Geometry of Curves and Surfaces

Thomas Walker

Spring 2024

Contents

1 Curves in Three-Dimensional Space

1.1 Regular Curves

Recall, the standard representation of Euclidean space, given by

$$\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, \text{ for } i = 1, \ldots, n\}$$

for $n \geq 1$, with

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

On this space we may introduce a notion of length, namely

$$|x| = \sqrt{\langle x, x \rangle}$$

is the length of $x \in \mathbb{R}^n$.

Definition 1.1.1. *A parameterised curve in $\mathbb{R}^n$ is a smooth map $\phi : [a, b] \to \mathbb{R}^n$. If additionally, $|\phi'(t)| \neq 0$ for $t \in [a, b]$, the curve is said to be regular.*

Example 1.1.2.

1. *The circle of radius $r > 0$ centred at the origin is a parameterised curve with $\phi : [0, 2\pi] \to \mathbb{R}^2$ given by*

$$\phi(t) = (r \cos(t), r \sin(t)).$$

Moreover, it is regular as

$$|\phi'(t)| = \sqrt{r^2 \sin^2(t) + r^2 \cos^2(t)} = r > 0.$$

2. *The parameterised curve represented $\phi : [-1, 1] \to \mathbb{R}^2$ where*

$$\phi(t) = \left(0, t^2\right),$$

is not smooth as $|\phi'(0)| = |(0, 0)| = 0$.

Dealing with regular curves ensures that the tangent of curves is well-defined.

Definition 1.1.3. *Let ϕ represent a regular curve. Then the tangent line to ϕ at $\phi(t_0)$ is given by*

$$L = \{\phi(t_0) = \phi'(t_0)s : s \in \mathbb{R}\}.$$

Example 1.1.4. *We note that being smooth is not sufficient to ensure that a tangent line is well-defined. Consider $\alpha : (-1, 1) \to \mathbb{R}^3$ given by*

$$\alpha(t) = \begin{cases} \left(\exp\left(-\frac{1}{t^2}, \exp\left(-\frac{1}{t^2}\right), 0\right)\right) & t > 0 \\ (0, 0, 0) & t = 0 \\ \left(-\exp\left(-\frac{1}{t^2}, \exp\left(-\frac{1}{t^2}\right), 0\right)\right) & t < 0. \end{cases}$$

Then α is smooth, however, the tangent line to α at $\alpha(0)$ is not well-defined at $\alpha'(0) = (0, 0, 0)$.

Remark 1.1.5. *When referring to a regular curve, we are referring to the image of the parameterization, that is $\phi([a, b])$. Consequently, different parameterisations, $\phi_1 : [a, b] \to \mathbb{R}^n$ and $\phi_2 : [c, d] \to \mathbb{R}^m$ correspond to the* *same regular curve if $\phi([a,b]) = \phi([c,d])$.*

Definition 1.1.6. *Let $\phi : [a,b] \to \mathbb{R}^n$ be a regular curve. Suppose that $f : [c,d] \to [a,b]$ is such that $|f'(x)| \neq 0$ for all $x \in [c,d]$ and $f(\{c,d\}) = \{a,b\}$. Then*

$$\phi \circ f : [c,d] \to \mathbb{R}^n$$

is a reparameterisation of ϕ. Moreover, $\phi \circ f$

Remark 1.1.7. *It is clear that a reparameterisation as given in Definition 1.1.6 is a regular curve.*

Definition 1.1.8. *Let $\phi : [a,b] \to \mathbb{R}^n$ be a regular curve. Then, its length is given by*

$$\ell(\phi) = \int_a^b |\phi'(t)| \, \mathrm{d}t.$$

Remark 1.1.9. *The motivation for Definition 1.1.8 can be seen by considering approximating the length of a two-dimensional line. For a two-dimensional line, one can choose points along the line and connect them with straight-line segments to obtain an approximation of the line's length. For t_i and t_{i+1} sufficiently closed to, the distance between the points $\phi(t_i)$ and $\phi(t_{i+1})$ on the line can be approximated by $|\phi(t_{i+1}) - \phi(t_i)|$, which can be further approximated by $|\phi'(t_i)||t_{i+1} - t_i|$. Consequently,*

$$\ell(\phi) \approx \sum_{i=0}^{n-1} |\phi'(t_i)||t_{i+1} - t_i|.$$

As $n \to \infty$ the approximation improves and deduce that

$$\ell(\phi) = \int_a^b |\phi'(t)| \, \mathrm{d}t.$$

Lemma 1.1.10. *The length of a regular curve is invariant under reparameterisations.*

Note there are numerous ways to parameterise a regular curve. Ideally, we would have a notion that would identify a parameterisation as the canonical parameterisation. The arc-length parameterisation of a regular curve requires that $|\phi'(t)| = 1$ for all t. Consequently,

$$\ell(\phi([a,b])) = b - a.$$

Lemma 1.1.11. *Any regular curve $\phi : [a,b] \to \mathbb{R}^n$ has an arc-length parameterisation.*

Remark 1.1.12. *The arc-length parameterisation of a regular curve is not unique. In particular, one can reverse the direction of an arc-length parameterisation and obtain a different arc-length parameterisation.*

Having investigated the length of a regular curve, we can now investigate another geometrical property of a regular curve, namely its curvature.

Definition 1.1.13. *Let $\phi : [0,L] \to \mathbb{R}^n$ be a regular curve parameterised by arc length. Then the curvature of*

ϕ at $\phi(t)$ is

$$k(t) = |\phi''(t)|.$$

The curvature vector at $\phi(t)$ is

$$\mathbf{k}(t) = \phi''(t).$$

Remark 1.1.14. *The curvature and curvature vector of a regular curve are independent of the parameterisation of ϕ by arc length.*

Henceforth, we assume that all regular curves are parameterised by arc length.

Proposition 1.1.15. *Let $\phi : [a, b] \to \mathbb{R}^n$ be a regular curve. Then $k(t) = 0$ for all $t \in [a, b]$ if and only if $\phi([a, b]) = 0$ is a straight line.*

Remark 1.1.16. *The curve must be parameterised by arc length in Proposition 1.1.15. Consider $\phi : [0, 1] \to \mathbb{R}^2$ given by $\phi(t) = (0, e^t)$. Then $\phi([0, 1])$ is a straight line, however,*

$$\phi''(t) = \big(0, e^t\big) \neq (0, 0).$$

Indeed,

$$|\phi'(t)| = e^t,$$

which is not equal to one for all t.

