Algebraic Topology

Thomas Walker

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Contents

1 Introduction

Algebraic topology is the study of spaces and their shape. More specifically, it aims to determine when spaces have the same shape.

1.1 Continuous Functions on Topological Spaces

Definition 1.1.1. *For topological spaces X and Y on the space $I = [0, 1]$, a homotopy is a continuous map $F : X \times I \to Y$ such that for every $t \in I$ the map $f_t : X \to Y$ given by*

$$f_t(x) = F(x, t)$$

is a continuous map.

Definition 1.1.2. *Continuous maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy $F : X \times I \to Y$ such that*

$$f_0(x) = F(x, 0)$$

and

$$f_1(x) = F(x, 1)$$

for $x \in X$. In such a case we write $f_0 \cong f_1$.

In other words, maps are homotopic if we can continuously deform one into the other. In particular, we note that $\cong$ defines an equivalence relation on the space of continuous maps.

Definition 1.1.3. *For a subspace $A \subseteq X$, a continuous map $r : X \to A$ such that $r(X) = A$ and $r|_A = \mathrm{id}_A$ is called a retraction of X onto A.*

Remark 1.1.4. *Let $r : X \to A$ be a retraction of X onto A. Then as $r(x) \in A$ for all $x \in X$ it follows that $r^2(x) = r(x)$ as $r|_A = \mathrm{id}_A$. Therefore, $r^2 = r$.*

Definition 1.1.5. *A deformation retraction of X onto $A \subseteq X$ is a retraction that is homotopic to the identity.*

Example 1.1.6. *Let X be a topological space, and consider $x_0 \in X$. Then the map $r : X \to \{x_0\}$ given by $r(x) = x_0$ is a retraction of X onto x_0. If X is a path-connected space then r is additionally a deformation retraction, however, if X is not path-connected then r need not be a deformation retraction. Consequently, we conclude that Definition 1.1.5 is strictly stronger than Definition 1.1.3.*

Definition 1.1.7. *A continuous map $f : X \to Y$ is a homotopy equivalence if there is a continuous map $g : Y \to X$ such that $fg \cong \mathrm{id}_Y$ and $gf \cong \mathrm{id}_X$. If there exists a homotopy equivalence between X and Y then they are said to be homotopy equivalent.*

Remark 1.1.8.

1. *We abbreviate the composition of function notation with standard multiplication notation.*
2. *A deformation retraction $f : X \to A$ is a homotopy equivalence. To see this $i : A \to X$ to be the inclusion map, then $fi = \mathrm{id}_A$ and $if = \mathrm{id}_X$.*

Definition 1.1.9. *A topological space is contractible if it is homotopy equivalent to a point.*

Definition 1.1.10. *A continuous map is null-homotopic if it is homotopic to a constant map.*

Lemma 1.1.11. *A topological space X is contractible if and only if id_X is null-homotopic.*

Proof. ($\Rightarrow$). We know that there exists $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ such that $gf \cong \mathrm{id}_X$. Note that $(gf)(x) = g(x_0)$ and thus is a constant map. Therefore, we conclude that $g(x_0) \cong \mathrm{id}_X$, which is to say that id_X is homotopic to a constant map and thus null-homotopic.
($\Leftarrow$). We know that $F : X \times I \to X$ exists such that $F(x, 0) = \mathrm{id}_X$ and $F(x, 1) = x_0$ for some $x_0 \in X$. That is, $\mathrm{id}_X \cong \mathrm{id}_{\{x_0\}}$. Let $f : X \to \{x_0\}$ be given by $f(x) = x_0$ and $g : \{x_0\} \to X$ be the inclusion map. Then $fg : \{x_0\} \to \{x_0\}$ is such that $fg(x) = x_0$ for every $x \in X$ meaning $fg = \mathrm{id}_{\{x_0\}}$. Furthermore, $gf : X \to X$ is such that $gf(x) = x_0$ for every $x \in X$ meaning $gf = \mathrm{id}_{\{x_0\}} \cong \mathrm{id}_X$. Therefore, X is homotopy equivalent to a point and thus contractible. ☐

1.2 Constructing Spaces

Let an n-cell be an n-dimensional object. So a 0-cell is a point, a 1-cell is a line and so on.

1. Start with a discrete set X^0 of 0-cells.
2. Inductively form an n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α through continuous maps $\phi : S^{n-1} \to X^{n-1}$. That is, X^n is the quotient space $X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha / \sim$ where (D^n_α) is a collection of n-disks under the identification that $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$.
3. One can continue indefinitely $X = \bigcup_n X^n$ or stop so that $X = X^n$. In the latter case, we say X has dimension n.

The above construction yields a topological space which we refer to as a CW complex. If the space is infinite-dimensional, then a subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Example 1.2.1. *The S^n has a CW structure with two cells, namely e^0 and e^n. More specifically, the n-cell is attached through the constant map $S^{n-1} \to e^0$ and can also be identified with $D^n/\partial D^n$. To gain a geometry intuition consider S^2, which is the two-dimensional surface of a sphere embedded in three-dimensional. In this case, D^n is a two-dimensional disk at ∂D^n is its perimeter, and so $D^n/\partial D^n$ can be thought of as connecting the perimeter of a disk at a point to form the surface of a sphere.*

Definition 1.2.2. *For a CW-complex X with finitely many cells, its Euler Characteristic is*

$$\chi(X) = |\{\textit{Even Cells}\}| - |\{\textit{Odd Cells}\}|.$$

Remark 1.2.3. *CW-complexes are Hausdorff.*

Example 1.2.4. *$\mathbb{R}P^n$ is the space of all line through the origin in $\mathbb{R}^{n+1}$. A line is determined by a non-zero vector up to scalar multiplication. $\mathbb{R}P^n$ is topologized as the quotient space $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation that says vectors are equivalent if they are non-zero scalar multiples of each other. If we restrict ourselves to vectors of norm 1 then we can equivalently state that $\mathbb{R}P^n$ is the quotient space $S^n/(v \sim -v)$, where S^n is the surface of a $(n+1)$-dimensional sphere. Which is equivalent to the quotient of the hemisphere D^n with the antipodal points of ∂D^n identified, which is just $\mathbb{R}P^{n-1}$ with an n-cell attached along the map*

$S^{n-1} \to \mathbb{R}P^{n-1}$. By induction we conclude that $\mathbb{R}P^n$ has the CW structure

$$e^0 \cup e^1 \cup \cdots \cup e^n,$$

where e^i is an i-cell.

2 The Fundamental Group

2.1 Intuition for the Fundamental Group

Consider two loops A and B which can be linked in different ways. To distinguish between linking mechanisms, suppose A has an associate front and back. Some examples of ways that A and B could interact include the following.

1. Loops and A and B could be separated.
2. Loop B could pass through A once. Either through the front or through the back of A.
3. Loop B could pass through the front of A twice.
4. Loop B could pass through the front of A and then through the back of A.

