Probability Theory

An update made by Thomas Walker on notes scribed by Ivan Kirev and Samuel Lam (1)
from a lecture series on Probability Theory given by Professor Igor Krasovsky.

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Part I. Measure Theory and Random Variables

1 Events, Probability and Random Variables

In developing an abstract mathematical framework to describe the likelihood of events happening in a random experiment we can formalise large-sample results, including the Law of Large Numbers and Central Limit Theorem.
Let us understand how to describe an experiment.

1. First, let the possible outcomes ω of the experiment be the sample space, Ω.
2. Let events A be subsets of the sample space Ω that we may observe. The collection of such subsets is denoted as $\mathscr{F}$.
3. Finally, assign a value $\mathbb{P}(A) \in [0,1]$ to each of the subsets $A \in \mathscr{F}$ to quantify how likely the event is to occur.

With this construction, we have a few problems to answer.

- What should the collection of events $\mathscr{F}$ include?
- How should we assign values to the events in $\mathscr{F}$?

It should be the case that we can either observe noting or something, that is, we should let $\mathscr{F}_* := \{\varnothing, \Omega\} \subseteq \mathscr{F}$. However, any useful experiment should be able to differentiate different observed outcomes, and so $\mathscr{F}$ should include more than just the whole set Ω as a potential result.
One could suggest letting $\mathscr{F}$ contain all possible subsets of Ω. We denote this $\mathscr{F} = \mathscr{F}^* := 2^\Omega$, and refer to it as the power set of Ω. For Ω a countable set, this is fine, and it will often be the case that $\mathscr{F} = 2^\Omega$. However, issues arise if Ω is uncountable, for example, if $\Omega = \mathbb{R}$.
Next, there are certain properties that $\mathbb{P}$ ought to satisfy to make sense. For example, it should be finitely additive. That is, if $A, B \in \mathscr{F}$ are disjoint outcomes, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

When one extends this property to countably many disjoint events, potential contradictions may arise if $\mathscr{F}$ doesn't have a certain structure. In the uncountable case, 2^Ω will often not possess such a structure. Therefore, we want to choose $\mathscr{F}$ such that it contains more information than $\mathscr{F}_*$ but is strictly smaller than 2^Ω.
At this point, one may be wondering why we need to extend the finite additivity property to the countable additivity property. Well, often one is interested in the long-term behaviour of a random experiment, such as the expected value of a dynamical system as it continues to run forward in time. Hence, questions regarding limits arise naturally, for which one needs to reason about countably many events rather than just a finite number.
Mathematicians, therefore, attempted to find a suitable criterion for $\mathscr{F}$ and $\mathbb{P}$ so that they would not give rise to contradictions, but still allow $\mathscr{F}$ to be large enough to be useful. The most successful attempt was, perhaps, made by Andrey Kolmogorov in 1933 when he devised the axioms of probability in his Foundations of the Theory of Probability. His work led to the development of the measure theory, which forms the foundations of our probability theory.

1.1 Algebras and σ-algebras

Let Ω be a set of points ω.

Definition 1.1.1 — Algebra and σ-algebra. A nonempty system of subsets of Ω is called an **algebra** $\mathscr{A}$ if

- $\Omega \in \mathscr{A}$,
- $A, B \in \mathscr{A}$ implies that $A \cup B \in A$, and

- $A \in \mathscr{A}$ implies that $A^c \in \mathscr{A}$.

In addition, if all countable unions $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$ whenever $A_1, A_2, \cdots \in \mathscr{A}$, then $\mathscr{A}$ is a σ-algebra.

Remark 1.1.2 Note that we can consider the complements of events to show that a σ-algebra (algebra) is also closed under countable (finite) intersections.

Definition 1.1.3 — Set function and measures.

- A set function $\mu : \mathscr{A} \longrightarrow [0, \infty]$ is finitely additive if for any disjoint $A, B \in \mathscr{A}$ we have
$$\mu(A \cup B) = \mu(A) + \mu(B).$$
Note that then for all $A, B \in \mathscr{A}$ we have
$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$
- Let $\mathscr{F}$ be a σ-algebra. A set function $\mu : \mathscr{F} \to [0, \infty]$ is called σ-additive if for any disjoint $A_1, A_2, \cdots \in \mathscr{F}$, it follows that
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$
Such a μ is called a measure on $\mathscr{F}$. A measure μ is a probability measure if $\mu(\Omega) = 1$. It can deduced be for any measure that $\mu(\varnothing) = 0$.
- A measure is called σ-finite if there exists a partition $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, where the Ω_k are pairwise disjoint, with $\mu(\Omega_k) < \infty$ for $k = 1, 2, \ldots$.

Definition 1.1.4 A probability space is a triple $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is a set called the sample space, $\mathscr{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P}$ is a probability measure on $\mathscr{F}$. An element of $\mathscr{F}$ is called an event.

Proposition 1.1.5 Probability measures have the following properties.

- $\mathbb{P}(\varnothing) = 0$.
- If $A, B \in \mathscr{F}$ then
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$
- If $A, B \in \mathscr{F}$ and $B \subseteq A$ then
$$\mathbb{P}(B) \leq \mathbb{P}(A).$$
- If $A_1, A_2, \cdots \in \mathscr{F}$, then
$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Proposition 1.1.6 Let $\mathbb{P}$ be a finitely additive set function defined over an algebra $\mathscr{A}$, with $\mathbb{P}(\Omega) = 1$. Then the following four conditions are equivalent.

1. $\mathbb{P}$ is σ-additive, and hence a probability measure.
2. $\mathbb{P}$ is continuous from below. That is, for any sets $A_1, A_2, \cdots \in \mathscr{A}$ such that $A_1 \subset A_2 \subset \ldots$ and

$\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

3. $\mathbb{P}$ is continuous from above. That is, for any sets $B_1, B_2, \ldots$ such that $B_1 \supset B_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} B_n \in \mathscr{A}$, we have

$$\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right).$$

4. $\mathbb{P}$ is continuous at $\varnothing$. That is, for any sets $B_1, B_2, \cdots \in \mathscr{A}$ such that $B_1 \supset B_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} B_n = \varnothing$, we have

$$\lim_{n\to\infty} \mathbb{P}(B_n) = 0.$$

Proof. $(1) \Rightarrow (2)$. Consider the sets $\tilde{A}_1 = A_1$, and $\tilde{A}_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then the sets $\tilde{A}_k$ are a collection of disjoint sets. Moreover,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \tilde{A}_n.$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right) \\
&\overset{(1)}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}\left(\tilde{A}_k\right) \\
&= \lim_{n\to\infty} \left(\mathbb{P}(A_1) + (\mathbb{P}(A_2) - \mathbb{P}(A_1)) + \cdots + (\mathbb{P}(A_n) - \mathbb{P}(A_{n-1}))\right) \\
&= \lim_{n\to\infty} \mathbb{P}(A_n).
\end{aligned}$$

$(2) \Rightarrow (3)$. Let $n \geq 1$ and consider the events $\tilde{B}_k = B_1 \setminus B_k$. Then

$$\mathbb{P}\left(\tilde{B}_n\right) = \mathbb{P}(B_1) - \mathbb{P}(B_n).$$

The sequence $\left(\tilde{B}_k\right)$ is an increasing sequence of events with

$$\bigcup_{n=1}^{\infty} \tilde{B}_n = B_1 \setminus \bigcap_{n=1}^{\infty} B_n.$$

By (2) it follows that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{B}_n\right) = \lim_{n\to\infty} \mathbb{P}\left(\tilde{B}_n\right).$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(B_n) &= \mathbb{P}(B_1) - \lim_{n\to\infty} \mathbb{P}\left(\tilde{B}_n\right) \\
&= \mathbb{P}(B_n) - \mathbb{P}\left(\bigcup_{n=1}^{\infty} \tilde{B}_n\right) \\
&= \mathbb{P}(B_1) - \mathbb{P}\left(B_1 \setminus \bigcap_{n=1}^{\infty} B_n\right) \\
&= \mathbb{P}(B_1) - \mathbb{P}(B_1) + \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) \\
&= \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right).
\end{aligned}$$

$(3) \Rightarrow (4)$. This is clear as $\mathbb{P}(\varnothing) = 0$.
$(4) \Rightarrow (1)$. Let $A_1, A_2, \cdots \in \mathscr{A}$ be pairwise disjoint with

$$A := \bigcup_{n=1}^{\infty} A_n \in \mathscr{A},$$

and

$$B_n := \bigcup_{i=n+1}^{\infty} A_i.$$

Note that $\bigcup_{i=1}^{n} A_i$ and B_n are disjoint and such that

$$A = \bigcup_{i=1}^{n} A_i \cup B_n.$$

Therefore, by finite additivity we have that

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) + \mathbb{P}(B_n).$$

The sequence of sets B_n is decreasing and such that $\bigcap_{n=1}^{\infty} B_n = \varnothing$. Therefore, by finite additivity and (4) it follows that

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{P}(A_i) &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \\
&= \lim_{n\to\infty} \left(\mathbb{P}\left(A\right) - \mathbb{P}(B_n)\right) \\
&= \mathbb{P}\left(A\right) - \lim_{n\to\infty} \mathbb{P}(B_n) \\
&= \mathbb{P}(A).
\end{aligned}$$

■

Proposition 1.1.7 Let μ be a finitely additive measure on an algebra $\mathscr{A}$ and let the sets $A_1, A_2, \cdots \in \mathscr{A}$ be pairwise disjoint with $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$. Then

$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A).$$

Example 1.1.8 — σ-algebras on Ω. Let Ω be a sample space. Then,

1. $\mathscr{F}_* = \{\varnothing, \Omega\}$, and
2. $\mathscr{F}^* = \{A : A \in \Omega\} = 2^{\Omega}$

are σ-algebras.

Lemma 1.1.9 For any collection $\mathscr{E}$ of subsets of Ω there exists a minimal algebra $a(\mathscr{E})$ and a minimal σ-algebra $\sigma(\mathscr{E})$ that contains all elements of $\mathscr{E}$. Equivalently, $a(\mathscr{E})$ (resp. $\sigma(\mathscr{E})$) is the intersection of all algebras (resp. σ-algebras) that contain $\mathscr{E}$.

Proof. Intersection, countable or uncountable, of algebras (resp. σ-algebras) containing $\mathscr{E}$ is an algebra (resp. σ-algebra) containing $\mathscr{E}$. ■

Remark 1.1.10 In the context of Lemma 1.1.9 we say that $\sigma(\mathscr{E})$ is generated by $\mathscr{E}$.

Exercise 1.1.11 Let $D = \{D_1, D_2, \dots\}$ be a countable partition of Ω such that $\Omega = \bigsqcup_{j=1}^{\infty} D_j$. Show that

$$\sigma(D) = \left\{ \bigcup_{j \in I} D_j : I \subset \mathbb{N} \right\}.$$

1.2 Measurable Spaces

Definition 1.2.1 A measurable space is a pair $(E, \mathscr{E})$, where E is a set and $\mathscr{E}$ is a σ-algebra on E.

Let $\mathbb{R} = (-\infty, \infty)$ be the real line and

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\},$$

for all a, b with $-\infty \leq a < b < \infty$. Let $\mathscr{A}$ be the algebra of subsets of $\mathbb{R}$ such that $A \in \mathscr{A}$ if for some $n < \infty$ we have

$$A = \bigcup_{i=1}^{n} (a_i, b_i].$$

Let $\mathscr{B}(\mathbb{R})$ be the smallest σ-algebra containing $\mathscr{A}$. Then for $a < b$ we observe that

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right],$$
$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right],$$
$$\{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a\right].$$

Thus in addition to containing intervals of the form $(a, b]$, the Borel σ-algebra also contains singleton set $\{a\}$ and intervals of the form

- (a, b),
- $[a, b]$,
- $[a, b)$,
- $(-\infty, b)$,
- $(-\infty, b]$, and
- (a, ∞).

Exercise 1.2.2 Show that $\mathscr{B}(\mathbb{R})$ is generated by the collection of (1) open intervals of the form (a, b), (2) closed intervals $[a, b]$, (3) half intervals, (4) intervals of the form $(-\infty, a]$ or $[a, \infty)$, (5) open sets and (6) closed sets with respect to the Euclidean metric.

Let $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$. That is, the set of ordered n-tuples $x = (x_1, \dots, x_n)$, where $x_k \in \mathbb{R}$ for $k = 1, \dots, n$. A rectangle then refers to a set of the form

$$I = I_1 \times \cdots I_n = \{x \in \mathbb{R}^n : x_k \in I_k, k = 1, \dots, n\}$$

where $I_k = (a_k, b_k]$ is known as a side of the rectangle. Let $\mathscr{I}$ be the collection of all rectangles I. The smallest σ-algebra $\sigma(\mathscr{I})$ generated by the system $\mathscr{I}$ is the Borel σ-algebra of subsets of $\mathbb{R}^n$.

Instead of the rectangles $I = I_1 \times \cdots I_n$ let us consider the rectangles $B = B_1 \times B_2 \times \cdots \times B_n$ with Borel sides. That is B_k is a Borel subset of the real line that appears in the k^{th} place in the direct product $\mathbb{R} \times \cdots \times \mathbb{R}$. The smallest σ-algebra containing all rectangles with Borel sides is denoted by

$$\mathscr{B}(\mathbb{R}) \otimes \cdots \otimes \mathscr{B}(\mathbb{R})$$

and called the direct product of the σ-algebras $\mathscr{B}(\mathbb{R})$. In fact,

$$\mathscr{B}(\mathbb{R}^n) = \mathscr{B}(\mathbb{R}) \otimes \cdots \otimes \mathscr{B}(\mathbb{R}).$$

In other words, σ-algebra generated by the rectangles $I = I_1 \times \cdots \times I_n$ coincides with the $sigma$-algebra generated by rectangles $B = B_1 \times \cdots \times B_n$ with Borel sides. We will now justify this.

Lemma 1.2.3 Let $\mathscr{E}$ be a class of subsets of Ω and let $\mathscr{B} \subseteq \Omega$, and define

$$\mathscr{E} \cap B = \{A \cap B : A \in \mathscr{E}\}.$$

Then

$$\sigma(\mathscr{E} \cap B) = \sigma(\mathscr{E}) \cap B.$$

It is clear that for $n = 1$, the σ-algebras $\mathscr{B}(\mathbb{R}^n)$ and $\mathscr{B}(\mathbb{R}) \otimes \cdots \otimes \mathscr{B}(\mathbb{R})$ are the same.

Lemma 1.2.4

$$\mathscr{B}\left(\mathbb{R}^2\right) = \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}).$$

Proof. $(\subset)$. For any open set, A we can write

$$A \subset \bigcup_{x \in A \cap \mathbb{Q}^2} R(x, \tau(x))$$

where $R(x, \tau)$ is the open square centered at x and of side length $\tau(x)$. As $A \cap \mathbb{Q}^2$ is countable and $R(x, \tau(x)) \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$, it follows that $A \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$.
$(\supset)$. For this inclusion, it is sufficient to check that $B_1 \times B_2 \in \mathscr{B}\left(\mathbb{R}^2\right)$ for any Borel sets B_1, B_2. Note that $B_1 \times \mathbb{R} \in \mathscr{B}(\mathbb{R}^2)$ since

$$B_1 \times \mathbb{R} \in \sigma(\{\text{open subsets of } \mathbb{R}\}) \times \mathbb{R} = \sigma(\{\text{open subsets of } \mathbb{R} \times \mathbb{R}\}).$$

Similarly, $\mathbb{R} \times B_2 \in \mathscr{B}(\mathbb{R}^2)$, and so $B_1 \times B_2 = (B_1 \times \mathbb{R}) \cap (\mathbb{R} \times B_2) \in \mathscr{B}(\mathbb{R}^2)$. ■

The case for any $n > 2$ can be discussed similarly to Lemma 1.2.4. The space $((\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$, on the other hand, requires a different approach. However, it is useful to outline this as $((\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ is consistently utilised for constructing probabilistic models of experiments with infinitely many steps. Let $\mathbb{R}^\infty = \{x = (x_1, x_2, \dots), x_k \in \mathbb{R}\}$.

Definition 1.2.5 A set $C \subset \mathbb{R}^\infty$ is called cylindrical if it is of the form

$$C = \left\{x \in \mathbb{R}^\infty : (x_1, x_2, ..., x_n) \in \tilde{C}_n\right\}$$

for some $n \geq 1$ and $\tilde{C}_n \in \mathscr{B}(\mathbb{R}^n)$.

Exercise 1.2.6 Show that the cylindrical sets form an algebra.

The σ-algebra generated by cylindrical sets is called the cylindrical σ-algebra and is denoted $\mathscr{B}(\mathbb{R}^\infty)$. One can verify that

$$\mathscr{B}(\mathbb{R}^\infty) = \sigma\left(\{A_1 \times A_2 \times \cdots \subset \mathbb{R}^\infty, A_k \in \mathscr{B}(\mathbb{R})\}\right).$$

Example 1.2.7 For all $c \in \mathbb{R}$, let

$$A = \left\{x \in \mathbb{R}^\infty : \limsup_n x_n = \inf_n \sup_{k>n} x_k > c\right\}.$$

It follows that $A \in \mathscr{B}(\mathbb{R}^\infty)$ because

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} \{x \in \mathbb{R}^\infty : x_k > c\}.$$

Similarly, letting

$$B = \left\{ x \in \mathbb{R}^\infty : \liminf_n x_n = \sup_n \inf_{k>n} x_k > c \right\},$$

we have that $B \in \mathscr{B}(\mathbb{R}^\infty)$ because

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} \{x \in \mathbb{R}^\infty : x_k > c\}.$$

Exercise 1.2.8 For $c \in \mathbb{R}$, show that $D = \{x \in \mathbb{R}^\infty : \lim_{n\to\infty} x_n = c\} \in \mathscr{B}(\mathbb{R}^\infty)$.

1.3 Probability Distributions

Throughout this section, we will consider the importance of non-decreasing functions for describing probability measures on measurable spaces.

Lemma 1.3.1 A non-decreasing function $g(x)$ on $\mathbb{R}$ is continuous up to possibly countably many discontinuities of the first kind. That is, for $\epsilon \searrow 0$ the limits $g(x+\epsilon)$ and $g(x-\epsilon)$ both exists but are distinct.

Note that it must be the case that $\lim_{\epsilon\searrow 0} (g(x+\epsilon) - g(x-\epsilon)) > 0$ due to the non-decreasing property of g. Lemma 1.3.1 is a positive result, as by construction our probability measures behave well in the countable domain.

Moreover, by Lemma 1.3.1 we deduce that the derivative of a non-decreasing function, $g(x)$, denoted $g'(x)$ exists Lebesgue a.e.

Exercise 1.3.2 Let $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P})$ be a probability space and denote $F(x) := \mathbb{P}((-\infty, x])$ for $x \in \mathbb{R}$. Show that,

- $F(x)$ is non-decreasing,
- $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$, and
- $F(x)$ is right-continuous for all $x \in \mathbb{R}$.

Definition 1.3.3 Every function $F : \mathbb{R} \longrightarrow [0,1]$ satisfying the above three conditions is called a **distribution function** on $\mathbb{R}$.

Thus to every probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, there corresponds a distribution function. In fact, the opposite is also true and there exists a one-to-one correspondence between distribution functions and probability measures.

Theorem 1.3.4 Let $F = F(x)$ be a distribution function on $\mathbb{R}$. Then there exists a unique probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that

$$\mathbb{P}(a, b] = F(b) - F(a),$$

for all a, b with $-\infty \le a < b < \infty$.

This relies on the following fundamental result in measure theory.

Theorem 1.3.5 — Caratheodory Theorem. Let μ_0 be a σ-additive (pre-)measure on $(\Omega, \mathscr{A})$, where $\mathscr{A}$ is an algebra of subsets of Ω. Then there exists a measure μ on $(\Omega, \sigma(\mathscr{A}))$, such that

$$\mu(A) = \mu_0(A)$$

for all $A \in \mathscr{A}$. If μ_0 is also σ-finite, then the measure μ is unique.

Our first remark Theorem 1.3.5 related to the completeness of the measure.

Definition 1.3.6 — Complete measure. A measure μ on a σ-algebra Σ on Ω is called complete if any subset of a set of measure zero (null sets) is measurable. That is if $A \in \Sigma$ is such that $\mu(A) = 0$. Then for any $B \subset A$ we have that $B \in \Sigma$ and $\mu(B) = 0$.

Requiring completeness helps to avoid any caveats in proving results relating to measures. The space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P})$ with $\mathbb{P}$ constructed from Theorem 1.3.4 is not complete as there are subsets of Borel sets that are not themselves Borel sets. Fortunately, one can enlarge the σ-algebra so that it includes all null sets. For a measure μ on Σ can be completed by extending Σ to

$$\overline{\Sigma} = \sigma(\Sigma \cup \{B \in \Omega : B \subset A \in \Sigma, \mu(A) = 0\}),$$

with $\mu(B) = 0$ for any B a subset of a null set. The completion of the measure obtained in Thereom 1.3.4 is called the *Lebesgue-Stiltjes measure*. In particular, the distribution function $F(x) = x$ corresponds to the Lebesgue measure on $\mathbb{R}$.

1.4 Measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$

Discrete/Atomic measures are measures $\mathbb{P}$ that have piecewise constant distributions $F = F(x)$. They change their values at the points $x_1, x_2, \dots$ such that $\Delta F(x_i) > 0$, where $\Delta F(x) = F(x) - F(x^-)$.

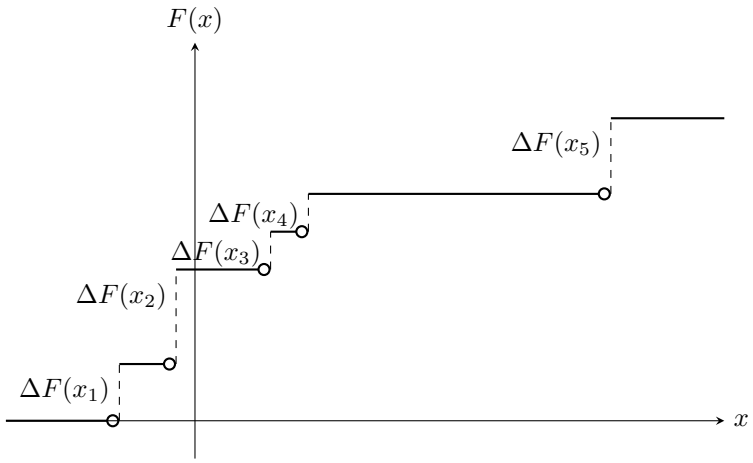


Figure 1: CDF of a (purely) atomic measure.

The measure is concentrated at the points $x_1, x_2, \dots$, known as **atoms**. We let $p_k := \mathbb{P}(\{x_k\}) = \Delta F(x_k) > 0$ and require that

$$\sum_k p_k = 1.$$

The set of numbers $(p_1, p_2, \dots)$ is called the **discrete probability distribution** and the corresponding distribution function

$$F(x) = F_{\text{disc}}(x) = \sum_{x_k \le x} p_k$$

is called **discrete**. Note that for $A \subseteq \mathbb{N}$ we have that

$$\mathbb{P}(A) = \sum_{k \in A} p_k.$$

Example 1.4.1

- The Discrete Uniform distribution has $p_k = \frac{1}{N}$ for $k = 1, \ldots, N$ for N fixed.
- The Bernoulli $\mathsf{B}(1, p)$ has $p_1 = p$ and $p_2 = 1 - p$ where $0 \leq p \leq 1$.
- The Binomial $\mathsf{B}(n, p)$ has $p_k = \binom{n}{k} p^k (1-p)^k$ for $k = 0, 1, \ldots, n$ and $0 \leq p \leq 1$.
- The Poisson $\mathsf{Po}(\lambda)$ has $p_k = e^{-\lambda} \frac{\lambda^k}{k!}$ for $\lambda > 0$ and $k = 0, 1, \ldots$.

Absolutely continuous measures.

Proposition 1.4.2 Let f be an integrable function [a] $f(x) \geq 0$ such that

$$F(x) = F_{\mathsf{ac}}(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

with respect to the Lebesgue measure. Then the set function $\mathbb{P}_{ac}(A) = \int_A f(t)\mathrm{d}t$ for $A \in \mathscr{F}$ is a measure. In particular, we say that f is a density of $\mathbb{P}_{\mathsf{ac}}$.

[a] see Chapter 2

Proof. First define the measure on half-open intervals as $\mathbb{P}_{ac}((a, b]) = \int_a^b f(t)\mathrm{d}t$, then use Theorem 1.3.5 to extend the measure to the σ-algebra. ■

Such measures are **absolutely continuous** with respect to the Lebesgue measure μ. In the sense that if $\mu(A) = 0$ then $\mathbb{P}_{\mathsf{ac}}(A) = 0$.

Theorem 1.4.3 — Radon-Nikodym. If $\mathbb{P}$ is a measure such that $\mu(A) = 0$ implies $\mathbb{P}(A) = 0$, then $\mathbb{P}$ has a density.

Note that there is a connection between the absolute continuity of measures and the absolute continuity of functions. If $\mathbb{P}$ is an absolutely continuous measure then $F_{ac}(x)$ is an absolutely continuous function, and $F'_{ac}(x) = f(x)$ almost everywhere.

Example 1.4.4

- The Uniform distribution on $[a, b]$ has density

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

- The Normal or Gaussian distribution has density

$$f(x) = (2\pi\sigma^2)^{-1/2} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)$$

for $x, m \in \mathbb{R}$ and $\sigma > 0$.

- The Gamma distribution has

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

for $x \geq 0$ and $a, \beta > 0$.

Singular continuous.

Definition 1.4.5 A measure ν is said to be concentrated on a measurable set A if $\nu(E) = 0$ for any $E \subset \mathbb{R} \setminus A$.

Singular continuous measures are those whose distribution functions are continuous but have all their points of increase on sets of zero Lebesgue measure. We have that $F(x) = F_{\mathsf{sc}}(x)$ is continuous at any x and $\mathbb{P}_{\mathsf{sc}}$ is concentrated on a set of Lebesgue measure zero. In particular, this distribution has no atoms. For x in this set, $F'_{\mathsf{sc}}(x) \neq 0$ or does not exist. Thus $F'_{sc}(x) = 0$ a.e and by continuity we have that $\mathbb{P}_{\mathsf{sc}}\{x\} = 0$ for each point $x \in \mathbb{R}$.

Example 1.4.6 — Cantor's Devil staircase. We consider the interval $[0, 1]$ and construct $F(x)$ by the following procedure originated by Cantor.

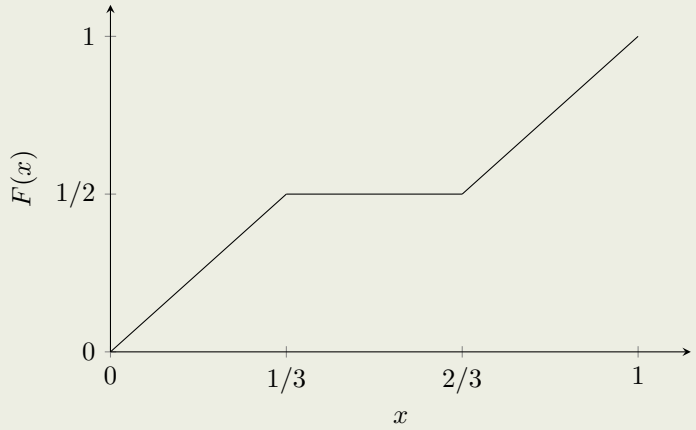


Figure 2: First step of constructing the devil staircase.

We divide $[0, 1]$ into thirds and put

$$F_2(x) = \begin{cases} 1/2 & x \in (\frac{1}{3}, \frac{2}{3}) \\ 1/4 & x \in (\frac{1}{9}, \frac{2}{9}) \\ 3/4 & x \in (\frac{7}{9}, \frac{8}{9}) \\ 0 & x = 0 \\ 1 & x = 1, \end{cases}$$

defining it in the intermediate intervals by linear interpolation. Then we divide each of the intervals $[0, 1/3]$ and $[2/3, 1]$ into three parts and define the function shown below with its values at other points determined by linear interpolation.