Proposition 1.1.17. *For any regular curve $\phi : [a, b] \to \mathbb{R}^n$, for every $t \in [a, b]$, the curvature vector $\mathbf{k}(t)$ is perpendicular to the tangent line to the curve $\phi([a, b])$ at $\phi(t)$.*

1.2 Frenet Frames

Definition 1.2.1. *Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parameterised by arc length.*

1. *The unit tangent vector to the curve at $\phi(t)$ is $T(t) = \phi'(t)$.*
2. *Provided $T'(t) \neq 0$ for $t \in [a, b]$, the principal normal vector to the curve $\phi(t)$ is*

$$N(t) = \frac{T'(t)}{|T'(t)|}.$$

3. *The binomial vector at $\phi(t)$ is $B(t) = T(t) \times N(t)$.*

Remark 1.2.2. *Note that by Proposition 1.1.17 $T'(t) = \mathbf{k}(t)$ is orthogonal to $T(t)$ and so $N(t)$ is orthonormal to $T(t)$.*

Definition 1.2.3. *The Frenet frame for a ϕ regular curve parameterised by arc length is (T, N, B), which is a positively oriented orthonormal basis of $\mathbb{R}^3$ at $\phi(t)$.*

Definition 1.2.4. *Let (T, N, B) be a Frenet frame. Then the smooth function $\tau : [a, b] \to \mathbb{R}$ such that*

$$B'(t) = -\tau(t)N(t)$$

is called the torsion of ϕ.

Remark 1.2.5. *For a Frenet frame (T, N, B), the following relationships hold.*

1. $T'(t) = k(t)N(t)$.
2. $B'(t) = -\tau(t)N(t)$.
3. $N'(t) = \tau(t)B(t) - k(t)T(t)$.

These relationships can also be represented by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Geometrically, the curvature is a measure of the curve's tendency not to be straight, and torsion is the measure of the curve's tendency to not be planar.

Proposition 1.2.6. *Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parameterised by arc length and suppose that $\phi''(t) \neq 0$ for all $t \in [a, b]$. Then ϕ is contained in a plane if and only if $\tau(t) \equiv 0$.*

Definition 1.2.7. *Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parameterised by arc length. Then a rigid motion of ϕ in $\mathbb{R}^3$ is a regular curve $\psi : [a, b] \to \mathbb{R}^3$ such that there are $g \in \mathrm{SO}(3)$ and $\mathbf{c} \in \mathbb{R}^3$ such that*

$$\psi = g \circ \phi + \mathbf{c}.$$

Theorem 1.2.8. *Assume that $k, \tau : [a, b] \to \mathbb{R}$ are smooth functions with $k > 0$. Then there exists a regular curve $\phi : [a, b] \to \mathbb{R}^3$, parameterised by arc length, such that it has curvature k and torsion τ. Moreover, ϕ is unique up to rigid motions of $\mathbb{R}^3$.*

Corollary 1.2.9. *If $\phi([a, b])$ is a regular curve in $\mathbb{R}^3$ that has torsion $\tau \equiv 0$, and curvature $k(t) \equiv c \in \mathbb{R}_{>0}$, then $\phi([a, b])$ is contained in a circle of radius $\frac{1}{c}$.*

1.3 Curves in $\mathbb{R}^2$

Consider $\phi : [a, b] \to \mathbb{R}^2$ a regular curve, not necessarily parameterised by arc-length, given by

$$\phi(t) = (x(t), y(t))$$

such that

$$\phi'(t) = (x'(t), y'(t))$$

and

$$n(t) = \frac{(-y'(t), x'(t))}{|\phi'(t)|}.$$

Note that $(\phi'(t), n(t))$ is a positively oriented orthonormal basis for $\mathbb{R}^2$. Moreover, let

$$\kappa(t) = \langle n(t), \phi''(t) \rangle.$$

If ϕ is parameterised by arc length, then $\kappa(t)$ is referred to as the signed curvature of ϕ at $\phi(t)$. As in this case, when the direction of the curve changes, the sign of $\kappa(t)$ changes. Still suppose that ϕ is parameterised by arc length, then we have that

$$\kappa(t) = x'(t)y''(t) - y'(t)x''(t).$$

Proposition 1.3.1. *For any regular curve $\phi : [a,b] \to \mathbb{R}^2$ we have*

$$\kappa(t) = \frac{\langle \phi''(t), n(t) \rangle}{|\phi'(t)|^2}.$$

Definition 1.3.2. *$\phi : [a,b] \to \mathbb{R}^2$ is a smooth closed curve if $\phi^{(k)}(a) = \phi^{(k)}(b)$ for $k \geq 0$.*

Remark 1.3.3. *If ϕ is a smooth closed curve, then then map $T(t) = \phi'(t) : [a,b] \to \mathbb{R}^2$ is also a smooth closed curve.*

Definition 1.3.4. *Let $\phi : [a,b] \to \mathbb{R}^2$ be a smooth closed curve. Then the winding number of ϕ about a point $p \in \mathbb{R}^2$ is the number of times ϕ rotates around p in a counter-clockwise direction.*

Remark 1.3.5. *As we can translate the plane, we can always consider the winding number of ϕ about the origin. We denote the winding number of ϕ about the origin with $w(\phi)$.*

Recall that $\mathbb{R}^2$ can be identified with the complex plane through $(x,y) \mapsto x + iy$. Thus, for a curve $\phi(t) = (x(t), y(t))$ we can represent it in the complex plane by $\phi(t) = x(t) + iy(t)$. Consequently, it can be shown that

$$w(\phi) = \frac{1}{2\pi} \int_a^b \frac{\langle \phi(t), (y'(t), -x'(t)) \rangle}{|\phi(t)|^2} \,\mathrm{d}t.$$

Theorem 1.3.6. *If $\phi : [a,b] \to \mathbb{R}^2$ is a closed regular curve parameterised by arc length, then the winding number of $T(t) = \phi'(t)$ is*

$$w(T) = \frac{1}{2\pi} \int_a^b \kappa(t) \,\mathrm{d}t.$$

Remark 1.3.7. *The winding number of $T(t) = \phi'(t)$ is called the index, or turning, number of ϕ and is denoted $\mathrm{Ind}(\phi)$. The index of a curve measures the number of times the unit tangent vector rotates around the origin as ϕ is traversed. Note that Theorem 1.3.6 shows that $\mathrm{Ind}(\phi)$ is robust against small perturbations of ϕ.*

2 Surfaces

2.1 Surfaces in $\mathbb{R}^3$

Definition 2.1.1. *A regular surface is a set $S \subset \mathbb{R}^3$ such that for all $p \in S$ there exists*

1. *an open neighbourhood $V \subseteq \mathbb{R}^3$ of p,*
2. *an open set $U \subseteq \mathbb{R}^2$, and*
3. *a smooth map $\phi : U \to \mathbb{R}^3$ such that*
   (a) *$\phi(U) = V \cap S$,*
   (b) *$\phi : U \to V \cap S$ is a homeomorphism, and*
   (c) *for all $q \in U$ the derivative of ϕ at q, denoted $\mathrm{d}\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$, is injective.*

Moreover, (ϕ, U) is the known as the chart, or local parameterisation, of S at p.

Remark 2.1.2. *Statements 3(a) and 3(b) of Definition 2.1.1 say that S locally resembles $\mathbb{R}^2$. Statement 3(c) of Definition 2.1.1 ensures the S is sufficiently regular near p. More specifically, let $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$ for $u, v \in U$. Suppose that $q = (u_0, v_0)$, then it can be shown that $\mathrm{d}\phi_q$ is injective if and only if the tangent vectors*

- *$\frac{\partial \phi}{\partial u}(u_0, v_0)$, and*
- *$\frac{\partial \phi}{\partial v}(u_0, v_0)$*

span a plane in $\mathbb{R}^3$.