Note how in example 4. the loops cancel each other out, meaning A and B remain separated. Consequently, there is an additive structure to linking mechanisms. Furthermore, it is clear that for any one of these examples, we can continuously deform the loops whilst maintaining the linking structure. Henceforth, let B_n denote a loop that has n-forward links with A, and let B_{-n} denote a loop with n-backward links with A, with B_0 denoting the loop that is separated from A. Intuitively it makes sense to define the addition of loops B_m and B_n as

$$B_m + B_n = B_{m+n}.$$

With extra work one can see that what we have here is a relationship between the additive group structure of the integers and loops in a topological space. Using this we can give an informal definition of the fundamental group.

Definition 2.1.1 (Informal). *The fundamental group of a space X has elements which are classes of equivalent loops in X that start and end at a fixed base point $x_0 \in X$. Loops are equivalent if one loop can be continuously deformed into the other.*

2.2 Constructions

2.2.1 Path Homotopy

Definition 2.2.1. *Let X be a topological space. Then a path is a continuous map $f : I \to X$, where $I = [0, 1]$.*

A loop now refers to a path where $f(0) = f(1)$. For a loop f we let $x_0 := f(0)$

Definition 2.2.2. *Paths f_0 and f_1 are homotopic if there exists a homotopy between them that preserves their endpoints. That is, there exists an $F : I \times I \to X$ such that the following statements hold.*

1. *$f_t(0) = f_0(0)$ and $f_t(1) = f_0(1)$ for $t \in I$.*
2. *$F(s, 0) = f_0(s)$ and $F(s, 1) = f_1(s)$ for $s \in I$.*

Remark 2.2.3. *For a path $f : I \to X$, we denote the set of paths that are homotopic to f with $[f]$. Indeed, this is the equivalence class of f under the equivalence of homotopy.*

Definition 2.2.4. *Let $f, g : I \to X$ be paths such that $f(1) = g(0)$. Then the product path $f \cdot g : I \to X$ is*

$$(f \cdot g)(s) = \begin{cases} f(2s) & s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Remark 2.2.5. *The product path transverses the path f and then traverses the path g. However, it does so twice as fast, such that it conforms to Definition 2.2.1. Intuitively, it should make sense that if we have paths f_0, f_1, g_0, g_1 on X for which $f_1 \cong f_0$ and $g_0 \cong g_1$ then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

A continuous function $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ can re-parameterized a path $f : I \to X$ into a f_ϕ that is equivalent to f through the homotopy $f\phi_t$ where

$$\phi_t(s) = (1-t)\phi(s) + ts.$$

Definition 2.2.6. *For $x \in X$ let $c_x : I \to X$ be the constant path $c(s) = x$.*

Definition 2.2.7. *For a path $f : I \to X$ let $f^{-1} : I \to X$ be the path defined by $f(s) = f(1-s)$.*

Lemma 2.2.8. *Let $f, g, h : I \to X$ be paths. Then the following statements hold.*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$.
2. $f \cdot c_{f(1)} \cong f$ *and* $c_{f(0)} \cdot f \cong f$.
3. $f \cdot f^{-1} \cong c_{f(0)}$ *and* $f^{-1} \cdot f \cong c_{f(1)}$.

Observe how from Lemma 2.2.8 a group structure on the set of homotopy classes emerges.

Definition 2.2.9. *Let $\pi_1(X, x_0)$ denote the set of homotopy classes $[f]$ of loops $f : I \to X$ with base point x_0.*

Proposition 2.2.10. *The set $\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and identity element $c_{x_0} : I \to X$.*

Remark 2.2.11. *The group of Proposition 2.2.10 is referred to as the fundamental group of X at x_0.*

Example 2.2.12. *Let X be a convex set in $\mathbb{R}$, and let $x_0 \in X$. Then $\pi_1(X, x_0) = 0$ because any two loops with base point x_0 are homotopic via*

$$f_t(s) = (1-t)f_0(s) + tf_1(s).$$

Note how the fundamental group is independent of the choice of x_0.

The dependence of $\pi(X, x_0)$ on x_0 can be illustrative of the topological properties of X.

Definition 2.2.13. *Assume the points $x_0, x_1 \in X$ are path connected with $h : I \to X$. Then, $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by*

$$\beta_h([f]) = \left[h \cdot f \cdot h^{-1}\right]$$

is well-defined. The map β_h is referred to as the change-of-base point map.

Exercise 2.2.14. *Show that $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism. Thus conclude that the fundamental group on a path-connected space is independent, up to isomorphism, of the base point.*

Definition 2.2.15. *A space X is simply connected if it is path-connected and $\pi_1(X) = 0$.*

Proposition 2.2.16. *A space X is simply connected if and only if there exists a unique homotopy class of paths between any points of X.*

2.2.2 The Fundamental Group of a Circle

Definition 2.2.17. *For a space X, its covering space is a set $\widetilde{X}$ with a continuous map $p : \widetilde{X} \to X$ such that for any $x \in X$ and open neighbourhood $x \in U \subseteq X$ the following hold.*

- $p^{-1}(U) = \bigcup_{j \in J} \widetilde{U}_j$ *where* $\widetilde{U}_j \subseteq \widetilde{X}$ *is open.*
- $\widetilde{U}_i \cap \widetilde{U}_j = \emptyset$ *for* $i \neq j$.
- $p|_{\widetilde{U}_j} : \widetilde{U}_j \to U$ *is a homeomorphism for all* $j \in J$.

In such a case U is said to be evenly covered with the sheets $\widetilde{U}_j$.

The set $\widetilde{X}$ can be thought of as an embedding of the topological space X. The conditions Definition 2.2.17 ensure that necessary topological properties are maintained in this embedding.

Definition 2.2.18. *For $p : \widetilde{X} \to X$ a covering space, the lift of a continuous map $f : Y \to X$ is a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that $p\tilde{f} = f$.*

Example 2.2.19. *The n-sphere is given by,*

$$S^n = \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}.$$

So S^1 is just the circle in $\mathbb{R}^2$. Using covering spaces we can study paths in S^1 with paths in $\mathbb{R}$. More specifically, consider the path in S^1 given by

$$\omega(s) = (\cos(2\pi s), \sin(2\pi s)).$$

Note that $[\omega]^n = [\omega_n]$ where

$$\omega_n(s) = (\cos(2n\pi s), \sin(2n\pi s)).$$

Furthermore, observe that

$$p\widetilde{\omega_n} = \omega_n$$

for $\widetilde{\omega_n} : I \to \mathbb{R}$ is given by $\widetilde{\omega_n}(s) = ns$ and

$$p(s) = (\cos(2\pi s), \sin(2\pi s)).$$

Hence, $\widetilde{\omega_n}$ is a lift of ω_n, and thus p provides a mechanism to study paths S^1 using paths of $\mathbb{R}$.

Lemma 2.2.20. *Let $f : I \to X$ be a path with $f(0) = x_0$ and suppose $p : \widetilde{X} \to X$ is a covering space. Then for each $\widetilde{x_0} \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \widetilde{X}$ such that $\tilde{f}(0) = \widetilde{x_0}$.*

Theorem 2.2.21. *Let $x_0 = (1, 0) \in S^1$. Then $\pi_1\left(S^1, x_0\right)$ is the infinite cycle group generated by the homotopy class of the loop $\omega : I \to S^1$ where*

$$\omega(s) = (\cos(2\pi s), \sin(2\pi s)).$$

From Theorem 2.2.21 we note that $\pi_1\left(S^1, x_0\right) \cong \mathbb{Z}$.