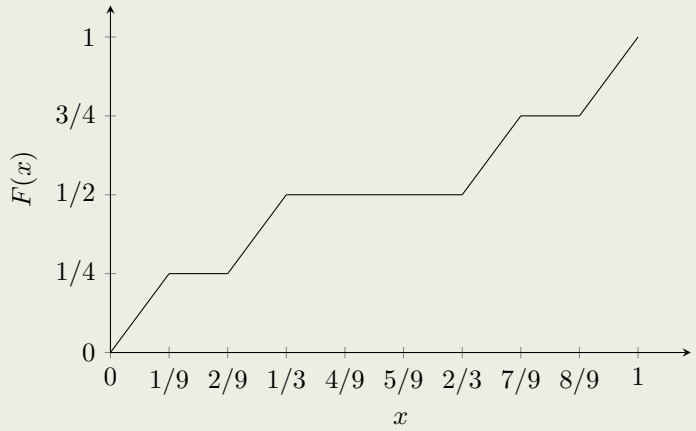


Figure 3: Second step of constructing the devil staircase.

Continuing with this process, we construct a sequence of functions $F_n(x)$, $n = 1, 2, \ldots$ which converges to a non-decreasing continuous function $F(x)$ (the Cantor function), whose points of increase form a set of Lebesgue measure zero. In fact, it is clear from the construction of $F(x)$ that the total

length of the intervals $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}), \dots$ on which the function is constant is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = 1.$$

Let N be the set of points of increase of the Cantor function $F(x)$. It follows from the sum above that $\mathsf{Leb}(N) = 0$. At the same time, if μ is the measure corresponding to the Cantor function $F(x)$, we have $\mu(N) = 1$. This measure is therefore singular with respect to the Lebesgue measure Leb.

Theorem 1.4.7 — Hahn decomposition. Any probability distribution has a representation of the form

$$F(x) = a_1 F_{\mathsf{disc}}(x) + a_2 F_{\mathsf{ac}}(x) + a_3 F_{\mathsf{sc}}(x)$$

for $a_1 + a_2 + a_3 = 1$.

1.5 Measures on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$

Distribution functions on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ are defined similarly. For example, when $n = 2$ we have

$$F(x, y) = \mathbb{P}((-\infty, x] \times (-\infty, y]).$$

For Probability spaces $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$, $(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$ the product measure on $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \otimes \mathscr{F}_2)$ is defined as follows.

1. Set
$$\mathbb{P}_0(A_1 \times A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2)$$
for $A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2$.
2. Then extend $\mathbb{P}_0$ to the algebra generated by $A_1 \times A_2$, and show that $\mathbb{P}_0$ is a σ-additive measure on this algebra.
3. Apply Theorem 1.3.5 to obtain the extension.

This extension is called the product measure and is denoted $\mathbb{P}_1 \otimes \mathbb{P}_2$.

1.6 Measures on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$

For the spaces $\mathbb{R}^n$, $n \geq 1$, the probability measures were constructed in the following way.

1. It was first defined for elementary sets of the form $(a, b]$.
2. The definition was then naturally extended to sets of the form $A = \sum_{i=1}^n (a_i, b_i]$.
3. The extension to sets in $\mathscr{B}(\mathbb{R}^n)$ was provided by Theorem 1.3.5.

A similar procedure of constructing probability measures also works for the space $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$. Let

$$\mathscr{I}_n(B) = \{x \in \mathbb{R}^\infty : (x_1, \dots, x_n) \in B\}$$

denote the cylinder set in $\mathbb{R}^\infty$ with base $B \in \mathscr{B}(\mathbb{R}^n)$. It is natural to take the cylinder sets as the elementary sets in $\mathbb{R}^\infty$ whose probabilities enable us to determine the probability measure on the sets of $\mathscr{B}(\mathbb{R}^\infty)$.

Definition 1.6.1 — Consistent Sequence. The sequence $\mathbb{P}_n$ of probability measures on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ is said to be **consistent** if for all $n = 1, 2, \dots$ and $B \in \mathscr{B}(\mathbb{R}^n)$ we have

$$\mathbb{P}_{n+1}(B \times \mathbb{R}) = \mathbb{P}_n(B).$$

The following theorem says that we can always construct a probability measure on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ from a consistent sequence of measures.

Theorem 1.6.2 — Kolmogorov Extension Theorem. For any consistent sequence $\mathbb{P}_n$ on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, there exists a unique probability measure $\mathbb{P}$ on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ such that

$$\mathbb{P}(\mathscr{I}_n(B)) = \mathbb{P}_n(B),$$

for $B \in \mathscr{B}(\mathbb{R}^n)$ and $n = 1, 2, \ldots$.

1.7 Random Variables

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 1.7.1 — Random variable. A real function $\xi : \Omega \to \mathbb{R}$ is an **$\mathscr{F}$-measurable function**, or a **random variable** if

$$\{\omega : \xi(\omega) \in B\} \in \mathscr{F}$$

for every $B \in \mathscr{B}(\mathbb{R})$. Equivalently,

$$\xi^{-1}(B) \equiv \{\omega : \xi(\omega) \in B\}$$

is a measurable set in Ω. When $(\Omega, \mathscr{F}) = (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, the $\mathscr{B}(\mathbb{R}^n)$-measurable functions are called **Borel functions**.

Random variables are used to *summarise* the abstract outcomes $\omega \in \Omega$ with a real number or vector.

Example 1.7.2 — Sum of two dice. Consider the experiment of throwing two independent fair six-faced dice. This can be represented by the probability space $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1) \otimes (\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$, where for $i = 1, 2$, $\Omega_i = \{1, 2, 3, 4, 5, 6\}$ is the outcome from dice i, $\mathscr{F}_i = 2^{\Omega_i}$ and $\mathbb{P}_i(\{j\}) \equiv 1/6$ for all $j \in \{1, 2, 3, 4, 5, 6\}$. We can then consider the function $X : \Omega_1 \times \Omega_2 \to \mathbb{R}$ by

$$X(\omega_1, \omega_2) = \omega_1 + \omega_2$$

which summarises the outcome of two dice by their sum. One can easily check that X is a measurable function by looking at the possible pre-images of X (exercise).

Exercise 1.7.3 In the context of Example 1.7.2 answer the following.

- What are the possible outcomes of X?
- What are the possible pre-images of X?

Remark 1.7.4 Let ξ be a random variable. Sets of the form $\{\omega : \xi(\omega) \in B\}$, for some $B \in \mathscr{B}(\mathbb{R})$, can be verified to form a σ-algebra. We call this the **σ-algebra generated by** ξ, and denote it by $\mathscr{F}_\xi \subset \mathscr{F}$.

Lemma 1.7.5 Let $\mathscr{D}$ be a collection of subsets on $\mathbb{R}$ such that $\sigma(\mathscr{D}) = \mathscr{B}(\mathbb{R})$. A necessary and sufficient condition that a function $\xi = \xi(\omega)$ is a random variable is that

$$\xi^{-1}(D) = \{\omega : \xi(\omega) \in D\} \in \mathscr{F}$$

for all $D \in \mathscr{D}$.

Corollary 1.7.6 A necessary and sufficient condition for $\xi = \xi(\omega)$ to be a random variable is that

$$\{\omega : \xi(\omega) < x\} \in \mathscr{F}$$

for every $x \in \mathbb{R}$, or that

$$\{\omega : \xi(\omega) \leq x\} \in \mathscr{F}$$

for every $x \in \mathbb{R}$.

Lemma 1.7.7 Let $\varphi = \varphi(x)$ be a Borel function and $\xi = \xi(\omega)$ a random variable. Then the composition $\eta = \varphi \circ \xi$ is also a random variable. In fact, it is $\mathscr{F}_\xi$- measurable.

Proof. The result follows directly from

$$\{w : \eta(\omega) \in B\} = \{w : \varphi(\xi(\omega)) \in B\} = \left\{w : \xi(\omega) \in \varphi^{-1}(B)\right\} \in \mathscr{F}$$

for $B \in \mathscr{B}(\mathbb{R})$, since $\varphi^{-1}(B) \in \mathscr{B}(\mathbb{R})$. ■

Example 1.7.8

- If ξ is a random variable and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function then $f(\xi)$ is a random variable.
- If ξ is a random variable, then so are $\xi^n, \xi^+ = \max(\xi, 0), \xi^- = -\min(\xi, 0)$, and $|\xi| = \xi^+ + \xi^-$.

Lemma 1.7.9 If ξ and η are random variables, then

$$\xi + \eta, \xi - \eta, \xi\eta, \xi/\eta, \max(\xi, \eta), \min(\xi, \eta)$$

are random variables (assuming that they are defined, that is non are of the form $\infty - \infty, \infty/\infty, a/0$).

Lemma 1.7.10 If for $n = 1, 2, \ldots$ the functions f_n are random variables and if for all ω we that

$$s(\omega) = \sup_n f_n(\omega)$$

exits, then $s(\omega)$ is a random variable. Similarly, we can replace $\sup_n$ with $\inf_n$ or $\lim_n$.

Definition 1.7.11 A random variable ξ is called **simple** if

$$\xi(\omega) = \sum_{j=1}^{n} x_j \chi_{D_j}(\omega)$$

for some $n \geq 1$, $D_1, \ldots, D_n$ being a partition of Ω consisting of measurable sets and

$$\chi_D(\omega) = \begin{cases} 1, & \omega \in D \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.7.12

- For every random variable $\xi = \xi(\omega)$ there is a sequence of simple random variables $\xi_1, \xi_2, \ldots$, such that $|\xi_n| \leq |\xi|$ and $\xi_n(\omega) \to \xi(\omega)$ as $n \to \infty$, for all $\omega \in \Omega$.
- For any random variable $\xi(\omega) \geq 0$ there exists a pointwise non-decreasing sequence of simple random variables $\xi_1(\omega) \leq \xi_2(\omega) \leq \cdots \leq \xi(\omega)$ such that

  $$\lim_{n\to\infty} \xi_n(\omega) = \xi(\omega)$$

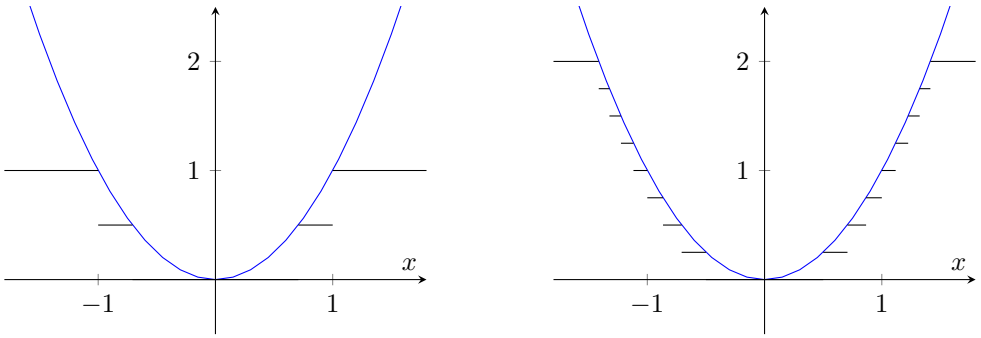
  for all $\omega \in \Omega$. Such a sequence is usually denoted $\xi_n \nearrow \xi$.

Proof. We begin by proving the second statement. For $n = 1, 2, \dots$, let

$$\xi_n(\omega) = \sum_{j=0}^{n2^n-1} \frac{j}{2^n} \chi_{\left\{\omega: \frac{j}{2^n} \le \xi(\omega) < \frac{j+1}{2^n}\right\}} + n\chi_{\{\omega:\xi(\omega)\ge n\}}$$

One can verify that the sequence $\xi_n(\omega)$ is such that $\xi_n \nearrow \xi$ for all $\omega \in \Omega$. The first statement follows from this if we merely observe that ξ can be represented in the form $\xi = \xi^+ - \xi^-$, where $\xi^+ = \max(\xi, 0)$ and $\xi^- = \max(-\xi, 0)$. ■

The below figures represent how one can build a simple function approximation to the function $f(x) = x^2$ for all $x \in \mathbb{R}$. For simplicity, only the first two steps are shown.



(a) $n = 1$ **(b)** $n = 2$

Figure 4: Approximation of x^2.

More generally, this construction gives a routine for proving statements related to random variables, known as the *four-step* proof.

1. First prove the statement for indicator functions.
2. Extend the statement for simple random variables by considering linearity.
3. Extend the statement to non-negative random variables by taking limits.
4. Extend the statement for any random variables by considering their positive and negative parts.

Lemma 1.7.13 Consider a measurable space $(\Omega, \mathscr{F})$ and a finite or countable decomposition $\mathscr{D} = \{D_1, D_2, \dots\}$ of the space Ω. Let $\xi = \xi(\omega)$ be a $\sigma(\mathscr{D})$-measurable random variable. Then ξ is representable in the form

$$\xi(\omega) = \sum_{k=1}^{\infty} \alpha_k \chi_{D_k}(\omega),$$

where $\alpha_k \in \mathbb{R}$. That is, $\xi(\omega)$ is constant on the elements D_k of the decomposition.

1.8 Distributions of random variables

Definition 1.8.1 A probability measure $\mathbb{P}_\xi$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with

$$\mathbb{P}_\xi(B) = \mathbb{P}\{\omega : \xi(\omega) \in B\},$$

for $B \in \mathscr{B}(\mathbb{R})$ is called the **probability distribution of** ξ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

It is clear that the above definition makes sense since $\mathbb{P}_\xi(B)$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Definition 1.8.2 The function

$$F_\xi(x) \equiv \mathbb{P}_\xi((-\infty, x]) = \mathbb{P}(\{\omega : \xi(\omega) \le x\}),$$

for $x \in \mathbb{R}$ is called the **distribution function** of ξ.

Example 1.8.3 — Sum of two dices - continued. One can verify that the probability distribution of X on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ satisfies

$$\mathbb{P}_X(\{j\}) = \frac{6 - |7 - j|}{36},$$

for $j \in \{2, 3, ..., 12\}$. Then we can extend the definition of $\mathbb{P}_X$ to other sets in $\mathscr{B}(\mathbb{R})$.

Notice that there are multiple random variables, on potentially different probability spaces, which give the same distribution function. Indeed, we can always construct a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ given a distribution function F_ξ. Therefore for any random variable ξ on a probability $(\Omega, \mathscr{F}, \mathbb{P})$, the identity random variable $I(\omega) = \omega$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_\xi)$ has same distribution as ξ. For example, consider the space $\big(\Omega, 2^\Omega, Q\big)$, with $\Omega = \{2, 3, ..., 12\}$ and $Q(\{j\}) = \mathbb{P}_X(\{j\})$ as defined above. Consider the random variable $\xi(j) = j$ for all $j \in \Omega \subseteq \mathbb{R}$. Then $Q_\xi(A) \equiv \mathbb{P}_X(A)$ for all $A \in \mathscr{B}(\mathbb{R})$.

Definition 1.8.4 — Extension to $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. The vector function

$$\xi : \Omega \to \mathbb{R}^n \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n)$$

is called a **random vector** if for any $B \subset \mathscr{B}(\mathbb{R}^n)$, we have $\xi^{-1}(B) \in \mathscr{F}$. As before we define $\mathbb{P}_\xi$ and we say that $\mathbb{P}_\xi = \mathbb{P}_{(\xi_1, \ldots, \xi_n)}$ is a **joint distribution** of $\xi_1, \ldots, \xi_n$, given by

$$F_\xi(x_1, \ldots, x_n) = \mathbb{P}(\omega : \xi_1 \leq x_1, \ldots, \xi_n \leq x_n).$$

Exercise 1.8.5 Show that the vector $\xi = (\xi_1, \ldots, \xi_n)$ is a random variable if and only if $\xi_1, \ldots, \xi_n$ are random variables.

For $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ we can similarly define random sequences $\xi = (\xi_1, \xi_2, \ldots)$.

1.9 Solution to Exercises

Exercise 1.1.11

Solution. Let $\mathscr{C} = \big\{\bigcup_{i \in I} D_i : I \subseteq \mathbb{N}\big\}$.

- As $\Omega = \bigcup_{i \in \mathbb{N}} D_i$ we have that $\Omega \in \mathscr{C}$.
- For $A \in \mathscr{C}$, there exists an index set $I \subseteq \mathbb{N}$ such that $A = \bigcup_{i \in I} D_i$. Let $I' = \mathbb{N} \setminus I$ and define $B = \bigcup_{i \in I'} D_i \in \mathscr{C}$. As $\mathscr{D}$ forms a partition, it is clear that $B = A^c$ and so $A^c \in \mathscr{C}$.
- Let $A_1, A_2, \cdots \in \mathscr{C}$. Then each A_i is a countable units of elements from $\mathscr{D}$. Therefore, $A = \bigcup_{i=1}^\infty A_i$ is also a countable union of elements from $\mathscr{D}$ and so $A \in \mathscr{C}$.

The above show that $\mathscr{C}$ is a σ-algebra. ■

Exercise 1.2.2

Solution.

1. As $(a, b] = \bigcap_{n=1}^\infty \big(a, b + \frac{1}{n}\big)$ it follows that the open intervals also generate $\mathscr{B}(\mathbb{R})$.
2. As $(a, b] = \bigcap_{n=1}^\infty \big[a + \frac{1}{n}, b\big]$ it follows that the closed intervals also generate $\mathscr{B}(\mathbb{R})$.
3. As $(a, b] = \bigcap_{n=1}^\infty \big[a + \frac{1}{n}, b + \frac{1}{n}\big)$ it follows that the right-open half intervals also generate $\mathscr{B}(\mathbb{R})$.
4. As $(a, b] = (-\infty, a]^c \cap (-\infty, b]$ it follows that intervals of the form $(-\infty, a]$ or $[a, \infty)$ also generate $\mathscr{B}(\mathbb{R})$.
5. As (a, b) it follows from (1) that open sets also generate $\mathscr{B}(\mathbb{R})$.

6. As $[a,b]$ are closed sets it follows from (2) that closed sets also generate $\mathscr{B}(\mathbb{R})$.

■

Exercise 1.2.6

Solution. The set $\{x \in \mathbb{R}^\infty : (x_1) \in \mathbb{R}\} = \mathbb{R}^\infty$. Similarly, the set $\{x \in \mathbb{R}^\infty : (x_1) \in \emptyset\} = \emptyset$.
Let $C = \left\{x \in \mathbb{R}^\infty : (x_1,\ldots,x_n) \in \tilde{C}_n\right\}$ with $\tilde{C}_n \in \mathscr{B}(\mathbb{R}^n)$. Then as $\left(\tilde{C}_n\right)^c \in \mathscr{B}(\mathbb{R}^n)$ it follows that $C^c = \left\{x \in \mathbb{R}^\infty : (x_1,\ldots,x_n) \in \left(\tilde{C}_n\right)^c\right\}$ is a cylindrical set.
Now consider the cylindrical sets $C_1,\ldots,C_m$ where $C_i = \left\{x \in \mathbb{R}^\infty : (x_1,\ldots,x_{n_i}) \in \tilde{C}_{n_i}\right\}$ for $\tilde{C}_{n_i} \in \mathscr{B}(\mathbb{R}^{n_i})$. Let $n = \max_{i=1,\ldots,m}(n_i)$, then for each $\tilde{C}_{n_i}$ we can define $\tilde{D}_n = \tilde{C}_{n_i} \cup \mathbb{R}^{n-n_i} \in \mathscr{B}(R^n)$. We note that

$$C_i = \left\{x \in \mathbb{R}^\infty : (x_1,\ldots,x_{n_i},\ldots,x_m) \in \tilde{D}_n\right\}.$$

Moreover,

$$\bigcup_{i=1}^m C_i = \left\{x \in \mathbb{R}^\infty : (x_1,\ldots,x_m) \in \bigcup_{k=1}^m \tilde{D}_{n_k}\right\}$$

where $\tilde{D}_{n_k} \in \mathscr{B}(\mathbb{R}^m)$ and so $\bigcup_{i=1}^m C - i$ is a cylindrical set.
Therefore, the cylindrical sets form an algebra.

■

Exercise 1.2.8

Solution. If $x \in D$ it follows that for all $\epsilon > 0$ there exists an N such that for $n \geq N$ we have that $|x_n - c| < \epsilon$. Therefore,

$$D = \bigcap_{k=1}^\infty \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty \left\{x \in \mathbb{R}^\infty : |x_n - c| < \frac{1}{k}\right\}.$$

The set $\bigcap_{n=N}^\infty \left\{x \in \mathbb{R}^\infty : |x_n - c| < \frac{1}{k}\right\}$ is equivalent to saying that $x_n \in \left(c - \frac{1}{k}, c + \frac{1}{k}\right)$ for all $n \geq N$ and x_m can be any real number for $1 \leq m < N$. Therefore, this is in $\mathscr{B}(\mathbb{R}^\infty)$ meaning that $D \in \mathscr{B}(\mathbb{R}^\infty)$.

■

Exercise 1.3.2

Solution.

1. Let $x \leq y$. Then $(-\infty, x] \subseteq (-\infty, y]$ so that by properties of the probability measure it follows that

   $$\mathbb{P}((-\infty, x]) \leq \mathbb{P}((-\infty, y])$$

   which implies that $F(x) \leq F(y)$.

2. Let (x_n) be a sequence such that $x_{n+1} \leq x_n$ and $x_n \to -\infty$ as $n \to \infty$. Then the sets $A_n = (-\infty, x_n]$ are a decreasing sequence of events. Therefore,

   $$\mathbb{P}\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} F(x_n).$$

   Note that $\bigcap_{n=1}^\infty A_n = \emptyset$ so that $\lim_{n\to\infty} F(x_n) = 0$. We can conclude that $\lim_{x\to-\infty} F(x) = 0$. Similarly, let (x_n) be a sequence such that $x_n \leq x_{n+1}$ and $x_n \to \infty$ as $n \to \infty$. Then the sets $A_n = (x_n, \infty)$ are a sequence of decreasing events. Therefore, as above

   $$0 = \mathbb{P}\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

   As $\mathbb{P}(A_n^c) = F(x_n)$, it follows that $1 = \lim_{n\to\infty} F(x_n)$ from which we conclude that $\lim_{x\to\infty} F(x) = 1$.

3. Let $x \in \mathbb{R}$ and $x_n \searrow x$ monotonically. Then the sets $A_n = (-\infty, x_n]$ are a sequence of decreasing events with the property that $\bigcap_{n=1}^{\infty} A_n = (-\infty, x]$. Therefore,

$$F(x) = \mathbb{P}((-\infty, x]) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} F(x_n).$$

More generally we see that $F(x) = \lim_{y\searrow x} F(y) = 0$ and so $F(x)$ is right-continuous.

■

Exercise 1.7.3 The possible outcomes of X are $X(\omega_1, \omega_2) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. The possible pre-images are summarised in Table 1.

Image	Pre-Image
2	$\{\{1,1\}\}$
3	$\{\{1,2\}\{2,1\}\}$
4	$\{\{1,3\},\{2,2\},\{3,1\}\}$
5	$\{\{1,4\},\{2,3\},\{3,2\},\{4,1\}\}$
6	$\{\{1,5\},\{2,4\},\{3,3\},\{4,2\},\{5,1\}\}$
7	$\{\{1,6\},\{2,5\},\{3,4\},\{4,3\},\{5,2\},\{6,1\}\}$
8	$\{\{2,6\},\{3,5\},\{4,4\},\{5,3\},\{6,2\}\}$
9	$\{\{3,6\},\{4,5\},\{5,4\},\{6,3\}\}$
10	$\{\{4,6\},\{5,5\},\{6,4\}\}$
11	$\{\{5,6\},\{6,5\}\}$
12	$\{\{6,6\}\}$

Table 1: Pre-Images of X

Solution. ■

Exercise 1.8.5

Solution. Let $\xi = (\xi_1, \ldots, \xi_n) : \Omega \to \mathbb{R}^n$ be a random vector, where each $\xi : \Omega \to \mathbb{R}$. Without loss of generality consider ξ_1 and $B \in \mathscr{B}(\mathbb{R})$. Then,

$$\begin{aligned}\xi_1^{-1}(B) &= \{\omega : \xi_1(\omega) \in B\} \\ &= \{\omega : \xi(\omega) \in B \times \mathbb{R} \times \cdots \times \mathbb{R}\}.\end{aligned}$$

As $B \times \mathbb{R} \times \cdots \times \mathbb{R} \in \mathscr{B}(\mathbb{R}^n)$ and ξ is a random variable we conclude that $\xi_1^{-1}(B) \in \mathscr{F}$ and hence a random variable. Conversely, if each of the ξ_i are random variables. Then for $B \in \mathscr{B}(\mathbb{R}^n)$, we can write $B = B_1 \times \cdots \times B_n$ for $B_i \in \mathscr{B}(\mathbb{R})$. Therefore,

$$\xi^{-1}(B) = \{\omega : \xi_1(\omega) \in B_1, \ldots, \xi_n(\omega) \in B_n\} = \bigcap_{i=1}^{n} \xi_i^{-1}(B_i).$$

Then as each ξ_i is a random variable we have that $\xi_i^{-1}(B_i) \in \mathscr{F}$. Which implies that $\xi^{-1}(B) \in \mathscr{F}$ and hence a random variable. ■

2 Expectation and Integrals

2.1 The Lebesgue Integral

First, we recall the construction of the Lebesgue integral. Define the expectation of a simple random variable $\xi = \sum_{j=1}^{n} x_j \chi_{D_j}$ as

$$\mathbb{E}(\xi) = \sum_{j=1}^{n} x_j \mathbb{P}(D_j),$$

where the sets D_j form a partition of Ω.

For an arbitrary non-negative random variable $\xi = \xi(\omega)$ we can construct a sequence of simple non-negative random variables $\{\xi_n\}_{n\geq 1}$ such that $\xi_n(\omega) \nearrow \xi(\omega)$, as $n \to \infty$ for each $\omega \in \Omega$. We then set $\mathbb{E}(\xi) = \lim_{n\to\infty} \mathbb{E}(\xi_n)$, which exists since $\mathbb{E}(\xi_n) \leq \mathbb{E}(\xi_{n+1})$ (possibly taking the value $+\infty$).

Definition 2.1.1 — Expectation. The **expectation** $\mathbb{E}(\xi)$ of a non-negative random variable ξ is the Lebesgue integral with respect to $\mathbb{P}$ given by

$$\mathbb{E}(\xi) := \lim_{n\to\infty} \mathbb{E}(\xi_n) = \int_\Omega \xi \mathrm{d}\mathbb{P} = \int_\Omega \xi(\omega)\mathbb{P}(\mathrm{d}\omega)$$

which may be infinite.

To see that this definition is consistent, one has to show it is independent of the choice of $\xi_n \nearrow \xi$.

Definition 2.1.2 An arbitrary random variable ξ is said to be integrable if $\mathbb{E}(|\xi|) < \infty$.

Remark 2.1.3 An integrable random variable may not be non-negative. However, the above is well-defined as $|\xi|$ is a non-negative random variable.

For an integrable we can define the expectation as $\mathbb{E}(\xi) = \mathbb{E}(\xi^+) - \mathbb{E}(\xi^-)$. This is well-defined as $\mathbb{E}(\xi^-) < \infty$ by assumption.

2.2 Properties

Here we observe some basic properties of the expectation.

Property 2.2.1 Let ξ and η be integrable random variables and let c be a constant. Then

- $\mathbb{E}(c) = c$,
- $\mathbb{E}(c\xi) = c\mathbb{E}(\xi)$,
- $\xi + \eta$ is integrable with $\mathbb{E}(\xi + \eta) = \mathbb{E}(\xi) + \mathbb{E}(\eta)$,
- $\xi \leq \eta$ implies that $\mathbb{E}(\xi) \leq \mathbb{E}(\eta)$,
- if $\xi = \eta$ a.e. with respect to $\mathbb{P}$, that is the equality holds up to sets of zero $\mathbb{P}$-measure, then $\mathbb{E}(\xi) = \mathbb{E}(\eta)$, and
- if $\xi \geq 0$ is such that $\mathbb{E}(\xi) = 0$, then $\xi = 0$ a.e.