To construct explicit examples of regular surfaces, we construct Proposition 2.1.5. To do so we require Theorem 2.1.4.

Definition 2.1.3. *For $n \geq 1$, let $U, V \subset \mathbb{R}^n$ be open. Then a map $f : U \to V$ is a $\mathcal{C}^k$-diffeomorphism, for $k \in \mathbb{N} \cup \{\infty\}$, if $f : U \to V$ is a homeomorphism, with $f : U \to V$ and $f^{-1} : V \to U$ in $\mathcal{C}^k$.*

Theorem 2.1.4 (Inverse Function Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ a $\mathcal{C}^k$-map, for $k \in \mathbb{N} \cup \{\infty\}$. Assume for some $p \in \Omega$ that $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then, there is a neighbourhood $U \subset \Omega$ of p such that $f : U \to f(U)$ is a $\mathcal{C}^k$-diffeomorphism.*

Proposition 2.1.5. *Let $\Omega \subset \mathbb{R}^3$ be open, $F : \Omega \to \mathbb{R}$ a smooth function, and $c \in \mathbb{R}^3$. Assume that $\nabla F(p) \neq 0$ for every $p \in \Omega$. Then*

$$S := F^{-1}(c) = \{(x, y, z) \in \Omega : F(x, y, z) = c\}$$

is a regular surface.

Remark 2.1.6. *Conversely, a set S is a regular level set if there exists an open set $\Omega \subset \mathbb{R}^3$, a smooth function $F : \Omega \to \mathbb{R}$ and a $c \in \mathbb{R}$ such that $S = F^{-1}(c)$ and $\nabla F(p) \neq 0$ for all $p \in S$.*

Proposition 2.1.7. *Let S be a regular surface, and let $p \in S$. Then there is a neighbourhood, $V \subset S$, of p such that V is the graph of a smooth function of the form*

1. $z = f(x, y)$,

2. $y = f(x, z)$, *or*

3. $x = f(y, z)$.

Remark 2.1.8. *It can be shown that locally a regular level set is the graph of a function. With Proposition 2.1.7 we have that a regular surface is locally the graph of some function.*

2.2 Tangent Vectors and Tangent Planes

Definition 2.2.1. *Let $S \subset \mathbb{R}^3$ be a regular surface. Then a tangent vector to S at p is a vector of the form $\alpha'(0)$, where $\alpha : (-\epsilon, \epsilon) \to S$ is a smooth map with $\alpha > 0$ and $\alpha(0) = p$.*

Definition 2.2.2. *Let $S \subset \mathbb{R}^3$ be a regular surface. Then the tangent plane of S at p is*

$$T_pS = \{\alpha'(0) : \alpha \text{ a tangent vector to } S \text{ at } p\}.$$

Geometrically, $Tp(S)$ is a plane in $\mathbb{R}^3$. Note that the tangent plane as given in Definition 2.2.2 is independent on the charts defining S. However, it is difficult to compute the tangent plane using Definition 2.2.2. To alleviate this challenge we formulate Theorem 2.2.3.

Theorem 2.2.3. *Let $S \subset \mathbb{R}^3$ be a regular surface. Let $p \in S$ and $\phi : U \to S$ a chart with $\phi(q) = p$. Then*

$$\mathrm{d}\phi_q\left(\mathbb{R}^2\right) = \operatorname{span}\left(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right) = T_pS.$$

One can think of the tangent plane as a planar approximation of a regular surface at a point.

Proposition 2.2.4. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, with $\nabla F \neq 0$ on $S := F^{-1}(0)$. Then for any $p \in S$ we have*

$$T_pS = \left\{v \in \mathbb{R}^3 : \langle v, \nabla F(p)\rangle = 0\right\} = (\nabla F(p))^{\perp}.$$

2.3 Smooth Maps on Surfaces

Definition 2.3.1. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces.*

1. *A map $F : S_1 \to \mathbb{R}^n$ is smooth if for every chart $\phi : U \to S_1$, the map $F \circ \phi : U \to \mathbb{R}^n$ is smooth.*

2. *A map $F : S_1 \to S_2$ is smooth, if it is smooth as a map $S_1 \to \mathbb{R}^3$. In such as case, its differential at a point $p \in S_1$ denoted $\mathrm{d}F_p : T_pS_1 \to T_{F(p)}S_2$ is defined as follows. For $v \in T_pS$ there exits a smooth map $\alpha : (-\epsilon, \epsilon) \to S_1$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. Consequently, $\beta := F \circ \alpha : (-\epsilon, \epsilon) \to S_2$ is a smooth map such that $\beta(0) = F(p)$. We let*

$$\mathrm{d}F_p(v) = \beta'(0)$$

Proposition 2.3.2. *The differential is given in statement 2. of Definition 2.3.1 is independent of the choice of α.*

Proposition 2.3.3. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces. Let $p \in S_1$ and $F : S_1 \to S_2$ be a smooth map.*

Then $\mathrm{d}F_p : T_pS_1 \to T_{F(p)}S_2$ is a linear map.

Exercise 2.3.4. *Let S be a regular surface, and suppose $f : S \to \mathbb{R}$ is a smooth function as in statement 1. of Definition 2.3.1. For $p \in S$ let $\mathrm{d}f_p : T_pS \to \mathbb{R}$ be such that if $\alpha'(0) \in T_pS$ then*

$$\mathrm{d}f_p\left(\alpha'(0)\right) = \frac{\mathrm{d}}{\mathrm{d}t} f(\alpha(t))\big|_{t=0}.$$

Show that $\mathrm{d}f_p$ is independent of the choice of α.

Proposition 2.3.5. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces. Let $f : S_1 \to S_2$ be a smooth map and let $p \in S_1$. If $\mathrm{d}f_p : T_pS_1 \to T_{f(p)}S_2$ is invertible then there exists a neighbourhood $V \subset S_1$ of p such that $f : V \to f(V)$ is a diffeomorphism.*

2.4 Normal Vectors, and the Gauss Map

The tangent plane at a point of a regular surface $S \subset \mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. Consequently, there exist two unit normal vectors to the tangent plane. For a regular level set S with respect to a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, the natural choice of unit normal vector to S at p is

$$N(p) = \frac{\nabla F(p)}{|\nabla F(p)|}.$$

Note that this is not canonical, as S is also the regular level set of $-F$ for which the above normal unit vector will point in the opposite direct. However, it does consistently define a unit normal on S that continuously depends on P.

