \in (0,1)} \left| -te^{-zt} \right| \\
&\le |x-y|.
\end{aligned}$$

It follows that

$$|Tf(x) - Tf(y)| \le |x-y| \tag{1}$$

which implies that $T\left(\bar{B}^{L^2(0,1)}\right) \subseteq \mathcal{C}^0(0,1)$. Hence, for a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \bar{B}^{L^2(0,1)}$ we have that $(Tf_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^0(0,1)$. As

$$Tf_n(0) = \int_0^1 f_n(t)\, dt \le \int_0^1 |f_n(t)|\, dt \le \|1\|_{L^2(0,1)} \|f_n\|_{L^2(0,1)} \le 1$$

for any $n \in \mathbb{N}$, it follows by using (1) that

$$|Tf_n(x)| \le |Tf_n(x) - Tf_n(0)| + |Tf_n(0)| \le |x| + 1 \le 2$$

for all $n \in \mathbb{N}$ and $x \in (0,1)$. Which implies that $(Tf_n)_{n\in\mathbb{N}}$ is bounded. Moreover, for any $\epsilon > 0$, let $\delta = \epsilon$ so that for any $x, y \in (0,1)$ such that $|x-y| < \delta$ we have

$$|Tf_n(x) - Tf_n(y)| \overset{(1)}{\le} |x-y| < \delta = \epsilon$$

for all $n \in \mathbb{N}$. This implies that the sequence $(Tf_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^0(0,1)$ is also equicontinuous. Therefore, it admits a convergent by the Arzela-Ascoli theorem and so $T\left(\bar{B}^{L^2(0,1)}\right)$ is pre-compact. Hence, the operator T is compact.

Problem Sheet 8 Problem 4

(i)

Let $(u_n) \subseteq F$ be a sequence that converges to u in H. Then for fixed v_i it follows that

$$\begin{aligned}
|(u, v_i) - (u_n, v_i)| &= |(u - u_n, v_i)| \\
&\overset{\text{C.S}}{\le} \|u - u_n\| \|v_i\|.
\end{aligned}$$

The right-hand side tends to 0 as $n \to \infty$ as $\|u - u_n\| \to 0$ by assumption and $\|v_i\|$ is a finite constant. Therefore, $0 = (u_n, v_i) \to (u, v_i)$ and so $(u, v_i) = 0$. Hence, $u \in F$ meaning F is closed.

(ii)

Expanding out the equation $M\Lambda = V$ we see that

$$\begin{pmatrix} \lambda_1(v_1, v_1) + \cdots + \lambda_n(v_1, v_n) \\ \vdots \\ \lambda_1(v_n, v_1) + \cdots + \lambda_n(v_n, v_n) \end{pmatrix} = \begin{pmatrix} (u, v_1) \\ \vdots \\ (u, v_n) \end{pmatrix}.$$

Using the properties of the inner product this is equivalent to

$$\begin{pmatrix} (v_1, \sum_{k=1}^n \lambda_k v_k) \\ \vdots \\ (v_n, \sum_{k=1}^n \lambda_k v_k) \end{pmatrix} = \begin{pmatrix} (v_1, u) \\ \vdots \\ (v_n, u) \end{pmatrix}.$$

In other words, we want to find $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\left(v_i, \sum_{k=1}^n \lambda_k v_k \right) = (v_i, u)$$

for all $i \in \{1, \ldots, n\}$. Which is equivalent to finding a $\tilde{u} \in \operatorname{span}(v_1, \ldots, v_n) =: M$ such that $(v, \tilde{u}) = (v, u)$ for all $v \in M$. In particular, $(u - \tilde{u}, v) = 0$ for all $v \in M$.

Claim 1. *Let H be a real Hilbert space. For a linearly independent set of vectors $\{v_1, \ldots, v_n\} \subseteq H$ the set $M := \operatorname{span}(v_1, \ldots, v_n) \subset H$ is a closed and convex linear subspace.*