2.2.1 Exchanging limits and expectations

We now outline some convergence theorems that facilitate the exchange of limits and summations under appropriate conditions.

Theorem 2.2.2 — Monotone convergence theorem (MCT). Let $0 \leq \xi_1 \leq \xi_2 \leq \cdots$ be random variables.

Then

$$\lim_{n\to\infty} \mathbb{E}(\xi_n) = \mathbb{E}\left[\lim_{n\to\infty} \xi_n\right]$$

exists, potentially being infinite.

Remark 2.2.3

- $0 \leq \xi_1 \leq \cdots$ can be replaced by $\eta \leq \xi_1 \leq \cdots$ with $\mathbb{E}(\eta) > -\infty$, as we just consider $\xi_n - \eta$ instead of ξ_n.
- Similarly, $0 \leq \xi_1 \leq \cdots$ can be replaced by $\cdots \leq \xi_2 \leq \xi_1 \leq \eta$, with $\mathbb{E}(\eta) < \infty$.

Corollary 2.2.4 Let $\{\eta_k\}_{k\geq 1}$ be a sequence of non-negative random variables. Then

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \eta_k\right) = \sum_{k=1}^{\infty} \mathbb{E}(\eta_k).$$

Theorem 2.2.5 — Fatou's Lemma. Let $\{\xi_n\}_{n\geq 1}$ be non-negative random variables. Then

$$\mathbb{E}\left(\liminf_n \xi_n\right) \leq \liminf_n \mathbb{E}(\xi_n).$$

Remark 2.2.6

- $\xi_n \geq 0$ can be replaced by $\xi_n \geq \eta$, if $\mathbb{E}[\eta] > -\infty$.
- If $\xi_n < \eta$, and $\mathbb{E}[\eta] < \infty$, then the statement holds for $\limsup$ instead.

Exercise 2.2.7 Prove Theorem 2.2.5

Theorem 2.2.8 — Lebesgue's Theorem or Dominated Convergence. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of random variables such that $\xi_n \to \xi$ (a.s.). If there exists an integrable random variable η such that $|\xi_n| \leq \eta$ for all n, then ξ is integrable, with

$$\mathbb{E}(\xi_n) \to \mathbb{E}(\xi),$$

and

$$\mathbb{E}\left(|\xi_n - \xi|\right) \to 0$$

as $n \to \infty$.

Corollary 2.2.9 Let $\eta, \xi, \xi_1, \ldots$ be random variables such that $|\xi_n| \leq \eta, \xi_n \to \xi$ (a.s.) and $\mathbb{E}\left(\eta^p\right) < \infty$ for some $p > 0$. Then $\mathbb{E}\left(|\xi|^p\right) < \infty$ and $\mathbb{E}\left(|\xi - \xi_n|^p\right) \to 0$ as $n \to \infty$.

Remark 2.2.10 In all the above theorems, the integral over Ω can be replaced by the integral over any measurable $\hat{A} \subset \Omega$.

2.2.2 Change of variables

Theorem 2.2.11 — Change of variables / Law of Unconscious Statistician (LOTUS). Let $\xi : \mathscr{F} \to \mathbb{R}$ be a random variable with probability distribution $\mathbb{P}_\xi$. If $g = g(x)$ is a Borel function, then for all

$A \in \mathscr{B}(\mathbb{R})$, we have

$$\int_A g(x)\mathrm{d}\mathbb{P}_\xi = \int_{\xi^{-1}(A)} g(\xi(\omega))\mathrm{d}\mathbb{P},$$

where both integrals exist or do not exist simultaneously. In particular, for $A = \mathbb{R}$ we obtain

$$\mathbb{E}[g(\xi(\omega))] = \int_\Omega g(\xi(\omega))\mathrm{d}\mathbb{P} = \int_{-\infty}^{\infty} g(x)\mathrm{d}\mathbb{P}_\xi \equiv \int_{-\infty}^{\infty} g(x)\mathrm{d}F_\xi.$$

Proof. We use the four-step proof. The result clearly holds for $g = \chi_B(x)$ for $B \in \mathscr{B}(\mathbb{R})$. Therefore, it also holds for simple $g(x)$ by the linearity of the integral. For any non-negative measurable function g we can consider a sequence of simple functions $g_n \nearrow g$. The result for g then follows from the monotone convergence theorem. For arbitrary measurable $g(x)$ we use $g(x) = g_+ - g_-$. ■

Remark 2.2.12 Theorem 2.2.11 guarantees that the expectation only depends on the probability distribution, and not on the underlying probability space.

1. If ξ is discrete ($\mathscr{F}_\xi$ is discrete) taking values $x_1, x_2, \ldots$ with probabilities $p_1, p_2, \ldots$ then

$$\mathbb{E}[g(\xi)] = \sum_j g(x_j)p_j.$$

2. If ξ is absolutely continuous (i.e. F_ξ is absolutely continuous) with density $f(x)$, then

$$\mathbb{E}[g(\xi)] = \int_{-\infty}^{\infty} g(x)f(x)\mathrm{d}x.$$

This provides a way to calculate expectations of $g(\xi)$ without being "conscious" of the actual distribution of $g(\xi)$. Thus we can make sense of the expectation of probability distributions **without** specifying its underlying probability space.

So is there any point in specifying the underlying probability space of a random variable instead of assuming it to be $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_\xi)$? Unfortunately the answer *can be* no, since there is a way to develop probability theory (and notions of random variables) without going through the Kolmogorov constructions. Indeed, the underlying probability space is not important in most of the applications; nevertheless, this formulation is still helpful in understanding more complicated random variables like stochastic processes.

2.3 Exchanging the Order of Integration

We now consider product measures, and more specifically expressing integration on product measures as repeated integrals.

Suppose $(X_1, \mathscr{M}_1, \mu_1)$ and $(X_2, \mathscr{M}_2, \mu_2)$ is a pair of measure spaces and consider the product measure $(\mu_1 \times \mu_2)$ on the space

$$X = X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}.$$

We assume that the two measure spaces are complete and σ-finite .

Given a set E in $\mathscr{M}$ we consider the *slices*

1. $E_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in E\}$, and
2. and $E^{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in E\}$.

Theorem 2.3.1 — Fubini's Theorem. In the setting above, suppose that $f(x_1, x_2)$ is an integrable function on $(X_1 \times X_2, \mu_1 \times \mu_2)$. Then the following hold.

- For almost every $x_2 \in X_2$, the slice $f^{x_2}(x_1) = f(x_1, x_2)$ is integrable on (X_1, μ_1).

- $\int_{X_1} f(x_1, x_2) \mathrm{d}\mu_1$ is an integrable function on X_2.
- We can exchange integrals as follows

$$\int_{X_2} \left(\int_{X_1} f(x_1, x_2) \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2 = \int_{X_1} \left(\int_{X_2} f(x_1, x_2) \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 = \int_{X_1 \times X_2} f \mathrm{d}\mu_1 \times \mu_2.$$

Remark 2.3.2 In general, the product space $(X, \mathscr{M}, \mu)$ is not complete. One can define the completion of this space, $\overline{\mathscr{M}}$, to be the collection of sets of the form $E \cup Z$, where $E \in \mathscr{M}$ and $Z \subset F$ with $F \in \mathscr{M}$ and $\mu(F) = 0$. Also, define $\overline{\mu}(E \cup Z) = \mu(E)$. Then,

- $\overline{\mathscr{M}}$ is the smallest $\sigma-$algebra containing $\mathscr{M}$ and all subsets of elements of $\mathscr{M}$ of measure zero, and
- the function $\overline{\mu}$ is a measure on $\overline{\mathscr{M}}$, and this measure is complete.

Theorem 2.3.1 continues to hold in this completed space.

In Theorem 2.3.1, we assume that the function f is integrable over the product space. We can relax this condition, by instead assuming that f is a non-negative measurable function.

Theorem 2.3.3 — Tonelli's Theorem. Suppose that $f(x_1, x_2) : X_1 \times X_2 \to [0, \infty]$ is a non-negative measurable function on $(X_1 \times X_2, \mu_1 \times \mu_2)$. Then

$$\int_{X_2} \left(\int_{X_1} f(x_1, x_2) \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2 = \int_{X_1} \left(\int_{X_2} f(x_1, x_2) \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 = \int_{X_1 \times X_2} f \mathrm{d}\mu_1 \times \mu_2.$$

Combining Fubini's theorem with Tonelli's theorem gives the following.

Theorem 2.3.4 — Fubini-Tonelli Theorem. If f is a measurable function, then

$$\int_{X_1} \left(\int_{X_2} |f(x_1, x_2)| \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 = \int_{X_2} \left(\int_{X_1} |f(x_1, x_2)| \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2 = \int_{X_1 \times X_2} |f| \mathrm{d}\mu_1 \times \mu_2.$$

Besides, if any one of these integrals is finite, then

$$\int_{X_1} \left(\int_{X_2} f(x_1, x_2) \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 = \int_{X_2} \left(\int_{X_1} f(x_1, x_2) \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2 = \int_{X_1 \times X_2} f \mathrm{d}\mu_1 \times \mu_2.$$

The absolute value of f in the conditions of Theorem 2.3.4 can be replaced by either the positive or the negative part of f. Theorem 2.3.4 is a generalisation of Theorem 2.3.3 as one can take the negative part of a non-negative function to zero whilst maintaining a finite integral. Informally all these conditions say that the double integral of f is well defined, though possibly infinite.

The advantage of the Theorem 2.3.4 over Theorem 2.3.1 is that the repeated integrals of the absolute value of $|f|$ may be easier to study than the double integral. As in Theorem 2.3.1, the single integrals may fail to be defined on a zero-measure set.

Proposition 2.3.5 — $\mathbb{E}$ and tail probabilities. Let ξ be a non-negative integrable random variable. Then

$$\mathbb{E}(\xi) = \int_{[0,\infty)} \mathbb{P}(\xi \geq x) \, \mathrm{d}x.$$

Proof. We have

$$\mathbb{E}(\xi) = \int_{[0,\infty)} x \, \mathrm{d}\mathbb{P}_\xi = \int_{[0,\infty)} \left(\int_0^x \mathrm{d}t \right) \mathrm{d}\mathbb{P}_\xi = \int_{[0,\infty)} \mathbb{P}(\xi \geq t) \, \mathrm{d}t, \tag{2.1}$$

where we applied Theorem 2.3.1 to

$$g(t,x) = \begin{cases} 1, & 0 \le t \le x, \\ 0, & \text{otherwise.} \end{cases}$$

■

Exercise 2.3.6 Generalise the above proof to prove that if $\xi \ge 0$ and $p \ge 1$ then

$$\mathbb{E}(\xi^p) = \int_0^\infty p y^{p-1} \mathbb{P}(\xi \ge y)\, dy. \tag{2.2}$$

Verify the formula 2.3.5 and 2.2 for some simple distributions, such as the exponential $\mathrm{Exp}(\lambda)$ distribution which has density $f(x) = \lambda e^{-\lambda x}$.

2.4 Jensen's Inequality and L^p Spaces

2.4.1 Convex Functions and Jensen Inequality

Definition 2.4.1 — Convexity.

- A set $\Omega \subseteq \mathbb{R}^n$ [a] is convex if for all $x, y \in \Omega$ and $\lambda \in [0,1]$ the point $(1-\lambda)x + \lambda y \in \Omega$.
- Let $E \subseteq \mathbb{R}^n$ be a convex set. A function $g : E \to \mathbb{R}$ is convex if, for all $x, y \in \Omega$ and $\lambda \in [0,1]$, we have
$$g((1-\lambda)x + \lambda y) \le (1-\lambda)g(x) + \lambda g(y).$$
- Let $E \subseteq \mathbb{R}^n$ be a convex set. A function $g : E \to \mathbb{R}$ is concave if $-g$ is convex.

[a] You can safely assume $n = 1$ for this section, and treat any gradient/Hessian as derivatives.

For the following, we assume $E = \mathbb{R}^n$ for simplicity, although removing this assumption won't affect results much.

Proposition 2.4.2 — Existence of subgradient. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for all $x_0 \in \mathbb{R}^n$, there is a vector $v \in \mathbb{R}^n$ (depending on x_0) such that for all $x \in \mathbb{R}^n$, we have

$$g(x) \ge g(x_0) + v^\top (x - x_0). \tag{2.3}$$

Any vector v satisfying (2.3) is called a subgradient of g at x_0.

With Proposition 2.4.2 we can prove Jensen's inequality for expectations.

Theorem 2.4.3 — Jensen's Inequality. Let ξ be an integrable random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a measurable, convex function. Then

$$g(\mathbb{E}[\xi]) \le \mathbb{E}[g(\xi)].$$

Proof. If $g(x)$ is convex then for each $x_0 \in \mathbb{R}$ there is a $v \in \mathbb{R}$ such that

$$g(x) \ge g(x_0) + v(x - x_0)$$

for all $x \in \mathbb{R}$. Putting $x = \xi$ and $x_0 = \mathbb{E}(\xi)$, we find that

$$g(\xi) \ge g(\mathbb{E}(\xi)) + v(\xi - \mathbb{E}(\xi))$$

and by the taking expectation of both sides we get $g(\mathbb{E}[\xi]) \le \mathbb{E}[g(\xi)]$. ■

Corollary 2.4.4 — Lyapunov's Inequality. If $0 < p < q < \infty$, then

$$(\mathbb{E}(|\xi|^p])^{1/p} \le (\mathbb{E}(|\xi|^q])^{1/q}. \tag{2.4}$$

Hint. *Consider* $f(x) = x^{q/p}$.

Proof. Let $r = q/p$. Then putting $\eta = |\xi|^p$ and applying Jensen's inequality to $g(x) = |x|^r$ (check that it is convex), we obtain $|\mathbb{E}(\eta)|^r \leq \mathbb{E}(|\eta|^r]$. That is,

$$(\mathbb{E}(|\xi|^p])^{q/p} \leq \mathbb{E}(|\xi|^q]$$

from which (2.4) follows. Consequently, if $\mathbb{E}(|\xi|^q) < \infty$ then $\mathbb{E}(|\xi|^p) = (\mathbb{E}(|\xi|^q))^{p/q} < \infty$. ■

Exercise 2.4.5 Prove that the composition of a convex non-decreasing function and a convex function is convex. Hence, conclude that $f(x) = |x|^r$ is a a convex function.

The following chain of inequalities among absolute moments

$$\mathbb{E}(|\xi|] \leq (\mathbb{E}(|\xi|^2])^{1/2} \leq \cdots \leq (\mathbb{E}(|\xi|^n])^{1/n} \leq \ldots$$

is a consequence of Lyapunov's inequality.

Remark 2.4.6 As a warning, Lyapunov inequality is only true when $\mathbb{P}$ is a **finite**, which is certainly the case for probability measures.

Definition 2.4.7 — Moments. Let $\mathbb{E}(|\xi|^p) < \infty$. For integers $0 \leq k \leq p$, we define the k^{th} moment to be $\mathbb{E}(\xi^k)$.

Definition 2.4.8 — L^p convergence. The sequence $\xi_1, \xi_2, \ldots$ of random variables converges in L^p to the random variable ξ if

$$(\mathbb{E}(|\xi_n - \xi|^p])^{1/p} \to 0$$

as $n \to \infty$.

Often one uses the notation $\|\xi\|_{L^p} = (\mathbb{E}(|\xi|^p))^{\frac{1}{p}}$. This is done to allude to the fact that $\|\cdot\|_{L^p}$ defines a norm on the vector space of functions with finite p^{th} moments, up to sets of $\mathbb{P}$-measure zero. More formally, we say that $\xi \in \mathscr{L}^p$ if $\mathbb{E}(|\xi|^p) < \infty$. Let $\xi \sim \eta$ if $\xi = \eta$ almost everywhere. Then $\|\cdot\|_{L^p}$ defines a norm on $L^p = \mathscr{L}^p/\sim$. This result is facilitated by the following inequalities.

Proposition 2.4.9 — Hölder's Inequality. Let $p \in [1, \infty]$ and let $q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$ then we let $q = \infty$ and vice-versa. If $\xi \in \mathscr{L}^p$ and $\eta \in \mathscr{L}^q$, then

$$\|\xi\eta\|_{\mathscr{L}^1(\Omega)} := \mathbb{E}(|\xi\eta|] \leq \|\xi\|_{L^p} \|\eta\|_{L^q}. \tag{2.5}$$

Proposition 2.4.10 — Minkowski's Inequality. If $\mathbb{E}(|\xi|^p) < \infty$ and $\mathbb{E}(|\eta|^p) < \infty$ for $1 \leq p \leq \infty$, then $\mathbb{E}(|\xi + \eta|^p) < \infty$ and

$$(\mathbb{E}(|\xi + \eta|^p))^{1/p} \leq (\mathbb{E}(|\xi|^p))^{1/p} + (\mathbb{E}(|\eta|^p))^{1/p}.$$

Remark 2.4.11 — Reverse Triangle Inequality. Note that if the sequence $(\xi_i)_{i \geq 1}$ converges in L^p to ξ, then

$$0 \leq |\,\|\xi_i\|_{L^p} - \|\xi\|_{L^p}\,| \leq \|\xi_i - \xi\|_{L^p} \to 0$$

as $i \to \infty$. That is, $\|\xi_i\|_{L^p} \to \|\xi\|_{L^p}$.

2.5 Tail Bounds

Most large sample results concern extreme events, for example, whether the value of a random variable deviates from its mean. This section builds the necessary tools to derive upper bounds for the probability of such events. These bounds are usually called "tail bounds" since they correspond to the "tail" of the densities of random variables. In particular, we will see how the tail bounds are related to the integrability of the random variables.

Example 2.5.1 — Tail bounds for specific random variables. As a motivation, consider random variables living on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with zero expectation. We would like to bound the probability of the tail event $\mathbb{P}(X > c)$ for $c \gg 1$ [a].

1. Let ξ_1 have a normal distribution $\mathbf{N}(0,1)$ with density

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

for $x \in \mathbb{R}$, such that

$$\begin{aligned}\mathbb{P}(\xi_1 > c) &= \int_c^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &\leq \int_c^\infty \frac{1}{\sqrt{2\pi}} \frac{x}{c} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{1}{c\sqrt{2\pi}} \exp\left(-\frac{c^2}{2}\right) \\ &=: r_1(c).\end{aligned}$$

2. Let ξ_2 have a double exponential (Laplace) distribution with density

$$f_2(x) = \frac{1}{2} e^{-|x|},$$

such that

$$\begin{aligned}\mathbb{P}(\xi_2 > c) &= \int_c^\infty \frac{1}{2} \exp(-x)\, dx \\ &= \frac{1}{2} \exp(-c) \\ &=: r_2(c).\end{aligned}$$

3. Let ξ_3 have a standard Cauchy distribution with density

$$f_3(x) = \frac{1}{\pi(1+x^2)}$$

such that

$$\begin{aligned}\mathbb{P}(\xi_3 > c) &= \frac{1}{2} - \frac{1}{\pi} \arctan c \\ &= \frac{1}{\pi} \arctan \frac{1}{c} \\ &\leq \frac{1}{\pi c} \\ &=: r_3(c).\end{aligned}$$

It is clear that $\mathbb{P}(X > c)$ decays much faster for the normal distribution than the double exponential distribution, and the Cauchy distribution admits the slowest decay. In particular, we have $r_3(c) \gg r_2(c) \gg r_1(c)$ in the sense that $r_1(c)/r_2(c) \to 0$ and $r_2(c)/r_3(c) \to 0$ when $c \to \infty$.

[a] This means much greater than. We will also use this when performing formal asymptotic analysis.

Exercise 2.5.2 — Moments for some distributions. Verify the following observations.

1. ξ_1 has zero odd moments, and has $(2k)^{\text{th}}$ moments $m_{1,k} = (2k-1)!! := (2k-1)\times\ldots\times 3\times 1 = \frac{(2k)!}{2^k k!}$

for all $k \in \mathbb{Z}_{\geq 1}$.

2. ξ_2 also has zero odd moments, and has $(2k)^{\text{th}}$ moments $m_{2,k} = (2k)!$.
3. ξ_3 has $(2k)^{\text{th}}$ moments $m_{3,k} = \infty$.

Therefore, we see that $\infty = m_{3,k} \gg m_{2,k} \gg m_{1,k}$ as $k \to \infty$ (in the sense that as $k \to \infty$ we have $m_{1,k}/m_{2,k} \to 0$). We therefore suspect that there is a connection between the tail bounds and the growth of moments.

To standardise the discussion of random variables, one often centralises the moments in the following way.

Definition 2.5.3 — Central Moments. The k^{th} central moment (for $k \geq 1$) of a random variable ξ is the expectation $\mathbb{E}\left((\xi - \mathbb{E}(\xi))^k\right)$ whenever $\mathbb{E}\left(|\xi|^k\right) < \infty$. In particular, the first central moment is zero. The main central moments of interest are the following.

- The 2^{nd} central moment $\mathbb{V}(\xi) := \mathbb{E}\left((\xi - \mathbb{E}(\xi))^2\right)$ is called the **variance**.
- The 3^{rd} central moment $\mathbb{E}\left((\xi - \mathbb{E}(\xi))^3\right)$ is called the **skewness**.
- The 4^{th} central moment $\mathbb{E}\left((\xi - \mathbb{E}(\xi))^4\right)$ is called the **kurtosis**.

With the above notions, we can state the central inequality which we use for deriving useful tail bounds.

Theorem 2.5.4 — Markov Inequality. Let ξ be a non-negative integrable random variable and $c > 0$ a constant. Then

$$\mathbb{P}(\xi \geq c) \leq \frac{\mathbb{E}(\xi)}{c}.$$

Proof. This follows directly from

$$\mathbb{E}(\xi) \geq \mathbb{E}\left(\xi \cdot \chi_{\xi \geq c}\right) \geq c\mathbb{E}\left(\chi_{\xi \geq c}\right) = c\mathbb{P}(\xi \geq c).$$

■

Remark 2.5.5 A generalisation of the Markov inequality considers ξ a random variable and g a non-negative Borel function. Let $c > 0$ be a constant and suppose that $\mathbb{E}(g(\xi))$ exists, then

$$\mathbb{P}(g(\xi) \geq c) \leq \frac{\mathbb{E}(g(\xi))}{c}. \tag{2.6}$$

Let us interrupt our discussion of tail bound by proving an interesting result regarding L^p norms. How powerful Markov's inequality (2.6) is for proving tail bounds for specific distribution depends on the integrability of ξ. As demonstrated in the following corollary.

Corollary 2.5.6 If $\xi L^p(\Omega)$ for $p \geq 1$ then for all $\varepsilon > 0$, we have

$$\mathbb{P}(|\xi - \mathbb{E}(\xi)| \geq \varepsilon) = \mathbb{P}(|\xi - \mathbb{E}(\xi)|^p \geq \varepsilon^p) \leq \frac{\mathbb{E}\left(|\xi - \mathbb{E}(\xi)|^p\right)}{\varepsilon^p}. \tag{2.7}$$

When $p = 2$ we obtain Chebyshev's Inequality, which states that

$$\mathbb{P}(|\xi - \mathbb{E}(\xi)| \geq \varepsilon) \leq \frac{\mathbb{V}(\xi)}{\varepsilon^2}. \tag{2.8}$$

2.5.1 Chernoff Bound and Moment Generating Function (MGF)

For the case when $\xi \in L^{\infty-}(\Omega)$ and the k^{th} moment does not grow "too quickly", one may choose an optimal p such that the right-hand side of (2.7) is minimised. This is rarely done in practice. Instead, we consider the moment-generating function.

Definition 2.5.7 — Moment Generating Function (MGF). The moment generating function of a random variable ξ is

$$M_\xi(t) = \mathbb{E}(\exp(tX)) = \int_{\mathbb{R}} e^{tx} dF_\xi(x).$$

A moment-generating function does not necessarily exist for all values of $t \in \mathbb{R}$. For example, a random variables ξ with Cauchy distribution has $M_\xi(t) = \infty$ for all $t \neq 0$, and is equal to 1 for $t = 0$. However, if we can show that $M_\xi(t) < \infty$ for a small neighbourhood of zero, say $t \in (-h, h)$, then have the following result.

Corollary 2.5.8 — Chernoff (Exponential Chebyshev) Inequality. Let ξ be a non-negative random variable, then for all $\varepsilon > 0$ and $t \in (0, h)$ we have

$$\mathbb{P}(\xi \geq \varepsilon) = \mathbb{P}(e^{t\xi} \geq e^{t\varepsilon}) \leq \frac{\mathbb{E}\left(e^{t\xi}\right)}{e^{t\varepsilon}} = \frac{M_X(t)}{e^{t\varepsilon}}. \tag{2.9}$$

Let us use the following example to illustrate how Markov's inequality can be used to prove tail bounds.

Example 2.5.9 — Tail bounds for Gaussian. Consider ξ following a standard normal distribution $\mathbf{N}(0, 1)$. The $(2k)^{\text{th}}$ moments of ξ is given by $(2k-1)!!$, and thus for all $k \geq 1$ and $c > 0$, we have

$$\mathbb{P}(X > c) \leq \frac{(2k-1)!!}{c^{2k}}. \tag{2.10}$$

Even though using larger k will lead to a faster rate of decay as $c \to \infty$, the numerator is also larger, so it is harder to use the bound for practical applications. We can obtain a much sharper bound than (2.10) by using the Chernoff bounds. Observe that

$$\mathbb{E}\left(e^{t\xi}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{tx - x^2/2}\, dx = \exp\frac{t^2}{2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-t)/2}\, dx = \exp\frac{t^2}{2}$$

so that by the Chernoff bound (2.9) we have

$$\mathbb{P}(\xi > c) \leq \exp\left(\frac{t^2}{2} - ct\right) = \exp\left(-\frac{c^2}{2} + \frac{1}{2}(t-c)^2\right). \tag{2.11}$$

Since (2.11) holds for all $t > 0$, we can choose the optimal t such that the right-hand side is minimised. In our case, we choose $c > 0$ to obtain

$$\mathbb{P}(\xi > c) \leq \exp\left(\frac{t^2}{2} - ct\right) = \exp\left(-\frac{c^2}{2}\right). \tag{2.12}$$

This is almost optimal compared to our previous Mill-ratio inequalities, in the sense that the right-hand side is off by a factor of C/λ, with C being a constant independent of c. It is also surprisingly useful in practice.

2.6 Solution to Exercises

Exercise 2.2.7

Solution. Let $f_n = \inf_{m \geq n} \xi_n$. Then clearly, $0 \leq f_n \leq f_{n+1}$ for all n. Therefore, by the monotone convergence theorem, we have that

$$\begin{aligned}
\mathbb{E}\left(\liminf_n \xi_n\right) &= \mathbb{E}\left(\lim_{n\to\infty} f_n\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left(f_n\right) \\
&= \liminf_n \mathbb{E}\left(f_n\right) \\
&\leq \liminf_n \mathbb{E}\left(\xi_n\right).
\end{aligned}$$

■

Exercise 2.3.6

Solution. Note that

$$\mathbb{E}\left(\xi^p\right) = \int_0^\infty t^p d\mathbb{P}_\xi(t) = \int_0^\infty \int_0^t py^{p-1}\, dy\, d\mathbb{P}_\xi(t).$$

Applying Fubini's theorem to

$$g(y,t) = py^{p-1}\mathbf{1}_{[0\le y\le t]}$$

allows us to interchange the integrals to conclude that

$$\begin{aligned}\mathbb{E}\left(\xi^p\right) &= \int_0^\infty \int_y^\infty py^{p-1}\, d\mathbb{P}_\xi(t)\, dy \\ &= \int_0^\infty py^{p-1}\mathbb{P}(\xi \ge y).\end{aligned}$$

■

Exercise 2.4.5

Solution. Let $f = g \circ h$, where g is a non-decreasing convex function and h is a convex function. Then for $x, y \in \Omega$ and $\lambda \in [0,1]$ we have

$$\begin{aligned}(g\circ h)((1-\lambda)x + \lambda y) &= g(h((1-\lambda)x+\lambda y)) \\ &\overset{(1)}{\le} g((1-\lambda)h(x) + \lambda h(y)) \\ &\overset{(2)}{\le} (1-\lambda)(g\circ h)(x) + \lambda(g\circ h)(y).\end{aligned}$$

Where in (1) we have used the fact that h is convex and g is non-decreasing, and in (2) we have used the fact that g is convex. Therefore, $g \circ h$ is a convex function. Hence, with $g(x) = x^r$ for $x \ge 0$ and $h(x) = |x|$, we deduce that $f(x) = |x|^r$ is convex. ■

Exercise 2.5.2

Solution.