Definition 2.4.1. *Let $S \subset \mathbb{R}^3$ be a regular surface. Given a chart $\phi : U \to S$ for S at p, with $q := \phi^{-1}(p)$, let*

$$N(p) = \frac{\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)}{\left|\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)\right|}$$

Remark 2.4.2. *Unlike the case when S is a regular level set, the unit normal of Definition 2.4.1 depends on the choice of the chart and so may not vary continuously across the surface S.*

Definition 2.4.3. *A regular surface S is orientable if there is a continuous choice of unit normal vector $N(p)$, as given in Definition 2.4.1, for $p \in S$.*

Example 2.4.4. *The Möbius strip is not an orientable surface.*

Definition 2.4.5. *Let $S \subset \mathbb{R}^3$ be a regular orientable surface. Then the continuous map $N : S \to \mathbb{S}^2$ given by $p \mapsto N(p)$ is called the Gauss map of S*

Remark 2.4.6. *If for a regular surface, there is a continuous choice of $N : S \to \mathbb{S}^2$, then the map $N : S \to \mathbb{R}^3$ is smooth.*

The derivative of the Gauss map, $\mathrm{d}N_p : T_pS \to T_{N(p)}\mathbb{S}^2$, can be viewed from different perspectives.

1. Note that the tangent plane $T_{N(p)}\mathbb{S}^2$ consists of the vectors orthogonal to $N(p)$, thus it is the same as T_pS. Therefore, we can view the derivative as a map $\mathrm{d}N_p : T_pS \to T_pS$.

2. For $\alpha'(0) \in T_pS$, since N is the unit normal we must have $\langle N(\alpha(t)), N(\alpha(t))\rangle = 1$. Consequently,

$$\begin{aligned}\langle \mathrm{d}N_p(\alpha'(0)), N_p(\alpha(0))\rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t} N_p(\alpha(t))\big|_{t=0}, N_p(\alpha(0))\right\rangle \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle N_p(\alpha(t)), N_p(\alpha(t))\rangle\big|_{t=0} \\ &= 0.\end{aligned}$$

   Thus, $\mathrm{d}N_p(\alpha'(0))$ is orthogonal to $N(\alpha(0))$, meaning $\mathrm{d}N_p(\alpha'(0)) \in T_pS$.

Recall that $\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\}$ forms a basis for T_pS. Therefore, to find a general form for $\mathrm{d}N_p$, each suffices to understand how $\mathrm{d}N_p$ acts on these vectors.

Proposition 2.4.7. *Let $S \subset \mathbb{R}^3$ be a regular surface, and let $\phi : U \to S$ be a chart for S at $p = \phi(q)$. Then*

$$\mathrm{d}N_p\left(\frac{\partial \phi}{\partial u}(q)\right) = \frac{\partial (N \circ \phi)}{\partial u}(q)$$

and

$$\mathrm{d}N_p\left(\frac{\partial \phi}{\partial v}(q)\right) = \frac{\partial (N \circ \phi)}{\partial v}(q).$$

3 Curvature

3.1 Second Fundamental Form

Definition 3.1.1. *Let $S \subset \mathbb{R}^3$ be an orientable regular surface. Then for $p \in S$ the map $A_p : T_pS \times T_pS \to \mathbb{R}$ given by*

$$A_p(X,Y) = -\langle X, \mathrm{d}N_p(Y)\rangle$$

is know as the second fundamental form of S at p.

Proposition 3.1.2. *For $S \subset \mathbb{R}^3$ a regular surface, the second fundamental form A_p is a symmetric bilinear form for every $p \in S$.*

3.2 Surface Curvature

Understanding the curvature of a surface amounts to studying the variation of N along its surface. It is clear that with $\mathrm{d}N_p$ we obtain A_p. Conversely, if we have an orthonormal basis $\{v, w\}$ for T_pS, then we can recover $\mathrm{d}N_p(X)$ through

$$\mathrm{d}N_p(X) = -A_p(v,X)v - A_p(w,X)w,$$

for $X \in T_pS$.

Exercise 3.2.1. *Let $\{v, w\}$ be an orthonormal basis for T_pS, where $S \subset \mathbb{R}^3$ is an orientable regular surface. Show that the matrix representation of $\mathrm{d}N_p$ in the basis $\{v, w\}$ is symmetric.*

From Exercise 3.2.1, it follows that $\mathrm{d}N_p$ is diagonalizable. In particular, T_pS has an orthonormal basis $\{X_1, X_2\}$ such that

- $\mathrm{d}N_p(X_1) = -\lambda_1 X_1$, and
- $\mathrm{d}N_p(X_2) = -\lambda_2 X_2$.

Moreover,

- $A_p(X_1, X_2) = \lambda_1$,
- $A_p(X_2, X_2) = \lambda_2$, and
- $A_p(X_1, X_2) = 0$.

The tangent vectors $X_1, X_2 \in T_pS$ are known as the principal directions at p, and $\lambda_1, \lambda_2 \in \mathbb{R}$ are known as the principal curvatures. Henceforth, we will use $\lambda_1(p)$ and $\lambda_2(p)$ to denote the principal curvatures of p, with $\lambda_1(p) \leq \lambda_2(p)$.

Lemma 3.2.2. *If $S \subset \mathbb{R}^3$ is a regular surface, then for $p \in S$ we have*

$$\lambda_1(p) = \min\{A_p(X,X) : X \in T_pS,\ |X| = 1\}$$

and

$$\lambda_2(p) = \max\{A_p(X,X) : X \in T_pS,\ |X| = 1\}.$$

Let $c : (-\epsilon, \epsilon) \to S$ be a regular curve in S with $c(0) = p$. Recall that the curvature of c at p is given by $\mathbf{k}(0) \in \mathbb{R}^3$. We can write

$$\mathbf{k}(0) = \langle \mathbf{k}(0), N(p)\rangle N(p) + \mathbf{k}_{\mathsf{tang}}(0),$$

where $\langle \mathbf{k}(0), N(p)\rangle N(p) \in T_pS^{\perp}$ and $\mathbf{k}_{\mathsf{tang}}(0) \in T_pS$.

Definition 3.2.3. *Let $c : (-\epsilon, \epsilon) \to S$ be a regular curve in S with $c(0) = p$. Then the normal curvature of S at p in the direction $c'(0)$ is*

$$k_n \langle \mathbf{k}(0), N(p) \rangle.$$

That is, the normal curvature is the length of the perpendicular component.

Remark 3.2.4. *Equivalently, if θ is the angle between $N(p)$ and $\frac{\mathbf{k}(0)}{k(0)}$, then*

$$k_n(0) = k(0)\cos(\theta).$$

Theorem 3.2.5. *Let $S \subset \mathbb{R}^3$ be an orientable regular surface. Let $p \in S$ and $v \in T_pS$ with $\|v\| = 1$. Then, the normal curvature of S at p in the direction of v satisfies*

$$k_n(p) = A_p(v, v).$$

Corollary 3.2.6. *Let $S \subset \mathbb{R}^3$ be an orientable regular surface. For every $p \in S$ the principal curvatures $\lambda_1(p)$ and $\lambda_2(p)$ are the minimum and maximum values of the normal curvature at p along all directions in T_pS respectively.*

Definition 3.2.7. *A point $p \in S$ is umbilical, if $\lambda_1(p) = \lambda_2(p)$.*

Proposition 3.2.8. *Let S be a connected orientable regular surface. If every point in S is umbilical, then either S is contained in a plane or S is contained in a sphere.*