Theorem 2.2.22. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proposition 2.2.23. *Let X and Y be path-connected topological spaces. For $x_0 \in X$ and $y_0 \in Y$ it follows that*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi(Y, y_0).$$

Corollary 2.2.24. *The torus $X = S^1 \times S^1$ has fundamental group*

$$\pi_1(X) = \pi_1\left(S^1\right) \times \pi_1\left(S^1\right) \cong \mathbb{Z}^2.$$

2.2.3 Induced Homomorphisms

Throughout, maps will be continuous unless stated otherwise. Let

$$f : (X, x_0) \to (Y, y_0)$$

be a continuous map with the property that $f(x_0) = y_0$. Note that $\phi : (X, x_0) \to (Y, y_0)$ induces a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where

$$\phi_*([f]) = [\phi f].$$

Proposition 2.2.25.

1. *For topological spaces X, Y and Z consider $\psi : X \to Y$ and $\phi : Y \to Z$. Then $(\phi\psi)_* = \phi_* \psi_*$.*
2. *Let X be a topological space, then $\mathrm{id}_X : X \to X$ induces the identity map on the fundamental group $\pi_1(X, x_0)$.*

Proposition 2.2.26. *Let $\phi : X \to Y$ be a homotopy equivalence. Then $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism for every $x_0 \in X$.*

Lemma 2.2.27. *For a topological space X and $x_0 \in X$ assume that*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha$$

such that

- *A_α is open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *$A_\alpha \cap A_\beta$ is path-connected for all $\alpha, \beta \in \Lambda$.*

Then if f is a loop in X at x_0 it follows that

$$[f] = [h_1] \ldots [h_m]$$

where the h_i are loops at x_0 and are contained in a single A_α.

Theorem 2.2.28 (Brouwer Fixed Point Theorem). *Let $h : D^2 \to D^2$ be a continuous map. Then there exists an $x \in D^2$ such that $h(x) = x$.*

Recall that D^n denotes the n-dimensional disk.

Theorem 2.2.29. *For $n \geq 2$ we have that $\pi_1(S^n) = 0$.*

2.3 Seifert-van Kampen

The Seifert-van Kampen theorem is a way to compute the fundamental groups of spaces that can be decomposed into simpler spaces.

2.3.1 Free Products

Example 2.3.1. *Consider the space X where circles A and B intersect at a single point x_0. We know that $\pi_1(A)$ and $\pi_1(B)$ isomorphic $\mathbb{Z}$.*

- *Let a^n denote n loops of A in one direction.*
- *Let a^{-n} denote n loops of A in the opposite direction.*
- *Let a^0 denote no loops of A.*

We adopt a similar notation for $\pi_1(B)$. Consequently, we can denote entries of $\pi_1(X)$ by $a^{n_1}b^{n_2}a^{n_3}$ say, which is the loop traverses A with n_1 loops, then traverses B with n_2 and then traverses A with n_3 times. We refer to such representations as words and, as expected, words form a group.

- *We can multiply words by concatenating them and then performing any simplifications at the point of joining.*
- *Inverses are formed by changing the sign of each exponent and reversing the ordering of the symbols.*
- *The identity is the empty word.*

We denote this group as $\mathbb{Z} * \mathbb{Z}$ and refer to it as the free product. The Seifert-van Kampen will tell us that for a space X its fundamental group is some free product involving the fundamental groups of the space's components.

Let S be an index set for a set of symbols, a_s. Let F_S denote the set of words constructed using the symbols a_s. That is, an element of F_S is of the form

$$x_1 \dots x_n$$

where each x_i is either a_s or a_s^{-1} and the word is reduced, that is all possible cancellations are made. The product of words is their concatenation with any subsequent cancellations made. With this, the set of words forms a group with the identity being the empty word. This group is referred to as the free group generated by S. The free product has a similar construction to the free group. Suppose that we have a collection of disjoint groups (G_α). Where disjoint only refers to the symbols of the group, rather than its structure. Consider

$$\underset{\alpha}{*} G_\alpha := \{g_1 \dots g_m : g_i \in G_{\alpha_i} \text{ not the identity, and } \alpha_i \neq \alpha_{i+1}\}.$$

On G_{α_i} let multiplication be

$$(g_1 \dots g_m)(h_1 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$$

with any simplifications made to ensure it is a reduced word. Let the inverse of $g_1 \dots g_m \in \underset{\alpha}{*} G_\alpha$ be

$$g_m^{-1} \dots g_1^{-1}$$

and let the identity be the empty word.

Exercise 2.3.2. *Show that multiplication on* $\underset{\alpha}{*} G_\alpha$ *is associative, so that* $\underset{\alpha}{*} G_\alpha$ *forms a group. We refer to* $\underset{\alpha}{*} G_\alpha$ *as the free product of the collection* (G_α).

Remark 2.3.3.

- *Each group in the collection* (G_α) *can be identified with a subgroup of the free product. With only the empty word being common to each of these subgroups which are otherwise disjoint.*
- *A collection of homomorphisms* $\phi_\alpha : G_\alpha \to H$ *extends uniquely to a homomorphism* $\phi : \underset{\alpha}{*} G_\alpha \to H$ *by*

$$\phi(g_1 \dots g_m) = \phi_{\alpha_1}(g_1) \dots \phi_{\alpha_m}(g_m).$$

Definition 2.3.4. *Any group* G *can be written as the quotient of some free group,* $G = F_S / \langle\langle R \rangle\rangle$, *where* $\langle\langle R \rangle\rangle$ *is the smallest normal subgroup of* F_S *containing* R. *This is called the presentation of* G *and is denoted by* $G = \langle S | R \rangle$.

Proposition 2.3.5. *Let* G *and* H *be groups with presentations* $\langle S_G | R_G \rangle$ *and* $\langle S_H | R_H \rangle$ *respectively. Then,*

$$G_H = \langle S_G \cup S_H | R_G \cup R_H \rangle.$$

A generalization of the free product is given by the amalgamated product. In particular, the amalgamated product facilitates dealing with a collection of groups that have overlapping symbols. More specifically, for groups G_0, G_1, G_2 let $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be homomorphisms. Then H is an amalgamated product of G_1 and G_2 over G_0 if it is a group with homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ such that the following statements hold.

- $h_1 f_1 = h_2 f_2$.
- They satisfy the universal property of Figure 1.

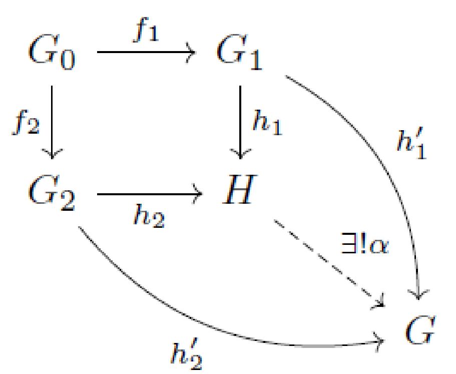

Figure 1: Amalgamated Product Universal Property

An Amalgamated product H is equivalent to $(G_1 * G_2)/N$ where N is the smallest subgroup of G_0 containing the elements g for which $(h_1 f_1)(h_2 f_2)^{-1}(g)$ is not the identity.