1. The odd moments are given by
$$\int_{-\infty}^\infty \frac{x^{2k+1}}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$
which is zero as the integrand is an odd function. For the even moments we can proceed by induction. For the base case we note that $\mathbb{E}\left(\xi^2\right) = \mathbb{V}(\xi) + \mathbb{E}(\xi) = 1 = (2(1)-1)!!$. Therefore, for $k \ge 1$ we have
$$\begin{aligned}\mathbb{E}\left(\xi^{2k}\right) &= \int_{-\infty}^\infty \frac{x^{2k}}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= 2\int_0^\infty \frac{x^{2k}}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &\overset{(1)}{=} 2\left(\left[-x^{2k-1}\exp\left(-\frac{x^2}{2}\right)\right]_0^\infty + \int_0^\infty (2k-1)x^{2k-2}\exp\left(-\frac{x^2}{2}\right)\right) \\ &= (2k-1)\int_{-\infty}^\infty x^{2k-2}\exp\left(-\frac{x^2}{2}\right), dx \\ &\overset{\text{Ind Hyp.}}{=} (2k-1)(2k-3)!! \\ &= (2k-1)!!.\end{aligned}$$
Where in (1) we have performed integration by parts with $u = x^{2k-1}$ and $\frac{dv}{dx} = x\exp\left(-\frac{x^2}{2}\right)$.

2. The odd moments are given by
$$\int_{-\infty}^{\infty} \frac{x}{2} \exp(-x)\, dx$$
which is zero as the integrand is an odd function. For the even moments, we can proceed by induction. The case for $k = 0$ is clear. For $k \geq 1$ we have
$$\begin{aligned}
\mathbb{E}\left(\xi^{2k}\right) &= \int_{-\infty}^{\infty} \frac{x^{2k}}{2} \exp(-x)\, dx \\
&= \int_{0}^{\infty} \frac{x^{2k}}{2} \exp(-x)\, dx \\
&= 2\left(\left[-x^{2k}\exp(-x)\right]_0^{\infty} + \int_0^{\infty} 2kx^{2k-1}\exp(-x)\, dx\right) \\
&= 2(2k)\int_0^{\infty} x^{2k-1}\exp(-x)\, dx \\
&= 2(2k)\left(\left[-x^{2k-1}\exp(-x)\right]_0^{\infty} + \int_0^{\infty} (2k-1)x^{2k-2}\exp(-x)\, dx\right) \\
&= 2(2k)(2k-1)\frac{(2k-2)!}{2} \\
&= (2k)!.
\end{aligned}$$

3. Recalling that $\mathbb{E}(|\xi|) = \infty$, we can use Corollary 2.4.4 to conclude that all higher-order moments are also infinite.

■

3 More on Random Variables

3.1 Transformation of Random Variables

Let us consider the problem of determining the distribution function of a random variable which is the function of other random variables. That is, let ξ be a random variable with distribution function $F_\xi(x)$ (and density $f_\xi(x)$, if it exists), and let $\varphi = \varphi(x)$ be a Borel function such that $\eta = \varphi(\xi)$. Then we would like to determine the distribution function of η. Proceeding directly we get that

$$F_\eta(y) = \mathbb{P}(\eta \le y) = \mathbb{P}(\xi \in \varphi^{-1}(-\infty, y]) = \int_{\varphi^{-1}(-\infty,y]} \mathrm{d}F_\xi, \tag{3.1}$$

which expresses the distribution function $F_\eta(y)$ in terms of $F_\xi(x)$ and φ.

Example 3.1.1

1. Location-scale family. Let $\eta = a\xi + b$ with $a > 0$. Then
$$F_\eta(y) = \mathbb{P}(\eta \le y) = \mathbb{P}\left(\xi \le \frac{y-b}{a}\right) = F_\xi\left(\frac{y-b}{a}\right).$$

2. χ^2_1 distribution. Let $\eta = \xi^2$. Then it is evident that $F_\eta(y) = 0$ if $y < 0$. While for $y \ge 0$, we have
$$\begin{aligned} F_\eta(y) = \mathbb{P}(\xi^2 \le y) &= \mathbb{P}\left(-\sqrt{y} \le \xi \le \sqrt{y}\right) \\ &= \mathbb{P}_\xi\left(-\infty, \sqrt{y}\right] - \mathbb{P}_\xi\left(-\infty, -\sqrt{y}\right) \\ &= F_\xi(\sqrt{y}) - F_\xi\left(-\sqrt{y}\right) + \mathbb{P}\left(\xi = -\sqrt{y}\right). \end{aligned}$$

Proposition 3.1.2 — Probability Integral Transform. Let ξ be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ with distribution function $F_\xi(x)$. Let U be a random variable on $([0,1], \mathscr{B}([0,1]), \mathsf{Leb})$ (Leb being the Lebesgue measure) such that it is uniformly distributed on $[0,1]$, that is $\mathbb{P}_U = \mathsf{Leb}$. Define the right inverse of F_ξ on $[0,1]$ as

$$F_\xi^{-1}(y) = \sup\left(\{x : F_\xi(x) < y\}\right). \tag{3.2}$$

and extend it so that $F_\xi(0) = -\infty$ and $F_\xi(1) = \infty$. Then ξ has the same distribution function as $F_\xi^{-1}(U)$. In such a case we say that $F_\xi^{-1}(U)$ is *equally distributed* as ξ, or $\xi \overset{d}{=} F_\xi^{-1}(U)$.

Hint. *Let us try to gain a better understanding of the right inverse.*

- *Begin by verifying that if F_ξ is strictly increasing (so that $F_\xi(x)$ is a continuous bijection from $(-\infty, \infty)$ to $(0,1)$), then the right inverse of F_ξ agrees with the inverse F_ξ^{-1}.*
  - *For this special case, we really have*
  $$F_{F_\xi^{-1}}(y) = \mathsf{Leb}\left(\left\{F_\xi^{-1}(U) \le y\right\}\right) = \mathsf{Leb}\left(\{U \le F_\xi(y)\}\right) = F_\xi(y). \tag{3.3}$$
  The actual proof won't differ too much.
- *One can prove a simpler version of this theorem, that the random variable $F_\xi(\xi)$ is equally distributed as U.*

Proof. The only real point to justify is the second equality of the above hint, that is $F_\xi^{-1}(u) \le y$ if and only if $u \le F_\xi(y)$, for general distribution functions F_ξ.
($\Leftarrow$) **Assume $u \le F_\xi(y)$.** Then clearly whenever $F_\xi(x) < u$ we have $F_\xi(x) < F_\xi(y)$ so that $x \le y$. Therefore, y is an upper bound of the set $\{x : F_\xi(x) < u\}$.

($\Rightarrow$) **Assume $F_\xi^{-1}(u) \le y$ but for contradiction that $u > F_\xi(y)$.** Note in such case we have $y \in \{x : F_\xi(x) < u\}$, so y is indeed the *maximum* of $\{x : F_\xi(x) < u\}$. In other words, for any $x > y$ we have $F_\xi(x) \ge u$. Consider an arbitrary monotonic decreasing sequence (x_n) with $x_n > y$ that converges to y. We then know that $\lim_{n\to\infty} F_\xi(x_n) = F(y)$ by right continuity, and that $F_\xi(y) \ge u$. This contradicts our assumption, so we must have $u \le F_\xi(y)$. ∎

Example 3.1.3 Let F be the distribution function for η which has a uniform distribution on $[0, 1/3] \cup [2/3, 1]$ with an atom at $2/3$.

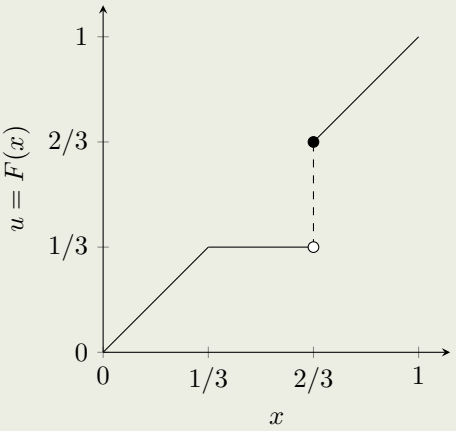


Figure 5: An example of a distribution function.

The inverse $F^{-1}(u)$ is given in Figure 6.

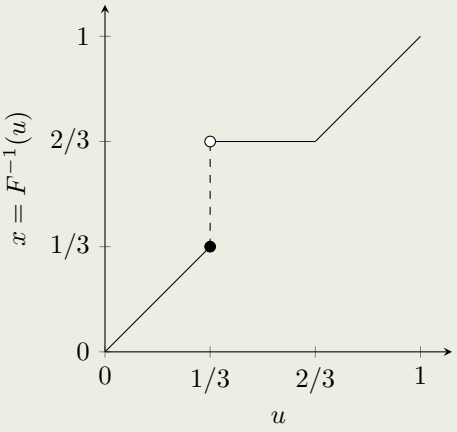


Figure 6: The pseudo-inverse of the previous example.

Now we turn to the problem of determining $f_\eta(y)$. Let us suppose that the range of ξ is a (finite or infinite) open interval $I = (a, b)$. Moreover, suppose that $\varphi = \varphi(x)$, with domain (a, b), is a strictly increasing or decreasing continuously differentiable function. We also suppose that $\varphi'(x) \neq 0$ for $x \in I$ so that we can write $h(y) = \varphi^{-1}(y)$. For definiteness suppose that φ is strictly increasing so that for $y \in \varphi(I)$ it follows that,

$$\begin{aligned} F_\eta(y) = \mathbb{P}(\eta \leq y) = \mathbb{P}(\varphi(\xi) \leq y) &= \mathbb{P}(\xi \leq \varphi^{-1}(y)) \\ &= \mathbb{P}(\xi \leq h(y)) \\ &= \int_{-\infty}^{h(y)} f_\xi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{y} f_\xi(h(z))h'(z)\,\mathrm{d}z. \end{aligned}$$

Therefore,

$$f_\eta(y) = f_\xi(h(y))h'(y).$$

On the other hand, if $\varphi(x)$ is strictly decreasing, then

$$f_\eta(y) = f_\xi(h(y))(-h'(y)).$$

In either case

$$f_\eta(y) = f_\xi(h(y))|h'(y)|.$$

Example 3.1.4 If $\eta = a\xi + b$ for $a \neq 0$, we have

$$h(y) = \frac{y-b}{a} \quad \text{and} \quad f_\eta(y) = \frac{1}{|a|} f_\xi\left(\frac{y-b}{a}\right).$$

If $\varphi = \varphi(x)$ is neither strictly increasing nor strictly decreasing, the above formula is not applicable. However, the following generalisation suffices for many applications.

Lemma 3.1.5 Let $\varphi = \varphi(x)$, defined on the set $\sum_{k=1}^n [a_k, b_k]$, be continuously differentiable and either strictly increasing or strictly decreasing on each open interval $I_k = (a_k, b_k)$, with $\varphi'(x) \neq 0$ for $x \in I_k$. Let $h_k = h_k(y)$ be the inverse of $\varphi(x)$ for $x \in I_k$. Then

$$f_\eta(y) = \sum_{k=1}^n f_\xi(h_k(y))|h_k'(y)| \cdot I_{D_k}(y),$$

where D_k is the domain of $h_k(y)$.

Example 3.1.6 Let $\eta = \xi^2$ with $I_1 = (-\infty, 0)$ and $I_2 = (0, \infty)$. Observe that $h_1(y) = -\sqrt{y}$ and $h_2(y) = \sqrt{y}$, so that

$$f_\eta(y) = \begin{cases} \frac{1}{2\sqrt{y}}[f_\xi(\sqrt{y}) + f_\xi(-\sqrt{y})] & y > 0 \\ 0 & y \leq 0. \end{cases}$$

If $\xi \sim N(0,1)$, then

$$f_{\xi^2}(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2} & y > 0 \\ 0 & y \leq 0. \end{cases}$$

Similar calculations show that

$$f_{|\xi|}(y) = \begin{cases} f_\xi(y) + f_\xi(-y) & y > 0 \\ 0 & y \leq 0, \end{cases}$$

and

$$f_{\sqrt{|\xi|}}(y) = \begin{cases} 2y(f_\xi(y^2) + f_\xi(-y^2)) & y > 0 \\ 0 & y \leq 0. \end{cases}$$

3.2 Independent Random Variables

Definition 3.2.1 — (Mutual) Independence. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a measure space.

- A finite collection of events $\{A_1, ..., A_n\}$ is independent if $\mathbb{P}(\cap_{i=1}^n A_i) = \prod_{i=1}^n \mathbb{P}(A_i)$. An infinite collection $\{A_1, A_2, \ldots\}$ is (mutually) independent if any finite sub-collection of events is independent.
- A finite collection of sub-σ-algebras $\{\mathscr{F}_1, \ldots, \mathscr{F}_n\}$ of $\mathscr{F}$ is (mutually) independent if for any $A_1 \in \mathscr{F}_1, \ldots, A_n \in \mathscr{F}_n$, we have $\mathbb{P}(\cap_{i=1}^n A_i) = \prod_{i=1}^n \mathbb{P}(A_i)$. An infinite collection $\{\mathscr{F}_1, \mathscr{F}_2, \ldots\}$ of sub-σ-algebras of $\mathscr{F}$ is (mutually) independent if any finite sub-collection is independent.
- A finite collection $\{\xi_1, \ldots, \xi_n\}$ of random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ is (mutually) independent if the collection of corresponding sub-σ-algebras $\{\sigma(\xi_1), \ldots, \sigma(\xi_n)\}$ is independent. In particular if $B_1, \ldots, B_n \in \mathscr{B}(\mathbb{R})$, then

$$\mathbb{P}(\xi_1 \in B_1, \ldots, \xi_n \in B_n) = \prod_{i=1}^n \mathbb{P}(\xi_i \in B_i) = \mathbb{P}_{\xi_i}(B_i).$$

An infinite collection $\{\xi_1, \xi_2, \ldots\}$ of random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ is (mutually) independent if the corresponding collection $\{\sigma(\xi_1), \sigma(\xi_2), \ldots\}$ of sub-σ-algebras is mutually independent.

Remark 3.2.2 Another notion of independence says that the collection of events $\{A_1, \ldots, A_n\}$, is *pairwise* independent if for all i, j with $i \neq j$ we have $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$. It is clear that mutual independence implies pairwise independence but not vice-versa. Pairwise independence can be analogously defined for collections of sub-σ-algebra or random variables. Mutual independence is far more applicable than pairwise independence in probability theory.

To establish the mutual independence of sub-σ-algebras, and hence random variables is made easier with the following.

Proposition 3.2.3 A necessary and sufficient condition for the random variables $\xi_1, \ldots, \xi_n$ to be independent is that

$$F_\xi(x_1, x_2, \ldots, x_n) = F_{\xi_1}(x_1) \ldots F_{\xi_n}(x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Corollary 3.2.4 If $\xi = (\xi_1, \ldots, \xi_n)$ has a density f_ξ, then each ξ_i has a density f_{ξ_i}. Furthermore, $\xi_1, \ldots, \xi_n$ are independent if and only if

$$f_\xi(x_1, x_2, \ldots, x_n) = f_{\xi_1}(x_1) \cdots f_{\xi_n}(x_n)$$

for all $(x_1, \ldots, x_n)$ except possibly for a Borel subset of $\mathbb{R}^n$ with Lebesgue measure zero.

Corollary 3.2.5 If $\xi_1, \ldots, \xi_n$ are independent and ξ_i has density f_{ξ_i}, for $i = 1, \ldots, n$, then ξ has a density f_ξ given by

$$f_\xi(x_1, x_2, \ldots, x_n) = f_{\xi_1}(x_1) \cdots f_{\xi_n}(x_n).$$

Remark 3.2.6 Even if $\xi_1, \ldots, \xi_n$ each have a density, it does not follow that $(\xi_1, \ldots, \xi_n)$ has a density.

Let ξ and η be independent random variables, so that $F_{(\xi,\eta)}(x, y) = F_\xi(x)F_\eta(y)$.For the random variable $\xi + \eta$. We get that

$$\begin{aligned} F_{\xi+\eta}(z) &= \int_{\{x,y:x+y\leq z\}} \mathrm{d}F_\xi(x) \cdot \mathrm{d}F_\eta(y) \\ &= \int_{\mathbb{R}^2} \chi_{x+y\leq z}\, \mathrm{d}F_\xi(x) \cdot \mathrm{d}F_\eta(y) \\ &= \int_{-\infty}^{\infty} \mathrm{d}F_\xi(x) \left\{ \int_{-\infty}^{\infty} \chi_{x+y\leq z}\, \mathrm{d}F_\eta(y) \right\} \\ &= \int_{-\infty}^{\infty} F_\eta(z-x)\, \mathrm{d}F_\xi(x). \end{aligned}$$

As this argument is symmetric we also get that

$$F_{\xi+\eta}(z) = \int_{-\infty}^{\infty} F_\xi(z-y)\mathrm{d}F_\eta(y).$$

Proposition 3.2.7 The distribution function $F_{\xi+\eta}$ of the sum of independent random variables is given by the convolution of their distribution functions. That is,

$$F_{(\xi,\eta)}(z) = F_\xi * F_\eta = \int_{-\infty}^{\infty} F_\xi(z-y)\mathrm{d}F_\eta(y) = \int_{-\infty}^{\infty} F_\eta(z-x)\mathrm{d}F_\xi(x).$$

Corollary 3.2.8 If ξ and η are independent absolutely continuous random variables, then the density

$f_{\xi+\eta}$ is given by the convolution of the densities,

$$f_{\xi+\eta} = f_\xi * f_\eta = \int_{-\infty}^{\infty} f_\xi(z-y) f_\eta(y) \mathrm{d}y = \int_{-\infty}^{\infty} f_\eta(z-x) f_\xi(x) \mathrm{d}x.$$

Example 3.2.9 — ξ, η - independent absolutely continuous random variables.

- Let $\xi \sim N(m_1, \sigma_1^2)$ and $\eta \sim N(m_2, \sigma_2^2)$, so that

  $$f_\xi(x) = \frac{1}{\sigma_1}\varphi\left(\frac{x-m_1}{\sigma_1}\right), \quad f_\eta(x) = \frac{1}{\sigma_2}\varphi\left(\frac{x-m_2}{\sigma_2}\right),$$

  where

  $$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

  Then

  $$f_{\xi+\eta}(z) = \int_{-\infty}^{\infty} f_\eta(z-x) f_\xi(x) \mathrm{d}x = \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}}\varphi\left(\frac{z-(m_1+m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right).$$

  Thus the sum of independent normal random variables is the normal random variable $N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

- Let $\xi_1, \dots \xi_n$ be independent $N(0,1)$. Then

  $$f_{\xi_1^2+\dots+\xi_n^2}(x) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} x^{(n/2)-1} e^{-x/2} & x > 0, \\ 0 & x \le 0. \end{cases}$$

  The random variable $\xi_1^2 + \dots + \xi_n^2$ is usually denoted by χ_n^2 and its distribution is the χ^2-distribution with n degrees of freedom.

Proposition 3.2.10 Let ξ and η be independent random variables with $\mathbb{E}(\xi) < \infty$ and $\mathbb{E}(\eta) < \infty$. Then $\mathbb{E}(\xi\eta) < \infty$ with $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$.

Proof. We utilise a four-step proof. First, assume that ξ and are non-negative $\eta \ge 0$ and define

- $\xi_n = \sum_{k=0}^{\infty} \frac{k}{n} \chi_{\{\frac{k}{n} \le \xi(\omega) < \frac{k+1}{n}\}}$, and
- $\eta_n = \sum_{k=0}^{\infty} \frac{k}{n} \chi_{\{\frac{k}{n} \le \eta(\omega) < \frac{k+1}{n}\}}$.

It follows that $\xi_n \le \xi$ and $\eta_n \le \eta$ with $|\xi - \xi_n| \le \frac{1}{n}$ and $|\eta - \eta_n| \le \frac{1}{n}$ for all n. Since ξ and η are integrable we can apply the dominated convergence theorem to deduce that

- $\lim_{n\to\infty} \mathbb{E}(\xi_n) = \mathbb{E}(\xi)$, and
- $\lim_{n\to\infty} \mathbb{E}(\eta_n) = \mathbb{E}(\eta)$.

Hence,

$$\begin{aligned} \mathbb{E}(\xi_n\eta_n) &\overset{(1)}{=} \sum_{i,j\ge 0} \frac{jk}{n^2} \mathbb{E}\left(\chi_{\{\frac{j}{n} \le \xi < \frac{j+1}{n}\}} \chi_{\{\frac{k}{n} \le \eta < \frac{k+1}{n}\}}\right) \\ &\overset{(2)}{=} \sum_{i,j\ge 0} \frac{jk}{n^2} \mathbb{E}\left(\chi_{\{\frac{j}{n} \le \xi < \frac{j+1}{n}\}}\right) \mathbb{E}\left(\chi_{\{\frac{k}{n} \le \eta < \frac{k+1}{n}\}}\right) \\ &= \mathbb{E}(\xi_n)\mathbb{E}(\eta_n), \end{aligned}$$

where (1) is an application of the monotone convergence theorem, and (2) follows from independence. Moreover,

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi_n\eta_n)| \le \mathbb{E}(|\xi\eta - \xi_n\eta_n|)$$

$$
\begin{aligned}
&= \mathbb{E}(|\xi(\eta-\eta_n)+\eta_n(\xi-\xi_n)|)\\
&\le \frac{1}{n}\mathbb{E}(|\xi|)+\frac{1}{n}\mathbb{E}\left(|\eta|+\frac{1}{n}\right)\\
&\to 0
\end{aligned}
$$

as $n\to\infty$. Therefore,

$$\mathbb{E}(\xi\eta)=\lim_{n\to\infty}\mathbb{E}(\xi_n\eta_n)=\lim_{n\to\infty}\mathbb{E}(\xi_n)\lim_{n\to\infty}\mathbb{E}(\eta_n)=\mathbb{E}(\xi)\mathbb{E}(\eta),$$

and $\mathbb{E}(\xi\eta)<\infty$. The general case follows by using the representations

- $\xi=\xi^+-\xi^-$, and
- $\eta=\eta^+-\eta^-$.

■

A converse is this result is given by the following.

Proposition 3.2.11 Integrable random variables ξ and η are independent if and only if for all Borel-measurable functions f and g we have $\mathbb{E}(f(\xi)g(\eta))=\mathbb{E}(f(\xi))\mathbb{E}(g(\eta))$.

Hint. *For $(\Leftarrow)$ we directly apply the assumption for $f=\chi_{A_1}$ and $g=\chi_{A_2}$, with $A_1=\xi_1^{-1}(B_1)$ for an arbitrary $B_1\in\mathscr{B}(\mathbb{R})$ and similarly for A_2. For $(\Rightarrow)$ we use a four-step proof similar to the above.*

3.3 Correlation

Here we quantify how unrelated two random variables are.

Definition 3.3.1 — Covariance. Let ξ and η be random variables defined on the same probability space. Provided that their expectations exist, their **covariance** is

$$\mathsf{Cov}[\xi,\eta]:=\mathbb{E}((\xi-\mathbb{E}(\xi))(\eta-\mathbb{E}(\eta))).$$

Remark 3.3.2 Note that

$$\mathbb{V}(\xi+\eta)=\mathbb{V}(\xi)+\mathbb{V}(\eta)+2\mathsf{Cov}(\xi,\eta).$$

Hence, $\mathsf{Cov}(\xi,\eta)=0$ implies that $\mathbb{V}(\xi+\eta)=\mathbb{V}(\xi)+\mathbb{V}(\eta)$.

Definition 3.3.3 — Uncorrelated variables. Random variables ξ and η are called **uncorrelated** if

$$\mathsf{Cov}(\xi,\eta)=0.$$

Corollary 3.3.4 Independent random variables are uncorrelated.

Proof. Using Proposition 3.2.10 we deduce that

$$\mathsf{Cov}(\xi,\eta)=\mathbb{E}(\xi\eta)-\mathbb{E}(\xi)\mathbb{E}(\eta)=0.$$

■

The converse is not true.

Example 3.3.5 Consider a random variable α which takes the values $\left\{0,\frac{\pi}{2},\pi\right\}$ uniformly. Then $\xi=\sin\alpha$ and $\eta=\cos\alpha$ are uncorrelated as $\mathbb{E}(\xi)=\frac{1}{3},\mathbb{E}(\eta)=0$. However, they are not independent since

$$\mathbb{P}(\xi=1,\eta=1)=0\neq\frac{1}{9}=\mathbb{P}(\xi=1)\mathbb{P}(\eta=1).$$

Moreover, the random variables ξ and η^2 are correlated, since $\mathbb{E}\left(\eta^2\right)=\frac{2}{3}$ and $\mathbb{E}(\xi)\mathbb{E}(\eta+1)=\frac{2}{9}$, but

$\mathsf{Cov}\left(\xi, \eta^2\right) = -\frac{2}{9}$.

Part II. Concepts of Convergence

4 Convergence in Probability

We now have sufficient tools from measure theory to get into the first serious topic of probability, limiting theorems. Given a random sequence $(\xi_1, \xi_2, \dots)$ with ξ_i *independently and identically distributed* (i.i.d.), we would like to study the deviation between the empirical mean S_n/n, where $S_n = \xi_1 + \cdots + \xi_n$, and the expectation $\mathbb{E}(\xi_1)$ as $n \to \infty$.

4.1 Definition and Properties

We have already encountered one form of convergence, namely L^p convergence. Here we introduce an alternative notion of convergence, known as convergence in probability.

Definition 4.1.1 A sequence $\xi_1, \xi_2, \dots$ of random variables from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ to $\mathbb{R}$ converges in probability, or in measure $\mathbb{P}$, to the random variable ξ, denoted by $\xi_n \xrightarrow{p} \xi$, if for every $\varepsilon > 0$ we have

$$\mathbb{P}(|\xi_n - \xi| > \varepsilon) \to 0,$$

as $n \to \infty$.

Proposition 4.1.2 Let $\xi, \xi_1, \xi_2, \dots$ be random variables on $(\Omega, \mathscr{F}, \mathbb{P})$. If $\xi_n \xrightarrow{L^p} \xi$, for $p \geq 1$, it follows that $\xi_n \xrightarrow{p} \xi$.

Proof. For the random variable ξ_n we have that $\eta_n = |\zeta_n - \zeta|^p$ is a non-negative random variable. So for any $\epsilon > 0$ by Markov's inequality, it follows that

$$\mathbb{P}\left(|\zeta_n - \zeta| \geq \epsilon\right) = \mathbb{P}\left(|\zeta_n - \zeta|^p \geq \epsilon^p\right) =\leq \frac{\mathbb{E}\left(|\zeta_n - \zeta|^p\right)}{\epsilon^p}.$$

Taking $n \to \infty$ the right-hand tends to zero by assumption, so we conclude that $\zeta_n \xrightarrow{P} \zeta$. ∎

Example 4.1.3 The converse of Proposition 4.1.2 is not true. Consider $\xi_n = n\chi_{\left(0, \frac{1}{n}\right]}(\omega)$. Then

$$\mathbb{P}(|\xi_n| > \epsilon) = \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

Therefore, $\xi_n \xrightarrow{P} 0$. However,

$$\mathbb{E}\left(|\xi_n|^p\right) = n^{p-1} \not\xrightarrow{n\to\infty} 0.$$

Therefore, $\xi_n \not\xrightarrow{L^p} 0$.

