# Fourier Analysis and Theory of Distribution*

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## Contents



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# 1 Orthonormal Systems in Euclidean Spaces

## 1.1 Euclidean Spaces

**Definition 1.1.1.** *A real Euclidean space $R$ is a linear space with a map $(\cdot,\cdot) : R \times R \to \mathbb{R}$ that satisfies the following statements.*

1. $(x, y) = (y, x)$.
2. $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$.
3. $(\lambda x, y) = \lambda(x, y)$ *for* $\lambda \in \mathbb{R}$.
4. $(x, x) \geq 0$ *with* $(x, x) = 0$ *if and only if* $x = 0$.

**Remark 1.1.2.**

1. *The map $(\cdot,\cdot)$ of Definition 1.1.1 is referred to as an inner product on $R$.*
2. *A Euclidean space $R$ is a normed vector space with*

$$\|x\| = \sqrt{(x, x)},$$

*and thus it is also a metric space with*

$$\rho(x, y) = \|x - y\|.$$

For the moment we will exclusively work with real Euclidean spaces.

**Definition 1.1.3.** *Let $R$ be a Euclidean space. A system of non-zero vectors $(x_\alpha)_{\alpha \in A} \subset R$ is orthogonal if $(x_\alpha, x_\beta) = 0$ for $\alpha \neq \beta$. In particular, it is orthonormal if in addition $\|x_\alpha\| = 1$ for all $\alpha \in A$.*

Given an orthogonal system $(x_\alpha)_{\alpha \in A}$, one can construct an orthonormal system $\left(\frac{x_\alpha}{\|x_\alpha\|}\right)_{\alpha \in A}$.

**Exercise 1.1.4.** *Let $(x_\alpha)_{\alpha \in A}$ be an orthogonal system of vectors. Show that $(x_\alpha)_{\alpha \in A}$ is linearly independent.*

**Definition 1.1.5.** *Let $R$ be a Euclidean space, with $(x_\alpha)_{\alpha \in A} \subset R$ an orthogonal system. Then $(x_\alpha)_{\alpha \in A}$ is complete if*

$$\overline{\operatorname{span}\left((x_\alpha)_{\alpha \in A} \subset R\right)} = R.$$

**Definition 1.1.6.** *If an orthogonal system $(x_\alpha)_{\alpha \in A}$ is complete, then it is said to be an orthogonal basis of $R$. In particular, it is an orthonormal basis of $R$ if in addition $\|x_\alpha\| = 1$ for all $\alpha \in A$.*

**Example 1.1.7.**

1. *$\mathbb{R}^n$ is a finite-dimensional real Euclidean space with inner product*

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

*An orthonormal basis of $\mathbb{R}^n$ is $(e_j)_{j=1,\dots,n}$ where*

$$e_j = (\underbrace{0, \dots, 1}_{j}, \dots, 0).$$

2. *The space*

$$\ell^2 = \left\{ x = (x_1, x_2, \dots) : \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

*is an infinite-dimensional real Euclidean space with inner product*

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

*Consider the system $(e_j)_{j\geq 1} \subseteq \ell^2$ where*

$$e_j = (\underbrace{0, \dots, 1}_{j}, \dots).$$

*Clearly $(e_j)_{j\geq 1}$ is orthogonal with $\|e_j\| = 1$. Next let $x \in \ell^2$ and consider $x^{(n)} = (x_1, \dots, x_n, 0, \dots)$. Then*

$$x^{(n)} = \sum_{j=1}^{n} x_j e_j,$$

*and*

$$\left\| x^{(n)} - x \right\| \overset{n\to\infty}{\longrightarrow} 0.$$

*Therefore, $(e_j)_{j\geq 1} \subset \ell^2$ is also complete and thus an orthonormal basis of $\ell^2$.*

**Exercise 1.1.8.** *The space $\mathcal{C}_2([-\pi, \pi])$ of continuous real-valued functions on $[-\pi, \pi]$ is a real Euclidean space with inner product*

$$(f, g) = \int_{-\pi}^{\pi} f(t) g(t) \, \mathrm{d}t.$$

*Show that*

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

*is an orthogonal basis of $\mathcal{C}_2([-\pi, \pi])$. Theorem 2.3.4 can be used without proof.*

**Definition 1.1.9.** *A space $R$ is separable if it contains a countably dense subset.*

**Example 1.1.10.** *The Euclidean spaces from Example 1.1.7 are separable.*

1. *$\mathbb{Q}^n \subseteq \mathbb{R}^n$ is countably dense and so $\mathbb{R}^n$ is a separable space.*

2. *The subset*

$$A := \{x = (x_1, \dots, x_n, 0, \dots) : x_i \in \mathbb{Q}, \, n \in \mathbb{N}\} \subset \ell^2$$

*is countable. Moreover, given any $x \in \ell^2$ and $\epsilon > 0$ let*

$$x^{(n)} := (x_1, \dots, x_n, 0, \dots).$$

*Since, $\sum_{k=1}^{\infty} x_k^2 < \infty$ it follows that*

$$\left\| x - x^{(n)} \right\|_{\ell^2} \overset{n\to\infty}{\longrightarrow} 0.$$

*In particular, there exists a $N \in \mathbb{N}$ such that $\left\| x - x^{(N)} \right\|_{\ell^2} < \frac{\epsilon}{2}$. As $\mathbb{Q} \subseteq \mathbb{R}$ is dense, there exists a $y \in A$ such that $\left\| y - x^{(N)} \right\|_{\ell^2} < \frac{\epsilon}{2}$. It follows that,*

$$\begin{aligned} \|y - x\|_{\ell^2} &\leq \left\| y - x^{(N)} \right\|_{\ell^2} + \left\| x^{(N)} - x \right\|_{\ell^2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\leq \epsilon. \end{aligned}$$

*This means that $A \subset \ell^2$ is countable and dense, meaning $\ell^2$ is separable.*

3. *Let*

$$A := \left\{ a_1 + \sum_{n=1}^{N} \left( b_n \cos(nt) + c_n \sin(nt) \right) : a_1, b_n, c_n \in \mathbb{Q},\ N \in \mathbb{N} \right\}.$$

*Let $f \in \mathcal{C}_2([-\pi, \pi])$ and let $\epsilon > 0$. Then as*

$$\{1\} \cup \{\cos(nt)\}_{n\in\mathbb{N}} \cup \{\sin(nt)\}_{n\in\mathbb{N}}$$

*is a complete orthogonal system on $\mathcal{C}_2([-\pi, \pi])$ it follows that*

$$f = a_1 + \sum_{n=1}^{\infty} \left( b_n \cos(nt) + c_n \sin(nt) \right)$$

*for some $a_1, b_n, c_n \in \mathbb{R}$. Let*

$$f^{(N)} := a_1 + \sum_{n=1}^{N} \left( b_n \cos(nt) + c_n \sin(nt) \right),$$

*then since $f^{(N)} \to f$ in $\mathcal{C}_2([-\pi, \pi])$, there exists a $N_0 \in \mathbb{N}$ such that*

$$\left\| f^{(N_0)} - f \right\|_{\mathcal{C}_2([-\pi,\pi])} < \frac{\epsilon}{2}.$$

*As $\mathbb{Q} \subset \mathbb{R}$ is dense, it follows that there exists a $\tilde{f} \in A$ such that*

$$\left\| \tilde{f} - f^{(N_0)} \right\|_{\mathcal{C}_2([-\pi,\pi])} < \frac{\epsilon}{2}.$$

*Therefore,*

$$\begin{aligned} \left\| \tilde{f} - f \right\|_{\mathcal{C}_2([-\pi,\pi])} &\leq \left\| \tilde{f} - f^{(N_0)} \right\|_{\mathcal{C}_2([-\pi,\pi])} + \left\| f^{(N_0)} - f \right\|_{\mathcal{C}_2([-\pi,\pi])} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

*This implies that $A \subseteq \mathcal{C}_2([-\pi, \pi])$ is dense. As $A$ is countable it follows that $\mathcal{C}_2([-\pi, \pi])$ is separable.*

**Lemma 1.1.11.** *Let $R$ be a separable Euclidean space. Then any orthogonal system in $R$ is at most countable.*

*Proof.* Without loss of generality let $(\varphi_\alpha)_{\alpha \in A} \subseteq R$ be an orthonormal system. Then for $\alpha \neq \beta$ observe that

$$\begin{aligned}\|\varphi_\alpha - \varphi_\beta\|^2 &= (\varphi_\alpha - \varphi_\beta, \varphi_\alpha - \varphi_\beta)\\ &= \|\varphi_\alpha\|^2 - 2(\varphi_\alpha, \varphi_\beta) + \|\varphi_\beta\|^2\\ &= 2.\end{aligned}$$

Therefore, the set of open balls $\left(B_{\frac{1}{2}}(\varphi_\alpha)\right)_{\alpha \in A}$ are disjoint. For a countably dense set $(\psi_n)_{n \in \mathbb{N}} \subseteq R$, there exists at least one $\psi_n$ in each $B_{\frac{1}{2}}(\varphi_\alpha)$, hence, there can be at most countably many such balls. Therefore, as the balls are centred on the $\varphi_\alpha$ it follows that the system $(\varphi_\alpha)_{\alpha \in A} \subseteq R$ is at most countable. □

**Theorem 1.1.12.** *Let $(f_n)_{n \in \mathbb{N}}$ be a linearly independent system in a Euclidean space $R$. Then there exists a system $(\varphi_n)_{n \in \mathbb{N}}$ such that the following statements hold.*

1. *$(\varphi_n)_{n \in \mathbb{N}}$ is orthonormal.*
2. *$\varphi_n = a_{n1} f_1 + \cdots + a_{nn} f_n$ where $a_{nk} \in \mathbb{R}$ and $a_{nn} \neq 0$ for $n \in \mathbb{N}$.*
3. *$f_n = b_{n1}\varphi_1 + \cdots + b_{nn}\varphi_n$ where $b_{nk} \in \mathbb{R}$ and $b_{nn} \neq 0$ for $n \in \mathbb{N}$.*

*Proof.* Let $\varphi_1 = a_{11} f_1$ where $a_{11} = \frac{\pm 1}{\sqrt{(f_1, f_1)}}$ and let $b_{11} = \frac{1}{a_{11}}$. Then $\|\varphi_1\| = 1$, $\varphi_n = a_{11} f_1$ and $f_1 = b_{11}\varphi_1$. Now suppose that $\{\varphi_1, \ldots, \varphi_{n-1}\}$ is constructed to satisfy statements 1, 2 and 3. Let

$$b_{nk} = \frac{(f_n, \varphi_k)}{(\varphi_k, \varphi_k)}$$

for $k = 1, \ldots, n-1$. Then letting

$$h_n := f_n - b_{n1}\varphi_1 - \cdots - b_{n,n-1}\varphi_{n-1},$$

it follows by the orthogonality of $\{\varphi_1, \ldots, \varphi_{n-1}\}$ that

$$(h_n, \varphi_k) = (f_n, \varphi_k) - b_{nk}(\varphi_k, \varphi_k) = 0$$

for $k = 1, \ldots, n-1$. Note that $h_n \neq 0$, as otherwise we would contradict the linear independent of $(f_n)_{n \in \mathbb{N}}$, so we let

$$\varphi_n = \frac{h_n}{(h_n, h_n)}.$$

Thus we have

$$f_n = b_{n1}\varphi_1 + \cdots + b_{n,n-1}\varphi_{n-1} + b_{nn}\varphi_n,$$

where $b_{nn} = (h_n, h_n)$. Moreover, using the induction hypothesis we have

$$\varphi_n = a_{n1} f_1 + \cdots + a_{nn} f_n$$

for $a_{nk} \in \mathbb{R}$ and $a_{nn} \neq 0$. Thus we conclude the proof by induction. □

**Remark 1.1.13.**

1. *The system $(\varphi_n)_{n \in \mathbb{N}}$ of Theorem 1.1.12 is unique up to multiplication by $\pm 1$.*
2. *Note that the subspaces produced by $(f_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ coincide, and so these systems are simultaneously complete or incomplete.*

**Corollary 1.1.14.** *A separable Euclidean space $R$ possess an orthonormal basis.*

*Proof.* As $R$ is separable there exists a subset $(\psi_n)_{n\in\mathbb{N}} \subset R$ that is countably dense. Without loss of generality one can assume that $(\psi_n)_{n\in\mathbb{N}}$ is linearly independent by removing elements $\psi_k$ that are represented as linear combinations of $(\psi_i)_{i=1,\dots,k-1}$. Therefore, applying Theorem 1.1.12 we obtain an orthonormal system $(\varphi_n)_{n\in\mathbb{N}} \subset R$ which is additionally an orthonormal basis as $(\psi_n)_{n\in\mathbb{N}} \subset R$ is dense. □

## 1.2 Closed Orthogonal Systems

For an $n$-dimensional Euclidean space $R$ with a basis $(e_j)_{j=1}^n \subset R$, any vector $x \in R$ can be written as

$$x = \sum_{k=1}^{n} c_k e_k$$

for $c_k \in \mathbb{R}$. More specifically, due to the orthogonality of the system $(e_j)_{j=1}^n$, it follows that $c_k = (x, e_k)$. Extrapolating to an infinite-dimensional Euclidean space suppose that $(\varphi_n)_{n\in\mathbb{N}} \subset R$ is an orthogonal system. For $f \in R$ consider the sequence $(c_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ where $c_k = (f, \varphi_k)$ are the Fourier coefficients of $f$ with respect to $(\varphi_n)_{n\in\mathbb{N}}$. The series $\sum_{k=1}^{\infty} c_k\varphi_k$ is thus known as the Fourier series of $f$ with respect to $(\varphi_n)_{n\in\mathbb{N}}$. Consequently, in the infinite-dimensional setting, we are confronted with questions regarding the convergence and subsequent limit of series.

**Proposition 1.2.1.** *Let $R$ be an infinite-dimensional Euclidean space with an orthogonal system $(\varphi_n)_{n\in\mathbb{N}}$. Let $f \in R$. For fixed $n \in \mathbb{N}$, let $(\alpha_k)_{k=1}^n \subset \mathbb{R}$ and $S_n^{(\alpha)} = \sum_{k=1}^n \alpha_k\varphi_k$. Then*

$$\left\| f - S_n^{(\alpha)} \right\| \geq \left\| f - S_n^{(c)} \right\|,$$

*where $S_n^{(c)} = \sum_{k=1}^n c_k\varphi_k$ and $c_k = (f, \varphi_k)$.*

*Proof.* Observe that

$$\begin{aligned}
\left\| f - S_n^{(\alpha)} \right\|^2 &= \left( f - \sum_{k=1}^n \alpha_k f_k, f - \sum_{k=1}^n \alpha_k f_k \right) \\
&= (f, f) - 2\left( f, \sum_{k=1}^n \alpha_k\varphi_k \right) + \left( \sum_{k=1}^n \alpha_k\varphi_k, \sum_{k=1}^n \alpha_k\varphi_k \right) \\
&= (f, f) - 2\sum_{k=1}^n \alpha_k c_k + \sum_{k=1}^n \alpha_k^2 \\
&= \|f\|^2 - \sum_{k=1}^n c_k^2 + \sum_{k=1}^n (\alpha_k - c_k)^2.
\end{aligned}$$

Thus the minimum is achieved when $\alpha_k = c_k$ for $k = 1, \dots, n$. In particular,

$$\left\| f - S_n^{(c)} \right\|^2 = \|f\|^2 - \sum_{k=1}^n c_k^2. \tag{1.2.1}$$

□

**Corollary 1.2.2.** *Let $R$ be an infinite-dimensional Euclidean space with an orthogonal system $(\varphi_n)_{n\in\mathbb{N}}$. Let $f \in R$ and $c_k = (f, \varphi_k)$ for $k \in \mathbb{N}$. Then*

$$\sum_{k=1}^n c_k^2 \leq \|f\|^2$$

*for every $n \in \mathbb{N}$, and so $\sum_{k=1}^{\infty} c_k^2$ converges with*

$$\sum_{k=1}^{\infty} c_k^2 \leq \|f\|^2. \tag{1.2.2}$$

*Proof.* From (1.2.1) it follows that

$$\|f\|^2 - \sum_{k=1}^{n} c_k^2 \geq 0$$

which implies that

$$\sum_{k=1}^{n} c_k^2 \leq \|f\|.$$

Taking the limit as $n \to \infty$ we deduce that

$$\sum_{k=1}^{\infty} c_k^2 \leq \|f\|^2.$$

□

**Remark 1.2.3.** *The inequality* (1.2.2) *is referred to as Bessel's inequality.*

**Exercise 1.2.4.** *With the notation of Proposition 1.2.1, show that $f - S_n^{(\alpha)}$ is orthogonal to $\operatorname{span}(\varphi_1, \ldots, \varphi_n)$ if and only if $\alpha_k = c_k$ for $k = 1, \ldots, n$.*

**Definition 1.2.5.** *An orthogonal system $(\varphi_n)_{n \in \mathbb{N}}$ is closed if for any $f \in R$ we have*

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2, \tag{1.2.3}$$

*where $c_k = (f, \varphi_k)$.*

**Remark 1.2.6.**

1. *Equation* (1.2.3) *is referred to as Parseval's equality.*
2. *With* (1.2.1)*, an orthogonal system being closed is equivalent to the partial sums of the Fourier series for $f \in R$ converging to $f$. That is,*

$$f = \sum_{k=1}^{\infty} c_k \varphi_k.$$

**Theorem 1.2.7.** *In a separable Euclidean space, an orthonormal system is complete if and only if it is closed.*

*Proof.* ($\Leftarrow$). Let $(\varphi_n)_{n \in \mathbb{N}} \subset R$ be a closed orthogonal system. Then the sequence of partial sums $\left(\sum_{k=1}^{n} c_k \varphi_k\right)_{n \in \mathbb{N}}$ where $c_k = (f, \varphi_k)$ converges to $f$. Therefore, linear combinations of $(\varphi_n)_{n \in \mathbb{N}}$ are dense in $R$, that is $(\varphi_n)_{n \in \mathbb{N}}$ is complete.
($\Rightarrow$). Using Lemma 1.1.11 any orthogonal system is countable, thus let $(\varphi_n)_{n \in \mathbb{N}} \subset R$ be a complete orthogonal system. Then every $f \in R$ can be approximated to any precision with a linear combination $\sum_{k=1}^{n} \alpha_k \varphi_k$. By Proposition 1.2.1, the partial sum $\sum_{k=1}^{n} c_k \varphi_k$ of the Fourier series provides no worse an approximation. Therefore, $\sum_{k=1}^{n} c_k f_k \overset{n \to \infty}{\longrightarrow} f$, meaning $(\varphi_n)_{n \in \mathbb{N}}$ is closed. □

**Example 1.2.8.** *Using Example 1.1.10 and Theorem 1.2.7, the orthonormal systems of Example 1.1.7 are closed.*

Fourier coefficients can be generalised to non-normalised systems. That is, let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthogonal system. One can the consider the normalised system $(\psi_n)_{n\in\mathbb{N}}$, where $\psi_n = \frac{\varphi_k}{\|\varphi_k\|}$. For any $f \in R$ we have

$$c_k = (f, \psi_k) = \frac{1}{\|\varphi_k\|}(f, \varphi_k).$$

Thus,

$$f = \sum_{k=1}^{\infty} c_k \psi_k = \sum_{k=1}^{\infty} a_k \varphi_k$$

where $a_k = \frac{(f,\varphi_k)}{\|\varphi_k\|^2}$. This is the Fourier series of $f$ with respect to $(\varphi_n)_{n\in\mathbb{N}}$, and using (1.2.2) it follows that

$$\sum_{k=1}^{\infty} a_k^2 \|\varphi_k\|^2 \leq \|f\|^2.$$

## 1.3 Complete Euclidean Spaces

**Definition 1.3.1.** *A complete Euclidean space $R$ is such that every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset R$ converges to some $x \in R$.*

For a sequence $(c_k)_{k\in\mathbb{N}} \subset R$ to be the Fourier coefficients of $f \in R$, it is necessary that $\sum_{k=1}^{\infty} c_k^2$ converges. If $R$ is a complete Euclidean space then the convergence of $\sum_{k=1}^{\infty} c_k^2$ is also sufficient to conclude that $(c_k)_{k\in\mathbb{N}}$ are the Fourier coefficients of $f \in R$.

**Theorem 1.3.2** (Riesz)**.** *Let $R$ be a complete Euclidean space and let $(\varphi_n)_{n\in\mathbb{N}} \subset R$ be an orthonormal system. Let $(c_k)_{k\in\mathbb{N}}$ be such that $\sum_{k=1}^{\infty} c_k^2$ converges. Then there is a $f \in R$ such that $c_k = (f, \varphi_k)$ and*

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

*Proof.* Let $f_n = \sum_{k=1}^{n} c_k \varphi_k$. Then by orthonormality it follows that $c_k = (f_n, \varphi_k)$ for $k = 1, \ldots, n$. Observe that

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}\varphi_{n+1} + \cdots + c_{n+p}\varphi_{n+p}\|^2 = \sum_{k=n+1}^{n+p} c_k^2.$$

Since, $\sum_{k=1}^{\infty} c_k^2$ converges it follows that $(f_n)_{n\in\mathbb{N}}$ is Cauchy. As $R$ is complete the sequence $(f_n)_{n\in\mathbb{N}}$ converges to some $f \in R$. Note that for $n \geq k$ we have

$$\begin{aligned} (f - f_n, \varphi_k) &\leq |(f - f_n, \varphi_k)| \\ &\leq \|f - f_n\| \|\varphi_k\| \\ &\overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

Therefore,

$$\begin{aligned} (f, \varphi_k) &\overset{n\geq k}{=} (f_n, \varphi_k) + (f - f_n, \varphi_k) \\ &= c_k + (f - f_n, \varphi_k) \\ &\overset{n\to\infty}{\longrightarrow} c_k + 0 \end{aligned}$$

for every $k \in \mathbb{N}$. Moreover,

$$\begin{aligned}
\|f\|^2 - \sum_{k=1}^{n} c_k^2 &= (f, f) - 2\sum_{k=1}^{n} c_k (f, \varphi_k) + \sum_{k=1}^{n} c_k^2 \\
&= \left( f - \sum_{k=1}^{n} c_k \varphi_k, f - \sum_{k=1}^{n} c_k \varphi_k \right) \\
&= \|f - f_n\|^2 \\
&\overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which implies that $\|f\|^2 = \sum_{k=1}^{\infty} c_k^2$. □

**Definition 1.3.3.** *A complete infinite-dimensional Euclidean space is known as a Hilbert space.*

**Remark 1.3.4.** *Euclidean spaces are isomorphic if there exists a bijective mapping between the spaces that preserves linear operations and the inner product. Finite-dimensional Euclidean spaces are isomorphic to $\mathbb{R}^n$ with*

$$(x, y) = \sum_{j=1}^{n} x_j y_j,$$

*and thus we only use Hilbert spaces to refer to infinite-dimensional spaces. Infinite-dimensional Euclidean spaces are not necessarily isomorphic.*

**Exercise 1.3.5.** *Show that $\ell^2$ and $\mathcal{C}_2([-\pi, \pi])$ are not isomorphic as Euclidean spaces, as $\ell^2$ is complete whereas $\mathcal{C}_2([-\pi, \pi])$ is not complete.*

**Proposition 1.3.6.** *Let $H$ be a separable Hilbert space. Then an orthonormal system $(\varphi_n)_{n\in\mathbb{N}} \subseteq H$ is complete if and only if there is no nonzero element in $H$ which is orthogonal to $\varphi_n$ for every $n \in \mathbb{N}$.*

*Proof.* ($\Rightarrow$). For $\varphi \in R \setminus \{0\}$, as $(\varphi_n)_{n\in\mathbb{N}} \subset R$ is complete it is in particular closed, by Theorem 1.2.7, and so

$$\|\varphi\|^2 = \sum_{k=1}^{\infty} c_k^2$$

where $c_k = (\varphi, \varphi_k)$. As $\varphi \neq 0$ it must be the case that $\|\varphi\|^2 > 0$, and so as $c_k^2 \geq 0$ there must exist some $k \in \mathbb{N}$ such that $c_k > 0$ which means that $c_k = (\varphi, \varphi_k) \neq 0$.
($\Leftarrow$). Suppose $(\varphi_n)_{n\in\mathbb{N}}$ were not complete, then, in particular, it is not closed, by Theorem 1.2.7, and so there exists a $\varphi \in R$ such that $\|\varphi\|^2 \neq \sum_{k=1}^{\infty} c_k^2$ where $c_k = (\varphi, \varphi_k)$. However, by (1.2.2) the series $\sum_{k=1}^{\infty} c_k^2 < \infty$ and so by Theorem 1.3.2 there exists a $\tilde{\varphi} \in R$ such that

$$\|\tilde{\varphi}\|^2 = \sum_{k=1}^{\infty} c_k^2.$$

with $c_k = (\tilde{\varphi}, \varphi_k)$. Thus, $(\tilde{\varphi}, \varphi_k) = (\varphi, \varphi_k)$ which implies that $(\tilde{\varphi} - \varphi, \varphi_k) = 0$ for all $k \in \mathbb{N}$. However, as $\|\tilde{\varphi}\| \neq \|\varphi\|$ we have $\tilde{\varphi} - \varphi \in R \setminus \{0\}$ which contradicts the assumption that no nonzero vector in $R$ exists that is orthogonal to $\varphi_k$ for all $k \in \mathbb{N}$. □

**Theorem 1.3.7.** *Separable Hilbert spaces are isomorphic.*

*Proof.* Let $H$ be a Hilbert space with $(\varphi_n)_{n\in\mathbb{N}} \subseteq H$ a complete orthonormal system, which is countable by Lemma 1.1.11. Let $\Phi : H \to \ell^2$ be the correspondence of $f \in H$ with its Fourier coefficients $(c_1, c_2, \dots)$. Since $\sum_{k=1}^{\infty} c_k^2 < \infty$, by (1.2.2), the correspondence $\Phi$ is well-defined. Using Theorem 1.3.2 for any $(c_1, c_2, \dots) \in \ell^2$ there is a $f \in H$ such that $\Phi(f) = (c_1, c_2, \dots)$. In particular, $\Phi$ provides a bijective correspondence. Furthermore, suppose $\Phi(f) = (c_k)_{k\in\mathbb{N}}$ and $\Phi(g) = (d_k)_{k\in\mathbb{N}}$. Then $\Phi(\lambda f) = (\lambda c_k)_{k\in\mathbb{N}} = \lambda\Phi(f)$ and $\Phi(f+g) = (c_k + d_k)_{k\in\mathbb{N}}$. Thus, $\|f+g\|^2 = \sum_{k=1}^{\infty}(c_k + d_k)^2$ using (1.2.3). Along with $\|f\|^2 = \sum_{k=1}^{\infty} c_k^2$ and $\|g\|^2 = \sum_{k=1}^{\infty} d_k^2$ it follows that

$$\begin{aligned}
\sum_{k=1}^{\infty} c_k^2 + 2\sum_{k=1}^{\infty} c_k d_k + \sum_{k=1}^{\infty} d_k^2 &= \sum_{k=1}^{\infty}(c_k + d_k)^2 \\
&= \|f+g\|^2 \\
&= (f+g, f+g) \\
&= \|f\|^2 + 2(f,g) + \|g\|^2 \\
&= \sum_{k=1}^{\infty} c_k^2 + 2(f,g) + \sum_{k=1}^{\infty} d_k^2,
\end{aligned}$$

which implies that $(f,g) = \sum_{k=1}^{\infty} c_k d_k$. Therefore, $\Phi$ is a bijection that is linear and preserves the inner product meaning it is an isomorphism between the spaces $H$ and $\ell^2$. As $H$ is arbitrary and $\ell^2$ is fixed, this is sufficient to show that any Hilbert spaces are isomorphic. $\square$

**Remark 1.3.8.** *From the proof of Theorem 1.3.7 we see that $\ell^2$ plays the same role for separable Hilbert spaces as $\mathbb{R}^n$ does for finite-dimensional Euclidean spaces.*

One can complete a Hilbert space to obtain separable Hilbert spaces. The completion of the Hilbert space $\mathcal{C}_2([-\pi,\pi])$ is $L^2([-\pi,\pi])$. Where $L^2([-\pi,\pi])$ is the space of equivalence classes, with respect to the Lebesgue measure, of real-valued functions, $f$, on $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} |f|^2 \,\mathrm{d}t < \infty.$$

The inner product on $L^2([-\pi,\pi])$ is given by

$$(f,g) = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t.$$

## 1.4 Complex Euclidean Spaces

A complex Euclidean space, $R$, is a linear space over $\mathbb{C}$, with a modified inner product. More specifically, a map $(\cdot,\cdot) : R \times R \to \mathbb{C}$ is an inner product over $\mathbb{C}$ if $(x,y) = \overline{(y,x)}$ and satisfies statements 2, 3 and 4 of Definition 1.1.1. It is important to observe that with this modification an inner product on $\mathbb{C}$ is no longer bilinear. More specifically, it is not linear in the second argument as

$$(x, \lambda y) = \bar{\lambda}(x,y)$$

for $x, y \in R$ and $\lambda \in \mathbb{C}$.

**Example 1.4.1.**

1. *The $n$-dimensional $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ with inner product*

$$(x,y) = \sum_{j=1}^{n} x_j \bar{y}_j$$

*is the $n$-dimensional Euclidean space over $\mathbb{C}$.*

2. *The space of sequences $x = (x_1, x_2, \dots)$ where $x_k \in \mathbb{C}$ and $\sum_{k=1}^{\infty} |x_k|^2 < \infty$, denoted $\ell^2$, with inner product*

$$(x, y) = \sum_{k=1}^{\infty} x_j \bar{y}_j$$

*is a complex Euclidean space.*

3. *The space of complex valued continuous functions on $[-\pi, \pi]$, denoted $\mathcal{C}_2([-\pi, \pi])$ with inner product*

$$(f, g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t$$

*is a complex Euclidean space.*

As for real Euclidean spaces, for $f \in R$, where $R$ is a complex Euclidean space, we can construct its Fourier series with respect to an orthogonal system $(\varphi_n)_{n\in\mathbb{N}}$ as

$$\sum_{k=1}^{\infty} a_k \varphi_k$$

where $a_k = \frac{(f,\varphi_k)}{\|\varphi_k\|^2}$ for $k \in \mathbb{N}$. The analogue of (1.2.2) for complex Euclidean spaces is

$$\sum_{k=1}^{\infty} \|\varphi_k\|^2 |a_k|^2 \leq \|f\|^2.$$

Results shown for real Euclidean statements have analogous formulations for complex Euclidean spaces, with only slight modifications.

## 1.5 Solution to Exercises

### Exercise 1.1.4

*Solution.* For any $n \in \mathbb{N}$ let $\left\{x_{\alpha(1)}, \dots, x_{\alpha(n)}\right\} \subset (x_\alpha)_{\alpha\in A}$ and suppose that

$$a_1 x_{\alpha(1)} + \cdots + a_n x_{\alpha(n)} = 0.$$

Then by orthogonality it follows that

$$0 = \left(x_{\alpha(1)}, a_1 x_{\alpha(1)} + \cdots + a_n x_{\alpha(n)}\right) = a_1 \left\|x_{\alpha(1)}\right\|^2.$$

As $x_{\alpha(1)} \neq 0$ it follows that $a_1 = 0$. More generally, $a_k = 0$ for $k = 1, \dots, n$. Therefore, $\left\{x_{\alpha(1)}, \dots, x_{\alpha(n)}\right\}$ is linearly independent which implies that $(x_\alpha)_{\alpha\in A}$ is linearly independent. ☐

### Exercise 1.1.8

*Solution.* For $n \in \mathbb{N}$ we have

$$\begin{aligned}(1, \cos(nt)) &= \int_{-\pi}^{\pi} \cos(nt)\,\mathrm{d}t \\ &= \left[\frac{1}{n}\sin(nt)\right]_{-\pi}^{\pi} \\ &= 0 - 0 \\ &= 0.\end{aligned}$$

For $n \in \mathbb{N}$ we have

$$\begin{aligned}(1, \sin(nt)) &= \int_{-\pi}^{\pi} \sin(nt)\,\mathrm{d}t \\ &= \left[-\frac{1}{n}\cos(nt)\right]_{-\pi}^{\pi} \\ &= \frac{1}{n} - \frac{1}{n} \\ &= 0.\end{aligned}$$

For $n \in \mathbb{N}$ we have

$$\begin{aligned}(\cos(nt), \sin(nt)) &= \int_{-\pi}^{\pi} \cos(nt)\sin(nt)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2nt)\,\mathrm{d}t \\ &= 0.\end{aligned}$$

For $n, m \in \mathbb{N}$ with $n \neq m$ we have

$$\begin{aligned}(\cos(nt), \sin(mt)) &= \int_{-\pi}^{\pi} \cos(nt)\sin(mt)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \sin((n+m)t) - \sin((n-m)t)\,\mathrm{d}t \\ &= 0.\end{aligned}$$

For $n, m \in \mathbb{N}$ for $n \neq m$ we have

$$\begin{aligned}(\cos(nt), \cos(mt)) &= \int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)t) + \cos((n-m)t)\,\mathrm{d}t \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}(\sin(nt), \sin(mt)) &= \int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)t) - \cos((n+m)t)\,\mathrm{d}t \\ &= 0.\end{aligned}$$

Thus the system is orthogonal. For $f \in \mathcal{C}_2(-\pi, \pi)$ if $f(-\pi) = f(\pi)$, then $f$ is a continuous and period function and so by Corollary 2.3.4 it follows that $f$ is the limit of a uniformly convergent sequence of functions in the trigonometric system. On the other hand, if $f(-\pi) \neq f(\pi)$ let $\epsilon > 0$ and consider $(g_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_2([-\pi, \pi])$ be given by

$$g_n(x) = \begin{cases} f(x) & x \in \left[-\pi, \pi - \frac{1}{n}\right) \\ f\left(x - \frac{1}{n}\right) + n\left(f(-\pi) - f\left(\pi - \frac{1}{n}\right)\right)\left(x - \pi + \frac{1}{n}\right) & x \in \left[\pi - \frac{1}{n}, \pi\right]. \end{cases}$$

That is, $g_n(x)$ coincides with $f(x)$ on $\left[-\pi, \pi - \frac{1}{n}\right]$ and then consists of a straight line segment such that $g_n(-\pi) = g_n(\pi)$. Thus, it is clear that $\|g_n - f\|_{\mathcal{C}_2([-\pi,\pi])} \to 0$ as $n \to \infty$. In particular, there exists a $N \in \mathbb{N}$ such that

$$\|g_n - f\|_{\mathcal{C}_2([-\pi,\pi])} < \frac{\epsilon}{2}$$

for $n \geq N$. Applying Corollary 2.3.4 to $g_n(x)$, there exists a sequence of trigonometric polynomials $\left(t_m^{(n)}\right)_{m\in\mathbb{N}}$ such that $\left\|t_m^{(n)} - g_n\right\| \to 0$ as $m \to \infty$. In particular, there exists an $m \geq M$ such that

$$\left\|t_m^{(n)} - g_n\right\|_{\mathcal{C}_2([-\pi,\pi])} < \frac{\epsilon}{2}.$$

Therefore, for $n \geq N$ it follows that

$$\left\|t_n^{(n)} - f\right\|_{\mathcal{C}_2([-\pi,\pi])} \leq \left\|t_n^{(n)} - g_n\right\|_{\mathcal{C}_2([-\pi,\pi])} + \|g_n - f\|_{\mathcal{C}_2([-\pi,\pi])} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, the trigonometric sequence $\left(t_n^{(n)}\right)_{n\in\mathbb{N}}$ converges to $f$. Therefore, the system

$$\{1\} \cup \{\cos(nt)\}_{n\in\mathbb{N}} \cup \{\sin(nt)\}_{n\in\mathbb{N}}$$

is complete and thus a basis of $\mathcal{C}_2([-\pi,\pi])$. $\square$

**Exercise 1.2.4**

*Solution.* For $S_n^{(\alpha)} = \sum_{k=1}^n \alpha_k \varphi_k$ it is clear that $f - S_n^{(\alpha)}$ is orthogonal to $\operatorname{span}(\varphi_1, \ldots, \varphi_n)$ if and only if $\left(f - S_n^{(\alpha)}, \varphi_k\right) = 0$ for each $k = 1, \ldots, n$ which is equivalent to $(f, \varphi_k) - \alpha_k \|\varphi_k\|^2 = 0$, that is $\alpha_k = (f, \varphi_k) = c_k$. $\square$

**Exercise 1.3.5**

*Solution.* Let $f_n : [-\pi,\pi] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & x \in \left[\frac{1}{n}, \pi\right] \\ nx & x \in \left(\frac{1}{n}, \frac{1}{n}\right) \\ -1 & x \in \left[-\pi, \frac{1}{n}\right]. \end{cases}$$

Note that for $m > n$ it follows that

$$\|f_m - f_n\|^2 = \int_{\frac{1}{n}}^{-\frac{1}{n}} |mx - nx| \, \mathrm{d}x = \frac{m-n}{n^2} \overset{n\to\infty}{\longrightarrow} 0,$$

which means that $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_2([-\pi,\pi])$ is Cauchy. Moreover, observe that

$$f_n(x) \overset{n\to\infty}{\longrightarrow} f(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

pointwise. Suppose that $f_n \to \varphi$ in $\mathcal{C}_2([-\pi,\pi])$. Suppose that $\varphi(0) > 0$. As $\varphi$ is continuous there exists a $\delta > 0$ such that $\varphi(x) > \frac{\varphi(0)}{2} > 0$ for $x \in (-\delta, \delta)$. Then for any $n \geq 1$ it follows that

$$\|f_n - \varphi\|^2 \geq \int_{-\delta}^{0} (f_n(x) - \varphi(x))^2 \, \mathrm{d}x > \left(\frac{\varphi(0)}{2}\right)^2 \delta > 0.$$

Therefore, $f_n \not\to \varphi$ in $\mathcal{C}_2([-\pi,\pi])$ if $\varphi(0) > 0$. Similar arguments shows that $\varphi(0) \not< 0$ and so $\varphi(0) = 0$. However, in such a case there exists a $\delta > 0$ such that $\varphi(x) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ for $x \in (-\delta, \delta)$. Which means that for $n \geq N$ where $N \in \mathbb{N}$ is such that $\frac{1}{N} < \delta$, it follows that

$$\|f_n - \varphi\|^2 \geq \int_{\frac{1}{n}}^{\delta} (f_n(x) - \varphi(x))^2 \, \mathrm{d}x \geq \frac{1}{2}\left(\delta - \frac{1}{n}\right) > \frac{1}{2}\frac{\delta}{2} > 0.$$

Therefore, $f_n \not\to \varphi$ in $\mathcal{C}_2([-\pi,\pi])$. Thus we conclude that $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_2([-\pi,\pi])$ is a Cauchy sequence that does not converge and so $\mathcal{C}_2([-\pi,\pi])$ is not complete. $\square$

# 2 Trigonometric Series

The space $L^2([-\pi,\pi])$ consists of all functions over $[-\pi,\pi]$ for which

$$\int_{-\pi}^{\pi} |f(t)|^2 \,\mathrm{d}t < \infty.$$

With the inner product

$$(f,g) = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t,$$

this space is a real Euclidean space. From Exercise 1.1.8 it follows that

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

is an orthogonal system of $L^2([-\pi,\pi])$. Moreover, assuming the conditions of Exercise 1.1.8 and the fact that $\mathcal{C}_2([-\pi,\pi])$ is a dense subset of $L^2([-\pi,\pi])$, it follows that the system is also complete. The corresponding orthonormal system is given by

$$\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos(nx)\right\}_{n\in\mathbb{N}} \cup \left\{\frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}.$$

## 2.1 Fourier Series

For $f \in L^2([-\pi,\pi])$, its Fourier series is given by

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx)), \tag{2.1.1}$$

where $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x$ for $k = 0,1,\dots$ and $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x$ for $k = 1,2,\dots$. Recall from Proposition 1.2.1 that the partial sum of (2.1.1) provides the best $L^2$-approximation of $f$ amongst trigonometric polynomials of the form

$$\alpha_0 + \sum_{k=1} \alpha_k\cos(kx) + \beta_k\sin(kx).$$

As the system is complete we have $\|S_n - f\| \to 0$ as $n \to \infty$ and so

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,\mathrm{d}x.$$

We can equally consider $L^2([-\pi,\pi])$ as a complex Euclidean space. In this space, we have the orthonormal basis $\left(e^{inx}\right)_{n\in\mathbb{Z}}$. Thus, the Fourier series for $f \in L^2([-\pi,\pi])$ is

$$\sum_{n\in\mathbb{Z}} c_n e^{inx},$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x$ for $n \in \mathbb{Z}$. Although we have seen that the Fourier series of $f$ converges in $L^2$, this does not provide the convergence of the series at specific points. To understand what guarantees are required for the Fourier series of $f$ at $x$ to converge to $f(x)$ for a given $x$, it will be more productive to consider $L^2([-\pi,\pi])$ as a real Euclidean space.