Theorem 2.3.6. *Given* $f_1 : G_0 \to G_1$ *and* $f_2 : G_0 \to G_2$, *there exists a unique (up to isomorphisms)* *amalgamated product which we denote by*

$$G_1 \underset{G_0}{*} G_2.$$

2.3.2 The Seifert-van Kampen Theorem

Theorem 2.3.7 (Seifert-van Kampen)**.** *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected. With $x_0 \in U_1 \cap U_2$ we have that*

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0) \cong (\pi_1(U_1, x_0) * \pi_1(U_2, x_0))/N,$$

where N is the normal closure of

$$\left\{(j_1)_*(\omega)\left((j_2)_*(\omega)\right)^{-1} : \omega \in \pi_1(U_1 \cap U_2, x_0)\right\}$$

with $j_i : U_1 \cap U_2 \hookrightarrow U_i$ satisfying Figure 2.

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & U_1 \arrow[d, hook, "j_1"] \\ U_2 \arrow[r, hook, "j_2"'] & X \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \pi_1(U_1 \cap U_2, x_0) \arrow[rr, "i_{1*}"] \arrow[d, "i_{2*}"'] & & \pi_1(U_1, x_0) \arrow[d, "j_{1*}"] \\ \pi_1(U_2, x_0) \arrow[rr, "j_{2*}"'] & & \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0) \end{tikzcd}$$

Figure 2: Seifert-van Kampen Diagram

Theorem 2.3.8 (Seifert-van Kampen, Strong version)**.** *Let X be a path-connected topological space such that*

- $X = \bigcup_\alpha A_\alpha$,
- $A_\alpha, A_\alpha \cap A_\beta, A_\alpha \cap A_\beta \cap A_\gamma$ *are open and path-connected for all α, β, γ, and*
- $x_0 \in \bigcap_\alpha A_\alpha$.

Then

$$\pi_1(X, x_0) \cong \underset{\alpha}{*} \pi_1(A_\alpha, x_0)/N$$

where $N \subseteq \underset{\alpha}{*}\pi_1(A_\alpha, x_0)$ is the normal closure of

$$\left\{(i_{\alpha\beta})_*(\omega)\left((i_{\beta\alpha})_*(\omega)\right)^{-1} : \omega \in \pi_1(A_\alpha \cap A_\beta)\right\}$$

with $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.

2.3.3 Application to CW-complexes

Consider a path-connected topological space X. Let Y be the space obtained by attaching the 2-cells $\left(e^2_\alpha\right)$ to X along the maps $\phi_\alpha : S^1 \to X$. Let ϕ'_α be the loops

$$\phi'_\alpha(s) = \phi_\alpha(\cos 2\pi s, \sin 2\pi s).$$

Note that ϕ'_α has base point $\phi'_\alpha(0)$. We would like to consider them as loops from a particular base point, x_0, so that we can work with fundamental groups. So for each α let γ_α be a path from x_0 to $\phi'_\alpha(0)$. Let $N \subseteq \pi(X, x_0)$ be the normal closure of elements of the form $\left[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}\right]$. Note that N is agnostic to the choice of path γ_α. As suppose instead that we choose η_α, then

$$\eta_\alpha \phi_\alpha (\eta_\alpha)^{-1} = \left(\eta_\alpha (\gamma_\alpha)^{-1}\right) \gamma_\alpha \phi_\alpha (\gamma_\alpha)^{-1} \left(\gamma_\alpha (\eta_\alpha)^{-1}\right).$$

Therefore, $\eta_\alpha\phi_\alpha(\eta_\alpha)^{-1}$ and $\gamma_\alpha\phi_\alpha(\gamma_\alpha)^{-1}$ are conjugate in $\pi_1(X, x_0)$.

Proposition 2.3.9. *The inclusion $i : X \hookrightarrow Y$ induces the surjection $i_* : \pi(X, x_0) \to \pi_1(Y, x_0)$ with $\ker(i_*) = N$ so that*

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N.$$

Corollary 2.3.10. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

2.4 Covering Spaces

2.4.1 Lifting Properties

Further utilising covering space we can translate algebraic facts about the fundamental group into geometrical language. Throughout this section, we will let $f : Y \to X$ be a continuous map, with lift $\tilde{f} : Y \to \widetilde{X}$. Meaning $\tilde{f}$ has the property that $p\tilde{f} = f$ where $p : \widetilde{X} \to X$ is a covering space.

Example 2.4.1. *Let $p : S^1 \to S^1$ be given by*

$$p(z) = z^n$$

where z is a complex number with $|z| = 1$ and n is a positive integer. One can think of this embedding S^1 into the boundary of a torus with n winds that are not self-intersection.

Suppose that Y is connected.

- The unique lifting property says that if two lifts $\tilde{f}_1$ and $\tilde{f}_2$ coincide at a point, then they coincide on all of Y.
- The homotopy lifting property says that if $f_t : Y \to X$ is a homotopy and $\tilde{f}_0$ is a lift of f_0, then there exists a unique homotopy $\tilde{f}_t : Y \to \widetilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Proposition 2.4.2. *Fix $x_0 \in X$ and $\widetilde{x_0} \in \widetilde{X}$ so that $p\left(\widetilde{x_0}\right) = x_0$, then the following statements hold.*

1. *$p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$ is injective.*
2. *$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$ consists of homotopy classes of loops starting at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x_0}$ are loops.*

Proposition 2.4.3. *Let X and $\widetilde{X}$ be path-connect and let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a covering space. Then the number of sheets of p equals the index of $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ in $\pi_1(X, x_0)$.*

Definition 2.4.4. *A topological space X has a property P locally if for each $x \in X$ and neighbourhood U of x there is an open neighbourhood $V \subseteq U$ that has property P.*

Proposition 2.4.5. *Let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a covering space and $f : (Y, y_0) \to (X, x_0)$ a continuous map, where Y is path-connected and locally path-connected. Then there is a lift $\tilde{f} : (Y, y_0) \to \left(\widetilde{X}, \widetilde{x_0}\right)$ if*

and only if
$$f_*\left(\pi_1(Y, y_0)\right) \subseteq p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right).$$

2.4.2 Classification of Covering Spaces

For a fixed space X there exists a classification of its different covering spaces.

Definition 2.4.6. *A covering space $p : \widetilde{X} \to X$ is a universal cover if $\widetilde{X}$ is simply-connected.*

Definition 2.4.7. *A topological space X is semi-locally simply connected if each $x \in X$ has a neighborhood U such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial for $i : U \hookrightarrow X$ being the inclusion map.*

Proposition 2.4.8. *If $p : \widetilde{X} \to X$ is a universal cover, then X is semi-locally simply connected.*

Theorem 2.4.9. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then there exists a universal cover $p : \widetilde{X} \to X$.*

We can construct this universal cover by noting the following. If $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ is a universal cover. Then, we have an equivalence between

1. points in $\widetilde{X}$,
2. $[\gamma]$ where γ is a path in $\widetilde{X}$ starting at $\widetilde{x_0}$, and
3. $[\gamma]$ where γ is a path in X starting at x_0.