Exercise 4.1.4 — Properties of convergence in probability. Let $(\xi_i)_{i\geq 1}$ and $(\eta_i)_{i\geq 1}$ be sequences of random variables on $(\Omega, \mathscr{F}, \mathbb{P})$, and let ξ, η be random variables on $(\Omega, \mathscr{F}, \mathbb{P})$.

1. Check that the limit of convergence in probability is almost surely unique. That is, if ξ_i converges in probability to ξ and ξ' then $\xi = \xi'$ almost surely.
2. Prove that if $\xi_i \xrightarrow{p} \xi$ and $\eta_i \xrightarrow{p} \eta$ then for all real numbers a, b we have $a\xi_i + b\eta_i \xrightarrow{p} a\xi + b\eta$.
3. Prove that if $\xi_i \xrightarrow{p} \xi$ and $\eta_i \xrightarrow{p} \eta$ then $\xi_i\eta_i \xrightarrow{p} \xi\eta$. Notice this is not necessarily true if we have L^p convergence. What's wrong with the argument, and can we refine the statements?

4. Show that if $\xi_i \xrightarrow{p} \xi$, $\eta_i \xrightarrow{p} \eta$ and $\varphi(x, y)$ is a continuous function, then

$$\varphi(\xi_i, \eta_i) \xrightarrow{p} \varphi(\xi, \eta).$$

4.2 Coin Flipping Example

We can motivate the notion of convergence in probability by considering flipping n independent coins. Firstly, consider the flipping of just one coin. Assume the outcomes are 0 and 1 with the probability of getting 1 being $p \in (0, 1)$. We hope to express this as a $\{0, 1\}$-valued random variable ξ on a suitable probability space $(\Omega, \mathscr{A}, \mathbb{P})$, such that $\mathbb{P}_\xi(\{0\}) = 1 - p$ and $\mathbb{P}_\xi(\{1\}) = p$. A more succinct way to write the above condition is

$$\mathbb{P}_\xi(\{x\}) = p^x(1 - p)^{1-x}, \tag{4.1}$$

for $x = 0, 1$. For the probability space, there are several choices.

- The natural choice would be
  - $\Omega = \{0, 1\}$,
  - $\mathscr{A} = 2^\Omega$, and
  - $\mathbb{P}(\{\omega\}) = p^\omega(1 - p)^{1-\omega}$.

  In this case, our desired random variable would be $\eta(\omega) = \omega$.
- A more complicated choice would be
  - $\Omega = [0, 1]$,
  - $\mathscr{A} = \mathscr{B}([0, 1])$, and
  - $\mathbb{P}(E) = \mathsf{Leb}(E)$.

  In this case, our desired random variable would be $\xi(\omega) = \chi_{(p,1]}(\omega)$.

Remark 4.2.1 The more complicated formulation represents how a computer simulates the flipping of a biased coin. First, it generates a random number $r \in [0, 1]$ from a uniform distribution (for instance, by using the `numpy.random.rand` function in Python), then returns 0 if $r < 1 - p$ and 1 otherwise.

In both cases we see that $\mathbb{P}_\xi(\{0\}) = 1 - p$ and $\mathbb{P}_\xi(\{1\}) = p$. In fact, from Theorem 2.2.11 we know that the distribution functions, and therefore the expectation, will not depend on our choice of probability spaces and random variables, as long as $\mathbb{P}_\xi$ satisfies (4.1).

How can we extend the experiment to the flipping of n coins? It would be wrong to define $\xi_1, ..., \xi_n$ on the same sample space $(\Omega, \mathscr{A}, \mathbb{P})$ such that $\xi_1 = ... = \xi_n$. This would correspond to the flipping of a single coin once and recording the result n times. It is clear that in such a construction the random variables wouldn't be independent. In fact, it will be hard to write down a large number of independent random variables defined on any of the sample spaces $(\Omega, \mathscr{A}, \mathbb{P})$ in the above example.

A standard way of describing n independent coin flips (or n independent trials in general) is to assume that the random variables $\xi_1, ..., \xi_n$ lie in different probability spaces. That is, ξ_1 is defined on $(\Omega_1, \mathscr{A}_1, \mathbb{P}_1)$, ξ_2 is defined on $(\Omega_2, \mathscr{A}_2, \mathbb{P}_2)$ and so on. Proceeding in this way requires us to operate in these different spaces simultaneously. Thus we need to consider the product space $\left(\Omega^{(n)}, \mathscr{A}^{(n)}, \mathbb{P}^{(n)}\right) = \otimes_{i=1}^n (\Omega_i, \mathscr{A}_n, \mathbb{P}_n)$. The sample space of this product space is

$$\Omega^{(n)} = \Omega_1 \times ... \times \Omega_n = \{\omega = (\omega_1, ..., \omega_n) \, : \omega_i \in \Omega_i, \text{ for } i = 1, \ldots, n\},$$

the collection of events are

$$\mathscr{A}^{(n)} = \sigma(\{A_1 \times ... \times A_n \, : A_i \in \mathscr{A}_i, \text{ for } i = 1, \ldots, n\}),$$

and the probability measure $\mathbb{P}^{(n)}$ satisfies

$$\mathbb{P}^{(n)}(A_1 \times ... \times A_n) = \prod_{i=1}^{n} \mathbb{P}_i(A_i).$$

Consequently, we define the family of projection functions onto the i^{th} component, $\text{proj}_i^{(n)} : \left(\Omega^{(n)}, \mathscr{A}^{(n)}\right) \to (\Omega_i, \mathscr{A}_i)$ such that $\text{proj}_i^{(n)}(\omega_1, ..., \omega_n) = \omega_i$. For convenience, we drop the superscript (n) if there is no ambiguity. Notice that the projection functions are measurable since the preimage of any sets in $\mathscr{A}_i$ is

$$\text{proj}_i^{-1}(A_i) = \Omega_1 \times ... \times \Omega_{i-1} \times A_i \times \Omega_{i+1}, ..., \Omega_n \in \mathscr{A}^{(n)}.$$

We can define the random variables $\tilde{\xi}_i : \left(\Omega^{(n)}, \mathscr{A}^{(n)}, \mathbb{P}^{(n)}\right) \to \{0, 1\}$ such that $\tilde{\xi}_i(\omega) = \xi_i(\text{proj}_i(\omega))$. These random variables are an accurate description of flipping n coins.

1. The marginal distribution of ξ_i, defined as the measure $A \mapsto \mathbb{P}_{\tilde{\xi}_i}(\Omega_1 \times ... \times A \times ... \times \Omega_n)$ satisfies (4.1).
2. The family $\left(\tilde{\xi}_i\right)$ of random variables are independent.

Exercise 4.2.2 Verify the above assertions.

Next, we want to extend the above construction to $n \to \infty$. More specifically, we want to consider the space $(\Omega, \mathscr{A}) = \otimes_{i=1}^{\infty}(\Omega_i, \mathscr{A}_i)$ with a suitable probability measure $\mathbb{P}$, so that we can discuss large-sample theorems. It should be the case that this probability measure satisfies

$$\mathbb{P}\left(A_1 \times ... \times A_n \times \Omega_{n+1} \times ...\right) = \prod_{i=1}^{n} \mathbb{P}_i(A_i). \tag{4.2}$$

If we use $\Omega_i \equiv \Omega$ and $\mathscr{A}_i \equiv \mathscr{A}$ in our above examples and assume the natural choice, then we can safely set

$$\mathbb{P}(\{(\omega_1, \omega_2, ...)\}) = \prod_{i=1}^{\infty} p^{\omega_i}(1-p)^{1-\omega_i} = p^{\sum \omega_i}(1-p)^{\sum(1-\omega_i)},$$

since the probability measure is well-defined for all singletons $\{(\omega_1, \omega_2, ...)\}$. If we use the example when $\Omega_i = [0, 1]$, we can check that our sequence of measures $(\mathbb{P}^{(n)})$ is consistent (see Definition 1.6.1) and we can apply the Kolmogorov Extension Theorem (Theorem 1.6.2) to define $\mathbb{P}$.
The above shows that we need not worry about specifying a single probability space to describe a sequence of independent experiments. **If we want to describe an infinite sequence of experiments with an underlying distribution $\mathbb{P}_\xi$, we can consider the infinite product space $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ equipped with the probability measure $\mathbb{P}$ as determined by the Kolmogorov extension theorem. The projections onto the i^{th} component proj_i are then random variables with distribution $\mathbb{P}_\xi$.** From now on, we abuse notation by not mentioning the underlying probability space, dropping the tilde sign above ξ and interpreting any operations in the above sense.

4.3 Bernoulli's Law of Large numbers

With ξ_k a $\{0, 1\}$-valued random variable, taking value 1 with probability $p \in (0, 1)$, let $S_n = \xi_1 + \cdots + \xi_n$. Then

$$\mathbb{E}(S_n) = \sum_{j=1}^{n} \mathbb{E}(\xi_n) = \sum_{j=1}^{n} (1 \cdot \mathbb{P}_{\xi_j}(\xi_j = 1) + 0 \cdot \mathbb{P}_{\xi_j}(\xi_j = 0)) = np.$$

Thus the mean value of S_n/n is equal to p. The question now is what does $|\frac{1}{n}S_n(\omega) - p|$ converge to for large n? Moreover, in what sense does this convergence occur? It cannot be that

$$\left|\frac{S_n(\omega)}{n} - p\right| \to 0$$

uniformly/pointwise in ω, because there is always an ω such that $\xi_i(\omega) = 1$ for all i, so $S_n(\omega)/n \equiv 1 \nrightarrow p$. Therefore, we must consider a weaker notion of convergence, as illustrated in the following exercise.

Exercise 4.3.1 Verify that $S_n \sim \mathrm{B}(n,p)$. Hence, show that $\|S_n/n - p\|_{L^2}^2 \to 0$ as $n \to \infty$.

With a more general analysis, we can show that we have L^p convergence for any $p \in [1,\infty)$. Moreover, by Chebyshev's inequality, it follows for any fixed $\epsilon > 0$ that

$$\mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \epsilon\right) \leq \frac{\mathbb{V}\left(\frac{S_n}{n}\right)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2}. \tag{4.3}$$

Hence, we can always make the probability $\mathbb{P}\left(|S_n/n - p| > \epsilon\right)$ arbitrarily small. That is, we have convergence in probability.
Stating the above formally gives us the Bernoulli Law of Large numbers.

Theorem 4.3.2 — Bernoulli Law of Large Numbers. Let ξ_1, ξ_2 be a sequence of independent and identically distributed Bernoulli random variables, with parameter $p \in (0,1)$. Then $\frac{S_n}{n}$ converges in probability to p.

4.4 Weak Law of Large Numbers

We can generalise the ideas of the previous section. Note, from Markov's inequality, if $\xi_n \to \xi$ in L^p ($p \geq 1$, see Definition 2.4.8), then we must have $\xi_n \xrightarrow{p} \xi$, because

$$\mathbb{P}(|\xi_n - \xi| > \varepsilon) \leq \frac{\|\xi_n - \xi\|_{L^p}^p}{\varepsilon^p} \to 0.$$

Let $\xi_1, \ldots, \xi_n$ be random variables and let

$$S_n^{(c)} = \sum_{j=1}^{n} (\xi_j - \mathbb{E}(\xi_j)).$$

Observe that $\mathbb{E}\left(S_n^{(c)}\right) = 0$. How can we use Chebyshev's inequality to make the weakest assumptions on the $\xi_1, ..., \xi_n$ such that $S_n^{(c)}$ converges? For simplicity, we can first assume that the $\xi_i \in L^2$ with $\mathbb{V}(\xi_i) \leq C$ for some constant C independent of i. Then, if we assume that the $\xi_1, ..., \xi_n$ are pairwise uncorrelated (which is a much weaker assumption than independence), then we have

$$\mathbb{V}(S_n) = \sum_{i=1}^{n} \mathbb{V}(\xi_i) \leq Cn$$

and hence we have the following.

Theorem 4.4.1 — L^2 Weak Law of Large Numbers. Let $\xi_1, \ldots, \xi_n$ be uncorrelated L^2 random variables such that $\mathbb{V}(\xi_j) \leq C$ for some $C > 0$ and all $n \geq 1$. Then

$$\frac{S_n^{(c)}}{n} \xrightarrow{p} 0.$$

Proof. Using Chebyshev's inequality and the fact the ξ_j are uncorrelated, it follows that for all $\epsilon > 0$ we have

$$\mathbb{P}\left(\left|\frac{S_n^{(c)}}{n}\right| > \varepsilon\right) \leq \frac{\mathbb{V}\left(\frac{S_n^{(c)}}{n}\right)}{\varepsilon^2} \leq \frac{C}{n\varepsilon^2} \xrightarrow{n\to\infty} 0.$$

■

Corollary 4.4.2 Let $\xi_1, \xi_2, \ldots$ be integrable, independent and identically distributed random variables, such that $\mathbb{V}[\xi_j] < \infty$. Then

$$\frac{S_n}{n} \xrightarrow{p} \mathbb{E}(\xi_1).$$

There are many applications of the L^2 weak law of large numbers. Some of which we will explore now.

Example 4.4.3 — Monte Carlo Integration. Consider f a measurable function on $[0,1]$ with $C := \int_0^1 |f(x)|^2 < \infty$, for example f couple be uniformly bounded by some number M. The integral

$$\theta = \int_0^1 f(x)\,dx,$$

is often intractable to compute in practice. However, if U is a random variable on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with uniform distribution $\mathsf{Unif}[0,1]$, then $\theta = \mathbb{E}(f(U))$. For example, one could take $U : \omega \in ([0,1], \mathscr{B}[0,1], \mathsf{Leb}) \mapsto \omega$. Therefore, an approach to estimating θ would be to use an "empirical mean".

1. Define the random variables $U_1(\omega), U_2(\omega), \ldots \overset{\text{iid}}{\sim} \mathsf{Unif}[0,1]$ using a construction as detailed in Section 4.2.
2. Then evaluate the empirical mean

$$\hat{\theta}_n(\omega) := \frac{f(U_1(\omega)) + \ldots + f(U_n(\omega))}{n}.$$

It is quite clear that $\mathbb{E}\left[\hat{\theta}_n\right] = \theta$. Moreover, since the random variables $f(U_i)$ are independent and identically distributed with finite variance $\mathbb{V}(f(U_i)) = \mathbb{E}\left((f(U_i))^2\right) - \theta^2 = C - \theta^2 < \infty$, the L^2 version of the WLLN tells us that $\hat{\theta}_n(\omega) \overset{n\to\infty}{\longrightarrow} \theta$ in probability.

Exercise 4.4.4 — L^2 Weak Law of Large Number for weakly correlated random variables. Let $\xi_1, \xi_2, \ldots$ be random variables on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{E}(\xi_i) = 0$ and $\mathbb{E}(\xi_i \xi_j) = r_{|i-j|}$, where $(r_k)_{k\geq 1}$ is a sequence of real numbers such that $r_k \overset{k\to\infty}{\longrightarrow} 0$. Let $S_n = \sum_{i=1}^n \xi_i$. Show that $S_n/n \overset{n\to\infty}{\longrightarrow} 0$ in probability.

Hint. *The idea is to expand*

$$\mathbb{V}\left(\frac{\xi_1 + \ldots + \xi_n}{n}\right) = \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E}(\xi_i \xi_j)$$

and to count how many of the terms $\mathbb{E}(\xi_i \xi_j)$ *satisfies* $|i-j| = k$.

If you have done some time series, then you will know from this exercise that any time average of a moving average or autoregressive $\mathsf{AR}(1)$ process has a time average converging to zero.

4.5 Local and Central Limit Theorem

We return to the coin-flipping scenario. Recall that $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, where $\xi_k \overset{\text{iid}}{\sim} \mathsf{B}(1,p)$ as constructed in Section 4.1. As discussed in the previous section, S_n tends to be close to np for large n.

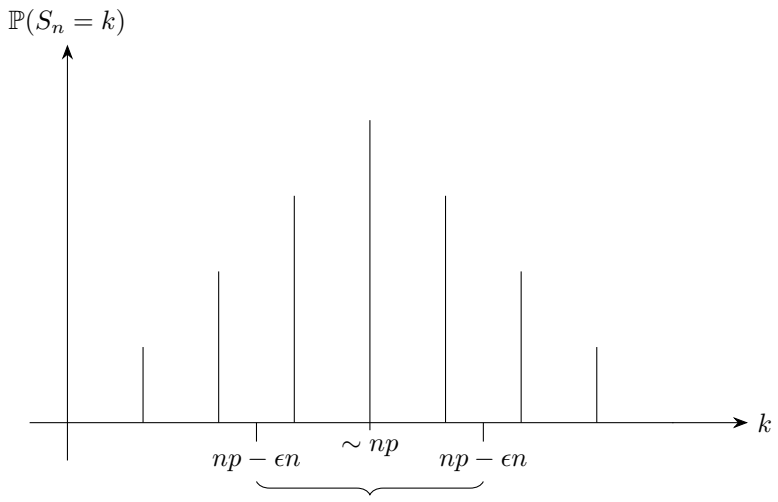


Figure 7: Concentration of measure $S_n^* \mathbb{P}$.

Specifically, let us take some interval $\mathscr{I}_n = (n(p-\varepsilon), n(p+\varepsilon))$. If we pick some sufficiently large ε, say $\varepsilon = n^\alpha$ where $\alpha > \frac{1}{2}$, then by Chebyshev's inequality (4.3) we know that

$$\mathbb{P}(S_n \in \mathscr{I}_n^c) = \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > n^{\alpha-1}\right) \leq \frac{p(1-p)}{n^{1+2\alpha-2}} = \frac{p(1-p)}{n^{2\alpha-1}} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.4}$$

These decaying bounds of $\mathbb{P}(S_n \in \mathscr{I}_n^c)$ no longer exist when $\alpha \leq \frac{1}{2}$. How much do we know about $\mathbb{P}(S_n \in \mathscr{I}_n^c)$ for this case? Will it tend to some non-trivial constant in $(0,1)$, or will it increase and tend to 1? We will see that the central limit theorem suggests that at the boundary $\alpha = 1/2$, the quantity $\mathbb{P}(S_n \in \mathscr{I}_n^c)$ tends to some non-trivial constant in $(0,1)$. This suggests that the rescaled mean $\frac{S_n}{\sqrt{n}}$ will "converge" in some way to a non-trivial distribution.
Before we delve into the main discussions, we define important order notations for a sequence f_n.

Definition 4.5.1 — Order notations. Consider two sequences f_n, g_n..

- Big O notation. We say that $g_n = O(f_n)$ as $n \to \infty$ if $|g_n/f_n|$ is bounded for sufficiently large n. That is, there exists constant $C > 0$ and N such that for all $n \geq N$ we have $|g_n| \leq C|f_n|$.
- Small o notation. We say that $g_n = o(f_n)$ as $n \to \infty$ if $|g_n/f_n| \to 0$ as $n \to \infty$. In other words, for all $\epsilon > 0$, there exists $N := N(\epsilon)$ such that for all $n \geq N$ we have $|g_n| \leq \epsilon |f_n|$. We sometimes write $g_n \ll f_n$ or $f_n \gg g_n$.
- Asymptotic equivalence. We write $g_n \sim f_n$ if $|g_n/f_n| \to 1$ as $n \to \infty$. Equivalently, we have $g_n = (1 + o(1))f_n$.
- Order. We write $g_n = \Theta(f_n) = \mathsf{ord}(f_n)$ if $g_n = O(f_n)$ but g_n is not $o(f_n)$.

Remark 4.5.2

- The use of equal sign is an abuse of notation.
- The definitions can be extended to any functions $f(x)$ defined on real or complex numbers, in such case, we can assume x tends to some point x_0 including ∞.
- We can also consider order notations for sequences of functions. Let $g_n = g_n(\alpha)$, $f_n = f_n(\alpha)$. We say $g_n(\alpha) = O(|f_n(\alpha)|)$ **uniformly** if the above definition holds for constants C and N independent of α. We also have analogous definition for $g_n(\alpha) = o(|f_n(\alpha)|)$.

With this, we can prove one of the most important results in asymptotic analysis.

Lemma 4.5.3 — Stirling's Approximation. As $n \to \infty$, we have

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n)).$$

With this, we can sketch an asymptotic analysis of the binomial coefficient. Let's say $n, k, n-k$ all tend to infinity, for example $k = np$ for $k \in (0,1)$, then we can use Stirling's formula to obtain

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \frac{(n/e)^n}{(k/e)^k((n-k)/e)^{n-k}} \frac{1+O(1/n)}{(1+O(1/n))(1+O(1/n))} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \exp\left(n\ln n - k\ln k - (n-k)\ln(n-k)\right) \frac{1+O(1/n)}{(1+O(1/n))(1+O(1/n))}.\end{aligned}$$

The purple term only gives a correction of $1 + O(1/n)$. Instead of seeing this through careful analysis, we can develop an intuition of why this is true by treating the $O(1/n)$ correction terms as being exactly equal to $1/n$. Then the denominator satisfies $(1+1/n)^{-2} = 1 - 2/n + \ldots = 1 + O(1/n)$. We conclude that $(1+1/n)(1-2/n) = 1 - 1/n + \ldots = 1 + O(1/n)$. Putting this together, we have

$$\binom{n}{k} = \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \exp\left(n\ln n - k\ln k - (n-k)\ln(n-k)\right)\left(1 + O\left(\frac{1}{n}\right)\right).$$

We recall the following result regarding the probability mass function of a binomial distribution $\mathsf{B}(n,p)$.

Exercise 4.5.4 — Monotonicity of Binomial probability. Show that $\mathbb{P}(S_n = k)$ is monotone in k below and above its point of maximum.

With this, we can prove the local limit theorem, which specifies the local asymptotics of a probability mass distribution at the point $S_n = k$.

Theorem 4.5.5 — Local Limit Theorem. For any $0 < p < 1$, we have

$$\max_{0\le k\le n}\left|\mathbb{P}(S_n = k) - \frac{1}{\sqrt{2\pi p(1-p)}\sqrt{n}} e^{-\frac{x^2}{2p(1-p)}}\right| = o\left(\frac{1}{\sqrt{n}}\right),$$

as $n \to \infty$ and where $x = x_{k,n} := \dfrac{k-np}{\sqrt{n}}$.

Proof. The subtlety here is that we cannot always apply Stirling's formula. We have to first consider the k that are "sufficiently close" to np. Specifically, we consider k such that

$$|x_{k,n}| \le \frac{A_n}{\sqrt{n}}$$

where $A_n = n^\epsilon$ for $\epsilon \in (0,1)$. Then we have $k = np + x\sqrt{n}$, so that

$$k = np + x\sqrt{n} = np(1 + O(A_n/n))$$

which implies that

$$n - k = n(1-p) - x\sqrt{n} = n(1-p)(1 + O(A_n/n)).$$

These inequalities ensure that both k and $n-k$ tend to infinity as $n \to \infty$, and we can safely use Stirling's approximation to show that

$$\mathbb{P}(S_n = k) = \underbrace{\frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}}}_{\text{(A)}} \underbrace{\exp\left(n\ln n - k(\ln k - \ln p) - (n-k)(\ln(n-k) - \ln(1-p))\right)}_{\text{(B)}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

We first analyse (A) and notice that

$$\begin{aligned}(\mathsf{A}) &= \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \\ &= \frac{1}{\sqrt{2\pi np(1-p)(1+O(A_n/n))(1+O(A_n/n))}} \\ &= \frac{1}{\sqrt{2\pi np(1-p)}}(1+O(A_n/n)).\end{aligned}$$

The correction factor can be obtained using similar arguments above. We can then analyse (B) by noticing that

$$\begin{aligned}(\mathsf{B}) &= \exp\left(n\ln n - k\left(\ln n + \ln\left(1+\frac{x}{p\sqrt{n}}\right)\right) - (n-k)\left(\ln n + \ln\left(1-\frac{x}{(1-p)\sqrt{n}}\right)\right)\right) \\ &= \exp\left(-\left((np+x\sqrt{n})\ln\left(1+\frac{x}{p\sqrt{n}}\right) + (n(1-p)-x\sqrt{n})\ln\left(1-\frac{x}{(1-p)\sqrt{n}}\right)\right)\right) \\ &= \exp\Bigg(-\Bigg[np\left(\frac{x}{p\sqrt{n}} - \frac{x^2}{2p^2n} + O\left(\frac{x^3}{n^{3/2}}\right)\right) + \frac{x^2}{p} + O\left(\frac{x^3}{n^{1/2}}\right)\Bigg) \\ &\quad + n(1-p)\left(-\frac{x}{(1-p)\sqrt{n}} - \frac{x^2}{2(1-p)^2n} + O\left(\frac{x^3}{n^{3/2}}\right)\right) + \frac{x^2}{(1-p)} + O\left(\frac{x^3}{n^{1/2}}\right)\Bigg]\Bigg) \\ &= \exp\left(-\frac{x^2}{2p(1-p)} + O\left(\frac{x^3}{\sqrt{n}}\right)\right) \\ &= \exp\left(-\frac{x^2}{2p(1-p)} + O\left(\frac{A_n^3}{n^2}\right)\right) \\ &= \exp\left(-\frac{x^2}{2p(1-p)}\right)\left(1+O\left(\frac{A_n^3}{n^2}\right)\right).\end{aligned}$$

Combining these we get that,

$$\mathbb{P}(S_n = k) = \frac{1}{\sqrt{2\pi p(1-p)n}}\exp\left(-\frac{x^2}{2p(1-p)}\right)\left(1+O\left(\frac{A_n}{n}\right)+O\left(\frac{A_n^3}{n^2}\right)\right).$$

We want to select $\varepsilon < \frac{2}{3}$ for $\frac{A_n^3}{n^2} \ll 1$. We select $\varepsilon = \frac{7}{12}$, so that $\frac{A_n^3}{n^2} = n^{-1/4}$ and $\frac{A_n}{n} = n^{-5/12}$, to yield

$$\max_{|x|\le A_n/\sqrt{n}} \mathbb{P}(S_n = k) = \frac{1}{\sqrt{2\pi p(1-p)n}}\exp\left(-\frac{x^2}{2p(1-p)}\right)\left(1+\underbrace{O\left(\frac{1}{n^{5/12}}\right)}_{=o(1/\sqrt{n})}\right). \tag{4.5}$$

We still have to consider the case when $x_{n,k}$ satisfies $|x| > \frac{A_n}{\sqrt{n}}$. Fortunately, both $\mathbb{P}(S_n = k)$ and the Gaussian tails are very small. Specifically,

$$\begin{aligned}&\max_{|x|>A_n/\sqrt{n}}\left|\mathbb{P}(S_n=k) - \frac{1}{\sqrt{2\pi p(1-p)n}}\exp\left(-\frac{x^2}{2p(1-p)}\right)\right| \\ &\le \max_{|x|>A_n/\sqrt{n}}|\mathbb{P}(S_n=k)| + \max_{|x|>A_n/\sqrt{n}}\left|\frac{1}{\sqrt{2\pi p(1-p)n}}\exp\left(-\frac{x^2}{2p(1-p)}\right)\right| \\ &\le \max\left(\mathbb{P}(S_n=\lfloor np+A_n\rfloor), \mathbb{P}(S_n=\lceil np-A_n\rceil)\right) + \frac{1}{\sqrt{2\pi p(1-p)n}}\exp\left(-\frac{A_n^2}{2np(1-p)}\right).\end{aligned}$$

The bound of the first term is a direct application of Exercise 4.5.4. Keeping the choice $A_n = n^{7/12}$, we see immediately that the second term is of $o(1/\sqrt{n})$. Now note that

$$n^{1/12} - n^{-1/2} = \frac{np+A_n-np-1}{\sqrt{n}} \le \frac{\lfloor np+A_n\rfloor - np}{\sqrt{n}} \le \frac{np+A_n-np}{\sqrt{n}} = n^{1/12}$$

so $x_{\lfloor np+A_n\rfloor,k} \sim n^{1/12}$, and this holds similarly for $x_{\lceil np+A_n\rceil,k}$, so the first term is also $o(1/\sqrt{n})$. Combining these results with (4.5) completes the proof. ■

The local limit theorem really tells us that

$$\mathbb{P}\left(\frac{S_n - np}{\sqrt{n}} = x\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2\pi p(1-p)}} e^{-\frac{x^2}{2p(1-p)}} + o(1)\right)$$

as $n \to \infty$. At first glance, you may find this result not useful, as it only tells us that the probability decays to zero at a rate of $O(1/\sqrt{n})$. However, since $\dfrac{S_n - np}{np(1-p)}$ seems to converge to a *continuous* distribution, we really should look at the **density** by ignoring the $1/\sqrt{n}$. The things inside the square bracket suggest that the density function of $n^{-1/2}(S_n - np)$ "converges" to a normal distribution $\mathbf{N}(0, p(1-p))$, which is equivalent to the distribution of $\dfrac{S_n - np}{np(1-p)}$ converging to standard normal $\mathbf{N}(0,1)$. The above heuristics can be formalised by adding the local probabilities and considering the cumulative distribution function. We therefore arrive at the central limit theorem (CLT).