Definition 3.2.9. *Let $S \subset \mathbb{R}^3$ be a regular surface. Let $p \in S$ have principal curvatures $\lambda_1(p)$ and $\lambda_2(p)$.*

1. *The Gaussian curvature of S at p is*

$$K(p) = \lambda_1(p)\lambda_2(p) = \det(\mathrm{d}N_p).$$

2. *The mean curvature of S at p is*

$$H(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2} = -\frac{1}{2}\mathrm{tr}(\mathrm{d}N_p).$$

Remark 3.2.10. *Note that the sign of Gaussian does not change if we change the orientation of S, that is use $-N$ instead of N. However, the sign of the mean curvatures changes.*

3.3 Elliptic, Hyperbolic and Parabolic Points

Suppose $S \subset \mathbb{R}^3$ is a regular surface, and let $\phi : U \to S$ be a local chart at $p \in S$ with $\phi(0,0) = p$. For $c, d \in \mathbb{R}$ consider $\gamma : (-\epsilon, \epsilon) : S$ where $\gamma(t) = \phi(ct, dt)$. As

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma'(t), N(\gamma(t)) \rangle = 0$$

we have

$$\langle \gamma''(t), N(\gamma(t)) \rangle + \langle \gamma'(t), \mathrm{d}N_{\gamma(t)}(\gamma'(t)) \rangle = 0,$$

which is equivalent to

$$A_{\phi(\gamma(t))}(\gamma'(t), \gamma'(t)) = \langle \gamma''(t), N(\gamma(t)) \rangle.$$

By calculating $\gamma'(t)$ and $\gamma''(t)$ explicitly, it follows that

$$A_p\left(c\frac{\partial \phi}{\partial u}(0,0) + d\frac{\partial \phi}{\partial v}(0,0), c\frac{\partial \phi}{\partial u}(0,0)+d\frac{\partial \phi}{\partial v}(0,0)\right)$$
$$= \left\langle c^2\frac{\partial^2 \phi}{\partial u^2}(0,0) + 2cd\frac{\partial^2 \phi}{\partial u \partial v}(0,0) + d^2\frac{\partial^2 \phi}{\partial v^2}(0,0), N(p)\right\rangle. \quad (3.3.1)$$

Suppose that $\phi(0,0) = (0,0,0)$, which can be achieved through a change of coordinates and rigid motion transformations, with

- $\frac{\partial \phi}{\partial u}(0,0) = (1,0,0)$,
- $\frac{\partial \phi}{\partial v}(0,0) = (0,1,0)$

being the principle directions. By (3.3.1) it follows that

$$\left\langle \frac{\partial^2 \phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2 \phi}{\partial u \partial v}(0,0)uv + \frac{\partial^2 \phi}{\partial v^2}(0,0)v^2, N(0,0,0)\right\rangle = A_{(0,0,0)}((u,v,0),(u,v,0)). \quad (3.3.2)$$

Noting that

$$N(0,0,0) = (1,0,0) \times (0,0,1),$$

it follows that the left-hand side of (3.3.2) is the third component of the quadratic term of the Taylor series expansion of $\phi(u,v)$ around $(0,0)$. Whereas, the right-hand side of (3.3.2) can be expressed as

$$A_{(0,0,0)}((u,v,0),(u,v,0)) = \lambda_1 u^2 + \lambda_2 v^2,$$

where λ_1 and λ_2 are the principal curvatures in the directions $(1,0,0)$ and $(0,1,0)$ respectively. Thus,

$$\phi(u,v) \approx \left(u, v, \frac{1}{2}\left(\lambda_1 u^2 + \lambda_2 v^2\right)\right).$$

That is, near $(0,0,0)$ the surface S can be approximated by the graph of

$$f(x,y) = \frac{1}{2}\left(\lambda_1 x^2 + \lambda_2 y^2\right).$$

With this we are able to understanding the local shape of a regular surface in terms of its principle curvatures.

Proposition 3.3.1. *Let $S \subset \mathbb{R}^3$ be a regular surface. For $p \in S$ the following statements hold.*

1. *If $K(p) > 0$, then there is a neighbourhood $V \subset \mathbb{R}^3$ of p such that $S \cap V$ lies on the same side of $p + T_pS$.*
2. *If $K(p) < 0$, then on any neighbourhood $V \subset \mathbb{R}^3$ of p, $S \cap V$ meets both sides of $p + T_pS$.*

Definition 3.3.2. *Let $S \subset \mathbb{R}^3$ be a regular surface.*

1. *A point $p \in S$ is elliptic if $K(p) > 0$.*
2. *A point $p \in S$ is hyperbolic if $K(p) < 0$.*
3. *A point $p \in S$ is parabolic if $K(p) = 0$ and $H(p) \neq 0$.*
4. *A point $p \in S$ is planar if $K(p) = H(p) = 0$.*

Remark 3.3.3. *The terminology of Definition 3.3.2 can also be expressed in terms of the principle curvatures.*

1. *A point $p \in S$ is elliptic if λ_1 and λ_2 have the same sign.*
2. *A point $p \in S$ is hyperbolic if λ_1 and λ_2 have the opposite sign.*
3. *A point $p \in S$ is parabolic if only one principle curvature is zero.*
4. *A point $p \in S$ is planar if $\lambda_1 = \lambda_2 = 0$.*

3.4 Curvature in Charts

Proposition 3.4.1. *Let $\phi : U \to S$ be a chart. Let*

$$g = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix}$$

and

$$A = \begin{pmatrix} A_{\phi(u,v)}(\phi_u, \phi_u) & A_{\phi(u,v)}(\phi_u, \phi_v) \\ A_{\phi(u,v)}(\phi_v, \phi_u) & A_{\phi(u,v)}(\phi_v, \phi_v) \end{pmatrix}.$$

With $\sigma = g^{-1}A$ it follows that

1. *$K(\phi(u,v)) = \det(\sigma) = \frac{\det(A)}{\det(g)}$, and*
2. *$H(\phi(u,v)) = \frac{1}{2}\operatorname{tr}(\sigma)$.*

Remark 3.4.2. *Note that the matrix A in Proposition 3.4.1 can also be represented as*

$$A = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix}$$

3.5 First Fundamental Form

Definition 3.5.1. *Let $S \subset \mathbb{R}^3$ be a regular surface. The first fundamental form at p is the map $g : T_pS \times T_pS \to \mathbb{R}$ given by*

$$g(u, w) = \langle v, w \rangle.$$

Let $\phi : U \to S$ be a chart for S at p, such that T_pS is spanned by $\left\{ \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v} \right\}$. Then

$$g = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix}.$$

Moreover, let $\alpha : [a, b] \to S$ be a path contained in $\phi(U)$, where $\phi : U \to S$ is a chart for S at p. Then $\alpha(t) = \phi(u(t), v(t))$ for some functions u and v. It follows that

$$\ell(\alpha([a,b])) = \int_a^b \sqrt{\begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}} \, \mathrm{d}t$$

Definition 3.5.2. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces. A smooth map $F : S_1 \to S_2$ is a local isometry if*

$$\langle \mathrm{d}F_p(X), \mathrm{d}F_p(Y) \rangle = \langle X, Y \rangle$$

for all $p \in S_1$ and $X, Y \in T_pS_1$. In particular, F is an isometry if it is a local isometry and bijective.

