Function Spaces and Applications

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Contents

Introduction

What is functional analysis? Essentially, it is linear algebra in infinite dimensions. There are two main sources of differences that arise as we move to infinite dimensions.

1. Norms are not equivalent. In finite dimensions they are equivalent.
   - Recall, that a norm is a function $\|\cdot\|$ on a vector space satisfying the following.
     (a) $\|\lambda x\| = |\lambda|\|x\|$.
     (b) $\|x+y\| = \|x\| + \|y\|$.
     (c) $\|x\| = 0$ if and only if $x = 0$.
   - We say norms are equivalent when there exists a constant c such that
   $$\frac{1}{c}\|\cdot\|_2 \leq \|\cdot\|_1 \leq c\|\cdot\|_2.$$
2. Linear operators. We can represent linear operators as matrices acting on vectors.
   $$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty} a_{1k}x_k \\ \vdots \\ \vdots \end{pmatrix}.$$
   From which questions about convergence arise.

1 Topological and Metric Spaces

1.1 Topological Spaces

Let X be a set.

Definition 1.1.1. *A subset $\mathcal{O}$ of $\mathcal{P}(X)$ is a topology if the following hold.*

1. $\emptyset, X \in \mathcal{O}$.
2. *For a family* $(O_i)_{i\in\mathcal{I}}$ *in* $\mathcal{O}$ *we have that* $\bigcup_{i\in\mathcal{I}} O_i \in \mathcal{O}$.
3. *For a family* $(O_i)_{i=1}^n$ *in* $\mathcal{O}$ *we have that* $\bigcap_{i=1}^n O_i \in \mathcal{O}$.

Elements of the topology are called open.

Definition 1.1.2. *A sequence (x_n) converges to x if for all $O \in \mathcal{O}$ such that $x \in O$ there exists an N such that $x_m \in O$ for $m > N$.*

Example 1.1.3. *Some examples of topologies for a set X include the following.*

- $\mathcal{O} = \{\emptyset, X\}$.
- $\mathcal{O} = \mathcal{P}(X)$.

1.2 Metric Spaces

Let X be a set.

Definition 1.2.1. *A metric is an application $d : X \times X \to [0, \infty)$ such that the following hold.*

1. *(Definiteness)* $d(x, y) = 0$ *if and only if* $x = y$.
2. *(Symmetry)* $d(x, y) = d(y, x)$.
3. *(Triangle Inequality)* $d(x, y) \leq d(x, z) + d(z, y)$.

A set X with a metric d is called a metric space, denoted (X, d).

Definition 1.2.2. *The ball with centre $x \in X$ and radius $r \geq 0$ is defined as*

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Definition 1.2.3. *An open set O is such that for all $x \in O$ there exists an $r > 0$ such that $B(x, r) \subset O$.*

Definition 1.2.4. *A closed set is the complement of an open set.*

Example 1.2.5. *Some sets and corresponding metrics are the following.*

- $\mathbb{R}^n$ *and* $d(x, y) = \sum_i |x_i - y_i|$.
- $\mathcal{C}([0, 1]; \mathbb{R})$ *and* $d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$.
  - *Where* $\mathcal{C}([0, 1]; \mathbb{R})$ *is the set of continuous functions from* $[0, 1] \to \mathbb{R}$.

Proposition 1.2.6. *Let (X, d) be a metric space, and let $\mathcal{O}$ be the set of open sets. Then $\mathcal{O}$ is a topology.*

Proof. Clearly, $X \in \mathcal{O}$, as for any $r > 0$ and $x \in X$ we have that $B_r(x) \subseteq X$. Note that $\emptyset \in \mathcal{O}$ by a tautology, as by definition there is no $x \in \emptyset$ and so the property required to be an open set holds trivially. Next let, $\{O_i\}_{i \in I} \subset \mathcal{O}$. Then for any $x \in \bigcup_{i \in I} O_i$ it follows that $x \in O_i$ for some i, and so there exists an r such that $B_r(x) \subset O_i \subset \bigcup_{i \in I} O_i$. Therefore, $\bigcup_{i \in I} O_i \in \mathcal{O}$. Similarly, Let $\{O_i\}_{i=1}^n \subset \mathcal{O}$. Then for any $x \in \bigcap_{i=1}^n O_i$ there exists an $r_i > 0$ such that $B_{r_i}(x) \subset O_i$ for each $i = 1, \dots, n$. Let $r = \min(r_1, \dots, r_n) > 0$, then $B_r(x) \subset \bigcap_{i=1}^n O_i$. Therefore, $\bigcap_{i=1}^n O_i \in \mathcal{O}$. With each of these, we conclude that $\mathcal{O}$ is a topology. $\square$

In a topology, the notion of convergence is that $x_n \overset{n \to \infty}{\longrightarrow} x$ if and only if for every open set $O \in \mathcal{O}$ such that $x \in O$ there exists an $N \in \mathbb{N}$ such that $x_n \in O$ for all $n \geq N$. In a metric space the notion of convergence is that $x_n \overset{n \to \infty}{\longrightarrow} x$ if and only if $d(x_n, x) \overset{n \to \infty}{\longrightarrow} 0$.

1.2.1 Sets

Let (X, d) be a metric space with $S \subset X$.

Definition 1.2.7.

1. *S is said to be closed if S^c is open.*
2. *The closure of S, denoted $\bar{S}$, is the smallest closed set which contains S.*
   (a) *$\bar{S} = \bigcap_{C \text{ closed}, C \supset S} C$.*
   (b) *Equivalently, we can say that for any $x \in \bar{S}$ there exists a sequence (x_n) with $x_n \in S$ such that $x_n \to x$.*
3. *The interior of S, denoted S°, is the largest open set contained in S.*
   (a) *$\mathring{S} = \bigcup_{O \text{ open}, O \subset S} O$.*
   (b) *Equivalent, for every $x \in \mathring{S}$ there exists an $r > 0$ such that $B(x, r) \subset S$.*

Definition 1.2.8. *A is dense if $\bar{A} = X$.*

Example 1.2.9.

- $\bar{\mathbb{Q}} = \mathbb{R}$.
- $\bar{\mathbb{Z}} = \mathbb{Z}$.

Proposition 1.2.10. *Let $A \subseteq X$. Then $A = \mathring{A}$ if and only if A is open in (X, d).*

Proof. $(\Rightarrow)$ If $A = \mathring{A}$ then A is open as $\mathring{A}$ is open.
$(\Leftarrow)$ If A is open then

$$\mathring{A} = A \cup \bigcup_{V \text{ open}, V \subseteq A} V$$

which implies that $A \subset \mathring{A}$. Therefore, as by definition, we have $\mathring{A} \subset A$ it follows that $A = \mathring{A}$. $\square$

Proposition 1.2.11. *Let $A \subseteq X$. Then $A = \bar{A}$ if and only if A is closed in (X, d).*

Proof. ($\Rightarrow$) If $A = \bar{A}$ then A is closed as $\bar{A}$ is closed.
($\Leftarrow$) If A is closed then

$$\bar{A} = A \cap \bigcap_{F \text{ closed}, A \subseteq F} F$$

which implies that $\bar{A} \subset A$ and hence $A = \bar{A}$. □

Definition 1.2.12. *$S \subset X$ is bounded if there exists an $x \in X$ and $r > 0$ such that $S \subset B(x, r)$.*

1.2.2 Continuity

Let (X, d) and (Y, d') be metric spaces. Let $f : X \to Y$.

Proposition 1.2.13. *For $x_0 \in X$ the following are equivalent.*

1. *For all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x_0, y) < \delta$ implies that*

$$d'(f(x_0), f(y)) < \epsilon.$$

2. *For any sequence such that $x_n \to x$, then $f(x_n) \to f(x_0)$.*

Remark 1.2.14. *If either of the above conditions holds, f is said to be continuous at x_0.*

Proposition 1.2.15. *The following are equivalent.*

1. *For any open set $O \subset Y$, the set $f^{-1}(O)$ is open in X.*
2. *f is continuous at any $x_0 \in X$.*

Remark 1.2.16. *If either of the above conditions holds, then f is continuous on X.*

The first proposition provides a local viewpoint of continuity, whilst the latter provides a global viewpoint.

Definition 1.2.17. *f is uniformly continuous on X if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $(x, y) \in X^2$ with $d(x, y) < \delta$ we have that $d'(f(x), f(y)) < \epsilon$.*

1.2.3 Completeness

Definition 1.2.18. *(x_n) is a convergent sequence if there exists x such that $d(x_n, x) \overset{n\to\infty}{\longrightarrow} 0$.*

Definition 1.2.19. *(x_n) is a Cauchy sequence if for any $\epsilon > 0$ there exists a N such that for $n, m > N$ then $d(x_n, x_m) < \epsilon$.*

Remark 1.2.20. *By the triangle inequality, a convergent sequence is a Cauchy sequence.*

Definition 1.2.21. *A metric space (X, d) is complete if Cauchy sequences in X are convergent with respect to d.*

Example 1.2.22.

1. *$\mathbb{Q}$ with $d(x,y) = |x-y|$ is not complete.*
   - *There exists a sequence (r_n) with $r_n \in \mathbb{Q}$ such that $\left|r_n - \sqrt{2}\right| \to 0$, but $\sqrt{2} \notin \mathbb{Q}$, and limits are unique.*
2. *$\mathbb{R}$ with $d(x,y) = |x-y|$ is complete.*

Theorem 1.2.23. *If (X,d) is a metric space then there exists a metric space (Y,d') such that*

1. *Y is complete,*
2. *there is an injection $i : X \to Y$, and*
3. *$d(x,y) = d'(i(x), i(y))$.*

Theorem 1.2.24 (Banach Fixed Point Theorem)**.** *Let (X,d) be a complete metric space. Let $f : X \to X$ be a contraction, that is there exists a $\kappa \in (0,1)$ such that $d(f(x), f(y)) \leq \kappa d(x,y)$ for any $x, y \in X$. Then f has a unique fixed point, that is there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.*

Proof. Let $x_1 \in X$ and consider the sequence (x_n) defined by $x_n = f(x_{n-1})$ for $n \geq 2$. It follows that

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \kappa d(x_{n-1}, x_n).$$

Proceeding by induction we conclude that $d(x_n, x_{n+1}) \leq \kappa^{n-1} d(x_1, x_2)$. Let $N \in \mathbb{N}$ and consider $l > k > N$. Then by the triangle inequality, it follows that

$$\begin{aligned} d(x_l, x_k) &\leq d(x_l, x_{l-1}) + d(x_{l-1}, x_{l-2}) + \cdots + d(x_{k+1}, x_k) \\ &\leq \left(\kappa^{l-2} + \kappa^{l-3} + \cdots + \kappa^{k-1}\right) d(x_1, x_2) \\ &\leq \left(\kappa^{l-1} + \kappa^{l-2} + \dots\right) d(x_1, x_2) \\ &= \frac{\kappa^{l-1}}{1-\kappa} d(x_1, x_2) \\ &\leq \frac{\kappa^{N}}{1-\kappa} d(x_1, x_2) \\ &\xrightarrow{N \to \infty} 0. \end{aligned}$$

Therefore, the sequence is Cauchy, and hence convergent to some $x_0 \in X$ as (X,d) is a complete metric space. Note that the contracting property of f implies it is continuous. As $x_n \to x_0$ to follows by the continuity of f that $f(x_n) \to f(x_0)$ and so by the uniqueness of limits $x_0 = f(x_0)$. Now suppose that there exists another fixed point $y \in X$ of f. Then

$$d(f(x), f(y)) = d(x,y)$$

which contradicts the contracting property of f. Therefore, the fixed point x_0 is unique. □

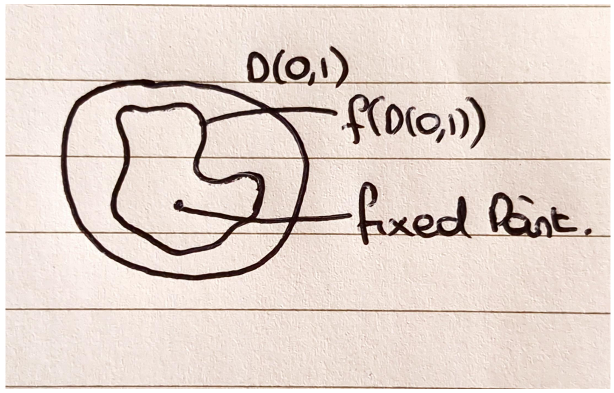


Figure 1: An Illustration of the Conditions Required for Banach's Fixed Point Theorem

Example 1.2.25. *Translations do not satisfy the conditions of this theorem as* $\kappa = 1$. *For example,* $f(x) = x+1$ *is such that* $|f(x) - f(y)| = |x - y|$, *and* $f(x) = x$ *has no solutions.*

1.2.4 Compactness

Theorem 1.2.26 ((Bolzano-Weierstrass))**.** *A bounded sequence of real numbers has a convergent subsequence. That is, if* (x_n) *is a sequence of* $\mathbb{R}$ *such that* $|x_n| \leq R$ *for some* $R > 0$. *Then there exists an extraction* φ *and* $y \in \mathbb{R}$ *such that* $x_{\varphi(n)} \to y$.

Remark 1.2.27. *An extraction* $\varphi : \mathbb{N} \to \mathbb{N}$ *is a strictly increasing function.*

For a metric space (X, d) and $S \subset X$. The **Bolzano-Weierstrass property** says that for all sequences (x_n) with $x_n \in S$ there exists a $y \in S$ and extraction φ such that $x_{\varphi(n)} \to y$ as $n \to \infty$.

Definition 1.2.28. *An open cover* $(O_i)_{i\in I}$ *is an open cover of* S *if the* O_i *is open for all* i *and* $S \subset \bigcup_{i\in I} O_i$.

Definition 1.2.29. *A subcover of an open cover* $(O_i)_{i\in I}$ *of* S *is a subset* $J \subset I$ *such that* $S \subset \bigcup_{i\in J} O_i$.

The **finite open cover property** says that for any open cover, you can extract a finite subcover.

Example 1.2.30. *If* $X = \mathbb{R}$ *and* $S = \mathbb{Z}$. *Then* $\mathbb{Z}$ *does not satisfy the FOCP.Choose* $O_i = \left(i - \frac{1}{10}, i + \frac{1}{10}\right)$ *for* $i \in \mathbb{N}$. *This is an open cover of* $\mathbb{Z}$ *with no finite subcover.*

Theorem 1.2.31. *The Bolzano-Weierstrass property and the finite open cover property are equivalent.*

Definition 1.2.32. *If either the Bolzano-Weierstrass or the finite cover property holds, then* S *is called compact.*

Example 1.2.33.

1. $\mathbb{Z} \subset \mathbb{R}$ *is not compact.*
2. $[a, b] \subset \mathbb{R}$ *is compact.*

3. $(a, b) \subset \mathbb{R}$ *is not compact.*

4. *Any finite subset* $S \subset \mathbb{R}$.

5. $\mathbb{Q} \subset \mathbb{R}$ *is not compact.*

Lemma 1.2.34. *If* $S \subset X$ *is compact then it is closed.*

Proof. Note that $S \subset X$ is closed if and only if $\bar{S}$. By definition $S \subset \bar{S}$ and so it suffices to show that $\bar{S} \subset S$. So choose $x \in \bar{S}$, then by the definition of the closure there exists a sequence (x_n) with $x_n \in S$ such that $x_n \to x$. By the BW property it follows that there exists an extraction φ and $y \in S$ such that $x_{\varphi(n)} \to y$. However, it must also be the case that $x_{\varphi(n)} \to x$, as any subsequence of a convergent sequence converges to the same limit. therefore, $x = y \in S$, which implies that $\bar{S} \subset S$ which completes the proof. $\square$

Theorem 1.2.35 (Heine-Borel)**.** *Consider the metric space* $(\mathbb{R}^d, d)$ *where* $d(x, y) = \sum_{i=1}^{d} |x_i - y_i|$. *Then compact sets of* $\mathbb{R}^d$ *are precisely the closed bounded sets.*

Remark 1.2.36.

1. *We note that compact implies being closed and that compact implies bounded.*

2. *Compact sets are the same for equivalent metrics.*

3. *Consider* $\mathbb{R}^d$ *with a norm* $\|\cdot\|$. *As all norms are equivalent in finite dimensions, the same conclusion holds in other finite dimensional normed vector spaces.*

Theorem 1.2.37. *If* $S \subset X$ *is compact, then the following hold.*

1. *Real-valued continuous functions in* S *achieve their supremum.*

2. *Real-valued continuous functions in* S *are uniformly continuous.*

Proof.

1. Let $M = \sup_{x \in S} f(x)$ and $f : S \to \mathbb{R}$ a continuous function.

   (a) If $M = \infty$, then there exists a sequence $(x_n) \subset S$ such that $f(x_n) \to \infty$. However, by compactness, we know there exists an extraction φ and $y \in S$ such that $x_{\varphi(n)} \to y$. Therefore, by continuity we have that $f(x_{\varphi(n)}) \to f(y) \in \mathbb{R}$ which contradicts $f(x_n) \to \infty$. Hence, we must have $M < \infty$.

   (b) If $M < \infty$, then choose $(x_n) \subset S$ such that $f(x_n) \to M$. Then by compactness there exists an extraction φ and $y \in S$ such that $x_{\varphi(n)} \to y$. By continuity, we have that $f(x_{\varphi(n)}) \to f(y)$ and so by the uniqueness of limits we conclude that $f(y) = M$.

2. Let $f : S \to \mathbb{R}$ be a continuous function. Suppose that it is not a uniformly continuous function. Then, there exists an $\epsilon > 0$ such that for $\delta = \frac{1}{n}$, for any $n \in \mathbb{N}$, there exists $x_n, y_n \in S$ such that $d(x_n, y_n) < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$.

   - By compactness, there exists an extraction φ and $\tilde{x} \in S$ such that $x_{\varphi(n)} \to \tilde{x}$.
   - Similarly, there exists an extraction ψ and $\tilde{y} \in S$ such that $y_{\psi(n)} \to \tilde{y}$. Given any $\tilde{\epsilon} > 0$, it follows for N sufficiently large with $n, m \geq N$ that

$$
\begin{aligned}
d(\tilde{x}, \tilde{y}) &\leq d\left(\tilde{x}, x_{\varphi(n)}\right) + d\left(\tilde{y}, x_{\varphi(n)}\right) \\
&\leq d\left(\tilde{x}, x_{\varphi(n)}\right) + d\left(x_{\varphi(n)}, y_{\psi(m)}\right) + d\left(y_{\psi(m)}, \tilde{y}\right) \\
&\leq \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} = \tilde{\epsilon}.
\end{aligned}
$$

Therefore, $d(\tilde{x}, \tilde{y}) = 0$ which implies that $\tilde{x} = \tilde{y}$. On the other hand, by the continuity of f we have that $f(x_{\varphi(n)}) \to f(\tilde{x})$ and $f(y_{\psi(n)}) \to f(\tilde{y})$ which implies that $|f(\tilde{x}) - f(\tilde{y})| \geq \epsilon$, which gives rise to a contradiction. Therefore, f is uniformly continuous.

□

Example 1.2.38. *The compactness condition of Theorem 1.2.37 is essential. Consider the space* $\mathcal{C}((0,1),\mathbb{R})$ *and the function* $f(x) = \sin\left(\frac{1}{x}\right) \in \mathcal{C}((0,1),\mathbb{R})$ *on this space. The function* $f(x)$ *is bounded and continuous on* $(0,1)$ *but it is not uniformly continuous.*

Example 1.2.39. *In infinite dimensions, these results break down. Let* $X = \mathcal{C}([0,1],\mathbb{R})$. *Then we have the following potential metrics*

- $d_1(f,g) = \sup_{x\in[0,1]} (|f(x) - g(x)|)$.
- $d_2(f,g) = \int_0^1 |f(x) - g(x)| dx$.

These are not equivalent, as for $f_n(x) = x^n$ *and* $g = 0$ *we have that*

- $d_1(f_n, 0) = 1$, *but*
- $d_2(f_n, 0) = \frac{1}{n}$.

With d_1 *the space* X *is complete but with* d_2 *the space* X *is not complete. See the figure below for an example of a sequence of functions in* X *that converge in* d_2 *to something not in* X.

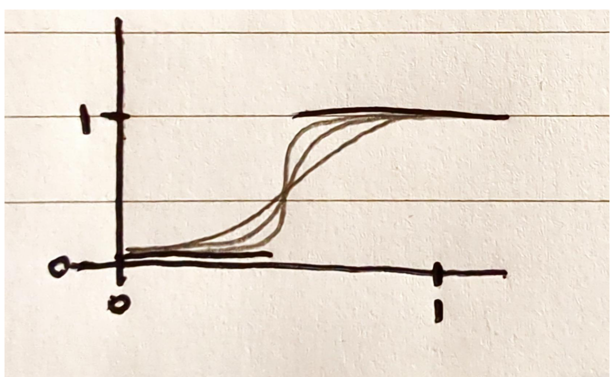


Figure 2: An example of how metrics in infinite dimensions need not be equivalent.

2 The Lebesgue Measure

2.1 Measure Spaces

Let X be a set.

Definition 2.1.1. *A σ-algebra, $\mathcal{A} \subset \mathcal{P}(X)$, satisfies the following.*

1. $X \in \mathcal{A}$.
2. *If* $A \in \mathcal{A}$, *then* $S^c \in \mathcal{A}$.
3. *If* $(S_i)_{i \in \mathbb{N}} \subset \mathcal{A}$ *then* $\bigcup_{i \in \mathbb{N}} S_i \in \mathcal{A}$.

Remark 2.1.2. *Combining 2. and 3. we get that $\mathcal{A}$ is closed under countable intersections.*

Definition 2.1.3. *A function $\mu : \mathcal{A} \to [0, \infty]$ is a measure if it satisfies the following.*

1. $\mu(\emptyset) = 0$.
2. *If* $(S_i)_{i \in \mathbb{N}} \subset \mathcal{A}$ *are such that* $S_i \cap S_j = \emptyset$ *for* $i \neq j$ *then*

$$\mu\left(\bigcup_{i \in \mathbb{N}} S_i\right) = \sum_{i \in \mathbb{N}} \mu(S_i).$$

Remark 2.1.4. *Property 2. is called the countable additivity, and can be thought of as a continuity property of the measure.*

- *The countable additivity property implies that if $(S_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ is an increasing sequence of sets then*

$$\lim_{j \to \infty} \mu(S_j) \to \mu\left(\bigcup_{j \in \mathbb{N}} S_j\right).$$

  This can be proved by applying countable additivity to the sets $E_j = S_{j+1} \setminus S_j$.

- *A similar result holds for a decreasing sequence of sets. Namely, if $(S_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ is a decreasing sequence of sets then we have that*

$$\lim_{j \to \infty} \mu(S_j) \to \mu\left(\bigcap_{j \in \mathbb{N}} S_j\right).$$

2.2 The Lebesgue Measure on $\mathbb{R}^d$

Theorem 2.2.1. *There exists a σ-algebra $\mathcal{A} \subset \mathcal{P}\left(\mathbb{R}^d\right)$ (the Lebesgue σ-algebra) and a measure μ (the Lebesgue measure) such that we have the following.*

1. *Open sets of $\mathbb{R}^d$, under the canonical metric, are in $\mathcal{A}$.*
2. *The rectangle* $R = \prod_{i=1}^d (a_i, b_i)$ *has measure* $\mu(R) = \prod_{i=1}^d (b_i - a_i)$.
3. *If* $A \in \mathcal{A}$ *with* $\mu(A) = 0$ *and* $B \subset A$ *then* $B \in \mathcal{A}$ *and* $\mu(B) = 0$.

Remark 2.2.2.

- *The countable intersection of open sets gives rise to many interesting sets, and so by countable additivity our σ-algebra captures a rich collection of sets.*
- *Point 2. tells us that μ extends our intuition on the size of sets in $\mathbb{R}^d$.*
- *Point 3. emphasises that the measure space is complete.*
- *Sets in the Lebesgue σ-algebra are called measurable sets.*
- *The Lebesgue measure is invariant under translations, that is for $x \in \mathbb{R}^d$ and A a measurable set we have*

$$\mu(A+x) = \mu(A).$$

- *For $\lambda \in \mathbb{R}$ and A a measurable set, the Lebesgue measure has the following scaling property,*

$$\mu(\lambda) = \lambda^d \mu(A).$$

Proposition 2.2.3. *A hyperplane in $\mathbb{R}^d$ has zero Lebesgue measure.*

Proof. A hyperplane in $\mathbb{R}^d$ is of the form

$$A_b = \left\{x \in \mathbb{R}^d : a_1x_1 + \cdots + a_dx_d = b,\ a_1, \ldots, a_d, b \in \mathbb{R}\right\}.$$

Due to the translational invariance of the Lebesgue measure we can consider

$$A := A_0 = \left\{x \in \mathbb{R}^d : a_1x_1 + \cdots + a_dx_d = 0,\ a_1, \ldots, a_d \in \mathbb{R}\right\}.$$

We will assume without loss of generality that each $a_i \neq 0$. We can isolate the graph of x_n by considering the continuous function

$$f(x_1, \ldots, x_{d-1}) = \frac{-(a_1x_1 + \cdots + a_{d-1}x_{d-1})}{a_d}.$$

Consider the compact set $K_j = \prod_{i=1}^{d-1}[-j, j] \subseteq \mathbb{R}^{d-1}$. Then as f is continuous, it is uniformly continuous on K_j. Therefore, for a given $\epsilon > 0$ we can partition K_j such that in each partition the variation of f is at most $\frac{\epsilon}{2^{j+d-1}j^{d-1}}$. Then

$$\mu(f(K_j)) = \frac{\epsilon}{2^{j+d-1}j^{d-1}}\mu(K_j) = \frac{\epsilon}{2^{j+d-1}j^{d-1}}(2j)^{d-1} = \frac{\epsilon}{2^j}.$$

As $A \subseteq \bigcup_{j=1}^{\infty} f(K_j)$ it follows that

$$\mu(A) \leq \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

Therefore, $\mu(A) = 0$ as $\epsilon > 0$ was arbitrary. $\square$

Definition 2.2.4. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is measurable if $f^{-1}((-\infty, a))$ is a measurable set for all $a \in \mathbb{R}$.*

Proposition 2.2.5.