Theorem 4.5.6 — de Moivre-Laplace CLT. For any $0 < p < 1$ and $x \in \mathbb{R}$ we have,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \le x\right) = \Phi(x),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y,$$

is the density of $\mathbf{N}(0,1)$.

Proof. (Sketch) We note that

$$\begin{aligned}\mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \le x\right) &= \mathbb{P}\left(S_n \le np + x\sqrt{np(1-p)}\right) \\ &= \sum_{k=0}^{\lfloor np - n^{7/12}\rfloor - 1} \mathbb{P}(S_n = k) + \sum_{k=\lfloor np - n^{7/12}\rfloor}^{\left\lfloor np + x\sqrt{np(1-p)}\right\rfloor} \mathbb{P}(S_n = k).\end{aligned}$$

Now the first term is a sum of a polynomial number of terms in exponentially small order $O(\exp(-n^{1/2}))$, so will vanish as $n \to \infty$. The second term is a Riemann sum. Writing

$$T_n = \left\{k : \left\lfloor np - n^{7/12}\right\rfloor \le k \le \left\lfloor np + x\sqrt{np(1-p)}\right\rfloor\right\}$$

we have

$$\begin{aligned}\sum_{k\in T_n} \mathbb{P}(S_n = k) &= \sum_{k\in T_n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi p(1-p)}} \exp\left(-\frac{1}{2}\left(\frac{k - np}{\sqrt{np(1-p)}}\right)^2\right) \\ &= \sum_{k\in T_n - np} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi p(1-p)}} \exp\left(-\frac{1}{2}\left(\frac{k/\sqrt{n}}{\sqrt{p(1-p)}}\right)^2\right).\end{aligned}$$

This is almost a Riemann sum on a partition of $(-\infty, x]$ with a mesh size of $1/\sqrt{n}$, missing some boundary terms. One can then show that the boundary terms lead to an $o(1)$ contribution, and conclude that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \le x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y.$$

We will omit the details here. ∎

Remark 4.5.7 It also holds, for all $a < b$, that

$$\lim_{n\to\infty} \mathbb{P}\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y.$$

We can let $a = -\epsilon\sqrt{\frac{n}{p(1-p)}}$ and $b = \epsilon\sqrt{\frac{n}{p(1-p)}}$ to conclude that

$$0 \leq \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \epsilon\right) = \lim_{n\to\infty} \left(\int_{-\infty}^{-\epsilon\sqrt{\frac{n}{p(1-p)}}} + \int_{\epsilon\sqrt{\frac{n}{p(1-p)}}}^{\infty}\right) \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y + o(1) = 0.$$

Hence, we obtain Theorem 4.3.2.

The de Moivre-Laplace CLT demonstrates that the sequence of random variables $\sqrt{n}((S_n/n) - p)$ converges *in distribution* (converges weakly) to a random variable with normal distribution $\mathsf{N}(0, p(1-p))$. We will formally define the notion of weak convergence in Chapters 5 and 6, as well as prove a generalised version of the central limit theorem.

Exercise 4.5.8 Using Theorem 4.5.6 prove Theorem 4.3.2.

4.6 Poisson Convergence

For completion, let us prove another result concerning convergence in distribution.

Theorem 4.6.1 — Poisson distribution. Fix k and let $p := p(n) \to 0$ as $n \to \infty$ s.t. $p(n) \cdot n \to \lambda > 0$. Then

$$\mathbb{P}(S_n = k) = \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right)^k \left(1 - \frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right)^{n-k} \longrightarrow \frac{1}{k!}\lambda^k e^{-\lambda}.$$

4.7 Solution to Exercises

Exercise 4.2.2

Solution.

1. It is clear that

$$\begin{aligned}\mathbb{P}_{\xi_i}(\{x\}) &= \mathbb{P}_{\tilde{\xi}_i}(\Omega_1 \times \cdots \times \{x\} \times \cdots \times \Omega_n) \\ &= \mathbb{P}^{(n)}(\Omega_1 \times \cdots \times \{x\} \times \cdots \times \Omega_n) \\ &= \mathbb{P}_i(\{x\}) \\ &= p^x(1-p)^{1-x}\end{aligned}$$

   for $x = 0, 1$.

2. By construction

$$\begin{aligned}\mathbb{P}\left(\tilde{\xi}_1 \in A_1, \ldots, \tilde{\xi}_n \in A_n\right) &= \mathbb{P}^{(n)}(A_1 \times \cdots \times A_n) \\ &= \prod_{i=1}^n \mathbb{P}_i(A_i).\end{aligned}$$

   Therefore, the $\left(\tilde{\xi}_i\right)$ are independent.

■

Exercise 4.3.1

Solution. As $S_n \sim \mathsf{B}(n,p)$ we know that $\mathbb{E}(S_n) = np$ and $\mathbb{V} = np(1-p)$. Therefore,

$$\begin{aligned}\mathbb{E}\left(\left|\frac{S_n}{n} - p\right|^2\right) &= \mathbb{E}\left(\left|\frac{S_n - np}{n}\right|^2\right)\\ &= \frac{1}{n^2}\mathbb{E}\left((S_n - np)^2\right)\\ &= \frac{1}{n^2}\mathbb{V}(S_n)\\ &= \frac{p(1-p)}{n}\\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

■

Exercise 4.1.4

Solution.

1. Consider the set $A_k = \left\{|\xi - \xi'| > \frac{1}{k}\right\}$. It is clear that $\{\xi \neq \xi'\} = \bigcup_{k=1}^{\infty} A_k$. Moreover,

$$\begin{aligned}\mathbb{P}(A_k) &\overset{(1)}{\leq} \mathbb{P}\left(|\xi - \xi_n| + |\xi_n - \xi'| > \frac{1}{k}\right)\\ &\overset{(2)}{\leq} \mathbb{P}\left(|\xi - \xi_n| > \frac{1}{2k}\right) + \mathbb{P}\left(|\xi_n - \xi'| > \frac{1}{2k}\right)\\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

   Where (1) is an application of the triangle inequality, and in (2) we use the fact that

$$\left\{|\xi - \xi_n| + |\xi_n - \xi'| > \frac{1}{k}\right\} \subseteq \left\{|\xi - \xi_n| > \frac{1}{2k}\right\} \cup \left\{|\xi_n - \xi'| > \frac{1}{2k}\right\}.$$

   Therefore, $\mathbb{P}(A_k) = 0$ which implies that

$$\begin{aligned}\mathbb{P}\left(\{\xi \neq \xi'\}\right) &= \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right)\\ &\overset{(1)}{=} \lim_{n\to\infty} \mathbb{P}(A_k)\\ &= 0.\end{aligned}$$

   Where in (1) we have used the continuity of the measure $\mathbb{P}$, as A_k is a sequence of increasing events. We conclude that $\xi = \xi'$ almost everywhere.

2. Using the triangle inequality we have that

$$|a\xi_n + b\eta_n - a\xi - b\eta| \leq |a||\xi_n - \xi| + |b||\eta_n - \eta|.$$

   Therefore,

$$\mathbb{P}(|a\xi_n + b\eta_n - a\xi - b\eta| \geq \epsilon) \leq \mathbb{P}(|a||\xi_n - \xi| + |b||\eta_n - \eta| \geq \epsilon).$$

   If ω satisfies the inequality on the right-hand side, it must be the case that either

   (a) $|a||\xi_n - \xi| \geq \frac{\epsilon}{2}$, or
   (b) $|b||\eta_n - \eta| \geq \frac{\epsilon}{2}$.

   Hence,

$$\mathbb{P}(|a||\xi_n - \xi| + |b||\eta_n - \eta| \geq \epsilon) \leq \mathbb{P}\left(|\xi_n - \xi| \geq \frac{\epsilon}{|a|}\right) + \mathbb{P}\left(|\eta_n - \eta| \geq \frac{\epsilon}{|b|}\right).$$

   Both terms on the right-hand side tend to 0 as $n \to \infty$ by assumption. Therefore,

$$\mathbb{P}(|a\xi_n + b\eta_n - a\xi - b\eta| \geq \epsilon) \overset{n\to\infty}{\longrightarrow} 0$$

   which implies that $a\xi_n + b\eta_n \xrightarrow{p} a\xi + b\eta$.

3. We proceed in the same way as in the previous parts to arrive at

$$\mathbb{P}(|\xi_i\eta_i - \xi\eta| \geq \epsilon) \leq \mathbb{P}\left(|\xi_i||\eta_i - \eta| \geq \frac{\epsilon}{2}\right) + \mathbb{P}\left(|\eta||\xi_i - \xi| \geq \frac{\epsilon}{2}\right). \tag{4.6}$$

Consider the first term on the right-hand side. We note that given an $M > 0$ it follows that

$$\left\{|\xi_i||\eta_i - \eta| \geq \frac{\epsilon}{2}\right\} \subseteq \{|\xi_i - \xi| \geq 1\} \cup \{|\xi| \geq M\} \cup \left\{|\eta_i - \eta| \geq \frac{\epsilon}{2(M+1)}\right\}.$$

One can observe this by considering the event than no of the events on the right-hand side hold, then

$$|\xi_i||\eta_i - \eta| \leq (|\xi_i - \xi| + |\xi|)\,|\eta_i - \eta| < (M+1)\frac{\epsilon}{2(M+1)} = \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(|\xi_i||\eta_i - \eta| \geq \frac{\epsilon}{2}\right) &\leq \mathbb{P}(|\xi_i - \xi| \geq 1) + \mathbb{P}(|\xi| \geq M) + \mathbb{P}\left(|\eta_i - \eta| \geq \frac{\epsilon}{2(M+1)}\right) \\ &\overset{i\to\infty}{\longrightarrow} \mathbb{P}(|\xi| \geq M) \\ &\overset{M\to\infty}{\longrightarrow} 0.\end{aligned}$$

Similarly, for the second term in the right-hand side of equation (4.6) we have that

$$\begin{aligned}\mathbb{P}\left(|\eta||\xi_i - \xi| \geq \frac{\epsilon}{2}\right) &\leq \mathbb{P}(|\eta| \geq M) + \mathbb{P}\left(|\eta_i - \eta| \geq \frac{\epsilon}{2M}\right) \\ &\overset{i\to\infty}{\longrightarrow} \mathbb{P}(|\eta| \geq M) \\ &\overset{M\to\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore, we can conclude that

$$\mathbb{P}(|\xi_i\eta_i - \xi\eta| \geq \epsilon) \overset{i\to\infty}{\longrightarrow} 0$$

which implies that $\xi_i\eta_i \xrightarrow{P} \xi\eta$.

4. Let $M \in \mathbb{R}$. On $[-M, M]^2$ the function $\varphi(x, y)$ is uniformly continuous. Therefore, given an $\epsilon > 0$ there exists a $\delta > 0$ such that for all $(x, y), (x', y') \in [-M, M]^2$ with $|(x, y) - (x', y')| < \delta$ we have

$$|\varphi(x, y) - \varphi(x', y')| < \epsilon.$$

Therefore,

$$\begin{aligned}\mathbb{P}\Big(|\varphi(\xi_i, \eta_i) - \varphi(\xi, \eta)| \geq \epsilon \Big| (\xi_i, \eta_i), (\xi, \eta) \in [-M, M]^2\Big) \\ &\leq \mathbb{P}\Big(|(\xi_i, \eta_i) - (\xi, \eta)| \geq \delta \Big| (\xi_i, \eta_i), (\xi, \eta) \in [-M, M]^2\Big) \\ &\leq \mathbb{P}\Big(|\xi_i - \xi| \geq \frac{\delta}{2} \Big| \xi_i, \xi \in [-M, M]\Big) + \mathbb{P}\left(|\eta_i - \eta| \geq \frac{\delta}{2} | \eta_i, \eta \in [-M, M]\right) \\ &\overset{i\to\infty}{\longrightarrow} 0.\end{aligned}$$

Sending $M \to \infty$ completes the proof.

■

Exercise 4.4.4

Solution. Note that as $(r_k)_{k\geq 1}$ converges to 0 it must be bounded. Suppose that $r_k \leq M$ for all $k \in \mathbb{N}$. Moreover, for any $\delta > 0$ we can find a $N_\delta \in \mathbb{N}$ such that $r_k \leq \delta$ for all $k \geq N_\delta$. Therefore, for an $\epsilon > 0$ and $n > N_\delta$ we have

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) &= \frac{\mathbb{V}(S_n)}{n^2\epsilon^2} \\ &= \frac{\sum_{i,j}^n \mathbb{E}(\xi_i\xi_j)}{n^2\epsilon^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(nr_0 + 2(n-1)r_1 + \cdots + 2(n-N_\delta)r_{N_\delta}) + (2(n-N_\delta-1)r_{N_\delta+1} + \cdots + 2(1)r_n}{n^2\epsilon^2} \\
&\leq \frac{2nN_\delta M + 2\delta(1 + \cdots + n - N_\delta - 1)}{n^2\epsilon^2} \\
&\leq \frac{2nN_\delta M + 2\delta\frac{1}{2}n(n-1)}{n^2\epsilon^2} \\
&\overset{n\to\infty}{\longrightarrow} \frac{\delta}{\epsilon^2}.
\end{aligned}
$$

As $\delta > 0$ we arbitrary for a fixed $\epsilon > 0$ we conclude that

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) \overset{n\to\infty}{\longrightarrow} 0.$$

■

Exercise 4.5.4

Solution. Consider the quotient

$$q = \frac{\binom{n}{k+1}p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{(n-k)p}{(k+1)(1-p)}.$$

When $q < 1$ the Binomial probability density function is decreasing and when $q > 1$ the Binomial probability density function (PDF) is increasing.

- The PDF is increasing for $k < p(n+1) - 1$.
- The PDF is decreasing for $k > p(n+1) - 1$.

When $p(n+1)$ is an integer the PDF is maximal for both $(n+1)p$ and $(n+1)p - 1$. ■

Exercise 4.5.8

Solution. Given a $\delta > 0$ there exists a $X_\delta > 0$ such that for all $x > X_\delta$ we have that

$$\Phi(x) - \Phi(-x) = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \geq 1 - \delta.$$

Observe that from CLT we have that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \leq x\right) = \Phi(x)$$

which implies that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{n} - p \leq \frac{x\sqrt{p(1-p)}}{\sqrt{n}}\right) = \Phi(x).$$

For fixed $\epsilon > 0$ let $\delta > 0$ and fix $x > X_\delta$ and choose $N \geq \frac{x^2 p(1-p)}{\epsilon^2}$. It follows that for $n \geq N$ we have that

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| < \epsilon\right) &\geq \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| \leq \frac{x\sqrt{p(1-p)}}{\sqrt{n}}\right) \\
&= \lim_{n\to\infty} \left(\mathbb{P}\left(\frac{S_n}{n} - p \leq \frac{x\sqrt{p(1-p)}}{\sqrt{n}}\right) - \mathbb{P}\left(\frac{S_n}{n} - p \leq -\frac{x\sqrt{p(1-p)}}{\sqrt{n}}\right)\right) \\
&= \Phi(x) - \Phi(-x) \\
&\geq 1 - \delta.
\end{aligned}
$$

Hence,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| \geq \epsilon\right) < \delta.$$

As δ was arbitrary we conclude that $\frac{S_n}{n} \xrightarrow{P} 0$. ■

5 Almost Sure Convergence

5.1 Definition

It will be useful now to briefly introduce almost sure convergence, and identify how it relates to convergence in probability and L^p convergence. The full theory of almost sure convergence will be discussed in Chapter 7.

Definition 5.1.1 — Almost sure convergence. A sequence $(\xi_n)_{n\geq 1}$ of random variables on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ converges $\mathbb{P}$-almost surely to the random variable ξ, denoted by $\xi_n \xrightarrow{a.e.} \xi$, if

$$\mathbb{P}\left(\left\{\omega : \xi_n(\omega) \overset{n\to\infty}{\not\longrightarrow} \xi(\omega)\right\}\right) = 0.$$

Further discussions will follow, for which the aim is to determine the following implications of convergence.

- Almost sure convergence and convergence in L^p both imply convergence in probability.
- Convergence in probability implies convergence in distribution.

$$\begin{array}{ccccc} & & \xrightarrow{L^p} & & \\ & & \Downarrow & & \\ \xrightarrow{a.e.} & \Longrightarrow & \xrightarrow{p} & \Longrightarrow & \xrightarrow{d} \end{array}$$

5.2 Connection to Convergence in Probability

To help draw the connections between these two forms of convergence we derive the Borel-Cantelli lemma.

5.2.1 Borel-Cantelli Lemma

Definition 5.2.1 — Infinitely often and eventually events. Let $A_1, A_2, \ldots$ be a sequence of events.

- Let
$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k.$$
If $\omega \in \limsup_{n\to\infty} A_n$, we say that A_n occurs infinitely often (i.o.).
- Let
$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k.$$
If $\omega \in \liminf_{n\to\infty} A_n$, we say that A_n occurs eventually (or almost except finitely often, a.e.f.o).

Exercise 5.2.2

1. Show that
$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n).$$
2. Describe the complement of $\limsup_{n\to\infty} A_n$.

Theorem 5.2.3 — Borel-Cantelli Lemma. Let $A_1, A_2, \ldots$ be a sequence of events.

1. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(A_n \text{ i.o.}) = 0$.
2. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and A_n are mutually independent then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

Proof.

1. By continuity of the measure we have

$$\mathbb{P}(A_n \text{ i.o.}) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) = 0.$$

2. Observe that $\{A_n \text{ i.o.}\}^c = \{A_n^c \text{ ev.}\} = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k^c$. Hence, we have

$$1 - \mathbb{P}(A_n \text{ i.o.}) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k\geq n} A_k^c\right).$$

By independence we have that

$$\mathbb{P}\left(\bigcap_{k\geq n} A_k^c\right) = \prod_{k\geq n} \mathbb{P}(A_k^c).$$

Note that $\log(1-x) \leq -x$ for $x \in [0,1)$, and thus

$$\log \mathbb{P}\left(\bigcap_{k\geq n} A_k^c\right) = \log \prod_{k\geq n} (1 - \mathbb{P}(A_k)) \leq -\sum_{k\geq n} \mathbb{P}(A_k) = -\infty,$$

That is, $\mathbb{P}\left(\bigcap_{k\geq n} A_k^c\right) = 0$ for all n.

■

Remark 5.2.4 Theorem 5.2.3 is an example of a zero-one law.

Example 5.2.5 — Infinite Monkey Theorem. For real numbers in $[0,1]$, we consider the event that its binary expansion contains a finite string of $\{0,1\}$ infinitely many times. Assume the desired string to be $(x_1, ..., x_m)$. We consider the sequence of events

$$A_n := \{\omega : \xi_{nm+1}(\omega) = x_1, \xi_{nm+2}(\omega) = x_2, ..., \xi_{nm+m}(\omega) = x_m\}$$

for $n \geq 0$ on $([0,1], \mathscr{B}([0,1]), \text{Leb})$. Where $\xi_k(\omega) = \omega_k$. A_n describes the event that the desired string appears starting from digit $nm+1$. These are mutually independent, given that the ξ_k are independent. As $\mathbb{P}(A_n) = \frac{1}{2^m}$ for all $n \in \mathbb{N}$ we observe that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, so by Theorem 5.2.3 we deduce that $\mathbb{P}(\{A_n \text{ i.o.}\}) = 1$. In other words, our finite string $\{0,1\}$ is guaranteed to appear infinitely often in the binary expansion of almost every real number in the interval $[0,1]$.

5.2.2 Application of the Borel-Cantelli Lemma

Proposition 5.2.6 A necessary and sufficient condition that $\xi_n \to \xi$ $\mathbb{P}$-almost surely is that

$$\mathbb{P}\left(\sup_{k\geq n} |\xi_k - \xi| \geq \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0,$$

for every $\varepsilon > 0$.

Proof. Note that $\xi_n(\omega) \not\to \xi(\omega)$ if and only if there exists an $\epsilon > 0$ such that

$$|\xi_n(\omega) - \xi(\omega)| \geq \epsilon$$

infinitely often. So let $A_n^\epsilon = \{\omega : |\xi_n - \xi| \geq \epsilon\}$ and $A^\epsilon = \limsup A_n^\epsilon$. It follows that

$$\{\omega : \xi_n(\omega) \not\to \xi(\omega)\} = \bigcup_{\varepsilon \geq 0} A^\varepsilon.$$

As the sets A^ϵ are nested, one can restrict ϵ to the form $\epsilon = \frac{1}{m}$ for some positive integer m, so that

$$\{\omega : \xi_n(\omega) \not\to \xi(\omega)\} = \bigcup_{m=1}^{\infty} A^{\frac{1}{m}}.$$

Hence, $\mathbb{P}(\{\omega : \xi_n \not\to \xi\}) = 0$ if and only if $\mathbb{P}\left(\bigcup_{m=1}^\infty A^{\frac{1}{m}}\right) = 0$. If this holds it follows for all $m \geq 1$ that

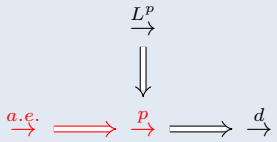
$$\mathbb{P}\left(A^{\frac{1}{m}}\right) \leq \mathbb{P}\left(\bigcup_{m=1}^{\infty} A^{\frac{1}{m}}\right) = 0.$$

Conversely, if for $m \geq 1$ we have $\mathbb{P}\left(A^{\frac{1}{m}}\right) = 0$ then by the continuity of the measure it follows that $\mathbb{P}\left(\bigcup_{m=1}^\infty A^{\frac{1}{m}}\right) = 0$. Now $\mathbb{P}\left(A^{\frac{1}{m}}\right) = 0$ for all $m \geq 1$ happens if and only if $\mathbb{P}\left(A^\epsilon\right) = 0$ for all $\epsilon > 0$. Therefore, as

$$\mathbb{P}(A^\varepsilon) = \mathbb{P}\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} A_k^\varepsilon\right) = \lim_n \mathbb{P}\left(\bigcup_{k\geq n} A_k^\varepsilon\right) = \mathbb{P}\left(\sup_{k\geq n}|\xi_k - \xi| \geq \varepsilon\right),$$

we complete the proof. ■

Corollary 5.2.7 Convergence almost surely implies convergence in measure.



Exercise 5.2.8 — Cauchy-like condition for almost sure convergence. Show that the sequence $\{\xi_n\}_{n\geq 1}$ converges almost surely if and only if, for all $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{k,l\geq n} |\xi_k - \xi_l| \geq \varepsilon\right) = 0,$$

or equivalently,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{k\geq 0} |\xi_{n+k} - \xi_n| \geq \varepsilon\right) = 0.$$

Example 5.2.9 — Typewriter Sequence. Consider the sequence $f_n := \chi_{A_n}$ for

$$A_n = \left[\frac{n}{2^k} - 1, \frac{n+1}{2^k} - 1\right]$$

whenever $k \geq 0$ and $2^k \leq n < 2^{k+1}$. The first few A_n are

- $[0, 1]$,

- $\left[0, \frac{1}{2}\right]$,
- $\left[\frac{1}{2}, 1\right]$,
- $\left[0, \frac{1}{4}\right]$,
- $\left[\frac{1}{4}, \frac{1}{2}\right]$,
- $\vdots$

Plotting these indicator functions one observes that they move from left to right over $[0, 1]$, half their width and repeat. Therefore, it is clear that in probability the function converges to 0. However, given that the indicator function moves from left to right infinitely many times, for all $\omega \in [0, 1]$, $f_n(\omega) = 1$ (and 0) infinitely often and so $f_n(\omega)$ does not converge almost surely.

There is a partial converse to the implication that almost sure converge implies convergence in probability. More specifically we will see that if a sequence converges in probability, then we can extract a subsequence that converges almost surely.

Lemma 5.2.10 A sufficient condition for $\xi_n \xrightarrow{a.s.} \xi$ is that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \varepsilon) < \infty$$

is satisfied for all $\varepsilon > 0$.

Proof. Denote the event $A_n^{\varepsilon} = \{\omega : |\xi_n(\omega) - \xi(\omega)| \geq \varepsilon\}$ as required. Since $\sum_{k \geq 1} \mathbb{P}(A^{\epsilon}) < \infty$, by Theorem 5.2.3 we know that $\mathbb{P}(A^{\varepsilon}) := \mathbb{P}(A_n^{\epsilon} \text{ i.o.}) = 0$. Following the arguments in Proposition 5.2.6 we know that $\mathbb{P}(\{\omega : \xi_n \not\to \xi\}) = 0$ as desired. ■

Corollary 5.2.11 Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive numbers such that $\varepsilon_n \downarrow 0$ as $n \to \infty$. Then if ξ_n converges to ξ in probability such that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \varepsilon_n) < \infty,$$

then $\xi_n \xrightarrow{a.s.} \xi$.

Proof. Fix an arbitrary $\epsilon > 0$. Choose N such that for all $n \geq N$ we have $\epsilon_n < \epsilon$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \epsilon) \leq \underbrace{\sum_{n=1}^{N-1} \mathbb{P}(|\xi_n - \xi| \geq \epsilon)}_{\leq N-1 < \infty} + \underbrace{\sum_{n=N}^{\infty} \mathbb{P}(|\xi_n - \xi| \geq \epsilon)}_{< \infty},$$

hence by Lemma 5.2.10 we have $\xi_n \xrightarrow{\text{a.s.}} \xi$ as $n \to \infty$. ■

Theorem 5.2.12 If $\xi_n \xrightarrow{p} \xi$, then there exists a subsequence such that

$$\xi_{n_k} \xrightarrow{a.s.} \xi.$$

Proof. Since $\lim_{n \to \infty} \mathbb{P}\left(|\xi_n - \xi| > \frac{1}{k}\right) = 0$ for all $k \geq 1$, we can choose a subsequence such that

$$\mathbb{P}\left(|\xi_n - \xi| > \frac{1}{k}\right) \leq 2^{-k}$$

for all $k \geq 1$. Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, by Corollary 5.2.11 we have $\xi_{n_k} \xrightarrow{a.s.} \xi$. ■

Corollary 5.2.13 If $\xi_1 \geq \xi_2 \geq \cdots \geq 0$ are random variables and $\xi_n \xrightarrow{p} 0$, then

$$\xi_n \xrightarrow{a.s.} 0.$$

Proof. Note that $\xi_n \xrightarrow{a.s} 0$ if and only if $\limsup \xi_n = 0$ almost surely. Let $\varepsilon > 0$ and let $A_n = \{\xi_n > \varepsilon\}$. Then by continuity,

$$\mathbb{P}(\limsup \xi_n > \varepsilon) \leq \mathbb{P}(\xi_n > \varepsilon \text{ i.o.}) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right).$$

Since A_n is non-increasing, the right hand side equals $\lim_{n\to\infty} \mathbb{P}(A_n)$ which is 0 since $\xi_n \xrightarrow{p} 0$. Thus

$$\mathbb{P}(\limsup \xi_n > \varepsilon) = 0$$

for all $\epsilon > 0$ and hence

$$\mathbb{P}(\xi_n \not\to \xi) \leq \sum_{m=1}^{\infty} \mathbb{P}\left(\limsup \xi_n > \frac{1}{m}\right) = 0.$$

■

5.3 Connection to L^p convergence

One can also see that Example 4.1.3 shows almost sure convergence does not guarantee L^p convergence. However, if we have almost sure convergence and convergence in mean, then we have L^1 convergence.