Remark 3.5.3. *If F is a local isometry between regular surfaces S_1 and S_2, then $\mathrm{d}F_p : T_pS_1 \to T_{F(p)}S_2$ is bijective. As T_pS_1 and $T_{F(p)}S_2$ are of the same dimension, it follows that $\mathrm{d}F_p$ is also surjective. Therefore, by Proposition 2.3.5, F is a diffeomorphism near p.*

Lemma 3.5.4. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces and let $F : S_1 \to S_2$ be a smooth map. Then F is a local isometry if and only if*

$$\ell(F \circ \alpha([a,b])) = \ell(\alpha([a,b]))$$

for every smooth map $\alpha : [a,b] \to S_1$.

3.6 Christoffel Symbols

Let $S \subset \mathbb{R}^3$. For a chart $\phi : U \to S$ of S recall that $\left\{\frac{\partial\phi}{\partial x_i}, \frac{\partial\phi}{\partial x_j}\right\}$ are linearly independent and span $T_{\phi(\cdot)}S$, where (x_1, x_2) is a point of U. As $N(\phi(\cdot))$ is orthogonal to these vectors it follows that $\left\{\frac{\partial\phi}{\partial x_i}, \frac{\partial\phi}{\partial x_j}, N\right\}$ is a basis for $\mathbb{R}^3$. Moreover, recall the matrices

$$g = \left(\left\langle \frac{\partial\phi}{\partial x_i}, \frac{\partial\phi}{\partial x_j} \right\rangle\right)_{i,j=1,2}$$

and

$$g = \left(\left\langle \frac{\partial^2\phi}{\partial x_i \partial x_j}, N \right\rangle\right)_{i,j=1,2}.$$

Definition 3.6.1. *The Christoffel symbols Γ_{ij}^k for $i, j, k = 1, 2$ are such that*

$$\frac{\partial^2\phi}{\partial x_i \partial x_j}\Gamma_{ij}^1 \frac{\partial\phi}{\partial x_1} + \Gamma_{ij}^2 \frac{\partial\phi}{\partial x_2} + A_{ij}N$$

Remark 3.6.2. *Note that $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $i, j, k = 1, 2$.*

Proposition 3.6.3. *For $S \subset \mathbb{R}^3$ a regular surface, the Christoffel symbols are determined by the first fundamental form $g = (g_{ij})_{i,j=1,2}$ and its partial derivatives.*

3.7 Theorema Egregium

Theorem 3.7.1 (Theorema Egregium)**.** *Gaussian curvature is an intrinsic quantity, that is, it only depends on the first fundamental form.*

Remark 3.7.2.

1. *Being intrinsic also means that Gaussian curvature is invariant under isometries.*
2. *Mean curvature, H, is not intrinsic.*

Corollary 3.7.3. *Assume $S_1, S_2 \subset \mathbb{R}^3$ are regular surface with Gaussian curvatures K_1 and K_2 respectively.*

If $F : S_1 \to S_2$ is a local isometry, then $K_2 \circ F = K_1$.

Corollary 3.7.4. *There is no local isometry from a plane to a sphere.*

Remark 3.7.5. *The sphere in Corollary 3.7.4 can be replaced with any compact surface. As for a compact surface S, there exists a point $p \in S$ such that $K(p) > 0$ and so any local isometry from a plane to S must avoid p.*

3.8 Surfaces of Constant Curvature

Theorem 3.8.1. *If $S \subset \mathbb{R}^3$ is a compact and connected regular surface with a constant positive Gaussian curvature, then S must be a sphere.*

3.9 Area of a Surface

Let $S \subset \mathbb{R}^3$ be a regular surface, with $\phi : U \to S$ a chart. A rectangle $(\delta_u, 0) \times (0, \delta_v) \subset U$ has area $\delta_u \delta_v$. Moreover, if δ_u and δ_v are small then $\phi((\delta_u, 0) \times (0, \delta_v)) \subset S$ is approximately a parallelogram with sides $\delta_u \phi_u$ and $\delta_v \phi_v$ which has area

$$|\delta_u \phi_u \times \delta_v \phi_v| = \delta_u \delta_v |\phi_u \times \phi_v|.$$

If $D \subset U$ is compact, then the area of $\phi(D)$ is given by

$$\text{area}(\phi(D)) = \int_D |\phi_u \times \phi_v| \, \mathrm{d}u \mathrm{d}v.$$

Proposition 3.9.1. *Let S be a regular surface, with $\phi : U \to S$ a chart for S and $D \subset U$ a compact set. Then area$(\phi(D))$ does not depend on the choice of ϕ, but only on the set $\phi(D)$.*

Lemma 3.9.2. *Let S be a regular surface, with $\phi : U \to S$ a chart for S, and $D \subset U$ a compact set. Then*

$$\text{area}(\phi(D)) = \int_D \sqrt{\det(g)} \, \mathrm{d}u \mathrm{d}v,$$

where g is the matrix corresponding to the fundamental form of S in the chart ϕ.

Let $f : S \to \mathbb{R}$ be a smooth function on S, with $\phi : U \to S$ a chart for S and $D \subset U$ compact. Then the integral of f on $\phi(D)$ is

$$\int_{\phi(D)} f \, \mathrm{d}A = \int_D f \circ \phi |\phi_u \times \phi_v| \, \mathrm{d}u \mathrm{d}v = \int_D f \circ \phi \sqrt{\det(g)} \, \mathrm{d}u \mathrm{d}v.$$

Following the arguments of Proposition 3.9.1, one can show that the integral of f on $\phi(D)$ is independent of ϕ and only dependent on $\phi(D)$. Provided S is a compact we can partition S as

$$S = S_1 \cup \cdots \cup S_k$$

such that the following hold.

- For $i \neq j$ we have $S_i \cap S_j \subset \partial S_i \cap \partial S_j$,
- For each i there is a chart $\phi_i : U_i \to S$ and compact set $D_i \subset U_i$ such that $S_i = \phi_i(D_i)$.
- For each i, the set ∂D_i has zero area.

Consequently, the integral of f over S can be calculated as

$$\int_S f \,\mathrm{d}A = \sum_{i=1}^{k} \int_{S_i} f \,\mathrm{d}A = \sum_{i=1}^{k} \int_{D_i} (f \circ \phi_i) \left| \frac{\partial \phi_i}{\partial u} \times \frac{\partial \phi_i}{\partial v} \right| \mathrm{d}u \mathrm{d}v.$$

The value of the integral is independent of the partition chosen for S.