1. *The composition of measurable functions is measurable.*
2. *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ for all x, then f is measurable. In other words, the function $\lim_{n\to\infty} f_n$ is measurable. Moreover, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf f_n$ are all measurable.*

3. *Sums and products of measurable functions are measurable.*
4. *Continuous functions are measurable.*

Definition 2.2.6. *A property is true almost everywhere or for almost any x if it is true on the complement of a zero-measure set.*

2.3 The Lebesgue Integral

2.3.1 The Integral of Simple Functions

Definition 2.3.1. *A simple function is of the form*

$$f = \sum_{i=1}^{N} c_i \mathbf{1}_{A_i}$$

where for each $i = 1, \ldots, N$ the $c_i \in \mathbb{R}$ and the A_i is a measurable set of $\mathbb{R}^d$ of finite measure.

The integral of a simple function is

$$\int_{\mathbb{R}^d} f(x)dx = \sum_{i=1}^{N} c_i \mu(A_i).$$

Similarly, for a measurable set S the integral of a simple function on S is

$$\int_S f(x)dx = \int_{\mathbb{R}^d} f(x) \mathbf{1}_S(x) dx.$$

Henceforth, we will often use the abbreviated notation

$$\int_{\mathbb{R}^d} f(x)dx = \int f\, dx$$

2.3.2 The Integral of Non-Negative Functions

Let $f : \mathbb{R}^d \to [0, \infty]$ be a non-negative function on $\mathbb{R}^d$. The integral of f is taken to be

$$\int f\, dx = \sup \left(\left\{ \int s\, dx : 0 \leq s \leq f,\ s \text{ a simple function} \right\} \right).$$

Proposition 2.3.2.

1. *If $\int f\, dx < \infty$ then $f < \infty$ almost everywhere.*
2. *If $\int f\, dx = 0$ then $f = 0$ almost everywhere.*

2.3.3 The Integral of Real-Valued Functions

A measurable function $f : \mathbb{R}^d \to (-\infty, \infty)$ admits the representation $f = f_+ - f_-$ where

- $f_+ = \max(0, f)$, and
- $f_- = \max(0, -f)$.

Consequently, we say that f is integrable, written $f \in L^1\left(\mathbb{R}^d\right)$, if $\int f_+ < \infty$ and $\int f_- < \infty$. The integral of an integrable function is taken to be

$$\int f\, dx = \int f_+\, dx - \int f_-\, dx.$$

Proposition 2.3.3.

1. *For $\alpha, \beta \in \mathbb{R}$ and $f, g \in L^1(\mathbb{R}^d)$ it follows that*

$$\int \alpha f + \beta f \, dx = \alpha \int f \, dx + \beta \int g \, dx.$$

2. *For $f \in L^1(\mathbb{R}^d)$ we have that*

$$\left| \int f dx \right| \leq \int |f| dx.$$

3. *A function $f \in L^1(\mathbb{R}^d)$ is $f = 0$ almost everywhere if and only if $\int_S f dx = 0$ for all measurable sets S.*

Proposition 2.3.4. *Let $f, g : A \to \mathbb{R}$ be measurable functions that satisfy $f \geq g$ almost everywhere in A. Then,*

$$\int_A f \leq \int_A g.$$

Proof. Suppose that f and g are non-negative measurable functions. Then for any simple function s such that $0 \leq s \leq f$ there is another simple function $\tilde{s}$ such that $0 \leq \tilde{s} \leq g$ such that $\int s = \int \tilde{s}$. Therefore,

$$\left\{ \int s \, dx : 0 \leq s \leq f, \ s \text{ a simple function} \right\} \subseteq \left\{ \int s \, dx : 0 \leq s \leq g, \ s \text{ a simple function} \right\}$$

which implies that

$$\sup \left(\left\{ \int s \, dx : 0 \leq s \leq f, \ s \text{ a simple function} \right\} \right) \leq \sup \left(\left\{ \int s \, dx : 0 \leq s \leq g, \ s \text{ a simple function} \right\} \right)$$

which then implies that $\int f \leq \int g$. For arbitrary measurable functions f and g we can write $f = f_+ - f_-$ and $g = g_+ - g_-$ where f_+, f_-, g_+, g_- are non-negative. As $f \leq g$ almost everywhere it follows that $f_+ \leq g_+$ almost everywhere and $g_- \leq f_-$ almost everywhere. Hence,

$$\int f = \int f_+ - \int f_- \leq \int g_+ - \int g_- = \int g.$$

$\square$

In light of Proposition 2.3.3 a reasonable suggestion for a distance on L^1 is $d(f, g) = \int |f - g| \, dx$. However, this is not a metric as if $f, g \in L^1$ are such that $d(f, g) = 0$ then we can only say that $f(x) = g(x)$ for almost all x.

- For continuous functions f and g such that $d(f, g) = 0$ we can conclude that $f(x) = g(x)$ for all x.

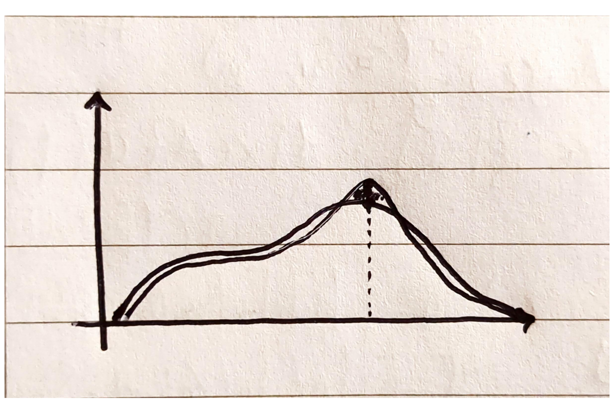


Figure 3: An illustration as to why continuous functions equal almost everywhere must be equal exactly.

To overcome this issue, we define equivalence classes. That is, for $f \in L^1$ we let

$$[f] = \left\{ g \in L^1 : f(x) = g(x) \text{ a.e.} \right\}.$$

Consequently, $d(f,g) = 0$ if and only if $[f] = [g]$. Abusing notation we will still speak of "functions" rather than "equivalence classes".

2.3.4 Connection to the Riemann Integral

Throughout let $I = [a,b]$ where $-\infty < a < b < \infty$.

Definition 2.3.5. *A set of points $\mathcal{P} = \{x_i\}_{i=0}^{N}$, for $N \in \mathbb{N}$, is called a partition of I if*

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b.$$

Definition 2.3.6. *A function $F : I \to \mathbb{R}$ is called a step function if there exists a partition $\mathcal{P}$ such that*

$$F(x) = \sum_{i=0}^{N-1} a_i \mathbf{1}_{[x_i, x_{i+1})}$$

where each $a_i \in \mathbb{R}$.

Definition 2.3.7. *For $f : I \to \mathbb{R}$ a bounded function and a partition $\mathcal{P} = \{x_i\}_{i=0}^{N-1}$ of I let*

- *the upper sum of f with respect to $\mathcal{P}$ be*

$$U_{\mathcal{P},I}(f) = \sum_{i=0}^{N-1} \left(\sup_{t \in [x_i, x_{i+1})} f(t) \right) (x_{i+1} - x_i),$$

- *and the lower sum of f with respect to $\mathcal{P}$ be*

$$L_{\mathcal{P},I}(f) = \sum_{i=0}^{N-1} \left(\inf_{t \in [x_i, x_{i+1})} f(t) \right) (x_{i+1} - x_i).$$

Definition 2.3.8. *A bounded function $f : I \to \mathbb{R}$ is said to be Riemann integrable if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of I such that*

$$|U_{\mathcal{P},I}(f) - L_{\mathcal{P},I}(f)| < \epsilon.$$

Proposition 2.3.9. *If f is Riemann integrable then*

$$\inf_{\mathcal{P}} U_{\mathcal{P},I}(f) = \sup_{\mathcal{P}} L_{\mathcal{P},I}(f).$$

Consequently, we denote the Riemann integral of a Riemann integrable function f as

$$\int_a^b f = \inf_{\mathcal{P}} U_{\mathcal{P},I}(f) = \sup_{\mathcal{P}} L_{\mathcal{P},I}(f).$$

Theorem 2.3.10. *Every Riemann integrable function on I is Lebesgue integrable and*

$$\int_a^b f(x)\,dx = \int_I f(x)\,dx.$$

Remark 2.3.11. *Therefore, all the facts and techniques we know surrounding Riemann integration, extend to Lebesgue integrals of Riemann integrable functions.*

With this equivalence, we can characterise the set of Riemann integrable functions using measure theory.

Theorem 2.3.12. *Let f be bounded on I. Then f is Riemann integrable on I if and only if it is continuous almost everywhere.*

One can readily extend the definition of Riemann integration to unbounded domains. In this case, we say that a function is Riemann integrable if the upper and lower sums are absolutely convergent and coincide. Similarly, for an unbounded function on a (finite or infinite) domain, we say that it is Riemann integrable if the upper and lower sums are absolutely convergent and coincide. We refer to both cases as improper Riemann integration.

Proposition 2.3.13. *If for a function f the improper Riemann integral absolutely converges, then f is also Lebesgue integrable and the two integrals coincide.*

2.4 Convergence of Functions and Convergence of Integrals

Example 2.4.1.

1. *Let $f_n = \mathbf{1}_{[n,n+1]}$ on $\mathbb{R}$. Then $\int f_n = 1$ with $f_n(x) \to f(x) = 0$ for all $x \in \mathbb{R}$. So that $\int f_n \not\to \int f$.*
2. *Let $f_n = n\mathbf{1}_{(0,\frac{1}{n})}$. Then $\int f_n = 1$ with $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. So that $\int f_n \not\to \int f$.*

Lemma 2.4.2. *If the $\operatorname{supp}(f_m) \subset K$ for K compact, and $\sup_x |f_n(x) - f(x)| \overset{n\to\infty}{\longrightarrow} 0$. Then,*

$$\int f_n dx \overset{n\to\infty}{\longrightarrow} \int f dx.$$

Proof. We note that as K is compact, $\mu(K) < \infty$. Therefore,

$$\begin{aligned}
\left|\int f_n dx - \int f dx\right| &= \left|\int (f_n - f) dx\right| \\
&\leq \int |f_n - f| dx \\
&\leq \int_K \sup_y |f_n(y) - f(y)| dx \\
&= \mu(K) \sup_y |f_n(y) - f(y)| \\
&\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore,

$$\int f_n dx \overset{n\to\infty}{\longrightarrow} \int f dx.$$

□

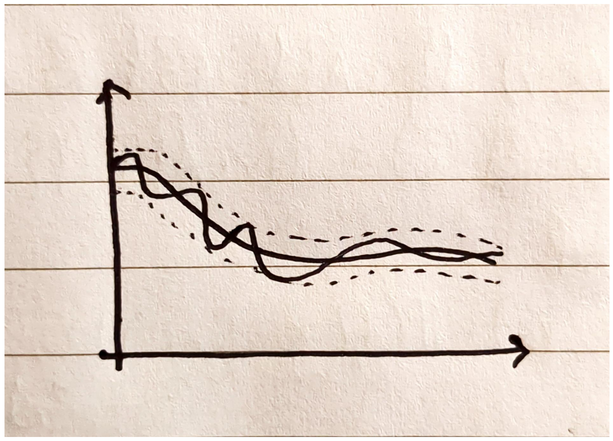


Figure 4: For the supremum between a sequence of functions and its limit to converge it must be the case that the functions lie within an ever-decreasing bounded region of the limit function.

Theorem 2.4.3 (Monotone Convergence Theorem)**.** *Let (f_n) be a sequence of non-negative measurable functions such that $f_{n+1}(x) \geq f_n(x)$ for almost all x. 1. Then $f_n(x) \to f(x) = \sup_n f_n(x)$ almost everywhere. 2. Furthermore, $\int f_n \to \int f$. 1. If the right-hand side is finite, then we also have convergence in L^1. That is,*

$$\int |f_n - f| \overset{n\to\infty}{\longrightarrow} 0.$$

Remark 2.4.4. *The monotonicity condition holds almost everywhere. The zero measure sets on which monotonicity may not hold can depend on n. However, the countable union of zero-measure sets is still a zero-measure set.*

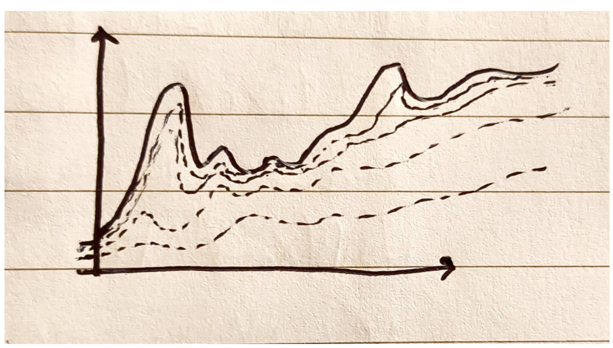


Figure 5: For a sequence of functions to converge monotonically from below to its limit, the graph of a function in the sequence must lie between the limiting function and the graph of the previous function in the sequence.

Theorem 2.4.5 (Dominated Convergence Theorem)**.** *Let (f_n) be a sequence of measurable functions such that the following hold.*

1. *$f_n(x) \to f(x)$ for almost all x.*
2. *There exists a $g \in L^1$ such that $|f_n(x)| \leq g(x)$ for almost any x.*

Then,

$$\int f_n \to \int f.$$

Example 2.4.6. *Recall Example 2.4.1 where we had pointwise convergence but not convergence of the integrals.*

1. *To apply DCT we would need* $g(x) = \sup_n(f_n(x)) = \mathbf{1}_{[0,\infty)}$ *which is not integrable.*
2. *To apply DCT we would need* $g(x) = \sup(f_n(x))...\frac{1}{2x}$.

MCT and DCT imply convergence in L^1 starting from pointwise convergence.

Example 2.4.7. *Pick a sequence* (x_n) *such that the following hold.*

1. x_n *is increasing.*
2. $x_{n+1} - x_n \to 0$ *as* $n \to \infty$.
3. $x_n \to \infty$.

For example, $x_n = \sqrt{n}$*. Let* $y_n \in [0,1)$ *such that* $x_n - y_m \in \mathbb{Z}$ *(*$y_n = x_n - \lfloor x_n \rfloor$*), then let* $f_m = \mathbf{1}_{(y_m, y_{m+1})}$ *(when there is a correction to be made when* $y_{m+1} < y_m$*). From this we have that*

$$\int f_m = y_{m+1} - y_m = x_{n+1} - x_n \overset{n\to\infty}{\longrightarrow} 0$$

and so convergence in the L^1 *sense. However,* $f_n(x) \not\to 0$ *for all* x *as the* y_m *continually traverse the interval* $[0,1)$.

Proposition 2.4.8. *If* $f_n \to f$ *in* L^1*, then there exists an extraction* φ *such that* $f_{\varphi(n)} \to f(x)$ *for almost all* x.

Theorem 2.4.9 (Fatou's Lemma)**.** *Let* (f_n) *be a sequence of non-negative measurable functions, then*

$$\liminf_n \int f_n \geq \int \liminf_n f_n.$$

3 Banach Spaces

3.1 Norms

Throughout let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$. For simplicity, we will assume it to be $\mathbb{R}$ throughout.

Definition 3.1.1. *A norm $\|\cdot\| : E \to [0, \infty)$ satisfies the following.*

1. $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in E$ *and* $\lambda \in \mathbb{R}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$.

Example 3.1.2. *The following are some examples of norms defined on vector spaces.*

1. *On $\mathbb{R}$, the map $|\cdot|$ is a norm.*
2. *On $\mathbb{R}^d$ the following are norms.*
   (a) $\|x\|_1 = \sum_{i=1}^{d} |x_i|$.
   (b) $\|x\|_\infty = \max_{i=1,\dots,d} |x_i|$.

Definition 3.1.3. *A vector space endowed with a norm is called a normed vector space.*

Remark 3.1.4. *To every norm $\|\cdot\|$ we can define the metric $d(x, y) = \|x - y\|$.*

Definition 3.1.5. *A complete, with respect to the induced metric, normed vector space is a Banach space.*

Definition 3.1.6. *Norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, are said to be equivalent if there exists a constant $C > 0$ such that*

$$\frac{1}{C}\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1.$$

Remark 3.1.7. *From a norm we get a metric, from which we define a topology, and thus establish a notion of convergence. Equivalent norms induce the same topology and notion of convergence.*

Theorem 3.1.8. *In finite dimensions, all norms are equivalent. In other words, if $\dim(E) < \infty$ then any norms on E are equivalent.*

Proof. Let $(e_i)_{1\leq i\leq d}$ be a basis of E. Define the norm

$$\left\|\sum_{i=1}^{d} x_i e_i\right\|_2 = \left(\sum_{i=1}^{d} |x_i|^2\right)^{\frac{1}{2}}.$$

Then consider another norm $\|\cdot\|$ on E. Firstly, we have that

$$\begin{aligned}
\|x\| &= \left\|\sum_{i=1}^{d} x_i e_i\right\| \\
&\overset{\text{T.I}}{\leq} \sum_{i=1}^{d} \|x_i e_i\| \\
&\overset{\text{Homo.}}{=} \sum_{i=1}^{d} |x_i|\|e_i\| \\
&\leq d \max_{1\leq i\leq d}(|x_i|) \max_{1\leq i\leq d}(\|e_i\|) \\
&\leq \left(d \max_{1\leq i\leq d}(\|e_i\|)\right)\|x\|_2 \\
&\leq M\|x\|_2.
\end{aligned}$$

Next consider the set

$$S = \{x \in E : \|x\|_2 = 1\}.$$

Then S is clearly bounded, and its closed as $\|\cdot\|_2$ is a continuous function. Therefore, S is compact by the Heine-Borel theorem. Note that the map $x \mapsto \|x\|$ is continuous for $(E, \|\cdot\|_2)$ as in the first part we showed it is bounded. Therefore, this map reaches its infimum, let's call it m. Observe that $m \neq 0$ as otherwise there would exist an $x \in S$ such that $\|x\| = 0$ which implies $x = 0$ which then implies $\|x\|_2 = 0$ which implies $x = 0$ which is a contradiction. Hence, $\|x\| \geq m > 0$ if $\|x\|_2$. Applying this to $y = \frac{x}{\|x\|_2}$ we conclude that for all $x \in E$ we have

$$\|x\| \geq m\|x\|_2.$$

Combining this with the first part we deduce that

$$m\|x\|_2 \leq \|x\| \leq M\|x\|_2$$

for all $x \in E$. Thus the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent. □

3.2 Spaces of Continuous Functions

We will consider functions on $\mathbb{R}^d$ or on open sets $\Omega \subset \mathbb{R}^d$.

Definition 3.2.1.

- *The set of bounded functions* $\Omega \to \mathbb{R}$ *is denoted* $\mathcal{B}(\Omega, \mathbb{R})$.
- *The set of continuous and bounded functions* $\Omega \to \mathbb{R}$ *is denoted* $\mathcal{C}^0(\Omega, \mathbb{R})$.

Remark 3.2.2.

- *As we are only really work with real functions, we will simply denotes these spaces as* $\mathcal{B}(\Omega)$ *and* $\mathcal{C}(\Omega)$ *respectively. Moreover, when the context is clear these function spaces may be denoted by* $\mathcal{B}$ *and* $\mathcal{C}^0$ *respectively. Sometimes* $\mathcal{C}^0$ *is also written as* $\mathcal{C}$.
- *These function spaces are vector spaces, usually equipped with the uniform norm.*

Definition 3.2.3. *The uniform norm is the map*

$$\|f\|_\infty = \sup_{x\in\Omega}(|f(x)|)$$

defined on $\mathcal{B}(\Omega)$ and $\mathcal{C}(\Omega)$.

Definition 3.2.4. *If $f_n \overset{\|\cdot\|_\infty}{\longrightarrow} f$ we say f_n converges to f uniformly.*

Theorem 3.2.5. *The uniform limit of continuous functions is continuous. In other words, if $(f_n) \in \mathcal{C}$ is such that $f_n \overset{\|\cdot\|_\infty}{\longrightarrow} f$, then f is continuous.*

Proof. Given $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\|f_n - f\|_\infty < \frac{\epsilon}{3}.$$

For x, as f_N is continuous there exists a $\delta > 0$ such that if $|x-y| < \delta$ then

$$|f_N(x) - f_N(y)| < \frac{\epsilon}{3}.$$

Therefore, for $|x-y| < \delta$ we have that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Where the first and the third differences are bounded by the uniform convergence, and the second difference is bounded by the continuity of f_N. This shows that f is continuous at x. $\square$

Theorem 3.2.6. *$\mathcal{B}(\Omega)$ and $\mathcal{C}(\Omega)$ are Banach spaces.*

Proof. We will only carry out the proof for $\mathcal{C}(\Omega)$. Consider a Cauchy sequence in $\mathcal{C}(\Omega)$.
<u>Step 1. Find a candidate for the limit.</u> For any x the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. As

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \overset{n\to\infty}{\longrightarrow} 0$$

we deduce that the sequence $(f_n(x))$ is a Cauchy sequence and hence convergent as $\mathbb{R}$ is complete. Note that $f \in \mathcal{B}$.
<u>Step 2. Show that (f_n) converges to f uniformly.</u> Choose $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for $n, m > N$ we have that $\|f_n - f_m\| < \infty$. Therefore, for all x we have that

$$|f_n(x) - f_m(x)| < \epsilon.$$

Sending $m \to \infty$ we conclude that $|f_n(x) - f(x)| < \epsilon$. Which implies that

$$\|f_n - f\| < \epsilon.$$

<u>Step 3. Show that $f \in \mathcal{C}$.</u> This follows from our previous theorem. $\square$

3.3 Spaces of Differentiable Functions

Recall, multi-index notation. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

- $|\alpha| = \alpha_1 + \cdots + \alpha_d$.
- $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$.

Definition 3.3.1. *The function space $\mathcal{C}^k(\Omega)$ contains functions on Ω which are k ties differentiable with continuous derivatives $\partial_x^\alpha f$ for all $|\alpha| \leq k$.*

The space $\mathcal{C}^k(\Omega)$ is a vector space which we endow with the norm

$$\|f\|_{\mathcal{C}^k} = \max_{|\alpha| \leq k} \|\partial_x^\alpha f\|_\infty .$$

With this norm, $\mathcal{C}^k(\Omega)$ is a normed vector space.

Theorem 3.3.2. *$\mathcal{C}^k(\Omega)$ with $\|\cdot\|_{\mathcal{C}^k}$ is a complete vector space, that is a Banach space.*

Proof. Start with a Cauchy sequence $(f_n) \in \mathcal{C}^k(\Omega)$.
Step 1. Find a candidate for the limit. For any α, the sequence $(\partial_x^\alpha f_n)$ is a Cauchy sequence for $(\mathcal{C}, \|\cdot\|_\infty)$. Therefore, by the previous theorem there exists a limit $f \in \mathcal{C}$ for (f_n) and there exists a limit $g_\alpha \in \mathcal{C}$ for $(\partial_x^\alpha f_n)$.
Step 2. Claim that $f \in \mathcal{C}^k$ and $g_\alpha = \partial_x^\alpha f$.

- For the case $k = 1$ and $d = 1$. We know that $f_n(x) \to f(x)$ and $\partial_x f_n(x) \to g(x)$ in $\|\cdot\|_\infty$. By the fundamental theorem of calculus (FTC) we have that

  $$f(x) - f(y) = \int_y^x \partial_x f_n(t) dt.$$

  When $n \to \infty$ as the integral of uniformly convergent function converges to the integral of the limit. We get that

  $$f(x) - f(y) = \int_y^x g(t) dt.$$

  So again by FTC, f is differentiable with derivative g.

- For the case $k \geq 2$ and $d = 1$. Use the previous case and proceed by induction.