**Remark 2.1.1.**

1. *Note that a function on $[-\pi,\pi]$ can be extended to a $2\pi$ periodic function on $\mathbb{R}$.*

2. *As $\cos(nx)$ and $\sin(nx)$ are bounded functions, the coefficients $a_k$ and $b_k$ exist for functions even in $L^1([-\pi,\pi])$. Recall that $L^2([-\pi,\pi]) \subseteq L^1([-\pi,\pi])$.*

**Exercise 2.1.2.** *For $l > 0$, show that*

$$\left\{\frac{1}{\sqrt{2l}}\right\} \cup \left\{\frac{1}{\sqrt{l}}\cos\left(\frac{n\pi}{l}x\right)\right\}_{n\in\mathbb{N}} \cup \left\{\frac{1}{\sqrt{l}}\sin\left(\frac{n\pi}{l}x\right)\right\}_{n\in\mathbb{N}}$$

*is an orthonormal system of $L^2(-l,l)$. Moreover, show that the Fourier series for $f \in L^2(-l,l)$ with respect to this system is*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{k\pi}{l}x\right) + b_k\sin\left(\frac{k\pi}{l}x\right)\right)$$

*where $a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi}{l}x\right)\,\mathrm{d}x$ and $b_k = \frac{1}{l}\int_{-l}^{l}\sin\left(\frac{k\pi}{l}x\right)\,\mathrm{d}x$.*

## 2.2 From Functions to Fourier Series

Let

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos(kx) + b_k\sin(kx)\right) \tag{2.2.1}$$

be the partial Fourier series of a function $f \in L^2([-\pi,\pi])$ at a point $x$.

**Exercise 2.2.1.** *Show that*

$$\frac{1}{2} + \sum_{k=1}^{n}\cos(ku) = \frac{\sin\left(\frac{2n+1}{2}u\right)}{2\sin\left(\frac{u}{2}\right)}$$

*and*

$$\sum_{k=1}^{n}\sin(ku) = \frac{\sin\left(\frac{n+1}{2}u\right)\sin\left(\frac{n}{2}u\right)}{\sin\left(\frac{u}{2}\right)}.$$

**Proposition 2.2.2.** *For $f \in L^2([-\pi,\pi])$ and $x \in [-\pi,\pi]$ we have*

$$S_n(x) = \int_{-\pi}^{\pi} f(x+z)D_n(z)\,\mathrm{d}z$$

*where*

$$D_n(z) = \frac{1}{2\pi}\frac{\sin\left(\frac{2n+1}{2}z\right)}{\sin\left(\frac{z}{2}\right)}$$

*is the Dirichlet Kernel.*

*Proof.* For $S_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))$ substitute in the formulas for $a_k$ and $b_k$ to obtain

$$\begin{aligned} S_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}(\cos(kx)\cos(kt) + \sin(kx)\sin(kt))\right)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x))\right)\,\mathrm{d}t. \end{aligned}$$

Note that the order of integration can be interchanged as the sums are finite. Using Exercise 2.2.1 it follows that

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin\left(\frac{2n+1}{2}(t-x)\right)}{2\sin\left(\frac{t-x}{2}\right)}.$$

Letting $z = t - x$ we have

$$S_n(x) = \int_{-\pi-x}^{\pi-x} f(x+z) \frac{1}{2\pi} \frac{\sin\left(\frac{2n+1}{2}z\right)}{\sin\left(\frac{z}{2}\right)} \, \mathrm{d}z \overset{(1)}{=} \int_{-\pi}^{\pi} f(x+z) D_n(z) \, \mathrm{d}z$$

where in $(1)$ we have used the fact that the integrand in $2\pi$-periodic. ☐

**Remark 2.2.3.** *Note that by Exercise 2.2.1 we have*

$$\begin{aligned} \int_{-\pi}^{\pi} D_n(z) \, \mathrm{d}z &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} + \sum_{k=1}^{n} \cos(kz) \, \mathrm{d}z \\ &= \frac{1}{2\pi} \left[ \frac{z}{2} + \sum_{k=1}^{n} \frac{1}{k} \sin(ku) \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}(2\pi) \\ &= 1. \end{aligned}$$

*Therefore, we can write*

$$S_n(x) - f(x) = \int_{-\pi}^{\pi} (f(x+z) - f(x)) D_n(z) \, \mathrm{d}z.$$

*Consequently, questions of convergence regarding the Fourier series at $x$ can be answered by studying the convergence property of the integral on the right-hand side.*

**Exercise 2.2.4.** *Show that*

$$\int_{-\pi}^{\pi} |D_n(z)| \, \mathrm{d}z = \frac{4}{\pi^2} \log(n) + O(1).$$

**Lemma 2.2.5.** *If $\varphi(x)$ is integrable on $[a,b]$ then*

$$\int_a^b \varphi(x) \sin(\gamma x) \, \mathrm{d}x \overset{\gamma \to \infty}{\longrightarrow} 0$$

*and*

$$\int_a^b \varphi(x) \cos(\gamma x) \, \mathrm{d}x \overset{\gamma \to \infty}{\longrightarrow} 0.$$

*Proof.* If $\varphi(x)$ is continuously differentiable, then we can integrate by parts to deduce that

$$\int_a^b \varphi(x) \sin(\gamma x) \, \mathrm{d}x = \left[ -\varphi(x) \frac{\cos(\gamma x)}{\gamma} \right]_a^b + \int_a^b \varphi'(x) \frac{\cos(\gamma x)}{\gamma} \, \mathrm{d}x \overset{\gamma \to \infty}{\longrightarrow} 0. \tag{2.2.2}$$

For $\varphi \in L^1([a,b])$ and $\epsilon > 0$, since continuously differentiable functions are dense in $L^1([a,b])$, there exists a continuously differentiable $\varphi_\epsilon$ such that

$$\int_a^b |\varphi(x) - \varphi_\epsilon(x)| \, \mathrm{d}x < \frac{\epsilon}{2}.$$

By (2.2.2), there exists a $\gamma_0$ such that for $\gamma > \gamma_0$ we have

$$\left| \int_a^b \varphi_\epsilon(x) \sin(\gamma x) \, \mathrm{d}x \right| < \frac{\epsilon}{2}.$$

Consequently for $\gamma > \gamma_0$ it follows that,

$$\begin{aligned}
\left|\int_a^b \varphi(x)\sin(\gamma x)\,\mathrm{d}x\right| &\leq \left|\int_a^b (\varphi(x)-\varphi_\epsilon(x))\sin(\gamma x)\,\mathrm{d}x\right| + \left|\int_a^b \varphi_\epsilon(x)\sin(\gamma x)\,\mathrm{d}x\right| \\
&\leq \frac{\epsilon}{2}+\frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Therefore,

$$\int_a^b \varphi(x)\sin(\gamma x)\,\mathrm{d}x \overset{\gamma\to\infty}{\longrightarrow} 0.$$

Similarly, one deduces that

$$\int_a^b \varphi(x)\cos(\gamma x)\,\mathrm{d}x \overset{\gamma\to\infty}{\longrightarrow} 0.$$

□

**Corollary 2.2.6.** *If $f \in L^1([-\pi,\pi])$ then its Fourier coefficients are such that $a_k, b_k \to 0$ as $k \to \infty$.*

*Proof.* Take $a = -\pi$, $b = \pi$ and $\gamma = k$ in Lemma 2.2.5. □

**Remark 2.2.7.** *If $\varphi \in \mathcal{C}^k([-\pi,\pi])$ then one can integrate by parts $k$-times to get that*

$$\int_a^b \varphi(x)\sin(\gamma x)\,\mathrm{d}x = O\left(\frac{1}{\gamma^k}\right).$$

*Thus the smoother a periodic function $f$ is, the faster its Fourier coefficients decay at $\infty$.*

**Exercise 2.2.8.** *Suppose $f$ is a $2\pi$ periodic and complex analytic function. Show that its Fourier coefficients exponentially decay.*

**Example 2.2.9.**

1. *Let*

$$f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & 1 \leq |x| \leq \pi. \end{cases}$$

*As $f(x)$ is an even function $b_k = 0$ for all $k \in \mathbb{N}$. On the other hand, for $k \geq 1$ we have*

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^1 (1-x)\cos(kx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\frac{1-\cos(k)}{k^2}\right)
\end{aligned}$$

*and for $k = 0$ we have*

$$a_0 = \frac{2}{\pi}\int_0^1 1-x\,\mathrm{d}x = \frac{2}{\pi}\left(\frac{1}{2}\right).$$

*Therefore, the Fourier series of $f(x)$ is*

$$\frac{1}{2\pi} + \sum_{k=1}^{\infty} \frac{2}{\pi} \left( \frac{1-\cos(k)}{k^2} \right) \cos(kx).$$

*Note that we can argue that this series converges for every $x$ as its terms are of order $\frac{1}{k^2}$.*

2. *Let*

$$g(x) = \begin{cases} 1 & -1 < x \leq 0 \\ -1 & 0 < x < 1 \\ 0 & 1 \leq |x| \leq \pi. \end{cases}$$

*As $g(x)$ is an odd function $a_k = 0$ for all $k \in \mathbb{N}$. Otherwise,*

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(kx)\, \mathrm{d}x \\ &= -\frac{2}{\pi} \int_0^1 \sin(kx)\, \mathrm{d}x \\ &= -\frac{2}{\pi} \left( -\frac{1}{k} \cos(k) + \frac{1}{k} \right) \\ &= \frac{2}{\pi} \left( \frac{\cos(k) - 1}{k} \right). \end{aligned}$$

*Therefore, the Fourier series of $g(x)$ is*

$$\sum_{k=1}^{\infty} \frac{2}{\pi} \left( \frac{\cos(k)-1}{k} \right) \sin(kx).$$

*In it is not clear that the series must converge. Instead, we will see later using Corollary 2.2.14 that converges for every $x$.*

**Exercise 2.2.10.** *Find the Fourier coefficients of $f(\theta) = \log\left(\left|2\sin\left(\frac{\theta}{2}\right)\right|\right)$.*

**Theorem 2.2.11.** *Let $f \in L^1(-\pi, \pi)$ be such that for a fixed $x$ and $\delta > 0$ we have*

$$\int_{-\delta}^{\delta} \left| \frac{f(x+t) - f(x)}{t} \right| \mathrm{d}t < \infty, \tag{2.2.3}$$

*then $S_n(x) \to f(x)$ as $n \to \infty$.*

*Proof.* Using Remark 2.2.3 observe that

$$\begin{aligned} S_n(x) - f(x) &= \int_{-\pi}^{\pi} (f(x+z) - f(x)) D_n(z)\, \mathrm{d}z \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x+z)-f(x)}{z} \frac{z}{\sin\left(\frac{z}{2}\right)} \sin\left(\frac{2n+1}{2} z\right) \mathrm{d}z. \end{aligned}$$

From (2.2.3) and the fact that $f \in L^1(-\pi, \pi)$, it follows that $\frac{f(x+z)-f(x)}{z}$ is integrable over $[-\pi, \pi]$. Therefore, $\frac{f(x+z)-f(x)}{z} \frac{z}{\sin\left(\frac{z}{2}\right)}$ is integrable over $[-\pi, \pi]$ and so applying Lemma 2.2.5 it follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x+z)-f(x)}{z} \frac{z}{\sin\left(\frac{z}{2}\right)} \sin\left(\frac{2n+1}{2} z\right) \mathrm{d}z \overset{n \to \infty}{\longrightarrow} 0.$$

Hence,

$$S_n(x) \overset{n\to\infty}{\longrightarrow} f(x).$$

□

**Remark 2.2.12.** *Equation* (2.2.3) *is known as Dini's condition. In particular, Dini's condition holds if $f$ is continuous at $x$ and the left or right derivative of $f$ at $x$ exists.*

**Theorem 2.2.13.** *Let $f \in L^1(-\pi,\pi)$ be such that for a fixed $x$ and $\delta > 0$ we have*

$$\int_{-\delta}^{0} \left| \frac{f(x+t) - f(x^-)}{t} \right| \mathrm{d}t < \infty \tag{2.2.4}$$

*and*

$$\int_{0}^{\delta} \left| \frac{f(x+t) - f(x^+)}{t} \right| \mathrm{d}t < \infty, \tag{2.2.5}$$

*then $S_n \to \frac{1}{2}\left(f(x^+) + f(x^-)\right)$ as $n \to \infty$.*

*Proof.* Using Remark 2.2.3 note that

$$S_n(x) - \frac{f(x^+) + f(x^-)}{2} = \int_{-\pi}^{0} \left( \frac{f(x+z) - f(x^-)}{2} \right) D_n(z)\,\mathrm{d}z + \int_{0}^{\pi} \left( \frac{f(x+z) - f(x^+)}{2} \right) D_n(z)\,\mathrm{d}z.$$

Then using on (2.2.4) and (2.2.5) with the fact that $f \in L^1(-\pi,\pi)$ it follows that $\frac{f(x+z)-(z^-)}{2}$ is integrable over $[-\pi,0]$ and $\frac{f(x+z)-f(x^+)}{2}$ is integrable over $[0,\pi]$. Consequently, $\frac{f(x+z)-(z^-)}{2}\frac{z}{\sin\left(\frac{z}{2}\right)}$ is integrable over $[-\pi,0]$ and $\frac{f(x+z)-f(x^+)}{2}\frac{z}{\sin\left(\frac{z}{2}\right)}$ is integrable over $[0,\pi]$. Therefore, applying Lemma 2.2.5 it follows that

$$\int_{-\pi}^{0} \left( \frac{f(x+z) - f(x^-)}{2} \right) D_n(z)\,\mathrm{d}z + \int_{0}^{\pi} \left( \frac{f(x+z) - f(x^+)}{2} \right) D_n(z)\,\mathrm{d}z \overset{n\to\infty}{\longrightarrow} = 0.$$

Hence,

$$S_n \overset{n\to\infty}{\longrightarrow} \frac{f(x^+) + f(x^-)}{2}.$$

□

**Corollary 2.2.14.** *Let $f$ be a bounded, $2\pi$-periodic function with discontinuities only of the first kind, that is $f(x^-)$ and $f(x^+)$ exist. Moreover, suppose that the left and right derivatives exist at each point. Then*

$$S_n(x) \to \begin{cases} f(x) & x \text{ is a point of continuity,} \\ \frac{f(x^+)+f(x^-)}{2} & x \text{ is a point of discontinuity.} \end{cases}$$

*Proof.* Note that as $f$ is bounded we have $f \in L^1(-\pi,\pi)$. Moreover, as the left and right derivatives of $f$ exist condition (2.2.3) is satisfied, and thus so are (2.2.4) and (2.2.5). Therefore, we can conclude by applying Theorem 2.2.11 at points of continuity and applying Theorem 2.2.13 at points of discontinuity. □

**Corollary 2.2.15.** *A continuous $2\pi$ periodic function is uniquely characterised by its Fourier coefficients.*

*Proof.* Let $f$ and $g$ be $2\pi$ periodic continuous functions with the same Fourier coefficients. It follows that the partial sum of the Fourier coefficients, $S_n(x)$, for $f-g$ is zero. Hence, as $f-g$ is still a $2\pi$ periodic continuous functions it follows by Corollary 2.2.14 that

$$(f-g)(x) = \lim_{n\to\infty} S_n(x) = 0.$$

Therefore, $f(x) = g(x)$. $\square$

**Remark 2.2.16.** *Note that* $D_n(z) = \frac{1}{2\pi}\frac{\sin\left(\frac{2n+1}{2}z\right)}{\sin\left(\frac{z}{2}\right)}$ *converges to* $\frac{2n+1}{2\pi}$ *as* $z \to 0$. *Moreover, the graph of* $D_n(z)$ *oscillates with higher frequency as* $n$ *gets larger.*

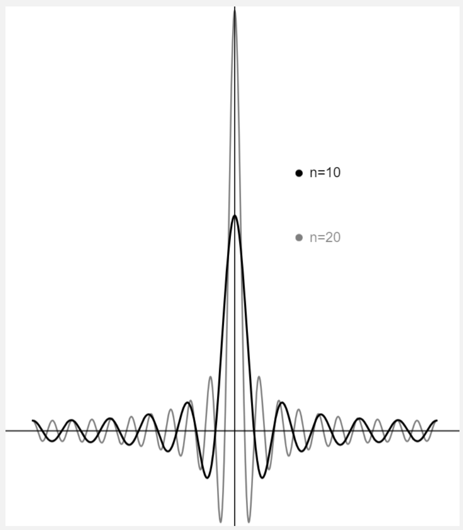


Figure 2.2.1: The graph of $D_n(z)$ for $n=10$ and $n=20$.

*Therefore, as* $n$ *gets large the main contribution to* $\int_{-\pi}^{\pi} f(x+z)D_n(z)\,\mathrm{d}z$ *comes from an ever smaller neighbourhood of* $z=0$. *With* (2.2.3) *this contribution converges towards* $f(x)$.

## 2.3 From Fourier Series to Functions

A continuous function $f$ with period $2\pi$ on $\mathbb{R}$ is uniquely determined by its Fourier series. However, as the Fourier series may not converge, we cannot naively use the sum of the series to determine the values of $f$. Instead, we consider the Fejér sums

$$\sigma_n(x) = \frac{1}{n}\left(S_0(x) + \cdots + S_{n-1}(x)\right) \tag{2.3.1}$$

where $S_k(x)$ is as in (2.2.1).

**Exercise 2.3.1.** *With* $\sigma_n(x)$ *as given by* (2.3.1), *show that*

$$\sigma_n(x) = \int_{-\pi}^{\pi} f(x+z)\Phi_n(z)\,\mathrm{d}z,$$

*where*

$$\Phi_n(z) = \frac{1}{2\pi n}\left(\frac{\sin\left(\frac{nz}{2}\right)}{\sin\left(\frac{z}{2}\right)}\right)^2$$

*is referred to as the Fejér kernel.*

**Lemma 2.3.2.** *Let $\Phi_n(z)$ be the Fejér kernel of a continuous function $f$ which is $2\pi$ periodic. Then the following statements hold.*

1. $\Phi_n(z) \geq 0$.
2. $\int_{-\pi}^{\pi} \Phi_n(z)\,\mathrm{d}z = 1$.
3. *For fixed $\delta > 0$ it follows that*

$$\int_{-\pi}^{-\delta} \Phi_n(z)\,\mathrm{d}z = \int_{\delta}^{\pi} \Phi_n(z)\,\mathrm{d}z = \eta_n(\delta) \overset{n\to\infty}{\longrightarrow} 0.$$

*Proof.*

1. This is clear.
2. As

$$\Phi_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(z).$$

it follows from Remark 2.2.3 that

$$\int_{-\pi}^{\pi} \Phi_n(z)\,\mathrm{d}z = 1.$$

3. For $\delta > 0$ note that $\sin\left(\frac{z}{2}\right) \geq \sin\left(\frac{\delta}{2}\right)$ for $x \in [\delta, \pi]$. Therefore,

$$\begin{aligned}\int_{\delta}^{\pi} \Phi_n(z)\,\mathrm{d}z &= \frac{1}{2\pi n}\int_{\delta}^{\pi}\left(\frac{\sin\left(\frac{nz}{2}\right)}{\sin\left(\frac{z}{2}\right)}\right)^2 \mathrm{d}z \\ &\leq \frac{1}{2\pi n}\int_{\delta}^{\pi} \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}\,\mathrm{d}z \\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Similarly,

$$\int_{-\pi}^{-\delta} \Phi_n(z)\,\mathrm{d}z \overset{n\to\infty}{\longrightarrow} 0.$$

□

**Theorem 2.3.3** (Fejér). *If $f$ is a continuous function with period $2\pi$, then the sequence $(\sigma_n)_{n\in\mathbb{N}}$ as given by* (2.3.1) *converges to $f$ uniformly on $\mathbb{R}$.*

*Proof.* Since $f$ is continuous and periodic on $\mathbb{R}$, it is bounded and uniformly continuous on $\mathbb{R}$. Thus, there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Moreover, for an $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(x')| < \frac{\epsilon}{2}$$

for $|x - x'| < 2\delta$. Write

$$\begin{aligned} f(x) - \sigma_n(x) &\overset{(1)}{=} \int_{-\pi}^{\pi} (f(x) - f(x+z))\Phi_n(z)\,\mathrm{d}z \\ &= \Bigg(\underbrace{\int_{-\pi}^{-\delta}}_{y_-} + \underbrace{\int_{-\delta}^{\delta}}_{y_0} + \underbrace{\int_{\delta}^{\pi}}_{y_+}\Bigg)(f(x) - f(x+z))\Phi_n(z)\,\mathrm{d}z,\end{aligned}$$

where $\Phi_n(z)$ is the Fejér kernel, and so in $(1)$ we use statement 2. of Lemma 2.3.2. Then,

$$|y_-| \leq 2M\eta_n(\delta),$$

and

$$|y_+| \leq 2M\eta_n(\delta)$$

where

$$\eta_n(\delta) = \int_\delta^\pi \Phi_n(z)\,\mathrm{d}z.$$

Moreover,

$$|y_0| \leq \frac{\epsilon}{2}\int_{-\delta}^{\delta} \Phi_n(z)\,\mathrm{d}z < \frac{\epsilon}{2}.$$

Where the second inequality follows from statement 1. and statement 2. of Lemma 2.3.2 which imply that $\int_{-\delta}^{\delta} \Phi_n(z)\,\mathrm{d}z \leq 1$. By statement 3. of Lemma 2.3.2 there exists a $n_0 = n_0(\delta(\epsilon))$ such that for $n \geq n_0$ we have $2M\eta_n(\delta) < \frac{\epsilon}{4}$. Therefore,

$$|f(x) - \sigma_n(x)| < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon$$

for $n \geq n_0$ and any $x \in \mathbb{R}$ which implies that $\sigma_n \overset{n\to\infty}{\longrightarrow} f$ uniformly on $\mathbb{R}$ □

**Corollary 2.3.4** (Weierstrass)**.** *Any continuous periodic function is a limit of a uniformly convergent sequence of trigonometric polynomials.*

**Remark 2.3.5.** *Theorem 2.3.3 gives an explicit sequence for Corollary 2.3.4, namely* $(\sigma_n)_{n\in\mathbb{N}}$.

**Corollary 2.3.6.** *The trigonometric system*

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

*is complete in* $L^2(-\pi,\pi)$.

*Proof.* As continuous functions are dense in $L^2(-\pi,\pi)$ and uniform convergence implies convergence in $L^2(-\pi,\pi)$, it follows by Corollary 2.3.4 that the system

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

is complete in $L^2(-\pi,\pi)$. □

**Remark 2.3.7.**

1. *Theorem 2.3.3 tells us that for* $f \in \mathcal{C}^0([-\pi,\pi])$*, the sequence* $(\sigma_n)_{n\in\mathbb{N}}$ *converges in the metric of* $\mathcal{C}^0([-\pi,\pi])$*, namely the supremum norm.*
2. *Although not in the statement of Theorem 2.3.3, we also have that if* $f \in L^1(-\pi,\pi)$ *then* $(\sigma_n)_{n\in\mathbb{N}}$ *converges to* $f$ *in the metric of* $L^1(-\pi,\pi)$*. Thus we deduce that* $f \in L^1(-\pi,\pi)$ *is uniquely determined by its Fourier coefficients. Indeed suppose that* $f, g \in L^1(-\pi,\pi)$ *have the same Fourier coefficients. Then the corresponding Fejèr sums of* $f - g$ *are zero. Therefore,* $f - g$ *is zero as the Fejèr sums converge to zero, hence,* $f = g$ *almost everywhere.*

## 2.4 Solution to Exercises

### Exercise 2.1.2

*Solution.* The system

$$\{1\} \cup \left\{\cos\left(\frac{n\pi}{l}x\right)\right\}_{n\in\mathbb{N}} \cup \left\{\sin\left(\frac{n\pi}{l}x\right)\right\}_{n\in\mathbb{N}}$$

is an orthogonal system of $L^2(-l,l)$ as after a re-scaling the orthogonality conditions are the same orthogonality conditions for

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

as a system of $L^2(-\pi,\pi)$, which we know the be orthogonal. In particular, we note that

$$\begin{cases} \|1\| = \sqrt{2l} \\ \left\|\cos\left(\frac{n\pi}{l}x\right)\right\| = \sqrt{l} \\ \left\|\sin\left(\frac{n\pi}{l}x\right)\right\| = \sqrt{l} \end{cases}$$

and so

$$\left\{\frac{1}{\sqrt{2l}}\right\} \cup \left\{\frac{1}{\sqrt{l}}\cos\left(\frac{n\pi}{l}x\right)\right\}_{n\in\mathbb{N}} \cup \left\{\frac{1}{\sqrt{l}}\sin\left(\frac{n\pi}{l}x\right)\right\}_{n\in\mathbb{N}}$$

is an orthonormal system of $L^2(-l,l)$. Moreover, the system is complete as $\{1\}\cup\{\cos(nx)\}_{n\in\mathbb{N}}\cup\{\sin(nx)\}_{n\in\mathbb{N}} \subseteq L^2(-\pi,\pi)$ is complete. For $f \in L^2(-l,l)$ its Fourier series with respect to this basis is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{l}x\right) + b_k \sin\left(\frac{k\pi}{l}x\right)$$

where $a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi}{l}x\right)\,\mathrm{d}x$ and $b_k = \frac{1}{l}\int_{-l}^{l} \sin\left(\frac{k\pi}{l}x\right)\,\mathrm{d}x$. ☐

### Exercise 2.2.1

*Solution.* Note that

$$1 + \sum_{k=1}^{n} \cos(ku) = \mathrm{Re}\left(\sum_{k=1}^{n} e^{iku}\right)$$

and

$$\sum_{k=1}^{n} \sin(ku) = \mathrm{Im}\left(\sum_{k=1}^{n} e^{iku}\right).$$

Observe that,

$$\begin{aligned}
\sum_{k=0}^{n} e^{iku} &= \frac{1-e^{i(n+1)u}}{1-e^{iu}} \\
&= \frac{\left(1-e^{i(n+1)u}\right)\left(1-e^{-iu}\right)}{\left(1-e^{iu}\right)\left(1-e^{-iu}\right)} \\
&= \frac{1-e^{i(n+1)u}-e^{-iu}+e^{inu}}{2-\left(e^{iu}+e^{-iu}\right)} \\
&= \frac{1-e^{i(n+1)u}-e^{-iu}+e^{inu}}{2-2\cos(u)}.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
1 + \sum_{k=1}^{n} \cos(ku) &= \frac{1-\cos(u)+(\cos(nu)-\cos((n+1)u))}{2\,(1-\cos(u))} \\
&= \frac{1}{2} + \frac{\sin\left(\frac{2n+1}{2}u\right)\sin\left(\frac{u}{2}\right)}{2\sin^2\left(\frac{u}{2}\right)} \\
&= \frac{1}{2} + \frac{\sin\left(\frac{2n+1}{2}u\right)}{2\sin\left(\frac{u}{2}\right)},
\end{aligned}$$

which upon rearrangement gives

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(ku) = \frac{\sin\left(\frac{2n+1}{2}u\right)}{2\sin\left(\frac{u}{2}\right)}.$$

On the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} \sin(ku) &= \frac{\sin(u) + \sin(nu) - \sin((n+1)u)}{2(1-\cos(u))} \\
&= \frac{2\sin\left(\frac{n+1}{2}u\right)\cos\left(\frac{n-1}{2}\right) - \sin((n+1)u)}{2\left(2\sin^2\left(\frac{u}{2}\right)\right)} \\
&= \frac{\sin\left(\frac{n+1}{2}u\right)\left(\cos\left(\frac{n-1}{2}\right) - \cos\left(\frac{n+1}{2}\right)\right)}{2\sin^2\left(\frac{u}{2}\right)} \\
&= \frac{2\sin\left(\frac{n+1}{2}u\right)\sin\left(\frac{n}{2}u\right)\sin\left(\frac{u}{2}\right)}{2\sin^2\left(\frac{u}{2}\right)} \\
&= \frac{\sin\left(\frac{n+1}{2}u\right)\sin\left(\frac{n}{2}u\right)}{\sin\left(\frac{u}{2}\right)}.
\end{aligned}$$

$\square$

**Exercise 2.2.4**

*Solution.* On the one hand, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_n(z)|\,\mathrm{d}z &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\sin\left(\frac{z}{2}\right)}\right| \mathrm{d}z \\
&= \frac{1}{\pi}\int_{0}^{\pi} \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\sin\left(\frac{z}{2}\right)}\right| \mathrm{d}z \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \left|\frac{\sin\left((2n+1)t\right)}{\sin(t)}\right| \mathrm{d}t \\
&\geq \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{\left|\sin\left((2n+1)t\right)\right|}{t}\,\mathrm{d}t \\
&= \frac{2}{\pi}\sum_{k=1}^{n}\int_{\frac{(k-1)\pi}{2n+1}}^{\frac{k\pi}{2n+1}} \frac{\left|\sin\left((2n+1)t\right)\right|}{t}\,\mathrm{d}t \\
&= \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin(x)|}{x}\,\mathrm{d}x \\
&= \frac{2}{\pi}\sum_{k=1}^{n}\int_{0}^{\pi} \frac{\sin(u)}{u+(k-1)\pi}\,\mathrm{d}u \\
&\geq \frac{2}{\pi}\sum_{k=2}^{n}\int_{0}^{\pi} \frac{\sin(u)}{(k-1)\pi}\,\mathrm{d}u \\
&\geq \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \\
&\geq \frac{4}{\pi^2}\log(n) + \frac{4}{\pi^2}\gamma,
\end{aligned}$$

where $\gamma$ is Euler's constant. On the other hand, we first observe that

$$\begin{aligned}
\left|\frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\sin\left(\frac{z}{2}\right)} - \frac{\sin(nz)}{\tan\left(\frac{z}{2}\right)}\right| &= \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)z\right) - \sin(nz)\cos\left(\frac{z}{2}\right)}{\sin\left(\frac{z}{2}\right)}\right| \\
&= \left|\frac{\sin(nz)\cos\left(\frac{z}{2}\right) - \sin\left(\frac{z}{2}\right)\cos(nz) - \sin(nz)\cos\left(\frac{z}{2}\right)}{\sin\left(\frac{z}{2}\right)}\right| \\
&= |\cos(nz)| \\
&\leq 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_n(z)| \, \mathrm{d}z &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\sin\left(\frac{z}{2}\right)}\right| \mathrm{d}z \\
&= \frac{1}{\pi}\int_{0}^{\pi} \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\sin\left(\frac{z}{2}\right)}\right| \mathrm{d}z \\
&\leq \frac{1}{\pi}\int_{0}^{\pi} 1 + \left|\frac{\sin(nz)}{\tan\left(\frac{z}{2}\right)}\right| \mathrm{d}z \\
&= 1 + \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \left|\frac{\sin(2nt)}{\tan(t)}\right| \mathrm{d}t \\
&\leq 1 + \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{|\sin(2nt)|}{\frac{1}{2}t} \, \mathrm{d}t \\
&= 1 + \frac{4}{\pi}\sum_{k=1}^{n}\int_{\frac{(k-1)\pi}{2n}}^{\frac{k\pi}{2n}} \frac{|\sin(2nt)|}{t} \, \mathrm{d}t \\
&= 1 + \frac{4}{\pi}\sum_{k=1}^{n}\int_{0}^{\pi} \frac{\sin(u)}{u+(k-1)\pi} \, \mathrm{d}u \\
&\leq 1 + \int_{0}^{\pi} \frac{\sin(u)}{u} \, \mathrm{d}u + \frac{4}{\pi^2}\sum_{k=1}^{n-1}\frac{1}{k} \\
&\leq 1 + \int_{0}^{\pi} \frac{\sin(u)}{u} \, \mathrm{d}u + \frac{4}{\pi^2}\gamma + \frac{4}{\pi^2}\log(n).
\end{aligned}$$

Hence,

$$\int_{-\pi}^{\pi} |D_n(z)| \, \mathrm{d}z = \frac{4}{\pi^2}\log(n) + O(1).$$

$\square$

**Exercise 2.2.8**

*Solution.* Recall that the complex Fourier coefficients of $F$ are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} \, \mathrm{d}x.$$