4 Geodesics

4.1 Geodesic Curvature

Let $S \subset \mathbb{R}^3$ be a regular surface, let $\gamma : [a, b] \to S$ be a regular curve, and let N be a unit normal to S. Then γ' is tangent to S and orthogonal to N, and so $\{\gamma', N \times \gamma', N\}$ is an orthonormal basis for $\mathbb{R}^3$, provided γ is parameterised by arc-length. Moreover, when γ is parameterised by arc length we can write the curvature vector, $\mathbf{k} = \gamma''(t)$, as

$$\mathbf{k} = k_n N + k_g (N \times \gamma'),$$

where

- $k_n = \langle \mathbf{k}, N \rangle$ is the normal curvature of γ at $\gamma(t)$, and
- $k_g = \langle \mathbf{k}, N \times \gamma' \rangle$ is the geodesic curvature of γ.

With this decomposition,

$$k^2 = k_n^2 + k_g^2.$$

If $k_g = 0$, then $\gamma(t)$ curves only in directions orthogonal to S. If $k_g \equiv 0$, then γ is called a geodesic.

Proposition 4.1.1. *Any local isometry of a regular surface in $\mathbb{R}^3$ maps geodesics to geodesics.*

Exercise 4.1.2. *Let $\gamma(t) = \phi(u(t), v(t))$. Derive the geodesic equations*

$$(g_{11}u' + g_{12}v')' = \frac{1}{2}\left((g_{11})_u (u')^2 + 2(g_{12})_u (u'v') + (g_{22})_u (v')^2\right)$$

and

$$(g_{21}u' + g_{22}v')' = \frac{1}{2}\left((g_{11})_v (u')^2 + 2(g_{12})_v (u'v') + (g_{22})_v (v')^2\right).$$

4.2 Minimising Arc Length

Definition 4.2.1. *Let $S \subset \mathbb{R}^3$ be a regular surface, and $\gamma : [0, L] \to S$ a smooth map. A variation of γ is a smooth map $[0, L] \times [-\epsilon, \epsilon] \to S$, say $(t, s) \mapsto \gamma_s(t)$, such that*

1. *$\gamma_0 \equiv \gamma$,*
2. *$\gamma_s(0) = \gamma(0)$ for all $s \in [-\epsilon, \epsilon]$, and*
3. *$\gamma_s(L) = \gamma(L)$ for all $s \in [-\epsilon, \epsilon]$.*

Proposition 4.2.2. *Let $S \subset \mathbb{R}^3$ be a regular surface. If $\gamma : [0, L] \to S$ is parameterised by arc length, and γ is the shortest regular curve from $\gamma(0)$ to $\gamma(L)$, then γ is a geodesic.*

Remark 4.2.3.

1. *Geodesics need not minimise arc length globally.*
2. *A geodesic between arbitrary points on a surface may not exist.*
3. *When a geodesic does exist between points it need not be unique.*

5 The Gauss-Bonnet Theorem

5.1 Local Gauss-Bonnet

Definition 5.1.1. *A regular surface with a boundary is a set $S \subset \mathbb{R}^3$ such that for every $p \in S$ one of the following statements holds.*

1. *There is a chart $\phi : U \to S$ for S at p.*
2. *There is a neighbourhood $U \subset \mathbb{R}^2$ of $(0,0)$, an open neighbourhood $V \subset \mathbb{R}^3$ of p, and a smooth map $\phi : U \to V$ such that the following statements are satisfied.*
   (a) *$\phi(0,0) = p$.*
   (b) *$\phi : \{(x,y) \in U : y \geq 0\} \to V \cap S$ is a homeomorphism.*
   (c) *For every $q \in U$, the map $\mathrm{d}\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.*

Remark 5.1.2.

1. *A regular surface with a boundary is a regular surface, with the additional requirement that at some points we have charts defined on a half disk with a boundary curve.*
2. *Points $p \in S$ satisfying statement 1. of Definition 5.1.1 are interior points of S.*
3. *Points $p \in S$ satisfying statement 2. of Definition 5.1.1 are boundary points of S.*
4. *The set of all boundary points of S is called the boundary of S and denoted ∂S. Note that ∂S is a union of regular curves.*

For a regular surface with a boundary, we can define a tangent space at each point using the same constructions used for regular surfaces. Namely,

$$T_pS = \{\alpha'(0) : \text{Where } \alpha : [0,\epsilon) \to S \text{ or } \alpha : (-\epsilon, 0] \to S \text{ is smooth with } \alpha(0) = p\}.$$

As before we also have that $T_pS = \mathrm{d}\phi_{(0,0)}\left(\mathbb{R}^2\right)$. Note that these are well-defined even for boundary points of S, so the notion of normal vectors at these points still makes sense. An orientation on S induces a unique orientation of ∂S. For a choice of unit normal vector $N : S \to \mathbb{R}^3$, a parameterisation $\gamma : [a,b] \to \partial S$ is positively oriented if $N \times \gamma'$ points into S. In particular, we say that ∂S is positively oriented.

Theorem 5.1.3 (Local Gauss-Bonnet)**.** *Let $S \subset \mathbb{R}^3$ be a regular surface with a boundary. Assume that $\phi : U \subset \mathbb{R}^2 \to S$ is a chart satisfying the following.*

1. *ϕ is smooth on a neighbourhood of $\bar{U}$, and $\bar{U}$ is diffeomorphic to a closed disc.*
2. *$S = \phi\left(\bar{U}\right)$ and $\partial S = \phi(\partial U)$.*

Then

$$\int_{\partial S} k_g \,\mathrm{d}s + \int_S K \,\mathrm{d}A = 2\pi. \tag{5.1.1}$$

Remark 5.1.4. *Note that if the surface lies in a plane then the curvature is zero, and so* (5.1.1) *becomes*

$$\int_{\partial S} k_g \,\mathrm{d}s.$$

Thus we recover Theorem 1.3.6.

5.2 Gauss-Bonnet for Triangles

Definition 5.2.1. *A curvilinear triangle in* $\mathbb{R}^2$ *is a continuous map* $\beta : \mathbb{R} \to \mathbb{R}^2$ *such that* $\beta(t) = \beta(t+3)$ *for all* $t \in \mathbb{R}$*, and for some* t_0, t_1, t_2 *with* $t_0 < t_1 < t_2 < t_0 + 3$ *the following statements hold.*

1. β *is regular on* (t_0, t_1), (t_1, t_2) *and* $(t_2, t_0 + 3)$.
2. $\beta : [t_0, t_0 + 3) \to \mathbb{R}^2$ *is injective.*
3. *The derivatives*
   - $\beta'\left(t_i^-\right) = \lim_{t \to t_i^-} \beta'(t)$, *and*
   - $\beta'\left(t_i^+\right) = \lim_{t \to t_i^+} \beta'(t)$

   exist for $i = 0, 1, 2$.
4. *Each collection* $\left\{\beta'\left(t_0^+\right), \beta'\left(t_3^-\right)\right\}$, $\left\{\beta'\left(t_1^+\right), \beta'\left(t_1^-\right)\right\}$ *and* $\left\{\beta'\left(t_2^+\right), \beta'\left(t_2^-\right)\right\}$ *are linearly independent.*

Remark 5.2.2.

1. *Definition 5.2.1 can be naturally generalised to curvilinear* n*-gons.*
2. *The points* $\beta(t_i)$ *for* $i = 0, 1, 2$ *in Definition 5.2.1 are the vertices of the curvilinear triangle* β.
3. *The curves* $\beta(t_0, t_1), \beta(t_1, t_2)$ *and* $\beta(t_2, t_0 + 3)$ *in Definition 5.2.1 are the edges of the curvilinear triangle* β.