- For the case $k \geq 2$ and $d \geq 2$. The case follows analogously to the first case, where we instead apply the FTC component-wise. That is

  $$|f_n(x) - f_n(x + Te_j)| = \int_0^T \partial_j f_n(x + sej) ds$$

  where e_j is the canonical j^{th} unit vector.

- For the case $k \geq 2$ and $d \geq 2$. Use the previous case and proceed by induction.

Step 3. f_n converges to f in $\mathcal{C}^k$. Given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|f_n - f_m\|_{\mathcal{C}^k} \leq \epsilon$ for $n, m \geq N$. This means that

$$\max_{|\alpha| \leq k} \|\partial_x^\alpha f_n - \partial_x^\alpha f_m\|_\infty \leq \epsilon.$$

Letting $m \to \infty$ such as to deduce that

$$\|\partial_x^\alpha - g_\alpha\| \leq \epsilon.$$

Previously we showed that $g_\alpha = \partial_n^\alpha f$. Therefore,

$$\|f_n - f\|_{\mathcal{C}^k} = \max_{|\alpha| \leq k} \|\partial_x^\alpha f_n - \partial_x^\alpha f\|_\infty \leq \epsilon.$$

□

Example 3.3.3. *Consider functions in $\mathcal{C}^1((-1, 1))$. The map*

$$\|f\| = \|\partial_x f\|_\infty$$

is not a norm, as it's not definite. For example, $\|1\| = 0$. The map

$$\|f\| = \|\partial_x f\|_\infty + |f(0)|$$

is a norm. As FTC tells us $f(x) = f(0) + \int_0^x f'(t)dt$ tells us that $\|f\| = 0$ if and only if $f = 0$. Moreover, with this norm the space $\mathcal{C}^1((-1,1))$ is a Banach space, that is it is complete with this norm.

3.4 Function Spaces on Compact Sets

In the previous sections, we considered spaces of real-valued functions defined on open sets $\Omega \subseteq \mathbb{R}^d$. Here we will suppose again that $\Omega \subset \mathbb{R}^d$ is open and bounded, and then consider spaces of real-valued functions defined on $\bar{\Omega}$.

Theorem 3.4.1. *The space $\mathcal{B}\left(\bar{\Omega}\right)$, with norm $\|\cdot\|_\infty$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence $\mathcal{B}\left(\bar{\Omega}\right)$. First observe that there exists an $N \in \mathbb{N}$ such that for every $m \geq N$ we have

$$\|f_N - f_m\|_\infty < 1.$$

As f_N is a bounded function it follows that $|f_N(x)| \leq M$ for all $x \in \bar{\Omega}$. Therefore, for sufficiently large m we have that $|f_m(x)| \leq M + 1$ for all $x \in \bar{\Omega}$. Next observe that as

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \overset{n\to\infty}{\longrightarrow} 0$$

the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, and hence convergent as $\mathbb{R}$ is complete. Let $f(x)$ be this limit. By our first observation, we conclude that $f(x) \leq M + 1$, as inequalities are preserved under limits. As this holds for all $x \in \bar{\Omega}$ we conclude that $f \in \mathcal{B}\left(\bar{\Omega}\right)$, hence, the space with the uniform norm is complete. $\square$

Theorem 3.4.2. *The space $\mathcal{C}^k\left(\bar{\Omega}\right)$, for $k \in \mathbb{N}$, with norm $\|\cdot\|_{\mathcal{C}^k}$ is a Banach space.*

Remark 3.4.3. *A f is in $\mathcal{C}^k\left(\bar{\Omega}\right)$ if for any points $x \in \partial\Omega$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, the $\partial^\alpha f(y)$ for $y \in x$ admits a limit when $y \in \Omega$. That is, there exists a $\beta \in \mathbb{R}$ such that for every sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq \Omega$ with $y_n \to x$ we have $\partial^\alpha f(y_n) \to \beta$.*

Corollary 3.4.4. *The space $\mathcal{C}^0\left(\bar{\Omega}\right)$ is a closed subset of $\mathcal{B}\left(\bar{\Omega}\right)$.*

Proof. Observe that continuous functions on compact domains are bounded so that $\mathcal{C}^0\left(\bar{\Omega}\right) \subseteq \mathcal{B}\left(\bar{\Omega}\right)$. Moreover,as $\|\cdot\|_\infty = \|\cdot\|_{\mathcal{C}^0}$ we know by Theorem 3.4.2 that $\left(\mathcal{C}^0\left(\bar{\Omega}\right), \|\cdot\|_\infty\right)$ is a Banach space. It is clear then that $\mathcal{C}^0\left(\bar{\Omega}\right)$ is a closed subset of $\mathcal{B}\left(\bar{\Omega}\right)$. $\square$

4 L^p Spaces

4.1 The L^p Norm

Functions are on $\mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ open.

Definition 4.1.1. *If f is a measurable function, its L^p norm is*

$$\|f\|_{L^p} = \left(\int |f(x)| \, dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$\|f\|_{L^\infty} = \inf \left\{ M > 0 : |f(x)| < M \text{ a.e.} \right\}.$$

Remark 4.1.2. *Integrals are of non-negative functions, and so are well-defined despite taking potentially infinite value.*

Definition 4.1.3. *The set L^p, more specifically $L^p\left(\mathbb{R}^d, \mathbb{R}\right)$, is the set of measurable functions, f, such that $\|f\|_{L^p} < \infty$.*

Remark 4.1.4.

- *For $\Omega \subseteq \mathbb{R}^d$ open, we can similarly define the $L^p(\Omega)$ space, where*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)| \, dx \right)^{\frac{1}{p}}.$$

- *Note that we are really consider $L^p(\Omega)$ as a space of equivalence classes rather than functions. That is, f and g are equivalent if and only if $f = g$ is almost everywhere.*

Proposition 4.1.5 (Young's Inequality)**.** *If $\frac{1}{p} + \frac{1}{q} = 1$, for $1 \leq p, q \leq \infty$, then for all $x, y > 0$ we have*

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} x^q.$$

Proof. Using the fact that $\log(\cdot)$ is a concave function we deduce that

$$\log\left(\frac{1}{p} x^p + \frac{1}{q} y^q \right) \geq \frac{1}{p} \log\left(x^p\right) + \frac{1}{q} \log\left(y^q\right).$$

Exponentiating both sides we get

$$\frac{1}{p} x^p + \frac{1}{q} x^q \geq xy.$$

□

Proposition 4.1.6 (Hölder's Inequality)**.** *For $\Omega \subseteq \mathbb{R}^d$ open, let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Proof. Let us first consider the case when $p = r$ so that $q = \infty$. As

$$\|g\|_{L^\infty(\Omega)} = \inf \left\{ M > 0 : |g(x)| < M \text{ a.e in } \Omega \right\}$$

we have that $|g| \leq \|g\|_{L^\infty(\Omega)}$ almost everywhere in Ω. Hence

$$\left(\int_\Omega |fg|^r\right)^{\frac{1}{r}} \leq \|g\|_{L^\infty(\Omega)} \left(\int_\Omega |f|^r\right)^{\frac{1}{r}} = \|f\|_{L^r(\Omega)} \|g\|_{L^\infty(\Omega)}.$$

For $r = 1$ and $1 < p < \infty$, it is clear from Young's Inequality that

$$\begin{aligned}\int_\Omega |fg| &\leq \int_\Omega \frac{1}{p}|f|^p + \frac{1}{q}|g|^q \\ &\leq \frac{1}{p}\int_\Omega |f|^p + \frac{1}{q}\int_\Omega |g|^q.\end{aligned}$$

If $\|f\|_{L^p(\Omega)} = 1$ and $\|g\|_{L^q(\Omega)} = 1$, then

$$\int_\Omega |fg| \leq \frac{1}{p}\|f\|_{L^p(\Omega)} + \frac{1}{q}\|g\|_{L^q(\Omega)} = 1.$$

Therefore, for arbitrary $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ we have that

$$\int_\Omega \left| \frac{f}{\|f\|_{L^p(\Omega)}} \frac{g}{\|g\|_{L^q(\Omega)}} \right| \leq 1$$

which implies that

$$\int_\Omega |fg| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

which is equivalent to $\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$. For $r \neq 1$, note that $\frac{1}{(\frac{p}{r})} + \frac{1}{(\frac{q}{r})} = 1$. Let $\tilde{p} = \frac{p}{r}$ and $\tilde{q} = \frac{q}{r}$. Then using our result for $r = 1$ we can deduce that

$$\begin{aligned}\||fg|^r\|_{L^1(\Omega)} &\leq \||f|^r\|_{L^{\tilde{p}}(\Omega)} \||g|^r\|_{L^{\tilde{q}}(\Omega} \\ &= \left(\int_\Omega |f|^{r\tilde{p}}\right)^{\frac{1}{\tilde{p}}} \left(\int_\Omega |g|^{r\tilde{q}}\right)^{\frac{1}{\tilde{q}}}.\end{aligned}$$

Therefore,

$$\left(\int_\Omega |fg|^r\right)^{\frac{1}{r}} \leq \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q\right)^{\frac{1}{q}}$$

and thus

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^r(\Omega)}.$$

□

Example 4.1.7. *If $p = q = 2$ and $r = 1$. Then*

$$\int |fg| \leq \left(\int f^2\right)^{\frac{1}{2}} \left(\int g^2\right)^{\frac{1}{2}}$$

and we recover the Cauchy-Schwarz inequality.

Proposition 4.1.8 (Minkowski's Inequality). *For $\Omega \subseteq \mathbb{R}^d(\Omega)$, if $f, g \in L^p(\Omega)$, then $f + g \in L^p(\Omega)$ and*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Proof. For $1 \leq p < \infty$ we have that

$$\begin{aligned}\|f+g\|_{L^p(\Omega)}^p &= \int_\Omega |f+g|^p \\ &\overset{\text{T.I}}{\leq} \int_\Omega |f||f+g|^{p-1} + \int_\Omega |g||f+g|^{p-1} \\ &\overset{\text{Prop. 4.1.6}}{\leq} \|f\|_{L^p(\Omega)}\|f+g\|_{L^p(\Omega)}^{p-1} + \|g\|_{L^p(\Omega)}\|f+g\|_{L^p(\Omega)}^{p-1}.\end{aligned}$$

Dividing both sides by $\|f+g\|_{L^p(\Omega)}^{p-1}$ we conclude that

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

When $p = \infty$ we note that if $m_f \in \{M > 0 : |f(x)| < M \text{ a.e. in } \Omega\}$ and $m_g \in \{M > 0 : |g(x)| < M \text{ a.e. in } \Omega\}$ then

$$|f(x)+g(x)| \leq |f(x)| + |g(x)| < m_f + m_g.$$

Taking infimums we conclude that

$$\|f+g\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}.$$

□

Theorem 4.1.9. *For $1 \leq p \leq \infty$ the map $\|\cdot\|_{L^p(\Omega)}$ defines a norm.*

Proof. Clearly, $\|f\|_{L^p(\Omega)} = 0$ if and only if f is almost everywhere 0, and thus equivalent to 0. Furthermore, for $\lambda \in \mathbb{R}$ we have $\|\lambda f\|_{L^p(\Omega)} = |\lambda|\|f\|_{L^p(\Omega)}$. The triangle inequality is Proposition 4.1.8. Therefore, $\|\cdot\|_{L^p(\Omega)}$ defines a norm. □

Proposition 4.1.10 (Generalised Minkowski Inequality)**.**

$$\left\|\int f(x,y)\,dy\right\|_{L^p_x} \leq \int \|f(x,y)\|_{L^p_x}\,dy.$$

Remark 4.1.11. *In the above, y can be thought of as the summation variable and x is the variable with respect to which we are computing the norms.*

Example 4.1.12. *Consider the function $f : \mathbb{R}^d \to \mathbb{R}$ given by*

$$f(x) = \frac{\mathbf{1}_{B_1(0)}}{|x|^\alpha}$$

where $\alpha \in \mathbb{R}^d$. Recall that

$$\int_{-1}^{1} \frac{1}{|x|^{\alpha p}}\,dx \begin{cases} = \infty & \alpha p \geq 1 \\ < \infty & \alpha p < 1. \end{cases}$$

This implies that $f \in L^p(\mathbb{R})$ if and only if $\alpha < \frac{d}{p}$. More generally in $\mathbb{R}^d$ as f is a radial function we know that $dx = C^{r-1}\,dr$ where C is the volume of the unit sphere in $\mathbb{R}^d$. Therefore, it follows that that

$$\left(\int_{B_1(0)} \frac{1}{|x|^{\alpha p}}\,dx\right)^{\frac{1}{p}} = C^{\frac{1}{p}}\left(\int_0^1 r^{d-1-\alpha p}\,dr\right)^{\frac{1}{p}}.$$

Consequently, $f \in L^p\left(\mathbb{R}^d\right)$ if and only if $\alpha < \frac{d}{p}$

L^p spaces can contain surprisingly exotic functions as its regularity is only formulated as an integral, which disregards behaviour at individual points.

Exercise 4.1.13.

1. *Find a function in $L^p(\mathbb{R})$ which is essentially unbounded on any $[n, n+1]$ for $n \in \mathbb{Z}$.*
2. *Find a function in $L^p((0,1))$ which is unbounded on any (a,b) for $a, b \in (0,1)$.*

4.2 Convergence

We have established that $(L^p, \|\cdot\|_{L^p})$ is a normed vector spaces. Consequently, we can start asking questions about convergence in this space, and how spaces with different values of p are related.

Theorem 4.2.1. *The space L^p with norm $\|\cdot\|_{L^p}$ is a Banach space.*

Proof. Given a convergent Cauchy sequence (f_n) with respect to $|\cdot|$, we can extract a subsequence (f_{n_k}) such that

$$|f_{n_k} - f_{n_{k+1}}| < \frac{1}{2^k}.$$

Moreover, as the sequence (f_n) is convergent, the limit of (f_{n_k}) coincides with the limit of (f_n). Hence, it suffices to consider a convergent Cauchy sequence (f_n) in L^p such that

$$\|f_{n+1} - f_n\| \leq \frac{1}{2^n}.$$

With this consider the following.

- $f = f_0 + \sum_{n=0}^{\infty}(f_{n+1} - f_n)$.
  - This is only formal now as we have no way to make sense of the convergence.
- $g = |f_0| + \sum_{n=0}^{\infty}|f_{n+1} - f_n|$.
  - The convergence here has a pointwise meaning as we are dealing with non-negative functions.
- $S_k f = f_0 + \sum_{n=0}^{k}(f_{n+1} - f_n)$.
- $S_k g = |f_0| + \sum_{n=0}^{k}|f_{n+1} - f_n|$.

Step 1: Show the candidate f is well-defined and in L^p. Observe that by Minkowski's inequality we have that

$$\|S_k g\|_{L^p} \leq \|f_0\| + \sum_{n=0}^{k}\|f_{n+1} - f_n\|_{L^p} \leq C + \sum_{n=0}^{k}\frac{1}{2^n} \leq \tilde{C} < \infty.$$

As $S_K \nearrow g$ pointwise, wee can conclude by the Monotone Convergence that

$$\int |g|^p = \lim_{k\to\infty}\int |S_k g|^p \leq \tilde{C}.$$

This implies that $g \in L^p$, and $g < \infty$ almost everywhere. Consequently $\sum_{n=0}^{\infty}|f_{n+1} - f_n|$ is absolutely convergent which implies that f is absolutely convergent. Therefore, as $|f| \leq |g|$ we conclude that $f \in L^p$.
Step 2. Show f_n converges to f in L^p. Note that

$$|f - S_k f| \leq |f| + |S_k f| \leq 2g$$

so that $|f - S_k f|^p \leq 2^p g^p$. Therefore, as $|f - S_k f|^p \to 0$ pointwise almost everywhere by the previous step, we can conclude by the Dominated Convergence theorem that

$$\|f - f_{k+1}\|_{L^p}^p = \int |f - S_k f|^p \to 0.$$

$\square$

Proposition 4.2.2. *If $\Omega \subset \mathbb{R}^d$ is bounded, then $L^p(\Omega) \subseteq L^q(\Omega)$ whenever $p \geq q$.*

Proof. Let $f \in L^p(\Omega)$. Note that $\frac{1}{q} = \frac{1}{p} + \frac{1}{\frac{pq}{p-q}}$. Let $r := \frac{pq}{p-q}$, then $\|\mathbf{1}_\Omega\|_{L^r} < \infty$ as Ω is bounded. Therefore, by Hölder's inequality

$$\|f\|_{L^q(\Omega)} = \|f\mathbf{1}_\Omega\|_{L^q(\Omega)} \leq \|f\|_{L^p(\Omega)}\|\mathbf{1}_\Omega\|_{L^r(\Omega)} < \infty.$$

Which implies that $f \in L^q(\Omega)$. $\square$

Example 4.2.3. *The condition that Ω in Proposition 4.2.2 is necessary for the inclusion to hold. Consider $\Omega = (1, \infty)$ and $f(x) = \frac{1}{x}$. Then*

$$\|f\|_{L^2((1,\infty))} \left(\int_1^\infty \frac{1}{|x|^2}\, dx\right)^{\frac{1}{2}} < \infty,$$

however,

$$\|f\|_{L^1((1,\infty))} = \int_1^\infty \frac{1}{x}\, dx = \infty.$$

Therefore, $L^1((1,\infty)) \not\subseteq L^2((1,\infty))$.

4.3 Convolution

Throughout, we will only be dealing with functions defined on $\mathbb{R}^d$. Let $\mathcal{C}_c^0$ denote the set of compactly supported continuous functions, with analogous definitions for $\mathcal{C}_c^k$ and $\mathcal{C}_c^\infty$.

Definition 4.3.1. *For $f \in L^1$ and $\phi \in \mathcal{C}_c^0$, their convolution is*

$$f \star \phi(x) = \int_{\mathbb{R}^d} f(y)\phi(x - y)\, dy.$$

Remark 4.3.2.

- *The above integral makes sense as the integrand is in L^1. Note $L^p \subset L^1$ locally. That is, if $f \in L^p$ and K is a compact set, then*

$$\int f\mathbf{1}_K\, dx \overset{\text{Hölder's}}{\leq} \|f\|_{L^p}\|\mathbf{1}_K\|_{L^q}$$

  for $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, as $\|\mathbf{1}_K\|_{L^q} < \infty$ we conclude that on K we have $f \in L^1$. Consequently, convolutions still make sense for $f \in L^p$ when ϕ has compact support.

- *If both $f, \phi \in \mathcal{C}_c^0$, then $f \star \phi = \phi \star f$.*

- *The convolution operation $(f, \phi) \mapsto f \star \phi$ is bilinear.*

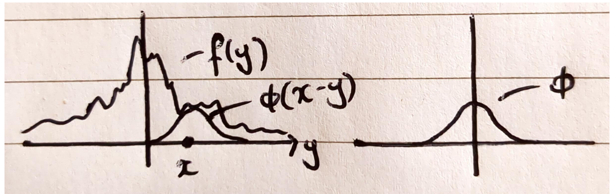


Figure 6: This illustration of the convolution shows how it can be interpreted as a smoothing operation for a rough function f, by taking a weighted average at x over the compact support of ϕ.

Definition 4.3.3. *For $f \in L^1$, we define $\operatorname{supp}(f)$ as the smallest closed set such that $f = 0$ almost everywhere in S^c.*

Definition 4.3.4. *For sets A and B let*

$$A + B = \{a + b : a \in A, b \in B\}.$$

Lemma 4.3.5. *For $f \in L^1$ and $\phi \in \mathcal{C}_c^0$ we have*

$$\operatorname{supp}(f \star \phi) \subset \operatorname{supp}(f) + \operatorname{supp}(\phi).$$

Intuition. Ideally, one would want to say that if $\int f(y)\phi(x-y)\,dy = (f \star \phi)(x) \neq 0$ then there exists a y such that $f(y) \neq 0$ and $\phi(x-y) \neq 0$. Therefore, $x = y + (x-y) \in \operatorname{supp}(f) + \operatorname{supp}(\phi)$. However, f here is really an equivalence class, and it doesn't make sense to talk about evaluating f at points. One instead has to work with small open sets. □

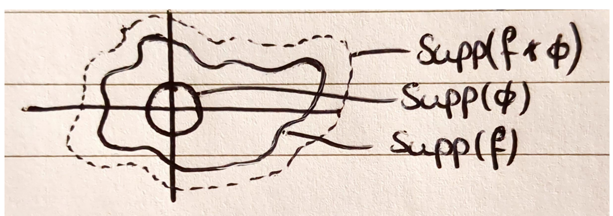


Figure 7: Thinking about a convolution as a weighted sum over a compact support, then graphically this is what we would expect the support of $f \star \phi$ to be.

Proposition 4.3.6. *If $f \in L^p$ and $\phi \in \mathcal{C}_c^0$ then $f \star \phi \in L^p$ and*

$$\|f \star \phi\|_{L^p} \le \|f\|_{L^p} \|\phi\|_{L^1}.$$

Proof. For $p = 1$ we can write

$$\begin{aligned}
\int |f \star \phi(x)|\,dx &= \int \left| \int f(y)\phi(x-y)\,dy \right| dx \\
&\overset{\text{T.I}}{\le} \iint |f(y)||\phi(x-y)|\,dy\,dx \\
&\overset{\text{Fubini.}}{=} \int |f(y)| \int |\phi(x-y)|\,dx\,dy \\
&= \|f\|_{L^1}\|\phi\|_{L^1}.
\end{aligned}$$

For the case when $p > 1$ we use the generalised Minkowski inequality to deduce that

$$\begin{aligned}\left\|\int f(x-y)\phi(y)\,dy\right\|_{L^p_x}^p &\le \int \|f(x-y)\phi(y)\|_{L^p_x}\,dy \\ &= \int |\phi(y)|\|f(x-y)\|_{L^p_x}\,dy \\ &= \|\phi\|_{L^1}\|f\|_{L^p}.\end{aligned}$$

Where in the last inequality we have pulled out $\|f\|_{L^p}$ as by translational invariance $\|f(x-y)\|_{L^p_x} = \|f(x)\|_{L^p_x}$, and so independent of y. □

Exercise 4.3.7.

1. *Show that if $f \in L^p$ and $\phi \in L^1$ then $f \star \phi \in L^p$.*
2. *Show that if $f \in L^1_{\text{loc}}$ and $\phi \in \mathcal{C}^0_c$ then $f \star \phi \in \mathcal{C}^0$.*
   - *The space L^p_{loc} is the space of functions for which for every compact set K, we have $\|f\mathbf{1}_K\|_{L^p} < \infty$.*

Proposition 4.3.8. *If $f \in L^1$ and $\phi \in \mathcal{C}^k_c$, then $f \star \phi \in \mathcal{C}^k$. What's more*

$$\partial^\alpha(f \star \phi) = f \star \partial^\alpha \phi$$

if $|\alpha| \le k$.

Proof. Consider

$$\frac{f \star \phi(x+h) - f \star \phi(x)}{h} = \int f(y)\frac{\phi(x-y+h) - \phi(x-y)}{h}\,dy.$$

By the mean value theorem, the quotient is bounded by its derivative. Which is a continuous function on a compact set and is therefore also bounded. As the quotient converges pointwise to $\phi'(x-y)$ as $h \to 0$ we can conclude by the Dominated Convergence theorem that

$$\partial_x(f \star \phi) = \int f(y)\phi'(x-y)\,dy = f \star \partial_x(\phi).$$

Continuing the argument by induction proves the proposition. □

Exercise 4.3.9. *Show that if $f \in L^1_{\text{loc}}$ and $\phi \in \mathcal{C}^k_c$ then $f \star g \in \mathcal{C}^k$. What's more*

$$\partial^\alpha(f \star \phi) = f \star \partial^\alpha \phi$$

if $|\alpha| \le k$.

4.4 Mollifer

For a function $\varphi \in \mathcal{C}^\infty_c$ be such that $\int \varphi = 1$ we define the sequence of mollifiers $\{\varphi_n\}_{n\in\mathbb{N}}$ where

$$\varphi_n(x) = n^d \varphi(nx).$$

Note that, $\text{supp}(\varphi_n) = \frac{1}{n}\text{supp}(\varphi)$, however, we maintain that $\int \varphi_n = 1$.

Remark 4.4.1. *Intuitively, $f \star \varphi_n$ should converge in some sense to f. As we understand that $f \star \varphi_n$ is like a weighted average of f over $\text{supp}(\varphi_n)$. As φ_n is increasingly concentrated on a smaller interval, we should* *expect some sort of convergence stated convergence.*

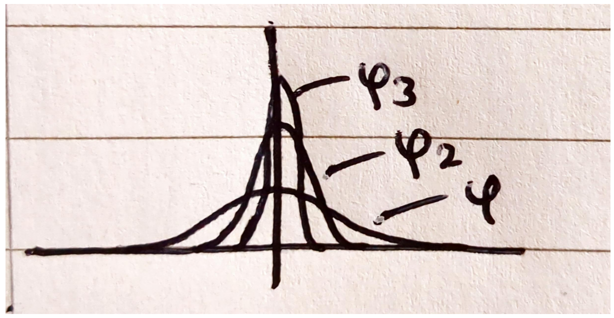


Figure 8: An graphical representation of a mollifer, φ, and subsequent φ_n.