As $f$ is complex analytic, there exists an $\eta > 0$ such that $f$ is complex analytic on the square $[-\pi, \pi] \times [-\eta, 0]$. In particular, the contour integral around the boundary of the square is zero. More specifically,

$$\oint_{\gamma} f(x)e^{-inx} \, \mathrm{d}x = 0,$$

where $\gamma$ is the clock-wise traversing of the boundary of the square. On the first vertical component of $\gamma$ we have the integral

$$I_1 = \int_0^{-\eta} f(\pi + iy)e^{-in(\pi+iy)} i \,\mathrm{d}y,$$

and along the second vertical component of $\gamma$ we have the integral

$$I_2 = \int_{-\eta}^{0} f(-\pi + iy)e^{-in(-\pi+iy)} i \,\mathrm{d}y.$$

As $f(x)$ and $e^{ix}$ are $2\pi$ periodic it follows that

$$I_2 = \int_{-\eta}^{0} f(\pi + iy)e^{-in(\pi+iy)} \,\mathrm{d}y = -I_1.$$

Therefore, $I_1$ and $I_2$ cancel each other out in the contour integral and so

$$0 = \int_{-\pi}^{\pi} f(x)e^{-inx} \,\mathrm{d}x + \int_{\pi}^{-\pi} f(x - i\eta)e^{-in(x-i\eta)} \,\mathrm{d}x.$$

Hence,

$$\begin{aligned}
|c_n| &= \left| \int_{-\pi}^{\pi} f(x - i\eta)e^{-in(x-i\eta)} \,\mathrm{d}x \right| \\
&= e^{-n\eta} \int_{-\pi}^{\pi} |f(x - i\eta)| \left|e^{-inx}\right| \,\mathrm{d}x \\
&\le e^{-n\eta}\Gamma,
\end{aligned}$$

where $\Gamma < \infty$ as $f$ is analytic and $\left|e^{-inx}\right| \le 1$. Therefore, the Fourier coefficient $c_n$ decays on the order of $e^{-n}$ as $n \to \infty$. $\square$

**Exercise 2.2.10**

*Solution.* Using the complex Fourier series we know that the Fourier coefficient $c_n$ is given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left( \left| 2\sin\left(\frac{\theta}{2}\right) \right| \right) e^{-in\theta} \,\mathrm{d}\theta.$$

In particular,

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_0^{\pi} \log\left(2\sin\left(\frac{\theta}{2}\right)\right) e^{-in\theta} \,\mathrm{d}\theta + \frac{1}{2\pi} \int_{-\pi}^{0} \log\left(-2\sin\left(\frac{\theta}{2}\right)\right) e^{-in\theta} \,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{\pi} \log\left(2\sin\left(\frac{\theta}{2}\right)\right) e^{-in\theta} \,\mathrm{d}\theta + \frac{1}{2\pi} \int_0^{\pi} \log\left(2\sin\left(\frac{\theta}{2}\right)\right) e^{in\theta} \,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{\pi} \log\left(2\sin\left(\frac{\theta}{2}\right)\right) (2\cos(n\theta)) \,\mathrm{d}\theta \\
&= \frac{1}{\pi} \int_0^{\pi} \log\left(2\sin\left(\frac{\theta}{2}\right)\right) \cos(n\theta) \,\mathrm{d}\theta \\
&= \frac{1}{\pi} \int_0^{\pi} \log(2)\cos(n\theta) \,\mathrm{d}\theta + \frac{1}{\pi} \int_0^{\pi} \log\left(\sin\left(\frac{\theta}{2}\right)\right) \cos(n\theta) \,\mathrm{d}\theta \\
&= \frac{1}{\pi} \int_0^{\pi} \log\left(\sin\left(\frac{\theta}{2}\right)\right) \cos(n\theta) \,\mathrm{d}\theta.
\end{aligned}$$

Note that it thus suffices to consider $n \geq 0$. In particular, for $n \neq 0$, through integration by parts, it follows that

$$\begin{aligned}
c_n &= \frac{1}{\pi}\left(\left[\frac{1}{n}\sin(n\theta)\log\left(\sin\left(\frac{\theta}{2}\right)\right)\right]_0^\pi - \frac{1}{n}\int_0^\pi \frac{\cos\left(\frac{\theta}{2}\right)\sin(n\theta)}{\sin\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta\right) \\
&= -\frac{1}{n\pi}\int_0^\pi \frac{\cos\left(\frac{\theta}{2}\right)\sin(n\theta)}{\sin\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta \\
&= -\frac{1}{n\pi}\int_0^\pi \frac{\frac{1}{2}\left(\sin\left(\left(\frac{1}{2}+n\right)\theta\right) - \sin\left(\left(\frac{1}{2}-n\right)\theta\right)\right)}{\sin\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta \\
&= -\frac{1}{n\pi}\int_0^\pi \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) + \sin\left(\left(n-\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta \\
&\overset{\text{Ex 2.2.1}}{=} -\frac{1}{n\pi}\int_0^\pi \int_0^\pi \frac{1}{2} + \sum_{k=1}^{n}\cos(k\theta) + \frac{1}{2} + \sum_{k=1}^{n-1}\cos(n\theta)\,\mathrm{d}\theta \\
&= -\frac{1}{n\pi}\int_0^\pi \mathrm{d}\theta \\
&= -\frac{1}{n}.
\end{aligned}$$

For $n = 0$ we have

$$\begin{aligned}
c_0 &= \frac{1}{\pi}\int_0^\pi \log\left(\sin\left(\frac{\theta}{2}\right)\right)\,\mathrm{d}\theta \\
&=^1 \frac{1}{\pi}\int_0^\pi -\log(2) - \sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k} \\
&= -\log(2).
\end{aligned}$$

□

**Exercise 2.3.1**

*Solution.* Using Proposition 2.2.2 we have

$$\begin{aligned}
\sigma_n(x) &= \frac{1}{n}\sum_{k=0}^{n-1} S_k(x) \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_{-\pi}^{\pi} f(x+z)\frac{1}{2\pi}\frac{\sin\left(\frac{2k+1}{2}z\right)}{\sin\left(\frac{z}{2}\right)}\,\mathrm{d}z \\
&= \int_{-\pi}^{\pi} f(x+z)\frac{1}{2\pi n}\frac{1}{\sin^2\left(\frac{z}{2}\right)}\sum_{k=0}^{n-1}\sin\left(\frac{2k+1}{2}z\right)\sin\left(\frac{z}{2}\right)\,\mathrm{d}z \\
&= \int_{-\pi}^{\pi} f(x+z)\frac{1}{2\pi n}\frac{1}{\sin^2\left(\frac{z}{2}\right)}\sum_{k=0}^{n-1}\frac{1}{2}\left(\cos(kz) - \cos((k+1)z)\right)\,\mathrm{d}z \\
&= \int_{-\pi}^{\pi} f(x+z)\frac{1}{2\pi n}\frac{1}{\sin^2\left(\frac{z}{2}\right)}\frac{1-\cos(nz)}{2}\,\mathrm{d}z \\
&= \int_{-\pi}^{\pi} f(x+z)\frac{1}{2\pi n}\left(\frac{\sin\left(\frac{n}{2}z\right)}{\sin\left(\frac{z}{2}\right)}\right)^2 \\
&= \int_{-\pi}^{\pi} f(x+z)\Phi_n(z)\,\mathrm{d}z
\end{aligned}$$

□

# 3 Fourier Transform

Thus far we have seen that a periodic, integrable function is represented by its Fourier coefficients. We now intend to generalise these arguments to non-periodic functions defined on $\mathbb{R}$. Our approach will be to use our previous work and a limiting argument. More specifically, we note that we can restrict a function $f$ defined on $\mathbb{R}$ to a function $f$ defined on $(-l, l)$. Through re-scaling, we can view this restriction as defined on $(-\pi, \pi)$. Thus we can leverage our previous work on periodic functions. By sending $l \to \infty$ one would expect that we obtain a representation for $f$ as a function on $\mathbb{R}$. The conditions under which the argument is productive are made explicit in the following section.

## 3.1 The Fourier Integral

Suppose $f \in L^1(\mathbb{R})$ satisfies (2.2.3) at each point in $(-l, l)$. Then we know that

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

where

$$a_k = \frac{1}{l} \int_{-l}^{l} f(t) \cos\left(\frac{k\pi t}{l}\right) \mathrm{d}t$$

and

$$b_k = \frac{1}{l} \int_{-l}^{l} f(t) \sin\left(\frac{k\pi t}{l}\right) \mathrm{d}t.$$

Consequently,

$$f(x) = \frac{1}{2l} \int_{-l}^{l} f(t) \, \mathrm{d}t + \frac{1}{l} \sum_{k=1}^{\infty} \int_{-l}^{l} f(t) \cos\left(\frac{k\pi(t-x)}{l}\right) \mathrm{d}t.$$

Letting $\lambda_k = \frac{\pi k}{l}$ and taking $l \to \infty$, one would expect to obtain the Fourier integral

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos(\lambda(t-x)) \, \mathrm{d}t \, \mathrm{d}\lambda.$$

This limit is entirely intuitive at present and it is not clear whether it should hold. One can think of the Fourier integral as a continuous analogue of the Fourier series. More specifically, one can re-write the Fourier integral as

$$f(x) = \int_0^{\infty} a_\lambda \cos(\lambda x) + b_\lambda \sin(\lambda x) \, \mathrm{d}x$$

where

$$a_\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\lambda t) \, \mathrm{d}t$$

and

$$b_\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\lambda t) \, \mathrm{d}t.$$

**Exercise 3.1.1.** *For $a > 0$ show that*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(az)}{z} \, \mathrm{d}z = 1.$$

**Theorem 3.1.2.** *Let $f \in L^1(\mathbb{R})$ and suppose it satisfies Dini's condition,* (2.2.3)*, at $x \in \mathbb{R}$. Then*

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos(\lambda(t-x)) \, \mathrm{d}t \, \mathrm{d}\lambda. \tag{3.1.1}$$

*Proof.* Let
$$\zeta(a) := \frac{1}{\pi}\int_0^a \int_{-\infty}^{\infty} f(t)\cos(\lambda(t-x))\,\mathrm{d}t\,\mathrm{d}\lambda.$$
As $f \in L^1(\mathbb{R})$, the double integral $\zeta(a)$ with absolute values converges. Hence, by Fubini's theorem, the order of integration can be exchanged such that
$$\begin{aligned}\zeta(a) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^a \cos(\lambda(t-x))\,\mathrm{d}\lambda\,\mathrm{d}t\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin(a(t-x))}{t-x}\,\mathrm{d}t\\ &\overset{z=t-x}{=} \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+z)\frac{\sin(az)}{z}\,\mathrm{d}z.\end{aligned}$$
Using Exercise 3.1.1 we can write
$$\begin{aligned}\zeta(a)-f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x+z)-f(x)}{z}\sin(az)\,\mathrm{d}z\\ &= \frac{1}{\pi}\int_{-N}^{N}\frac{f(x+z)-f(x)}{z}\sin(az)\,\mathrm{d}z + \frac{1}{\pi}\int_{|z|\geq N}\frac{f(x+z)}{z}\sin(az)\,\mathrm{d}z\\ &\quad -\frac{f(x)}{\pi}\int_{|z|\geq N}\frac{\sin(az)}{z}\,\mathrm{d}z.\end{aligned}$$
As $f \in L^1(\mathbb{R})$ it follows that the second integral converges to zero as $N \to \infty$. Similarly, using Exercise 3.1.1 the third integral converges to zero as $N \to \infty$. Thus there exists an $N_0 \in \mathbb{N}$ such that the absolute value of the second and third integrals is less than $\frac{\epsilon}{3}$ for $N \geq N_0$. By (2.2.3) and Lemma 2.2.5 the first integral converges to zero as $a \to \infty$. Hence, there exists some $A > 0$ such that for $a > A$ the absolute value of the integral is less the $\frac{\epsilon}{3}$. Hence, for $N \geq N_0$ and $a \geq A$ we have
$$|\zeta(a)-f(x)| < \epsilon.$$
Therefore, $\zeta(a) \to f(x)$ as $a \to \infty$. □

As we did for the Fourier series, we can consider the Fourier integral over the $L^1(\mathbb{R})$ as a complex Euclidean space. Doing so, under appropriate conditions, leads to the inverse Fourier transform. Let $f \in L^1(\mathbb{R})$ and suppose it satisfies (2.2.3) at $x \in \mathbb{R}$. Then as $\cos(\cdot)$ is an even function we can write the Fourier integral as
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos(\lambda(t-x))\,\mathrm{d}t\,\mathrm{d}\lambda.$$
Similarly, as $\sin(\cdot)$ is an odd function and $\int_{-\infty}^{\infty} f(t)\sin(\lambda(t-x))\,\mathrm{d}t$ exists as $f \in L^1(\mathbb{R})$ it follows that
$$\frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N}\int_{-\infty}^{\infty} f(t)\sin(\lambda(t-x))\,\mathrm{d}t\,\mathrm{d}\lambda = 0.$$
Therefore, if $f \in L^1(\mathbb{R})$ satisfies (2.2.3) at $x \in \mathbb{R}$ its complex Fourier integral is given by
$$f(x) = \frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N}\int_{-\infty}^{\infty} f(t)e^{-i\lambda(t-x)}\,\mathrm{d}t\,\mathrm{d}\lambda. \tag{3.1.2}$$

**Definition 3.1.3.** *Let $f \in L^1(\mathbb{R})$. The Fourier transform of $f$ is*
$$g(\lambda) = F[f](\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\,\mathrm{d}t.$$

Note that the Fourier transform of $f$ exists provided that $f \in L^1(\mathbb{R})$. However, if additionally $f$ satisfies (2.2.3) at $x \in \mathbb{R}$ then (3.1.2) also holds.

**Definition 3.1.4.** *Let $f \in L^1(\mathbb{R})$, and suppose that $f$ satisfies (2.2.3) at $x \in \mathbb{R}$. Then the inverse Fourier transform of $f$ at $x$ is*

$$f(x) = \frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^{N} g(\lambda) e^{i\lambda x} \, \mathrm{d}\lambda.$$

**Remark 3.1.5.** *The Fourier transform exists for any $f \in L^1(\mathbb{R})$, whereas the inverse Fourier transform exists only for $f \in L^1(\mathbb{R})$ that additionally satisfies Dini's condition. This is similar to how Fourier coefficients can be defined for any $f \in L(-\pi, \pi)$, whereas, the Fourier series only converges for $f$ which satisfies Dini's condition.*

**Theorem 3.1.6.** *Let $f \in L^1(\mathbb{R})$. If $g(\lambda) = F[f] \equiv 0$, then $f(x) = 0$ almost everywhere.*

*Proof.* Observe that

$$\begin{aligned} 0 &= g(\lambda) \\ &= \int_{-\infty}^{\infty} f(z) e^{-i\lambda z} \, \mathrm{d}z \\ &\overset{z=x+t}{=} \int_{-\infty}^{\infty} f(x+t) e^{-i\lambda(x+t)} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x+t) e^{-i\lambda x} \, \mathrm{d}x. \end{aligned}$$

Let $\varphi(x) := \int_0^\mu f(x+t) \, \mathrm{d}t$ for fixed $\mu > 0$. Note that $\varphi \in L^1(\mathbb{R})$ and by Fubini's theorem we have

$$\begin{aligned} F[\varphi] &= \int_{-\infty}^{\infty} \varphi(x) e^{-i\lambda x} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \int_0^\mu f(x+t) \, \mathrm{d}t \, \mathrm{d}x \\ &= \int_0^\mu \int_{-\infty}^{\infty} f(x+t) e^{-i\lambda x} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\mu 0 \, \mathrm{d}t \\ &= 0. \end{aligned}$$

Moreover, it is clear that on any finite interval $\varphi$ is absolutely continuous, and thus its derivative exists almost everywhere which implies that it satisfies Dini's condition (2.2.3) almost everywhere. Therefore, using the inversion formula and the fact that $F[\varphi] \equiv 0$, it follows that $\varphi(x) = 0$ almost everywhere, but as $\varphi(x)$ is continuous this means that $\varphi \equiv 0$. Therefore,

$$\int_0^\mu f(t) \, \mathrm{d}t = 0$$

for all $\mu \in \mathbb{R}$ which implies that $f(x) = 0$ almost everywhere. $\square$

**Example 3.1.7.**

1. *Let* $f(x) = e^{-\gamma|x|}$ *for* $\gamma > 0$. *Then*

$$\begin{aligned}
g(\lambda) &= \int_{-\infty}^{\infty} e^{-\gamma|x|} e^{-i\lambda x}\,\mathrm{d}x \\
&= 2\int_0^{\infty} e^{-\gamma x}\cos(\lambda x)\,\mathrm{d}x \\
&= 2\left(\left[e^{-\gamma x}\left(-\frac{1}{\lambda}\sin(\lambda x)\right)\right]_0^{\infty} - \int_0^{\infty}\frac{\gamma}{\lambda}e^{-\gamma x}\sin(\lambda x)\,\mathrm{d}x\right) \\
&= -\frac{2\gamma}{\lambda}\int_0^{\infty} e^{-\gamma x}\sin(\lambda x)\,\mathrm{d}x \\
&= -\frac{2\gamma}{\lambda}\left(\left[e^{-\gamma x}\left(\frac{1}{\lambda}\cos(\lambda x)\right)\right]_0^{\infty} + \int_0^{\infty}\frac{\gamma}{\lambda}e^{-\gamma x}\cos(\lambda x)\,\mathrm{d}x\right) \\
&= -\frac{2\gamma}{\lambda}\left(-\frac{1}{\lambda} + \frac{\gamma}{2\lambda}g(\lambda)\right),
\end{aligned}$$

*and so*

$$g(\lambda) = \frac{2\gamma}{\lambda^2 + \gamma^2}.$$

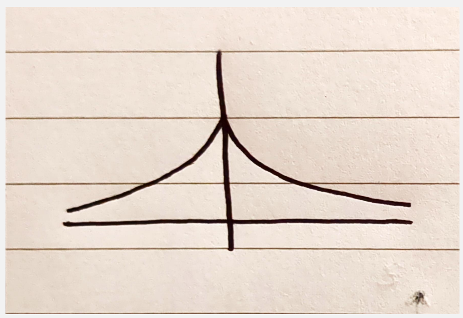

Figure 3.1.1: Graph of $f(x)$.

2. *Let*

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a. \end{cases}$$

*Then*

$$\begin{aligned}
g(\lambda) &= \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x \\
&= \int_{-a}^{a} e^{-i\lambda x}\,\mathrm{d}x \\
&= \frac{2\sin(\lambda a)}{\lambda}.
\end{aligned}$$

*Note that* $g(\lambda) \notin L^1(\mathbb{R})$.

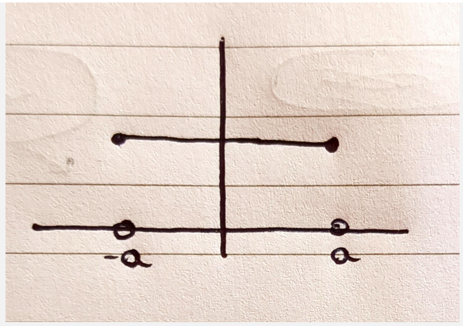


Figure 3.1.2: Graph of $f(x)$.

3. *Let* $f(x) = \frac{1}{x^2+a^2}$ *for* $a > 0$. *For* $\lambda < 0$ *let*

$$\gamma = \gamma_1 \cup \gamma_R$$

*where* $\gamma_1 = [-R, R]$ *and* $\gamma_R = \left\{Re^{i\theta} : \theta \in [0, \pi]\right\}$ *for* $R > a$. *Then*

$$\operatorname{Res}\left(\frac{1}{x^2+a^2}e^{-i\lambda x}, ia\right) = \oint_\gamma \frac{1}{z^2+a^2}e^{-i\lambda z}\,\mathrm{d}z.$$

*Observe that*

$$\begin{aligned}\left|\int_{\gamma_R} \frac{1}{x^2+a^2}e^{-i\lambda x}\,\mathrm{d}x\right| &= \left|\int_0^\pi \frac{1}{\left(Re^{i\theta}\right)^2+a^2}e^{-i\lambda Re^{i\theta}}iRe^{i\theta}\,\mathrm{d}\theta\right| \\ &\leq \frac{R}{R^2-a^2}\int_0^\pi e^{\lambda R\sin\theta}\,\mathrm{d}\theta \\ &\overset{R\to\infty}{\longrightarrow} 0.\end{aligned}$$

*Moreover,*

$$\operatorname{Res}\left(\frac{1}{x^2+a^2}e^{-i\lambda x}, ia\right) = \lim_{z\to ia}\frac{(x-ia)}{x^2+a^2}e^{-i\lambda x} = \frac{\pi}{a}e^{a\lambda}.$$

*Hence,*

$$\frac{\pi}{a}e^{a\lambda} = \int_{-\infty}^{\infty}\frac{1}{x^2+a^2}e^{-i\lambda x}\,\mathrm{d}x = g(\lambda).$$

*Similarly, for* $\lambda > 0$ *letting* $\gamma_R = \left\{Re^{i\theta} : \theta \in [0, -\pi]\right\}$ *it follows that*

$$\frac{\pi}{a}e^{-a\lambda} = g(\lambda).$$

*Therefore,*

$$g(\lambda) = \frac{\pi}{a}e^{-a|\lambda|}.$$

*On the other hand, one can see that* $g(\lambda) = \frac{\pi}{a}e^{-a|\lambda|}$ *using the inversion formula and statement 1.*

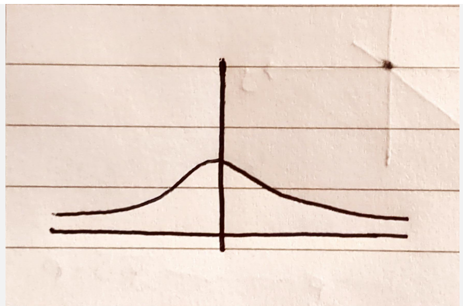

Figure 3.1.3: Graph of $f(x)$.

*Note the rate of convergence for each example. Continuity in statement 1. yields $\frac{1}{\lambda^2}$ convergence, the discontinuity in statement 2. means we only get $\frac{1}{\lambda}$ rate of converges, and the analyticity of statement 3 means we get exponential convergence.*

4. *Let $f(x) = e^{-ax^2}$ for $a > 0$. Consider the contour given by*

$$\underbrace{[-R, R]}_{\gamma_1} \cup \underbrace{[R, R + i\epsilon]}_{\gamma_2} \cup \underbrace{[R + i\epsilon, -R + i\epsilon]}_{\gamma_3} \cup \underbrace{[-R + i\epsilon, -R]}_{\gamma_4}.$$

*Note that*

$$\begin{aligned} \left| \int_{\gamma_2} e^{-az^2} \, \mathrm{d}z \right| &= \left| \int_0^\epsilon e^{-a(R+yi)} i \, \mathrm{d}y \right| \\ &\leq e^{-aR^2} \int_0^\epsilon \left| e^{-a\left(-y^2 + 2Ryi\right)} \right| \mathrm{d}y \\ &\leq e^{-aR^2} \epsilon e^{a\epsilon^2} \\ &\xrightarrow{R\to\infty} 0. \end{aligned}$$

*Similarly,*

$$\left| \int_{\gamma_4} e^{-az^2} \, \mathrm{d}z \right| \xrightarrow{R\to\infty} 0.$$

*Furthermore,*

$$\begin{aligned} \int_{\gamma_3} e^{-az^2} \, \mathrm{d}z &= \int_R^{-R} e^{-a(x+\epsilon i)^2} \, \mathrm{d}x \\ &= -e^{a\epsilon^2} \int_{-R}^R e^{-ax^2} e^{-2ax\epsilon i} \, \mathrm{d}x. \end{aligned}$$

*Therefore, as*

$$\int_{\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4} e^{-az^2} \, \mathrm{d}z = 0$$

*it follows that*

$$0 = \int_{-\infty}^{\infty} e^{-ax^2} \, \mathrm{d}x - e^{a\epsilon^2} \int_{-\infty}^{\infty} e^{-ax^2} e^{-2ax\epsilon i} \, \mathrm{d}x.$$

*In particular, letting $\epsilon = -\frac{\lambda}{2a}$ it follows that*

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{\lambda x i} \, \mathrm{d}x = e^{-\frac{\lambda^2}{4a}} \int_{-\infty}^{\infty} e^{-ax^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} e^{-\frac{\lambda^2}{4a}}.$$

*Therefore,*

$$g(\lambda) = F[f](\lambda) = e^{-\frac{\lambda^2}{4a}}\sqrt{\frac{\pi}{a}}.$$

*Observe that for $a = \frac{1}{2}$ we have*

$$F\left[e^{-\frac{x^2}{2}}\right] = \sqrt{2\pi}e^{-\frac{\lambda^2}{2}}.$$

## 3.2 Properties of the Fourier Transform

**Lemma 3.2.1.** *Let $(f_n)_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ and $f \in L^1(\mathbb{R})$. Suppose that $f_n \to f$ in $L^1(\mathbb{R})$. Then $g_n(\lambda) := F[f_n] \to F[f]$ uniformly on $\mathbb{R}$.*

*Proof.* Observe that

$$\begin{aligned}
|g_n(\lambda) - F[f](\lambda)| &= \left|\int_{-\infty}^{\infty} (f_n(x) - f(x))e^{-i\lambda x}\,\mathrm{d}x\right| \\
&\le \int_{-\infty}^{\infty} |f_n(x) - f(x)|\left|e^{-i\lambda x}\right|\,\mathrm{d}x \\
&\le \int_{-\infty}^{\infty} |f_n(x) - f(x)|\,\mathrm{d}x \\
&= \|f_n - f\|_{L^1} \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

□

**Lemma 3.2.2.** *Let $f \in L^1(\mathbb{R})$. Then $g(\lambda) = F[f]$ is a bounded and continuous function, with $g(\lambda) \to 0$ as $|\lambda| \to \infty$.*

*Proof.* As $f \in L^1(\mathbb{R})$ it follows that

$$|g(\lambda)| \le \int_{-\infty}^{\infty} |f(x)|\left|e^{-i\lambda x}\right|\,\mathrm{d}x \le \|f\|_{L^1} < \infty.$$

Suppose $f(x) = \mathbf{1}_{[a,b]}$. Then

$$F[f] = \int_a^b e^{-i\lambda x}\,\mathrm{d}x = \frac{e^{-i\lambda b} - e^{-i\lambda a}}{-i\lambda},$$

which is continuous and decays to zero as $|\lambda| \to \infty$. Since, $F[\cdot]$ is a linear operation, it follows that the Fourier transform of any step function is continuous and decays to zero as $|\lambda| \to \infty$. As step functions are dense in $L^1(\mathbb{R})$, for any $f \in L^1(\mathbb{R})$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of step functions such that $f_n \to f$ in $L^1(\mathbb{R})$. Using Lemma 3.2.1 we have that $F[f_n] \to F[f] = g(\lambda)$ uniformly in $\lambda \in \mathbb{R}$. Therefore, $g(\lambda)$ is continuous as it is the uniform limit of continuous functions. Given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\|f - f_N\|_{L^1} < \frac{\epsilon}{2}.$$

Moreover, there exists a $\lambda_0 \in \mathbb{R}$ such that $|F[f_N]| < \frac{\epsilon}{2}$ for $|\lambda| > \lambda_0$. Therefore, for $\lambda > \lambda_0$ it follows that

$$\begin{aligned}
|g(\lambda)| &\le |g(\lambda) - F[f_N]| + |F[f_N]| \\
&\le \int_{-\infty}^{\infty} |f(x) - f_n(x)|\left|e^{-i\lambda x}\right|\,\mathrm{d}x + |F[f_N]| \\
&\le \|f - f_N\|_{L^1} + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $g(\lambda) \to 0$ as $|\lambda| \to \infty$. $\square$

**Lemma 3.2.3.** *Let $f \in L^1(\mathbb{R})$. Then $g(\lambda) = F[f]$ is uniformly continuous on $\mathbb{R}$.*

*Proof.* Fix $\epsilon > 0$. As $f \in L^1(\mathbb{R})$ there exists $R > 0$ such that

$$\int_{|x|>R} |f(x)| \, \mathrm{d}x \le \frac{\epsilon}{4}.$$

As

$$\left|e^{-i\delta x} - 1\right| = \left|2\sin\left(\frac{\delta t}{2}\right)\right|$$

it follows that for $|x| < R$, there exists a $\delta_1 > 0$ such that for $\delta < \delta_1$ we have

$$\left|e^{-i\delta x} - 1\right| \le \frac{\epsilon}{2\|f\|_{L^1(\mathbb{R})}}.$$

Therefore for $\delta < \delta_1$ we have,

$$\begin{aligned}
|g(\lambda+\delta) - g(\lambda)| &= \left|\int_{-\infty}^{\infty} f(x)\left(e^{-i(\lambda+\delta)x} - e^{-i\lambda x}\right) \mathrm{d}x\right| \\
&\le \int_{-\infty}^{\infty} |f(x)| \left|e^{-i\delta x} - 1\right| \mathrm{d}x \\
&= \int_{|x|\le R} |f(x)| \left|e^{-i\delta x} - 1\right| \mathrm{d}x + \int_{|x|>R} |f(x)| \left|e^{-\delta x} - 1\right| \mathrm{d}x \\
&\le \frac{\epsilon}{2\|f\|_{L^1(\mathbb{R})}} \int_{|x|<R} |f(x)| \, \mathrm{d}x + 2\int_{|x|>R} |f(x)| \, \mathrm{d}x \\
&\le \frac{\epsilon}{2\|f\|_{L^1(\mathbb{R})}} \|f\|_{L^1(\mathbb{R})} + 2\frac{\epsilon}{4} \\
&= \epsilon.
\end{aligned}$$

Therefore, $g$ is uniformly continuous. $\square$

**Exercise 3.2.4.** *The statement of Lemma 3.2.3 holds more generally. Show that if $f$ is a real and continuous function such that $f(x) \to 0$ as $|x| \to \infty$, then $f$ is uniformly continuous.*

**Lemma 3.2.5.** *Let $f, f' = \frac{\mathrm{d}f}{\mathrm{d}t} \in L^1(\mathbb{R})$, with $f$ absolutely continuous on any finite interval. Then $F[f'] = i\lambda F[f]$.*

*Proof.* The function $f$ admits a representation

$$f(x) = f(0) + \int_0^x f'(t) \, \mathrm{d}t.$$

As $f' \in L^1(\mathbb{R})$ it follows that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, however, as $f \in L^1(\mathbb{R})$ it must be the case that both of these are zero. Using the integration by parts formula observe that

$$\begin{aligned}
F[f'] &= \int_{-\infty}^{\infty} f'(x)e^{-i\lambda x} \, \mathrm{d}x \\
&= \left[f(x)e^{-i\lambda x}\right]_{-\infty}^{\infty} + i\lambda \int_{-\infty}^{\infty} f(x)e^{-i\lambda x} \, \mathrm{d}x \\
&= i\lambda F[f].
\end{aligned}$$

$\square$

**Remark 3.2.6.** *Let $f$ is $n$-times differentiable, that is $f^{(n)}$ exists with each $f, f^{(1)}, \dots, f^{(n-1)}$ absolutely continuous. Moreover, suppose that $f, f', \dots, f^{(n)} \in L^1(\mathbb{R})$. Then by integrating by parts, and using Lemma 3.2.5, we obtain*

$$F\left[f^{(n)}\right] = (i\lambda)^n F[f].$$

*In particular,*

$$|F[f]| = \frac{\left|F\left[f^{(n)}\right]\right|}{|\lambda|^n} \leq \frac{C}{|\lambda|^n} \overset{|\lambda|\to\infty}{\longrightarrow} 0,$$

*where the inequality follows by the assumption that $f^{(n)} \in L^1(\mathbb{R})$. Hence, the smoother $f$ is the faster $F[f]$ decays at infinity. The converse also holds, namely the faster $f$ decays at infinity the smoother $F[f]$ is.*

**Exercise 3.2.7.** *Suppose $f$ is twice differentiable with $f, f', f'' \in L^1(\mathbb{R})$. Show that $F[f] \in L^1(\mathbb{R})$.*

**Lemma 3.2.8.**

1. *Suppose $f(x), xf(x) \in L^1(\mathbb{R})$. Then $g(\lambda) = F[f](\lambda)$ is differentiable with*

$$g'(\lambda) = F[-ixf].$$

2. *Suppose $f(x), xf(x), \dots, x^p f(x) \in L^1(\mathbb{R})$. Then $g(\lambda) = F[f](\lambda)$ is $p$-times differentiable with*

$$g^{(p)}(\lambda) = F\left[(-ix)^p f\right].$$

*Proof.*

1. Observe that
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x = -i\int_{-\infty}^{\infty} xf(x)e^{-i\lambda x}\,\mathrm{d}x. \tag{3.2.1}$$
Since $xf \in L^1(\mathbb{R})$, we know that $g'(\lambda)$ exists and thus it must be given by (3.2.1).

2. Follows similar arguments made for statement 1.

□

**Remark 3.2.9.** *Note that from statement 2. of Lemma 3.2.8 it follows that if $x^p f(x) \in L^1(\mathbb{R})$ for all $p \in \mathbb{N}$, then $g(\lambda)$ is infinitely differentiable.*

**Lemma 3.2.10.** *If $e^{\delta|x|}f(x) \in L^1(\mathbb{R})$ for some $\delta > 0$, then $g(\zeta)$ is an analytic function in a neighbourhood of $\mathbb{R}$.*

*Proof.* The integral

$$\int_{-\infty}^{\infty} f(x)e^{ix\zeta}\,\mathrm{d}x$$

where $\zeta = \lambda + i\mu$, uniformly converges for $|\mu| < \delta$. Therefore,

$$g(\zeta) = \int_{-\infty}^{\infty} f(x)e^{ix\zeta}\,\mathrm{d}x$$

is analytic in a neighbourhood of $\mathbb{R}$.

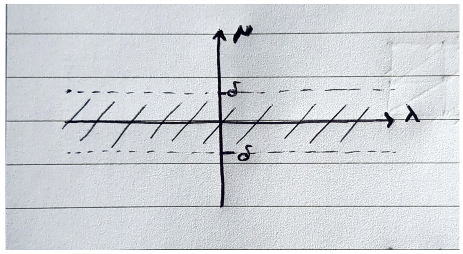


Figure 3.2.1:

$\square$

### 3.2.1 Convolution

**Definition 3.2.11.** *Let $f_1, f_2 \in L^1(\mathbb{R})$. Then*

$$f(x) = (f_1 \star f_2)(x) := \int_{-\infty}^{\infty} f_1(y) f_2(x-y)\, \mathrm{d}y$$

*is the convolution of $f_1$ and $f_2$.*

**Remark 3.2.12.** *Note that $(\cdot \star \cdot) : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a well-defined operation. More specifically,*

$$\begin{aligned}
\int_{-\infty}^{\infty} |(f_1 \star f_2)(x)|\, \mathrm{d}x &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_1(y)||f_2(x-y)|\, \mathrm{d}y\, \mathrm{d}x \\
&\overset{\text{Fubini.}}{=} \int_{-\infty}^{\infty} |f_1(y)| \left( \int_{-\infty}^{\infty} |f_2(x-y)|\, \mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-\infty}^{\infty} |f_1(y)| \|f_2\|_{L^1(\mathbb{R})}\, \mathrm{d}y \\
&= \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})} \\
&< \infty.
\end{aligned}$$

**Theorem 3.2.13.** *Let $f_1, f_2 \in L^1(\mathbb{R})$. Then*

$$F[f_1 \star f_2] = F[f_1] F[f_2].$$

*Proof.* Using Fubini's theorem

$$\begin{aligned}
\int_{-\infty}^{\infty} (f_1 \star f_2)(x) e^{-i\lambda x}\, \mathrm{d}x &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(y) f_2(x-y) e^{-i\lambda x}\, \mathrm{d}y\, \mathrm{d}x \\
&\overset{t=x-y}{=} \int_{-\infty}^{\infty} f_1(y) \int_{-\infty}^{\infty} f_2(x-y) e^{-i\lambda x}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} f_1(y) \int_{-\infty}^{\infty} f_2(t) e^{-i\lambda t} e^{-i\lambda y}\, \mathrm{d}t\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} f_1(y) e^{-i\lambda y}\, \mathrm{d}y \int_{-\infty}^{\infty} f_2(t) e^{-i\lambda t}\, \mathrm{d}t \\
&= F[f_1] F[f_2].
\end{aligned}$$

$\square$

### 3.2.2 The Heat Equation

The discussed properties of the Fourier transform are significant for their application to solving differential equations. Consider the linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = \varphi(x), \tag{3.2.2}$$

which has constant coefficients for non-zero derivatives of $y(x)$. If $y, \varphi \in L^1(\mathbb{R})$, an application of the Fourier transform to (3.2.2) yields

$$(i\lambda)^n z(\lambda) + a_1 (i\lambda)^{n-1} z(\lambda) + \cdots + a_n z(\lambda) = F[\varphi](\lambda) \tag{3.2.3}$$

where $z(\lambda) = F[y](\lambda)$. Equation (3.2.3) is significantly easier to solve than (3.2.2).