Therefore, we can define $\widetilde{X}$ to be the space of homotopy classes of paths starting at x_0. We can also show that the collection $\mathcal{U}$ containing the path-connected open sets $U \subset X$ such that i_* is trivial forms a basis for the topological space X. Subsequently, if for a given set $U \in \mathcal{U}$ and path γ in X from x_0 to a point in U we let
$$U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ a path in } U \text{ with } \eta(0) = \gamma(1)\}$$
we form a basis of $\widetilde{X}$. One can then show that the map $p : \widetilde{X} \to X$ with $U_{[\gamma]} \mapsto U$ gives our desired covering space.

Proposition 2.4.10. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x_0})) = H$ for some base point x_0 and where $p : \widetilde{X} \to X$ is a covering space.*

Definition 2.4.11. *Covering spaces*
$$p_1 : \widetilde{X}_1 \to X$$
and
$$p_2 : \widetilde{X}_2 \to X$$
are isomorphic if there exists a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 f = p_1$.

Proposition 2.4.12. *Let X be a path-connected, locally path-connected and let $x_0 \in X$. Path-connected covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are isomorphic through $f : \widetilde{X}_1 \to \widetilde{X}_2$ that maps $\widetilde{x_1} \in p^{-1}(x_0)$*

to $\widetilde{x_2} \in p_2^{-1}(x_0)$ if and only if

$$(p_1)_*\left(\pi_1\left(\widetilde{X}_1, \widetilde{x_1}\right)\right) = (p_2)_*\left(\pi_1\left(\widetilde{X}_2, \widetilde{x_2}\right)\right).$$

Such an isomorphism is called a base point isomorphism.

Theorem 2.4.13 (Galois Correspondence)**.** *Let X be a path-connected, locally path-connected, and semi-locally simply-connected, and let $x_0 \in X$. Then the following statements hold.*

1. *Path-connected covering spaces up to base point preserving isomorphisms $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ are in bijection with subgroups $H \subseteq \pi_1(X, x_0)$.*
2. *Path connected covering spaces $p : \widetilde{X} \to X$ up to isomorphisms are in bijection with conjugancy classes $H \subseteq \pi_1(X, x_0)$.*

2.4.3 Deck Transformations

Definition 2.4.14. *Let $p : \widetilde{X} \to X$ be a covering space. A deck transformation is an automorphism on $\widetilde{X}$. The group of deck transformations is denoted by $G\left(\widetilde{X}\right)$.*

Example 2.4.15. *Recall Example 2.4.1 In this case, $G\left(\widetilde{X}\right)$ is $\mathbb{Z}_n$ corresponding to the rotations of $\frac{2\pi}{n}$.*

A consequence of the unique lifting property is that if $\widetilde{X}$ is path-connected then $f \in G\left(\widetilde{X}\right)$ is determined by where it sends a single point. Consequently, the identity is the only deck transformation with a fixed point.

Definition 2.4.16. *A covering space $p : \widetilde{X} \to X$ is normal if for each $x \in X$ and every pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is an $f \in G\left(\widetilde{X}\right)$ such that $f(\tilde{x}) = \tilde{x}'$.*

Remark 2.4.17. *Example 2.4.1 is normal.*

Proposition 2.4.18. *Let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a path-connected covering space, and X be path-connected and locally path-connected. Then $p : \widetilde{X} \to X$ is normal if and only if $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ is a normal subgroup.*

Proposition 2.4.19. *Let $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ be a covering space, let X be path-connected and locally path-connected, and let $\widetilde{X}$ be path-connected. Let $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normalizer of H. Then $G\left(\widetilde{X}\right)$ is isomorphic to $N(H)/H$.*

- *If $\tilde{X}$ is normal, then*

$$G\left(\widetilde{X}\right) \cong \pi_1(X, x_0)/H.$$

- *If $\widetilde{X}$ is the universal cover, then*
$$G\left(\widetilde{X}\right) \cong \pi_1(X, x_0).$$

3 Homology

3.1 Δ-Complexes

3.1.1 Motivation

Just as we considered base point preserving homotopies of the form $\phi : I \to X$ with the fundamental group $\pi_1(X, x_0)$. We can also consider higher homotopies groups $\pi_n(X, x_0)$ which are groups of base point preserving homotopies of the form $\phi : I^n \to X$ where $\phi(\partial I^n) = x_0$. This will be useful to probe higher dimensional spaces, as currently with the fundamental group we cannot distinguish between S^n's for $n \geq 2$. More specifically, we see by Theorem 2.2.29 that the fundamental group is dependent only on the 2-skeleton of X. More generally, $\pi_n(X)$ depends only on the $(n+1)$-skeleton of X.

Theorem 3.1.1. *$\pi_i(S^n) = 0$ for $i < n$ and $\mathbb{Z}$ for $i = n$.*

3.1.2 Geometry in Higher-Dimensions

Definition 3.1.2.

- *An n-simplex in $\mathbb{R}^m$ is the convex hull of a set V of $n+1$ points in $\mathbb{R}^m$ that are not all contained in an affine $(n-1)$-dimensional subspace of $\mathbb{R}^m$. The standard n-simplex is given by*

$$\left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, x_0 + \cdots + x_n = 1\right\}.$$

- *An ordered n-simplex is an n-simplex where the vertices have some order defined on them. We denote this by $[v_0, \ldots, v_n]$. The standard ordered n-simplex is $[e_1, \ldots, e_{n+1}]$, which is the convex hull of the standard basis of $\mathbb{R}^{n+1}$ with the natural ordering imposed on the vertices. We denote this simplex by Δ^n.*
- *For the n-simplex $[v_0, \ldots, v_n]$ in $\mathbb{R}^m$ let $L = \mathrm{Sp}(v_0, \ldots, v_n)$. Then there is a unique affine morphism $L \to \mathbb{R}^{n+1}$ defined by $v_i \mapsto e_{i+1}$ for $i = 0, \ldots, n$. More specifically, we have a homeomorphism from $[v_0, \ldots, v_n]$ to Δ^n which preserves the ordering.*
- *The faces of $[v_0, \ldots, v_n]$ are defined to be $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ for $i = 0, \ldots, n$, where $\hat{v}_i$ means this vertex is omitted.*
- *The boundary of a simplex Δ is the union of all the faces.*
- *The interior of a simplex Δ is $\mathring{\Delta} = \Delta \setminus \partial\Delta$.*

Definition 3.1.3. *A Δ-complex structure on a topological space X is a collection of maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ for $\alpha \in A$ and $n(\alpha) \in \mathbb{N}$ with the following properties.*

1. *$\sigma_\alpha|_{\mathring{\Delta}^{n(\alpha)}}$ is injective for all $\alpha \in A$, and for $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha\left(\mathring{\Delta}^{n(\alpha)}\right)$.*
2. *The restriction of σ_α to faces is equal to σ_β for some $\beta \in A$ and $n(\beta) = n(\alpha) - 1$.*
3. *$U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{n(\alpha)}$ for all $\alpha \in A$.*

3.2 Homologies

3.2.1 Simplicial Homology

In practice, homotopy groups are difficult to compute. An alternative, more manageable, group is the homology group. For X a Δ-complex let the free abelian group of the $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ for $n(\alpha) = n$ be referred to as

n-chains, denoted $\Delta_n(X)$. Hence, we denote elements of $\Delta_n(X)$ as

$$\sum_{\alpha \in A, n(\alpha)=n} c_\alpha \sigma_\alpha$$

for $c_\alpha \in \mathbb{Z}$ with finitely many of the c_α non-zero.