Theorem 4.4.2.

1. *If $f \in \mathcal{C}_c^0$, then $f \star \varphi_n \overset{n\to\infty}{\longrightarrow} f$ under the uniform topology on $\mathcal{C}$.*
2. *If $f \in L^p$, for $1 \leq p < \infty$, then $f \star \varphi_n \overset{n\to\infty}{\longrightarrow} f$ in L^p.*

Proof.

1. Given an $\epsilon > 0$ by the uniform continuity of f there exists a $\delta > 0$ such that for $|x - y| < \delta$ we have that $|f(x) - f(y)| < \epsilon$. Note that

$$\begin{aligned} f \star \varphi_n(x) - f(x) &= \int \varphi_n(x-y) f(y)\, dy - f(x) \\ &= \int \varphi(x-y)_n (f(y) - f(x))\, dy. \end{aligned}$$

   The last inequality follows from the fact that $f(x)$ is independent of y and $\int \varphi_n(x-y)\, dy = 1$. We can choose $N \in \mathbb{N}$ such that for $x, y \in \operatorname{supp}(\varphi_N)$ we have $|x - y| < \delta$. Therefore, for $n \geq N$ we have

$$|f \star \varphi_n(x) - f(x)| \leq \epsilon \int |\varphi_n(x-y)|\, dy \leq \epsilon C.$$

   Hence, we have uniform convergence.

2. Let $f \in L^p$. Using the fact that $\mathcal{C}_c^0$ is dense in L^p, given an $\epsilon > 0$ there exists a $g \in \mathcal{C}_c^0$ and $h \in L^p$ such that
   - $f = g + h$,
   - $g \in \mathcal{C}^0$,
   - $h \in L^p$, and
   - $\|h\|_{L^p} \leq \epsilon$.

   Hence,

$$f \star \varphi_n - f = g \star \varphi_n - g + h \star \varphi_n - h$$

   so that

$$\|f \star \varphi_n - f\|_{L^p} \leq \|g \star \varphi_n - g\|_{L^p} + \|h \star \varphi_n\|_{L^p} + \|h\|_{L^p}.$$

   The second and third term on the right-hand side are less than or equal to ϵ by construction. The function in the first term has compact support, that is independent of n, and so $g \star \varphi_n - g \overset{\text{Unif.}}{\longrightarrow} 0$. Therefore, $\|g \star \varphi_n - g\|_{L^p} \overset{n\to\infty}{\longrightarrow} 0$.

$\square$

Corollary 4.4.3. *$\mathcal{C}_c^\infty$ is dense in L^p.*

Theorem 4.4.2 breaks down for $p = \infty$. If it were true then we could choose $f \in L^\infty \setminus \mathcal{C}^0$ and find a sequence $(f_n) \in \mathcal{C}^\infty$ such that $f_n \to f$ in L^∞. However, for continuous functions $\|f\|_{L^\infty} = \|f\|_{\mathcal{C}^0}$. Therefore, the sequence is convergent in $\mathcal{C}^0$ with the uniform topology, which implies that $f \in \mathcal{C}^0$, which is a contradiction.

4.5 Solution to Exercises

Exercise 4.1.13

Solution.

1. For a given p let $f_n(x) = \mathbf{1}_{[n,n+1]}\frac{1}{|x-n|^{\frac{1}{2p}}}$. By Example 4.1.12 the L^p-norm of f_n is finite and is independent of n since the measure is translationally invariant. Therefore,

$$f(x) = \sum_{n \in \mathbb{Z}} \frac{1}{n^2} f_n(x)$$

is absolutely convergent. As L^p is complete we deduce that $f \in L^p$. Notice that p is unbounded at $n \in \mathbb{Z}$ and so satisfies the requirements of the exercise.

2. As $\mathbb{Q} \cap [0,1]$ is countable we can enumerate it as $\{q_n\}_{n \in \mathbb{N}}$. As before we can define the function

$$f(x) = \mathbf{1}_{[0,1]} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - q_n|^{\frac{1}{2p}}}.$$

Note that $f \in L^p$ by similar arguments and satisfies the requirement of the exercise as $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$ and f is unbounded at all q_n.

$\square$

Exercise 4.3.7

Solution.

1. Observe that

$$\|\phi \star f\|_{L_x^p}^p = \left\| \int \phi(y) f(x-y)\, dy \right\|_{L_x^p}^p.$$

Applying the generalised Minkowski inequality we deduce that

$$\begin{aligned}
\|\phi \star d\|_{L_x^p}^p &\le \int \|\phi(y) f(x-y)\|_{L_x^p}\, dy \\
&= \int |\phi(y)| \|f\|_{L_x^p}\, dy \\
&= \|\phi\|_{L^1} \|f\|_{L^p}
\end{aligned}$$

Therefore, $\phi \star f \in L^p$ as $\|\phi\|_{L^1}$ and $\|f\|_{L^p}$ are finite by assumption.

2. Fix $x \in \mathbb{R}^d$. For $z \in \mathbb{R}^d$ observe that

$$|\phi \star f(x) - \phi \star f(z)| \le \int |f(y)||\phi(x-y) - \phi(z-y)|\, dy.$$

Assume that $\mathrm{supp}(\phi) \subseteq B_R$, so that $\mathrm{supp}(\phi(\cdot - x)) \subseteq B_R(x)$ and $\mathrm{supp}(\phi(\cdot - z)) \subseteq B_R(z)$. Suppose that $|z - x| = \delta$ so that

$$\mathrm{supp}(\phi(\cdot - x) - \phi(\cdot - z)) \subseteq B_{R+2\delta}(x).$$

Then

$$|\phi \star f(x) - \phi \star f(z)| \leq \int_{B_{R+2\delta}(x)} |f(y)||\phi(x-y) - \phi(z-y)|\, dy.$$

As $\phi(\cdot - x) - \phi(\cdot - z)$ is continuous and compactly support, it is also uniformly continuous on $B_{R+2\delta}(x)$. Therefore, for $\epsilon > 0$ there exists a $\delta > \delta_0 > 0$ such that $|\tilde{y} - \bar{y}| < \delta_0$ implies that

$$|\phi(\tilde{y}) - \phi(\bar{y})| < \epsilon.$$

Hence,

$$|\phi \star f(x) - \phi \star f(z)| \leq \epsilon \int_{B_{R+2\delta}(x)} |f(y)|\, dy.$$

Thus we have continuity, but we do not have uniform continuity as the right-hand side is dependent on x.

□

Exercise 4.3.9

Solution. We proceed for $k = 1$. Let $G(x) := \phi \star f(x)$. fix $i \in \{1, \ldots, d\}$ and $x \in \mathbb{R}^d$. Consider

$$\frac{G(x + h_n \cdot \mathbf{e}_i) - G(x)}{h_n} = \int \underbrace{\frac{\phi(x + h_n \cdot \mathbf{e}_i - y) - \phi(x-y)}{h_n}}_{F_n^x(y)} g(y)\, dy$$

where $h_n \to 0$. We know that $F_n^x(y)$ is supported on $B_R(x)$ for R sufficiently large. As $\phi \in \mathcal{C}_c^k\left(\mathbb{R}^d\right)$ we know that

$$f(y) F_n^x(y) \overset{n\to\infty}{\longrightarrow} \partial_i \phi(x-y) f(y)$$

pointwise almost everywhere. Moreover,

$$|f(y) F_n^x(y)| \leq |f(y)| \, \|\phi\|_{\mathcal{C}^1(B_R)}.$$

As $f \in L^1_{\text{loc}}$ we know the right-hand side of the above is in $L^1(B_R)$. Hence, by the dominated convergence theorem

$$\lim_{n\to\infty} \left(\frac{G(x + h_n \cdot \mathbf{e}_i) - G(x)}{h_n} \right) = \int \partial_i \phi(x-y) f(y)\, dy.$$

It follows that $\partial_i(f \star \phi)(x) = (\partial_i \phi) \star f(x)$ for all $i \in \{1, \ldots, d\}$ and all $x \in \mathbb{R}^d$. As $\partial_1 \phi \in \mathcal{C}_c^0$ it follows that $\partial_i(f \star \phi) \in \mathcal{C}^0\left(\mathbb{R}^d\right)$. Proceed by induction to complete the proof. □

5 ℓ^p Spaces

In this section, we will briefly explore ℓ^p spaces which can be thought of as a discrete analogue of L^p, but with some key differences.

5.1 ℓ^p Norm

Definition 5.1.1. *For $1 \leq p < \infty$ define the real vector space*

$$\ell_p = \left\{ x = \{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R} : \sum_{n\in\mathbb{N}} |x_k|^p < \infty \right\}.$$

When $p = \infty$ define the real vector space

$$\ell^\infty = \left\{ (x_k) : \sup_k |x_k| < \infty \right\}.$$

So for $1 \leq p < \infty$ the space ℓ^p consists of absolutely summable sequences. Whereas ℓ^∞ deals with bounded sequences, which is a significant distinction between the spaces.

Definition 5.1.2. *For $1 \leq p < \infty$ let $\|\cdot\|_{\ell^p} : \ell^p \to \mathbb{R}$ be such that*

$$\|x\| = \left(\sum_{k\in\mathbb{N}} |x_k|^p \right)^{\frac{1}{p}}.$$

For $p = \infty$ let $\|\cdot\|_{\ell^\infty} : \ell^\infty \to \mathbb{R}$ be such that

$$\|x\|_{\ell^\infty} = \sup_k |x_k|.$$

Remark 5.1.3. *If $f = \sum_{k=0}^{\infty} \mathbf{1}_{[k,k+1]}$, for $c_k \in \mathbb{R}$, then*

$$\|f\|_{L^p} = \|(c_k)\|_{\ell^p}.$$

Proposition 5.1.4 (Hölder's Inequality)**.** *Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x \in \ell^p$ and $y \in \ell^q$ it follows that*

$$\|fg\|_{\ell^1} \leq \|f\|_{\ell^p} \|g\|_{\ell^q}.$$

Proof. For $1 \leq p < \infty$, it is clear from Young's Inequality that

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| &\leq \sum_{k=1}^{\infty} \left(\frac{1}{p}|x_k|^p + \frac{1}{q}|y_k|^q \right) \\ &\leq \frac{1}{p} \sum_{k=1}^{\infty} |x_k|^p + \frac{1}{q} \sum_{k=1}^{\infty} |y_k|^q. \end{aligned}$$

If $\frac{1}{p} + \frac{1}{q} = 1$, $\|x\|_{\ell^p} = 1$ and $\|y\|_{\ell^q} = 1$, then

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \frac{1}{p}\|x\|_{\ell^p} + \frac{1}{q}\|y\|_{\ell^q} = 1.$$

Therefore, for arbitrary $x \in \ell^p$ and $y \in \ell^q$ we have that

$$\sum_{k=1}^{\infty} \left| \frac{x_k}{\|x\|_{\ell^p}} \frac{y_k}{\|y\|_{\ell^q}} \right| \leq 1$$

which implies that

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_{\ell^p} \|y\|_{\ell^q}$$

which is equivalent to $\|xy\|_{\ell^1} \leq \|x\|_{\ell^p} \|y\|_{\ell^q}$. When $p = \infty$, then $q = 1$ and

$$\begin{aligned} \|xy\|_{\ell^1} &= \sum_{k=1}^{\infty} |x_k y_k| \\ &\leq \sum_{k=1}^{\infty} \left(\sup_k |x_k| \right) |y_k| \\ &= \|x\|_{\ell^\infty} \sum_{k=1}^{\infty} |y_k| \\ &= \|x\|_{\ell^\infty} \|y\|_{\ell^1} \end{aligned}$$

as required. □

Proposition 5.1.5 (Minkowski's Inequality)**.** *If $x, y \in \ell^p$ then $x + y \in \ell^p$ and*

$$\|x + y\|_{\ell^p} \leq \|x\|_{\ell^p} + \|y\|_{\ell^p}.$$

Proof. When $1 \leq p < \infty$ we have that

$$\begin{aligned} \|x + y\|_{\ell^p}^p &= \sum_{k=1}^{\infty} |x_k + y_k|^p \\ &\overset{\text{T.I}}{\leq} \sum_{k=1}^{\infty} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{\infty} |y_k||x_k + y_k|^{p-1} \\ &\overset{\text{Prop. 5.1.4}}{\leq} \|x\|_{\ell^p} \|x + y\|_{\ell^p}^{p-1} + \|y\|_{\ell^p} \|x + y\|_{\ell^p}^{p-1}. \end{aligned}$$

Dividing both sides by $\|x + y\|_{\ell^p}^{p-1}$ we conclude that

$$\|x + y\|_{\ell^p} \leq \|x\|_{\ell^p} + \|y\|_{L^p}.$$

When $p = \infty$ then

$$\begin{aligned} \|x + y\|_{\ell^\infty} &= \sup_k \left(|x_k + y_k| \right) \\ &\leq \sup_k |x_k| + \sup_k |y_k| \\ &= \|x\|_{\ell^\infty} + \|y\|_{\ell^\infty}. \end{aligned}$$

□

Theorem 5.1.6. *For $1 \leq p \leq \infty$ the map $\|\cdot\|_{\ell^p}$ defines norm.*

Proof. Clearly, $\|x\|_{\ell^p} = 0$ if and only if $x_k = 0$ for all $k \in \mathbb{N}$. Furthermore, for $\lambda \in \mathbb{R}$ we have $\|\lambda x\|_{\ell^p} = |\lambda| \|x\|_{\ell^p}$. The triangle inequality is Proposition 5.1.5. Therefore, $\|\cdot\|_{\ell^p}$ defines a norm on ℓ^p. □

Consequently, we can consider ℓ^p as normed vector spaces with norm $\|\cdot\|_{\ell^p}$.

5.2 Convergence

We have established that $(\ell^p, \|\cdot\|_{\ell^p})$ is a normed vector spaces. Consequently, we can start asking questions about convergence in this space, and how spaces with different values of p are related.

Theorem 5.2.1. *For $1 \le p \le \infty$, the space ℓ^p is a Banach space.*

Proof. Consider the case when $1 \le p < \infty$. Let $\left\{x^{(n)}\right\}_{n\in\mathbb{N}} \subseteq \ell^p$ be a Cauchy sequence. Then given an $\epsilon > 0$, there exists a $\tilde{N} \in \mathbb{N}$ such that for all $n, m \ge \tilde{N}$ we have

$$\left\|x^{(n)} - x^{(m)}\right\|_{\ell^p} < \epsilon$$

therefore for any $k \in \mathbb{N}$ we have

$$\left|x_k^{(n)} - x_k^{(m)}\right|^p < \epsilon^p$$

hence the sequence $\left\{x_k^{(n)}\right\}_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{R}$ and therefore converges, denote this limit $x_k^{(\infty)}$. The sequence $\left\{x^{(n)}\right\}_{n\in\mathbb{N}}$ is Cauchy and thus bounded so that for some M we have that

$$\left\|x^{(n)}\right\|_{\ell^p} \le M$$

for all $n \in \mathbb{N}$. Therefore, as for any N we have

$$\left(\sum_{k=1}^{N}\left|x_k^{(\infty)}\right|^p\right)^{\frac{1}{p}} = \lim_{n\to\infty}\left(\sum_{k=1}^{N}\left|x_k^{(n)}\right|^p\right)^{\frac{1}{p}} \le \lim_{n\to\infty}\left\|x^{(n)}\right\|_{\ell^p} \le M.$$

Sending $n \to \infty$ preserves the limit, so

$$\left\|x^{(\infty)}\right\|_{\ell^p} \le M$$

meaning $x^{(\infty)} \in \ell^p$. Recall, that for any $n, m \ge \tilde{N}$ we have

$$\left\|x^{(n)} - x^{(m)}\right\|_{\ell^p} < \epsilon.$$

Therefore, for any N we have

$$\left(\sum_{k=1}^{N}\left|x_k^{(n)} - x_k^{(m)}\right|^p\right)^{\frac{1}{p}} < \epsilon.$$

Sending $m \to \infty$ we deduce that

$$\left(\sum_{k=1}^{N}\left|x_k^{(n)} - x_k^{(\infty)}\right|^p\right)^{\frac{1}{p}} < \epsilon.$$

Sending $N \to \infty$ we conclude that, $x^{(n)} \to x^{(\infty)}$ in ℓ^p. Hence $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space when $1 \le p < \infty$. Now consider the case when $p = \infty$. Let $\left\{x^{(n)}\right\} \subseteq \ell^\infty$ be a Cauchy sequence. As $\left|x_k^{(n)} - x_k^{(m)}\right| \le \left\|x^{(n)} - x^{(m)}\right\|_{\ell^\infty}$, it follows that $\left\{x_k^{(n)}\right\}_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Therefore, as before, we can define a sequence $x^{(\infty)}$, where $x_k^{(\infty)} = \lim_{n\to\infty} = x_k^{(n)}$. Moreover, for any N we have

$$\sup_{k=1,\ldots,N}\left|x_k^{(\infty)}\right| = \lim_{n\to\infty}\sup_{k=1,\ldots,N}\left|x_k^{(n)}\right| \le \lim_{n\to\infty}\left\|x^{(n)}\right\|_{\ell^\infty}.$$

As the sequence $\left\{x^{(n)}\right\}$ is Cauchy it is bounded, which is preserved under the limit of the right-hand side above. Therefore, $x^{(\infty)}$ is bounded and thus is ℓ^∞. Furthermore, as $\left\{x_k^{(n)}\right\}_{k\in\mathbb{N}}$ is Cauchy there exists an N such that

$$\left|x_k^{(n)} - x_k^{(m)}\right| < \frac{\epsilon}{2}$$

sending $m \to \infty$ we get that

$$\left| x_k^{(n)} - x_k^{(\infty)} \right| \leq \frac{\epsilon}{2}$$

then taking the supremum over k we can conclude that

$$\left\| x^{(n)} - x^{(\infty)} \right\|_{\ell^\infty} < \epsilon$$

which shows that $x^{(n)} \to x^{(\infty)}$ in ℓ^∞. Hence, $(\ell^\infty, \|\cdot\|_{\ell^\infty})$ is a Banach space. □

Proposition 5.2.2. *If $p \leq q$ then $\ell^p \subseteq \ell^q$.*

Proof. Clearly, if $p = \infty$ then $q = \infty$ and so the inclusion holds trivially. Similarly, $\ell^p \subseteq \ell^\infty$ for all p as all absolutely summable sequences are bounded. For $1 \leq p < \infty$ Let $x \in \ell^p$ and consider $p \leq q < \infty$. We know that

$$\|x\|_{\ell^p} = \sum_{k=0}^{\infty} |x_k|^p < \infty.$$

It must be the case that $|x_k|^p \to \infty$ as $k \to \infty$. More specifically there exists a K for which $|x_k| < 1$. Therefore, $|x_k|^q \leq |x_k|^p$ for $k \geq K$. Therefore, for $N \geq K$ we have that

$$\begin{aligned}
\|(x_k)\|_{\ell^q} &= \sum_{k=0}^{\infty} |x_k|^q \\
&= \sum_{k=0}^{K-1} |x_k|^q + \lim_{N\to\infty} \sum_{k=K}^{N} |x_k|^p \\
&\leq \sum_{k=0}^{K-1} |x_k|^q + \|(x_k)\|_p \\
&< \infty.
\end{aligned}$$

Therefore, $x \in \ell^q$. □

Remark 5.2.3. *Note the difference between Proposition 5.2.2 and Proposition 4.2.2.*

6 Linear Maps

6.1 Continuous Maps

Let E and F be normed vector spaces. Then we will denote the set of continuous maps from E and F by $\mathcal{L}(E, F)$.

Proposition 6.1.1. *Let E and F be normed vector spaces, and consider $T \in \mathcal{L}(E, F)$. Then the following are equivalent.*

- *T is continuous at zero.*
- *T is continuous on E.*
- *T is bounded, that is*
$$\|T\|_{E \to F} := \sup_{x \neq 0} \frac{\|Tx\|_F}{\|x\|_E} < \infty.$$

Proposition 6.1.2. *The space $\mathcal{L}(E, F)$, endowed with $\|\cdot\|_{E \to F}$, is a normed vector space. Moreover, if F is a Banach space, then $\mathcal{L}(E, F)$ is a Banach space.*

Proof. Let $(T_n) \subseteq \mathcal{L}(E; F)$ be a Cauchy sequence. Fix $0 \neq x \in E$. Given an $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|T_n - T_m\|_{\mathcal{L}(E,F)} < \frac{\epsilon}{\|x\|_E}$ for all $n, m \geq N$. Hence,

$$\|T_n(x) - T_m(x)\|_F \leq \|T_n - T_m\|_{\mathcal{L}(E,F)} \|x\|_E < \epsilon.$$

Therefore, the sequence $(T_n(x)) \subseteq F$ is Cauchy which implies that $T_n(x) \to y_x \in F$. Let $T \in \mathcal{L}(E, F)$ be defined as $T(x) = y_x$. For $x_1, x_2 \in E$ and $\lambda \in R$, we note that

$$\begin{aligned} T(x_1 + \lambda x_2) &= \lim_{n \to \infty} F_n(x_1 + \lambda x_2) \\ &= \lim_{n \to \infty} F_n(x_1) + \lambda \lim_{n \to \infty} F_n(x_2) \\ &= T(x_1) + \lambda T(x_2). \end{aligned}$$

Therefore, $T : E \to F$ is linear. As the sequence $(T_n) \subseteq \mathcal{L}(E; F)$ is Cauchy it is bounded. That is, there exists a $M > 0$ such that for all $n \in \mathbb{N}$ we have

$$\|T_n\|_{\mathcal{L}(E,F)} \leq M.$$

Moreover, for any $x \in E$, with $\|x\|_F = 1$, and $\epsilon > 0$, there exists a $N_x \in \mathbb{N}$ such that

$$\|T_n(x) - T(x)\|_F < \epsilon$$

for $n \geq N_x$. Therefore, for $n \geq N_x$ we deduce that

$$\|Tx\|_F \leq \|T_n(x) - T(x)\| + \|T_n(x)\|_F \leq \epsilon + M.$$

Which implies that $\|T\|_{\mathcal{L}(E,F)} < \infty$. Moreover,

$$\|T_n(x) - T(x)\|_F < \epsilon$$

for $n \geq N_x$ implies that

$$\|T_n - T\|_{\mathcal{L}(E,F)} < \epsilon,$$

so that $T_n \to T$ in $\mathcal{L}(E, F)$. $\square$

6.2 Dual Spaces

Throughout, let E be a Banach space.

Definition 6.2.1. *A linear form is a linear map of the form $E \to \mathbb{R}$ (or $\mathbb{C}$).*

Definition 6.2.2. *The dual of E, denoted E', is the set of continuous linear forms. That is, $E' = \mathcal{L}(E, \mathbb{R})$.*

Example 6.2.3. *For $E = \mathbb{R}^N$. An example of a linear form is $(x_1, \dots, x_N) \mapsto x_i$. In fact, any linear form on $\mathbb{R}^N$ can be written as*

$$x := (x_1, \dots, x_n) \mapsto x \cdot y = \sum_{i=1}^{N} x_i y_i.$$

Exercise 6.2.4. *Show that for $p \in (1, \infty)$, we have that $(\ell^p)' = \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 6.2.5 (Hahn-Banach). *Let $G \subset E$ be a linear subspace, and $g \in \mathcal{L}(G, \mathbb{R})$ be bounded. Then there exists an extension $f \in E'$ such that*

- *$f = g$ on G, and*
- *$\|f\|_{E \to \mathbb{R}} = \|g\|_{G \to \mathbb{R}}$.*

Proof. Let $P = \{h : D(h) \subset E \to \mathbb{R} : \text{Satisfying } 1 - 5\}$.

1. $D(h)$ is a linear subspace.
2. $h \in \mathcal{L}(D(h), \mathbb{R})$.
3. $D(h) \supset D(g) = G$.
4. $h = g$ on G.
5. $\|h\|_{D(h) \to \mathbb{R}} = \|g\|_{G \to \mathbb{R}}$.

Let us introduce an order relation, $\leq$ on P where $h_1 \leq h_2$ if and only if the following hold.

1. $D(h_1) \supset D(h_2)$.
2. $h_2 = h_1$ on $D(h_1)$.