**Example 3.2.14.** *The heat equation is the partial differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{3.2.4}$$

*where $u = u(x,t)$ represent the temperature at positive $x \in \mathbb{R}$ for time $t \geq 0$. Suppose that $u_0(x) = u(x,0)$ is given. Moreover, assume that $u_0, u_0', u_0'' \in L^1(\mathbb{R})$. To make progress on solving* (3.2.4) *one assumes the following conditions are satisfied.*

1. $u(x,t), \frac{\partial}{\partial x} u(x,t), \frac{\partial^2}{\partial x^2} u(x,t) \in L^1(\mathbb{R})$ *for all* $t \geq 0$.
2. *For any $T$ there exists a $f_T(x) \in L^1(\mathbb{R})$ such that*

$$\left| \frac{\partial}{\partial t} u(x,t) \right| \leq f_T(x)$$

*for all $0 \leq t \leq T$.*

*Using assumption 1. we can apply the Fourier transform to the right-hand side of* (3.2.4) *to get*

$$F\left[\frac{\partial^2}{\partial x^2} u\right] = -\lambda^2 v(\lambda, t)$$

*where $v(\lambda,t) = F[u]$. Using assumption 2. we can apply the dominated convergence theorem to deduce that the Fourier transform of the left-hand side of* (3.2.4) *is given by*

$$\begin{aligned} F\left[\frac{\partial u}{\partial t}\right] &= \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\lambda x} \, \mathrm{d}x \\ &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-i\lambda x} \, \mathrm{d}x \\ &= \frac{\partial}{\partial t} v(\lambda, t) \end{aligned}$$

*In particular, we are viewing $\frac{\partial u}{\partial t}$ as a limit to apply the dominated convergence theorem. Thus*

$$-\lambda^2 v = \frac{\partial v}{\partial t}$$

*to which a solution satisfying the initial conditions is given by*

$$v(\lambda, t) = \exp\left(-\lambda^2 t\right) v_0(\lambda)$$

*where* $v_0(\lambda) = F[u_0]$. *Noting that* $\exp\left(-\lambda^2 t\right) = F\left[\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right)\right]$ *we can use Theorem 3.2.13 to see that*

$$v(\lambda, t) = F\left[\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right)\right] F[u_0] = F\left[\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right) \star u_0(x)\right].$$

*Therefore,*

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\exp\left(-\frac{\mu^2}{4t}\right)u_0(x-\mu)\,\mathrm{d}\mu,$$

*which is known as the Poisson integral for the solution to* (3.2.4).

## 3.3 Schwartz Functions

**Definition 3.3.1.** *Let* $\mathcal{S}^\infty$ *denote the set of functions,* $f$*, on* $\mathbb{R}$ *that are infinitely differentiable and such that for any* $p, q \in \mathbb{N}$*, there exists a constant* $C(p,q,f)$ *such that*

$$\left|x^p f^{(q)}(x)\right| < C(p,q,f)$$

*for all* $x \in \mathbb{R}$.

**Remark 3.3.2.** *A function* $f \in \mathcal{S}^\infty$*, as in Definition 3.3.1, is known as a Schwartz function.*

**Lemma 3.3.3.** *If* $f \in \mathcal{S}^\infty$*, then* $g = F[f] \in \mathcal{S}^\infty$.

*Proof.* Note that

$$\left|x^p f^{(q)}(x)\right| \leq \frac{C(p+2,q,f)}{x^2}$$

which implies that $x^p f^{(q)}(x) \in L^1(\mathbb{R})$ for every $p, q \in \mathbb{N}$. Therefore, $g = F[f]$ is infinitely differentiable by Remark 3.2.9. Moreover, letting $p = 0$ we have $f^{(q)} \in L^1(\mathbb{R})$ and so it follows by Remark 3.2.6 that $g(\lambda)$ tends to zero as $|\lambda| \to \infty$ faster than $\frac{1}{|\lambda|^q}$ for every $q \in \mathbb{N}$. Next, note that $F\left[\left((-ix)^p f\right)^{(q)}\right](\lambda) \to 0$ as $|\lambda| \to \infty$ by Lemma 3.2.2. Similarly, as $g^{(p)}$ is the Fourier transform of $(-ix)^p f \in L^1(\mathbb{R})$, we know by by Lemma 3.2.2 that $g^{(p)}(\lambda) \to 0$ as $|\lambda| \to \infty$. Therefore as,

$$\begin{aligned} F\left[\left((-ix)^p f\right)^{(q)}\right] &\overset{\text{Rem 3.2.6}}{=} (i\lambda)^q F\left[(-ix)^p f\right] \\ &\overset{\text{Lem 3.2.8}}{=} (i\lambda)^q g^{(p)}(\lambda) \end{aligned}$$

it must be the case that $g^{(p)}(\lambda)$ decays to zero faster than $\frac{1}{|\lambda|^q}$ as $|\lambda| \to \infty$. Thus if $f \in \mathcal{S}^\infty$ it follows that $g(\lambda) = F[f] \in \mathcal{S}^\infty$. $\square$

**Remark 3.3.4.** *Note the for* $f \in \mathcal{S}^\infty$ (2.2.3) *is satisfied and so the inverse Fourier transform of Definition 3.1.4 holds. In particular, in addition to Lemma 3.3.3 we also have that if* $F[f] \in \mathcal{S}^\infty$ *then* $f \in \mathcal{S}^\infty$*. Moreover, as Schwartz functions are continuous this correspondence is unique. Thus, the Fourier transform provides a bijection on* $\mathcal{S}^\infty$.

**Example 3.3.5.** *Consider* $f(x) = e^{-x^2}$*. Then* $f^{(n)}(x) = p_n(x)e^{-x^2}$ *where* $p_n(x)$ *is some polynomial. Note that for any* $k \in \mathbb{N}$ *we have*

$$e^{x^2} = \sum_{l=0}^{\infty}\frac{\left(x^2\right)^l}{l!} \geq \frac{\left(x^2\right)^k}{k!}$$

*so that $|x|^{-k}k! \geq |x|^k e^{-x^2}$. In particular, for $|x| \geq 1$ we have that $\left|x^k e^{-x^2}\right| \leq \frac{k!}{|x|^k} \leq k!$and for $|x| \leq 1$ as $x^k e^{x-^2}$ is continuous it follows that it is bounded on $|x| \leq 1$. Therefore, there exists a $M \in \mathbb{R}$ such that*

$$\left|x^k e^{-x^2}\right| \leq M$$

*for $x \in \mathbb{R}$. Therefore, there exists a $C = C(p, q, f)$ such that*

$$\left|x^p f^{(q)}(x)\right| < C(p, q, f)$$

*for all $x \in \mathbb{R}$, which implies that $f \in \mathcal{S}^\infty$. Indeed from statement 4 of Example 3.1.7 we have that $F[f] = \sqrt{2\pi}e^{-\frac{\lambda^2}{2}} = \sqrt{2\pi}f(\lambda) \in \mathcal{S}^\infty$, which verifies the conclusion of Lemma 3.3.3.*

**Theorem 3.3.6.** *The class $\mathcal{S}^\infty$ is dense in $L^p(\mathbb{R})$ for every $p \in [1, \infty)$.*

## 3.4 Fourier Transform in $L^2(\mathbb{R})$

Throughout this section, we will consider $L^2(\mathbb{R})$ as a complex Euclidean space. Thus we recall that for $f \in L^2([-\pi, \pi]) \subset L^1([-\pi, \pi])$ we defined the Fourier coefficients as

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x$$

for $n \in \mathbb{Z}$. Moreover, the map $f \to (c_n)_{n \in \mathbb{Z}}$ can be seen as a map $L^2([-\pi, \pi]) \to \ell^2$, that satisfies Parseval's equality,

$$2\pi \sum_{n \in \mathbb{Z}} |c_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x.$$

To extend the Fourier transform to $L^2(\mathbb{R})$ requires additional work as $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$ and so we cannot utilise the work of the previous section.

**Theorem 3.4.1** (Plancherel)**.** *For $f \in L^2(\mathbb{R})$, we have*

$$g_N(\lambda) = \int_{-N}^{N} f(x) e^{-i\lambda x} \, \mathrm{d}x \in L^2(\mathbb{R})$$

*for any $N$. Moreover, as $N \to \infty$ the function $g_N(\lambda)$ converges in $L^2(\mathbb{R})$ to some $g \in L^2(\mathbb{R})$ with,*

$$\int_{-\infty}^{\infty} |g(\lambda)|^2 \, \mathrm{d}\lambda = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x. \tag{3.4.1}$$

*If additionally $f \in L^1(\mathbb{R})$, then $g$ coincides with the usual Fourier transform of $f \in L^1(\mathbb{R})$.*

*Proof.* Step 1: Show the result for functions in $\mathcal{S}^\infty$.
Let $f_1, f_2 \in \mathcal{S}^\infty$, with $g_1, g_2$ denoting their Fourier transforms. By Lemma 3.3.3 we have $g_1, g_2 \in \mathcal{S}^\infty$. Applying the inverse Fourier transform and Fubini's theorem we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)} \, \mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} g_1(\lambda) e^{i\lambda x} \, \mathrm{d}\lambda \right) \overline{f_2(x)} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(\lambda) \overline{\int_{-\infty}^{\infty} f_2(x) e^{-i\lambda x} \, \mathrm{d}x} \, \mathrm{d}\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(\lambda) \overline{g_2(\lambda)} \, \mathrm{d}\lambda.
\end{aligned}$$

Setting $f_1 = f_2$ gives (3.4.1).
Step 2: Show the result for functions in $L^2(\mathbb{R})$ with compact support.
Let $f \in L^2(\mathbb{R})$ be such that $f(x) = 0$ for $x \notin [-a, a]$ for some $a > 0$. Then $f \in L^2(-a, a)$ which implies that $f \in L^1(-a, a)$, and as $f(x) = 0$ for $x \notin [-a, a]$ we have that $f \in L^1(\mathbb{R})$. Consequently, the Fourier transform of $f$ is exists and is given by

$$g(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x.$$

Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^\infty$ be such that $f_n(x) = 0$ for $x \notin [-a, a]$, and such that $f_n \to f$ in $L^2(-a, a)$. This exists due to Theorem 3.3.6. We note that $f_n \to f$ in $L^2(-a, a)$ implies that $f_n \to f$ in $L^1(-a, a)$, and so we also have $f_n \to f$ in $L^1(\mathbb{R})$. Therefore, using Lemma 3.2.1 we have that $g_n = F[f_n] \to g$ uniformly on $\mathbb{R}$. Note that $g_n - g_m \in \mathcal{S}_\infty$, and so using step 1 we have

$$\int_{-\infty}^{\infty} |g_n(\lambda) - g_m(\lambda)|^2\,\mathrm{d}\lambda = 2\pi \int_{-\infty}^{\infty} |f_n(x) - f_m(x)|^2\,\mathrm{d}x.$$

In particular, this means that $(g_n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(\mathbb{R})$ as $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(\mathbb{R})$. Thus, $g_n$ converges in $L^2(\mathbb{R})$, more specifically it must converge to $g$. Therefore, from step 1. we have $\|f_n\|_{L^2}^2 = \frac{1}{2\pi}\|g_n\|_{L^2}$ to which we can now take the limit as $n \to \infty$ to deduce (3.4.1) for $f \in L^2(\mathbb{R})$ with compact support.
Step 3: Show the result for functions in $L^2(\mathbb{R})$.
For $f \in L^2(\mathbb{R})$ let

$$f_N(x) := \begin{cases} f(x) & |x| \le N \\ 0 & |x| > N. \end{cases}$$

Clearly, $\|f - f_N\|_{L^2} \to 0$ as $N \to \infty$. By similar arguments as made before we know that $f_N \in L^1(\mathbb{R})$ and so its Fourier transform exists and is given by

$$g_N(\lambda) = \int_{-\infty}^{\infty} f_N(x)e^{-i\lambda x}\,\mathrm{d}x = \int_{-N}^{N} f(x)e^{-i\lambda x}\,\mathrm{d}x.$$

In particular, by step 2. we know that

$$\|f_N - f_M\|_{L^2}^2 = \frac{1}{2\pi}\|g_N - g_M\|_{L^2}^2,$$

and so $g_N$ converges in $L^2(\mathbb{R})$ to some $g \in L^2(\mathbb{R})$. Therefore, we can take the limit of $\|f_N\|_{L^2}^2 = \frac{1}{2\pi}\|g_N\|_{L^2}^2$, given by step 2, as $n \to \infty$ to deduce (3.4.1) for $f \in L^2(\mathbb{R})$.
Step 4: Coinciding with Fourier transform for functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the usual Fourier transform of $f$ exists,

$$\tilde{g}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x.$$

Since $f_N \to f$, from step 3, in $L^1(\mathbb{R})$ it follows that $g_N \to \tilde{g}$ uniformly on $\mathbb{R}$ by Lemma 3.2.1. However, we know that $g_N \to g$, from step 3, and so it must be the case that $g = \tilde{g}$. □

**Remark 3.4.2.**

1. *The function $g \in L^2(\mathbb{R})$ of Theorem 3.4.1 is called the Fourier transform of $f \in L^2(\mathbb{R})$. Indeed, if $f \in L^1(\mathbb{R})$ then $g$ as in Theorem 3.4.1 coincides with the Fourier transform of $f$ as given by Definition 3.1.3.*

2. *From* (3.4.1) *we see that as a linear operator in $L^2(\mathbb{R})$, the Fourier transform preserve norms, up to $2\pi$.*

**Corollary 3.4.3.** *For any $f_1, f_2 \in L^2(\mathbb{R})$ we have*

$$\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(\lambda)\overline{g_2(\lambda)}\,\mathrm{d}\lambda.$$

*Proof.* Using Theorem 3.4.1 let $g_1, g_2 \in L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} |f_1(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_1(\lambda)|\,\mathrm{d}\lambda$$

and

$$\int_{-\infty}^{\infty} |f_2(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_2(\lambda)|\,\mathrm{d}\lambda.$$

In particular, note that $f_1 + f_2 \in L^2(\mathbb{R})$, and so through the algebra of limits we have

$$\int_{-\infty}^{\infty} |f_1(x) + f_2(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_1(\lambda) + g_2(\lambda)|^2\,\mathrm{d}\lambda. \tag{3.4.2}$$

On the one hand,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_1(x) + f_2(x)|^2\,\mathrm{d}x &= \int_{-\infty}^{\infty} (f_1(x) + f_2(x))\,\overline{(f_1(x) + f_2(x))}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} |f_1(x)|^2 + f_1(x)\overline{f_2(x)} + \overline{f_1(x)}f_2(x) + |f_2(x)|^2\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_1(\lambda)|^2\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_2(\lambda)|^2\,\mathrm{d}\lambda + \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)} + \overline{f_1(x)}f_2(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_1(\lambda)|^2\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_2(\lambda)|^2\,\mathrm{d}\lambda + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{Re}\left(f_1(x)\overline{f_2(x)}\right)\,\mathrm{d}x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{-\infty}^{\infty} |g_1(\lambda) + g_2(\lambda)|^2\,\mathrm{d}\lambda &= \int_{-\infty}^{\infty} |g_1(\lambda)|^2 + |g_2(\lambda)|^2 + \left(g_1(\lambda)\overline{g_2(\lambda)} + \overline{g_1(\lambda)}g_2(\lambda)\right)\,\mathrm{d}\lambda \\
&= \int_{-\infty}^{\infty} |g_1(\lambda)|^2 + |g_2(\lambda)|^2 + \frac{1}{2}\mathrm{Re}\left(g_1(\lambda)\overline{g_2(\lambda)}\right)\,\mathrm{d}\lambda.
\end{aligned}$$

Thus returning to (3.4.2) it follows that

$$\mathrm{Re}\left(\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,\mathrm{d}x\right) = \frac{1}{2\pi}\mathrm{Re}\left(\int_{-\infty}^{\infty} g_1(\lambda)\overline{g_2(\lambda)}\,\mathrm{d}\lambda\right).$$

Similarly, noting that

$$\int_{-\infty}^{\infty} |f_1(x) + if_2(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g_1(\lambda) + ig_2(\lambda)|^2\,\mathrm{d}\lambda$$

it follows that

$$\mathrm{Im}\left(\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,\mathrm{d}x\right) = \frac{1}{2\pi}\mathrm{Im}\left(\int_{-\infty}^{\infty} g_1(\lambda)\overline{g_2(\lambda)}\,\mathrm{d}\lambda\right).$$

Therefore,

$$\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(\lambda)\overline{g_2(\lambda)}\,\mathrm{d}\lambda.$$

□

## 3.5 Laplace Transform

The Laplace transform aims to extend the Fourier transform beyond integrable functions.

**Definition 3.5.1.** *Let $\mathcal{L}$ be the class of functions, $f$, that satisfy the following statements.*

1. *$f(x)$ satisfies Dini's condition.*
2. *$f(x) = 0$ for $x < 0$.*
3. *$|f(x)| < Ce^{\gamma_0 x}$ for some $C, \gamma_0 > 0$.*

For $f \in \mathcal{L}$ let

$$g(s) := \int_{-\infty}^{\infty} f(x)e^{-isx}\,\mathrm{d}x,$$

where $s = \lambda + i\mu$ for $\lambda, \mu \in \mathbb{R}$. Despite $f$ being potentially exponentially large and not integrable, it follows that from condition 3 of Definition 3.5.1 that

$$g(s) = \int_0^{\infty} f(x)e^{\mu x}e^{-i\lambda x}\,\mathrm{d}x$$

exists and defines an analytic function of $s$ in the half plane $\mathrm{Im}(s) = \mu < -\gamma_0$. In particular, for fixed $\mu < -\gamma_0$, $g(s)$ is the Fourier transform of $f(x)e^{\mu x}$, thus using condition 1 of Definition 3.5.1 we can apply the inverse Fourier transform to deduce that

$$f(x)e^{\mu x} = \frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N} g(s)e^{i\lambda x}\,\mathrm{d}\lambda.$$

With the change of variables $p = is$, letting $\Phi(p) = g(s)$ and $\partial = -\mu$, we obtain

$$f(x) = \frac{1}{2\pi i}\int_{\partial - i\infty}^{\partial + i\infty} \Phi(p)e^{px}\,\mathrm{d}p$$

where $\partial > \gamma_0$ and

$$\Phi(p) = \int_0^{\infty} f(x)e^{-px}\,\mathrm{d}x. \tag{3.5.1}$$

We note that $\Phi(p)$ is analytic for $\mathrm{Re}(p) > \gamma_0$, as when $\mathrm{Re}(p) > \gamma_0$ we have that $\mathrm{Im}(s) < -\gamma_0$ for which we observed above that ensures that $\Phi(p) = g(s)$ is analytic.

**Definition 3.5.2.** *For $f \in \mathcal{L}$, the function $\Phi(p)$ as given by* (3.5.1) *is the Laplace transform of $f(x)$.*

### 3.5.1 Application to Ordinary Differential Equations

As before we consider the application of the Laplace transform to ordinary differential equations. Suppose

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x) \tag{3.5.2}$$

with $a_1, \ldots, a_n \in \mathbb{C}$ has the initial conditions $y^k(0) = y_k$ for $k = 0, \ldots, n-1$. Assuming that $b(x) \in \mathcal{L}$ we seek a solution such that $y^{(k)} \in \mathcal{L}$ for $k = 0, \ldots, n$. Let

$$Y(p) = \int_0^{\infty} y(x)e^{-px}\,\mathrm{d}x$$

and

$$B(p) = \int_0^{\infty} b(x)e^{-px}\,\mathrm{d}x.$$

Using integration by parts, and an inductive argument it follows that

$$\int_0^{\infty} y^{(k)}(x)e^{-px} = p^k Y(p) - y_{k-1} - py_{k-2} - \cdots - p^{k-1}y_0$$

for $k = 1, \dots, n$. Thus applying the Laplace transform to (3.5.2) yields

$$Q(p) + R(p)Y(p) = B(p)$$

where

$$R(p) = p^n + a_1 p^{n-1} + \cdots + a_n,$$

and $Q(p)$ is a polynomial of degree $n-1$ dependent on $y_0, \dots, y_{n-1}$. Consequently, one can show that

$$y(x) = \frac{1}{2\pi i} \int_{\partial - i\infty}^{\partial + i\infty} \frac{B(p) - Q(p)}{R(p)} e^{ipx} \, \mathrm{d}p,$$

which can then be computed using residues. This method for obtaining a solution to a linear differential equation with constant coefficients is known as the operator method.

**Exercise 3.5.3.** *Using the Laplace transform, solve the differential equation*

$$y^{(3)}(x) + y(x) = 1$$

*for $y(x) \in \mathbb{R}$ satisfying the initial conditions $y(0) = y'(0) = y''(0) = 0$.*

## 3.6 Fourier-Stiltjes Transform

Recall that for $f \in L^1(\mathbb{R})$ the Fourier transform is given by

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) \, \mathrm{d}x,$$

which as a Lebesgue-Stiltjes integral can be written as

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x} \, \mathrm{d}F(x), \tag{3.6.1}$$

where

$$F(x) = \int_{-\infty}^{x} f(t) \, \mathrm{d}t. \tag{3.6.2}$$

**Definition 3.6.1.** *A function $F(x)$ is of bounded variation on $[a, b]$ if*

$$V_a^b F := \sup \left( \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right) < \infty,$$

*where the supremum is over finite division of $[a, b]$ of the form*

$$a_0 = x_0 \leq \cdots \leq x_n = b,$$

*where $n \in \mathbb{N}$ can vary. Similarly, a function $F(x)$ is of bounded variation on $\mathbb{R}$ if*

$$V_{-\infty}^{\infty} F = \mathrm{Var}(F) := \lim_{a \to -\infty, b \to \infty} V_a^b F < \infty.$$

**Remark 3.6.2.**

1. *A function of bounded variation can be written as a difference of monotone functions.*
2. *A function bounded variation is differentiable almost everywhere. Indeed, a function $F$ of bounded variation can be written as*
$$F = \varphi(x) + \psi(x) + \eta(x)$$

*where $\varphi(x)$ is absolutely continuous, $\psi(x)$ is singular continuous, and $\eta(x)$ is a jump function. Hence, $F'(x) = \varphi'(x)$ almost everywhere, as $\psi'(x) = \eta'(x) = 0$ almost everywhere.*

**Example 3.6.3.**

1. *If $F(x)$ is a monotonically increasing function on $[a, b]$ then*

$$V_a^b F = F(b) - F(a).$$

2. *If $F(x)$ is a differentiable function on $[a, b]$ then*

$$V_a^b = \int_a^b F'(x)\,\mathrm{d}x.$$

For (3.6.1), we note that $F(x)$ is absolutely continuous with a bounded variation on $\mathbb{R}$, indeed

$$\mathrm{Var}(F) = \int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty.$$

However, we note that (3.6.1) is well-defined even if $F(x)$ is not directly of the form (3.6.2). It is sufficient for $F(x)$ to be of bounded variation on $\mathbb{R}$, for (3.6.2) to be well-defined.

**Definition 3.6.4.** *For $F(x)$ a function of bounded variation on $\mathbb{R}$, the function*

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F(x)$$

*is the Fourier-Stiltjes transform of $F(x)$.*

**Example 3.6.5.** *For $x_1 < x_2 < \cdots < x_n$ and $a_1, \ldots, a_n \in \mathbb{R}$ consider the step function*

$$F(x) = \begin{cases} \sum_{x_k < x} a_k & x \geq x_1 \\ 0 & x \leq x_1. \end{cases}$$

*For $a < x_1$ and $x_n < b$ we have*

$$\begin{aligned} \int_a^b e^{-i\lambda x}\,\mathrm{d}F(x) &= \sum_{k=1}^{n} e^{-i\lambda x_k} \left(F(x_k)_+ - F(x_k)_-\right) \\ &= e^{-i\lambda x_1}(a_1 - 0) + \sum_{k=2}^{n} e^{-i\lambda x_k} \left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{k-1} a_i\right) \\ &= e^{-i\lambda x_1} a_1 + \sum_{k=2}^{n} e^{-i\lambda x_k} a_k \\ &= \sum_{k=1}^{n} e^{-i\lambda x_k}. \end{aligned}$$

*Sending $a \to -\infty$ and $b \to \infty$ it follows that*

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F(x) = \sum_{k=1}^{n} e^{-i\lambda x_k} a_k.$$

**Lemma 3.6.6.** *The Fourier-Stiltjes transform of a function $F$ of bounded variation on $\mathbb{R}$, is bounded and continuous on $\mathbb{R}$.*

*Proof.* The Lebesgue-Stiltjes measure of an interval corresponding to $V_{-\infty}^{x}F$ is greater than or equal to the measure of the interval corresponding to $F(x)$. Therefore, the following can be deduced.

1. Note that
$$|g(\lambda)| \leq \int_{-\infty}^{\infty} \left|e^{-i\lambda x}\right| \mathrm{d}Fx = \int_{-\infty}^{\infty} \mathrm{d}V_{-\infty}^{x}F < \infty.$$
So $g(\lambda)$ is bounded.

2. Note that
$$|g(\lambda_1) - g(\lambda_2)| \leq \underbrace{\int_{-N}^{N} \left|e^{-i\lambda_1 x} - e^{-i\lambda_2 x}\right| \mathrm{d}V_{-\infty}^{x}F}_{I_1} + \underbrace{\int_{|x|\geq N} \left|e^{-i\lambda_1 x} - e^{-i\lambda_2 x}\right| \mathrm{d}V_{-\infty}^{x}F}_{I_2}.$$
Since $F$ is of bounded variation and $\left|e^{-i\lambda_1 x} - e^{-i\lambda_2 x}\right|$ the integral $I_2$ can be made arbitrarily small for large $N$ uniformly over $\lambda_1$ and $\lambda_2$. With this fixed $N$, we note that
$$\left|e^{-i\lambda_1 x} - e^{-i\lambda_2 x}\right| = \left|2\sin\left(\frac{(\lambda_1 - \lambda_2)\,x}{2}\right)\right| \overset{|\lambda_1-\lambda_2|\to 0}{\longrightarrow} 0.$$
Hence, $I_1 \to 0$ as $|\lambda_1 - \lambda_2| \to 0$. Therefore, $g$ is uniformly continuous.

□

**Example 3.6.7.** *Unlike the Fourier transform, the Fourier-Stiltjes transform does not necessarily decay as $|\lambda| \to \infty$. Indeed consider*
$$F(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$
*Then*
$$g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F(x) = e^{-i\lambda\cdot 0}\left(F\left(0_+\right) - F\left(0_-\right)\right) = 1,$$
*for all $\lambda$.*

**Exercise 3.6.8.** *Let $F(x) \in \mathcal{S}^\infty$ have Fourier-Stiltjes transform $g(\lambda)$. Show that,*
$$F(b) - F(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)\frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}\,\mathrm{d}\lambda$$
*for $a < b$.*

### 3.6.1 Convolution

Recall the convolution of $f_1, f_2 \in L^1(\mathbb{R})$ as given by Definition 3.2.11. Now let
$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$
and
$$F_j(x) = \int_{-\infty}^{x} f_j(t)\,\mathrm{d}t$$

for $j = 1, 2$. Using the absolute integrability fo $f, f_1$ and $f_2$, it follows by Fubini's theorem that

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_1(t-y) f_2(y)\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{x} f_1(t-y)\,\mathrm{d}t \right) f_2(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} F_1(x-y)\,\mathrm{d}F_2(y).
\end{aligned}
$$

However, the resulting integral is well-defined more generally, not just when $F_1$ and $F_2$ are absolutely continuous as is the case here. Indeed, a function of bounded variation $F$ is Borel measurable. Thus, the integral of $F_1$ with respect to $F_2$ is well-defined provided $F_1$ is of bounded variation. Moreover, the integral is finite provided $F_2$ is of bounded variation.

**Definition 3.6.9.** *For $F_1, F_2$ functions of bounded variation on $\mathbb{R}$, their convolution is given by*

$$
F(x) = F_1 \star F_2 := \int_{-\infty}^{\infty} F_1(x-y)\,\mathrm{d}F_2(y).
$$

**Lemma 3.6.10.** *The function $F_1 \star F_2$, as in Definition 3.6.9, is of bounded variation on $\mathbb{R}$.*

*Proof.* Observe that

$$
\begin{aligned}
|F(x_1) - F(x_2)| &= \left| \int_{-\infty}^{\infty} (F_1(x_1-y) - F_1(x_2-y))\,\mathrm{d}F_2(y) \right| \\
&\leq \int_{-\infty}^{\infty} |F_1(x_1-y) - F_1(x_2-y)|\,\mathrm{d}V_{-\infty}^{y} F_2.
\end{aligned}
$$

Which implies that

$$
\mathrm{Var}(F) \leq \mathrm{Var}(F_1)\mathrm{Var}(F_2) < \infty.
$$

□

**Theorem 3.6.11.** *Let $F = F_1 \star F_2$, where $F_1$ and $F_2$ are of bounded variation on $\mathbb{R}$. Let $g, g_1, g_2$ be their respective Fourier-Stiltjes transform. Then*

$$
g(\lambda) = g_1(\lambda) g_2(\lambda).
$$

*Proof.* Let

$$
a = x_0 \leq x_1 \leq \cdots \leq x_n = b.
$$

Then for any $\lambda$, since $e^{-i\lambda x}$ is continuous, the Lebesgue-Stiltjes integral coincides with the Riemann-Stiltjes integral

$$
\begin{aligned}
\int_a^b e^{-i\lambda x}\,\mathrm{d}F(x) &= \lim_{\max(\Delta x_k) \to 0} \sum_{k=1}^{n} e^{-i\lambda x_k} \left(F\left(x_k\right) - F\left(x_{k-1}\right)\right) \\
&= \lim_{\max(\Delta x_k) \to 0} \int_{-\infty}^{\infty} \sum_{k=1}^{n} e^{-i\lambda(x_k - y)} \left(F_1\left(x_k - y\right) - F_1\left(x_{k-1} - y\right)\right) e^{-i\lambda y}\,\mathrm{d}F_2(y).
\end{aligned}
$$

That is,

$$
\int_a^b e^{-i\lambda x}\,\mathrm{d}F(x) = \int_{-\infty}^{\infty} \int_{a-y}^{b-y} e^{-i\lambda x}\,\mathrm{d}F_1(x) e^{-i\lambda y}\,\mathrm{d}F_2(y),
$$

where the limit has been brought into the integral using the dominated convergence theorem. By another application of the dominated convergence theorem, it follows by taking the limit $a \to -\infty$ and $b \to \infty$ that

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F(x) = \int_{-\infty}^{\infty} e^{-i\lambda y}\,\mathrm{d}F_2(y) \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F_1(x).$$

In other words, $g(\lambda) = g_2(\lambda)g_1(\lambda)$. $\square$

## 3.7 Application to Probability

Let $\zeta$ and $\eta$ be independent random variables with distribution functions $F_1$ and $F_2$ respectively. Then $F = F_1 \star F_2$ is the distribution function of $\zeta + \eta$. In probability theory, the Fourier-Stiltjes transform is known as the method of characteristic functions. That is,

$$g_1(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F_1$$

is known as the characteristic function of $\zeta + \eta$. Consequently, we have that the characteristic function of $\zeta + \eta$ is the product of the characteristic functions of $\zeta$ and $\eta$.

## 3.8 Solution to Exercises

### Exercise 3.1.1

*Solution.* With $\gamma_1 = [r, R]$ $\gamma_2 = [-R, -r]$, $\gamma_r = \left\{re^{i\theta} : \theta \in [\pi, 0]\right\}$ and $\gamma_R = \left\{Re^{i\theta} : \theta \in [0, \pi]\right\}$, let

$$\gamma = \gamma_1 \cup \gamma_R \cup \gamma_2 \cup \gamma_r.$$

Then as $\frac{e^{iaz}}{z}$ is analytic in $\gamma$ it follows that

$$0 = \oint_{\gamma} \frac{e^{iaz}}{z}\,\mathrm{d}z.$$

Note that

$$\begin{aligned}
\left|\int_{\gamma_R} \frac{e^{iaz}}{z}\,\mathrm{d}z\right| &= \left|\int_0^{\pi} \frac{e^{iaRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}\,\mathrm{d}\theta\right| \\
&= \left|\int_0^{\pi} e^{-aR\sin\theta} e^{iaR\cos\theta}\,\mathrm{d}\theta\right| \\
&\leq \int_0^{\pi} e^{-aR\sin\theta}\,\mathrm{d}\theta \\
&\overset{R\to\infty}{\longrightarrow} 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\gamma_r} \frac{e^{iaz}}{z}\,\mathrm{d}z &= \int_{\pi}^{0} ie^{iare^{i\theta}}\,\mathrm{d}\theta \\
&\overset{r\to 0}{\longrightarrow} \int_{\pi}^{0} i\,\mathrm{d}\theta \\
&= -i\pi.
\end{aligned}$$

Therefore, sending $R \to \infty$ and $r \to 0$ it follows that

$$0 = \int_0^{\infty} \frac{e^{iaz}}{z}\,\mathrm{d}z + 0 + \int_{-\infty}^{0} \frac{e^{iaz}}{z}\,\mathrm{d}z - i\pi.$$

Looking at the imaginary parts we have

$$0=\int_{-\infty}^{\infty}\frac{\sin(az)}{z}\,\mathrm{d}z-\pi$$

and so

$$1=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(az)}{z}\,\mathrm{d}z.$$

□

**Exercise 3.2.4**

*Solution.* Given $\epsilon>0$, let $M\in\mathbb{R}$ be such that $|f(x)|<\frac{\epsilon}{3}$ for $|x|\geq M$. Then since $f$ is continuous, it follows that $f$ is uniformly continuous on $[-M,M]$. In particular, let $\delta>0$ be such that $|f(x)-f(y)|<\frac{\epsilon}{3}$ for $x,y\in[-M,M]$ with $|x-y|<\delta$. Let $x,y\in\mathbb{R}$ be such that $|x-y|<\delta$.