Proof. It is clear that M is a linear subspace, from which the convexity of M easily follows. Suppose that we have a sequence $(m_k)_{k\in\mathbb{N}} \subseteq M$ that is convergent in H to m. Note that due to the linear independence of $\{v_1, \ldots, v_n\}$ we have a bijection from M to $\mathbb{R}^n$ given by $m_k = \sum_{i=1}^n x_i^{(k)} v_i \mapsto \left(x_1^{(k)}, \ldots, x_n^{(k)}\right)$. Since norms in finite dimensions are equivalent, it follows that the sequences $\left(x_i^{(k)}\right)$ are Cauchy for each $i \in \{1, \ldots, n\}$ in $\mathbb{R}$ as $(m_k)_{k\in\mathbb{N}}$ is Cauchy in M. Therefore, as $\mathbb{R}$ is complete it follows that each sequence $\left(x_i^{(k)}\right)_{k\in\mathbb{N}}$ converges to some $x_i \in \mathbb{R}$. Hence, $(m_k)_{k\in\mathbb{N}}$ converges to $m := \sum_{i=1}^n x_i v_i \in M$. Therefore, M is closed. $\square$

Claim 2. *Let H be a Hilbert space. Let $K \subset H$ be a closed and convex linear subspace. Then for $f \in H$, its projection onto K, as given by the Hilbert Projection theorem, is characterised by the unique vector $u \in K$ such that*

$$(f - u, v) = 0 \tag{2}$$

for all $v \in K$.

Proof. Recall, that the original characterisation of the projection of f onto K is the unique vector $u \in K$ such that

$$\|f - u\| = \min_{v\in K} \|f - v\|. \tag{3}$$

Suppose that $u \in K$ satisfies (2). Then for $v \in K$ as $u - v \in K$ we that $(f - u, u - v) = 0$ by (2). Hence,

$$\begin{aligned} \|f - v\|^2 &= \|f - u + u - v\|^2 \\ &= \|f - u\|^2 + 2(f - u, u - v) + \|u - v\|^2 \\ &= \|f - u\|^2 + \|u - v\|^2. \end{aligned}$$

In particular, this implies that $\|f - v\|^2 \geq \|f - u\|^2$ for all $v \in K$. Conversely, suppose that (3) is satisfied for u. Then for $v \in K$ and $t \in \mathbb{R}$, as K is a linear subspace, we have that $u + tv \in K$ and so $\|f - u\|^2 \leq \|f - (u + tv)\|^2$. Therefore,

$$0 \leq \|f - (u + tv)\|^2 - \|f - u\|^2 = 2t(u - f, v) + t^2\|v\|^2 =: g(t).$$

If $(u-f,v)\neq 0$, then as $g(t)$ is minimised by $t=-\frac{(u-f,v)}{\|v\|^2}$ we get a minimum of

$$g\left(-\frac{(u-f,v)}{\|v\|^2}\right)=-2\frac{(u-f,v)^2}{\|v\|^2}+\frac{(f-u,v)^2}{\|v\|^2}=-\frac{(u-f,v)^2}{\|v\|^2}<0.$$

This is a contradiction and so it must be the case that $(f-u,v)=0$. $\square$

Using Claim 1 we can apply Claim 2 to deduce that $\tilde{u}$ is the unique projection of u onto M, which we denote $P_M u\in M$. As the set $\{v_1,\dots,v_n\}$ is linearly independent we can write

$$P_M u=\lambda_1 v_1+\cdots+\lambda_n v_n$$

for unique $\lambda_1,\dots,\lambda_n\in\mathbb{R}^n$. Hence, there is a unique solution for $\Lambda\in\mathbb{R}^n$ satisfying $M\Lambda=V$.

(iii)

Note that if $u\in F$, it is clear that for any $m=\sum_{i=1}^n \mu_i v_i\in M$ we have

$$\left(u,\sum_{i=1}^n \mu_i v_i\right)=\sum_{i=1}^n \mu_i(u,v_i)=0.$$

Thus $u\in M^\perp$ and $F\subseteq M^\perp$. On the other hand, let $\tilde{m}\in M^\perp$. Then as $v_i\in M$ for all $i\in\{1,\dots,n\}$, it follows that $(\tilde{m},v_i)=0$ for all $i\in\{1,\dots,n\}$. Hence, $\tilde{m}\in F$ and so $F=M^\perp$. As M is closed, it follows by using Problem Sheet 8 Problem $2(iii)$ that $F^\perp=\left(M^\perp\right)^\perp=M$. Hence using by part (ii) we get that $P^\perp u=P_M u=\sum_{i=1}^n\lambda_i v_i$. Now let $\tilde{u}=u-P^\perp u$. We are now going to show that $Pu=\tilde{u}$. As $v_i\in M=F^\perp$ we can use the characterisation of $P^\perp u$ given in Claim 2to deduce that