Step 1: P is inductive.
Let $Q \subset P$ be a totally ordered subset. Then let $(h, D(h))$ be defined by $D(h) = \bigcup_{q \in Q} D(q)$ and $h(x) = q(x)$ if $D(q)$. This is well-defined, and h is an upper bound of Q, implying P is inductive.
Step 2: Apply Zorn's Lemma.
By Zorn's lemma there exists a maximal element f.
Step 3: Show that $D(f) = E$.
Proceed by contradiction, and assume that $D(f) \neq E$. Then choose $x_0 \in E \setminus D(f)$. Define $(h, D(h))$ by $D(h) = D(f) + \mathbb{R}x_0$ and $h(x + tx_0) = f(x) + \alpha t$ for $(x, t) \in D(f) \times \mathbb{R}$. Let $C_0 = \|g\|_{G \to \mathbb{R}}$. We want to choose α such that

$$|f(x) + t\alpha| \leq C_0 \|x + tx_0\|.$$

By positive homogeneity we note that $|f(x) + t\alpha| = |t| \left| f\left(\frac{x}{t}\right) + \alpha \right|$, so it suffices to consider $t = \pm 1$. Thus it suffices to require that

$$\begin{cases} f(x) + \alpha \leq C_0 \|x + x_0\| \\ f(x) - \alpha \leq C_0 \|x + x_0\| \end{cases}$$

which is equivalent to

$$\sup_{y\in D(h)} f(y) - C_0\|y+x_0\| \leq \alpha \leq \inf_{\zeta\in D(h)} C_0\|\zeta+x_0\| - f(\zeta).$$

For such an α to exists we need

$$f(y) - C_0\|y+x_0\| \leq C_0\|\zeta+x_0\| - f(\zeta)$$

for all y, ζ. Which happens if and only if

$$f(y-\zeta) = f(y) - f(\zeta) \leq C_0\|\zeta+x_0\| + C_0\|y+x_0\|$$

which is true since

$$h(y-\zeta) \leq C_0\|y-\zeta\| \leq C_0(\|y+x_0\| + \|\zeta+x_0\|)$$

by the triangle inequality. $\square$

6.3 Applications of the Hahn-Banach Theorem

Theorem 6.3.1. *If E is a normed vector space and $x \in E$, then there exits a $\rho \in E'$ such that*

$$\|x\|_E = \frac{\rho(x)}{\|\rho\|_{E'}}$$

where $\|\rho\|_{E'} = \|\rho\|_{E\to\mathbb{R}}$.

Proof. Define ρ on $\mathbb{R}x$ by $\rho(tx) = t$. Extend ρ to E by the Hahn-Banach theorem. Then

$$\|\rho\|_{\mathbb{R}x} = \sup \frac{t\|x\|}{\|tx\|} = 1.$$

So by Hahn-Banach we have that $\|\rho\|_E = 1$. Furthermore, $\rho(x) = \|x\|_E$. $\square$

Remark 6.3.2.

- *Equivalently, we can say that there exists a $\rho \in E'$ with $\|\rho\|_{E'} = 1$ such that $\|x\|_E = \rho(x)$.*
- *In finite dimensions, say with $E = \mathbb{R}^n$. Any linear form can written as $\rho_y : \mathbb{R}^N \to \mathbb{R}$ where $x \mapsto x \cdot y = \sum_{i=1}^N x_i y_i$. Note that*

  $$\|\rho_y\| = \sup_{x\neq 0} \frac{x\cdot y}{\|x\|} = \|y\|$$

  by Cauchy-Schwarz. Furthermore,

  $$\frac{|x\cdot y|}{\|y\|} = \frac{|\rho_y(x)|}{\|y\|} = \|x\|$$

  if and only if y is parallel to x.

Theorem 6.3.3. *Let E be a normed vector space with $F \subset E$ a linear subspace. Then if $\bar{F} \neq E$, it follows that there exists a $\rho \in E'$ such that $\rho \neq 0$ and*

$$\rho(x) := \langle \rho, x\rangle = 0$$

for all $x \in F$.

Proof. Let $v \in E \setminus \bar{F}$ and define $\tilde{F} = F + \operatorname{span}(v)$. Note that for each $u \in \tilde{F}$ we can write $u = f + \lambda v$ uniquely, for $f \in F$ and $\lambda \in \mathbb{R}$. Let $g : \tilde{F} \to \mathbb{R}$ be defined by

$$u \mapsto \lambda.$$

Note that $\rho(u)0$ for all $u \in F$. As $v \notin \bar{F}$ there exists an $\epsilon > 0$ such that $\|v - f\|_E \geq \epsilon > 0$ for all $f \in F$. As F is a linear subspace we note that $f \in F$ if and only if $-\frac{f}{\lambda} \in F$. So we can equivalently say that $\|v + \frac{f}{\lambda}\|_E \geq \epsilon > 0$ for all $f \in F$. Hence, for $u \in \tilde{F}$ we have that

$$\frac{|g(u)|}{\|u\|_E} = \frac{|\lambda|}{\|\lambda v + f\|_E} = \frac{1}{|\lambda|} \frac{|\lambda|}{\left\|v + \frac{f}{\lambda}\right\|_E} \leq \frac{1}{\epsilon}.$$

As g is clearly linear, it follows that $g \in \left(\tilde{F}\right)'$. Therefore, by Theorem 6.2.5 this can be extended to $\rho \in E'$. ☐

6.4 Riesz Representation Theorem

For $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we say that p and q are dual, and usually denote $q = p'$. Let $f \in L^{p'}$ and consider the linear form $\rho_f : L^p \to \mathbb{R}$ where

$$\varphi \mapsto \int f \varphi \, dx.$$

Note that by Hölder's inequality this is well-defined and bounded,

$$|\rho_f(\varphi)| = \left| \int f \varphi \right| \leq \|f\|_{L^{p'}} \|\varphi\|_{L^p}.$$

Consequently, $\rho_f \in (L^p)'$. Moreover,

$$\|\rho_f\|_{(L^p)'} \leq \|f\|_{L^{p'}}.$$

Exercise 6.4.1. *Show that we actually have*

$$\|\rho_f\|_{(L^p)'} = \|f\|_{L^{p'}}.$$

Theorem 6.4.2 (Riesz Representation Theorem)**.** *If $1 \leq p < \infty$, then any element of $(L^p)'$ can be represented as ρ_f for some $f \in L^{p'}$.*

Remark 6.4.3. *The same holds if L^p is replaced with ℓ^p.*

The statement of Theorem 6.4.2 breakdown for $p = \infty$. One can see how for the space ℓ^p. Observe that

$$\underbrace{\left| \sum_{n=0}^{\infty} x_n y_n \right|}_{\rho_y(x)} \leq \|x\|_{\ell^\infty} \|y\|_{\ell^1}.$$

Which means that ℓ^1 provides linear forms on ℓ^∞, that is $\rho \in (\ell^\infty)'$. Now let $X \subset \ell^\infty$ of sequences with a limit. Then define ρ on X by $\rho((x_n)) = \lim_{n\to\infty} x_m$. By the Hahn-Banach theorem, ρ can be extended to ℓ^∞. Hence, we get a $\rho \in (\ell^\infty)'$ such that $\rho(x) = \lim_{n\to\infty}(x_n)$ if (x_n) converges. Suppose $\rho_y(x) = \sum x_n y_n$. Then there exists N such that $\sum_{n \geq N} |y_n| < \epsilon$. Choose (x_n) such that

$$x_n = \begin{cases} 0 & n < N \\ 1 & n \geq N. \end{cases}$$

Then

$$1 = \rho(x) = |\rho_y((x_n))| = \left| \sum_{n \geq N} y_n \right| \leq \epsilon.$$

Therefore, ρ cannot be equal to ρ_y for any y, and so the statement of Theorem 6.4.2 cannot hold.

Exercise 6.4.4. *Show that the f in the statement of Theorem 6.4.2 is unique, up to equality almost everywhere.*

6.5 Bi-dual Space

For E a Banach space the bi-dual of E is the dual of E', namely E''. On E'' we define the norm

$$\|f\|_{E''} = \sup_{0 \neq \rho \in E'} \frac{|f(\rho)|}{\|\rho\|_{E'}}.$$

There is a natural map from $\Phi : E \to E''$, with $x \mapsto f_x$ where $f_x : E' \to \mathbb{R}$ is such that $\rho \mapsto \rho(x)$.

Exercise 6.5.1. *Verify that f_x is linear.*

Observe that

$$\begin{aligned} \|f_x\|_{E''} &= \sup_{0 \neq \rho \in E'} \frac{|f_x(\rho)|}{\|\rho\|_{E'}} \\ &\overset{(1)}{=} \sup_{0 \neq \rho \in E'} \frac{|\rho(x)|}{\|\rho\|_{E'}} \\ &= \|x\|_E. \end{aligned}$$

To justify (1) recall that $|\rho(x)| \leq \|\rho\|_{E'} \|x\|_E$. However, by the Hahn-Banach theorem, we can construct a ρ that achieves this supremum. Hence, (1) is justified. Consequently, $f_x \in E''$ and Φ is well-defined. Moreover, we deduce that Φ is an isometry, which implies that Φ is an injective linear operator. If Φ is also surjective, we call E a reflexive space.

Example 6.5.2.

1. *On $\mathbb{R}^d$ with the Euclidean norm, any linear form is bounded and can be represented as*

   $$\rho_\zeta(x) = \langle \zeta, x \rangle$$

   for some $\zeta \in \mathbb{R}^d$. Furthermore,

   $$\|\rho_\zeta\|_{(\mathbb{R}^d)'} = \sup_{x \neq 0} \frac{\langle \zeta, x \rangle}{\|x\|} = \|\zeta\|.$$

   Hence, $E \simeq E'$. It is easy to check then that Φ is an isomorphism. Consequently, $\mathbb{R}^d$ with the Euclidean norm is reflexive.

2. *Consider L^p for $1 < p < \infty$. By the Riesz Representation theorem, $(L^p)' \simeq L^{p'}$. Consequently,*

   $$(L^p)'' \simeq \left(L^{p'}\right)' \simeq L^p. \tag{6.5.1}$$

   Therefore, L^p is reflexive for $1 < p < \infty$.

3. *For $p \in \{1, \infty\}$, the space L^p is not reflexive. Note that although the first equality in* (6.5.1) *holds for $p = 1$, however, $p' = \infty$ and so the second equality does not hold.*

4. *The same conclusions made for L^p made above hold for ℓ^p.*

6.6 Solution to Exercises

Exercise 6.2.4

Solution. Let $p \in (1, \infty)$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $v \in \ell^q$ let $T(v) : \ell^p \to \mathbb{R}$ be given by

$$u \mapsto \sum_{n \in \mathbb{N}} v_n u_n.$$

Step 1: Show that for $v \in \ell^q$ the map $T(v) : \ell^p \to \mathbb{R}$ is well-defined.
Observe that

$$\begin{aligned} |T(v)u| &= \left| \sum_{n \in \mathbb{N}} v_n u_n \right| \\ &\leq \sum_{n \in \mathbb{N}} |v_n u_n| \\ &\leq \|v\|_{\ell^q} \|u\|_{\ell^p} \\ &< \infty. \end{aligned}$$

Therefore, $T(v)$ is well-defined.
Let $T : \ell^q \to (\ell^p)'$ be given by $v \mapsto T(v)$.
Step 2: Show that $T : \ell^q \to (\ell^p)'$ is well-defined and continuous.
The map $v \mapsto T(v)$ is well-defined as from Step 1 we know that $T(v) \in (\ell^p)'$. For $v^1, v^2 \in \ell^p$, $\lambda \in \mathbb{R}$, and fixed $u \in \ell^p$ we have that

$$\begin{aligned} T\left(v^1 + \lambda v^2\right)(u) &= \sum_{n \in \mathbb{N}} \left(v_n^1 + \lambda v_n^2\right) u_n \\ &= \sum_{n \in \mathbb{N}} v_n^1 u_n + \lambda \sum_{n \in \mathbb{N}} v_n^2 u_n \\ &= T\left(v^1\right)(u) + \lambda T\left(v^2\right)(u). \end{aligned}$$

Hence $v \mapsto T(v)$ is linear. Next observe that for $0 \neq u \in \ell^p$ we have that

$$\|T(v)\|_{(\ell^p)'} \leq \frac{|T(v)(u)|}{\|u\|_{\ell^p}} \leq \frac{\sum_{n \in \mathbb{N}} |v_n u_n|}{\|u\|_{\ell^p}} \leq \frac{\|v\|_{\ell^q} \|u\|_{\ell^p}}{\|u\|_{\ell^p}} = \|v\|_{\ell^q}.$$

Therefore,

$$\|T\|_{\ell^q \to (\ell^p)'} = \sup_{0 \neq v \in \ell^q} \frac{\|T(v)\|_{(\ell^q)'}}{\|v\|_{\ell^p}} \leq 1$$

which implies that the map is bounded and hence continuous as it is also linear.
Step 3: Show that T is injective.
Suppose that for $u, v \in \ell^q$ we have that $T(u) = T(v)$. For $i \in \mathbb{N}$, consider $e^i \in \ell^p$ where

$$e_n^i = \begin{cases} 1 & n = i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $e^i \in \ell^p$ and $u_i = T(u)\left(e^i\right) = T(v)\left(e^i\right) = v_i$. Therefore, $u = v$ and so $v \mapsto T(v)$ is injective.
Step 4: Show that T is surjective.

Let $\xi \in (\ell^p)'$. For $u \in \ell^p$ let $u^N = (u_n \mathbf{1}_{n \le N})_{n\in\mathbb{N}}$. Observe that

$$\begin{aligned} T(v)\left(u^N\right) &= \sum_{n=1}^{N} v_n u_n \\ &= \sum_{n \ge N+1} \xi(e_n) u_n \\ &= \xi\left(\sum_{n=1}^{N} u_n e_n\right) \\ &= \xi\left(u^N\right) \end{aligned}$$

which implies $\left|T(v)u^N\right| \le \|\xi\|_{(\ell^p)'} \left\|u^N\right\|_{\ell^p}$. Moreover,

$$\left\|u^N - u\right\|_{\ell^p}^p = \sum_{n=N+1}^{\infty} |u_n|^p \overset{N\to\infty}{\longrightarrow} 0.$$

Hence,

$$\left|T(v)u^N - \xi\left(u\right)\right| = \left|\xi\left(u^N - u\right)\right| \le \|\xi\|_{(\ell^p)'} \left\|u^N - u\right\|_{\ell^p}.$$

Therefore, $T(v)u^N \to \xi(u)$ in $\mathbb{R}$ as $N \to \infty$. As $T(v)u^N \to T(v)u$ as $N \to \infty$ by the continuity of T, it follows using the uniqueness of limits that $T(v)u = \xi(u)$. As this holds for any $u \in \ell^p$ it follows that $T(v) = \xi$ in the $(\ell^p)'$ sense. As $\xi \in (\ell^p)'$ was arbitrary we conclude that T is surjective.
Step 5: Deduce that $(\ell^p)' = \ell^q$.
The map T is a bijective and continuous map, so $(\ell^p)' = \ell^q$. $\square$

Exercise 6.4.1

Solution. Let $\varphi = \operatorname{sgn}(f(x))|f|^{p'-1}$. Then

$$\begin{aligned} \frac{|\rho_f(\varphi)|}{\|\varphi\|_{L^p}} &= \frac{\int |f|^{p'}}{\left(\int |f|^{p(p'-1)}\right)^{\frac{1}{p}}} \\ &= \frac{\int |f|^{p'}}{\left(\int |f|^{p'}\right)^{\frac{1}{p}}} \\ &= \left(\int |f|^{p'}\right)^{1-\frac{1}{p}} \\ &= \|f\|_{L^{p'}}. \end{aligned}$$

Hence, $\|\rho_f\|_{(L^p)'} = \|f\|_{L^{p'}}$. $\square$

Exercise 6.4.4

Solution. Suppose that for $f, g \in L^{p'}$ we have that $\rho_f = \rho_g$. It follows that

$$\int f\varphi\, dx = \int g\varphi\, dx$$

for all $\varphi \in L^p$, in particular,

$$\int (f-g)\varphi\, dx = 0$$

for all $\varphi \in L^p$. Letting $\varphi = \mathbf{1}_{[-n,n]^d}$we deduce that $h_n = (f-g)\mathbf{1}_{[-n,n]^d} = 0$ almost everywhere. We observe that $h_n \to (f-g)$ pointwise and so by the dominated convergence theorem we deduce that

$$0 = \lim_{n\to\infty} \int h_n\, dx = \int f - g\, dx$$

which implies that $f = g$ almost everywhere. $\square$

Exercise 6.5.1

Solution. Note that for $\rho, \varphi \in E'$ and $\lambda \in \mathbb{R}$ we have that

$$\begin{aligned} f_x(\rho + \lambda\varphi)(x) &= (\rho + \lambda\varphi)(x) \\ &= \rho(x) + \lambda\varphi(x) \\ &= f_x(\rho) + \lambda f_x(\varphi). \end{aligned}$$

Hence, f_x is linear. □

7 Compactness in Normed Vector Spaces

7.1 Compact Sets

In metric spaces, (X, d), we have two equivalent properties, namely the Bolzano-Weierstrass property and the Open-Covering property. If $K \subset X$ then (K, d) is a metric space and K is said to be compact if the Bolzano-Weierstrass or the Open-Covering property holds for (K, d). Where in (K, d) open sets are intersections of open sets for X with K.
In finite dimensional vector spaces the Heine-Borel theorem characterises compact spaces.

Lemma 7.1.1. *Let E be a normed vector space, and let $M \subset E$ be a closed subspace where $M \neq E$. Then for all $\epsilon > 0$ there exists $u \in E$ such that,*

1. *$\|u\| = 1$, and*
2. *$\operatorname{dist}(u, M) \geq 1 - \epsilon$.*

Proof. Pick $v \in E \setminus M$. Then $d = \operatorname{dist}(v, M) > 0$ as $v \notin M$ and M is closed. So there exists a $m_0 \in M$ such that

$$d \leq \|v - m_0\| \leq \frac{d}{1-\epsilon}.$$

Now let $u = \frac{v - m_0}{\|v - m_0\|}$. It is clear that $\|u\| = 1$. Moreover, for $m \in M$ then

$$\begin{aligned}
\|u - m\| &= \left\| \frac{v - m_0}{\|v - m_0\|} - m \right\| \\
&= \frac{1}{\|v - m_0\|} \|v - m_0 - \|v - m_0\| m\| \\
&\geq \frac{1-\epsilon}{d} \|v - m'\|
\end{aligned}$$

where m' is some element of M. Hence as $\|v - m'\| \geq d$ we have that

$$\|u - m\| \geq 1 - \epsilon.$$

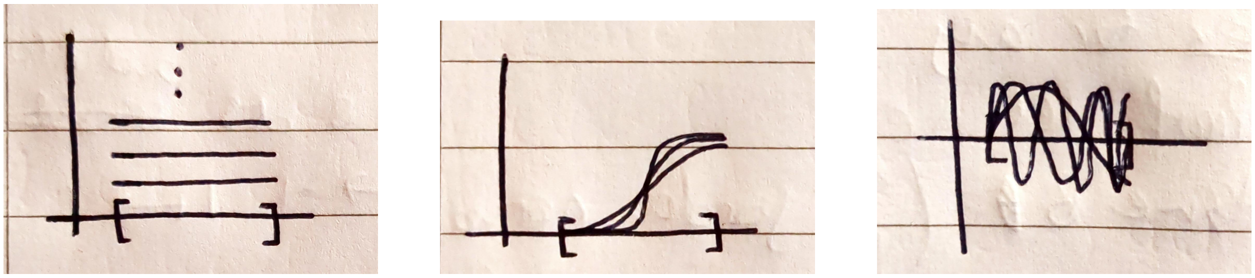
$\square$

Example 7.1.2. *If $E = \mathbb{R}^d$ with the Euclidean norm, and $M \subset E$ is a subspace where $M \neq E$. Then one considers the line orthogonal to M passing through the origin. Choosing a point where this line intersects the unit ball will provide a satisfactory vector u.*

Theorem 7.1.3 (Riesz)**.** *If E is a normed vector space of infinite dimension, the closed unit ball is not compact.*

Proof. Choose u_0 a vector of norm 1. Then by Lemma 7.1.1 there exists a vector $u_!$ such that $\|u_1\| = 1$ and $\operatorname{dist}(u_1, \operatorname{span}(u_0)) \geq 1 - \epsilon$. Continuing, there exists a u_n such that $\|u_n\| = 1$ and $\operatorname{dist}(u_n, \operatorname{span}(u_0, \ldots, u_{n-1})) \geq 1 - \epsilon$. Then the sequence (u_n) is such that $\|u_n - u_m\| \geq 1 - \epsilon$ for all $n \neq m$. Therefore, the sequence has no convergent subsequence and so does not satisfy the Bolzano-Weierstrass property. Therefore, the closed unit ball is not compact. $\square$

Therefore, extending our notions of compactness to infinite dimensions fails rather fundamentally. Theorem 7.1.7 gives us a characterisation of compactness for the set of continuous functions on the closure of open and bounded sets Ω, denoted $\mathcal{C}^0\left(\bar{\Omega}\right)$.



(a) A sequence of functions that is unbounded.

(b) A sequence of functions converging to a step function.

(c) A sequence of functions that oscillate at an ever-increasing rate.

Figure 9: Examples illustrating some necessary conditions for sequences of functions to admit convergent subsequences

From Figure 9a we note that we must require a sequence of functions to be bounded to admit a convergence subsequence. Similarly, Figures 9b and 9c show that we must have a condition which ensures the derivatives of these functions are bounded.

Definition 7.1.4. *A sequence $(f_n) \subseteq \mathcal{C}^0\left(\bar{\Omega}\right)$ is bounded, with constant C, if*

$$\|f_n\|_\infty \leq C.$$

Definition 7.1.5. *A sequence $(f_n) \subseteq \mathcal{C}^0\left(\bar{\Omega}\right)$ is equicontinuous if for all $x \in \bar{\Omega}$ and for all $\epsilon > 0$ there exists a $\delta > 0$ such that for $x, y \in \bar{\Omega}$ with $|x - y| < \delta$ we have that $|f_n(x) - f_n(y)| < \epsilon$ for all n.*

Example 7.1.6. *Let $f_n : \overline{B_{\mathbb{R}^d}(0,1)} \to \mathbb{R}$ be given by $f_n(x) = e^{-n\|x\|}$. Clearly, the f_n are continuous as $x \mapsto e^{-x}$ and $x \mapsto \|x\|$ are continuous and $f_n(x)$ is their composition. Moreover, the functions are bounded. However, let $x = 0$ and $\epsilon = \frac{1}{2}$. Then for any $\delta > 0$ we can choose $y \in \overline{B_{\mathbb{R}^d}(0,1)}$ such that $\|y\| = \frac{\delta}{2}$. Then*

$$|f_n(x) - f_n(y)| = \left|1 - e^{-\frac{n\delta}{2}}\right| \overset{n\to\infty}{\longrightarrow} 1.$$

Therefore, there exists an $n \in \mathbb{N}$ such that

$$|f_n(x) - f_n(y)| \geq \frac{\epsilon}{2}$$

and so the sequence is not equicontinuous.

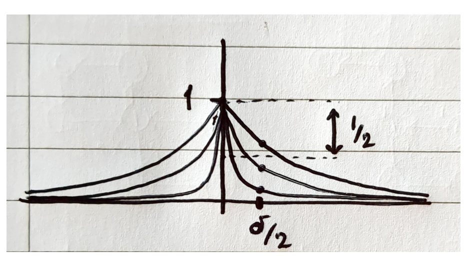


Figure 10: Intuitively the functions referenced in Example 7.1.6 are not equicontinuous, as the gradients of the function near the origin diverge as n gets large.

Theorem 7.1.7 (Arzela-Ascoli). *Consider a sequence $(f_n) \subseteq \mathcal{C}^0\left(\bar{\Omega}\right)$ such that the sequence is bounded, with constant C, and equicontinuous. Then the sequence (f_n) admits a convergent subsequence.*

Proof. To simplify the proof we consider (f_n) to be uniformly equicontinuous. In this case, we do not need $\bar{\Omega}$ to be bounded. However, if we relax uniform equicontinuity to equicontinuity we require $\bar{\Omega}$ to be bounded. That is, for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in \bar{\Omega}$ and $n \in \mathbb{N}$ then $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$.
Step 1: Finding a dense set of points.
Arrange the rational in $\bar{\Omega}$ into a sequence $(r_n)_{n\in\mathbb{N}}$.
Step 2: Apply the (Cantor) diagonal argument.
Let $\varphi_1 : \mathbb{N} \to \mathbb{N}$ be such that $f_{\varphi(m)}(r_1)$ converges. This is possible since the sequence $(f_n(r_1))$ is bounded and so has a convergent subsequence. Now let $(f_{\varphi_2(n)})$ be a subsequence of $\left(f_{\varphi_1(n)}\right)$ such that $f_{\varphi_2(n)}(r_2)$ converges. Again we can do this as the sequences are bounded and so admit convergent subsequences. Note that we also have that $f_{\varphi_2(n)}(r_1)$ converges as $(f_{\varphi_2(n)}) \subseteq (f_{\varphi_1(n)})$. Continue in this way to define φ_k such that $\left(f_{\varphi_k(n)}\right)$ is a subsequence of $\left(f_{\varphi_{k-1}(n)}\right)$ and $f_{\varphi_k(n)}(r_k)$ converges. Again note that $f_{\varphi_k(n)}(r_j)$ converges for all $j = 1, \dots, k-1$. Now set $\varphi(n) = \varphi_n(n)$. Then $f_{\varphi(n)}(r_j)$ converges for any j as $\left(f_{\varphi(n)}\right) \subseteq \left(f_{\varphi_j(n)}\right)$ for all j.
Step 3: The Candidate.Let $f(r) = \lim_{n\to\infty} f_{\varphi(n)}(r)$ for all $r \in \mathbb{Q} \cap \bar{\Omega}$.
Step 4: f is "uniformly continuous".
For any $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f_\varphi(n)(x) - f_\varphi(n)(y)| < \epsilon$ for all $n \in \mathbb{N}$. If $(x, y) = (r, s) \in \mathbb{Q}^2$ then passing to the limit we get that $|f(r) - f(s)| < \epsilon$ if $|r - s| < \delta$. Then using the uniform equicontinuity of the sequence (f_n) we can extend f to $\bar{\Omega}$ by letting $f(x) = \lim_{r\to x} f(r)$.
Step 5: $f_{\varphi(m)}$ converges to f in $\|\cdot\|_\infty$.
Fix $\epsilon > 0$.