- If $x,y\in[-M,M]$ we have
$$|f(x)-f(y)|<\frac{\epsilon}{3}<\epsilon.$$
- If $|x|,|y|>M$, then
$$|f(x)-f(y)|<|f(x)|+|f(y)|<\frac{\epsilon}{3}+\frac{\epsilon}{3}<\epsilon.$$
- If $x\in[M-\delta,M]$ and $y>M$, then
$$\begin{aligned}|f(x)-f(y)|&\leq|f(x)-f(M)|+|f(M)-f(y)|\\&\leq\frac{\epsilon}{3}+|f(M)|+|f(y)|\\&<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\&=\epsilon.\end{aligned}$$

Therefore, in any case, for $x,y\in\mathbb{R}$ with $|x-y|<\delta$ we have $|f(x)-f(y)|<\epsilon$, which implies that $f$ is uniformly continuous. □

**Exercise 3.2.7**

*Solution.* Using Remark 3.2.6 we have that

$$|F[f](\lambda)|\leq\frac{C}{\lambda^2}$$

for all $\lambda\in\mathbb{R}$. Therefore, as $\frac{1}{\lambda^2}\in L^1(\mathbb{R})$ it follows that $F[f]\in L^1(\mathbb{R})$. □

**Exercise 3.5.3**

*Proof.* Note that

$$\int_0^{\infty}y^{(3)}(x)e^{-px}\,\mathrm{d}x=p^3Y(p)-y''(0)-py'(0)-p^2y(0)=p^3Y(p),$$

and

$$\int_0^{\infty}e^{-px}\,\mathrm{d}x=\frac{1}{p}.$$

Hence,

$$\left(p^3+1\right)Y(p)=\frac{1}{p}.$$

Thus,

$$y(x)=\frac{1}{2\pi i}\int_{\partial-i\infty}^{\partial+i\infty}\frac{e^{ipx}}{p\left(p^3+1\right)}\,\mathrm{d}p.$$

for $\partial > 0$ larger than the real component of any pole. For $x > 0$, taking the left semi-circle contour and using Jordan's lemma it follows that

$$\begin{aligned} y(x) &= \operatorname{Res}\left(\frac{e^{ipx}}{p\left(p^3+1\right)}, 0\right) + \operatorname{Res}\left(\frac{e^{ipx}}{p\left(p^3+1\right)}, -1\right) \\ &\quad + \operatorname{Res}\left(\frac{e^{ipx}}{p\left(p^3+1\right)}, e^{\frac{\pi i}{3}}\right) + \operatorname{Res}\left(\frac{e^{ipx}}{p\left(p^3+1\right)}, e^{\frac{-\pi i}{3}}\right) \\ &= 1 - \frac{1}{3}e^{-x} - \frac{1}{3}e^{\frac{1}{2}x}\left(e^{i\frac{\sqrt{3}}{2}x} + e^{-i\frac{\sqrt{3}}{2}x}\right) \\ &= 1 - \frac{1}{3}e^{-x} - \frac{2}{3}e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right). \end{aligned}$$

For $x < 0$, taking the right semi-circle contour and using Jordan's lemma it follows that

$$y(x) = 0.$$

□

**Exercise 3.6.8**

*Solution.* We have

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F(x), \tag{3.8.1}$$

thus for fixed $\rho$ we have

$$g(\lambda)e^{i\rho\lambda} = \int_{-\infty}^{\infty} e^{-i(x-\rho)\lambda}\,\mathrm{d}F(x) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}F(x+\rho) \tag{3.8.2}$$

Subtracting (3.8.1) from (3.8.2) it follows that

$$g(\lambda)\left(e^{i\rho\lambda} - 1\right) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left(\mathrm{d}F(x+\rho) - \mathrm{d}F(x)\right).$$

Letting $G(x) = F(x+\rho) - F(x)$ we have

$$g(\lambda)\left(e^{i\rho\lambda} - 1\right) = \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}G(x).$$

As $|x| \to \infty$ note that $G(x) \to 0$, and so

$$\begin{aligned} g(\lambda)\left(e^{i\rho\lambda} - 1\right) &= \int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}G(x) \\ &= \left(\left[e^{-i\lambda x}G(x)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\lambda e^{-i\lambda x}G(x)\,\mathrm{d}x\right) \\ &= i\lambda\int_{-\infty}^{\infty} e^{-i\lambda x}G(x)\,\mathrm{d}x. \end{aligned}$$

Since, $G(x) \in \mathcal{S}^\infty$ we can apply the inverse Fourier transform to deduce that

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)\frac{e^{i\rho\lambda} - 1}{i\lambda}e^{i\lambda x}\,\mathrm{d}x.$$

Setting $\rho = b - a$ and $x = a$ it follows that

$$\begin{aligned} F(b) - F(a) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)\frac{e^{i(b-a)\lambda} - 1}{i\lambda}e^{ia\lambda}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)\frac{e^{ib\lambda} - e^{ia\lambda}}{i\lambda}\,\mathrm{d}\lambda. \end{aligned}$$

□

# 4 Linear Functionals on Normed Linear Spaces

## 4.1 Linear Functionals

**Definition 4.1.1.** *Let $L$ be a linear space. Then $f : L \to \mathbb{C}$, or $\mathbb{R}$, is a functional on $L$. Moreover, it is linear if*

$$f(x+y) = f(x) + f(y)$$

*and*

$$f(\alpha x) = \alpha(x)$$

*for all $x, y \in L$ and $\alpha \in \mathbb{C}$, or $\mathbb{R}$.*

**Definition 4.1.2.** *For $f$ a linear functional on a linear space $L$, the kernel of $f$ is*

$$\ker(f) := \{x \in L : f(x) = 0\}.$$

**Lemma 4.1.3.** *The codimension of the kernel of a linear functional defined on a linear space is one.*

*Proof.* Let $f : L \to \mathbb{C}$ be a non-zero functional. Then there exists an $x_0 \in L$ such that $f(x_0) \neq 0$. In particular, using the linearity of $f$ we can assume without loss of generality that $f(x_0) = 1$. Note that for $x \in L$ we have $f(x - f(x)x_0) = f(x) - f(x)f(x_0) = 0$, which implies that $x - f(x)x_0 \in \ker(f)$. Hence, we can write $x = f(x)x_0 + y$ for some $y \in \ker(f)$. Moreover, suppose that $x = \lambda x_0 + \tilde{y}$ for some $\lambda \in \mathbb{C}$ and $\tilde{y} \in \ker(f)$. Then,

$$0 = f((\lambda - f(x))x_0 + \tilde{y} - y) = (\lambda - f(x))f(x_0) + f(\tilde{y}) - f(y) = \lambda - f(x),$$

and so $\lambda = f(x)$. This implies that $\tilde{y} = y$ and so the representation $x = f(x)x_0 + y$ is unique. Thus, we deduce that $L/\ker(f) = \operatorname{span}(x_0)$ and so the codimension of the kernel of $f$ is $1$. $\square$

**Lemma 4.1.4.** *Suppose that $f$ is a non-zero linear functional on a linear space $L$. Then $f$ is uniquely determined by $\{x \in L : f(x) = 1\}$.*

*Proof.* Let $f : L \to \mathbb{C}$ be a linear functional and let $E_f := \{x \in L : f(x) = 1\}$. For $x \in L$ note that $f\left(\frac{x}{f(x)}\right) = 1$ and so $\frac{x}{f(x)} \in E_f$. Thus $f$ is determined by $E_f$. For another linear functional $\tilde{f} : L \to \mathbb{C}$ suppose that $E_f = E_{\tilde{f}}$. For $x \in L$, as $\frac{x}{f(x)} \in E_f$ it follows that $\frac{x}{f(x)} \in E_{\tilde{f}}$. Therefore,

$$1 = \tilde{f}\left(\frac{x}{f(x)}\right) = \frac{\tilde{f}(x)}{f(x)}$$

which implies that $f(x) = \tilde{f}(x)$. As $x \in L$ was arbitrary, we deduce that $f \equiv \tilde{f}$. Therefore, $E_f$ uniquely determines a linear functional. $\square$

**Definition 4.1.5.** *A function $\|\cdot\| : E \to \mathbb{R}$ on a linear space $E$ is a norm if the following statements are satisfied.*

1. *$\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$.*
2. *$\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{C}$, or $\alpha \in \mathbb{R}$ if $E$ is a real linear space.*
3. *$\|x + y\| \leq \|x\| + \|y\|$*

*A linear space $E$ with a norm $\|\cdot\|$ is a normed linear space.*

**Remark 4.1.6.** *Note that a normed vector space $E$ is a metric space with*

$$\rho(x, y) = \|x - y\|.$$

*Thus a normed vector is induced with a topology.*

Henceforth, $E$ will denote a normed linear space.

**Definition 4.1.7.** *A functional $f$ on $E$ is continuous if for any $x_0 \in E$ and $\epsilon > 0$, there exists a neighbourhood $U$ of $x_0$ such that*

$$|f(x) - f(x_0)| < \epsilon$$

*for $x \in U$.*

**Exercise 4.1.8.** *Suppose that $E$ is a finite-dimensional normed vector space. Show that any linear functional is continuous.*

**Lemma 4.1.9.** *If a linear functional is continuous at some $x \in E$ then it is continuous on $E$.*

*Proof.* Suppose a linear functional $f : E \to \mathbb{C}$ is continuous as $x \in E$. Let $y \in E$ and $\epsilon > 0$. As $f$ is continuous at $x$, there exists a neighbourhood $U \subset E$ of $x$ such that $|f(x) - f(t)| < \epsilon$ for $t \in U$. Let $V := U + (y - x)$. Then $V$ is a neighbourhood of $y$, such that for $z \in V$ we have $z + x - y \in U$ and so

$$\epsilon > |f(x) - f(z + x - y)| = |f(y) - f(z)|,$$

where in the second equality we have used the linearity of $f$. It follows that $f$ is continuous at $y \in E$, and thus $f$ is continuous on $E$. ☐

**Theorem 4.1.10.** *A linear functional $f$ on $E$ is continuous if and only if there is a neighbourhood of $0 \in E$ on which $f$ is bounded.*

*Proof.* ($\Rightarrow$). As $f$ is continuous on $E$ it is continuous at $0 \in E$. As $f(0) = 0$, it follows that for any $\epsilon > 0$ there exists a neighbourhood $U \subset E$ of $0 \in E$ such that $|f(x)| < \epsilon$ for $x \in U$.
($\Leftarrow$). Let $V \subset E$ be a neighbourhood of $0 \in E$ such that $|f(x)| < c$ for $x \in V$. For $\epsilon > 0$, then using the linearity of $f$ we have

$$\left| f\left(\frac{\epsilon}{c}x\right) \right| = \frac{\epsilon}{c}|f(x)| < \frac{\epsilon}{c}c = \epsilon.$$

Hence, $\frac{\epsilon}{c}V \subset E$ is a neighbourhood of $0 \in E$ such that $|f(x) - f(0)| = |f(x)| < \epsilon$. Therefore, $f$ is continuous at $0 \in E$ which implies that it is continuous on $E$ be Lemma 4.1.9. ☐

**Corollary 4.1.11.** *A linear functional $f$ on $E$ is continuous if and only if it is bounded on $\{x \in E : \|x\| \leq 1\}$.*

*Proof.* Any neighbourhood of $0 \in E$ contains a ball of sufficiently small radius. Being bounded on this ball is equivalent to being bounded on $\{x \in E : \|x\| \leq 1\}$ through linearity. Therefore, using Theorem 4.1.10, a linear functional is continuous if and only if it is bounded on $\{x \in E : \|x\| \leq 1\}$. ☐

**Exercise 4.1.12.** *Let $E$ be a normed linear space and let $f$ be a linear functional on $E$. Show that the following are equivalent.*

1. *$f$ is continuous on $E$.*
2. *There exists an open set $U \subset E$ and a $t \in \mathbb{R}$ such that $t \notin f(U)$.*

3. *The kernel of $f$ is closed in $E$.*
4. *$f$ is bounded on any bounded subset of $E$.*

**Definition 4.1.13.** *Let $f$ be a continuous linear functional on $E$. Then the norm of $f$ is*

$$\|f\| := \sup_{\|x\|\leq 1} |f(x)|.$$

*Equivalently,*

$$\|f\| = \sup_{x\in E\setminus\{0\}} \frac{|f(x)|}{\|x\|}.$$

**Remark 4.1.14.**

1. *Note that Definition 4.1.13 is well-defined due to Corollary 4.1.11.*
2. *From Definition 4.1.13, it is clear that*
$$|f(x)| \leq \|f\|\|x\| \tag{4.1.1}$$
*for all $x \in E$.*

**Example 4.1.15.**

1. *Consider Euclidean space $\mathbb{R}^n$, with $a \in \mathbb{R}^n$ defining the linear functional $f(x) = (x, a)$. Using Cauchy-Schwartz we have*
$$|f(x)| = |(x, a)| \leq \|x\|\|a\|$$
*and so $f$ is bounded on the unit ball by $\|a\|$. Thus, $f$ is continuous. In particular, it follows that*
$$\sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x)|}{\|x\|} \leq \|a\|.$$
*However, we note that for $x = a$ we have*
$$\frac{|f(x)|}{\|x\|} = \frac{\|a\|^2}{\|a\|} = \|a\|.$$
*Therefore,*
$$\sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x)|}{\|x\|} = \|a\|,$$
*and so $\|f\| = \|a\|$.*
   - *More generally, for $a \in X$, where $X$ is a Euclidean space, consider the linear functional $F(x) = (x, a)$. Then $F$ is continuous with $\|F\| = \|a\|$.*
2. *Consider the space $\mathcal{C}([a, b])$ with norm $\|x\| = \max_{t\in[a,b]} |x(t)|$. Then*
$$I(x) := \int_a^b x(t)\,\mathrm{d}t$$

*is a linear functional of* $\mathcal{C}([a,b])$. *In particular,*

$$\begin{aligned}|I(x)| &= \left|\int_a^b x(t)\,\mathrm{d}t\right| \\ &\leq (b-a)\max_{t\in[a,b]}|x(t)| \\ &= \|x\|(b-a).\end{aligned}$$

*We note that equality is reached when* $x \in \mathcal{C}([a,b])$ *is constant, and so we conclude that* $I(x)$ *is a continuous linear functional with norm* $b-a$.

3. *For* $y_0 \in \mathcal{C}([a,b])$ *consider the linear functional*

$$F(x) := \int_a^b x(t)y_0(t)\,\mathrm{d}t.$$

*Then,*

$$\begin{aligned}|F(x)| &= \left|\int_a^b x(t)y_0(t)\,\mathrm{d}t\right| \\ &\leq \int_a^b |x(t)||y_0(t)|\,\mathrm{d}t \\ &\leq \|x\|\int_a^b |y_0(t)|\,\mathrm{d}t.\end{aligned}$$

*Therefore,* $F$ *is bounded by* $\int_a^b |y_0(t)|\,\mathrm{d}t \leq \|y_0\|(b-a) < \infty$ *on the unit ball of* $\mathcal{C}([a,b])$ *and so is a continuous linear functional. Moreover,*

$$\|F\| = \sup_{x\in\mathcal{C}([a,b])\setminus\{0\}} \frac{|F(x)|}{\|x\|} \leq \int_a^b |y_0(t)|\,\mathrm{d}t.$$

*If* $y_0(t) \equiv 0$, *then it is clear that* $\|F\| = \int_a^b |y_0(t)|\,\mathrm{d}t = 0$. *So suppose* $y_0(t) \neq 0$ *and let* $x_n(t) = \frac{y_0(t)}{|y_0(t)|+\frac{1}{n}}$,*which is continuous, as* $|y_0(t)| + \frac{1}{n} \neq 0$. *Observe that*

$$\begin{aligned}|F(x_n)| &= \left|\int_a^b \frac{y_0(t)^2}{|y_0(t)|+\frac{1}{n}}\right|\mathrm{d}t \\ &= \int_a^b \frac{y_0(t)^2}{|y_0(t)|+\frac{1}{n}}\,\mathrm{d}t.\end{aligned}$$

*In particular,* $\frac{y_0(t)^2}{|y_0(t)|+\frac{1}{n}} \to |y_0(t)|$ *as* $n\to\infty$ *with* $\frac{y_0(t)^2}{|y_0(t)|+\frac{1}{n}} \leq |y_0(t)|$ *which is integrable. Therefore, by the dominated convergence theorem it follows that*

$$\lim_{n\to\infty}|F(x_n)| = \lim_{n\to\infty}\int_a^b \frac{y_0(t)^2}{|y_0(t)|+\frac{1}{n}}\,\mathrm{d}t = \int_a^b |y_0(t)|\,\mathrm{d}t.$$

*Which implies that* $\|F\| \geq \int_a^b |y_0(t)|\,\mathrm{d}t$ *and so*

$$\|F\| = \int_a^b |y_0(t)|\,\mathrm{d}t.$$

4. *Let $t_0 \in [a,b]$, and consider the linear functional $\delta_{t_0}(x) := x(t_0)$ on $\mathcal{C}([a,b])$. Then*

$$|\delta_{t_0}(x)| = |x(t_0)| \leq \|x\|$$

*with equality when $x$ is constant. Therefore, $\delta_{t_0}(x)$ is continuous with $\|\delta_{t_0}\| = 1$.*

Suppose $f$ is a linear functional on the normed vector space $E$, and consider the hyperplane $F = \{x \in E : f(x) = 1\}$. The distance from the origin to $F$ is given by

$$d := \inf_{x \in F} \|x\|.$$

Using (4.1.1) on $F$ we have that $\|x\| \geq \frac{1}{\|f\|}$ and so

$$d \geq \frac{1}{\|f\|}.$$

On the other hand, by Definition 4.1.13, for all $\epsilon > 0$ there exists an $x_\epsilon \in F$ such that $1 = f(x_\epsilon) > (\|f\| - \epsilon)\|x_\epsilon\|$. Consequently,

$$d < \frac{1}{\|f\| - \epsilon}$$

which implies that

$$d = \frac{1}{\|f\|}.$$

Thus, we have a geometric interpretation of the norm of a linear functional, namely, it is the reciprocal of the distance between the origin and the unit level-set of the functional.

**Definition 4.1.16.** *Let $p$ be a non-negative functional on a linear space $L$. Then $p$ is convex if*

$$p(x+y) \leq p(x) + p(y)$$

*and*

$$p(\alpha x) = |\alpha| p(x)$$

*for all $x, y \in L$ and $\alpha \in \mathbb{C}$.*

**Remark 4.1.17.** *A norm is a convex functional.*

**Theorem 4.1.18** (Hanh-Banach)**.** *Let $L$ be a linear space, and let $p$ be a convex functional on $L$. Suppose that $f_0$ is a linear functional on a subspace $L_0 \subset L$ and is such that $|f_0(x)| \leq p(x)$ for all $x \in L_0$. Then there exists a linear functional, $f$, on $L$ such that the following are satisfied.*

1. *$f(x) = f_0(x)$ for all $x \in L_0$.*
2. *$|f(x)| \leq p(x)$ for all $x \in L$.*

**Theorem 4.1.19** (Hanh-Banach on Normed Linear Spaces)**.** *Let $E$ be a normed linear space, and let $f_0$ be a continuous linear functional defined on a subspace $E_0 \subset E$. Then there exists a continuous linear functional on $E$ such that the following are satisfied.*

1. *$f(x) = f_0(x)$ for all $x \in E_0$.*
2. *$\|f\|_{E \to \mathbb{C}} = \|f_0\|_{E_0 \to \mathbb{C}}$.*

*Proof.* Let $\|f_0\|_{E_0 \to \mathbb{C}} = c$. Observe that $p(x) := c \cdot \|x\|$ is a convex functional of $E$ such that $|f_0(x)| \leq p(x)$. Applying Theorem 4.1.18 we obtain a linear functional $f$ on $E$ such that $f(x) = f_0(x)$ for all $x \in E_0$ and $|f(x)| \leq c\|x\|$. In particular, since $\|f_0\|_{E_0 \to \mathbb{C}} = c$ it must be the case that $\|f\|_{E \to \mathbb{C}} = c$ as $E_0 \subset E$. $\square$

**Corollary 4.1.20.** *Let $E$ be a normed linear space, and let $x_0 \in E \setminus \{0\}$. Then there exists a linear functional $f$ on $E$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

*Proof.* Let $E_0 = \{\alpha x_0 : \alpha \in \mathbb{C}\}$, and let $f_0 : E_0 \to \mathbb{C}$ be given by $\alpha x_0 \mapsto \alpha\|x_0\|$. Clearly, $\|f_0\|_{E_0 \to \mathbb{C}} = 1$, and so by Theorem 4.1.19, there exists a functional $f : E \to \mathbb{C}$ such that

$$\|f\|_{E \to \mathbb{C}} = \|f_0\|_{E_0 \to \mathbb{C}} = 1$$

and

$$f(x_0) = f_0(x_0) = 1.$$

□

## 4.2 The Adjoint Space

So far we have been considering linear functionals individually. However, we can also view linear functionals as a space in their own right. Throughout, $E$ is a linear space, with $f_1, f_2$ linear functionals on $E$.

**Definition 4.2.1.** *The sum of $f_1, f_2$ is given by $f(x) = f_1(x) + f_2(x)$ for all $x \in E$. Similarly, the product of $f_1$ by $\alpha \in \mathbb{C}$ is given by $f(x) = \alpha f_1(x)$ for all $x \in E$.*

**Remark 4.2.2.**

1. *With the operations of Definition 4.2.1, the space of linear functionals satisfies the axioms of a linear space.*
2. *In particular, if $E$ is a normed space then $f_1 + f_2$, and $\alpha f_1$ are continuous if $f_1$ and $f_2$ are continuous.*

**Definition 4.2.3.** *For a normed linear space $E$, the adjoint space to $E$ denoted $E^*$, is the space of continuous linear functional on $E$ with operations as given by Definition 4.2.1.*

**Exercise 4.2.4.** *Verify that the map given in Definition 4.1.13 is a norm on $E^*$.*

**Definition 4.2.5.** *With Exercise 4.2.4, we have that $E^*$ is a normed linear space. The induced topology on this space is known as the strong topology on $E^*$.*

**Theorem 4.2.6.** *For a normed linear space $E$, the adjoint space is complete.*

*Proof.* Let $(f_n)_{n\in\mathbb{N}} \subseteq E^*$ be a Cauchy sequence. In particular, for $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \epsilon$ for every $m > n \geq N$. Therefore,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|\|x\| < \epsilon\|x\| \tag{4.2.1}$$

for all $x \in E$. Hence, for fixed $x \in E$ the sequence $(f_n(x))_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is Cauchy and thus convergent as $\mathbb{C}$ is complete. Let $f : E \to \mathbb{C}$ be given by $f(x) := \lim_{n\to\infty} f_n(x)$. Observe that

$$\begin{aligned} f(\alpha x + \beta y) &= \lim_{n\to\infty} f_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} (\alpha f_n(x) + \beta f_n(y)) \\ &= \alpha f(x) + \beta f(y), \end{aligned}$$

and so $f$ is linear. Moreover, taking $m \to \infty$ in (4.2.1) it follows that

$$|f_n(x) - f(x)| \le \epsilon \|x\|, \tag{4.2.2}$$

which implies that $f_n - f$ is bounded on $\{x \in E : \|x\| \le 1\}$. Using Corollary 4.1.11 we deduce that $f_n - f$ is continuous and thus $f$ is continuous as $f_n$ is continuous. Moreover, from (4.2.2) we have

$$\|f - f_n\| \le \epsilon$$

for all $n \ge N$, that is $f_n \to f$ in $E^*$, and so $E^*$ is complete. $\square$

**Remark 4.2.7.** *Note that in Theorem 4.2.6 we do not require that $E$ is complete. A consequence of this is explored in Corollary 4.2.8.*

For a linear space $E$, we denote by $\bar{E}$ its completion. That is, $\bar{E}$ is $E$ along with the limits of all Cauchy sequences in $E$. Furthermore, linear spaces $E$ and $F$ are isometric, written $E = F$, if there exists an isomorphism, that is a bijective map preserving linear operations, that also preserves the norm.

**Corollary 4.2.8.** *Let $E$ be a linear space. Then $E^*$ and $\left(\bar{E}\right)^*$ are isometric.*

*Proof.* Note that $E \subseteq \bar{E}$ is everywhere dense, and so $f \in E^*$ is uniquely extended to a functional $\bar{f}$ on $\bar{E}$ using continuity. In particular, as $E \subset \bar{E}$ it is clear that $\|f\| = \left\|\bar{f}\right\|$. Conversely, any functional $\bar{f}$ on $\bar{E}$ restricts to a functional $f$ on $E$, which is again such that $\left\|\bar{f}\right\| = \|f\|$. Therefore, the correspondence of $f$ to $\bar{f}$ is a bijective correspondence that preserves the norm. That is, $E^* = \left(\bar{E}\right)^*$. $\square$

**Example 4.2.9.** *Consider a $n$-dimensional linear space $E$. Let $\{e_1, \dots, e_n\}$ be a basis in $E$, so that any $x \in E$ can be written as*

$$x = \sum_{j=1}^{n} x_j e_j$$

*for $x_j \in \mathbb{R}$. Thus, for a linear functional $f : E \to \mathbb{R}$ we have*

$$f(x) = \sum_{j=1}^{n} x_j f(e_j).$$

*In particular, this means that $f$ is determined by $\{f(e_1), \dots, f(e_n)\}$. Consider the linear functions $g_1, \dots, g_n$ given by*

$$g_k(e_j) = \begin{cases} 1 & j = k \\ 0 & j \ne k. \end{cases}$$

*It is clear that $\{g_1, \dots, g_n\}$ is linearly independent. Moreover, observe that*

$$f(x) = \sum_{j=1}^{n} f_j g_j(x),$$

*where $f_j := f(e_j)$. It follows that $\{g_1, \dots, g_n\}$ forms a basis for $E^*$ which means that $E^*$ is also $n$-dimensional.*

**Exercise 4.2.10.** *Let $E$ be a $n$-dimensional linear space.*

1. *Show that the norm $\|x\| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}$ on $E$ induces the norm $\|f\| = \left(\sum_{j=1}^{n} |f_j|^2\right)^{\frac{1}{2}}$ on $E^*$.*

2. *For $p > 1$, show that the norm $\|x\| = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$ on $E$ induces the norm $\|f\| = \left(\sum_{j=1}^{n} |f_j|^q\right)^{\frac{1}{q}}$, where $\frac{1}{p} + \frac{1}{q} = 1$, on $E^*$.*
3. *Show that the norm $\|x\| = \sum_{j=1}^{n} |x_j|$ on $E$ induces the norm $\|f\| = \sup_{1\leq j\leq n} |f_j|$ on $E^*$.*
4. *Show that the norm $\|x\| = \sup_{1\leq j\leq n} |x_j|$ on $E$ induces the norm $\|f\| = \sum_{i=1}^{n} |f_j|$ on $E^*$.*

**Remark 4.2.11.** *As $E$ is finite-dimensional, all the norms identified in Exercise 4.2.10 induce the same topology.*

**Lemma 4.2.12.** *Consider the space*

$$C_0 := \left\{x = (x_1, x_2, \dots) : \lim_{n\to\infty} x_n = 0\right\}$$

*with norm $\|x\| = \sup_n |x_n|$. Then $(C_0^*, \|\cdot\|)$ is isometric to $\ell^1$.*

*Proof.* Let $f = (f_1, f_2, \dots) \in \ell^1$, so that $\sum_{n=1}^{\infty} |f_n| = \|f\|_{\ell^1} < \infty$. Then we can define the linear map $\hat{f} : C_0 \to \mathbb{C}$ by

$$\hat{f}(x) = \sum_{n=1}^{\infty} f_n x_n.$$

Observe that

$$\left|\hat{f}(x)\right| \leq \|x\| \sum_{n=1}^{\infty} |f_n| = \|x\|\|f\|_{\ell^1}$$

which shows that $\hat{f}$ is bounded and thus it must be continuous as it is linear. In particular, $\left\|\hat{f}\right\| \leq \|f\|_{\ell^1}$. On the other hand, consider

$$x^{(N)} = \sum_{n=1}^{N} \frac{f_n}{|f_n|} e_n,$$

where $e_n = (\underbrace{0, \dots, 1}_{n}, 0 \dots)$, and $\frac{f_n}{|f_n|}$ is set to zero in the case when $f_n = 0$. Clearly, $x^{(N)} \in C_0$ and $\left\|x^{(N)}\right\| \leq 1$. Moreover,

$$\hat{f}\left(x^{(N)}\right) = \sum_{n=1}^{n} \hat{f}(e_n) \frac{f_n}{|f_n|} = \sum_{n=1}^{N} |f_n|.$$

Therefore,

$$\lim_{N\to\infty} \hat{f}\left(x^{(N)}\right) = \|f\|_{\ell^1}$$

which implies that $\left\|\hat{f}\right\| \geq \|f\|_{\ell^1}$. Thus it follows that $\left\|\hat{f}\right\| = \|f\|_{\ell^1}$. Which means that the map $\varphi : \ell^1 \to C_0^*$ given by $f \mapsto \hat{f}$ preserves the norm between $\ell^1$ and $C_0^*$. Furthermore, if $\varphi(f_1) = \varphi(f_2)$, then

$$(f_1)_n = \varphi(f_1)(e_n) = \varphi(f_2)(e_n) = (f_2)_n.$$

Hence, $f_1 = f_2$ which shows that $\varphi$ is injective. Now, let $\hat{f} \in C_0^*$. For any $x = (x_1, x_2, \dots) \in C_0$ one can write

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

as

$$\left\|x - \sum_{n=1}^{N} x_n e_n\right\| = \sup_{n>N} |x_n| \overset{N\to\infty}{\longrightarrow} 0.$$

Thus using the continuity of $\hat{f}$ we deduce that

$$\hat{f}(x) = \sum_{n=1}^{\infty} x_n \hat{f}(e_n).$$

Now set

$$x^{(N)} = \sum_{n=1}^{N} \frac{\hat{f}(e_n)}{\left|\hat{f}(e_n)\right|} e_n.$$

Then

$$\sum_{n=1}^{N} \left|\hat{f}(e_n)\right| = \sum_{n=1}^{N} \frac{\hat{f}(e_n)\hat{f}(e_n)}{\left|\hat{f}(e_n)\right|} = \hat{f}\left(x^{(N)}\right) \leq \left\|\hat{f}\right\|,$$

where the inequality follows as $x^{(N)} \in C_0$ and $\|x^{(N)}\| \leq 1$. Therefore, $\sum_{n=1}^{\infty} \left|\hat{f}(e_n)\right| < \infty$ and so $\left(\hat{f}(e_n)\right)_{n\in\mathbb{N}} \in \ell^1$. In particular, $\varphi\left(\left(\hat{f}(e_n)\right)_{n\in\mathbb{N}}\right) = \hat{f}$ so we deduce that $\varphi$ is surjective and thus an isometry between $(C_0^*, \|\cdot\|)$ and $\ell^1$. □

**Lemma 4.2.13.** *Consider the space*

$$m := \left\{x = (x_1, x_2, \dots) : \sup_{n\in\mathbb{N}} |x_n| < \infty\right\}$$

*with norm* $\|x\| = \sup_{n\in\mathbb{N}} \|x_n\|$. *Then* $\left(\left(\ell^1\right)^*, \|\cdot\|_{\ell^1}\right)$ *is isometric to* $m$.

*Proof.* Let $f := (f_1, f_2, \dots) \in m$. Let $\hat{f} : \ell^1 \to \mathbb{C}$ be given by

$$\hat{f}(x) = \sum_{n=1}^{\infty} x_n f_n.$$

Clearly, $\hat{f}$ is linear. Furthermore,

$$\begin{aligned} \left\|\hat{f}\right\| &= \sup_{x\in\ell^1\setminus\{0\}} \frac{\left|\sum_{n=1}^{\infty} x_n f_n\right|}{\|x\|_{\ell^1}} \\ &\leq \sup_{x\in\ell^1\setminus\{0\}} \frac{\sum_{n=1}^{\infty} |x_n||f_n|}{\|x\|_{\ell^1}} \\ &\leq \sup_{x\in\ell^1\setminus\{0\}} \frac{\sup_n |f_n| \|x\|_{\ell^1}}{\|x\|_{\ell^1}} \\ &= \sup_n |f_n|. \end{aligned} \tag{4.2.3}$$

Hence, $\hat{f}$ is bounded as $(f_n)_{n\in\mathbb{N}} \in m$ and thus continuous which means that $\hat{f} \in \left(\ell^1\right)^*$. Moreover, let

$$x = \operatorname{sgn}(f_m) e_m$$

where $m$ is such that $|f_m| = \sup_n |f_n|$ and $e_m = (\underbrace{0, \dots, 1}_{m}, \dots)$. Then,

$$\begin{aligned} \frac{\left|\hat{f}(x)\right|}{\|x\|_{\ell^1}} &= \frac{|\operatorname{sgn}(f_m) f_m|}{|\operatorname{sgn}(f_m)|} \\ &= |f_m| \\ &= \sup_n |f_n|. \end{aligned}$$

Hence, $\left\|\hat{f}\right\| = \sup_n |f_n|$. On the other hand, for $\hat{f} \in \left(\ell^1\right)^*$ one can write

$$\hat{f}(x) = \sum_{n=1}^{\infty} f_n x_n$$

where $f_n = \hat{f}(e_n)$. As $\hat{f} \in \left(\ell^1\right)^*$ we know that $\left\|\hat{f}\right\| = M < \infty$. Thus, letting $x = e_n$ we note that

$$|f_n| = |\hat{f}(e_n)| \leq \left\|\hat{f}\right\| \|e_n\| = \left\|\hat{f}\right\|$$

and so $\sup_n |f_n| \leq M < \infty$ which implies that $f := (f_1, f_2, \dots) \in m$. Furthermore, using (4.2.3) we deduce that $\sup_n |f_n| = \left\|\hat{f}\right\|$. Therefore, the map $f \mapsto \hat{f}$ is an isometry. □

**Exercise 4.2.14.** *Consider the space $\ell^p$, for $p > 1$, of sequences $x = (x_n)_{n \in \mathbb{N}}$ such that*

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty.$$

*Show that $\left(\ell^p\right)^*$ is isometric to $\ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 4.2.15.** *Let $H$ be a Hilbert space. Then for any $f \in H^*$ there is a unique $x_0 \in H$ such that $f(x) = (x, x_0)$ for every $x \in H$ and $\|f\| = \|x_0\|$. Conversely, for any $x_0 \in H$ if $f(x) = (x, x_0)$ for every $x \in H$ then $f \in H^*$ with $\|f\| = \|x_0\|$. Consequently, the map $f \mapsto x_0$ is an isometry between $H$ and $H^*$, with conjugate-linear isomorphism $\lambda x_0 \mapsto \bar{\lambda} f$ if $H$ is complex.*

*Proof.* $(\Leftarrow)$. For any $x_0$ consider the map $f : H \to \mathbb{C}$ given by $f(x) = (x, x_0)$. Observe that

$$|f(x)| = |(x, x_0)| \leq \|x\| \|x_0\|.$$

In particular, as $|f(x_0)| = \|x_0\|^2$ we have $\|f\| = \|x_0\|$. Thus, $f$ is bounded and as $f$ is linear by the properties of the inner product it follows that $f$ is continuous.
$(\Rightarrow)$. If $f = 0$, then $f(x) = (x, x_0)$ with $x_0 = 0$. Suppose instead that $f \neq 0$ and let

$$H_0 := \ker(f) = \{x : f(x) = 0\}.$$

By Lemma 4.1.3, the codimension of $H_0$ is one. Moreover, as $f$ is continuous we have that $H_0$ is closed by statement 3 of Exercise 4.1.12. Consequently, for some $y_0 \in H_0^{\perp}$, we can write any $x \in H$ as $x = \lambda y_0 + y$ for some $\lambda \in \mathbb{C}$ and $y \in H_0$. We can assume without loss of generality that $\|y_0\| = 1$. Now let $x_0 = \overline{f(y_0)} y_0$, then for any $x \in H$ note that $f(x) = \lambda f(y_0)$, and thus

$$\begin{aligned}
(x, x_0) &= \left(x, \overline{f(y_0)} y_0\right) \\
&= \left(\lambda y_0, \overline{f(y_0)} y_0\right) \\
&= \lambda f(y_0)(y_0, y_0) \\
&= \lambda f(y_0) \\
&= f(x).
\end{aligned}$$

Now suppose there exists another $x_0' \in H$ such that $f(x) = (x, x_0')$. Then

$$0 = (x, x_0 - x_0')$$

for every $x \in H$. In particular, this holds for $x = x_0 - x_0'$ which implies that $\|x_0 - x_0'\| = 0$ and so $x_0 = x_0'$. □

**Remark 4.2.16.** *From Theorem 4.2.15, we have that $H = H^*$ in the sense of linear spaces. Thus, for $E$ a linear space that is not complete as its completion $\bar{E}$ is a Hilbert space, it follows that $\bar{E} = \left(\bar{E}\right)^* = E^*$.*

### 4.2.1 Second Adjoint Space

For a linear normed space $E$, the adjoint space $E^*$ is itself a linear normed space. Thus we can consider its adjoint space. More specifically, fix $x_0 \in E$ and let $\varphi_{x_0} : E^* \to \mathbb{C}$ be given by

$$\varphi_{x_0}(f) = f(x_0). \tag{4.2.4}$$

Note that $\varphi_{x_0}$ is linear as

$$\begin{aligned}\varphi_{x_0}\left(f_1 + \lambda f_2\right) &= \left(f_1 + \lambda f_2\right)(x_0) \\ &= f_1(x_0) + \lambda f_2(x_0) \\ &= \varphi_{x_0}(f_1) + \lambda \varphi_{x_0}(f_2).\end{aligned}$$

Moreover, note that

$$|\varphi_{x_0}(f)| = |f(x_0)| \leq \|f\|\|x_0\|.$$

Which shows that $\varphi_{x_0}$ is bounded on the closed unit ball of $E^*$, and so by Corollary 4.1.11 we have that $\varphi_{x_0} \in \left(E^*\right)^* = E^{**}$.

**Definition 4.2.17.** *The map $\pi : E \to E^{**}$ given by $x \mapsto \varphi_x$, in the sense of* (4.2.4)*, is called the natural map of $E$ into $E^{**}$.*

**Exercise 4.2.18.** *The natural map, as given by Definition 4.2.17, is an isomorphism between $E$ and $\pi(E) \subseteq E^{**}$.*

**Lemma 4.2.19.** *Let $E$ be a normed linear space. Then the natural map, as given by Definition 4.2.17, is an isometry between $E$ and $\pi(E) \subseteq E^{**}$.*

*Proof.* For $x \in E$ let $\|x\|$ be its norm in $E$ and $\|x\|_2 = \|\varphi_x\|$ be the norm in $E^{**}$ of its image under the natural map. Let $f \in E^* \setminus \{0\}$. Then $|f(x)| \leq \|f\|\|x\|$, and so

$$\|x\| \geq \frac{|f(x)|}{\|f\|}.$$

Taking the supremum over $f \in E^*$ we deduce that

$$\|x\| \geq \sup_{f \in E^* \setminus \{0\}} \frac{|f(x)|}{\|f\|} = \|x\|_2.$$

On the other hand, for any $x_0 \in E \setminus \{0\}$ by Corollary 4.1.20 there exists a $f_0 \in E^* \setminus \{0\}$ such that $|f_0(x_0)| = \|f_0\|\|x_0\|$. In particular, taking $x_0 = x$ it follows that

$$\|x\|_2 = \sup_{f \in E^* \setminus \{0\}} \frac{|f(x)|}{\|f\|} \geq \|x\|.$$

Therefore, $\|x\| = \|x\|_2$, and so in conjunction with Exercise 4.2.18 we have that the natural map is an isometry between $E$ and $\pi(E) \subseteq E^{**}$. ☐

**Definition 4.2.20.** *A normed linear space $E$ is reflexive if $\pi(E) = E^{**}$, where $\pi$ is the natural map between $E$ and $E^{**}$.*

**Example 4.2.21.**

1. *Finite dimensional Euclidean spaces and Hilbert spaces are reflexive. Indeed, in these cases, we also have* $E = E^*$.