Theorem 5.3.1 Let ξ_n be a sequence of non-negative random variables such that $\xi_n \xrightarrow{\text{a.s.}} \xi$ and $\mathbb{E}(\xi_n) \to \mathbb{E}(\xi) < \infty$. Then $\xi_n \xrightarrow{L^1} \xi$, that is,

$$\mathbb{E}(|\xi_n - \xi|) \to 0$$

as $n \to \infty$.

Proof. We have $\mathbb{E}(|\xi_n|) < \infty$ for sufficiently large n and therefore for such n we have

$$\begin{aligned}\mathbb{E}(|\xi_n - \xi|) &= \mathbb{E}(\xi - \xi_n)\chi_{\xi\geq\xi_n} + \mathbb{E}(\xi_n - \xi)\chi_{\xi_n>\xi} \\ &= 2\mathbb{E}(\xi - \xi_n)\chi_{\xi\geq\xi_n} + \mathbb{E}(\xi_n - \xi).\end{aligned}$$

The second term on the right-hand side tends to zero by assumption. For the first term we note that $0 \leq (\xi - \xi_n)\chi_{\xi\geq\xi_n} \leq \xi$ so by the dominated convergence theorem it follows that $\mathbb{E}(\xi - \xi_n)\chi_{\xi\geq\xi_n} \to 0$. ■

Example 5.3.2 Let $\xi_1, \ldots, \xi_n$ be independent $\{0,1\}$-valued random variables on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}(\xi_n = 1) = \frac{1}{n}$. For example, $(\Omega, \mathscr{F}) = (\{0,1\}, \mathscr{B}(\{0,1\}))^{\mathbb{N}}$ and $\mathbb{P}$ being an appropriate infinite product measure. Then

$$\mathbb{E}\left(|\xi_n - 0|^p\right) = \frac{1}{n} \to 0,$$

so $\xi_n \xrightarrow{L^p} 0$. However,

$$\{\omega : \xi_n \to 0\} = \{\xi_n = 0 \text{ eventually}\} = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k\geq n} \{\xi_k = 0\}}_{\text{increasing sequence of sets}}.$$

Therefore,

$$\mathbb{P}(\xi_n \to 0) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k\geq n} \{\xi_k = 0\}\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \prod_{k\geq n} \mathbb{P}(\xi_k = 0) \\
&= \lim_{n\to\infty} \prod_{k\geq n} \left(1 - \frac{1}{k}\right) \\
&= 0.
\end{aligned}
$$

Indeed,

$$
\prod_{k\geq n} \left(1 - \frac{1}{k}\right) = \lim_{N\to\infty} \prod_{k=n}^{N} \frac{k-1}{k} = \lim_{N\to\infty} \frac{n-1}{n}\frac{n}{n+1}\cdots\frac{N-1}{N} = 0.
$$

Thus ξ_n does not converge almost surely to 0.

Example 5.3.3 Example 5.2.9 also shows that L^p convergence does imply convergence almost surely.

5.4 Strong Law of Large Numbers

Definition 5.4.1 Let $\xi_1, \xi_2, \ldots$ be a sequence of integrable random variables and let $S_n = \xi_1 + \cdots + \xi_n$. Then the sequence $(\xi_n)_{n\geq 1}$ satisfies the strong law of large numbers if

$$
\frac{S_n - \mathbb{E}(S_n)}{n} \xrightarrow{\text{a.s.}} 0.
$$

Note that the strong law of large numbers properties is stronger than the weak law of large numbers, as now we require convergence almost surely which is stronger than convergence in probability.
Recall that for i.i.d. ξ_n with $\mathbb{V}(\xi_1) < \infty$ we showed the L^2 weak law of large number. Imposing a stronger moment assumption we arrive at a strong law of large numbers.

Proposition 5.4.2 — Cantelli's Strong Law of Large Numbers. Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{E}\left(\xi_1^4\right) < \infty$. Then

$$
\frac{S_n - \mathbb{E}(S_n)}{n} \xrightarrow{\text{a.s.}} 0.
$$

Proof. Without loss of generality, we centralise the random variables by subtracting its mean, that is $\mathbb{E}(\xi_1) = 0$. Note by Chebyshev's inequality that

$$
\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \epsilon\right) < \frac{\mathbb{E}\left(\left|\frac{S_n}{n}\right|^4\right)}{\epsilon^4}.
$$

In the expansion of forth moments, the only non-vanishing terms are terms of the form $\mathbb{E}\left(\xi_j^4\right)$ and $\mathbb{E}\left(\xi_i^2\xi_j^2\right) = \mathbb{E}\left(\xi_i^2\right)\mathbb{E}\left(\xi_j^2\right)$ with $i \neq j$, noticing that the odd moments of ξ_i vanishes. We therefore have the expansion

$$
\mathbb{E}\left(S_n^4\right) = \mathbb{E}\left(\sum_{k=1}^{n} \xi_k^4 + \binom{4}{2,2} \sum_{j,k=1, j<k} \xi_j^2\xi_k^2\right)
$$

where $\binom{4}{2,2} = 4!/(2!)^2 = 6$ comes from the multinomial theorem. Now notice that there are $\frac{n(n-1)}{2}$ unique ways to choose the indices (j,k) such that $j < k$, therefore we have, from the i.i.d assumption,

$$
\mathbb{E}\left(\left|\frac{S_n}{n}\right|^4\right) = \frac{1}{n^4}\left(n\mathbb{E}\left(\xi_1^4\right) + 3n(n-1)(\mathbb{E}\left(\xi_1^2\right))^2\right)
$$

From Corollary 2.4.4 we have that $\mathbb{E}\left(\xi_1^2\right)^{1/2} \leq \mathbb{E}\left(\xi_1^4\right)^{1/4}$, and so

$$
\mathbb{E}\left(\left|\frac{S_n}{n}\right|^4\right) \leq \frac{3n^2 - 2n}{n^4}\mathbb{E}\left(\xi_1^4\right) \lesssim \frac{1}{n^2},
$$

and therefore $\sum_{n\geq 1}\mathbb{P}\left(\left|\frac{S_n}{n}\right|\geq\epsilon\right)\lesssim\sum_{n\geq 1}\frac{1}{n^2}<\infty$. Thus we can conclude by applying Lemma 5.2.10. ■

We will now work towards Kolmogorov's strong law of large numbers.

Proposition 5.4.3 — Kolmogorov's maximal inequality. Let $\xi_1,\xi_2,\dots$ be independent random variables with finite variances. Then for all $n\geq 1$ and $x>0$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq n}|S_k-\mathbb{E}(S_k)|\geq x\right)\leq\frac{\mathbb{V}(S_n)}{x^2}.$$

Proof. Without loss of generality, we can consider $\mathbb{E}(\xi_1)=0$. Consider the event $A=\{\omega:\max_{1\leq k\leq n}|S_k(\omega)|\geq x\}$ and the events

$$A_k=\{\omega:|S_j|<x\, j=1,\dots,k-1\,|S_k|\geq x\}$$

for $k=1,\dots,n$. Note that the sets A_k are mutually disjoint and are such that $A=\bigcup_{k=1}^n A_k$. Therefore,

$$\begin{aligned}
\mathbb{E}\left(S_n^2\right)&\geq\mathbb{E}\left(S_n^2\chi_A\right)\\
&\overset{\mathbb{E}}{\sum_{k=1}}\left(S_n^2\chi_{A_k}\right)\\
&=\sum_{k=1}^n\mathbb{E}\left(S_k+\xi_{k+1}+\dots\xi_n\chi_{A_k}\right)\\
&=\sum_{k=1}^n\underbrace{\mathbb{E}\left(S_k^2\chi_{A_k}\right)}_{\geq x^2\mathbb{P}(A_k)}+\underbrace{2\mathbb{E}\left((S_k\chi_{A_k})(\xi_{k+1}+\ldots+\xi_n)\right)}_{=2\mathbb{E}\left(S_k\chi_{A_k}\right)\mathbb{E}(\xi_{k+1}+\ldots+\xi_n)=0}+\underbrace{\mathbb{E}\left((\xi_{k+1}+\ldots+\xi_n)^2\right)}_{\geq 0}\\
&\geq x^2\sum_{k=1}^n\mathbb{P}(A_k)\\
&=x^2\mathbb{P}(A).
\end{aligned}$$

Therefore,

$$\mathbb{V}\left(S_n^2\right)=\mathbb{E}\left(S_n^2\right)\geq x^2\mathbb{P}(A).$$

■

Remark 5.4.4 Note that Proposition 5.4.3 is stronger than Chebyshev's inequality, since Chebyshev's inequality only gives

$$\mathbb{P}\left(|S_k-\mathbb{E}(S_k)|\geq x\right)\leq\frac{\mathbb{V}(S_n)}{x^2}.$$

Which when combined with a union bound gives

$$\mathbb{P}\left(\max_{1\leq k\leq n}|S_k-\mathbb{E}(S_k)|\geq x\right)=\mathbb{P}\left(\bigcup_{k=1}^n\{|S_k-\mathbb{E}(S_k)|\geq x\}\right)\leq\frac{n\mathbb{V}(S_n)}{x^2}.$$

Hence, Proposition 5.4.3 removes the factor of n in the bound.

Lemma 5.4.5 Suppose $\xi_1,\xi_2,\dots$ is a series of real-valued independent random variables with $\mathbb{E}(\xi_i)=0$ for all i. If $\sum_{i\geq 1}\mathbb{V}(\xi_i)<\infty$, then $\sum_{i\geq 1}\xi_i$ converges almost surely.

Proof. Note that

$$\begin{aligned}
0\leq\sup_{m,n\geq k}|S_n-S_m|&\leq\sup_{m,n\geq k}\left(|S_n-S_k|+|S_k-S_m|\right)\\
&=2\sup_{n\geq k}|S_n-S_k|=:2\sigma_k.
\end{aligned}$$

By Proposition 5.4.3, we have that

$$\begin{aligned}\mathbb{P}(\sigma_k \geq x) &= \lim_{m\to\infty} \mathbb{P}\left(\max_{m\geq n\geq k} |S_n - S_k| \geq x\right) \\ &\leq \frac{1}{x^2} \lim_{m\to\infty} \sum_{n=k+1}^{m} \mathbb{V}(\xi_n) \\ &= \frac{1}{x^2} \sum_{n=k+1}^{\infty} \mathbb{V}(\xi_n) \\ &\overset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \mathbb{P}\left(\sup_{n,m\geq k} |S_n - S_m| \geq \epsilon\right) = 0$$

and so we conclude by using the result of Exercise 5.2.8. ■

Theorem 5.4.6 Let $\xi_1, \xi_2, \ldots$ be a series of independent random variables. If $\sum_{n=1}^{\infty} \mathbb{E}(\xi_n)$ converges and $\sum_{n=1}^{\infty} \mathbb{V}(\xi_n)$ converges, then $\sum_{n=1}^{\infty} \xi_n$ converges almost surely.

Proof. Consider

$$\sum_{i=1}^{\infty} \xi_i = \sum_{i=1}^{\infty} (\xi_i - \mathbb{E}(\xi_i)) + \sum_{i=1}^{\infty} \mathbb{E}(\xi_i).$$

By assumption we know that $\sum_{i=1}^{\infty} \mathbb{E}(\xi_i)$ converges, and $\sum_{i=1}^{\infty} (\xi_i - \mathbb{E}(\xi_i))$ converges by Lemma 5.4.5. ■

In our aim to prove Kolmogorov's strong law of large numbers, we require some results regarding the convergence of weighted averages.

Lemma 5.4.7 — Toeplitz. Let $\{a_n\}$ be a sequence of non-negative numbers. Let $b_n = \sum_{i=1}^{n} a_i$ so that $b_1 = a_1 > 0$, and $b_n \uparrow \infty$ as $n \to \infty$. Let $\{x_n\}_{n\geq 1}$ be a sequence of numbers converging to x. Then

$$\frac{1}{b_n} \sum_{j=1}^{n} a_j x_j \to x.$$

In particular, if $a_n = 1$, then

$$\frac{x_1 + \cdots + x_n}{n} \to x.$$

Proof. Fix $\epsilon > 0$. Choose $N_0 := N_0(\epsilon)$ such that for all $n \geq N_0$ we have $|x_j - x| < \epsilon/2$. Now choose $N_1 > N_0$, which depends on N_0, such that $\frac{1}{b_{N_1}} \sum_{j=1}^{N_0} |x_j - x| < \frac{\epsilon}{2}$, which exists since $|x_j - x|$ is bounded for $j = 1, \ldots, N_0$. Then for any $n > N_1$, we have

$$\begin{aligned}\left|\frac{1}{b_n} \sum_{j=1}^{n} a_j x_j - x\right| &\leq \left|\frac{1}{b_n} \sum_{j=1}^{N_0} a_j (x_j - x)\right| + \left|\frac{1}{b_n} \sum_{j=N_0+1}^{n} a_j (x_j - x)\right| \\ &\leq \frac{1}{b_{N_1}} \left|\sum_{j=1}^{N} a_j (x_j - x)\right| + \left|\frac{1}{b_n} \sum_{j=N_0+1}^{n} a_j (x_j - x)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \underbrace{\left(\frac{1}{b_n} \sum_{j=N_0+1}^{n} a_j\right)}_{\leq 1} \leq \epsilon.\end{aligned}$$

■

Exercise 5.4.8 Suppose $(\xi_i)_{i\geq 1}$ is a sequence of independent random variables with common mean m and variance $\mathbb{V}(\xi_k) = k\eta(k)$ with the condition that $\mathbb{V}[\xi_k] \to \infty$, $\eta(k) > 0$ and $\eta(k) \searrow 0$ as $k \to \infty$. Using Lemma 5.4.7 prove that the sequence satisfies the weak law of large numbers. That is, show that $n^{-1}\sum_{i=1}^{n} \xi_i \to m$ as $n \to \infty$ in L^2 and in probability.

Lemma 5.4.9 — Kronecker. Let $(a_n), (b_n)$ be as in Lemma 5.4.7 and let $\{x_n\}$ be a sequence of numbers such that $\sum x_n$ converges. Then

$$\frac{1}{b_n}\sum_{j=1}^{n} b_j x_j \to 0$$

as $n \to \infty$. In particular, if $b_n = n$, $x_n = \frac{y_n}{n}$ and $\sum \frac{y_n}{n}$ converges, then

$$\frac{y_1 + \cdots + y_n}{n} \to 0$$

as $n \to \infty$.

Proof. We let $b_0 = S_0 = 0$ and $S_n = \sum_{j=1}^{n} x_j$. Then

$$\begin{aligned}\sum_{j=1}^{n} b_j x_j = \sum_{j=1}^{n} b_j(S_j - S_{j-1}) &= b_n S_n - b_0 S_0 - \sum_{j=1}^{n} S_j(b_j - b_{j-1}) \\ &= b_n S_n - b_0 S_0 - \sum_{j=1}^{n} a_j S_j.\end{aligned}$$

Dividing b_n gives

$$\frac{1}{b_n}\sum_{j=1}^{n} b_j x_j = S_n - \underbrace{\frac{b_0 S_0}{b_n}}_{\to 0} - \frac{1}{b_n}\sum_{j=1}^{n} a_j S_j.$$

So when $n \to \infty$, we see that $b_n^{-1}\sum_{j=1}^{n} b_j x_j \overset{n\to\infty}{\to} 0$ by Lemama 5.4.7. ■

With this, we may prove Kolmogorov's lemma.

Theorem 5.4.10 — Kolmogorov. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with $\mathbb{E}\left(\xi_i^2\right) < \infty$ for all i. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $b_n \nearrow \infty$ and

$$\sum_{n\geq 1} \frac{\mathbb{V}(\xi_n)}{b_n^2} < \infty.$$

Then

$$\frac{S_n - \mathbb{E}(S_n)}{b_n} \xrightarrow{a.s.} 0.$$

When $b_n = n$ we obtain a strong law of large numbers.

Proof. Observe that

$$\frac{S_n - \mathbb{E}(S_n)}{b_n} = \frac{1}{b_n}\sum_{i=1}^{n} b_k \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k}. \tag{5.1}$$

Now

$$\mathbb{V}\left(\sum_{k=1}^{n} \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k}\right)\sum_{k=1}^{n} = \mathbb{V}\left(\frac{\xi_k - \mathbb{E}(\xi_k)}{b_k}\right) = \sum_{k=1} \frac{\mathbb{V}(\xi_k)}{b_k^2} < \infty.$$

Therefore, by Theorem 5.4.6

$$\sum_{k\geq 1} \frac{\xi_k - \mathbb{E}(\xi_k)}{b_k}$$

converges almost surely. Hence, applying Lemma 5.4.9 to (5.1) completes the proof. ■

Exercise 5.4.11 For ξ an integrable non-negative random variable, show that

$$\sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n) \leq \mathbb{E}(\xi) \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n).$$

Theorem 5.4.12 — Kolmogorov's Strong Law of Large Numbers. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables with $\mathbb{E}(|\xi_1|) < \infty$. Then

$$\frac{S_n}{n} \xrightarrow{\text{a.s}} \mathbb{E}(\xi_1)$$

as $n \to \infty$.

Proof. Without loss of generality assume that $\mathbb{E}(\xi_1) = 0$. By Exercise 5.4.11 it follows that $\sum_{n\geq 1} \mathbb{P}(|\xi_n| \geq n) \leq \mathbb{E}(|\xi_1|) < \infty$. By the first Borel-Cantelli lemma, we know that $\mathbb{P}(|\xi_n| \geq n \text{ i.o.}) = 0$. That is, $|\xi_n| < n$ eventually $\mathbb{P}$-almost everywhere. So letting $\tilde{\xi}_n = \xi_n \chi_{|\xi_n|<n}$, we have

$$\frac{S_n}{n} \xrightarrow{n\to\infty} 0 \text{ a.s.}$$

if and only if

$$\frac{1}{n}\sum_{i=1}^{n} \tilde{\xi}_i \xrightarrow{n\to\infty} 0 \text{ a.s.}$$

Note that by dominated convergence theorem it follows that $\mathbb{E}\left(\tilde{\xi}_n\right) \xrightarrow{n\to\infty} \mathbb{E}(\xi_1) = 0$. Therefore, using Lemma 5.4.7 with $x_n = \mathbb{E}\left(\tilde{\xi}_n\right)$ it follows that $\frac{1}{n}\sum_{i=1^n} \mathbb{E}\left(\tilde{\xi}_i\right) \to 0$ as $n \to \infty$. Therefore

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0$$

if and only if

$$\frac{1}{n}\sum_{i=1}^{n} \left(\tilde{\xi}_i - \mathbb{E}\left(\tilde{\xi}_i\right)\right) \xrightarrow{\text{a.s.}} 0. \tag{5.2}$$

By Lemma 5.4.9 we know that (5.2) holds if $\sum_{n=1}^{\infty} \frac{\tilde{\xi}_n - \mathbb{E}(\tilde{\xi}_n)}{n}$ converges. Using Theorem 5.4.6 this is the case if $\sum_{n=1}^{\infty} \frac{\mathbb{V}(\tilde{\xi}_n - \mathbb{E}(\tilde{\xi}_n))}{n^2}$ converges. Hence, observe that

$$\begin{aligned}
\mathbb{V}\left(\sum_{n\geq 1} \frac{\tilde{\xi}_n - \mathbb{E}\left(\tilde{\xi}_n\right)}{n}\right) &\leq \sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\xi_1^2 \chi_{|\xi_1|<n}\right)}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{E}\left(\xi_n^2 \chi_{\{|\xi_1| \in [k-1,k)\}}\right) \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left(\xi_n^2 \chi_{|\xi_1| \in [k-1,k)}\right) \underbrace{\left(\sum_{n=k}^{\infty} \frac{1}{n^2}\right)}_{\leq 2/k} \\
&= 2\sum_{k=1}^{\infty} \mathbb{E}\left(|\xi| \underbrace{\frac{|\xi|}{k}}_{\leq 1} \chi_{\{|\xi_1| \in [k-1,k)\}}\right) \\
&\leq 2\mathbb{E}(|\xi_1|) < \infty.
\end{aligned}$$

Hence, (5.2) holds which completes the proof. ∎

Example 5.4.13 Consider the probability space $([0,1), \mathscr{B}([0,1)), \text{Leb})$. An $\omega \in [0,1)$ has the binary representation

$$\omega = \frac{\omega_1}{2} + \frac{\omega_2}{2^2} + \cdots = 0.\omega_1\omega_2\ldots$$

where $\omega_j \in \{0,1\}$. Let $\xi_j(\omega) = \omega_j$. Then for $(x_1,\ldots,x_n) \in \{0,1\}^n$ consider

$$\begin{aligned} A_{(x_1,\ldots,x_n)} &= \{\omega : \xi_1 = x_1, \ldots, \xi_n = x_n\} \\ &= \left\{\omega : \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} \leq \omega < \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^n}\right\}. \end{aligned}$$

so $\mathbb{P}\left(A_{(x_1,\ldots,x_n)}\right) = \frac{1}{2^n}$. Therefore, as $\xi_1, \xi_2, \ldots$ are i.i.d Bernoulli random variables with $\mathbb{P}(\xi_1 = 1) = \frac{1}{2}$ by the strong law of large numbers we conclude that

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k \xrightarrow{\text{a.s}} \mathbb{E}(\xi_1) = \frac{1}{2}.$$

That is, for almost every number in $[0,1)$ the proportion of 0s and 1s in its binary expansion tends to $\frac{1}{2}$. We call such numbers normal.

5.5 Kolmogorov's 0-1 Law

Definition 5.5.1 Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

1. Let $\mathscr{F}_1, \mathscr{F}_2, \ldots$ be sub-σ-algebra of $\mathscr{F}$. Define the σ-algebra generated by their union as

$$\bigvee_{i=1}^{n} \mathscr{F}_i = \sigma\left(\bigcup_{i=1}^{n} \mathscr{F}_i\right),$$

with the natural extension for when $n = \infty$.

2. Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Then

$$\sigma(\xi_1, \ldots, \xi_n) = \bigvee_{i=1}^{n} \sigma(\xi_i),$$

with the natural extension for when $n = \infty$.

Definition 5.5.2 Under settings of Definition 5.5.1, define $\mathscr{F}_n^p = \sigma(\xi_n, \ldots, \xi_p)$ for $p \geq n$ and $\mathscr{F}_n^\infty = \sigma(\xi_n, \ldots)$. For a sequence of random variables $\xi_1, \xi_2, \ldots$, the σ-algebra

$$\mathscr{T} = \bigcap_{n=1}^{\infty} \mathscr{F}_n^\infty$$

is called the tail σ-algebra. Events of $\mathscr{T}$ are called tail events.

Example 5.5.3

- For some $B \in \mathscr{B}(\mathbb{R})$, the event $\{\xi_n \in B \text{ i.o.}\} \in \mathscr{T}$ as

$$\{\xi_n \in B \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \{\xi_k \in B\} \in \mathscr{T}.$$

- Similarly, $\{\sum_{n=1}^{\infty} \xi_n \text{ converges}\} \in \mathscr{T}$.

- The event $\{\xi_{10} \in B\}$ may not be in $\mathscr{T}$, since its occurrence may be affected by changing a finite number of ξ_n, namely changing ξ_{10}. Similarly, $\{\omega : \xi_n \notin \mathscr{B} \text{ for all } n\}$ is not in $\mathscr{T}$ since its occurrence may be affected by just a single ξ_n.

Lemma 5.5.4 If $\mathscr{A}, \mathscr{B} \subset \mathscr{F}$ are independent, that is for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, then $\sigma(\mathscr{A})$ and $\sigma(\mathscr{B})$ are independent.

Proof. Consider $A \in \mathscr{A}$ and the measures

- $\mathbb{P}_A^{(1)}(B) = \mathbb{P}(AB)$, and
- $\mathbb{P}_A^{(2)}(B) = \mathbb{P}(A)\mathbb{P}(B)$.

These measures coincide on $\mathscr{B}$, and so by Theorem 1.3.5 they coincide on $\sigma(\mathscr{B})$. That is $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathscr{A}$ and $B \in \sigma(\mathscr{B})$. Now let $B \in \sigma(\mathscr{B})$ and consider the measures

- $Q_B^{(1)}(A) = \mathbb{P}(AB)$, and
- $Q_B^{(2)}(A) = \mathbb{P}(A)\mathbb{P}(B)$.

Similarly we conclude that using Theorem 1.3.5 that for all $A \in \sigma(\mathscr{A})$ and $B \in \sigma(\mathscr{B})$ that $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$. ■

Lemma 5.5.5 Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables. Then, $\mathscr{T}$ is independent of itself.

Proof. Note that $\mathscr{F}_1^n$ is independent of $\mathscr{F}_{n+1}^{n+k}$ for all k as the random variables ξ_i are independent for all i. Hence, $\mathscr{F}_1^n$ is independent with $\bigcup_{k=1}^{\infty} \mathscr{F}_{n+1}^{n+k}$. So by using Lemma 5.5.4 we deduce that $\mathscr{F}_1^n$ is independent with $\mathscr{F}_{n+1}^{\infty}$. As $\mathscr{T} \subset \mathscr{F}_{n+1}^{\infty}$ it follow that $\mathscr{F}_1^n$ is independent of $\mathscr{T}$. It follows that $\bigcup_{n=2}^{\infty} \mathscr{F}_1^n$ is independent with $\mathscr{T}$, and so by using Lemma 5.5.4 we have that $\mathscr{F}_1^{\infty}$ is independent with $\mathscr{T}$. However, as $\mathscr{T} \subset \mathscr{F}_1^{\infty}$ it follows that $\mathscr{T}$ is independent of $\mathscr{T}$. ■

Theorem 5.5.6 — Kolmogorov's Zero-One Law. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables, and let $A \in \mathscr{T}$. Then $\mathbb{P}(A) \in \{0, 1\}$.

Proof. Using Lemma 5.5.5 we have $\mathbb{P}(A) = \mathbb{P}(A \cap A) = (\mathbb{P}(A))^2$. Hence, $\mathbb{P}(A)$ must have a value of zero or one. ■

Example 5.5.7 Given independent sequence $\xi_1, \xi_2, \ldots$ and $B_1, B_2, \ldots \in \mathscr{B}(\mathbb{R})$, then $\{\xi_n \in B_n \text{ i.o.}\} = \cap_{n=1}^{\infty} \cup_{k \geq n} \{\xi_k \in B_k\} \in \mathscr{T}$. Now consider the independent events $A_1, A_2, \ldots$, then the sequence of random variables $\chi_{A_1}, \chi_{A_2}, \ldots$ are independent, and that the event $\limsup_n A_n := \{\chi_{A_i} = 1 \text{ i.o.}\}$ is a tail event associated with this independent sequence of random variables. Kolmogorov's Zero-One Law then says that $\limsup_n A_n$ must have probability zero and one, which coincides with our observation from the Borel-Cantelli lemmas.

5.6 Law of Iterated Logarithms

Definition 5.6.1 — Rate of convergence.