Definition 3.2.1. *The boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is given by*

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}.$$

In particular, we let $\partial_0 = 0$.

Note that $\partial_{n-1} \circ \partial_n = 0$. This is because the composed map gives us the n-simplices restricted to the $(n-2)$-ordered faces by removing two vertices from the n-simplices. Removing these vertices in two different orders results in each restriction appearing twice in the sum. Due to the changing signs, these cancel each other and thus we are left with the zero map.

3.2.2 The Algebraic Situation

A chain complex of abelian groups $(C_\bullet, \partial)$ is the chain

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0, \tag{3.2.1}$$

where each C_i is an abelian group and the ∂_n are homomorphisms such that $\partial_n \circ \partial_{n+1} = 0$ for all n. Note that

$$Z_n := \ker(\partial_n) \subseteq C_n$$

and

$$B_n := \mathrm{im}(\partial_{n+1}) \subseteq C_n.$$

Elements of Z_n are referred to as cycles and elements of B_n are referred to as boundaries. For $b \in B_n$ we have that $b = \partial_{n+1}(a)$ for some $a \in C_{n+1}$, so that $\partial_n(b) = (\partial_n \circ \partial_{n+1})(a) = 0$ and so $B_n \subseteq Z_n$. We note that requiring ∂_n such that $B_n \subseteq Z_n$ is equivalent to requiring that $\partial_n \circ \partial_{n+1} = 0$.

Definition 3.2.2. *The n^{th} homology group of a chain complex, as given by* (3.2.1), *is*

$$H_n(C_\bullet, \partial) = Z_n/B_n.$$

Definition 3.2.3. *The n^{th} simplicial homology group of a chain complex, as given by* (3.2.1), *is*

$$H_n^\Delta(X) = H_n(\Delta_\bullet(X), \partial) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

Elements of H_n are cosets referred to as homology classes.

Example 3.2.4. *Consider $X = T$ the torus with Δ-complex as illustrated in Figure 3. Namely, T has a vertex, three edges (a, b, c), and two 2-simplices U and L.*

- $\partial_1 = v - v = 0$*, so that* $H_0^\Delta(T) = \mathbb{Z}$
- $\partial_2(U) = \partial_2(L) = a + b - c$*. As* $\Delta_1(T)$ *has basis* $\{a, b, a + b - c\}$ *it follows that* $H_1^\Delta(T) = \mathbb{Z} \oplus \mathbb{Z}$ *to correspond to the homology classes* $[a]$ *and* $[b]$*.*
- $\partial_2(pU + qL) = 0$ *only if* $p = -q$*. So* $\ker(\partial_2)$ *is generated by* $U - L$ *and hence* $H_2^\Delta(T) = \mathbb{Z}$*.*

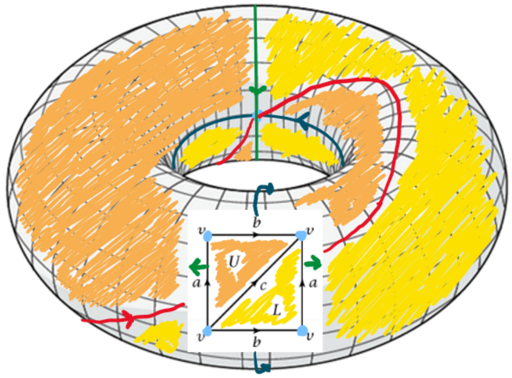

Figure 3: Torus

3.2.3 Singular Homology

Definition 3.2.5. *A singular n-simplex in a topological space X is a continuous map $\sigma : \Delta^n \to X$.*

The σ does not have to be a nice map, in particular, it can have singularities. Thus, a singular n-simplex is more general than a Δ-complex as the image of σ does not necessarily have to be a simplex. The free abelian group of the singular n-simplices in X is denoted $C_n(X)$. Elements of $C_n(X)$ are finite singular n-chains of the form

$$\sum_i n_i \sigma_i$$

for $n_i \in \mathbb{Z}$ and σ_i a singular n-simplex. As the space of singular n-simplicies is larger, when working with $C_n(X)$ in practice we are less likely to encounter finitely generated groups as we did in the case of simplicial homology.

Definition 3.2.6. *The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_1,\ldots,\hat{v}_i,\ldots,v_n]},$$

where σ is a n-simplex.

Observe that $\partial_n \circ \partial_{n+1} = 0$, and so

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \tag{3.2.2}$$

is a chain complex.

Definition 3.2.7. *The n^{th} singular homology group of the chain complex* (3.2.2) *is*

$$H_n(X) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

Remark 3.2.8. *If X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$.*

A simplicial homology can be constructed from a singular homology. Let X be an arbitrary space, and define the simplicial complex $S(X)$ as the following Δ-complex.

1. One n-simplex Δ^n_σ for each n-simplex $\sigma : \Delta^n \to X$.

2. Attach Δ^n_σ to the restrictions of σ to the $(n-1)$-simplices of $\partial\Delta^n$.

From this construction we have that $H_n^{\Delta}(S(X)) = H_n(X)$ for all n.

Proposition 3.2.9. *For a topological space $X = \bigcup_\alpha X_\alpha$, where the X_α are path-connected components, we have that*

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

Proposition 3.2.10. *If X is a non-empty, path-connected topological space, then*

$$H_0(X) \cong \mathbb{Z}.$$

From Proposition 3.2.9 and Proposition 3.2.10 we deduce that for an arbitrary space X the o^{th} singular homology group, $H_0(X)$, is a direct sum of $\mathbb{Z}$.

Proposition 3.2.11. *If X is a point, then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}$$

3.2.4 Reduced Homology Group

Definition 3.2.12. *The reduced homology group $\widetilde{H_n}(X)$ is the homology group of the augmented chain*

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0,$$

where $\epsilon : C_0(X) \to \mathbb{Z}$ is given by

$$\epsilon\left(\sum_i n_i \sigma_i\right) = \sum_i n_i.$$

Therefore, $H_n(X) \cong \widetilde{H_n}(X)$ for $n \geq 1$, but,

$$H_0(X) \cong \widetilde{H_0}(X) \oplus \mathbb{Z}.$$

This is useful as now we have that the point topological is trivial for all dimensions. The singular homology groups were only trivial for the point topological space in dimensions greater than zero.