2. *For the space* $C_0$*, sequences converging to zero, we have* $(C_0)^* = \ell_1$ *from Lemma 4.2.12 and* $(C_0)^{**} = m$ *for Lemma 4.2.13. Therefore,* $\ell^0$ *is not reflexive.*

3. *The space* $\mathcal{C}([a,b])$ *is not reflexive.*

4. *Using Exercise 4.2.14, it follows that for* $p > 1$ *the space* $\ell^p$ *is reflexive. More specifically,* $(\ell^p)^* = \ell^q$ *and so* $(\ell^p)^{**} = \ell^p$*. In particular, if* $p \neq 2$ *then* $\ell^p \neq (\ell^p)^*$*. For* $p = 2$ *we do indeed have* $\ell^2 = \left(\ell^2\right)^2$*, which is to be expected as* $\ell^2$ *is a Hilbert space.*

## 4.3 Linear Topological Spaces

**Definition 4.3.1.** *A set* $E$ *is a linear topological space if the following statements hold.*

1. $E$ *is a linear space over the real or complex numbers.*

2. $E$ *is a topological space.*

3. *Linear operations are continuous in* $E$.

**Remark 4.3.2.** *Statement 3 of Definition 4.3.1 means that the following statements hold.*

1. *If* $z_0 = x_0 + y_0$*, then for any neighbourhood* $U$ *of* $z$ *there are neighbourhoods* $V$ *of* $x_0$ *and* $W$ *of* $y_0$*, such that* $V + W \subseteq U$.

2. *If* $\alpha_0 x_0 = y_0$*, then for any neighbourhood* $U$ *of* $y_0$ *there is a neighbourhood* $V$ *of* $x_0$ *and an* $\epsilon$*-neighbourhood of* $\alpha$ *such that for* $|\alpha - \alpha_0| < \epsilon$ *and* $x \in V$ *we have* $\alpha x \in U$.

*Consequently, the topology on* $E$ *is fully defined by specifying a set of neighbourhoods of zero. Indeed let* $x \in E$ *and* $U$ *be a neighbourhood of zero, then* $U + x$ *is said to be a neighbourhood of* $x$*. Refer to Section 6.1 for more details.*

**Exercise 4.3.3.** *Let* $E$ *be a linear topological space.*

1. *If* $U$ *and* $V$ *are open in* $E$*, then*

$$U + V := \{u + v : u \in U, v \in V\}$$

*is open.*

2. *If* $U$ *is open in* $E$*, then*

$$\alpha U := \{\alpha u : u \in U\}$$

*is open for* $\alpha \neq 0$.

3. *If* $F$ *is closed in* $E$*, then* $\alpha F$ *is closed for all* $\alpha$.

4. *Let* $U$ *open with* $0 \in U$*. Then there exists a* $W$ *open with* $0 \in W$*,* $W = -W$ *and* $W + W \subseteq U$.

5. *If* $F \subset E$ *is closed, and* $x \in E \setminus F$*, then* $x$ *and* $F$ *have non-intersecting neighbourhoods.*

**Proposition 4.3.4.** *The point $x_0 = 0$ is closed if and only if the intersection of all neighbourhoods of $0$ does not contain non-zero elements.*

*Proof.* $(\Rightarrow)$. Let $x \in E \setminus \{0\}$ be in every neighbourhood of $0$. Since $\{0\}$ is closed, by statement 5 of Exercise 4.3.3, there exist disjoint neighbourhoods of $x$ and $0$ which is a contradiction.
$(\Leftarrow)$. Let $x \in E \setminus \{0\}$. Then there exists a neighbourhood $U$ of $0$ such that $x \notin U$. The set $V_x := E \setminus U$ is closed with $0 \notin V_x$. Therefore, by statement 5 of Exercise 4.3.3 there exists a neighbourhood $U_x$ such that $V_x \subseteq U_x$ and $0 \notin U_x$. Note that $E \setminus \{0\} = \bigcup_{x \in E \setminus \{0\}} U_x$ is an open set. Therefore, $\{0\}$ is closed. ☐

**Proposition 4.3.5.** *If $\{x_0\} \subseteq E$ is closed, then $E$ is Hausdorff.*

*Proof.* If $\{0\}$ is closed then $\{x\}$ is closed for all $x \in E$. Therefore, by statement 5 Exercise 4.3.3 for any $x, y \in E$ distinct we can find disjoint neighbourhoods. Therefore, $E$ is Hausdorff. ☐

**Example 4.3.6.**

1. *A normed linear space is a linear topological space where the topology is induced by the norm. Indeed the linear operations are continuous due to the properties of the norm.*
2. *Let $\mathcal{K}([a, b])$ be the space of continuously differentiable functions on $(a, b)$. For $m \in \mathbb{N}$ and $\epsilon > 0$ let*
$$U_{m,\epsilon} := \left\{ \varphi \in \mathcal{K}([a,b]) : \left| \varphi^{(k)}(x) \right| < \epsilon \text{ for every } k = 0, 1 \ldots, m \right\}.$$
*Using the $(U_{m,\epsilon})_{k \in \mathbb{N}, \epsilon > 0}$ we can induce a topology on $\mathcal{K}([a, b])$ that leads to a linear topological space.*

**Definition 4.3.7.** *Let $E$ be a linear topological space. Then $M \subset E$ is bounded if for any neighbourhood $U$ of $0$ there exists a $n > 0$ such that*
$$M \subset \lambda U$$
*for all $|\lambda| \geq n$.*

**Remark 4.3.8.** *If $E$ is additionally a normed linear space, then Definition 4.3.7 coincides with boundedness in the norm.*

**Exercise 4.3.9.** *Show that a set $A \subseteq E$ is bounded if and only if for any sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ and any $(\epsilon_n)_{n \in \mathbb{R}_{>0}}$ such that $\epsilon_n \to 0$ we have that $(\epsilon_n x_n)_{n \in \mathbb{N}}$ converges to zero.*

Note that Definition 4.1.7 holds on a linear topological space as well.

**Lemma 4.3.10.** *Let $E$ be a linear topological space. Then a linear functional $f$ on $E$ is continuous if and only if there is a neighbourhood of $0$ on which $f$ is bounded.*

*Proof.* Follows in the same way as the proof for Theorem 4.1.10. ☐

**Lemma 4.3.11.** *If $f_1, f_2$ are continuous functions on $E$, and $\alpha \neq 0$ then $f_1 + f_2$ and $\alpha f_1$ are continuous.*

*Proof.* This follows directly from statement 1 and statement 2 of Exercise 4.3.3. ☐

Many of the notions we encountered for normed linear spaces extended to linear topological spaces. For example, for a linear topological space $E$, the adjoint space $E^*$, as in Definition 4.2.3, is well-defined by Lemma 4.3.11. However, $E$ is only equipped with a topology which does not necessarily correspond to a norm. Thus, we cannot define a topology on $E^*$ as we did in the case of normed linear spaces, we instead appeal to systems of neighbourhoods, Definition 6.1.1. For a normed linear space $E$ on the adjoint space $E^*$, as constructed using the norm of $E$, the system of neighbourhoods

$$U_\epsilon = \{f \in E^* : |f(x)| < \epsilon \text{ for } x \in B\}$$

where $B = \{x : \|x\| \leq 1\}$ is an open base. Indeed, for any open neighbourhood $U$ of $0 \in E^*$ in the topology induced by the norm $\|\cdot\|$ on $E^*$, there exists an $\epsilon > 0$ such that

$$V := \{f \in E^* : \|f\| < \epsilon\} \subseteq U.$$

In particular, as $|f(x)| \leq \|x\|\|f\|$ it is clear that $V = U_\epsilon$. Hence, the collection $\mathcal{B} = (U_\epsilon)_{\epsilon > 0}$ is an open base, as in the sense of statement 2 Remark 6.1.5, of the strong topology on $E^*$ for a normed linear space $E$. This motivates Definition 4.3.12 which induces the strong topology on the adjoint space of a linear topological space.

**Definition 4.3.12.** *For a linear topological space $E$, the strong topology on $E^*$ is induced by the local base of $E^*$ where open sets are given by*

$$U_{\epsilon,A} = \{f \in E^* : |f(x)| < \epsilon \text{ for } x \in A\},$$

*for $\epsilon > 0$ and $A$ a bounded set in $E$.*

**Remark 4.3.13.**

1. *Indeed,*

$$U_{\min(\epsilon_1,\epsilon_2),A_1 \cup A_2} \subseteq U_{\epsilon_1,A_1} \cap U_{\epsilon_2,A_2}$$

*which equivalently shows that the system of neighbourhoods of Definition 4.3.12 is a local base. Moreover, page 42 of [1] shows that the defining system of Definition 4.3.12 makes $E^*$ a linear topological space.*

2. *The sets $U_{\epsilon,A}$ are neighbourhood of zero in $E^*$. However, we can translate the sets to obtain neighbourhood for arbitrary points in $E^*$.*

**Exercise 4.3.14.** *Verify that if $E$ is a normed linear space, then Definition 4.3.12 coincides with Definition 4.2.5.*

Having endowed $E^*$ with a topology we can consider the second adjoint and define the natural map. However, with the lack of norms on these spaces, we no longer have the notion of an isometry.

**Definition 4.3.15.** *A linear topological space $E$ is reflexive if $\pi$ is continuous and $\pi(E) = E^{**}$.*

## 4.4 Weak Convergence

### 4.4.1 Topological Spaces

**Exercise 4.4.1.** *Let $E$ be a linear topological space. Let $\epsilon > 0$ and $f_1, \ldots, f_n \in E^*$ for $n \in \mathbb{N}$. Show that*

$$U := \{x \in E : |f_j(x)| < \epsilon,\ j = 1, \ldots, n\}$$

*is an open neighbourhood of $0$ in $E$. Show also that the system of open neighbourhoods of the form $U$ is defining.*

**Definition 4.4.2.** *The topology generated by the local base of Exercise 4.4.1 is called the weak topology on* $E$.

**Remark 4.4.3.**

1. *We note that since the sets of Exercise 4.4.1 are open in* $E$*, the topology generated by them is weaker than the original topology on* $E$*. In particular, the weak topology is the weakest topology on* $E$ *such that all* $f \in E^*$ *are continuous.*

2. *Moreover,* $E$ *with the weak topology is a linear topological space since linear operation is continuous.*

3. *Convergence in* $E$ *under the weak topology is referred to as weak convergence, whereas convergence under the original topology is referred to as strong convergence. In particular, for* $(x_n)_{n\in\mathbb{N}} \subset E$*, if* $x_n \to x$ *strongly then* $x_n \to x$ *weakly. One often writes* $x_n \xrightarrow{w} x$ *to denote weak convergence.*

**Proposition 4.4.4.** *A sequence* $(x_n)_{n\in\mathbb{N}} \subset E$ *converges in the weak topology to* $x_0 \in E$ *if and only if for all* $f \in E^*$ *the sequence* $(f(x_n))_{n\in\mathbb{N}} \subset \mathbb{C}$ *converges to* $f(x_0)$.

*Proof.* Without loss of generality, we can consider $x_0 = 0$.
($\Rightarrow$). For any $U$ there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. Consequently, for any fixed $f \in E^*$ we have $|f(x_n)| < \epsilon$ for $n \geq N$, and so $f(x_n) \to 0 = f(0)$ as $n \to \infty$.
($\Leftarrow$). Let

$$U = \{x : |f_j(x)| < \epsilon,\, j = 1, \ldots, n\}$$

be a weak neighbourhood of $0 \in E$. For each $j = 1, \ldots, k$, there exists an $N_j \in \mathbb{N}$ such that $|f_j(x_n)| < \epsilon$ for all $n \geq N_j$. Letting $N := \max_{j=1,\ldots,n}(N_j)$ it follows that for all $n \geq N$ we have $x_n \in U$. Hence, $(x_n)_{n\in\mathbb{N}}$ converges in the weak topology. $\square$

### 4.4.2 Normed Spaces

**Theorem 4.4.5.** *Let* $E$ *be a linear normed space. If* $(x_n)_{n\in\mathbb{N}} \subset E$ *is weakly convergent, then there exists a* $C > 0$ *such that*

$$\|x_n\| \leq C$$

*for all* $n \in \mathbb{N}$.

*Proof.* Let

$$A_{k,n} := \{f \in E^* : |f(x_n)| \leq k\} \subseteq E^*$$

for $k, n \in \mathbb{N}$. Since $f(x_n)$ for fixed $x_n$ is a continuous function in $f$, the sets $A_{k,n}$ are closed, and thus $A_k := \bigcap_{n=1}^{\infty} A_{k,n}$ is closed. Since $(x_n)_{n\in\mathbb{N}}$ is weakly convergent, the sequence $(f(x_n))_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is bounded for each $f \in E^*$. Therefore, each $f \in E^*$ is in some $A_k$ which implies that $E^* = \bigcup_{k=1}^{\infty} A_k$. Since $E^*$ is complete it cannot be represented as a countable union of nowhere-dense sets, by Baire's theorem, hence, for some $k = k_0$ we must have that $A_{k_0}$ is dense in some $B_\epsilon(f_0)$. As $A_{k_0}$ is closed we have $B_\epsilon(f_0) \subset A_{k_0}$ which implies that the sequence $(x_n)_{n\in\mathbb{N}} \subseteq E^{**}$ is bounded on $B_\epsilon(f_0)$, and in particular it therefore must be bounded on the unit ball around $0$. Since $E$ and $\pi(E) \subseteq E^{**}$ are isometric, Lemma 4.2.19, it follows that $(x_n)_{n\in\mathbb{N}} \subset E$ is also bounded. $\square$

**Proposition 4.4.6.** *For a normed linear space* $E$*, we have that* $A \subseteq E$ *is bounded if and only if any* $f \in E^*$ *is bounded on* $A$.

*Proof.* ($\Rightarrow$). Note that for any $f \in E^*$ we have $|f(x)| \leq \|f\|\|x\|$ for each $x \in E$. Since, $A$ is bounded, for $x \in A$ we have that $|f(x)| \leq C\|f\|$ where $C = \sup_{x\in A}\|x\|$. Therefore, $f$ is bounded on $A$.

($\Leftarrow$). Suppose that $A$ is not bounded. Then there exists an unbounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$. Now consider

$$A_{k,n} := \{f \in E^* : |f(x_n)| \leq k\}.$$

Then $A_k := \bigcap_{n\in\mathbb{N}} A_{k,n}$ is the collection of linear functionals that are bounded by $k$ on $(x_n)_{n\in\mathbb{N}} \subseteq A$. Therefore, by assumption, we have that $E^* = \bigcup_{k\in\mathbb{N}} A_k$. Therefore, by the same arguments as those made in Theorem 4.4.5 we deduce that $(x_n)_{n\in\mathbb{N}}$ is bounded, which is a contradiction. ☐

**Theorem 4.4.7.** *Let $E$ be a linear normed space. Then $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$ if the following statements hold.*

1. *$(x_n)_{n\in\mathbb{N}}$ is bounded in norm. That is, for some $c > 0$ we have $\|x_n\| \leq c$ for every $n \in \mathbb{N}$.*
2. *$f(x_n) \to f(x)$ for any $f \in \Delta$, where $\Delta \subseteq E^*$ is a complete system in $E^*$ with respect to the strong topology.*

*Proof.* If $\varphi$ is a finite linear combination of elements of $\Delta$, it follows by condition 2 that $\varphi(x_n) \to \varphi(x)$. Now consider $\varphi$ a general element of $E^*$. Then since $\Delta \subseteq E^*$ is complete it follows that there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subseteq E^*$ such that $\varphi_k \to \varphi$ in the norm of $E^*$. In particular, for fixed $\epsilon > 0$ there exists a $K \in \mathbb{N}$ such that

$$\|\varphi_k - \varphi\| < \frac{\epsilon}{3}$$

for $k \geq K$. From our initial remarks there exists a $N \in \mathbb{N}$ such that

$$|\varphi_K(x_n) - \varphi_K(x)| < \frac{\epsilon}{3}$$

for $n \geq N$. Therefore, for $n \geq N$ it follows that

$$\begin{aligned}
|\varphi(x_n) - \varphi(x)| &\leq |\varphi(x_n) - \varphi_K(x_n)| + |\varphi_K(x_n) - \varphi_K(x)| + |\varphi_K(x) - \varphi(x)| \\
&\leq \|\varphi - \varphi_K\| \, \|x_n\| + |\varphi_K(x_n) - \varphi_K(x_n)| + \|\varphi_K - \varphi\| \, \|x\| \\
&\leq \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c \\
&= \epsilon.
\end{aligned}$$

Therefore, $\varphi(x_n) \to \varphi(x)$ for all $\varphi \in E^*$ and so $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x$ by Proposition 4.4.4. ☐

**Proposition 4.4.8.** *Let $E$ be a finite-dimensional Euclidean space. Then weak convergence and strong convergence are equivalent.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}} \subset E$ weakly converges to $x \in E$. Let $\{e_1, \ldots, e_n\} \subset E$ be a basis. Then

$$x_k = \sum_{j=1}^{n} x_k^{(j)} e_j$$

for each $k \in \mathbb{N}$, and

$$x = \sum_{j=1}^{n} x^{(j)} e_j.$$

As the inner product $(\cdot, e_j)$ is a continuous linear functional for each $j = 1, \ldots, n$ it follows that

$$x_k^{(j)} = (x_k, e_j) \overset{k\to\infty}{\longrightarrow} (x, e_j) = x^{(j)},$$

meaning $x_k \to x$ coordinate-wise. Therefore,

$$\|x_k - x\| = \sqrt{\sum_{j=1}^{n} \left(x_k^{(j)} - x^{(j)}\right)} \overset{k\to\infty}{\longrightarrow} 0,$$

which means $x_k \to x$ strongly. The converse holds even in infinite dimensions. ☐

**Example 4.4.9.**

1. *Consider the space $\ell^2$. If $(x_k)_{k\in\mathbb{N}}$ is bounded, and $(x_k, e_j) = x_k^{(j)} \to x^{(j)} = (x, e_j)$ for $j = 1, 2, \dots$ where $e_j = (\underbrace{0, \dots, 1}_{j}, \dots)$. Then as $\left(\ell^2\right)^* = \ell^2$ and $(e_j)_{j\in\mathbb{N}} \subseteq \ell^2$ is complete system, it follows from Theorem 4.4.7 that $x_n \overset{w}{\to} x$. However, consider $(e_j)_{j\in\mathbb{N}} \subseteq \ell^2$ as a sequence. It does not converge strongly to any limit, however, for any $f \in \left(\ell^2\right)^*$ we can write $f(x) = (x, a)$ for some $a = (a_1, a_2, \dots, ) \in \ell^2$. So $f(e_j) = \bar{a}_j \to 0$ as $j \to \infty$ since $\sum_{j-1}^{\infty} |a_j|^2 < \infty$. Therefore, $e_j \overset{w}{\to} 0$, thus strong and weak convergence do not coincide in $\ell^2$.*

2. *Consider the space $\mathcal{C}([a,b])$ with supremum norm. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}([a,b])$ be such that $x_n \overset{w}{\to} x$. Then by Theorem 4.4.5 the sequence $(x_n)_{n\in\mathbb{N}}$ bounded in norm. For $t_0 \in [a,b]$, consider the functional $\delta_{t_0} \in \mathcal{C}([a,b])^*$ given by $\delta_{t_0}(x) = x(t_0)$. Then since $x_n(t) \overset{w}{\to} x(t)$ it follows that $\delta_{t_0}(x_n) \to \delta_{t_0}(x)$ which implies that $x_n(t_0) \to x(t_0)$. Therefore, we conclude that $x_n \overset{w}{\to} x$ when there exists a $C > 0$ such that $|x_n(t)| \leq C$ for all $t \in [a,b]$ and $n \in \mathbb{N}$, that is the sequence $\left(x_n(t)\right)_{n\in\mathbb{N}}$ is uniformly bounded. Moreover, the sequence converges pointwise.*

**Theorem 4.4.10.** *Let $(x_n)_{n\in\mathbb{N}} \subseteq H$, where $H$ is a Hilbert space, converge weakly to $x \in H$ with $\|x_n\| \to \|x\|$ as $n \to \infty$. Show that $x_n \to x$ strongly.*

*Proof.* By the Theorem 1.3.2, for any $f \in H^*$ we can write $f(x) = (x, z)$ for unique $z \in H$. As $H$ and $H^*$ are isometric, it follows that for any $z \in H^*$ we have that $(x_n, z) \to (x, z)$ as $n \to \infty$ because $f(x_n) \to f(x)$ as $n \to \infty$. In particular, $(x_n, x) \to (x, x)$ as $n \to \infty$. Therefore,

$$\begin{aligned} \|x_n - x\|^2 &= (x_n, x_n) - (x_n, x) - (x, x_n) + (x, x) \\ &\overset{n\to\infty}{\longrightarrow} (x, x) - (x, x) - \overline{(x, x)} + (x, x) \\ &= 0. \end{aligned}$$

Where we have used $(x_n, x_n) = \|x_n\|^2 \to \|x\|^2 = (x, x)$. Therefore, $x_n \to x$ strongly. ☐

### 4.4.3 Adjoint Space

**Definition 4.4.11.** *For a linear topological space $E$, the weak-$*$ topology on $E^*$ is induced by the local base of $E^*$ where open sets are given by*

$$U_{\epsilon,A} = \left\{ f \in E^* : |f(x)| < \epsilon \text{ for } x \in A \right\},$$

*where $\epsilon > 0$ and $A$ is a finite set in $E$.*

**Remark 4.4.12.**

1. *The weak-$*$ topology on $E^*$ is weaker than the strong topology on $E^*$. Indeed, the strong topology on $E^*$ is characterised by neighbourhoods of the same form as those in Definition 4.4.11 but where $A$ is instead a bounded set. Thus, as any finite set is bounded it follows that the weak-* topology on $E^*$ is weaker than the strong topology on $E^*$.*

2. *Convergence with respect to the weak-$*$ topology is referred to as weak-$*$ convergence.*

**Proposition 4.4.13.** *A sequence $(f_n)_{n\in\mathbb{N}} \subset E^*$ converges $*$-weakly denoted $f_n \overset{w^*}{\longrightarrow} f$ if and only if for all $x \in E$ the sequence $(f_n(x)) \subset \mathbb{C}$ converges to $f(x)$.*

**Remark 4.4.14.** *Clearly, if $f_n \to f$ strongly then $f_n \xrightarrow{w^*} f$.*

**Theorem 4.4.15.** *Let $E$ be a Banach space. Then if $(f_n)_{n\in\mathbb{N}} \subset E^*$ is $*$-weakly convergent, then there exists a $C > 0$ such that*

$$\|f_n\| \leq C$$

*for all $n \in \mathbb{N}$.*

*Proof.* We proceed as in the case of Theorem 4.4.5 with the sets

$$A_{k,n} := \{x \in E : |f_n(x)| \leq k\}\,.$$

Where now the application of Baire's theorem is justified as $E$ is Banach and thus complete. □

**Theorem 4.4.16.** *Let $E$ be a Banach space. Then $(f_n)_{n\in\mathbb{N}} \subset E^*$ is $*$-weakly convergent to $f \in E^*$ if the following statements hold.*

1. *$(f_n)_{n\in\mathbb{N}}$ is bounded in norm.*
2. *$(f_n, x) \to (f, x)$ for any $x \in \Delta$ where $\Delta \subset E$ is a complete system in $E$ with respect to the strong topology.*

*Proof.* We proceed as in the case of Theorem 4.4.7, arriving

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f - n(x_K)| + |f_n(x_K) - f(x_K)| + |f(x_K) - f(x)| \\
&\leq \|f_n\|\|x - x_k\| + |f_n(x_K) - f(x_K)| + \|f\|\|x_K - x\| \\
&\leq \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c \\
&= \epsilon.
\end{aligned}$$

□

**Example 4.4.17.** *Consider the space $\mathcal{C}([a,b])$ and the $\delta$-function given by $\delta(x) = x(0)$. Let $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}([a,b])$ be such that the following statements hold for all $n \in \mathbb{N}$.*

1. *$\varphi_n(t) \geq 0$ with $\varphi_n(t) = 0$ for $|t| > \frac{1}{n}$.*
2. *$\int_a^b \varphi_n(t)\,\mathrm{d}t = 1$.*

*Then for any $x \in \mathcal{C}([a,b])$ consider*

$$\begin{aligned}
\Phi_n(x) &:= \int_a^b \varphi_n(t)x(t)\,\mathrm{d}t \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} \varphi_n(t)x(t)\,\mathrm{d}t \\
&\overset{n\to\infty}{\longrightarrow} x(0) \\
&= \delta(x).
\end{aligned}$$

*As $(\Phi_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}([a,b])^*$ it follows that $\Phi_n \xrightarrow{w^*} \delta$. That is, the $\delta$-function can be represented as a limit, in the weak-* sense, of functions $(\varphi_n)_{n\in\mathbb{N}}$.*

**Remark 4.4.18.** *Considering $E^*$ as a linear topological space, we can also consider the weak topology on $E^*$. We note that the weak-$*$ topology on $E^*$ is weaker than the weak topology on $E^*$, and they coincide where $E$ is reflexive.*

**Theorem 4.4.19** (Banach-Alaoglu)**.** *For a separable normed linear space $E$, the closed unit ball in $E^*$ is compact with respect to the weak-* topology.*

**Corollary 4.4.20.** *Let $E$ be a separable normed linear space. Then a bounded sequence $(x_n)_{n\in\mathbb{N}} \subset E^*$ has a $*$-weakly convergent subsequence.*

## 4.5 Countably-Normed Spaces

**Definition 4.5.1.** *Let $E$ be a linear space and let $\|\cdot\|_1$, $\|\cdot\|_2$ be norms on $E$. If for any sequence $(x_n)_{n\in\mathbb{N}} \subset E$ that is Cauchy in $\|\cdot\|_1$ and $\|\cdot\|_2$ we have that convergence to $x \in E$ in $\|\cdot\|_1$ means convergence to $x \in E$ in $\|\cdot\|_2$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be compatible.*

**Definition 4.5.2.** *A linear space $E$ is countably-normed if a countable system of pairwise compatible norms is given on $E$.*

The topology for a countable-normed space is generated by the defining system of neighbourhoods of $0 \in E$ given by

$$U_{r,\epsilon} := \{x \in E : \|x\|_0 < \epsilon, \dots, \|x\|_r < \epsilon\} \tag{4.5.1}$$

for $\epsilon > 0$ and $r \in \mathbb{N}$.

**Exercise 4.5.3.** *Verify that a countably-normed space $E$ is a topological linear space with the topology generated by* (4.5.1)*.*

**Lemma 4.5.4.** *Let $E$ be a countably-normed linear space. Then $(x_n)_{n\in\mathbb{N}} \subset E$ converges to $x \in E$ if and only if $x_n \to x$ with respect to each norm.*

*Proof.* ($\Rightarrow$). Without loss of generality suppose that $(x_n)_{n\in\mathbb{N}} \subseteq E$ converges to $0 \in E$. Then fix $m \in \mathbb{N}$ and $\epsilon > 0$. For $U_{m,\epsilon}$ there exists an $N \in \mathbb{N}$ such that $x_n \in U_{m,\epsilon}$ for all $n \geq N$. In particular, $\|x_n\|_m \leq \epsilon$ for all $n \geq N$. Thus, $x_n \to 0$ with respect to $\|\cdot\|_m$.
($\Leftarrow$). Without loss of generality suppose that $(x_n)_{n\in\mathbb{N}} \subset E$ converges to $0$ with respect to $\|\cdot\|_m$ for each $m \in \mathbb{N}$. Then for any $m \in \mathbb{N}$ and $\epsilon > 0$ let $N = \max_{j=1,\dots,m}(N_j)$ where $N_j$ is such that $\|x_n\|_j \leq \epsilon$ for $n \geq N_j$. It follows that $x_n \in U_{m,\epsilon}$ for all $n \geq N$. This implies that $x_n \to 0$ in $E$. $\square$

The norms on a countably-normed space $E$ can always be considered such that

$$\|x\|_k \leq \|x\|_l \tag{4.5.2}$$

for $k < l$ and all $x \in E$. Indeed if this is not the case, then we can consider instead $(\|\cdot\|'_k)_{k\in\mathbb{N}}$ where

$$\|x\|'_k = \sup_{i=0,\dots,k}(\|x\|_i),$$

without affecting the generated topology on $E$.

**Lemma 4.5.5.** *A countably-normed space $E$ is metrizable.*

*Proof.* Consider $\rho : E \times E \to \mathbb{R}$ given by

$$\rho(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

Note that $\rho(x,y)$ is well-defined, since

$$\begin{aligned}
|\rho(x,y)| &\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n} \\
&\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \\
&< \infty.
\end{aligned}$$

Clearly, $\rho(x,y) = \rho(y,x)$. Moreover, $\rho(x,y) \geq 0$ and $\rho(x,y) = 0$ if and only if $\|x-y\|_n = 0$ for all $n \in \mathbb{N}$ which happens if and only if $x = y$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{x}{1+x} = \frac{1}{(1+x)^2} > 0$$

for $x \geq 0$. Therefore, as

$$\|x-z\|_n \leq \|x-y\|_n + \|y-z\|_n,$$

it follows that

$$\begin{aligned}
\frac{\|x-z\|_n}{1+\|x-z\|_n} &\leq \frac{\|x-y\|_n}{1+\|x-y\|_n+\|y-z\|_n} + \frac{\|y-z\|_n}{1+\|x-y\|_n+\|y-z\|_n} \\
&\leq \frac{\|x-y\|_n}{1+\|x-y\|_n} + \frac{\|y-z\|_n}{1+\|y-z\|_n}.
\end{aligned}$$

Therefore, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$, and thus $\rho$ is a metric. Now suppose that $(x_m)_{m\in\mathbb{N}} \subset E$ converges to $x \in E$ with respect to the topology on $E$, and fix an $\epsilon > 0$. Note that

$$\frac{x}{1+x} = \frac{1+x-1}{1+x} = 1 - \frac{1}{1+x} < 1$$

for $x > 0$ and

$$\frac{x}{1+x} \to 0$$

as $x \searrow 0$. Therefore, $\frac{\|x-y\|_n}{1+\|x-y\|_n} < 1$ for all $x, y \in E$ and $n \in \mathbb{N}$. As $\sum_{n=0}^{\infty} \frac{1}{2^n} < \infty$ there exists an $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}$. Moreover, by Lemma 4.5.4 we have that $(x_m)_{m\in\mathbb{N}}$ converges with respect to each norm $\|\cdot\|_n$. In particular, for each $n \in \{1, \ldots, N\}$ there exists a $M_n \in \mathbb{N}$ such that

$$\frac{\|x-x_m\|_n}{1+\|x-x_m\|_n} < \frac{2^N}{2^{N+1}-1} \frac{\epsilon}{2}$$

for $m \geq M_n$. Taking $M = \max_{n=0,\dots,N}(M_n)$, then for $n \geq M$ we have

$$\begin{aligned}
\rho(x, x_n) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|x - x_n\|_n}{1 + \|x - x_n\|_n} \\
&= \sum_{n=0}^{N} \frac{1}{2^n} \frac{\|x - x_n\|_n}{1 + \|x - x_n\|_n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\|x - x_n\|_n}{1 + \|x - x_n\|_n} \\
&\leq \sum_{n=0}^{N} \frac{1}{2^n} \frac{\|x - x_n\|_n}{1 + \|x - x_n\|_n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\
&\leq \frac{\epsilon}{2} \frac{2^N}{2^{N+1} - 1} \sum_{n=0}^{N} \frac{1}{2^n} + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $(x_m)_{m\in\mathbb{N}}$ converges to $x \in E$ with respect to $\rho$. Conversely, let $(x_m)_{m\in\mathbb{N}} \subset E$ converge with respect to $\rho$. Then for $n \in \mathbb{N}$ and $\epsilon > 0$ there exists a $M \in \mathbb{N}$ such that for $m \geq M$ we have $\rho(x, x_m) < \frac{\epsilon}{(1+2^n\epsilon)}$. Hence, for $m \geq M$ we have

$$\begin{aligned}
\frac{1}{2^n} \frac{\|x - x_m\|_n}{1 + \|x - x_m\|_n} &\leq \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|x - x_k\|_n}{1 + \|x - x_k\|_n} \\
&= \rho(x, x_m) \\
&< \frac{\epsilon}{(1 + 2^n\epsilon)}.
\end{aligned}$$

Therefore, $\|x - x_m\|_n < \epsilon$ for $m \geq M$. This implies that $(x_m)_{m\in\mathbb{N}}$ converges to $x \in E$ with respect to $\|\cdot\|_n$ for every $n \in \mathbb{N}$. So by Lemma 4.5.4 it follows that $(x_m)_{m\in\mathbb{N}} \subset E$ converges to $x \in E$ with the topology of $E$. In conclusion, we have that topology induced by $\rho$ on $E$ is equivalent to the inherent topology of $E$, and thus $E$ is metrizable. ☐

**Remark 4.5.6.** *Despite a countably normed space being metrizable, it is not necessarily a normed space.*

**Exercise 4.5.7.** *To show the metric of Lemma 4.5.5 induced the same topology, it was shown that convergence under the metric coincides with convergence in the underlying topology. Equivalently show that the metric of Lemma 4.5.5 induces the same topology by showing the open sets under the metric coincide with the open sets defined on the original topology.*

**Example 4.5.8.**

1. *The space $\mathcal{K}([a, b])$ is a countably normed space with*

$$\|f\|_m := \sup_{t\in[a,b], 0\leq k\leq m} \left| f^{(k)}(t) \right|$$

*for $m = 0, 1, \dots$. Indeed, one can verify the compatibility by verifying the compatibility of $\|\cdot\|_p$ and $\|\cdot\|_{p+1}$. On the one hand suppose that $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{K}([a, b])$ converges to zero with respect to $\|\cdot\|_p$ and is Cauchy with respect to $\|\cdot\|_{p+1}$. Then this means that $\varphi_n^{(k)}(x)$ converges uniformly to zero as $n \to \infty$ for $k = 0, 1 \dots, p$, and converges uniformly to some $\theta(x)$ for $k = p + 1$. However, it must be the case* *that $\theta(x) = 0$ and so $\|\varphi_n\|_{p+1} \to 0$. Conversely, if $\|\varphi\|_{p+1} \to 0$ then indeed*

$$\|\varphi_n\|_p \leq \|\varphi_n\|_{p+1} \to 0.$$

*Therefore, $\|\cdot\|_p$ and $\|\cdot\|_{p+1}$ are compatible.*

2. *The space $\mathcal{S}^\infty$ is a countably normed space with*

$$\|f\|_m := \sup_{k,q \leq m} \left| t^k f^{(q)}(t) \right|$$

*for $m = 0, 1, \ldots$. Indeed, suppose that $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^\infty$ converges to zero with respect to $\|\cdot\|_{m_1}$ and is Cauchy with respect to $\|\cdot\|_{m_2}$. Then,*

$$|\varphi_n(t)| \leq \|\varphi_n\|_{m_1} \overset{n\to\infty}{\longrightarrow} 0,$$

*which means that $(\varphi_n)_{n\in\mathbb{N}}$ uniformly converges to zero. Thus, since the sequence of derivatives $\left(\varphi_n^{(q)}\right)_{n\in\mathbb{N}}$ is Cauchy for each $q \leq m_2$ it converges and in particular, it must converge to zero. Therefore,*

$$\|\varphi_n\|_{m_2} \overset{n\to\infty}{\longrightarrow} 0,$$

*and so the norms are compatible.*

**Proposition 4.5.9.** *Let $E$ be a countable-normed linear space. Let $f$ be a linear functional on $E$. Then $f$ is continuous on $E$ if and only if $f$ is continuous with respect to $\|\cdot\|_k$ for some $k \in \mathbb{N}$.*