- Choose $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_m(y)| < \frac{\epsilon}{3}$.
- Choose N such that for all $x, y \in \bar{\Omega}$ there exists a $j \in \{1, \dots, N\}$ such that $|x - r_j| < \delta$.
- Choose M such that for all $j \in \{1, \dots, N\}$ if $m > M$, then $\left|f_{\varphi(m)}(r_j) - f(r_j)\right| < \frac{\epsilon}{3}$.

The if $x \in \bar{\Omega}$, choose j_0 such that $|r_{j_0} - x| < \delta$. If $n > M$ then

$$\begin{aligned}
\left|f(x) - f_{\varphi(n)}(x)\right| &\leq |f(x) - f(r_j)| + \left|f(r_j) - f_{\varphi(n)}(r_j)\right| + \left|f_{\varphi(n)}(r_j) - f_{\varphi(n)}(x)\right| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq \epsilon.
\end{aligned}$$

□

Example 7.1.8. *Consider the functions $(f_n) \subseteq \mathcal{C}^0\left(\overline{B_{\mathbb{R}^d}(0,1)}\right)$ from Example 7.1.6. Suppose that $f_{\varphi(n)} \to f$ in $\mathcal{C}^0(0,1)$. Then this implies the pointwise convergence of $f_{\varphi(n)}(x) \to f(x)$. However,*

$$f_{\varphi(n)}(x) = e^{-\varphi(n)\|x\|} \longrightarrow \begin{cases} 1 & x = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

which is not a continuous function. Therefore, there cannot exist a convergent subsequence $\left(f_{\varphi(n)}\right)$ in $\mathcal{C}^0(0,1)$. Recall, that the sequence (f_n) was shown not to be equicontinuous. Hence, the requirement of equicontinuity in Theorem 7.1.7 is a necessary condition.

7.2 Compact Operators

Recall that $\mathcal{L}(E, F)$ is the set of bounded linear operators $E \to F$. Moreover,

$$\|T\|_{E\to F} = \sup_{0\neq x\in E} \frac{\|Tx\|_F}{\|x\|_E}.$$

In other words,

$$\|Tx\|_F \leq \|T\|_{E\to F}\|x\|_E.$$

Definition 7.2.1. *A set $S \subset X$ is pre-compact if $\bar{S}$ is compact.*

Definition 7.2.2. *The operator $T \in \mathcal{L}(E,F)$ is compact if $T\left(\bar{B}^E\right)$ is pre-compact, where $\bar{B}^E := \{x \in E : \|x\| \leq 1\}$.*

Example 7.2.3.

1. *For a Banach space E, the operator $\mathrm{Id} : E \to E$ is compact if and only if $\dim(E) < \infty$. Therefore, in some sense, pre-compact operators must shrink sets on which it is applied.*
2. *Consider $\mathrm{Id} : \mathcal{C}^1\left(\bar{\Omega}\right) \to \mathcal{C}^0\left(\bar{\Omega}\right)$. The unit ball consists of functions $f \in \mathcal{C}^1$ such that $\|f\|_\infty + \sum_{i=1}^d \|\partial_i f\|_\infty \leq 1$. In particular,*
$$|f(x) - f(y)| \leq C\|x-y\|$$
for any $x, y \in \bar{\Omega}$ by the mean value theorem. Therefore, Theorem 7.1.7 applies and so the image of the unit ball is compact.
3. *For $T : E \to F$ where $\dim(F) < \infty$, the image of the unit ball is bounded and so by Heine-Borel its closure is compact and hence the set is pre-compact. Therefore, T is compact.*
4. *Let $T : L^p(0,1) \to \mathcal{C}^0(0,1)$ where $f \mapsto \int K(x,y)f(y)\,dy$ for $K \in \mathcal{C}^1\left([0,1]^2\right)$. This is well-defined by Holder's inequality. Moreover,*
$$\begin{aligned} |Tf(x) - Tf(x')| &= \left|\int_0^1 (K(x,y) - K(x',y))\,f(y)\,dy\right| \\ &\overset{(1)}{\leq} \int_0^1 |K(x,y) - K(x',y)|\,|f(y)|\,dy \\ &\overset{(2)}{\leq} C\,|x-x'|\,\|f\|_{L^p}, \end{aligned}$$
where (1) is the generalised triangle inequality, and (2) follows from Holder's inequality and the mean value theorem applied to K. Therefore, by Theorem 7.1.7 the operator is compact.

Theorem 7.2.4. *The set of compact operators, $\mathcal{K}(E,F)$, is closed in $\mathcal{L}(E,F)$.*

Proof. Choose T_i a sequence of $\mathcal{K}(E,F)$ converging to $T \in \mathcal{L}(E,F)$. Choose a sequence (x_j) of $\bar{B}^E$. We can use a diagonal argument to find a $\varphi : \mathbb{N} \to \mathbb{N}$ such that $T_i\left(x_{\varphi(j)}\right)$ converges as $j \to \infty$ for all i. We can write

$$\left\|Tx_{\varphi(n)} - Tx_{\varphi(m)}\right\| \leq \left\|Tx_{\varphi(n)} - T_k x_{\varphi(n)}\right\| + \left\|T_k x_{\varphi(n)} - T_k x_{\varphi(m)}\right\| + \left\|T_k x_{\varphi(m)} - Tx_{\varphi(m)}\right\|$$

where the first term can be made small for large k as $\|Tx_{\varphi(n)} - T_k x_{\varphi(n)}\| \leq \|T - T_k\|\left\|x_{\varphi(n)}\right\|$ where $\|T - T_k\| \to 0$ and $\left\|x_{\varphi(n)}\right\| \leq 1$, similarly for the third term. The second term can be made small by the fact that $\left(T_k x_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is convergent. Hence, we deduce that $\left(Tx_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is Cauchy, and thus it converges as F is a Banach space. Therefore, $T\left(\bar{B}^E\right)$ is pre-compact and thus $T \in \mathcal{K}(E,F)$. □

Corollary 7.2.5. *Let $T_n : E \to F$ be such that $\dim(T_n(E)) < \infty$, these are also known as finite rank operators. If $T_n \to T$, then T is compact.*

Proof. For each n we have that $T_n\left(\bar{B}^E\right) \subseteq T_n(E)$ and so $\dim\left(T_n\left(\bar{B}^E\right)\right) < \infty$. Hence, $T_n\left(\bar{B}^E\right)$ is pre-compact as it is bounded, which implies that T_n is compact. Therefore, if $T_n \to T$ exists, Theorem 7.2.4 says that T is compact. □

Example 7.2.6. *Let $T : \ell^2 \to \ell^2$ be given by $(x_m) \mapsto (c_m x_m)$. One can think of this operator as the matrix*

$$\begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & c_3 & \\ 0 & & & \ddots \end{pmatrix}.$$

- *T is bounded if and only if $|c_n| \leq C$ for all n.*
- *T is compact if and only if $c_n \to 0$. To see the suppose that $c_n \to 0$. Then define the operator T_k by the matrix*

$$T_k = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_k \end{pmatrix}.$$

Observe that

$$\begin{aligned} \|T - T_k\|_{\ell^2 \to \ell^2} &= \sup_{0 \neq x \in \ell^2} \frac{\|(T - T_k)(x)\|_{\ell^2}}{\|x\|_{\ell^2}} \\ &= \sup_{0 \neq x \in \ell^2} \frac{\sqrt{\sum_{i=k+1}^{\infty} |c_i x_i|^2}}{\|x\|_{\ell^2}} \\ &\leq \sup_{0 \neq x \in \ell^2} \frac{\sup_{i \in \mathbb{N}} |c_i| \|x\|_{\ell^2}}{\|x\|_{\ell^2}} \\ &= \sup_{i \in \mathbb{N}} |c_i|. \end{aligned}$$

Hence, it is clear that $T_k \to T$ and so by Corollary 7.2.5, the operator T is compact. For the converse assume T is compact and suppose that $c_n \not\to 0$ as $n \to \infty$. Then for some $\epsilon > 0$ there exists a sequence $\varphi(n)$ such that $\left|c_{\varphi(n)}\right| \geq \epsilon$ for all $n \in \mathbb{N}$. Let $x^{(n)}$ be the sequence where $x_i^{(n)} = \delta_{i\varphi(n)}$. It follows that $\left\|x^{(n)}\right\|_{\ell^2} = 1$ for all $n \in \mathbb{N}$ and

$$\left\|Tx^{(n)} - Tx^{(m)}\right\|_{\ell^2} \geq \sqrt{2}\epsilon$$

for all $n \neq m$. Hence, the sequence $\left(Tx^{(n)}\right)_{n \in \mathbb{N}} \in T\left(\bar{B}^E\right)$ has no convergent subsequence and so $T\left(\bar{B}^E\right)$ is not pre-compact. This contradicts T being compact, therefore, we must have that $c_n \to 0$ as $n \to \infty$.

8 Hilbert Spaces

Throughout let H be a real vector space.

8.1 Inner Product

Definition 8.1.1. *An inner product on H is an application $(\cdot,\cdot) : H \times H \to \mathbb{R}$ that satisfies the following.*

1. *It is bilinear. That is,*
$$(ax + by, z) = a(x, z) + b(y, z)$$
and
$$(z, ax + by) = a(z, x) + b(z, y)$$
for all $x, y, z \in H$ and $a, b \in \mathbb{R}$.
2. *It is symmetric. That is, $(x, y) = (y, x)$ for all $x, y \in H$.*
3. *It is positive definite. That is $(x, x) \geq 0$ for all $x \in H$ and $(x, x) = 0$ if and only if $x = 0$.*

Remark 8.1.2. *Elements $x, y \in H$ are said to be orthogonal if $(x, y) = 0$.*

Lemma 8.1.3 (Cauchy-Schwarz)**.** *For $x, y \in H$ we have that*
$$|(x, y)| \leq \sqrt{(x, x)}\sqrt{(y, y,)}. \tag{8.1.1}$$

Proof. The map $t \mapsto (x + ty, x + ty)$ is a non-negative polynomial in t. Hence, its discriminant is non-negative, which is equivalent to the statement of the lemma. □

Remark 8.1.4. *Note that* (8.1.1) *holds if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$.*

Proposition 8.1.5. *If $(\cdot,\cdot)$ is an inner product on H, then*
$$\|x\| = \sqrt{(x, x)} \tag{8.1.2}$$
defines a norm on H.

Proof. Clearly, $\|x\| = 0$ if and only if $x = 0$ by the positive definiteness of the inner product. Similarly, homogeneity follows from the bilinearity of the inner product. Moreover, using the Cauchy-Schwarz inequality we get that
$$\begin{aligned}\|x + y\|^2 &= (x + y, x + y)\\ &= (x, x) + 2(x, y) + (y, y)\\ &= \|x\|^2 + 2(x, y) + \|y\|^2\\ &\leq \|x\|^2 + \|x\|\|y\| + \|y\|^2\\ &= (\|x\| + \|y\|)^2\end{aligned}$$
which implies that the triangle inequality holds for $\|\cdot\|$. Hence, $\|\cdot\|$ defines a norm. □

For a norm $\|\cdot\|$ given by (8.1.2) for some inner product $(\cdot,\cdot)$, the following identities hold.

- Parallelogram law,
$$\left\|\frac{u + v}{2}\right\|^2 + \left\|\frac{u - v}{2}\right\|^2 = \frac{\|u\|^2 + \|v\|^2}{2}.$$

- Polarization identity,

$$(u,v) = \frac{1}{2}\left(\|u+v\|^2 - \|u\|^2 - \|v\|^2\right).$$

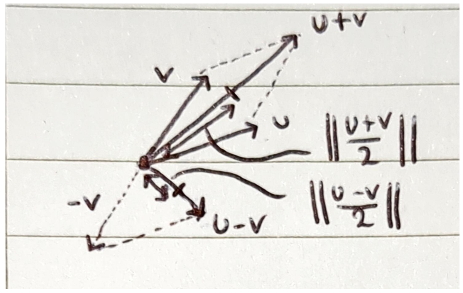


Figure 11: Parallelogram Law

Definition 8.1.6. *A Hilbert space is a normed vector space whose norm is given by an inner product as in* (8.1.2), *moreover, the space is complete.*

Remark 8.1.7. *We only consider real Hilbert spaces, however, the theory can be extended to complex vector spaces by requiring the inner product to also be conjugate symmetric. That is,*

- $x \mapsto \langle x, y\rangle$ *for all* y *is linear, and*
- $y \mapsto \langle x, y\rangle$ *for all* x *is anti-linear.*

In other words, $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Example 8.1.8.

1. $\mathbb{R}^d$ *with the Euclidean inner product*

$$(x,y) = \sum_{i=1}^{d} x_i y_i$$

is a real Hilbert space. $\mathbb{C}^d$ *with inner product*

$$(x,y) = \sum_{i=1}^{d} x_i \overline{y}_i$$

is a complex Hilbert space.

2. $\ell^2(\mathbb{N})$ *with the inner product*

$$((x_n),(y_n)) = \sum_{n=1}^{\infty} x_n y_n$$

is a real Hilbert space.

3. $L^2(\Omega)$ *with the inner product*

$$(f,g) = \int f(x)g(x)\,dx$$

is a real Hilbert space.

- *$L^p(\Omega)$ for $p \neq 2$ is not a Hilbert space.*

In each of these cases, the inner product defines the usual norms defined on these spaces.

8.2 Projection

Theorem 8.2.1. *Let H be a Hilbert space. Let $K \subset H$ be a closed convex set. Then for every $f \in H$ there exists a unique $u \in K$ such that*

$$\|f-u\| = \min_{v \in K} \|f-v\| = \operatorname{dist}(f, K). \tag{8.2.1}$$

Moreover, u is characterised by the property that $u \in K$ and

$$(f-u, v-u) \leq 0 \tag{8.2.2}$$

for all $v \in K$.

Proof. Step 1: Existence of $\min_v \|f-v\|$.
Consider a sequence $(v_n) \subset K$ such that

$$d_n = \|f - v_n\| \to d = \in_{v \in K} \|f-v\|.$$

Applying the parallelogram identity to $\|f-v_n\|$ and $\|f-v_m\|$ we deduce that

$$\left\|f - \frac{v_n + v_m}{2}\right\|^2 + \left\|\frac{v_n - v_m}{2}\right\|^2 = \frac{1}{2}\left(d_n^2 + d_m^2\right)$$

which implies that

$$\left\|\frac{v_n - v_m}{2}\right\|^2 \leq \frac{1}{2}\left(d_n^2 + d_m^2\right) - d^2 \overset{n,m \to \infty}{\longrightarrow} 0.$$

Hence (v_n) is Cauchy, which implies that it is convergent to some $v \in H$. Passing to the limit we conclude that

$$\|f-v\| = \min_{v \in K} \|f-u\|.$$

Step 2: Equivalence of the characterisations.
Assume that u satisfies (8.2.1) and consider a $v \in K$. By the convexity of K it follows that

$$(1-t)u + tv \in K$$

for all $t \in [0,1]$. Therefore,

$$\|f - ((1-t)u + tv)\|^2 \geq \|f-u\|^2.$$

The left-hand side is polynomial in t and can be expanded as

$$\|f-u\|^2 - 2t(f-u, v-u) + O\left(t^2\right).$$

As $t \to 0$, the assumption of (8.2.1) can only hold if $(f-u, v-u) \leq 0$. Conversely, suppose that (8.2.2) holds, then for all $v \in K$ it follows that

$$\|u-f\|^2 - \|v-f\|^2 = 2(f-u, v-u) - \|u-v\|^2 \leq 0$$

which implies that $\|u-f\| \leq \|v-f\|$ for all $v \in K$.
Step 3: Uniqueness.
Suppose u_1 and u_2 satisfy (8.2.2), then

1. $(f-u_1, v-u_1) \leq 0$ for all $v \in K$, and

2. $(f - u_2, v - u_2) \le 0$ for all $v \in K$.

Choosing $v = u_2$ and $v = u_1$ in the first and second conditions respectively it follows that

1. $(f - u_1, u_2 - u_1) \le 0$, and
2. $(f - u_2, u_1 - u_2) \le 0$.

Adding these together it follows that $\|u_1 - u_2\|^2 \le 0$ which implies that $u_1 = u_2$. $\square$

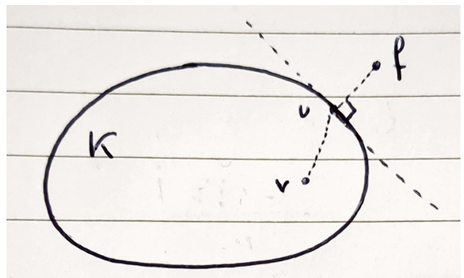


Figure 12: An illustration of the condition stated in (8.2.2)

Proposition 8.2.2. *An alternative characterisation of u in Theorem 8.2.1 when K is additionally a linear subspace of H, is $u \in K$ and*

$$(f - u, v) = 0 \tag{8.2.3}$$

for all $v \in K$

Proof. Suppose that $\tilde{u} \in K$ satisfies (8.2.3). Then for $v \in K$ we have that $\tilde{u} - v \in K$ so that

$$\begin{aligned}
\|f - v\|^2 &= \|f - \tilde{u} + \tilde{u} - v\|^2 \\
&= \|f - \tilde{u}\|^2 + 2(f - \tilde{u}, \tilde{u} - v) + \|\tilde{u} - v\|^2 \\
&\overset{(8.2.3)}{=} \|f - \tilde{u}\|^2 + \|\tilde{u} - v\|^2.
\end{aligned}$$

In particular, this implies that $\|f - v\|^2 \ge \|f - \tilde{u}\|^2$. Conversely, suppose that (8.2.1) is satisfied for $\tilde{u}$. Then for $v \in K$ and $t \in \mathbb{R}$, as K is a linear subspace of H, we have that $\tilde{u} + tv \in K$ and so $\|f - \tilde{u}\|^2 \le \|f - (\tilde{u} + tv)\|^2$. Therefore,

$$0 \le \|f - (u + tv)\|^2 - \|f - u\|^2 = 2t(u - f, v) + t^2\|v\|^2 =: g(t).$$

If $(u - f, v) \ne 0$, then $g(t)$ is minimised by $t = -\frac{(u-f,v)}{\|v\|^2}$, giving a minimum value

$$g\left(-\frac{(u - f, v)}{\|v\|^2}\right) = -2\frac{(u - f, v)^2}{\|v\|^2} + (f - u, v)^2$$

which is strictly negative as we are assuming $(u - f, v) \ne 0$. This is a contradiction and so it must be the case that $(f - u, v) = 0$. $\square$

Remark 8.2.3.

1. *Suppose that M is a closed linear subspace. Then the $P : H \to M$ given by $f \mapsto u$, as in Theorem 8.2.1, is a linear operator. It is similarly characterised by the property that $Pf \in M$ and*

$$\|f - Pf\| = \min_{v \in M} \|f - v\|.$$

Equivalently, it can be characterised by the property that $Pf \in M$ and

$$(f - Pf, v) = 0$$

for all $v \in M$. It follows that $(f - Pf, Pf) = 0$, and so we recover a Pythagoras type relation

$$\|f\|^2 = \|f - Pf\|^2 + \|Pf\|^2.$$

2. *Convexity is necessary for the uniqueness statement of Theorem 8.2.1. Consider $H = \mathbb{R}^2$, $f = (0, 0)$ and K the annulus with centre $(0, 0)$. Although the distance from f to K is well-defined, the projection of f to K is not unique.*

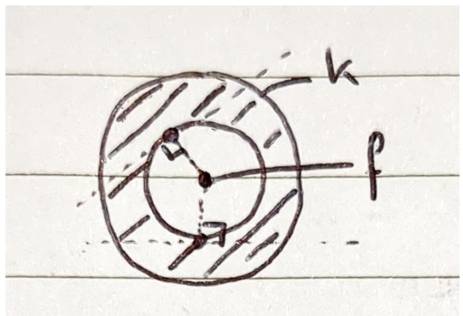


Figure 13: A non-convex set that does not satisfy the uniqueness statement of Theorem 8.2.1. Note that the angle between v and f is obtuse.

For a linear subspace F of a Hilbert space H we can consider the set

$$F^\perp = \{y \in H : (y, x) = 0 \text{ for all } x \in F\}$$

often referred to as the orthogonal complement of F in H.

Proposition 8.2.4. *Let F be a closed subspace of a Hilbert space H. Then $H = F \oplus F^\perp$. In particular, for $v \in H$ we have that $v = Pv + P^\perp v$, where Pv is the projection of v onto F, and $P^\perp v$ is the projection of v onto $F^\perp$.*

Proof. □

Corollary 8.2.5. *Let F be a closed subspace of a Hilbert space H. Then for $v \in H$ it follows that*

$$\|v\|_H^2 = \|Pv\|_H^2 + \left\|P^\perp v\right\|_H^2,$$

where Pv is the projection of v onto F and $P^\perp v$ is the projection of v onto $F^\perp$.

Proof. □

Corollary 8.2.6. *For every closed and non-empty subspace F of a Hilbert space H, there exists a unique linear map $\pi : H \to F$ such that*

1. $\|\pi\|_{H \to H} = \sup_{0 \neq x \in H} \frac{\|\pi x\|_H}{\|x\|_H} = 1$,

2. $\pi^2 = \pi$, *and*

3. $\ker(\pi) = F^{\perp}$.

Proof. For $v \in H$, let $\pi(v) = Pv$. Then using Corollary 8.2.5 it is clear that $\|v\|_H \geq \|\pi(v)\|_H$. Hence,

$$\|\pi\|_{H\to H} = \sup_{0\neq v\in H} \frac{\|\pi(v)\|_H}{\|v\|_H} \leq 1.$$

However, as for $v \in F \setminus \{0\}$ we have $\|v\| = \|\pi(v)\|$ it follows that $\|\pi\|_{H\to H} = 1$. Moreover, as $Pu \in F$ it is clear that $P(Pu) = u$ and so $\pi^2 = \pi$. Next if $v \in F^{\perp}$, then $\pi(v) = 0$ and so $v \in \ker(\pi)$. On the other hand, if $\pi(v) = 0$, it follows that $v = 0 + v$, and so $v \in F^{\perp}$. $\square$

Exercise 8.2.7. *Let F and G be linear subspaces of a Hilbert space H, with $F \subseteq G$. Prove the following statements.*

1. $H^{\perp} = \{0\}$ *and* $\{0\}^{\perp} = H$.
2. $F^{\perp}$ *is a closed linear subspace of* H.
3. $G^{\perp} \subseteq F^{\perp}$.
4. $\left(F^{\perp}\right)^{\perp} = \bar{F}$.
5. *If in addition F and G are closed, show that the following hold.*
   (a) $F \cap G = \left(F^{\perp} + G^{\perp}\right)^{\perp}$.
   (b) $F^{\perp} \cap G^{\perp} = (F + G)^{\perp}$.
   (c) $(F \cap G)^{\perp} = \overline{F^{\perp} + G^{\perp}}$.
   (d) $\left(F^{\perp} \cap G^{\perp}\right)^{\perp} = \overline{F + G}$.

8.3 The Dual Space

Observe that for any $u \in H$, the map $\varphi_u : H \to \mathbb{R}$ given by $v \mapsto (u, v)$ is in the dual space of H, denoted H^*. Moreover, using the Cauchy-Schawarz inequality we can show that the map $H \to H^*$ given by $u \mapsto \varphi_u$ is an isometry. If $\dim(H) < \infty$, then it follows by arguments involving linear algebra, that any element of H^* is of the form φ_u for some $u \in H$.