Let $S \subset \mathbb{R}^3$ be a regular surface with chart $\phi : U \to S$, unit normal vector N, and curvilinear triangle T. Then $\phi(T)$ is a curvilinear triangle in S, with edges being the image of the edges of T under ϕ, and vertices being the image of the vertices of T under ϕ. For such a curvilinear triangle, we can parameterise adjacent edges with

$$\gamma_{\text{in}} : (-\epsilon, 0] \to S$$

and

$$\gamma_{\text{out}} : [0, \epsilon) \to S.$$

Consequently, their tangent vectors meet at an angle $\theta \in (-\pi, \pi)$ given by

$$\cos(\theta) = \frac{\langle \gamma_{\text{in}}'(0), \gamma_{\text{out}}'(0) \rangle}{|\gamma_{\text{in}}'(0)||\gamma_{\text{out}}(0)|}$$

and satisfying the convention

- $\theta > 0$ if $\{\gamma_{\text{in}}'(0), \gamma_{\text{out}}'(0), N\}$ is a positive basis for $\mathbb{R}^3$, and
- $\theta < 0$ if $\{\gamma_{\text{in}}'(0), \gamma_{\text{out}}'(0), N\}$ is a negative basis for $\mathbb{R}^3$.

The angle θ is referred to as the exterior angle at the vertex, and $\pi - \theta$ is referred to as the interior angle at the vertex.

Theorem 5.2.3 (Gauss-Bonnet for Triangle)**.** *Let* $S' \subset \mathbb{R}^3$ *be a regular surface and* S *a curvilinear triangle in* S'. *If* ∂S *has edges* γ_i *for* $i = 1, 2, 3$, *each parameterised by arc length, that meet with exterior angle* θ_i *for* $i = 1, 2, 3$, *then*

$$\sum_{i=1}^{3} \int_{\gamma_i} k_g \, \mathrm{d}s + \sum_{i=1}^{3} \theta_i + \int_S K \, \mathrm{d}A = 2\pi.$$

Remark 5.2.4. *Theorem 5.2.3 provides an interpretation of Gaussian curvature. Let S be a surface and T a triangle in S with edges being geodesics of S. Let α_i be the interior angles of T, then Theorem 5.2.3 says that*

$$\int_T K \,\mathrm{d}A = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Consequently, we have the following relationship between surface curvature and interior angles of a triangle.

1. *If S is a plane, then $\sum_{i=1}^{3} \alpha_i = \pi$.*
2. *If S is a unit sphere, then*
$$\sum_{i=1}^{3} \alpha_i - \pi = \int_T 1 \,\mathrm{d}A = \text{area}(T).$$
3. *If S has negative Gaussian curvature, then*
$$\sum_{i=1}^{3} \alpha_i - \pi = \int_T K \,\mathrm{d}A < 0,$$
which implies that $\sum_{i=1}^{3} \alpha_i < \pi$.

5.3 Triangulation and the Euler Characteristic

Definition 5.3.1. *Let $S \subset \mathbb{R}^3$ be a compact regular surface. A triangulation of S is a partition of S into curvilinear triangles $\{T_1, \ldots, T_n\}$ such that the following are satisfied.*

1. *$S = \bigcup_{i=1}^{n} T_i$.*
2. *For $i \neq j$, if $T_i \cap T_j \neq \emptyset$ then $T_i \cap T_j$ is either a vertex or an edge.*
3. *For any edge that lies in the interior of S, there are exactly two distinct triangles sharing this edge. Similarly, for any edge of the triangulation which lies on ∂S, there is a unique triangle of the partition containing that edge.*

Definition 5.3.2. *Let $S \subset \mathbb{R}^3$ be a compact regular surface with a triangulation $(T_i)_{i=1}^{n}$. Then the Euler characteristic of S is*

$$\chi\left(\bigcup_{i=1}^{n} T_i\right) = V - E + F,$$

where

- *F is the number of faces, that is triangles,*
- *E is the number of distinct faces, and*
- *V is the number of distinct vertices.*

Remark 5.3.3. *It can be shown that every compact surface S admits a triangulation, and thus has an Euler characteristic. Moreover, the Euler characteristic of the surface is independent of the chosen triangulation. Consequently, we use $\chi(S)$ to denote the Euler characteristic of a compact surface S.*

A surface of genus g, Σ_g, is obtained by attaching g toruses, in a manner such that Σ_g has g holes. One can observe that

- $V_{\Sigma_g} = V_{\Sigma_{g-1}} + V_{\Sigma_1} - 3$,

- $E_{\Sigma_g} = E_{\Sigma_{g-1}} + E_{\Sigma_1} - 3$, and
- $F_{\Sigma_g} = F_{\Sigma_{g-1}} + F_{\Sigma_1} - 2$.

Consequently,

$$\chi(\Sigma_g) = \chi(\Sigma_{g-1}) + \chi(\Sigma_1) - 2$$

and so through induction it follows that

$$\chi(\Sigma_g) = 2 - 2g.$$

Proposition 5.3.4. *If S and S' are homeomorphic surfaces, then $\chi(S) = \chi(S')$.*

Theorem 5.3.5. *Let S be a compact, connected and orientable surface with a boundary. Then S is homeomorphic to Σ_g for some g, determined by $\chi(S) = 2 - 2g$.*

5.4 Gauss-Bonnet Theorem

Theorem 5.4.1 (Gauss-Bonnet)**.** *Let $S \subset \mathbb{R}^3$ be a compact and orientable regular surface, possibly with a boundary. Then*

$$\int_{\partial S} k_g \, \mathrm{d}s + \int_S K \, \mathrm{d}A = 2\pi\chi(S),$$

where ∂S is parameterised by arc length, and is positive oriented. In particular, if $\partial S = \emptyset$, then

$$\int_S K \, \mathrm{d}A = 2\pi\chi(S).$$

Remark 5.4.2. *When $\partial S = \emptyset$, then Theorem 5.4.1 is the surface analogue of Theorem 1.3.6.*

Corollary 5.4.3. *Let S be a compact and connected regular surface without a boundary. Suppose that $K \geq 0$ on S, then S is homeomorphic to a sphere. Equivalently, if S has genus $g > 0$, then the Gaussian curvature of S must be negative at a point on S.*

Corollary 5.4.4. *Let S be a compact and connected regular surface without boundary. Suppose that $K > 0$ on S. Then if γ_1 and γ_2 are simple, that is with no self-intersections, closed geodesics on S it follows that $\gamma_1 \cap \gamma_2 \neq \emptyset$.*

Corollary 5.4.5. *Let S be a regular surface diffeomorphic to a disk, and satisfies $K \leq 0$. Assume that γ_1 and γ_2 are geodesics in S that are not part of the same geodesic such that $\gamma_1(0) = \gamma_2(0) = p \in S$. Then $\gamma_1 \cap \gamma_2 = \{p\}$.*