*Proof.* ($\Rightarrow$). Using Theorem 4.1.10 there exists some neighbourhood $U$ of $0 \in E$ such that $f$ is bounded on $U$. By construction of the topology of $E$ this means there exists some $\epsilon > 0$ and $k \geq 0$ such that

$$B_{k,\epsilon} := \{x : \|x\|_k < \epsilon\} \subset U.$$

Consequently, $f$ is bounded on $B_{k,\epsilon}$ and in particular $f$ is continuous with respect to $\|\cdot\|_k$ by Theorem 4.1.10. ($\Leftarrow$). By Theorem 4.1.10, $f$ is bounded on a neighbourhood of $0 \in E$, say $B_{k,\epsilon} := \{x : \|x\|_k < \epsilon\}$. Using the convention that $\|\cdot\|_l < \|\cdot\|_k$ for all $l < k$ it follows that $B_{l,\epsilon} \subseteq B_{k,\epsilon}$ for all $l < k$. Hence, $f$ is also bounded on each $B_{l,\epsilon}$ for $l < k$. Therefore, $f$ is bounded on

$$U := \{x : \|x\|_0 < \epsilon, \ldots, \|x\|_k < \epsilon\}.$$

Thus using Theorem 4.1.10 it follows that $f$ is continuous on $E$. ☐

**Corollary 4.5.10.** *Let $E$ be a countably-normed linear space. Then*

$$E^* = \bigcup_{n=0}^{\infty} E_n^*$$

*where $E_n^*$ is the space of continuous linear functionals on $E$ with respect to $\|\cdot\|_n$. In particular, assuming that* (4.5.2) *holds, we have that*

$$E_0^* \subset \ldots E_n^* \subset \ldots.$$

**Definition 4.5.11.** *Let $E$ be a countably-normed linear space. Let $f \in E^*$. Then the smallest $n$ such that $f \in E_n^*$ is referred to as the order of $f$.*

**Remark 4.5.12.** *From Corollary 4.5.10, any $f \in E^*$ has finite order.*

## 4.6 Solution to Exercises

### Exercise 4.1.8

*Solution.* Let $\{e_1, \ldots, e_n\}$ be a basis for $E$, such that for any $x \in E$ one can write

$$x = \sum_{i=1}^{n} x_i e_i$$

for $x_i \in \mathbb{R}$. As norms on finite-dimensional spaces are equivalent, we can assume without loss of generality that the norm on $E$ is given by

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

Let $f : E \to \mathbb{R}$ be a linear functional and let $x \in E$. Given $\epsilon > 0$, let $U := B_\delta(x)$, where $\delta = \frac{\epsilon}{\max_{i=1,\ldots,n}|f(e_i)|}$. Then, for $y \in U$ it follows that

$$\begin{aligned}
|f(x) - f(y)| &= \left|\sum_{i=1}^{n}(x_i - y_i) f(e_i)\right| \\
&\leq \max_{i=1,\ldots,n} |f(e_i)| \sum_{i=1}^{n} |x_i - y_i| \\
&= \|x - y\| \max_{i=1,\ldots,n} |f(e_i)| \\
&< \frac{\epsilon}{\max_{i=1,\ldots,n} |f(e_i)|} \max_{i=1,\ldots,n} |f(e_i)| \\
&= \epsilon.
\end{aligned}$$

Therefore, $f$ is continuous at $x \in E$. □

### Exercise 4.1.12

*Solution.* $(1) \Rightarrow (2)$. Since $f$ is continuous it is bounded on an open neighbourhood $U$ of $0$. In particular,

$$|f(x)| \leq M$$

for all $x \in U$ and for some $M > 0$. Hence, for $t = M + 1$ we have that $t \notin f(U)$.
$(2) \Rightarrow (1)$. Without loss of generality, we can suppose that $U$ is a neighbourhood of $0$. Moreover, we can suppose that $U = \{x : \|x\| < \epsilon\}$ for some $\epsilon > 0$. In particular, if $x \in U$ then for $\alpha \in [-1, 1]$ we have

$$\|\alpha x\| = |\alpha|\|x\| \leq \|x\| < \epsilon,$$

which implies that $\alpha x \in U$. Therefore, if $t \notin f(U)$ it must also be the case that $\frac{1}{\alpha} t \notin f(U)$, when $\alpha \neq 0$. It follows that $|f(x)| \leq |t|$ for all $x \in U$ which implies that $f$ is continuous.
$(1) \Rightarrow (3)$. Observe that $\{0\}$ is closed, and so $\ker(f) = f^{-1}(\{0\})$ is closed.
$(3) \Rightarrow (2)$. The set $U := \mathbb{C} \setminus \ker(f)$ is open and such that $0 \notin f(U)$.
$(1) \Rightarrow (4)$. Let $U \subseteq E$ be a bounded set, that is

$$U \subseteq \{x \in E : \|x\| \leq R\}$$

for some $R > 0$. A continuous linear functional is bounded on $\{x \in E : \|x\| \leq 1\}$, that is

$$|f(x)| \leq M$$

for $x \in \{x \in E : \|x\| \leq 1\}$ and some $M > 0$. Therefore, for $x \in U$ we have

$$|f(x)| = \|x\| \left| f\left(\frac{x}{\|x\|}\right) \right| \leq RM.$$

Hence, $f$ is bounded on $U$.
$(4) \Rightarrow (1)$. As $U = \{x \in E : \|x\| \leq 1\}$ is a bounded set we have that $f$ on the unit ball and therefore continuous by Corollary 4.1.11 ☐

**Exercise 4.2.4**

*Solution.*

1. $\|f\| \geq 0$ with $\|f\| = 0$ if and only if $f = 0$.
2. Clearly, $\|\alpha f\| = |\alpha|\|f\|$ for $\alpha \in \mathbb{R}$.
3. For $f_1, f_2 \in E^*$ we have

$$\|f_1 + f_2\| = \sup_{x \in E \setminus \{0\}} \frac{|f_1(x) + f_2(x)|}{\|x\|} \leq \sup_{x \in E \setminus \{0\}} \frac{|f_1(x)| + |f_2(x)|}{\|x\|} = \|f_1\| + \|f_2\|.$$

☐

**Exercise 4.2.10**

*Solution.* Throughout let $\{e_1, \ldots, e_n\}$ be a basis for $E$, such that

$$x = \sum_{i=1}^{n} x_i e_i$$

and

$$f(x) = \sum_{i=1}^{n} f_i x_i,$$

where $f_i := f(e_i)$.

1. Observe that

$$\begin{aligned}
\|f\| &= \sup_{x \setminus \{0\}} \frac{|f(x)|}{\|x\|} \\
&= \sup_{x \in \setminus \{0\}} \frac{|\sum_{i=1}^{n} f_j x_j|}{\|x\|} \\
&\leq \sup_{x \in E \setminus \{0\}} \frac{\left(\sum_{i=1}^{n} |f_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}}{\|x\|} \\
&= \left(\sum_{i=1}^{n} |f_i|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

With equality when $x = (f_1, \ldots, f_n)$, and so

$$\|f\| = \left(\sum_{i=1}^{n} |f_i|^2\right)^{\frac{1}{2}}.$$

2. Observe that

$$
\begin{aligned}
\|f\| &= \sup_{x\setminus\{0\}} \frac{|f(x)|}{\|x\|} \\
&= \sup_{x\in\setminus\{0\}} \frac{|\sum_{i=1}^{n} f_j x_j|}{\|x\|} \\
&\le \sup_{x\in E\setminus\{0\}} \frac{\left(\sum_{i=1}^{n} |f_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}}{\|x\|} \\
&= \left(\sum_{i=1}^{n} |f_i|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

With equality when $x = \big(\mathrm{sgn}(f_1)|f_1|^{q-1}, \ldots, \mathrm{sgn}(f_n)|f_n|^{q-1}\big)$, and so

$$
\|f\| = \left(\sum_{i=1}^{n} |f_i|^q\right)^{\frac{1}{q}}.
$$

3. Observe that

$$
\begin{aligned}
\|f\| &= \sup_{x\setminus\{0\}} \frac{|f(x)|}{\|x\|} \\
&= \sup_{x\in\setminus\{0\}} \frac{|\sum_{i=1}^{n} f_j x_j|}{\|x\|} \\
&\le \sup_{x\in E\setminus\{0\}} \frac{\max_{i=1,\ldots,n} |f_j| \sum_{i=1}^{n} |x_i|}{\|x\|} \\
&= \max_{i=1,\ldots,n} |f_i|.
\end{aligned}
$$

Suppose that $|f_j| = \max_{i=1,\ldots,n} |f_i|$. Then equality arises when $x = (x_1, \ldots, x_n)$ where

$$
x_i = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}
$$

for $i = 1, \ldots, n$. So

$$
\|f\| = \max_{i=1,\ldots,n} |f_i|.
$$

4. Observe that

$$
\begin{aligned}
\|f\| &= \sup_{x\setminus\{0\}} \frac{|f(x)|}{\|x\|} \\
&= \sup_{x\in\setminus\{0\}} \frac{|\sum_{i=1}^{n} f_j x_j|}{\|x\|} \\
&\le \sup_{x\in E\setminus\{0\}} \frac{\max_{i=1,\ldots,n} |x_i| \sum_{i=1}^{n} |f_i|}{\|x\|} \\
&= \sum_{i=1}^{n} |f_i|.
\end{aligned}
$$

With equality when $x = (\mathrm{sgn}(f_1), \ldots, \mathrm{sgn}(f_n))$ and so

$$
\|f\| = \sum_{i=1}^{n} |f_i|.
$$

$\square$

**Exercise 4.2.14**

*Solution.* Let $(f_1, f_2, \dots)\ell^q$, so that $(\sum_{n=1}^{\infty} |f_n|^q)^{\frac{1}{q}} = \|f\|_{\ell^q} < \infty$. Let $\hat{f} : \ell^p \to \mathbb{C}$ be given by

$$\hat{f}(x) = \sum_{n=1}^{\infty} f_n x_n.$$

Clearly $\hat{f}$ is linear. Furthermore,

$$\begin{aligned}\left\|\hat{f}\right\| &= \sup_{x\in\ell^p\setminus\{0\}} \frac{|\sum_{n=1}^{\infty} f_n x_n|}{\|x\|_{\ell^p}} \\ &\leq \sup_{x\in\ell^p\setminus\{0\}} \frac{\|x\|_{\ell^p}\|f\|_{\ell^q}}{\|x\|_{\ell^p}} \\ &= \|f\|_{\ell^p}.\end{aligned}$$

Hence, $\hat{f}$ is bounded which means that $\hat{f} \in (\ell^p)^*$. More specifically, let

$$x = \left(\operatorname{sgn}(f_1)|f_1|^{1-q}, \operatorname{sgn}(f_1)|f_2|^{q-1}, \dots\right).$$

Note that

$$\begin{aligned}\|x\|_{\ell^p}^p &= \left|\sum_{n=1}^{\infty} |f_n|^{p(q-1)}\right| \\ &= \sum_{n=1}^{\infty} |f_n|^q \\ &= \|f\|_{\ell^q}^q \\ &< \infty,\end{aligned}$$

so that $x \in \ell^p$. Moreover,

$$\begin{aligned}\left|\hat{f}(x)\right| &= \left|\sum_{n=1}^{\infty} \operatorname{sgn}(f_n) f_n |f_n|^{q-1}\right| \\ &= \sum_{n=1}^{\infty} |f_n|^q \\ &= \|f\|_{\ell^q}^q,\end{aligned}$$

so that

$$\begin{aligned}\frac{\left|\hat{f}(x)\right|}{\|x\|} &= \frac{\|f\|_{\ell^q}^q}{\|f\|_{\ell^q}^{\frac{q}{p}}} \\ &= \|f\|_{\ell^q}^{q-\frac{q}{p}} \\ &= \|f\|_{\ell^q},\end{aligned}$$

which implies that $\left\|\hat{f}\right\| = \|f\|_{\ell^q}$. Conversely, let $\hat{f} \in (\ell^p)^*$. Then one can write

$$\hat{f}(x) = \sum_{n=1}^{\infty} f_n x_n$$

where $f_n = \hat{f}(e_n)$. Let $f = (f_1, f_2, \dots)$. As $\hat{f} \in (\ell^p)^*$ we know that $\left\|\hat{f}\right\| = M < \infty$. In particular, letting

$$x = \left(\operatorname{sgn}(f_1)|f_1|^{1-q}, \operatorname{sgn}(f_1)|f_2|^{q-1}, \dots\right)$$

we deduce from our above computations that

$$\|f\|_{\ell^q} \leq M < \infty,$$

that is $f \in \ell^q$. As before we deduce that $M = \|\hat{f}\|$, thus the map $f \mapsto \hat{f}$ is an isometry. $\square$

**Exercise 4.2.18**

*Solution.* Let $x_1, x_2 \in E$ with $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}
\varphi_{x_1+\lambda x_2}(f) &= f\left(x_1 + \lambda x_2\right) \\
&= f(x_1) + \lambda f(x_2) \\
&= \varphi_{x_1}(f) + \lambda \varphi_{x_2}(f) \\
&= \left(\varphi_{x_1} + \lambda \varphi_{x_2}\right)(f).
\end{aligned}$$

Therefore, $x \mapsto \varphi_x$ is a linear map. Suppose $x \in \ker(\pi)$. Then $f(x) = 0$ for all $f \in E^*$. Which means that $x = 0$ and so $\ker(\pi) = \{0\}$ meaning $\pi$ is injective. Therefore, we conclude that $\pi$ is an isomorphism onto it image $\pi(E) \subseteq E^{**}$. $\square$

**Exercise 4.3.3**

*Solution.*

1. For fixed $u \in U$, the map $\varphi(v) = v - u$ is linear and thus continuous. Thus, $u + V = \varphi^{-1}(V)$ is an open set. Therefore,
$$U + V = \bigcup_{u \in U} u + V$$
is open.

2. For $\alpha \neq 0$, the map $\varphi(u) = \frac{1}{\alpha}u$ is linear and thus continuous. Thus, $\alpha U = \varphi^{-1}(U)$ is an open set.

3. If $\alpha = 0$, then $\alpha F$ is closed since its complement $E$ is open. If $\alpha \neq 0$ and $G$ is the complement of $F$, then $\alpha G$ is the complement of $\alpha F$. Since $G$ is open it follows by statement 2 that $\alpha G$ is open meaning $\alpha F$ is closed.

4. Since $0 + 0 \in U$ there exists neighbourhoods $V_1$ and $V_2$ of $0$ such that $V_1 + V_2 \subseteq U$ by statement 1 of Remark 6.1.5. By statement 2 of Exercise 4.3.3 the sets $-V_1$ and $-V_2$ are open and also contain $0$ since $-0 = 0$. Therefore,
$$W := V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$$
is an open set containing $0$. In particular, we note that if $w \in W$ then $-w \in W$ so that $-W = W$. Moreover, for $w_1, w_2 \in W$ we have that $w_1 \in V_1$ and $w_2 \in V_2$ so that
$$w_1 + w_2 \in V_1 + V_2 \subseteq U,$$
meaning $W + W \subseteq U$.

5. Note that $E \setminus F - x$ is an open set containing zero. Therefore, by statement 4 of Exercise 4.3.3 there exists an open set $V_x'$ of zero such that $V_x' + V_x' \subseteq E \setminus F - x$. Similarly, as $V_x'$ is an open set containing zero, by statement 4 of Exercise 4.3.3 there exists an open set $V_x$ of zero such that $V_x + V_x \subseteq V_x'$. In particular, we have that
$$V_x + V_x + V_x + V_x \subseteq E \setminus F - x,$$
or equivalently
$$x + V_x + V_x + V_x + V_x \subseteq E \setminus F.$$

Since, $0 \in V_x$ it follows that

$$x + V_x + V_x + V_x \subseteq E \setminus F.$$

Now suppose that

$$(x + V_x + V_x) \cap (F + V_x) \neq \emptyset.$$

Then there exists $u_1, u_2, u_3 \in V_x$ and $f \in F$ such that $x + u_1 + u_2 = f + u_3$ which implies that $f = x + u_1 + u_2 - u_3$. However, since $-u_3 \in V_x$ it follows that

$$f \in x + V_x + V_x + V_x,$$

which is a contradiction. Therefore, $x + V_x + V_x$ and $F + V_x$ are non-intersecting neighbourhoods of $x$ and $F$ respectively.

□

**Exercise 4.3.9**

*Solution.* ($\Rightarrow$). Let $U$ be a neighbourhood of $0$. Let $k$ be such that $A \subseteq \lambda U$ for every $|\lambda| \geq k$. Equivalently, $\frac{1}{\lambda} x_n \in U$ for every $|\lambda| \geq k$. As there exists a $N \in \mathbb{N}$ such that $\epsilon_n \leq \frac{1}{k}$ for $n \geq N$ it follows that $\epsilon_n x_n \in U$ for every $n \geq N$. Hence, $\epsilon_n x_n \to 0$ as $n \to \infty$.
($\Leftarrow$). Let $x \in A$. Take $x_n = x$ for every $n \in \mathbb{N}$ and $(\epsilon_n)_{n \in \mathbb{N}}$ such that $\epsilon_n \to 0$ as $n \to \infty$. Then given an open neighbourhood $U$ of $0$ there exists a $N \in \mathbb{N}$ such that $\epsilon_n x \in U$. In particular, this means that $\alpha x \in U$ for $0 < \alpha < \sup_{n \geq N}(\epsilon_n)$. Let $k > \frac{1}{\sup_{n \geq N} \epsilon_n}$. Then $x \in \lambda U$ for every $|\lambda| \geq k$. As $k$ is independent of $x$ it follows that $A \subseteq \lambda U$ for every $|\lambda| \geq k$. Therefore, $A$ is bounded. □

**Exercise 4.3.14**

*Solution.* Assuming the topology of Definition 4.3.12, we have already seen that $(U_{\epsilon,B})_{\epsilon>0}$ is an open base for the strong topology as given by Definition 4.2.5. On the other hand, suppose $E$ is a normed linear space with $E^*$ having the strong topology as given by Definition 4.2.5. Consider the set $U_{\epsilon,A}$. Then for $g \in U_{\epsilon,A}$ it must be the case that $|g(x)| \leq \delta < \epsilon$. Otherwise, the function $\frac{1}{\epsilon - |g(x)|}$ would be continuous and unbounded on $A$, which cannot be the case. Then as the set $A$ is bounded, there exists an $R$ such that $\|x\| \leq R$ for every $x \in A$. Therefore, one can take the open set

$$U := \left\{ f \in E^* : \|f\| < \frac{\epsilon - \delta}{R} \right\}$$

and observe that

$$\begin{aligned} |g(x) + f(x)| &\leq |g(x)| + |f(x)| \\ &\leq \delta + \|f\| \|x\| \\ &< \delta + \frac{\epsilon - \delta}{R} R \\ &= \epsilon. \end{aligned}$$

Thus, $U + g \subseteq U_{\epsilon,A}$ which means that $U_{\epsilon,A}$ is open. □

**Exercise 4.4.1**

*Solution.* Since the pre-images of open sets under continuous functionals is open, it follows that $U$ is the finite intersection of open sets and thus is itself open. Suppose

$$U_1 = \{x \in E : |f_j(x)| < \epsilon_1,\, j = 1, \ldots, n_1\}$$

and

$$U_2 = \{x \in E : |f_j(x)| < \epsilon_2\, j = n_1 + 1, \ldots, n_2\}.$$

Then,

$$U_1 \cap U_2 \supseteq \{x \in E : |f_j(x)| \leq \min(\epsilon_1, \epsilon_2),\, j = 1, \ldots, n_2\}.$$

Therefore, the system is an open base. □

**Exercise 4.5.3**

*Solution.* It suffices to check that linear operations are continuous on $E$.

- For fixed $\lambda \neq 0$ consider $f : E \to E$ given by $f(x) = \lambda x$. For $x_0 \in E$ consider the neighbourhood

$$U_{r,\epsilon} + f(x_0).$$

  Note that for $x \in U_{r,\frac{\epsilon}{|\lambda|}}$ it follows that

$$\begin{aligned} \|f(x) - f(x_0)\|_l &= \|\lambda x - \lambda x_0\|_l \\ &= |\lambda|\|x - x_0\|_l \\ &\leq |\lambda|\frac{\epsilon}{|\lambda|} = \epsilon, \end{aligned}$$

  for $l = 0, \ldots, r$, which implies that $f(x) \in U_{r,\epsilon} + f(x_0)$. Therefore, $f$ is continuous at $x_0$ and hence continuous on $E$. If $\lambda = 0$ then $f(x) = 0$ which is continuous.

- For fixed $y \in E$ consider $f : E \to E$ given by $f(x) = x + y$. For $x_0 \in E$ consider the neighbourhood

$$U_{r,\epsilon} + f(x_0).$$

  Note that for $x \in U_{r,\epsilon}$ it follows that

$$\begin{aligned} \|f(x) - f(x_0)\|_l &= \|x - x_0\|_l \\ &\leq \epsilon, \end{aligned}$$

  for $l = 0, \ldots, r$, which implies that $f(x) \in U_{r,\epsilon} + f(x_0)$. Therefore, $f$ is continuous at $x_0$ and hence continuous on $E$.

□

**Exercise 4.5.7**

*Solution.* Let $N_\rho$ be an open neighbourhood with respect to the metric $\rho$. For $x_0 \in N_\rho$ consider the neighbourhood $N_\rho - x_0$ of $0$. Then there exists a neighbourhood

$$B = \{x \in E : \rho(x, 0) < \epsilon\}$$

of $0$ such that $B \subseteq N_\rho - x_0$. Let $r \in \mathbb{N}$ be such that $\sum_{n=r+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}$. Then for $x \in U_{r,\frac{\epsilon}{4}}$ it follows that

$$\begin{aligned} \rho(x, 0) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|x\|_n}{1 + \|x\|_n} \\ &= \sum_{n=0}^{r} \frac{1}{2^n} \frac{\|x\|_n}{1 + \|x\|_n} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} \frac{\|x\|_n}{1 + \|x\|_n} \\ &\leq \sum_{n=0}^{r} \frac{1}{2^n} \frac{\|x\|_n}{1 + \|x\|_n} + \frac{\epsilon}{2} \\ &\leq \sum_{n=0}^{r} \frac{1}{2^n} \frac{\frac{\epsilon}{4}}{1 + \frac{\epsilon}{4}} + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{4} \sum_{n=0}^{r} \frac{1}{2^n} + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $U_{r,\frac{\epsilon}{4}} \subseteq B$, implying that $B$ is open in the topology of the countably normed space. Conversely, consider

$$U_{r,\epsilon} = \{x \in E : \|x\|_j < \epsilon \text{ for } j = 0, \ldots, r\}$$

and

$$B = \{x \in E : \rho(x, 0) < \tilde{\epsilon}\}.$$

For $x \in B$ we have $\frac{1}{2^n} \frac{\|x\|_n}{1+\|x\|_n} < \tilde{\epsilon}$ for every $n \in \mathbb{N}$, in particular, $(1 - 2^n\tilde{\epsilon}) \|x\|_n < 2^n\tilde{\epsilon}$. Therefore, for sufficiently small $\tilde{\epsilon}$, so that $1 - 2^n\tilde{\epsilon} > 0$ for $n = 0, \ldots, r$, we have that

$$\|x\|_n < \frac{2^n\tilde{\epsilon}}{1 - 2^n\tilde{\epsilon}}.$$

As $\tilde{\epsilon} \searrow 0$ it follows that $\|x\|_n \searrow 0$ for $n = 0, \ldots, r$. Therefore, for $\tilde{\epsilon}$ sufficiently small we have $\|x\|_n < \epsilon$ for $n = 0, \ldots, r$ which implies that $x \in U_{r,\epsilon}$. Hence, $B \subseteq U_{r,\epsilon}$ meaning $U_{r,\epsilon}$ is open in the topology of the metric. $\square$

# 5 Distributions

Consider $f : \mathbb{R} \to \mathbb{R}$ a locally integrable function and $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support. Then

$$T(f) = (f, \varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x,$$

is a well-defined linear functional on the space of functions $\varphi$. However, the space of all linear functionals on $\varphi$ extends beyond those we can identify with $f$. In particular, shrinking the space of functions $\varphi$ expands the space of functionals on such functions.

## 5.1 The Space of Test Functions

For $A \subset \mathbb{R}$, let $\mathcal{C}_c^\infty(A)$ denote the linear space of infinitely differentiable functions on $A$ with compact support. We succinctly write $\mathcal{D}(A) = \mathcal{C}_c^\infty(A)$ and let $\mathcal{D} = \mathcal{D}(\mathbb{R})$.

**Example 5.1.1.** *Consider*

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{(b-x)(x-a)}\right) & x \in (a,b) \\ 0 & \textit{otherwise.} \end{cases}$$

*Then* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$.

**Definition 5.1.2.** *The linear space $\mathcal{D}$ is referred to as the space of test functions, with elements of $\mathcal{D}$ known as test functions.*

Let $\mathcal{D}_m \subset \mathcal{D}$ consist of the test functions vanishing outside $[-m, m]$ such that $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an increasing sequence of sets with $\mathcal{D} = \bigcup_{m\in\mathbb{N}} \mathcal{D}_m$. Note that $\mathcal{D}_m$ is a countably-normed space with

$$\|\varphi\|_n^{(m)} = \sup_{0\le k\le n, |t|\le m} \left|\varphi^{(k)}(t)\right|$$

for $n \in \mathbb{N}$. A set $U$ is a neighbourhood of $0$ in $\mathcal{D}$ if for all $m$ we have that $U \cap \mathcal{D}_m$ is a neighbourhood of $0$ in $\mathcal{D}_m$. The topology on $\mathcal{D}$ induced by these neighbourhoods makes $\mathcal{D}$ a topological linear space.

**Lemma 5.1.3.** *The sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ converges to $\varphi \in \mathcal{D}$ if and only if the following statements hold.*

1. *There is an interval $[a, b]$ such that for all $n \in \mathbb{N}$ we have $\varphi_n(x) = 0$ for $x \in \mathbb{R} \setminus [a, b]$.*
2. *For fixed $k \in \mathbb{N}$, the sequence $\left(\varphi_n^{(k)}(x)\right)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges to $\varphi^{(k)}(x)$ uniformly.*

*Proof.* $(\Rightarrow)$. Suppose that $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ converges to $0$ in $\mathcal{D}$.

- Suppose that 1 does not hold. Then for each $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ with $|x_{n_j}| > j$ such that $\varphi_{n_j}(x_{n_j}) =: \epsilon_j > 0$. Since $\varphi_{n_j} \to 0$, every neighbourhood of $0$ contains a tail of this sequence. However, let $U$ be a neighbourhood of $0$ containing distributions such that if $\psi \in \mathcal{D}_1$ then $\|\psi\|_0^{(1)} < \frac{\epsilon_0}{2}$, if $\psi \in \mathcal{D}_2 \setminus \mathcal{D}_1$ then $\|\psi\|_0^{(2)} < \frac{1}{2}\min(\epsilon_0, \epsilon_1)$ and so on. By construction, the set $U$ cannot contain a tail of $\varphi_{n_j}$, which is a contradiction.
- Note that statement 1 implies that for every $n \in \mathbb{N}$ we have that $\varphi_n \in \mathcal{D}_m$ for some $m \in \mathbb{N}$. Since $\varphi_n$ converges in $\mathcal{D}$ it also converges in $\mathcal{D}_m$. In particular, it converges with respect to the norm of $\mathcal{D}_m$ by Lemma 4.5.4. Hence, $\left(\varphi_n^{(k)}\right) \to 0$ uniformly for each $k \in \mathbb{N}$.

$(\Leftarrow)$. Suppose that for $(\varphi_n)_{n\in\mathbb{N}}$ we have that $\varphi_n(x) = 0$ for $x \in \mathbb{R} \setminus [-m, m]$ and every $n \in \mathbb{N}$. Moreover, suppose that $\left(\varphi_n^{(k)}(x)\right)_{n\in\mathbb{N}}$ converges uniformly to $0$ for every $k \in \mathbb{N}$. Then it follows that $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_m$ and converges to $0$ in $\mathcal{D}_m$. Therefore, for any neighbourhood $U$ of $0$ in $\mathcal{D}$, there exists a $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$\varphi_n \in U \cap \mathcal{D}_m \subseteq U.$$

Thus, $\varphi_n \to \varphi$ in $\mathcal{D}$. ☐

**Definition 5.1.4.** *A function $f$ is a distribution, or generalised function if $f \in \mathcal{D}^*$. We use $\mathcal{D}' = \mathcal{D}^*$ to denote the space of distributions.*

**Lemma 5.1.5.** *A linear functional $f$ on $\mathcal{D}$ is continuous if and only if $f(\varphi_n) \to f(\varphi)$ if $\varphi_n \to \varphi$ in $\mathcal{D}$.*

*Proof.* ($\Rightarrow$). Without loss of generality suppose that $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ is such that $\varphi_n \to 0$. In particular, by statement 1 of Lemma 5.1.3 there exists a $M > 0$ such that for all $n \in \mathbb{N}$ we have $\varphi_n(x) = 0$ for all $x \in \mathbb{R} \setminus [-M, M]$. Given an $\epsilon > 0$, by the continuity of $f$, there exists an open neighbourhood $U$ of $0 \in \mathcal{D}$ such that $|f(\varphi)| < \epsilon$ for all $\varphi \in \mathcal{D}$. In particular $U \cap \mathcal{D}_M$ is open so that there exists a $\delta > 0$ such that

$$U_{r,\delta} = \left\{\varphi \in \mathcal{D} : \|\varphi\|_0^{(M)} < \delta, \dots, \|\varphi\|_r^{(M)} < \delta\right\} \subseteq U.$$

By statement 2 of Lemma 5.1.3 there exists a $N \in \mathbb{N}$ such that for each $k = 0, \dots, r$ we have $\|\varphi_n\|_k^{(M)} < \delta$ for $n \geq N$. Therefore,

$$\varphi_n \in U_{r,\delta} \subseteq U,$$

and so $|f(\varphi_n)| < \epsilon$. Thus, $f(\varphi_n) \to 0$.
($\Leftarrow$). Suppose $f$ is not continuous at $0$. Then for some $\epsilon > 0$ it follows that for any neighbourhood $U$ of $0$ in $\mathcal{D}$ there exists a $\varphi \in U$ such that $|f(\varphi)| \geq \epsilon$. Note that

$$U_n := \left\{\varphi : \|\varphi\|_k^{(M)} < \frac{1}{n} \text{ for all } k \in \mathbb{N}\right\}$$

is an open neighbourhood of $0$ in $\mathcal{D}$. Let $\varphi_n \in U_n$ be such that $|f(\varphi_n)| \geq \epsilon$, in particular we can choose $\varphi_n$ such that $\varphi_n(x) = 0$ for $x \in \mathbb{R} \setminus [-M, M]$. Since, $\|\varphi_n\|_k^{(M)} \to 0$ as $n \to \infty$ for any $k \in \mathbb{N}$ it follows that $\left(\varphi_n^{(k)}(x)\right)_{n\in\mathbb{N}}$ converges uniformly to $0$ for each $k \in \mathbb{N}$. Therefore, $\varphi_n \to 0$ in $\mathcal{D}$ by Lemma 5.1.3. However, this is a contradiction, as this would imply that $f(\varphi_n) \to 0$ which is not the case. Hence, $f$ must be continuous at $0$. ☐

**Remark 5.1.6.** *By linearity, it is sufficient to check the criterion of Lemma 5.1.5 for $\varphi_n \to 0$ in $\mathcal{D}$.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. With $f$ one can identify the linear functional

$$(f, \varphi) := \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x.$$

In particular, if $\varphi_n \to \varphi$, then by statement 1 of Lemma 5.1.3 there exists a compact set $K \subseteq \mathbb{R}$ such that

$$(f, \varphi_n) = \int_K f(x)\varphi_n(x)\,\mathrm{d}x$$

for every $n \in \mathbb{N}$. By statement 2 of Lemma 5.1.3, the sequence $(\varphi_n(x))_{n\in\mathbb{N}}$ uniformly converges to $\varphi(x)$ on $K$. As $\varphi$ is bounded, it follows that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is uniformly bounded on $\mathbb{R}$. Therefore, as $f$ is locally integrable it follows that $|(f, \varphi_n)| \leq M$ for every $n \in \mathbb{N}$. So one can use the dominated convergence theorem to deduce that $(f, \varphi_n) \to (f, \varphi)$ as $n \to \infty$. Thus, using Lemma 5.1.5 we deduce that $(f, \cdot)$ is a continuous and hence defines a distribution. Distributions that can be identified with an $f$ in such a way are referred to as regular, whilst the other distributions are referred to as singular.

**Example 5.1.7.** *The following are singular distributions.*

1. *The $\delta$-function, which is given by $\delta(\varphi) = \varphi(0)$. Similarly, the distribution $\delta(x-a)$ given by $\delta(x-a)(\varphi) = \varphi(a)$ is singular. Indeed, suppose that $\delta(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x$ for some locally integrable function $f$. Then for $\varphi \in \mathcal{D}$ with $0 \notin \operatorname{supp}(\varphi)$, it follows that*
$$0 = \int_{\operatorname{supp}(\varphi)} f(x)\varphi(x)\,\mathrm{d}x.$$
*This implies that $f(x) = 0$ almost everywhere on $\mathbb{R}$. Hence, if $\varphi \in \mathcal{D}$ is such that $0 \in \operatorname{supp}(\varphi)$ it follows that*
$$0 = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x = \delta(\varphi) = \varphi(0) > 0,$$
*which is a contradiction. Therefore, the $\delta$-function is a singular distribution.*

2. *Recall that $\frac{1}{x}$ is not integrable at $0$. However,*
$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{1}{x}\varphi(x)\,\mathrm{d}x$$
*exists for $\varphi(x) \in \mathcal{D}$. Indeed, for $\varphi \in \mathcal{D}$, there exists an $R > 0$ such that $\varphi(x) = 0$ on $x \in \mathbb{R} \setminus [-R, R]$. Thus,*
$$\begin{aligned} f_{\frac{1}{x}}(\varphi(x)) &:= \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{1}{x}\varphi(x)\,\mathrm{d}x \\ &= \lim_{\epsilon \searrow 0} \int_{[-R,R]\setminus[-\epsilon,\epsilon]} \frac{1}{x}\varphi(x)\,\mathrm{d}x \\ &= \lim_{\epsilon \searrow 0} \int_{[-R,R]\setminus[-\epsilon,\epsilon]} \frac{\varphi(x)-\varphi(0)}{x}\,\mathrm{d}x + \varphi(0)\lim_{\epsilon \searrow 0} \int_{[-R,R]\setminus[-\epsilon,\epsilon]} \frac{1}{x}\,\mathrm{d}x \\ &= \int_{[-R,R]} \frac{\varphi(x)-\varphi(0)}{x}\,\mathrm{d}x + 0. \end{aligned}$$
*Moreover, through integration by parts*
$$f_{\frac{1}{x}}(\varphi) = -\int_{[-R,R]} \varphi'(x)\log(|x|)\,\mathrm{d}x$$
*which implies that*
$$\left|f_{\frac{1}{x}}(\varphi)\right| \le C(R) \sup_{|x|\le R} |\varphi'(x)|\,.$$
*Thus, if $\varphi \to 0$ in $\mathcal{D}$ then $\left|f_{\frac{1}{x}}(\varphi)\right| \to 0$ by statement 2 of Lemma 5.1.3, and so by Lemma 5.1.5 it follows that $f_{\frac{1}{x}}$ is continuous.*

**Lemma 5.1.8.** *A linear functional $f$ on $\mathcal{D}$ is continuous if and only if $f$ is continuous as a linear function on $\mathcal{D}_m$ for every $m \in \mathbb{N}$.*

*Proof.* ($\Rightarrow$). Suppose that there exists a $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $c > 0$ there exists a $\varphi_{n,c} \in \mathcal{D}_m$ such that
$$|\varphi_{n,c}| > c\,\|\varphi_{n,c}\|_n^{(m)}\,.$$
Let
$$\psi_{n,c} := \frac{\varphi_{n,c}}{|f(\varphi_{n,c})|},$$

then $1 > c\,\|\psi_{n,c}\|_n^{(m)}$. In particular,

$$\|\psi_{n,n}\|_n^{(m)} < \frac{1}{n}$$

which implies that $\psi_{n,n} \to \mathcal{D}$ by Lemma 5.1.3. However, $|f(\psi_{n,n})| = 1 \not\to 0$, which contradicts Lemma 5.1.5.
($\Leftarrow$). Let $\varphi_n \to \varphi$ in $\mathcal{D}$. Then, using statement 1. of Lemma 5.1.3, there exists an interval $[a, b]$ such that $\varphi_n(x) = 0$ for $x \in \mathbb{R} \setminus [a, b]$ and every $n \in \mathbb{N}$. In particular, there exists an $m$ such that $\varphi_n(x) = 0$ for $x \in \mathbb{R} \setminus [-m, m]$ for every $n$. Therefore, $\varphi_n \to \varphi$ in $\mathcal{D}_m$. Hence, $f(\varphi_n) \to f(\varphi)$ as $f$ is continuous on $\mathcal{D}_m$. Thus, $f$ is continuous using Lemma 5.1.5. ☐

One can show that on $\mathcal{D}'$, the convergence of sequences under the strong and weak-$*$ topology coincide. This motivates the Definition 5.1.9 for convergence in $\mathcal{D}'$.