Theorem 8.3.1 (Riesz-Frechet Representation Theorem)**.** *For any $\varphi \in H^*$, there exists a $u \in H$ such that $\varphi = \varphi_u$ and $\|\varphi\|_{H^*} = \|u\|_H$.*

Proof. For $\varphi \in H^*$, let $M = \varphi^1(\{0\})$. By the continuity of φ we know that M is a closed subspace. If $\varphi = 0$ then $M = H$, so we assume instead that there exists a $g_0 \in H \setminus M$. Let P_M be the projection on M, and defined $g_1 = P_M g_0$ and $g = \frac{g_0 - g_1}{\|g_0 - g_1\|}$. Then g is such that $\|g\| = 1$ and $(g, v) = 0$ for all $v \in M$. In particular, this means that $g \notin M$ which implies that $\varphi(g) \neq 0$. Now consider $\varphi(u - \lambda g) = 0$. If $\lambda = \frac{\varphi(u)}{\varphi(g)}$, then $u - \lambda g \in \ker(\varphi) = M$ and so $(g, u - \lambda g) = 0$. Expanding this we deduce that

$$(g, u) - \lambda(g, g) = 0$$

which implies that

$$(g, u) = \frac{\varphi(u)}{\varphi(g)}$$

and so $\varphi(u) = \varphi(g)(g, u)$. It follows that $\varphi = \varphi_{\varphi(g)g}$. $\square$

Remark 8.3.2. *As $u \mapsto \varphi_u$ is an isometry it is injective. As Theorem 8.3.1 shows that $u \mapsto \varphi_u$ is injective, we have that $H = H^*$ for H a Hilbert spaces. As $(L^p)' = L^{p'}$ by Theorem 6.4.2, it follows that L^p is a Hilbert space if and only if $p = p'$, which is only true for $p = 2$.*

Theorem 8.3.3 (Lax-Milgram)**.** *Let H be a real Hilbert space. Assume $a : H \times H \to \mathbb{R}$ is such that the following hold.*

1. *It is bilinear, that is $a(x, \cdot)$ and $a(\cdot, y)$ are linear for all $x, y \in H$.*
2. *It is continuous, that is $|a(x, y)| \leq C\|x\|\|y\|$ for all $x, y \in H$.*
3. *It is coercive, that is $|a(x, x)| \geq c\|x\|^2$ for all $x \in H$.*

Then for $f \in H$ there exists a unique u such that

$$a(u, v) = \langle f, v\rangle$$

for all $v \in H$.

Proof. Step 1: The linear operator associated with a.
For fixed u, we look at $v \mapsto a(u, v) \in H^*$. By Theorem 8.3.1 there exists $A(u)$ such that

$$a(u, v) = \langle A(u), v\rangle$$

for all $v \in H$. Observe that $A : H \to H$ is linear. Moreover, A is bounded as

$$|\langle A(u), v\rangle| = |a(u, v)| \leq C\|u\|\|u\|$$

and so continuous. Furthermore, A is non-degenerate as

$$\|u\|\|Au\| \geq \langle Au, u\rangle = a(u, u) \geq c\|u\|^2$$

and so $\|Au\| \geq c\|u\|$.
Step 2: Solving $Au = f$.

1. A is injective as $\|Au\| \geq c\|u\|$.
2. The range of A, denoted $\mathrm{Ran}(A)$, is given by $A(H)$. Let $(g_n)_{n\in\mathbb{N}} \subset \mathrm{Ran}(A)$ such that $g_n \to g$ in H. We know that there exists a u_n such that $Au_n = g_n$. In particular, $A(u_n - u_m) = g_n - g_m$. Hence, by coercivity it follows that
$$\|u_n - u_m\| \leq \frac{1}{c}\|g_n - g_m\|.$$
Therefore, as $(g_n)_{n\in\mathbb{N}}$ converges it is Cauchy and so $(u_n)_{n\in\mathbb{N}}$ is Cauchy. Using completeness it follows that $u_n \to u$ in $\mathbb{R}$. Passing to the limit we deduce that $Au_n = g_n \to g = Au$ where $Au \in \mathrm{Ran}(A)$. Thus we conclude that $\mathrm{Ran}(A)$ is closed.
3. Suppose that $\mathrm{Ran}(A)$ is not dense. Then its orthogonal is non-zero. That is, there exists a $v \neq 0$ such that $\langle Au, v\rangle = 0$ for all $u, v \in H$. In particular, choosing $u = v$ we obtain
$$0 = \langle Av, v\rangle \geq c\|v\|^2$$
which is a contradiction. Therefore, $\mathrm{Ran}(A)$ is dense.

Using 2. and 3. it is clear that $\mathrm{Ran}(A) = H$ and hence surjective. Combining with 1. we get that A is bijective and so a unique solution $u \in H$ to $Au = f$ exists. ☐

Remark 8.3.4.

1. *Note that* $\langle f, u\rangle \in \varphi(u)$ *where* $\varphi \in H^*$. *So taking* $a(u,v) = \langle u, v\rangle$ *the problem solved by Theorem 8.3.3 is equivalent to the problem solved by Theorem8.3.1. Hence, one can view Theorem 8.3.3 as an extension of Theorem 8.3.1.*

2. *Note that* a *is not symmetric and so in general not an inner product.*

Theorem 8.3.3 has applications in partial differential equations. For a domain $\Omega \subset \mathbb{R}^d$ and $f \in \mathcal{C}^0_\infty$. The Dirichlet problem is to solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking the inner product of the first equation with $\varphi \in \mathcal{C}_c^\infty(\Omega)$ yields

$$-\int_\Omega (\Delta u) \cdot \varphi \, dx = \int_\Omega f \cdot \varphi \, dx.$$

Integrating by parts gives

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \varphi \, dx \tag{8.3.1}$$

as φ vanishes on $\partial\Omega$. Note that the right-hand of (8.3.1) is the inner of f and φ on $L^2(\Omega)$ and the left-hand side is of the form $a(u, \varphi)$. The idea now is to then use Theorem 8.3.3 to solve the Dirichlet problem. To do this H needs to be chosen such that a satisfies the conditions of Theorem 8.3.3.

8.4 Hilbert Sums and Orthonormal Bases

If H is a finite-dimensional Hilbert space, there exists a bases $(e_n)_{n=1}^N \subset H$ such that for any $x \in H$ we can write

$$x = \sum_{i=1}^N x_i e_i$$

for some $x_i \in \mathbb{R}$. In particular, if $(e_n)_{n=1}^N$ is an orthonormal basis it follows that

$$\|x\|^2 = \sum_{i=1}^n \|x_n\|^2. \tag{8.4.1}$$

We would like to generalise the idea of bases to infinite dimensional Hilbert spaces. Using the relation (8.4.1), which holds for orthonormal bases, this generalisation amounts to understanding the convergence of sums.

Definition 8.4.1. *Let* $(E_n)_{n\in\mathbb{N}}$ *be a sequence of closed subspaces of a Hilbert space* H. *Then* H *is a Hilbert sum of the* $(E_n)_{n\in\mathbb{N}}$, *written* $H = \bigoplus_{n=1}^\infty E_n$ *if the following hold.*

1. *The* E_n *are mutually orthogonal. That is* $\langle x, y\rangle = 0$ *if* $x \in E_n$ *and* $y \in E_m$ *for* $n \neq m$.

2. *The subspace* $\operatorname{span}\left(\bigcup_{n=1}^\infty E_n\right)$ *is dense in* H.

Remark 8.4.2. *Throughout, the span of a set of vectors refers to all finite linear combinations of the vectors.*

Lemma 8.4.3. *Let* $(v_n)_{n\in\mathbb{N}} \subset H$ *be such that* $(v_n, v_m) = 0$ *for* $n \neq m$ *and* $\sum_{n=1}^\infty \|v_n\|^2 < \infty$. *Then*

$S_n = \sum_{k=1}^n v_k$ converges, to S say. Furthermore,

$$\|S\|^2 = \sum_{k=1}^{\infty} \|v_k\|^2.$$

Proof. For $n < m$, by orthogonality we have that

$$\|S_n - S_m\|^2 = \sum_{k=n+1}^{m} \|v_k\|^2. \tag{8.4.2}$$

Since, $\sum_{k=1}^{\infty} \|v_k\|^2 < \infty$, using (8.4.2) it follows that $(S_n)_{n\in\mathbb{N}}$ is Cauchy. Therefore, by completeness $(S_n)_{n\in\mathbb{N}}$ has a limit, say S. Furthermore, using (8.4.1) we know that $\|S_n\|^2 = \sum_{k=1}^{n} \|v_k\|^2$ and so passing to the limit we deduce that

$$\|S\|^2 = \sum_{k=1}^{\infty} \|v_k\|^2.$$

$\square$

Theorem 8.4.4. *Assume that $H = \bigoplus_{n=1}^{\infty} E_n$ is a Hilbert sum of the closed subspaces $(E_n)_{n\in\mathbb{N}}$. For $u \in H$, let $u_n = P_{E_n} u$ and $S_n = \sum_{k=1}^{n} u_k$. Then $S_n \to u$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} \|u_n\|^2 = \|u\|^2. \tag{8.4.3}$$

Proof. Step 1: Show that the limit exists.
On the one hand,

$$\|S_n\|^2 = \sum_{k=1}^{n} \|u_k\|^2$$

using (8.4.1). On the other hand, as $u_n = P_{E_n} u$ we have that

$$(u, u_n) = \|u_n\|^2$$

which implies that $(u, S_n) = \sum_{k=1}^{n} \|u_k\|^2$ using the orthogonality of the $E_1, \ldots, E_n$. Therefore, using the Cauchy-Schwarz inequality it follows that

$$\|S_n\|^2 = (u, S_n) \leq \|u\|\|S_n\|.$$

Which implies that

$$\left(\sum_{k=1}^{n} \|v_k\|^2\right)^{\frac{1}{2}} = \|S_n\| \leq \|u\|.$$

Passing to the limit it follows that

$$\sum_{k=1}^{\infty} \|v_k\|^2 \leq \|u\|^2 < \infty.$$

Hence, the conditions of Lemma 8.4.3 and thus we deduce that S_n converges to S and

$$\|S\|^2 = \sum_{k=1}^{\infty} \|v_k\|^2.$$

Step 2: Identification of the limit.
Note that $(u - S_n, v) = 0$ for all $v \in E_m$ where $m \leq n$, by the characterisation of the projection. Letting $n \to \infty$ it follows that $(u - S, v) = 0$ for all $v \in E_m$ where $m \in \mathbb{N}$. By linearity it follows that $(u - S, v) = 0$ for all $v \in \operatorname{span}(\bigcup_{m\in\mathbb{N}} E_m)$. Moreover, by the density of $\operatorname{span}\left(\bigcup_{m\in\mathbb{N}} E_m\right)$ it follows that $(u - S, v) = 0$ for all $v \in H$. Therefore, $u = S$. $\square$

Remark 8.4.5.

1. *The equation* (8.4.3) *is often referred to as the Bessel-Parseval identity.*
2. *The vector* S_n *in Theorem 8.4.4 is the projection of* u *on* $\mathrm{span}\left(\bigcup_{k=1}^n E_n\right)$ *and so the convergence* $S_n \to u$ *is expected. Moreover,* (8.4.1) *is reasonable due to the orthogonality assumptions we impose on the* $(E_n)_{n\in\mathbb{N}}$.
3. *Henceforth, we write* $\sum_{n=1}^\infty u_n = u$ *to mean* $\lim_{n\to\infty} S_n = u$.

Definition 8.4.6. *A sequence* $(e_n)_{n\in\mathbb{N}} \subset H$ *orthonormal basis if the following hold.*

1. $(e_n, e_m) = \delta_{nm}$.
2. $\overline{\mathrm{span}(\{e_n\}_{n\in\mathbb{N}})} = H$.

Remark 8.4.7. *An orthonormal basis of a Hilbert is sometimes referred to as a Hilbert basis.*

Exercise 8.4.8. *Let* H *be a Hilbert space and for* $0 \neq v \in H$ *let* $V := \mathrm{span}(v)$. *Show that* V *is a closed linear subspace of* H. *Moreover, for* $u \in H$ *show that* $P_V u = \frac{(u,v)}{\|v\|^2} v$.

Corollary 8.4.9. *If* $(e_n)_{n\in\mathbb{N}} \subset H$ *is an orthonormal basis, then for all* $u \in H$ *we have*

$$u = \sum_{n=1}^\infty (u, e_n) e_n$$

and

$$\|u\|^2 = \sum_{n=1}^\infty |(u, e_n)|^2 .$$

Proof. Consider the subspaces $(E_n)_{n\in\mathbb{N}}$ of H given by $E_n = \mathrm{span}(e_n)$. By Exercise 8.4.8 the subspace E_n is closed and $u_n = P_{E_n} = (u, e_n)e_n$. Moreover, if $x \in E_n$ and $y \in E_m$, for $n \neq m$, we have that $x = \lambda e_n$ and $y = \mu e_n$. Using the orthogonality of $(e_n)_{n\in\mathbb{N}}$ it follows that that

$$\langle x, y\rangle = \lambda\mu\langle e_n, e_m\rangle = 0.$$

Similarly, as $\{e_n\}_{n\in\mathbb{N}} \subseteq \bigcup_{n\in\mathbb{N}} E_n$ we have that

$$H = \overline{\mathrm{span}\left(\{e_n\}_{n\in\mathbb{N}}\right)} \subseteq \overline{\mathrm{span}\left(\bigcup_{n\in\mathbb{N}} E_n\right)} \subseteq H.$$

Which implies that $\overline{\mathrm{span}\left(\bigcup_{n\in\mathbb{N}} E_n\right)} = H$ and so $\mathrm{span}\left(\bigcup_{n\in\mathbb{N}} E_n\right)$ is dense. Therefore, we can apply Theorem 8.4.4 to conclude that

$$u = \sum_{n=1}^\infty (u, e_n) e_n$$

and

$$\|u\|^2 = \sum_{n=1}^\infty \|(u, e_n)e_n\|^2 = \sum_{n=1}^\infty |(u, e_n)|^2.$$

$\square$

Definition 8.4.10. *A Hilbert space H is separable if it admits a countably dense subset.*

Theorem 8.4.11. *H is a separable metric space if and only if H has an orthonormal basis.*

Proof. ($\Leftarrow$). Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H and consider the subset

$$F = \left\{ \sum_{k=1}^{n} r_k e_k : r_k \in \mathbb{Q},\, n \in \mathbb{N} \right\} \subset H.$$

Let $u \in H$ and $\epsilon > 0$. By Corollary 8.4.9 we know that $u = \sum_{k=1}^{\infty}(u, e_k)e_k$ and

$$\sum_{k=1}^{\infty} |(u, e_k)|^2 = \|u\|^{\infty} < \infty.$$

Hence, we can find an $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |(u, e_k)|^2 < \frac{\epsilon}{2}.$$

Moreover, for $k \leq N$ we can find $r_k \in \mathbb{Q}$ such that $|(u, e_n) - r_k|^2 < \frac{\epsilon}{2N}$. Let

$$\tilde{u} = \sum_{k=1}^{N} r_k e_k \in F,$$

it follows that

$$\begin{aligned}
\|u - \tilde{u}\|^2 &= \left\| \sum_{k=1}^{\infty}(u, e_k)e_k - \sum_{k=1}^{N} r_k e_k \right\|^2 \\
&= \left\| \sum_{k=1}^{N} ((u, e_k) - r_k) + \sum_{k=N+1}^{\infty} (u, e_k)e_k \right\|^2 \\
&\overset{\text{Cor 8.4.9}}{=} \sum_{k=1}^{N} |(u, e_k) - r_k|^2 + \sum_{k=N+1}^{\infty} |(u, e_k)|^2 \\
&< \sum_{k=1}^{N} \frac{\epsilon}{2N} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, F is a countable dense subset of H.
($\Rightarrow$). Let $\{u_n\}_{n\in\mathbb{N}} \subset H$ is a countably dense subset. Construct the sequence $(e_n)_{n\in\mathbb{N}}$ in the following way.

1. $E_1 := \mathrm{span}(u_1)$, and let $e_1 = \frac{u_1}{\|u_1\|}$.
2. $E_2 := \mathrm{span}(u_1, u_2)$ and choose e_2 such that $\{e_1, e_2\}$ is an orthonormal basis of E_2.
   - Note that we assume that u_1 and u_2 are not aligned. We label the subset $\{u_n\}_{n\in\mathbb{N}}$ in this way as the subset is countably dense.
3. For general $k \in \mathbb{N}$, let $E_k := \mathrm{span}(u_1, \ldots, u_k)$ and choose e_k such that $\{e_1, \ldots, e_k\}$ is an orthonormal basis of E_k.
   - Again we can assume that the $u_1, \ldots, u_k$ are not aligned by the fact that $\{u_n\}_{n\in\mathbb{N}}$ is countably dense.

The sequence $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H. $\square$

Remark 8.4.12. *Let H and H' be separable real Hilbert spaces then we know that orthonormal bases $(e_n)_{n\in\mathbb{N}} \subset H$ and $(e'_n)_{n\in\mathbb{N}} \subset H'$ exist. Hence, we can define a map $J : H \to H'$ by*

$$\sum_{n=1}^{\infty} x_n e_n \mapsto \sum_{n=1}^{\infty} x_n e'_n.$$

This is an isometric isomorphism. In particular, fixing $H = \ell^2$ the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ given

$$e_n = (\underbrace{0, \ldots, 1}_{n}, 0, \ldots).$$

The above arguments that any separable real Hilbert space has the same structure of ℓ^2. One may think then that we can characterise all properties of general Hilbert spaces by investigating ℓ^2. After all the isometric isomorphism captures all the structural information regarding the inner product and norm. However, certain interesting Hilbert spaces have additional structures that are not captured within this isometric isomorphism.

Example 8.4.13. *Let $H = L^2(0, 2\pi)$ be a complex Hilbert space and consider $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$. It follows that*

$$(e_n, e_m) = \frac{1}{2\pi} \int_0^{2\pi} e^{inx} e^{imx} \, dx = \delta_{nm}.$$

With additional computations one can show that $\overline{\operatorname{span}(\{e_n\}_{n\in\mathbb{N}})} = H$. With this it follows that $(e_n)_{n\in\mathbb{N}} \subset H$ is an orthonormal basis of H.

8.5 Linear Operators

8.5.1 Adjoint Operators

Consider the finite-dimensional real Hilbert space $H = \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times m}$. Then

$$\langle Mx, y \rangle = \langle x, M^\top y \rangle.$$

For $H = L^2\left(\mathbb{R}^d\right)$ consider

$$(Lu)(x) = \int K(x, y) u(y) \, dy,$$

where $K(x, y)$ is sufficiently smooth and decays fast enough such that the map $u \mapsto Lu$ is well-defined. Then under sufficient assumptions, we can write

$$\begin{aligned}
\langle Lu, v \rangle &= \int \left(\int K(x, y) u(y) \, dy \right) u(x) \, dx \\
&\overset{(1)}{=} \int \left(\int K(x, y) u(x) \, dx \right) u(y) \, dy \\
&= \langle u, L^* v \rangle,
\end{aligned}$$

where

$$(L^* u)(x) = \int K(y, x) u(y) \, dy.$$

For a general Hilbert space, suppose $L \in \mathcal{L}(H, H)$. Then the map $u \mapsto \langle Lu, v \rangle$ is bounded for all $v \in H$. Moreover, it can be represented as $\langle u, \varphi_v \rangle$ for some φ_v using Theorem 8.3.1. Denoting $\varphi_v = L^* v$, then we get the identity

$$\langle Lu, v \rangle = \langle u, L^* v \rangle.$$

One can check that L^* is linear and

$$\|L^*\|_{H\to H} = \|L\|_{H\to H}.$$

Such a linear operator L^* is called the adjoint of L.

8.6 Solution to Exercises

Exercise 8.2.7

Solution. ☐

Exercise 8.4.8

Proof. Let $(\lambda_k v)_{k\in\mathbb{N}} \subset V$ be a sequence converging to $u \in H$. Note that there is a bijection between V and $\mathbb{R}$, namely $\lambda v \mapsto \lambda$. As metrics are equivalent in finite dimensions it follows that $\lambda_k \to \lambda \in \mathbb{R}$, and so $\lambda_k v \to \lambda v \in V$. Hence, V is closed. Consequently, we can write $H = V \oplus V^{\perp}$ using Proposition 8.2.4. In particular, for $u \in H$ we have that $u = \lambda v + P_{V^\perp} v$, where $P_V v = \lambda v \in V$ and $P_{V^\perp} v \in V^{\perp}$. Therefore, $(u, v) = \lambda(v, v)$ which implies that $\lambda = \frac{(u,v)}{\|v\|^2}$. ☐

9 Appendix

9.1 Ordered Sets

Let P be a set. Then $\leq$ is said to define a partial order relation on P if it satisfies the following.

- Reflexivity, that is $a \leq a$ for all $a \in P$.
- Anti-symmetry, that is $a \leq b$ and $b \leq a$ implies that $a = b$ for all $a, b \in P$.
- Transitivity, that is $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in P$.

Definition 9.1.1. *A subset $S \subset P$ is totally ordered if $a \leq b$ or $b \leq a$ for any $a, b \in S$.*

Definition 9.1.2. *If $Q \subset P$, then $c \in P$ is an upper bounded for Q if $a \leq c$ for all $a \in Q$.*

Definition 9.1.3. *An element $m \in S \subset P$ is maximal if $m \leq x$ for $x \in S$ implies that $m = x$.*

Definition 9.1.4. *P is inductive if any totally ordered subset Q has an upper bound.*

Lemma 9.1.5 (Zorn's Lemma)**.** *Every non-empty ordered set that is inductive has a maximal element.*

9.2 Hardy's Inequality

Theorem 9.2.1 (Hardy's Inequality)**.** *Let $1 < p \leq \infty$ and let $f \in L^p(0, \infty)$. Then there exists a $C_p > 0$ such that*

$$\left\| \frac{f(x)}{x} \right\|_{L^p} \leq C_p \left\| f'(x) \right\|_{L^p}.$$

Equivalently, if $F(x) = \int_0^x f(t)\, dt$ then

$$\left\| \frac{F(x)}{x} \right\|_{L^p} \leq C_p \left\| f \right\|_{L^p}.$$

Proof. *Step 1: Let $f \in \mathcal{C}_c^\infty(0, \infty)$ with $f(x) \geq 0$ for all $x \in\in (0, \infty)$. Let $F(x) = \frac{1}{x} \int_0^x f(t)\, dt$. Show that $F \in \mathcal{C}^1(0, \infty)$ and that $xF' = f - F$.*
Note that by the fundamental theorem of calculus

$$F'(x) = \frac{1}{x} f(x) - \frac{1}{x^2} F(x)$$

and so $xF' = f - F$. It is clear that F and F' are continuous. We now show that F and F' are bounded to complete the step. As f is a bounded function the only concerns of unboundedness arise for the $\frac{1}{x}$ terms as $x \to 0$. Recall, that $f \in \mathcal{C}_c^\infty(0, \infty)$. Hence, $\operatorname{supp}(f) = K$ is a compact set of $(0, \infty)$. Note that as $K \subseteq \mathbb{R}$ this implies that K is closed. Suppose that for every $\epsilon > 0$ the set $[0, \epsilon] \cap K \neq \emptyset$. Then there exists a sequence $(x_n) \subseteq K$ such that $x_n \to 0$ as $n \to \infty$. As K is closed this would imply that $0 \in K$ which contradicts $K \subseteq (0, \infty)$. Therefore, there exists an $\epsilon > 0$ such that $[0, \epsilon] \cap K = \emptyset$. Consequently, $f(x) = 0$ for all $x \in [0, \epsilon]$. Therefore, $\int_0^x f(x)\, dx = 0$ for all $x \in [0, \epsilon]$. This implies that $\frac{1}{x} \int_0^x f(x)\, dx$ on $[0, \epsilon]$. One carries out a similar argument to show that F' is bounded near 0. Thus we have that F and F' are continuous and bounded which implies that $F \in \mathcal{C}^1(0, \infty)$.
Step 2: Show that $\int_0^\infty F(x)^p\, dx = -p \int_0^\infty xF(x)^{p-1} F'(x)\, dx$.

To set aside questions regarding convergence for the moment we will first consider the integral $I_R = \int_0^R F(x)^p \, dx$. Performing integration by parts with $u = F(x)^p$ and $\frac{dv}{dx} = 1$ we deduce that

$$\int_0^R F(x)^p \, dx = [xF(x)^p]_0^R - \int_0^R pxF(x)^{p-1}F'(x) \, dx.$$

Letting K be the compact support of f we know that K is bounded and so for sufficiently large R it follows that

$$\int_K f(x) \, dx \int_0^R f(x) \, dx = \int_0^\infty f(x) \, dx.$$

As f is bounded on K it follows that

$$\int_0^\infty f(x) \, dx \leq M$$

for some $M > 0$ which implies that $xF(x)^p \leq \frac{M^p}{x^{p-1}}$. Hence,

$$[xF(x)^p]_0^R \overset{R\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\int_0^\infty F(x)^p \, dx = -p \int_0^\infty xF(x)^{p-1}F'(x) \, dx$$

as the integrand on the right-hand side is well-defined as $R \to \infty$ as F and F' are bounded.

Step 3: Deduce that $\|F\|_{L^p}^p \leq C_p \|f\|_{L^p}$.