3.2.5 Homotopy Invariance

Definition 3.2.13. *For chain complexes $(A_\bullet, \partial)$ and $(B_\bullet, \partial)$ a chain map $f : (A_\bullet, \partial) \to (B_\bullet, \partial)$ is collection of homomorphisms $f_n : A_n \to B_n$ such that $\partial \circ f_n = f_{n+1} \circ \partial$.*

A chain map gives a mechanism to transfer between chain complexes. For topological spaces X and Y let $f : X \to Y$ be a map and let $f_\# : C_n(X) \to C_n(Y)$ by

$$f_\#(\sigma) = f \circ \sigma$$

for $\sigma : \Delta^n \to X$ and extend it linearly to elements of $C_n(X)$. Observe that $f_\# \circ \partial = \partial \circ f_\#$ meaning $f_\#$ is a chain map between $(C_\bullet(X), \partial)$ and $(C_\bullet(Y), \partial)$, where cycles are mapped to cycles and boundaries to boundaries. Therefore, $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_* = g_* \circ f_*$.
- $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

Theorem 3.2.14. *If $f, g : X \to Y$ are homotopic then $f_* = g_*$.*

Corollary 3.2.15. *If $f : X \to Y$ is a homotopy equivalence, then f_* is an isomorphism.*

Similar results hold for reduced homology groups, as given in Definition 3.2.12. That is the induced homomorphism between reduced homology groups is invariant under homotopy. In this case, the induced homomorphism is additionally required to be the identity on the added $\mathbb{Z}$ groups. To note the invariance we proceed as before but with the added observation that $f_\# \epsilon = \epsilon f_\#$.

3.3 Exact Sequences and Excision

3.3.1 Exact Sequences

Relationships between $H_n(X), H_n(A)$ and $H_n(X/A)$ for $A \subseteq X$ are useful to establish as often CW-complexes are built inductively from subspaces.

Definition 3.3.1. *A sequence of group homomorphisms of abelian groups,*

$$\dots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} \dots,$$

is exact at A_n if $\ker(\alpha_n) = \operatorname{im}(\alpha_{n+1})$. A sequence

$$\dots \xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots,$$

is exact if it is exact at A_n for all n.

Remark 3.3.2. *For an exact sequence, the following statements hold.*

1. *$\alpha_n \circ \alpha_{n+1} = 0$ meaning it is a chain complex.*
2. *Its homology groups are trivial.*

Definition 3.3.3. *Let $A \subseteq X$, for X a topological space. Then A is a strong deformation retract to X if there is a deformation retraction $r : X \to A$, and a map $F : I \times X \to X$ such that for $x \in X, a \in A$ and $t \in I$ we have*

1. *$F(0, x) = x$,*
2. *$F(1, x) = r(x)$,*
3. *$F(t, a) = a$.*

Definition 3.3.4. *Let $\emptyset \neq A \subseteq X$ be closed, for X a topological space. Then (X, A) is a good pair if A has a neighbourhood in X that is a strong deformation retract to A.*

Theorem 3.3.5. *Let (X, A) be a good pair, then there is an exact sequence*

$$\cdots \longrightarrow \widetilde{H_1}(A) \xrightarrow{i_*} \widetilde{H_1}(X) \xrightarrow{j_*} \widetilde{H_1}(X/A) \xrightarrow{\partial} \widetilde{H_0}(A) \xrightarrow{i_*} \widetilde{H_0}(X) \xrightarrow{j_*} \widetilde{H_0}(X/A) \longrightarrow 0,$$

where $i : A \hookrightarrow X$ is the inclusion map and $j : X \to X/A$ is the quotient map.

Corollary 3.3.6. *In the setting of Theorem 3.3.5 we have that*

$$\widetilde{H_i}\left(S^n\right) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Corollary 3.3.7. *There exists no retraction $r : D^n \to \partial D^n$.*

Theorem 3.3.8 (Brouwer Fixed Point Theorem)**.** *Every continuous map $f : D^n \to D^n$ has a fixed point.*

3.3.2 Relative Homology Group

To add flexibility to our present framework, we can introduce methods that ignore to certain data and structures. More specifically, for a topological space X and $A \subseteq X$ let

$$C_n(X, A) = C_n(X)/C_n(A),$$

such to ignore the chains in the subspace A. Let $\partial : C_n(X) \to C_{n-1}(X)$ be the boundary map so that

$$\partial(\sigma : \Delta^n \to A) \in \partial(C_n(A)) \subseteq C_{n-1}(A).$$

This induces the homomorphism $\partial : C_n(X, A) \to C_{n-1}(X, A)$ with $\partial \circ \partial = 0$ so that we have the chain complex

$$\cdots \longrightarrow C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \longrightarrow \dots$$

for which the following statements hold.

- The homology groups are the relative homology groups, $H_n(X, A)$.
- The relative n-chains are $C_n(X, A)$.
- The relative n-cycles are elements $[\alpha]$ of $\ker(\partial) \subseteq C_n(X, A)$ such that $\partial(\alpha) \in C_{n-1}(A)$.
- The relative n-boundaries are elements $[\alpha]$ of $\operatorname{im}(\partial) \subseteq C_n(X, A)$ such that $\alpha = \partial(\beta) + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Definition 3.3.9. *A short exact sequence of chain complexes is*

$$0 \to (A_\bullet, \delta) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \longrightarrow 0$$

where i and j are chain maps such that

$$0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \longrightarrow 0,$$

is a short exact sequence for every n.

Lemma 3.3.10 (Zig-Zag)**.** *Suppose*

$$0 \to (A_\bullet, \delta) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \longrightarrow 0$$

is a short exact sequence of chain complexes. Then we can construct a long exact sequence of homology *groups*

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to \dots$$

Theorem 3.3.11. *The sequence*

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to \dots$$

is exact.

Consider the short exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \longrightarrow 0,$$

for i the inclusion map and j the quotient map. Using Lemma 3.3.10 we obtain a long exact sequence of homology groups,

$$\dots \to H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X,A) \to 0,$$

where $\partial([\alpha]) = [\partial(\alpha)]$. Similarly, using the short exact sequence of the augmented chain complex we get

$$\dots \to \widetilde{H_n}(A) \to \widetilde{H_n}(X) \to \widetilde{H_n}(X,A) \to \widetilde{H_{n-1}}(A) \to \widetilde{H_{n-1}}(X) \to \widetilde{H_{n-1}}(X,A) \to \dots.$$

A map $f : X \to Y$ such that $f(A) \subseteq B$ induces the chain map $f_\# : C_n(X,A) \to C_n(Y,B)$, and the homomorphism $f_* : H_n(X,A) \to H_n(Y,B)$. Such induced maps have the property that $(f \circ g)_* = f_* \circ g_*$.

Definition 3.3.12. *A homotopy between maps $f,g : (X,A) \to (Y,B)$ is a map $F : I \times X \to Y$ such that*

1. $F(0,x) = f(x)$,
2. $F(1,x) = g(x)$, *and*
3. $F(s,a) \in B$,

for all $x \in X$, $S \in I$ and $a \in A$.