$$0=\left(u-P^\perp u,v_i\right)=\left(\tilde{u},v_i\right).$$

Which implies that $\tilde{u}\in F$. Moreover, for $v\in F$ we have

$$(u-\tilde{u},v)=\left(u-\left(u-P^\perp u\right),v\right)=\left(P^\perp u,v\right)\overset{(1)}{=}0,$$

where (1) follows from the fact that $P^\perp u\in F^\perp$. Hence, as F is a closed and convex linear subspace we can use Claim 2 to deduce that $Pu=\tilde{u}=u-P^\perp u$.

Problem Sheet 9 Problem 5

Step 1: Show that S is closed.
Let $(f_n)_{n\in\mathbb{N}}\subset S$ be a sequence converging to $f\in H$. Let $f\mathbf{1}_E$ be non-zero on a set $F\subset E$. Note that as $F\subset E$ it must be the case that each f_n is almost everywhere zero in F, hence,

$$\begin{aligned}\|f-f_n\|^2_{L^2(\mathbb{R}^d)}&=\int_{\mathbb{R}^d}(f-f_n)^2\\&=\int_F(f-f_n)^2+\int_{\mathbb{R}^d\setminus F}(f-f_n)^2\\&\geq\int_F f^2.\end{aligned}$$

If F has non-zero measure then $\int_F f^2=c>0$ as $f^2>0$ on F. This then implies that $\|f-f_n\|_{L^2(\mathbb{R}^2)}\geq c$ for all $n\in\mathbb{N}$, which contradicts the assumption that $\|f-f_n\|_{L^2(\mathbb{R}^d)}\to 0$ as $n\to\infty$. Therefore, F has zero measure meaning f is zero almost everywhere in E as we know f is zero on $E\setminus F$. Hence, $f\in S$ meaning S is closed.
Step 2: Show that S is linear, and in particular convex.
Let $f,g\in S$ and $0\neq\lambda\in\mathbb{R}$. Then by construction $f\mathbf{1}_E$ and $g\mathbf{1}_E$ are non-zero on zero-measures sets $F_1,F_2\subset E$ respectively. Suppose that $(f+\lambda g)\mathbf{1}_E$ is non-zero on the set $F\subset E$. It is clear that if $(f+\lambda g)\mathbf{1}_E\neq 0$ then

either $f\mathbf{1}_E \neq 0$ or $g\mathbf{1}_E \neq 0$. Hence $F \subset F_1 \cup F_2$ which implies that F also has zero-measure as $F_1 \cup F_2$ has zero measure. Therefore, $f + \lambda g \in S$ and the set S is linear, which in particular means it is convex.

Step 3: For $f \in L^2\left(\mathbb{R}^d\right)$, find its projection, Pf, onto S.

For $f \in L^2\left(\mathbb{R}^d\right)$ let

$$(Pf)(x) = \begin{cases} f(x) & x \in \mathbb{R}^d \setminus E \\ 0 & x \in E. \end{cases}$$

Clearly, $(Pf) \in S$. Moreover, for $g \in S$ we have that

$$\begin{aligned} (f - Pf, g) &= \int_{\mathbb{R}^d} (f - Pf) \cdot g \\ &= \int_E (f - Pf) \cdot g + \int_{\mathbb{R}^d \setminus E} (f - Pf) \cdot g \\ &= \int_E f \cdot g \\ &\leq \int_E |f||g| \\ &\overset{\text{Hölder's}}{\leq} \|f\|_{L^2(E)} \|g\|_{L^2(E)} \\ &= 0. \end{aligned}$$

Therefore, $(f - Pf, g) \leq 0$ for all $g \in S$, hence as we know S is a linear subspace from Step 2 it is clear that $(f - Pf, g - Pf) \leq 0$ for all $g \in S$. Thus, Pf is the projection of f onto S.

Step 4: For $f \in L^2\left(\mathbb{R}^d\right)$ find its projection, $P^{\perp} f$, onto $S^{\perp}$.

As S is closed we can use Problem Sheet 8 Problem 3 part (iii) to conclude that

$$P^{\perp} f = f - Pf = \begin{cases} 0 & x \in \mathbb{R}^d \setminus E \\ f(x) & x \in E. \end{cases}$$