Using the expression $xF' = f - F$ deduce in Step 1 and the expression deduce in Step 2 we deduce that

$$\begin{aligned}\int_0^\infty F(x)^p \, dx &= -p \int_0^\infty xF(x)^{p-1}F'(x) \, dx \\ &= -p \int_0^\infty F(x)^{p-1}(f(x) - F(x)) \, dx \\ &= p \int_0^\infty F(x)^p \, dx - p \int_0^\infty F(x)^{p-1} f(x) \, dx.\end{aligned}$$

Therefore,

$$\int_0^\infty F(x)^p \, dx = \frac{p}{p-1} \int_0^\infty F(x)^{p-1} f(x) \, dx.$$

As $f(x) \geq 0$ for all $x \in (0, \infty)$ it follows that $F(x) \geq 0$ for all $x \in (0, \infty)$. Therefore,

$$\begin{aligned}\|F\|_{L^p}^p &= \int_0^\infty |F(x)|^p \, dx \\ &= \int_0^\infty F(x)^p \, dx \\ &= \frac{p}{p-1} \int_0^\infty F(x)^{p-1} f(x) \, dx.\end{aligned}$$

Let p' be such that $1 = \frac{1}{p} + \frac{1}{p'}$, which implies that $p' = \frac{p}{p-1}$. Then by applying Holder's inequality to the right-hand side we deduce that

$$\begin{aligned}\int_0^\infty F(x)^p \, dx = \|F\|_{L^p}^p &\leq \frac{p}{p-1} \|f\|_{L^p} \left\|F^{p-1}\right\|_{L^{p'}} \\ &= \frac{p}{p-1} \|f\|_{L^p} \left(\int_0^\infty \left(F(x)^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\ &= \frac{p}{p-1} \|f\|_{L^p} \left(\int_0^\infty F(x)^p \, dx\right)^{1-\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\|F\|_{L^p} \leq \frac{p}{p-1}\|f\|_{L^p}.$$

Step 4: Extend the result to general $g \in \mathcal{C}_c^\infty(0,\infty)$.
For $g \in \mathcal{C}_c^\infty(0,\infty)$, note that $|g|$ is still a continuous function with compact support. As the continuous differentiability of f in the previous steps is not used the claims still hold true for $|g|$ as $|g(x)| \geq 0$ for all $x \in (0,\infty)$. Therefore,

$$\left\|\frac{1}{x}\int_0^x |g(x)|\,dx\right\|_{L^p} \leq \frac{p}{p-1}\||g|\|_{L^p}.$$

As $\||g|\|_{L^p}$ and $\frac{1}{x}\int_0^x g(t)\,dt \leq \frac{1}{x}\int_0^x |g(t)|\,dt$ for all $t \in (0,\infty)$ we deduce that

$$\|G\|_{L^p} \leq \frac{p}{p-1}\|g\|_{L^p}$$

where $G(x) := \frac{1}{x}\int_0^x g(t)\,dt$.
Step 5: Extend the result to $f \in L^p(0,\infty)$.
Recall that $\mathcal{C}_c^\infty(0,\infty)$ is dense in $L^p(0,\infty)$. Therefore, given $f \in L^p(0,\infty)$ there exists a sequence $(f_n) \subset \mathcal{C}_c^\infty(0,\infty)$ such that $f_n \xrightarrow{L^p} f$. Letting $F_n(x) = \frac{1}{x}\int_0^x f_n(t)\,dt$ we observe that

$$\begin{aligned}
\|F_n(x) - F(x)\|_{L_x^p} &= \left(\int_0^\infty \left|\int_0^x \frac{1}{x} f_n(t) - f(t)\,dt\right|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left|\int_0^1 f_n(xt) - f(xt)\,dt\right|^p dx\right)^{\frac{1}{p}} \\
&\overset{(1)}{\leq} \int_0^1 \left(\int_0^\infty |f_n(xt) - f(xt)|\,dx\right)^{\frac{1}{p}} dt \\
&= \int_0^1 \frac{1}{t^{\frac{1}{p}}}\|f_n - f\|_{L^p}\,dx \\
&\overset{(2)}{=} M\|f_n - f\|_{L^p}.
\end{aligned}$$

Where (1) follows from Minkowski's integral inequality[1], and (2) follows from the fact that $p > 1$ and so the integral is finite. Therefore, $F_n \xrightarrow{L^p} F$. As $f_n \in \mathcal{C}_c^\infty(0,\infty)$ we know that the inequality $\|F_n\|_{L^p} \leq C_p\|f_n\|_{L^p}$ holds. Sending $n \to \infty$ we preserve this inequality as we have convergence in L^p and so $\|F\|_{L^p} \leq C_p\|f\|_{L^p}$. Which completes the proof. ☐

9.3 Hölder Spaces

Definition 9.3.1. *For an open set $\Omega \subset \mathbb{R}^d$, the $\alpha \in (0,1)$ Hölder space denoted $\mathcal{C}^\alpha\left(\bar{\Omega}\right)$ is the set of continuous functions $f \in \mathcal{C}^0\left(\bar{\Omega}\right)$ such that*

$$\sup_{x\neq y,(x,y)\in\Omega^2} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

The norm on $\mathcal{C}^\alpha\left(\bar{\Omega}\right)$ is defined to be

$$\|f\|_{\mathcal{C}^\alpha(\bar{\Omega})} = \|f\|_\infty + \sup_{x\neq y,(x,y)\in\Omega^2} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

[1] https://en.wikipedia.org/wiki/Minkowski_inequality

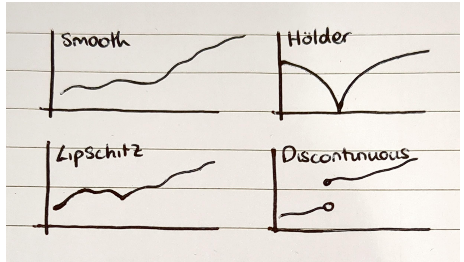


Figure 14: Smooth functions are the strongest class of continuous functions. Lipschitz continuous functions have joins where the gradients at the joins are finite. Lipschitz continuous functions can be thought of as Höder continuous with $\alpha = 1$. Höder continuous functions for $\alpha \in (0,1)$ can have cusps where the gradient at the cusp is potentially unbounded. Discontinuous function contains jumps that do not satisfy the conditions of the previous spaces.

Theorem 9.3.2. *The space* $\left(\mathcal{C}^\alpha\left(\bar{\Omega}\right), \|\cdot\|_{\mathcal{C}^\alpha(\bar{\Omega})}\right)$ *is a Banach space.*

Proof. For $(f_n) \subseteq \mathcal{C}^\alpha\left(\bar{\Omega}\right)$ a Cauchy sequence, it is clear that $(f_n) \subseteq \mathcal{C}^0\left(\bar{\Omega}\right)$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$. As $\left(\mathcal{C}^0\left(\bar{\Omega}\right), \|\cdot\|_\infty\right)$ is a Banach space we know that $f_n \to f \in \mathcal{C}^0\left(\bar{\Omega}\right)$. It remains to show that $f \in \mathcal{C}^\alpha\left(\bar{\Omega}\right)$ and $f_n \to f$ in $\mathcal{C}^\alpha\left(\bar{\Omega}\right)$. For any $(x,y) \in \Omega^2$ with $x \neq y$, let $\delta = |x-y|$. Then as $f_n \to f$ in $\|\cdot\|_\infty$ it follows that there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\delta^\alpha}{2}$$

for all $x \in \Omega$. Therefore, for $n \geq N$ it follows that

$$\begin{aligned}
\frac{|f(x)-f(y)|}{|x-y|^\alpha} &\leq \frac{|f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|}{|x-y|^\alpha} \\
&= \frac{|f(x)-f_n(x)| + |f_n(y)-f(y)|}{\delta^\alpha} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \\
&\leq \frac{\frac{\delta^\alpha}{2} + \frac{\delta^\alpha}{2}}{\delta^\alpha} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \\
&= 1 + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha}.
\end{aligned}$$

As $(f_n) \in \mathcal{C}^\alpha\left(\bar{\Omega}\right)$ is Cauchy we know that the sequence (f_n) is bounded and so $\frac{|f_n(x)-f_n(y)|}{|x-y|} \leq C$ for all n and $(x,y) \in \Omega^2$. Therefore,

$$\sup_{x \neq y, (x,y) \in \Omega^2} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq 1 + C$$

and so $f \in \mathcal{C}^\alpha\left(\bar{\Omega}\right)$. By similar arguments we show that given an $\epsilon > 0$ and $(x,y) \in \Omega^2$ there exits a $N \in \mathbb{N}$ such that for $n \geq N$ we have that

$$\frac{|f(x) - f_n(x) - (f(y) - f_n(y))|}{|x-y|^\alpha} \leq \frac{\epsilon}{2}.$$

Therefore,

$$\sup_{x \neq y (x,y) \in \Omega^2} \frac{|f(x) - f_n(x) - (f(y) - f_n(y))|}{|x-y|^\alpha} \leq \frac{\epsilon}{2}.$$

Moreover, there exists a $M \in \mathbb{N}$ such that for $n \geq M$ we have that $\|f - f_n\|_\infty \leq \frac{\epsilon}{2}$ by the fact that $f_n \to f$ in $\|\cdot\|_\infty$. Therefore,

$$\|f - f_n\|_{\mathcal{C}^\alpha(\bar{\Omega})} = \|f - f_n\|_\infty + \sup_{x \neq y (x,y) \in \Omega^2} \frac{|f(x) - f_n(x) - (f(y) - f_n(y))|}{|x-y|^\alpha} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $f_n \to f$ in $\mathcal{C}^\alpha\left(\bar{\Omega}\right)$. □

Example 9.3.3. *Let $p \in (1, \infty]$ and consider the operator $T : L^p(0,1) \to \mathcal{C}^{1-\frac{1}{p}}(0,1)$ given by*

$$Tf(x) = \int_0^x f(z)\,dz$$

for $x \in [0,1]$. We first show that T is a well-defined operator. For $x < y$ we have that,

$$\begin{aligned}
|Tf(x) - Tf(y)| &= \left|\int_0^x f(z)\,dz - \int_0^y f(z)\,dz\right| \\
&= \left|\int_y^x f(z)\,dz\right| \\
&\overset{T.I}{\leq} \int_0^1 \mathbf{1}_{[x,y]}|f(z)|\,dz \\
&\overset{Holders}{\leq} \left\|\mathbf{1}_{[x,y]}\right\|_{L^{p'}(0,1)} \|f\|_{L^p(0,1)} \\
&= \|f\|_{L^p(0,1)}|x-y|^{1-\frac{1}{p}}.
\end{aligned}$$

Hence, for $1 - \frac{1}{p} > 0$ we have that $Tf \in \mathcal{C}^0(0,1)$. Moreover, we have that

$$\begin{aligned}
\|Tf\|_{\mathcal{C}^0(0,1)} &= \sup_{x\in(0,1)} \left|\int_0^x f(z)\,dz\right| \\
&\leq \int_0^1 |f(z)|\,dz \\
&\leq \|\mathbf{1}\|_{L^{p'}(0,1)}\|f\|_{L^p(0,1)} \\
&= \|f\|_{L^p(0,1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Tf\|_{\mathcal{C}^{1-\frac{1}{p}}(0,1)} &= \|Tf\|_{\mathcal{C}^0(0,1)} + \sup_{x\neq y,(x,y)\in(0,1)^2} \frac{|Tf(x) - Tf(y)|}{|x-y|^{1-\frac{1}{p}}} \\
&\leq \|f\|_{L^p(0,1)} + \|f\|_{L^p(0,1)} < \infty.
\end{aligned}$$

Thus $Tf \in \mathcal{C}^{1-\frac{1}{p}}(0,1)$ and the operator T is well-defined. Moreover, this show that

$$\|T\|_{L^p(0,1)\to\mathcal{C}^{1-\frac{1}{p}}(0,1)} \leq 2.$$

Therefore, as T is a linear map we also deduce that T is continuous. Note that for all $f \in \bar{B}^{L^p(0,1)}$ we have that

$$|Tf(x) - Tf(y)| \leq |x-y|^{1-\frac{1}{p}},$$

hence, $T\left(\bar{B}^{L^p(0,1)}\right) \subseteq \mathcal{C}^0(0,1)$. Moreover, any sequence $(Tf_n) \subseteq T\left(\bar{B}^{L^p(0,1)}\right) \subseteq \mathcal{C}^0(0,1)$ is bounded and equicontinuous and so by Theorem 7.1.7 admits a convergent subsequence. Thus $T\left(\bar{B}^{L^p(0,1)}\right)$ is pre-compact, implying that $T : L^p(0,1) \to L^p(0,1)$ is a compact operator.

9.4 Weak Convergence in Hilbert Spaces

Definition 9.4.1. *Let H be a Hilbert space. A sequence $(x_n)_{n\in\mathbb{N}} \subset H$ weakly converges to $x \in H$ if*

$$(x_n, y) \to (x, y)$$

for all $y \in H$.

Remark 9.4.2.

1. *Symbolically one writes $x_n \rightharpoonup x$ to say that the sequence $(x_n)_{n\in\mathbb{N}} \subset H$ converges weakly to $x \in H$.*
2. *If $x_n \to x$ in the usual sense, then as*

$$|(x_n, y) - (x, y)| \leq \|x - x_n\|\|y\|$$

by Cauchy-Schwarz, it follows that $x_n \rightharpoonup x$.

Example 9.4.3. *In a finite-dimensional Euclidean space, the notions of strong and weak convergence are equivalent. In Remark 9.4.2 2. we saw that strong convergence implies weak convergence using the Cauchy-Schwarz inequality. Conversely, consider the finite-dimensional Euclidean space $\mathbb{R}^d$ and suppose that $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ converges weakly to $x \in \mathbb{R}^d$. Then it follows that $(x_n, e_i) \overset{n\to\infty}{\longrightarrow} (x, e_i)$ where e_i is the $i^{th} \in \mathbb{R}^d$ is the i^{th} coordinate vector. This implies that $x_n^{(i)} \overset{n\to\infty}{\longrightarrow} x^{(i)}$ for each $i \in \{1, \ldots, d\}$. Consequently,*

$$\|x_n - x\| \leq \sum_{i=1}^{d} \left| x_n^{(i)} - x^{(i)} \right| \overset{n\to\infty}{\longrightarrow} 0,$$

and so $x_n \to x$ strongly.

Theorem 9.4.4. *Let H be a Hilbert space. Then every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset H$ has a weakly convergent subsequence.*

Proof. Let $M > 0$ be such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. It follows by Cauchy-Schwarz that for fixed $m \in \mathbb{N}$ the sequence $(x_n, x_m)_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded. Therefore, it has a convergent subsequence. By Cantor's diagonal argument we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ such that $(x_{n_k}, x_m)_{k\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ converges for every $m \in \mathbb{N}$ as $k \to \infty$. Consequently, for $y' \in \operatorname{span}(\{x_n\}_{n\in\mathbb{N}}) =: S$ it follows that $(x_{n_k}, y')_{k\in\mathbb{N}}$ converges as $k \to \infty$. Now consider $y \in \bar{S}$. For $y' \in S$ it follows that

$$\begin{aligned} \left|(x_{n_j} - x_{n_k}, y)\right| &\leq \left|(x_{n_j}, y - y')\right| + \left|(x_{n_j} - x_{n_k}, y')\right| + |(x_{n_k}, y' - y)| \\ &\leq 2M\,\|y - y'\| + \left|(x_{n_j} - x_{n_k}, y')\right|. \end{aligned}$$

Hence, given $\epsilon > 0$, let $y' \in S$ be such that $\|y' - y\| < \frac{\epsilon}{4M}$, and let j, k be large enough such that $\left|(x_{n_j} - x_{n_k}, j)\right|\| < \frac{\epsilon}{2}$. It follows that

$$\left|(x_{n_j} - x_{n_k}, y)\right| < \epsilon,$$

and so $\left|(x_{n_j} - x_{n_k}, y)\right| \to 0$ as $j, k \to \infty$. This implies that for $y \in \bar{S}$ the sequence (x_{n_k}, y) is Cauchy, and so has a limit. Let $Ly := \lim_{k\to\infty} (x_{n_k}, y)$. It is clear that $L : \bar{S} \to \mathbb{R}$ is linear. We also note that L is bounded using Cauchy-Schwarz and the fact that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Therefore, by Theorem 8.3.1 there exists an $x \in \bar{S}$ such that $(x, y) = Ly$ for all $y \in \bar{S}$. Now as $\bar{S}$ is closed we can write $H = \bar{S} \oplus \bar{S}^{\perp}$ by Proposition 8.2.4. Hence, for any $y \in H$ we can write $y = y_1 + y_2$, where $y_1 \in \bar{S}$ and $y_2 \in \bar{S}^{\perp}$. It follows that $(x_n, y) = (x_n, y_1)$ for all $n \in \mathbb{N}$. In particular, we have shown that $(x_{n_k}, y_1)_{k\in\mathbb{N}}$ converges for any $y_1 \in \bar{S}$ and so it follows that $(x_{n_k}, y)_{k\in\mathbb{N}}$ converges for any $y \in H$. Thus we have that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges weakly. ☐

Corollary 9.4.5. *Let H be a Hilbert space. If $(x_n)_{n\in\mathbb{N}} \subset H$ converges weakly to x, then*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Moreover, $\lim_{n\to\infty}\|x_n\| = \|x\|$ if and only if $x_n \to x$ strongly in H.

Proof. As

$$0 \leq (x_n - x, x_n - x) = \|x_n\|^2 - 2(x_n, x) + \|x\|^2 \tag{9.4.1}$$

and $(x_n, x) \to (x, x)$ as $n \to \infty$, it follows that

$$0 \leq \liminf \|x_n\|^2 - \|x\|^2.$$

Moreover, it is clear from (9.4.1) that if $\lim_{n\to\infty}\|x_n\| = \|x\|$ then $(x_n - x, x_n - x) \to 0$ which implies strong convergence. Conversely, by the triangle inequality, we know that $\|x_n - x\| \geq |\|x_n\| - \|x\||$, and so strong convergence implies $\lim_{n\to\infty}\|x_n\| = \|x\|$. ☐

Definition 9.4.6. *Let H be a Hilbert space. A family $(e_n)_{n\in\mathbb{N}} \subset H$ is orthonormal if*

$$(e_n, e_m) = \delta_{nm}$$

for every $n, m \in \mathbb{N}$. If additionally,

$$x = \sum_{n\in\mathbb{N}} (x, e_n)e_n$$

for every $x \in H$, then the family is complete.

Example 9.4.7. *Consider the Hilbert space $L^2((-\pi,\pi))$ and the family $E = (e_n)_{n\in\mathbb{N}}$*

1. *$e_1 = \frac{1}{\sqrt{2\pi}}$,*
2. *$e_{2n} = \frac{1}{\sqrt{\pi}}\sin(nx)$, and*
3. *$e_{2n+1}\frac{1}{\sqrt{\pi}}\cos(nx)$*

for $n \geq 1$. One can show that E is an orthonormal family. Moreover, one can consider E as an orthonormal sequence in the infinite-dimensional Hilbert space $H = L^2((-\pi,\pi))$. Suppose that $(e_n)_{n\mathbb{N}}$ did not converge weakly to zero. Then we can choose a subsequence and an $x \in H$ such that

$$|(x, e_n)| \geq \epsilon \tag{9.4.2}$$

for all $n \in \mathbb{N}$ and some $\epsilon > 0$. Consider $E_m = \operatorname{span}(e_m)$, which is a closed subspace of H as it is finite-dimensional. Hence, by Proposition 8.2.4 $x = \lambda e_m + y$ for unique $\lambda \in \mathbb{R}$ and $y \in E_m^{\perp}$, where in particular λe_m is the projection of x onto E_m. Considering (x, e_m) we see that $\lambda = (x, e_m)$, and so $(x, e_m)e_m$ is the projection of x onto E_m. Similarly,

$$\sum_{n=1}^{N} (x, e_n)e_n$$

is the projection of x onto $E_{1,\ldots,N} := \operatorname{span}(e_1, \ldots, e_N)$. Thus using (9.4.2) it follows that

$$\|x\|^2 = \left\| x - \sum_{n=1}^{N}(x, e_n)e_n \right\|^2 + \left\| \sum_{n=1}^{N}(x, e_n)e_n \right\|^2 \geq \sum_{n=1}^{N}(x, e_n)^2 \geq N\epsilon^2$$

which is contradicts $\|x\|^2 < \infty$. Thus we conclude that $e_n \rightharpoonup 0$. In particular, we have shown that in the setting of Corollary 9.4.5 we cannot ask for equality. Moreover, $(e_n)_{n\in\mathbb{N}}$ is an example of a sequence that converges weakly, but whose norm does not converge to the norm of the limit, and so we do not have strong convergence.

Corollary 9.4.8 (Banach-Saks)**.** *Let H be a Hilbert Space. Let $(x_n)_{n\in\mathbb{N}}$ be such that $\|x_n\| \le K$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\left(x_{n_j}\right)_{j\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ and $x \in H$ such that*

$$\frac{1}{k}\sum_{j=1}^{k} x_{n_j} \overset{k\to\infty}{\longrightarrow} x$$

in H.

Proof. Let x be the weak limit of a subsequence $(x_{n_i})_{i\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ as given by Theorem 9.4.4. Now consider the sequence $(y_i)_{i\in\mathbb{N}}$ given by $y_i := x_{n_i} - x$. It is clear that $y_i \rightharpoonup 0$ and $\|y_i\| \le K'$ for some fixed K'. Consequently, one can choose a subsequence $\left(y_{i_j}\right)$ successively such that

$$\left|\left(y_{i_l}, y_{i_j}\right)\right| \le \frac{1}{j}$$

for $l < j$. This is because for $j \in \mathbb{N}$ we have that $(y_{i_l}, y_i) \overset{i\to\infty}{\longrightarrow} 0$ for each $l < j-1$. Hence, there exists an I such that

$$|(y_{i_l}, y_i)| \le \frac{1}{j}$$

for all $l < j$ and $i \ge I$. Thus, we can let $i_j = \max\left(I, i_{j-1}\right)$. Therefore,

$$\begin{aligned}
\left\|\frac{1}{k}\sum_{j=1}^{k} y_{i_j}\right\|^2 &= \frac{1}{k^2}\sum_{l,j=1}^{k}\left(y_{i_l}, y_{i_j}\right) \\
&= \frac{1}{k^2}\left(\sum_{j=1}^{k}\left(\left(y_{i_j}, y_{i_j}\right) + 2\sum_{l=1}^{j-1}\left(y_{i_l}, y_{i_j}\right)\right)\right) \\
&\le \frac{1}{k^2}\left(k\left(K'\right)^2 + 2\sum_{j=1}^{k} j\frac{1}{j}\right) \\
&\le \frac{\left(K'\right)^2 + 2}{k} \\
&\overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

□

Lemma 9.4.9. *Let H be a Hilbert space. Then every weakly convergent sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is bounded.*

Proof. Consider the sequence of linear functions $(L_n)_{n\in\mathbb{N}}$ given by $L_n y := (x_n, y)$. Now suppose that $(L_n)_{n\in\mathbb{N}}$ is not bounded on any closed ball of H. Then there exists a sequence $(K_i)_{i\in\mathbb{N}}$ of closed balls such that

1. $K_i := \{y : |y - y_i| \le r_i\}$,
2. $K_{i+1} \subseteq K_i$, and
3. $r_i \to 0$.

Moreover, there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ with $|L_{n_i}y| > i$ for all $y \in K_i$. Note that the $(y_i)_{i\in\mathbb{N}}$ form a Cauchy sequence and so have a limit $y_0 \in H$. As $y_0 \in \bigcap_{i=1}^{\infty} K_i$ it follows that $|L_{n_i}y_0| > i$ for all $i \in \mathbb{N}$. This contradicts the weak convergence of $(x_{n_i})_{i\in\mathbb{N}}$, and so there must exist a closed ball on which the linear functions $(L_n)_{n\in\mathbb{N}}$ are bounded. It follows by the linearity of the L_n that the set of linear functions $(L_n)_{n\in\mathbb{N}}$ is bounded on the closed unit ball, that is $\|L_n y\| = \|(x_n, y)\| \leq M$ for some $M > 0$ and for all $n \in \mathbb{N}$. In particular, letting $y = \frac{x_n}{\|x_n\|}$ it follows that

$$\|x_n\| = \left(x_n, \frac{x_n}{\|x_n\|}\right) \leq M$$

for all $n \in \mathbb{N}$, hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. □

Corollary 9.4.10. *Let H be a Hilbert space. If $K \subset H$ is closed and convex, then K is closed with respect to weak convergence.*

Proof. Let $(x_n)_{\mathbb{N}} \subset K$ be weakly convergent to $x \in H$. Then by Lemma 9.4.9 the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, and by Corollary 9.4.8 there exists a subsequence $\left(x_{n_j}\right)_{j\in\mathbb{N}}$ such that

$$\frac{1}{k}\sum_{j=1}^{k} x_{n_j} \to x.$$

As K is convex we know that $\frac{1}{k}\sum_{j=1}^{k} x_{n_j} \in K$ for all j, so because K is closed it follows that $x \in K$. □