**Definition 5.1.9.** *A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}'$ converges to $f \in \mathcal{D}'$ if $f_n(\varphi) \to f(\varphi)$ for any $\varphi \in \mathcal{D}$.*

## 5.2 Derivative of a Distribution

Suppose that $f$ is a continuously differentiable function, and let

$$T(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x,$$

where $\varphi$ is differentiable with compact support. Then integrating by parts it follows that

$$\int_{-\infty}^{\infty} f'(x)\varphi(x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,\mathrm{d}x. \tag{5.2.1}$$

It is natural to consider the left-hand side of (5.2.1), as the derivative of $T$, namely $T'(\varphi)$. Consequently, as the right-hand side of (5.2.1) does not require the assumption that $f$ is differentiable, it provides a means by which to define a derivative more generally.

**Definition 5.2.1.** *For $f \in \mathcal{D}'$, its derivative is given by*

$$f'(\varphi) = -f(\varphi').$$

*Similarly,*

$$f^{(k)}(\varphi) = (-1)^k f\left(\varphi^{(k)}\right)$$

*for $k = 1, 2, \ldots$.*

With this we see that if $f_n \to f$ in $\mathcal{D}'$, then $f_n^{(k)} \to f^{(k)}$ in $\mathcal{D}'$.

**Example 5.2.2.**

1. *If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then its derivative is identified with the distribution of the corresponding induced distribution.*
2. *Let*

$$h(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0, \end{cases}$$

*such that*

$$(h, \varphi) = \int_0^{\infty} \varphi(x), \mathrm{d}x.$$

*Then*

$$\begin{aligned}(h', \varphi) &= -(h, \varphi') \\ &= -\int_0^\infty \varphi'(x)\,\mathrm{d}x \\ &= \varphi(0).\end{aligned}$$

*Therefore, $h' = \delta$.*

3. *Using 1. and 2. we see that if $f : \mathbb{R} \to \mathbb{R}$ is a function with jumps at $(x_i)_{i\in\mathbb{N}}$ equal to $(h_i)_{i\in\mathbb{N}}$ and continuously differentiable everywhere else, then its distributional derivative is the sum of the ordinary derivative at the points where it exists, and $\sum_{i=1}^\infty h_i\delta(x - x_i)$.*

4. *The distributional derivative of $\delta$ is*
$$\delta'(\varphi) = -\varphi'(0).$$

5. *Consider*
$$f(x) = \sum_{n=1}^\infty \frac{\sin(nx)}{n} = \begin{cases} \frac{\pi - x}{2} & 0 < x \le \pi \\ -\frac{\pi+x}{2} & -\pi \le x < 0 \\ 0 & x = 0, \end{cases}$$
*extended as a $2\pi$ periodic function on $\mathbb{R}$. Using the right-hand side it follows from 3. that*
$$f' = -\frac{1}{2} + \pi \sum_{k=-\infty}^\infty \delta(x - 2\pi k).$$
*However, using the left-hand side it follows that*
$$f' = \sum_{n=1}^\infty \cos(nx)$$
*in the sense of distributions. Therefore,*
$$\sum_{n=-\infty}^\infty e^{inx} = 2\pi \sum_{k=-\infty}^\infty \delta(x - 2\pi k).$$

### 5.2.1 Application to Differential Equations

To understand how distributions can be applied to solve differential equations it will be useful to let $\mathcal{D}^{(1)}$ denote the linear subspace of $\mathcal{D}$ consisting of distributions $\varphi \in \mathcal{D}$ that are the derivative of some distribution $\psi \in \mathcal{D}$.

**Lemma 5.2.3.** *Let $\varphi \in \mathcal{D}$. Then $\varphi \in \mathcal{D}^{(1)}$ if and only if $\int_{-\infty}^\infty \varphi(x)\,\mathrm{d}x = 0$.*

*Proof.* $(\Rightarrow)$. Let $\varphi(x) = \psi'(x)$ for $\psi \in \mathcal{D}$. Then
$$\int_{-\infty}^\infty \varphi(x)\,\mathrm{d}x = [\psi(x)]_{-\infty}^\infty = 0.$$
$(\Leftarrow)$. Let
$$\psi(x) := \int_{-\infty}^x \varphi(t)\,\mathrm{d}t.$$
Note that $\psi(x)$ is infinitely differentiable, with $\psi' = \varphi$. As $\varphi$ has compact support, there exists a $K$ such that $\varphi(x) = 0$ for $x \notin [-K, K]$. In particular, as $\int_{-\infty}^\infty \varphi(t)\,\mathrm{d}t = 0$ we have that $\psi(x) = 0$ for $x \notin [-K, K]$. Therefore, the support of $\psi$ is bounded and thus must also be compact. Thus, $\psi \in \mathcal{D}$ which means $\varphi \in \mathcal{D}^{(1)}$. $\square$

**Remark 5.2.4.** *Lemma 5.2.3 can be interpreted as saying that the kernel of the functional $f \equiv 1$ is $\mathcal{D}^{(1)}$. Hence, using the general theory of linear functionals on linear spaces, it follows that any $\varphi \in \mathcal{D}$ can be represented as*

$$\varphi = c\varphi_0 + \varphi_1$$

*for some $\varphi_1 \in \mathcal{D}^{(1)}$, $c \in \mathbb{C}$, with $\varphi_0$ a fixed element of $\mathcal{D} \setminus \mathcal{D}^{(1)}$ that satisfies*

$$(f, \varphi_0) = \int_{-\infty}^{\infty} \varphi_0(x)\,\mathrm{d}x = 1.$$

*Note that $c = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x$ so we deduce that $\varphi_1 = \varphi - c\varphi_0$.*

**Theorem 5.2.5.** *In $\mathcal{D}'$, the only solutions to the equation $y' = 0$ are constant solutions.*

*Proof.* With $y' = 0$ it follows that

$$0 = (y', \varphi) = (y, -\varphi') \tag{5.2.2}$$

for all $\varphi \in \mathcal{D}$. In particular, (5.2.2) defines the linear functional $y$ on $\mathcal{D}^{(1)}$. To determine the linear functional $y$ on $\mathcal{D}$, it suffices to determine $y$ on $\varphi_0$ from Remark 5.2.4. Indeed, let $(y, \varphi_0) = \alpha$ for some $\alpha \in \mathbb{C}$, then

$$\begin{aligned} (y, \varphi) &= (y, c\varphi_0 + \varphi_1) \\ &= c(y, \varphi_0) + (y, \varphi_1) \\ &\overset{(5.2.2)}{=} c(y, \varphi_0) \\ &= c\alpha \\ &= \int_{-\infty}^{\infty} \alpha\varphi(x)\,\mathrm{d}x. \end{aligned}$$

Hence, $y = \alpha$ on $\mathcal{D}$. ☐

**Corollary 5.2.6.** *Let $f, g \in \mathcal{D}'$. Then if $f' = g'$ it follows that $f - g = c$, where $c$ is a constant.*

**Theorem 5.2.7.** *The differential equation $y' = f$ for $f \in \mathcal{D}'$ has a solution $y \in \mathcal{D}'$.*

*Proof.* With $y' = f$ we have

$$(f, \varphi) = (y', \varphi) = (y, -\varphi') \tag{5.2.3}$$

for all $\varphi \in \mathcal{D}$. From (5.2.3), the linear functional $y$ is defined on all of $\mathcal{D}^{(1)}$. In particular, for $\varphi_1 \in \mathcal{D}^{(1)}$, we have

$$(y, \varphi_1) = \left(f, -\int_{-\infty}^{x} \varphi_1(t)\,\mathrm{d}t\right).$$

Let $\varphi_0 \in \mathcal{D} \setminus \mathcal{D}^{(1)}$ be as in Remark 5.2.4, and set $(y, \varphi_0) = 0$. Then for $\varphi \in \mathcal{D}$ we have

$$\begin{aligned} (y, \varphi) &= (y, \varphi_1) \\ &= \left(f, -\int_{-\infty}^{x} \varphi_1(t)\,\mathrm{d}t\right) \end{aligned} \tag{5.2.4}$$

where $\varphi_1 = \varphi - c\varphi_0$. Note that $y$ as given by (5.2.4) is linear. Moreover, let $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be such that $\varphi_n \to 0$. Let $\varphi_n = c^{(n)}\varphi_0 + \varphi_1^{(n)}$ for each $n \in \mathbb{N}$. As $\varphi^{(n)} \in \mathcal{D}^{(1)}$, it follows that there exists a $\psi^{(n)} \in \mathcal{D}$ such that

$\left(\psi^{(n)}\right)' = \varphi_1^{(n)}$. Thus,

$$\begin{aligned}
|(y, \varphi_n)| &= \left|\left(f, -\int_{-\infty}^{x} \varphi_1^{(n)}(t)\,\mathrm{d}t\right)\right| \\
&= \left|(f, -\int_{-\infty}^{x} \left(\psi^{(n)}(t)\right)'\,\mathrm{d}t\right| \\
&= \left|\left(f, \psi^{(n)}\right)\right|
\end{aligned}$$

As $\varphi_n \to 0$ it is clear that $\psi^{(n)} \to 0$ and so as $f$ is continuous we have $\left|\left(f, \psi^{(n)}\right)\right| \to 0$. Hence, $|(y, \varphi_n)| \to 0$ which implies that $y$ is also continuous. Observing that

$$\begin{aligned}
(y', \varphi) &= (y, -\varphi') \\
&\overset{(5.2.4)}{=} \left(f, \int_{-\infty}^{x} \varphi'(t)\,\mathrm{d}t\right) \\
&= (f, \varphi)
\end{aligned}$$

it follows that $y' = f$ and so $y \in \mathcal{D}'$ is a solution to the differential equation. □

**Remark 5.2.8.** *By Corollary 5.2.6 it follows that the solution given by Theorem 5.2.7 is unique up to an additive constant.*

**Theorem 5.2.9.** *Consider a system of differential equations given by*

$$y_j' = \sum_{k=1}^{n} a_{jk}(x) y_k \tag{5.2.5}$$

*for $j = 1, \ldots, n$, where the $a_{jk}$ are infinitely differentiable functions. Then all solutions to (5.2.5) in $\mathcal{D}'$ are regular and coincide with the classical solutions.*

**Theorem 5.2.10.** *Consider a system of differential equations given by*

$$y_j' = \sum_{k=1}^{n} a_{jk}(x) y_k + f_j \tag{5.2.6}$$

*for $j = 1, \ldots, n$, where the $a_{jk}$ are infinitely differentiable functions and $f_j \in \mathcal{D}'$. Then a solution $(y_j)_{j=1}^{n} \subset \mathcal{D}'$ to (5.2.6) exists and is unique up to an arbitrary solution of (5.2.5). Moreover, if $f_j$ for $j = 1, \ldots, n$ are classical ordinary functions then the solution to (5.2.6) is also classical.*

**Remark 5.2.11.** *By a regular distribution, we refer to the distributions that can be identified by a function $f$ through the equation*

$$T(\varphi) = (f, \varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x.$$

*Note that we could have alternatively defined a correspondence between functions $f$ and distributions through the integral*

$$\int_{-\infty}^{\infty} \overline{f(x)}\varphi(x)\,\mathrm{d}x.$$

*Equally, we could have considered*

$$\int_{-\infty}^{\infty} f(x)\overline{\varphi(x)}\,\mathrm{d}x$$

*or*

$$\int_{-\infty}^{\infty} \overline{f(x)\varphi(x)}\,\mathrm{d}x.$$

*Each of which would have provided a different way of embedding ordinary functions into distributions.*

## 5.3 Functions of Several Variables

Let $\varphi(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ have partial derivatives of all orders with respect to each of the $n$ variables, and vanish outside some $[a_1, b_1] \times \cdots \times [a_n, b_n]$. One can introduce a topology on this linear space such that $\varphi_k \to \varphi$ if there exists some $B := [a_1, b_1] \times \cdots \times [a_n, b_n]$ such that $\varphi_k(x_1, \ldots, x_n) = 0$ on $B$ for all $k \in \mathbb{N}$, and

$$\frac{\partial^r \varphi_k}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \to \frac{\partial^r}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

for $r = \sum_{j=1}^n \alpha_j$ uniformly on $B$ for any $(\alpha_1, \ldots, \alpha_n) \subset \mathbb{N}^n$. We denote this space $\mathcal{D}(\mathbb{R}^n) = \mathcal{C}_c^\infty(\mathbb{R}^n)$.

**Definition 5.3.1.** *A linear continuous functional on $\mathcal{D}(\mathbb{R}^n)$ is referred to as a distribution of $n$-variables. The space of distributions is denoted $\mathcal{D}'(\mathbb{R}^n)$.*

Locally integrable functions, $f$, on $\mathbb{R}^n$ correspond to the regular distributions

$$(f, \varphi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

All the results derived for single-variable distributions can be extended to the $n$-variable case. For example, the derivative of an $n$-variable distribution is given by

$$\left(\frac{\partial^r f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \varphi\right) = (-1)^r \left(f, \frac{\partial^r \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}\right).$$

## 5.4 Functions on the Unit Circle

Consider the unit circle in the complex plane, namely

$$\Pi := \left\{x \in \mathbb{C} : x = e^{i\theta},\, 0 \le \theta < 2\pi\right\}.$$

Just as we have consider functions defined on $\mathbb{R}$, we can consider functions defined on $\Pi$. Note how we can view functions defined on $\Pi$ as periodic functions defined on $\mathbb{R}$. For the linear space of infinitely differentiable functions defined on $\Pi$, we can consider a topology where $\varphi_n \to \varphi$ if $\varphi_n^{(k)}(x) \to \varphi^{(k)}(x)$ uniformly on $\Pi$ for every $k = 0, 1, \ldots$. As $\Pi$ is bounded the property that the functions of $\mathcal{D}(\Pi)$ have compact support is implicit. Note how we had to explicitly require this property for $\mathcal{D}$. We denote this space as $\mathcal{D}(\Pi)$.

**Definition 5.4.1.** *An element $f \in \mathcal{D}(\Pi)^*$ is referred to as a distribution on the unit circle.*

**Definition 5.4.2.** *An element $f \in \mathcal{D}'$ is a periodic distribution with period $a$ if*

$$(f, \varphi(x - a)) = (f, \varphi(x))$$

*for every $\varphi \in \mathcal{D}(\mathbb{R})$.*

## 5.5 Tempered Distributions

Let $\mathcal{S} = \mathcal{S}^\infty$, be the space of Schwartz functions on $\mathbb{R}$ as given by Definition 3.3.1.

**Exercise 5.5.1.** *Show that $\mathcal{S}$ is a countably-normed space with norms*

$$\|\varphi\|_n = \sum_{p+q=n} \sup_{x\in\mathbb{R}, 0\le j\le p, 0\le k\le q} \left| \left(1+|x|^j\right) \varphi^{(k)}(x) \right|$$

*for $n = 0, 1, \dots$.*

**Lemma 5.5.2.** *A sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ converges to $\varphi \in \mathcal{S}$ if and only if for any $q = 0, 1, \dots$ the sequence $\left(\varphi_n^{(q)}\right)_{n\in\mathbb{N}}$ converges uniformly on any bounded interval, and*

$$\left| x^p \varphi_n^{(q)}(x) \right| < C_{p,q}$$

*holds for some constant $C_{p,q} > 0$ independent of $n$.*

*Proof.* From Lemma 4.5.4 we have that $\varphi_n \to \varphi$ in $\mathcal{S}$ if and only if $\varphi_n \to \varphi$ with respect to each $\|\cdot\|_n$. Since $\mathcal{S}$ is a linear space, it suffices to consider $\varphi = 0$.
($\Rightarrow$). Let $p, q \in \{0, 1, \dots\}$ and $m = p + q$. Then since $\|\varphi_n\|_m \to 0$ it follows that

$$\sup_{x\in\mathbb{R}} \left| \varphi_n^{(q)}(x) \right| \le \|\varphi_n\|_m \overset{n\to\infty}{\longrightarrow} 0$$

and

$$\sup_{x\in\mathbb{R}} \left| x^p \varphi_n^{(q)}(x) \right| \le \|\varphi_n\|_m \overset{n\to\infty}{\longrightarrow} 0. \tag{5.5.1}$$

In particular, (5.5.1) implies that there exists a $C_{p,q} > 0$ independent on $n$ such that

$$\left| x^p \varphi_n^{(q)}(x) \right| \le C_{p,q}$$

for all $x \in \mathbb{R}$.
($\Leftarrow$). Let $m \in \mathbb{N}$. Let $j, k \in \mathbb{N}$ be such that $i+k \le m$. Let $\epsilon > 0$. Let $\tilde{x} > \max\left(\frac{2(m+1)^2}{\epsilon} C_{1,k}, \frac{2(m+1)^2}{\epsilon} C_{j+1,k}, 1\right)$.
Let $N \in \mathbb{N}$ be such that

$$\left| \varphi_n^{(k)}(x) \right| \le \frac{\epsilon}{2\tilde{x}^j (m+1)^2}.$$

for $|x| \le \tilde{x}$. Then for $n \ge N$ we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \left| \left(1+|x|^j\right) \varphi_n^{(q)}(x) \right| &= \max\left( \sup_{|x|\le\tilde{x}} \left| \left(1+|x|^j\right) \varphi_n^{(q)}(x) \right|, \sup_{|x|>\tilde{x}} \left| \left(1+|x|^j\right) \varphi_n^{(q)}(x) \right| \right) \\
&\le \max\left( \sup_{|x|\le\tilde{x}} \left( 2\tilde{x}^j \frac{\epsilon}{2\tilde{x}^j (m+1)^2} \right), \sup_{|x|>\tilde{x}} \left( \frac{|x| \left| \varphi_n^{(q)}(x) \right|}{|x|} + \frac{|x|^{j+1} \left| \varphi_n^{(q)}(x) \right|}{|x|} \right) \right) \\
&\le \max\left( \frac{\epsilon}{(m+1)^2}, \frac{C_{1,k}}{\tilde{x}} + \frac{C_{j+1,k}}{\tilde{x}} \right) \\
&\le \max\left( \frac{\epsilon}{(m+1)^2}, \frac{\epsilon}{2(m+1)^2}, \frac{\epsilon}{2(m+1)^2} \right) \\
&= \frac{\epsilon}{(m+1)^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\varphi_n\|_m &= \sum_{p+q=m} \sup_{x\in\mathbb{R},0\le j\le p,0\le k\le q} \left|\left(1+|x|^j\right)\varphi_n^{(k)}(x)\right| \\ &\le \sum_{p+q=m} \frac{\epsilon}{(m+1)^2} \\ &= (m+1)^2\frac{\epsilon}{(m+1)^2} \\ &= \epsilon.\end{aligned}$$

Therefore, $\varphi_n \to 0$ with respect to $\|\cdot\|_m$. Since $m \in \mathbb{N}$ was arbitrary it follows by Lemma 4.5.4 that $\varphi_n \to 0$ in $\mathcal{S}$. ☐

**Definition 5.5.3.** *A linear continuous functional on $\mathcal{S}$ is referred to as a tempered distribution. The space of tempered distribution is denoted $\mathcal{S}'$.*

As before regular functionals on $\mathcal{S}$ can be identified with a function $f$ through

$$\int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x = (f,\varphi).$$

**Example 5.5.4.** *As $\mathcal{D} \subset \mathcal{S}$ it follows that $\mathcal{S}' \subset \mathcal{D}'$. In particular, these inclusions are strict. Indeed, $e^{x^2} \in \mathcal{D}'$ since $e^{x^2}$ is locally integrable. However, from Example 3.3.5 we know that $e^{-x^2} \in \mathcal{S}^\infty$. Therefore, since*

$$\int_{\mathbb{R}} e^{x^2}e^{-x^2}\,\mathrm{d}x = \infty,$$

*the regular distribution of $e^{x^2}$ cannot be a tempered distribution.*

**Lemma 5.5.5.** *A linear functional $f$ on $\mathcal{S}$ is continuous if and only if $f(\varphi_n) \to f(\varphi)$ whenever $\varphi_n \to \varphi$ in $\mathcal{S}$.*

**Definition 5.5.6.** *A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}'$ converges to $f \in \mathcal{S}'$ if $f_n(\varphi) \to f(\varphi)$ for all $\varphi \in \mathcal{S}$.*

## 5.6 Fourier Transform

**Definition 5.6.1.** *The Fourier transform of a distribution $f \in \mathcal{S}'$ is the distribution $F[f] = g \in \mathcal{S}'$ given by*

$$(g,\varphi) = (f, F[\varphi])$$

*for all $\varphi \in \mathcal{S}$.*

Note that

$$\begin{aligned}(g,\varphi_1+\lambda\varphi_2) &= (f, F[\varphi_1+\lambda\varphi_2]) \\ &\overset{(1)}{=} (f, F[\varphi_1]+\lambda F[\varphi_2]) \\ &\overset{(2)}{=} (f, F[\varphi_1]) + \lambda(f, F[\varphi_2]) \\ &= (g\varphi_1) + \lambda(g,\varphi_2),\end{aligned}$$

where in $(1)$ the linearity of the Fourier transform on $\mathcal{S}$ is used, and in $(2)$ the linearity of $f$ is used. Thus, we deduce that $g$ is linear. Moreover, suppose that $\varphi_n \to \varphi$ in $\mathcal{S}$. In particular, note that $F[\varphi_n] \in \mathcal{S}$ and

$F[\varphi_n] \to F[\varphi]$ in $\mathcal{S}$. Therefore, as $f$ is a distribution on $\mathcal{S}$ it follows that

$$\begin{aligned}(g, \varphi_n) &= (f, F[\varphi_n]) \\ &\to (f, F[\varphi]) \\ &= (g, \varphi).\end{aligned}$$

Thus, we deduce that $g$ as given by Definition 5.6.1 is indeed a distribution on $\mathcal{S}$.

**Exercise 5.6.2.** *As $L^1(\mathbb{R}) \subset \mathcal{S}'$, as regular distributions, it should be the case that Definition 5.6.1 extends Definition 3.1.3. Verify that this is indeed the case.*

**Example 5.6.3.** *Let $\psi = F[\varphi]$.*

1. *Let $f(x) = c$, where $c \in \mathbb{R}$. Then*

$$\begin{aligned}(F[c], \varphi) &= (f, \psi) \\ &= c \int_{-\infty}^{\infty} \psi(x)\, \mathrm{d}x \\ &= 2\pi c \varphi(0),\end{aligned}$$

   *where for the last equality we have used the inverse Fourier transform of $\varphi$. Hence, $F[c] = 2\pi c\delta(x)$.*

2. *Let $f(x) = e^{iax}$. Then*

$$\begin{aligned}\left(F\left[e^{iax}\right], \varphi\right) &= (f, \psi) \\ &= \int_{-\infty}^{\infty} e^{iax}\psi(x)\, \mathrm{d}x \\ &= 2\pi\varphi(a),\end{aligned}$$

   *where in the last equality we have used the inverse Fourier transform on $\varphi$. Thus $F\left[e^{iax}\right] = 2\pi\delta(x-a)$.*

3. *Let $f(x) = \delta(x-a)$. Then*

$$\begin{aligned}(F\left[\delta(x-a)\right], \varphi) &= (f, \psi) \\ &= \psi(a) \\ &= \int_{-\infty}^{\infty} \varphi(x) e^{-iax}\, \mathrm{d}x.\end{aligned}$$

   *Thus, $F\left[\delta(x-a)\right] = e^{-iax}$.*

4. *Recall, the distribution $f_{\frac{1}{x}}$ given by*

$$\left(f_{\frac{1}{x}}, \varphi\right) = \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\varphi(x)}{x}\, \mathrm{d}x.$$

*Suppose* $F\left[f_{\frac{1}{x}}\right] = g$. *Then*

$$\begin{aligned}(g', \varphi) &= (g, -\varphi') \\ &= (f, -F[\varphi']) \\ &= (f, -i\lambda\psi(\lambda)) \\ &= -i \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{1}{\lambda}\lambda\psi(\lambda)\,\mathrm{d}\lambda \\ &= -i\int_{-\infty}^{\infty} \psi(\lambda)\,\mathrm{d}\lambda \\ &= -2\pi i\varphi(0).\end{aligned}$$

*As* $(\mathrm{sgn})' = 2\delta$, *it follows by Corollary 5.2.6 that*

$$g = -\pi i\mathrm{sgn}(x) + c,$$

*for some* $c \in \mathbb{R}$. *By considering even test functions it follows that* $c = 0$ *and so*

$$F\left[f_{\frac{1}{x}}\right] = -i\pi\mathrm{sgn}(x).$$

**Definition 5.6.4.** *Let* $\mathcal{Z}$ *be the linear space consisting of entire functions* $\psi$ *such that for all* $q = 0, 1, \ldots$ *there exists* $C_q(\psi), a(\psi) > 0$ *such that*

$$|\lambda|^q|\psi(\lambda)| \le C_q(\psi)\exp\left(a(\psi)|\mathrm{Im}(\lambda)|\right). \tag{5.6.1}$$

**Lemma 5.6.5.** *The Fourier transform is a bijection between* $\mathcal{D}$ *and* $\mathcal{Z}$, *which preserves linear operations.*

*Proof.* Let $\varphi \in \mathcal{D}$, then

$$\begin{aligned}\psi(\lambda) &:= F[\varphi](\lambda) \\ &= \int_{-\infty}^{\infty} e^{-i\lambda x}\varphi(x)\,\mathrm{d}x \\ &\overset{(1)}{=} \int_{-a}^{a} e^{-i\lambda x}\varphi(x)\,\mathrm{d}x,\end{aligned}$$

where $(1)$ follows as $\varphi$ has compact support. As $e^{-i\lambda x}\varphi(x)$ is analytic in $\lambda$ and continuous in $x$, it follows that $\psi(\lambda)$ extends to an entire function. Moreover, by integration by parts we obtain

$$|\lambda|^q|\psi(\lambda)| = \left|\int_{-a}^{a} \varphi^{(q)}(x)e^{-ix\lambda}\,\mathrm{d}x\right| \le C_q\exp(a|\mathrm{Im}(\lambda)|),$$

which means that $\psi \in \mathcal{Z}$. Conversely, let $\psi \in \mathcal{Z}$ and consider

$$\varphi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\lambda)e^{i\lambda x}\,\mathrm{d}\lambda,$$

which converges absolutely and uniformly for $x \in \mathbb{R}$ by taking $\mathrm{Im}(\lambda) = 0$ in (5.6.1). Similarly,

$$\varphi^{(q)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\lambda)^q\psi(\lambda)e^{i\lambda x}\,\mathrm{d}\lambda$$

for $q = 1, 2, \ldots$ is absolutely and uniformly convergent and so $\varphi$ is infinitely differentiable. Now for $x > a(\psi) =: a$, where $a(\psi)$ comes from (5.6.1), consider the integral of $\psi(\lambda)e^{i\lambda x}$ over the contour $\gamma^{A,\tau} = \gamma_1^{A,\tau} \cup \gamma_2^{A,\tau} \cup \gamma_3^{A,\tau} \cup \gamma_4^{A,\tau}$

where

$$\begin{cases} \gamma_1^{A,\tau} := \{\lambda = \sigma : \sigma \in [-A, A]\} \\ \gamma_2^{A,\tau} := \{\lambda = A + i\eta : \eta \in [0, \tau]\} \\ \gamma_3^{A,\tau} := \{\lambda = \sigma + i\tau : x \in [A, -A]\} \\ \gamma_4^{A,\tau} := \{\lambda = -A + i\eta : \eta \in [\tau, 0]\}. \end{cases}$$

Observe that

$$\begin{aligned} \left| \int_{\gamma_2^{A,\tau}} \psi(\lambda) e^{i\lambda x} \, \mathrm{d}\lambda \right| &= \left| \int_0^\tau \psi(A + i\eta) e^{i(Ai + i\eta)x} \, \mathrm{d}\eta \right| \\ &\overset{(5.6.1)}{\leq} \int_0^\tau \frac{C_1(\psi) \exp(a\eta)}{\sqrt{A^2 + \eta^2}} \, \mathrm{d}\eta \\ &\overset{A\to\infty}{\longrightarrow} 0. \end{aligned}$$

Similarly,

$$\left| \int_{\gamma_4^{A,\tau}} \psi(\lambda) e^{i\lambda x} \, \mathrm{d}\lambda \right| \overset{A\to\infty}{\longrightarrow} 0.$$

Therefore, as

$$\oint_\gamma \psi(\lambda) e^{i\lambda x} \, \mathrm{d}\lambda = 0$$

it follows that,

$$\begin{aligned} \varphi(x) &= -\int_{\gamma_3^{A,\tau}} \psi(\lambda) e^{i\lambda x} \, \mathrm{d}\sigma \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\sigma + i\tau) e^{i\sigma x - \tau x} \, \mathrm{d}\sigma \end{aligned}$$

for $\tau > 0$. With $s = \sigma + i\tau$, using (5.6.1) for $q = 0$ and $q = 2$, we obtain

$$\begin{aligned} |\psi(\lambda)| &\leq e^{a|\tau|} \min\left( C_0, \frac{C_2}{|s|^2} \right) \\ &\leq C \frac{e^{a|\tau|}}{1 + |s|^2} \\ &\leq C \frac{e^{a|\tau|}}{1 + \sigma^2}, \end{aligned}$$

where $C$ is just some constant. Hence,

$$\begin{aligned} |\varphi(x)| &\leq \frac{C}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(a-x)\tau}}{1 + \sigma^2} \, \mathrm{d}\sigma \\ &\leq C' e^{-(x-a)\tau}, \end{aligned}$$

where $C'$ is a constant independent of $\tau > 0$. Since, $\tau > 0$ and $x > a$, by taking $\tau \to \infty$ we deduce that $|\varphi(x)| = 0$. A similar argument shows that $|\varphi(x)| = 0$ for $x < -a$. Therefore, $\varphi$ has a compact support and because it is infinitely differentiable we have $\varphi \in \mathcal{D}$. Moreover, $\varphi \in \mathcal{D}$ is the unique test function such that $F[\varphi] = \psi$. In conclusion $F : \mathcal{D} \to \mathcal{Z}$ is a bijection. □

**Definition 5.6.6.** *The Fourier transform of a distribution $f \in \mathcal{D}'$ is the distribution $g = F[f] \in \mathcal{Z}^* = \mathcal{Z}'$ given by*

$$(g, \varphi) = (f, F[\varphi])$$

*for all $\varphi \in \mathcal{Z}$.*

## 5.7 Solution to Exercises

### Exercise 5.5.1

*Solution.* On the one hand,

$$\begin{aligned}\sum_{p+q=n} \sup_{x\in\mathbb{R},0\le j\le p,0\le k\le q} \left|\left(1+|x|^j\right)\varphi^{(k)}(x)\right| &\ge \sum_{p+q=n} \sup_{x\in\mathbb{R},0\le j\le p,0\le k\le q} |x|^j \left|\varphi^{(k)}(x)\right| \\ &\ge \sup_{0\le j,k\le n} \left|x^j\varphi^{(k)}(x)\right|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{p+q=n} \sup_{x\in\mathbb{R},0\le j\le p,0\le k\le q} \left|\left(1+|x|^j\right)\varphi^{(k)}(x)\right| &\le \sum_{p+q=n} \sup_{x\in\mathbb{R},0\le j\le p,0\le k\le q} \left|\left(1+|x|+\cdots+|x|^j\right)\varphi^{(k)}(x)\right| \\ &\le \sum_{p+q=n} \sup_{x\in\mathbb{R},0\le j\le p,0\le k\le q} \left|(1+|x|)^j\varphi^{(k)}(x)\right| \\ &\le (n+1)^2 \sup_{0\le j,k\le n} \left|x^j\varphi^{(k)}(x)\right|.\end{aligned}$$

Therefore, $\|\cdot\|_n$ is equivalent to the norm from statement 2 of Example 4.5.8 and thus $\mathcal{S}$ with the norms $\|\cdot\|_n$ is a countably normed space. □

### Exercise 5.6.2

*Solution.* Indeed for $f\in\mathcal{S}$ and $\varphi\in\mathcal{S}$ then

$$\begin{aligned}(F[f],\varphi) &= \int_{-\infty}^{\infty} F[f](z)\varphi(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-izx}f(x)\,\mathrm{d}x\right)\varphi(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty}\varphi(z)e^{ixz}\,\mathrm{d}z\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x)F[\varphi](x)\,\mathrm{d}x \\ &= (f,F[\varphi]). \qquad (5.7.1)\end{aligned}$$

Using the density of $\mathcal{S}$ in $L^1(\mathbb{R})$, under an appropriate limit it follows that (5.7.1) holds for all $f\in L^1(\mathbb{R})$. □

# 6 Appendix

## 6.1 Constructing Topologies from Neighbourhoods

**Definition 6.1.1.** *Let $T$ be a non-empty set. For $x \in T$, a system of neighbourhoods $N(x)$ is a collection of subsets of $T$ such that the following hold.*

1. *$N(x)$ is not empty.*
2. *If $U \in N(x)$ then $x \in U$.*
3. *If $U, V \in N(x)$ then $U \cap V \in N(x)$.*
4. *If $U \in N(x)$ then there exists a $V \in N(x)$ such that $V \subset U$ and $V = \bigcup_{y \in V} V_y$ where $V_y \in N(y)$.*

*An element $U \in N(x)$ is referred to as a neighbourhood of $x$.*

**Definition 6.1.2.** *When we have a system of neighbourhoods for the elements of set $T$, we say a subset $S \subseteq T$ is open if either $S = \emptyset$ or for every $s \in S$ there exists a $U \in N(s)$ such that $U \subseteq S$.*

**Remark 6.1.3.** *Note that a neighbourhood itself may not be an open set.*

**Lemma 6.1.4.** *The collection of open sets given by Definition 6.1.2 defines a topology on $T$.*

*Proof.*

- The empty set is open.
- Let $(S_k)_{k \in \mathbb{N}}$ be a collection of open sets. Then for each $s \in \bigcup_{k \in \mathbb{N}} S_k$, there exists a $k' \in \mathbb{N}$ such that $s \in S_{k'}$. Hence, as $S_{k'}$ is open there exists a $U \in N(s)$ such that $U \subseteq S_{k'} \subseteq \bigcup_{k \in \mathbb{N}} S_k$. Therefore, $\bigcup_{k \in \mathbb{N}} S_k$ is open.
- Let $(S_k)_{k=1}^n$ be open sets. Then for $s \in \bigcap_{k=1}^n S_k$, there exists a $U_k \in N(s)$ such that $U_k \subseteq S_k$ for each $k = 1, \dots, n$. From statement 3 Definition 6.1.1 the set $U := \bigcap_{k=1}^n U_k$ is open, and in particular is such that $s \in U \subseteq \bigcap_{k=1}^n S_k$. Therefore, $\bigcap_{k=1}^n S_k$ is open.

$\square$

**Remark 6.1.5.**

1. *The topology of Lemma 6.1.4 is denoted $\tau$, and is referred to as the topology induced by the defining system of neighbourhoods $\{N(x) : x \in T\}$.*
2. *Note that the set $V$ of statement 4 of Definition 6.1.1 is an open set in the sense of Definition 6.1.2. The collection of such open sets $\mathcal{B}$ with $\emptyset$ is an open base of $\tau$. Namely, $\mathcal{B} \subseteq \tau$ with each $A \in \tau$ a union of sets from $\mathcal{B}$.*
3. *With this construction of $\tau$, we can extend the notion of a neighbourhood of $x$ to mean any set $U$ such that $x \in U$ and $U$ contains an open set containing $x$. Note that if $U$ and $V$ are neighbourhoods of $x$ then so is $U \cap V$.*

## References

[1] I. M. Gelfand and G. E. Shilov. *Generalized Functions Vol 2 Spaces Of Fundamental And Generalized Functions*. 1968. URL: http://archive.org/details/gelfand-shilov-generalized-functions-vol-2-spaces-of-fundamental-and-generalized-functions (visited on 02/22/2024).