Proposition 3.3.13. *If $f,g : (X,A) \to (Y,B)$ are homotopic then $f_* = g_*$.*

Now consider the short exact sequence

$$(A,B) \to (X,B) \to (X,A)$$

for the triple (X,A,B) where X is a topological space and $B \subset A \subset X$. This induces the short exact sequence of chain complexes

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

which in turn yields the long exact sequence

$$\dots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to H_{n-1}(X,B) \to H_{n-1}(X,A) \to \dots$$

3.3.3 Excision

Theorem 3.3.14. *Let X be a topological space with $Z \subset A \subset X$ subspaces such that the closure of Z is contained in the interior of A, $\bar{Z} \subset \dot{A}$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X,A)$ induces the isomorphism*

$$H_n(X \setminus Z, A \setminus Z) \cong H_n(X,A)$$

for all n. Similarly, if $A, B \subseteq X$ are such that $\dot{A} \cup \dot{B} = X$, then the inclusion induces the isomorphism

$$H_n(B, A \cap B) \cong H_n(X, A)$$

for all n.

Theorem 3.3.14 gives conditions for which the impact of relative groups $H_n(X \setminus A)$ may be ignored.

Corollary 3.3.15. *Let $(X_\alpha)_{\alpha \in A}$ be a collection of topological spaces and $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair for all $\alpha \in A$. Let $\bigvee_\alpha X_\alpha$ denote the wedge sum with respect to the x_α. Then there is an isomorphism*

$$\widetilde{H_n}\left(\bigsqcup_\alpha X - \alpha\right) = \bigoplus_\alpha \widetilde{H_n}(X_\alpha) \cong \widetilde{H_n}\left(\bigvee_\alpha X_\alpha\right).$$

Theorem 3.3.16. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, non-empty. Then, if U and V are homeomorphic then $m = n$.*

3.3.4 Naturality

Theorem 3.3.17. *Let $(A_\bullet, \partial)$, $(B_\bullet, \partial)$, $(C_\bullet, \partial)$, $(A'_\bullet, \partial)$, $(B'_\bullet, \partial)$, and $(C'_\bullet, \partial)$ be chain complexes that satisfy the commutative diagram depicted in Figure 4, where each row is a short exact sequence. Then the induced diagram depicted in Figure 5 is commutative.*

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & B_\bullet \arrow[r, "j"] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "i'"] & B'_\bullet \arrow[r, "j'"] & C'_\bullet \arrow[r] & 0
\end{tikzcd},$$

Figure 4: Commutative diagram for the short exact sequences of Theorem 3.3.17

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & \mathrm{H}_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_{n-1}(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_{n-1}(C) \arrow[r] \arrow[d, "\gamma_*"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(A') \arrow[r, "i'_*"] & \mathrm{H}_n(B') \arrow[r, "j'_*"] & \mathrm{H}_n(C') \arrow[r, "\partial"] & \mathrm{H}_{n-1}(A') \arrow[r, "i'_*"] & \mathrm{H}_{n-1}(B') \arrow[r, "j'_*"] & \mathrm{H}_{n-1}(C') \arrow[r] & \dots
\end{tikzcd}$$

Figure 5: Commutative diagram for the induced long exact sequences of Theorem 3.3.17

3.4 Mayer-Vietoris Sequences

3.4.1 The Sequence

Theorem 3.4.1. *Let X be a topological space with $A, B \subseteq X$ such that $\dot{A} \cup \dot{B} = X$. Let*

- $i_1 : A \cap B \hookrightarrow A$,
- $i_2 : A \cap B \hookrightarrow B$,
- $j_1 : A \hookrightarrow X$, *and*

- $j_2 : B \hookrightarrow X$

be inclusion maps. Then we have the exact sequence

$$\cdots \to H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \xrightarrow{\Psi} H_1(X) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} H_0(X) \to 0,$$

where

1. $\Phi(x) = ((i_1)_*(x), -(i_2)_*(x))$, *and*
2. $\Psi(x, y) = (j_1)_*(x) + (j_2)_*(y)$.

Remark 3.4.2.

1. *If $A \cap B \neq \emptyset$ then we have an analogous sequence for the augmented chain complex.*
2. *Theorem 3.4.1 utility lies in inductive arguments. If one has results for A, B and $A \cap B$ one can argue by induction using Theorem 3.4.1 that the result is holds for $A \cup B$.*

3.4.2 The Applications

Definition 3.4.3. *A continuous map $\phi : X \to Y$ between topological spaces is an embedding if it is homeomorphic to its image.*

Proposition 3.4.4. *For $h : D^k \to S^n$ an embedding. Then*

$$\widetilde{H}_i\left(S^n \setminus h\left(D^k\right)\right) = 0$$

for all i.

Proposition 3.4.5. *For $h : S^k \to S^n$ an embedding for $k < n$ it follows that*

$$\widetilde{H}_i\left(S^n \setminus h\left(S^k\right)\right) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.4.6. *Let $h : S^1 \to S^2$ be an embedding. Then $S^2 \setminus h\left(S^1\right)$ consists of exactly two path-connected components.*

Theorem 3.4.7 (Jordan Curve)**.** *Let $h : S^{n-1} \to \mathbb{R}^n$ be an embedding. Then $\mathbb{R}^n \setminus h\left(S^{n-1}\right)$ consists of exactly two path-connected components.*

3.5 Degree

For the continuous map $f : S^n \to S^n$ let $f_* : H_n\left(S^n\right) \to H_n\left(S^n\right)$ be the induced homomorphism. As f_* is a homomorphism from the infinite cycle group to itself, it follows that

$$f_*(\alpha) = d\alpha$$

for some $d \in \mathbb{Z}$. The integer d is known as the degree of f and is denoted $\deg(f)$.

Proposition 3.5.1.

1. $\deg(\mathrm{id}_{S_n}) = 1$.
2. $\deg(f) = 0$ *if* f *is not surjective.*
3. *If* $f \cong g$ *then* $\deg(f) = \deg(g)$.
4. $\deg(fg) = \deg(f)\deg(g)$. *In particular, if* f *is a homotopy equivalence then* $\deg(f) = \pm 1$.
5. *For* $R_i(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$ *the reflection map, we have that* $\deg(R_i) = -1$.
6. *For* $-\mathrm{id}_{S^n}(x) = -x$ *we have that* $\deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$.
7. *If* $f : S^n \to S^n$ *has no fixed points then* $\deg(f) = (-1)^{n+1}$.

Proposition 3.5.2. *If* n *is even, then* $\mathbb{Z}/2\mathbb{Z}$ *is the only non-trivial group that can act freely by homeomorphisms on* S^n.

Definition 3.5.3. *A vector field on* S^n *is a continuous map* $v : S^n \to \mathbb{R}^{n+1}$ *such that for each* $x \in S^n$, $v(x)$ *is tangent to* S^n *at* x.

Theorem 3.5.4. S^n *admits a continuous vector field* $v : S^n \to \mathbb{R}^{n+1}$ *that is nowhere zero if and only if* n *is odd.*