1. A function $\varphi^*(n)$ is called upper for S_n if $S_n \leq \varphi^*(n)$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$, with probability 1.
2. A function $\varphi_*(n)$ is called lower for S_n if $S_n > \varphi_*(n)$ for infinitely many n with probability 1.

Remark 5.6.2 If a function $\psi(n)$ is such that for all $\epsilon > 0$ the function $(1+\epsilon)\psi(n)$ is upper for S_n and the function $(1-\epsilon)\psi(n)$ is lower for S_n, then the function $\psi(n)$ is an optimal rate of convergence for S_n.

Example 5.6.3 Let $\xi_1, \xi_2, \dots$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = \frac{1}{2}$ and let $S_n = \xi_1 + \cdots + \xi_n$. We know that $\frac{S_n}{n} \to 0$. Moreover, since

$$\sum \frac{1}{(n(\log n)^{2\epsilon+1})} < \infty$$

for all $\epsilon > 0$, it follows that $\frac{S_n}{\sqrt{n(\log n)^{1+2\epsilon}}} \xrightarrow{a.s.} 0$ for all $\epsilon > 0$ by Theorem 5.4.10. However, by the central limit theorem

$$\frac{S_n}{\sqrt{n}} \overset{d}{\not\to} 0.$$

Consider some $\varphi(n)$.

- Then
$$\left\{\limsup \frac{S_n}{\varphi(n)} \leq 1\right\} = \left\{\lim_{n\to\infty} \sup_{m\geq n} \frac{S_m}{\varphi(m)} \leq 1\right\}$$
which means that for all $\epsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that $\sup_{m\geq n} \frac{S_m}{\varphi(m)} \leq 1+\varepsilon$ for all $n \geq n_1$. Equivalently, $S_m \leq (1+\varepsilon)\varphi(m)$ for all $m \geq_1$. Therefore, if $\mathbb{P}\big(\limsup \frac{S_n}{\varphi(n)} \leq 1\big) = 1$, it follows that $(1+\varepsilon)\varphi(n)$ is upper for S_n for all $\varepsilon > 0$.
- Similarly,
$$\left\{\limsup \frac{S_n}{\varphi(n)} \geq 1\right\} = \left\{\lim_{n\to\infty} \sup_{m\geq n} \frac{S_m}{\varphi(m)} \geq 1\right\}$$
means that for all $\epsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that $\sup_{m\geq n} \frac{S_m}{\varphi(m)} \geq 1-\varepsilon$ for all $n > n_1$. Equivalently, $S_m \geq (1-\varepsilon)\varphi(m)$ for infinitely many m. So if $\mathbb{P}\left(\limsup \frac{S_n}{\varphi(n)} \geq 1\right) = 1$ then $(1-\varepsilon)\varphi(n)$ is lower for S_n for all $\varepsilon > 0$.

Theorem 5.6.4 — Law of Iterated Logarithm. Let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables with $\mathbb{E}(\xi_1) = 0$ and $\mathbb{E}\left(\xi_1^2\right) = \sigma^2 > 0$. Then

$$\mathbb{P}\left(\limsup \frac{S_n}{\psi(n)} = 1\right) = 1,$$

where

$$\psi(n) = \sqrt{2\sigma^2 n \log(\log n)}.$$

That is, for all $\varepsilon > 0$, the function $(1+\varepsilon)\psi$ is upper and the function $(1-\varepsilon)\psi$ is lower for S_n.

5.7 Solution to Exercises

Exercise 5.2.2

Solution.

- Recall that
$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k.$$
Note that
$$\bigcap_{k\geq n} A_k \subseteq \bigcap_{k\geq n+1} A_k$$

so that by the continuity of the measure it follows that

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k\geq n} A_k\right).$$

Moreover, as $\bigcap_{k\geq n} A_k \subseteq A_k$ for all $k \geq n$ it follows that

$$\mathbb{P}\left(\bigcap_{k\geq n}\right) \leq \inf_{k\geq n} \mathbb{P}(A_k)$$

for all $m \geq n$. Therefore,

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}\left(A_n\right).$$

- Using de Morgan's law it follows that

$$(\limsup A_n)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\right)^c = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k^c = \liminf A_n^c.$$

■

Exercise 5.2.8

Solution. For $n \in \mathbb{N}$ and $\epsilon > 0$ consider the set

$$B_n^{\epsilon} = \left\{\omega : \sup_{k,l\geq n} |\xi_k(\omega) - \xi_l(\omega)| \geq \epsilon\right\}.$$

It is clear that if $\xi_n(\omega) \not\to \xi(\omega)$ then there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $k, l \geq n$ such that $|\xi_k(\omega) - \xi_l(\omega)| \geq \epsilon$. Otherwise, $(\xi_n(\omega))_{n\in\mathbb{N}}$ would be Cauchy and therefore convergent. Therefore,

$$\{\omega : \xi_n(\omega) \not\to \xi(\omega)\} \subseteq \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}.$$

On the other hand, if $\omega \in \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}$ for some $m \in \mathbb{N}$ this implies that for all $n \in \mathbb{N}$ there exists $k, l \geq n$ such that $|\xi_k(\omega) - \xi_l(\omega)| \geq \frac{1}{2m}$. Hence, $\xi_n(\omega) \not\to \xi_(\omega)$. Therefore,

$$\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}} \subseteq \{\omega : \xi_n(\omega) \not\to \xi(\omega)\}.$$

Note that for any $m' \in \mathbb{N}$ we have

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m'}}\right) \leq \mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) \leq \sum_{m=1}^{\infty} \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right).$$

So that $\mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) = 0$ if and only if $\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) = 0$ for all $m \in \mathbb{N}$. As the sets $B_n^{\frac{1}{m}}$ are decreasing in n we know that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n^{\frac{1}{m}}\right) = \lim_{n\to\infty} \mathbb{P}\left(B_n^{\frac{1}{m}}\right) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{k,l\geq n} |\xi_k - \xi_l| \geq \frac{1}{m}\right)$$

for all $m \in \mathbb{N}$. Therefore, $\xi_n \to \xi$ almost surely if and only if

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{k,l\geq n} |\xi_k - \xi_l| \geq \epsilon\right) = 0$$

for all $\epsilon > 0$. ■

Exercise 5.4.8

Solution. Let $S_n = \frac{1}{n}\sum_{k=1}^{n} \xi_k$. Then

$$\mathbb{V}(S_n) = \frac{1}{n^2}\sum_{k=1}^{n} k\eta(k). \tag{5.3}$$

In the notation of Lemma 5.4.7 let $a_k = k$ and $x_k = \eta(k)$ so that

$$b_k = \sum_{i=1}^{n} a_i = \frac{1}{2}n(n+1) \leq \frac{1}{2}n^2.$$

Consequently,

$$\frac{2}{n^2}\sum_{k=1}^{n} k\eta(k) \leq \frac{1}{b_n}\sum_{k=1}^{n} k\eta(k) \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore, using (5.3) it is clear that $\mathbb{V}(S_n) \to 0$ as $n \to \infty$, hence, $S_n \to m$ in L^2. Moreover for $\epsilon > 0$, using Chebyshev's inequality we observe that

$$\mathbb{P}\left(|S_n - m|\right) \leq \frac{\mathbb{V}(S_n)}{\epsilon^2}.$$

Hence, $S_n \to m$ in probability as well. ■

Exercise 5.4.11

Solution. Proceeding directly from $\sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n)$ it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n) &= \sum_{n=1}^{\infty}\sum_{k\geq n} \mathbb{P}(k \leq \xi < k+1) \\
&= \sum_{k=1}^{\infty} k\mathbb{P}(k \leq \xi < k+1) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left(k\chi_{[k,k+1)}\right) \\
&\leq \sum_{k=0}^{\infty} \mathbb{E}\left(\xi\chi_{[k,k+1)}\right) \\
&= \mathbb{E}(\xi) \\
&\leq \sum_{k=0}^{\infty} \mathbb{E}\left((k+1)\chi_{[k,k+1)}\right) \\
&= 1 + \sum_{n=1}^{\infty} \mathbb{P}(\xi \geq n).
\end{aligned}$$

■

6 Convergence in Distribution

In this section, we focus on the weak convergence of measures defined $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

6.1 Weak Convergence

Definition 6.1.1 — Weak convergence. For $n = 1, 2, \ldots, \infty$, let $\xi_n : (\Omega_n, \mathscr{F}_n, \mathbb{P}_n) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be a random variable. Then $\xi_n \to \xi_\infty$ weakly as $n \to \infty$ if

$$\lim_{n\to\infty} (\mathbb{E}_{\mathbb{P}_n}(f(\xi_n))) = \mathbb{E}_{\mathbb{P}}(f(\xi))$$

for all $f \in C_b(X)$.

It is important to note that we do not need to specify the probability space of ξ_n when establishing convergence in distribution, what matters is the distribution of ξ_n. Weak convergence is often taken to be the definition of convergence in distribution, however, convergence in distribution has also been defined differently. Specifically, we say that $\xi_n \to \xi$ in distribution if the distribution function $F_{\xi_n}(x) \to F_\xi(x)$ pointwise for all x where F_ξ is continuous. To show that weak convergence is equivalent to our usual definition of convergence in distribution, we have to show that weak convergence can be formulated on a single probability space.

Theorem 6.1.2 Suppose μ, μ_1, μ_2 are probability measures, such that the corresponding distribution functions, F_n, converge pointwise to F at the points of continuity of F. Then there exists random variables $\xi, \xi_1, \xi_2, \ldots$ defined on a single probability space $(\Omega', \mathscr{F}', \mathbb{P}')$ such that ξ has distribution μ, ξ_n has distribution μ_n, and $\xi_n \to \xi$ $\mathbb{P}'$-almost surely.

Proof. Let $(\Omega', \mathscr{F}', \mathbb{P}') = ([0,1], \mathscr{B}([0,1], \mathsf{Leb}))$. Note that the probability measures μ and μ_n induce distribution functions $F_n, F : \mathbb{R} \to [0,1]$. Let F^{-1} and F_n^{-1} be their right inverses as defined in equation (3.2), then by probability integral transform (Proposition 3.1.2), they have same distribution as μ and the μ_n's respectively. It remains to prove that $F_n^{-1}(u) \to F^{-1}(u)$ almost surely as $n \to \infty$. In fact, we can prove the required limit for any u such that the preimage of $\{u\}$ under F is finite (that is a singleton or empty set). As F is increasing and right continuous there are only countably many u such that the above condition doesn't hold. Establishing the above limit will complete the proof.

Let u be a point such that the preimage of $\{u\}$ under F is finite. We make two observations.

- If $x < F^{-1}(u)$, then using the argument in the proof of Proposition 3.1.2 we have $F(x) < u$. If x is a point of continuity of F, then $F_n(x) \to F(x)$ by assumption, which implies that $F_n(x) < u$ for sufficiently large n. For such n we have that $x \leq F_n^{-1}(u)$ which implies that $x \leq \liminf F_n^{-1}(u)$. Therefore, we can choose a sequence (x_k) of points of continuity of F such that $x_k \nearrow F^{-1}(u)$. Thus we obtain that $F^{-1}(u) \leq \liminf F_n^{-1}(u)$.
- If $x > F^{-1}(u)$, then $F(x) \geq u$. In fact we must have $F(x) > u$, for if $F(x) = u$ then $F(y) = u$ for any $y \in [F^{-1}(u), x]$, contradicting the assumption that the preimage of singleton $\{u\}$ is finite. Repeating the above arguments yields $F^{-1}(u) \geq \limsup_{n\to\infty} F_n^{-1}(u)$.

Combining the observations we have $F_n^{-1}(u) \to F^{-1}(u)$ as desired. ∎

Theorem 6.1.3 Let $\xi, \xi_1, \xi_2, \cdots : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be integrable random variables. Then the following are equivalent.

1. $\xi_n \to \xi$ weakly.
2. $\limsup \mathbb{P}(\xi_n \in E) \leq \mathbb{P}(\xi \in E)$ for any closed set $E \subseteq \mathbb{R}$, .
3. $\liminf \mathbb{P}(\xi_n \in O) \geq \mathbb{P}(\xi \in O)$ for any open set $O \subseteq \mathbb{R}$.
4. $\lim \mathbb{P}(\xi_n \in C) = \mathbb{P}(\xi \in C)$ for any C such that $\mathbb{P}(\xi \in \partial C) = 0$.

5. Let $F_{\xi_n}(x)$ be the distribution function of ξ_n and similarly for $F_\xi(x)$, then $F_{\xi_n}(x) \to F_\xi(x)$ (pointwise) at any point of continuity of $F_\xi(x)$.
6. *For all bounded Lipschitz functions, f, it follows that $\mathbb{E}(f(\xi_n)) \to \mathbb{E}(f(\mu))$.

Remark 6.1.4 Point 6. of Theorem 6.1.3 is not examinable, and consequently we will not include it in our proof but is useful to note for upcoming results.

Proof. (1) $\implies$ (2). Let $E \subseteq \mathbb{R}$ be a closed set and consider the function $f(x) = \chi_E(x)$. Let

$$g(t) = \begin{cases} 1 & t \leq 0 \\ 1-t & 0 \leq t \leq 1 \\ 0 & t \geq 1. \end{cases}$$

Then define

$$f_\epsilon(x) = g\left(\frac{1}{\epsilon}\rho(x,E)\right)$$

where

$$\rho(x,E) = \inf\{|x-y| : y \in E\}.$$

Note that $E_\epsilon := \{x : \rho(x,E) < \epsilon\}$ forms a decreasing sequence of sets as $\epsilon \searrow 0$ such that $E_\epsilon \searrow E$. Observe that,

$$\mathbb{P}_n(\xi_n \in E) = \int f \, d\mathbb{P}_n \leq \int f_\epsilon \, d\mathbb{P}_n.$$

Consequently,

$$\begin{aligned} \limsup \mathbb{P}_n(\xi_n \in E) &\leq \limsup \int f_\epsilon \, d\mathbb{P}_n \\ &\overset{(1)}{=} \int f_\epsilon \, d\mathbb{P} \\ &\leq \mathbb{P}(\xi \in E_\epsilon) \\ &\xrightarrow{\epsilon \searrow 0} \mathbb{P}(\xi \in E). \end{aligned}$$

Where (1) follows from the fact that $\xi_n \to \xi$ weakly.

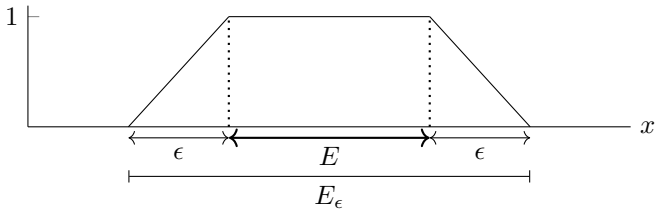


Figure 8: An illustration of how f_ϵ is a continuous and bounded approximation of f.

(2) $\implies$ (3). This is clear after observing that $O = \mathbb{R} \setminus E$ for E a closed set. Moreover, it is clear from this that (2) and (3) are equivalent.
(3) $\implies$ (4). Recall that $\bar{C} = C \cup \partial C$ and $\mathring{C} = C \setminus \partial C$. As $\mathbb{P}(\xi \in \partial C) = 0$ it follows that

- $\limsup \mathbb{P}_n(\xi_n \in C) \leq \limsup \mathbb{P}_n\left(\xi_n \in \bar{C}\right) \leq \mathbb{P}\left(\xi \in \bar{C}\right) = \mathbb{P}(\xi \in C)$, and
- $\liminf \mathbb{P}_n(\xi_n \in C) \geq \liminf \mathbb{P}_n\left(\xi_n \in \mathring{C}\right) \geq \mathbb{P}\left(\xi \in \mathring{C}\right) = \mathbb{P}(\xi \in C)$.

Therefore, $\lim \mathbb{P}_n(\xi_n \in C) = \mathbb{P}(\xi \in C)$.
(4) $\implies$ (5). This is clear.
(5) $\implies$ (1). Let f be a bounded continuous function. Let $X, X_1, X_2, \ldots$ be the random variables given

by Theorem 6.1.2, defined on the probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. It follows that $f(X_n) \to f(X)$ almost surely with respect to $\mathbb{P}'$. Therefore, using the dominated convergence theorem we conclude that

$$\mathbb{E}_n(f(\xi_n)) = \mathbb{E}(f(\xi_n)) \to \mathbb{E}(f(X)) = \mathbb{E}(f(\xi)).$$

Hence, $\xi_n \to \xi$ weakly. ■

Remark 6.1.5 Henceforth, we can establish that a sequence of random variables, (ξ_n), converges in distribution to a random ξ using the notion of weak convergence or (5) of Theorem 6.1.3. In any case, we will denote this convergence as $\xi_n \xrightarrow{d} \xi$.

Corollary 6.1.6 Suppose that the random variables ξ_n, ξ have densities $f_n(x), f(x)$, respectively. Also let $f_n(x) \to f(x)$ for any x. Then $\xi_n \xrightarrow{d} \xi$.

Proof. It is sufficient to show that

$$F_n(x) = \int_{-\infty}^{x} f_n(y)\,\mathrm{d}y \to F(x) = \int_{-\infty}^{x} f(y)\,\mathrm{d}y$$

for any x. Using

$$|F_n(x) - F(x)| \le \int_{-\infty}^{x} |f_n(y) - f(y)|\,\mathrm{d}y,$$

we need to check that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(y) - f(y)|\,\mathrm{d}y = 0.$$

Note that $a = a_+ - a_-$ and $|a| = a_+ + a_-$ for any real a. Since f_n and f are densities, they integrate to 1, so for each n

$$0 = \int_{-\infty}^{\infty} (f(y) - f_n(y))\,\mathrm{d}y = \int_{-\infty}^{\infty} [(f(y) - f_n(y))_+ - (f(y) - f_n(y))_-]\,\mathrm{d}y.$$

Then

$$\int_{-\infty}^{\infty} |f(y) - f_n(y)|\,\mathrm{d}y = 2\int_{\infty}^{\infty} (f(y) - f_n(y))_+\,\mathrm{d}y,$$

which goes to zero by the dominated convergence theorem. Indeed

$$0 \le (f(y) - f_n(y))_+ \le f(y),$$

for any n and the function $f(y)$ is integrable. ■

Exercise 6.1.7 — Uniform convergence of distribution function.

1. Let $F_n \to F$ and suppose that F is continuous. Show that F_n converges *uniformly* to F. That is to say that as $n \to \infty$ we have
$$\sup_x |F_n(x) - F(x)| \to 0.$$

2. Give an example of distribution functions $F_n(x), F(x)$ such that $F_n(x) \xrightarrow{d} F(x)$, but
$$\sup_x |F_n(x) - F(x)| \not\to 0,$$
as $n \to \infty$.

3. Give an example of probability measures $\mathbb{P}, \mathbb{P}_n$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, $n \ge 1$ such that $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$, but convergence $\mathbb{P}_n(B) \to \mathbb{P}(B)$ need not hold for all Borel sets $B \in \mathscr{B}(\mathbb{R})$.

6.2 Connection to Convergence in Probability

We will show convergence in probability is a strictly stronger notion than convergence in distribution. However, there does exist a partial converse that we will explore.

Theorem 6.2.1 Consider random variables $\xi, \xi_1, \ldots$ and $\eta, \eta_1, \ldots$ from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. If $\xi_n \overset{n\to\infty}{\longrightarrow} \xi$ in probability then $\eta_n \overset{n\to\infty}{\longrightarrow} \xi$ in distribution.

$$\begin{array}{ccccc} & & \overset{L^p}{\to} & & \\ & & \Downarrow & & \\ \overset{a.e.}{\to} & \Longrightarrow & \overset{p}{\to} & \Longrightarrow & \overset{d}{\to} \end{array}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that

$$|f(x)| \leq C$$

for all $x \in \mathbb{R}$. Choose M such that $\mathbb{P}(|\xi| > M) \leq \frac{\epsilon}{6C}$. Note that for $x \in [-N, N]$ the function f is uniformly continuous. Therefore, there exists a $\delta > 0$ such that for $|x - y| < \delta$, with $x, y \in [-N, N]$, we have that

$$|f(x) - f(y)| \leq \frac{\epsilon}{3}.$$

Moreover, there exists an N such that

$$\mathbb{P}(|\xi_n - \xi| > \delta) < \frac{\epsilon}{6C}$$

for $n \geq N$. Hence for $n \geq N$ it follows that,

$$\begin{aligned} \mathbb{E}(|f(\xi_n) - f(\xi)|) \leq & \mathbb{E}\left(|f(\xi_n) - f(\xi)| | |\xi_n - \xi| \leq \delta, |\xi| \leq N\right) \\ & + \mathbb{E}\left(|f(\xi_n) - f(\xi)| | |\xi_n - \xi| \leq \delta, |\xi| > N\right) \\ & + \mathbb{E}\left(|f(\xi_n) - f(\xi)| | |\xi_n - \xi| > \delta\right) \mathbb{P}(|\xi_n - \xi| > \delta) \\ \leq & \frac{\epsilon}{3} + 2C\frac{\epsilon}{6C} + 2C\frac{\epsilon}{6C} \\ = & \epsilon. \end{aligned}$$

■

Proposition 6.2.2 Let $\xi_1, \xi_2, \ldots$ be random variables such that $\xi_n \to \xi := c$ in distribution, where c is a constant. It follows that $\xi_n \to \xi = c$ in probability.

Proof. Given an $\epsilon > 0$, the set $E_\epsilon = \mathbb{R} \setminus B_\epsilon(c)$ is closed. Therefore, by Theorem 6.1.3 we conclude that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(|\xi_n - c| \geq \epsilon) &= \lim_{n\to\infty} \mathbb{P}(\xi_n \in E_\epsilon) \\ &= \limsup_n \mathbb{P}(\xi_n \in E_\epsilon) \\ &\leq \mathbb{P}(\xi \in E_\epsilon) \\ &= 0. \end{aligned}$$

Therefore, $\xi_n \to \xi = c$ in probability. ■

Example 6.2.3 Note that convergence in distribution does not imply convergence in probability. Consider a real-valued random variable X that is symmetric about zero, e.g. $\xi \sim \mathsf{N}(0, 1)$. Then the sequence $\xi_n := (-1)^{n+1}\xi$ converges in distribution, as they are identically distributed, but not in probability.

Proposition 6.2.4 — Continuous Mapping Theorem. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function with points of discontinuity, U_φ. Let $\xi, \xi_1, \xi_2, \cdots : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be random variables where $\mathbb{P}(\xi \in U_\varphi) = 0$ and $\xi_n \overset{n\to\infty}{\longrightarrow} \xi$ in distribution, then $\varphi(\xi_n) \overset{n\to\infty}{\longrightarrow} \varphi(\xi)$.

Proof. Let $E \subseteq \mathbb{R}$ be a closed set. Let $x \in \overline{\varphi^{-1}(E)} \setminus U_\varphi$. By definition, there exists a sequence $(x_n) \subseteq \varphi^{-1}(E)$ such that $x_n \to x$. As φ is continuous at x it follows that $\varphi(x_n) \to \varphi(x)$. As $(\varphi(x_n)) \subseteq E$ and E is closed, it follows that $\varphi(x) \in E$ which implies that $x \in \varphi^{-1}(E)$. Therefore,

$$\overline{\varphi^{-1}(E)} \subseteq \varphi^{-1}(E) \cup U_\varphi.$$

Therefore,

$$\begin{aligned}
\limsup \mathbb{P}(\varphi(\xi_n) \in E) &= \limsup \mathbb{P}\left(\xi_n \in \varphi^{-1}(E)\right) \\
&\overset{(1)}{\leq} \limsup \mathbb{P}\left(\xi_n \in \overline{\varphi^{-1}(E)}\right) \\
&\overset{\text{Thm 6.1.3 5}}{\leq} \mathbb{P}\left(\xi \in \overline{\varphi^{-1}(E)}\right) \\
&\leq \mathbb{P}\left(\xi \in \varphi^{-1}(E)\right) + \mathbb{P}(U_\varphi) \\
&= \mathbb{P}\left(\xi \in \varphi^{-1}(E)\right).
\end{aligned}$$

Where (1) is justified as $\varphi^{-1}(E) \subseteq \overline{\varphi^{-1}(E)}$. Therefore, by point 5 from Theorem 6.1.3 we conclude that $\varphi(\xi_n) \to \varphi(\xi)$ in distribution. ■

Theorem 6.2.5 Let $\xi, \xi_1, \xi_2, \cdots : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be random variables such that $\xi_n \to \xi$ in distribution. Moreover, let $\eta_1, \eta_2, \cdots : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be random variables such that $\mathbb{P}(|\xi_n - \eta_n| > \epsilon) \to 0$ as $n \to \infty$ for all $\epsilon > 0$. Then $\eta_n \to \xi$ in distribution.

The proof Theorem 6.2.5 relies on point 6. of Theorem 6.1.3, and so we will omit it here.

Corollary 6.2.6 Consider the random variables $\xi, \xi_1, \ldots$ and $\eta_1, \ldots$ from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Assume $\xi_n \overset{n\to\infty}{\longrightarrow} \xi$ in distribution and $\eta_n \overset{n\to\infty}{\longrightarrow} c$ in probability, with c being a constant. Show that for the random variables $T_n : \omega \in \Omega \mapsto (\xi_n(\omega), \eta_n(\omega))$ and $T : \omega \in \Omega \mapsto (\xi_n(\omega), c)$ we have that $T_n \overset{n\to\infty}{\longrightarrow} T$ in distribution.

One can refer to [2] for proof of these results.

Theorem 6.2.7 — Slutsky's Theorem. Consider the random variables $\xi, \xi_1, \ldots$ and $\eta_1, \ldots$ from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Assume $\xi_n \overset{n\to\infty}{\longrightarrow} \xi$ in distribution and $\eta_n \overset{n\to\infty}{\longrightarrow} c$ in probability, with c being a constant. Then the following hold.

1. $\xi_n + \eta_n \overset{n\to\infty}{\longrightarrow} \xi + c$ in distribution.
2. $\xi_n \eta_n \overset{n\to\infty}{\longrightarrow} c\xi$ in distribution.
3. If $c \neq 0$, then $\frac{\xi_n}{\eta_n} \overset{n\to\infty}{\longrightarrow} \frac{\xi}{c}$ in distribution.

Proof. Using Corollary 6.2.6 we can apply Proposition 6.2.4 to $\varphi(x, y) = x+y$, $\varphi(x, y) = xy$ and $\varphi(x, y) = \frac{x}{y}$ repsectively. ■

Example 6.2.8 Consider a real-valued random variable, ξ, that is symmetric about zero. Let $\xi_n = \xi$ and $\eta_n = (-1)^{n+1}\xi$. Then $\xi_n + \eta_n$ takes the form $(2\xi, 0, 2\xi, 0, \ldots)$ which does not converge in distribution. Hence, it is not true in general that both $\xi_n \overset{n\to\infty}{\longrightarrow} \xi$ and $\eta_n \overset{n\to\infty}{\longrightarrow} \eta$ in distribution implies $\xi_n + \eta_n \overset{n\to\infty}{\longrightarrow} \xi + \eta$ in distribution.

6.3 Relative Compactness and Tightness

Soon we will prove the Central Limit Theorem by looking at characteristic functions. To establish the relationship between characteristic functions and measures, we need some tools relating to the collections of measures and clarify the meaning of *relatively compact* and *tight* collections. These concepts are also useful for working with stochastic processes and ergodic theory.

Definition 6.3.1 A family of probability measures $\mathscr{P} = \{\mathbb{P}_\alpha, \alpha \in A\}$, with the corresponding set of distribution functions F_α is called relatively compact if every sequence of measures from $\mathscr{P}$ contains a subsequence that weakly converges to a probability measure.

Remark 6.3.2 We emphasise that in this definition the limit measure is to be a probability measure, although it need not belong to the original class $\mathscr{P}$. In fact, $\mathscr{P}$ is relatively compact if its closure with respect to the Levi-Prokhorov metric is compact.

Example 6.3.3 The collection consisting of a weakly convergent sequence of measures is relatively compact.

Lemma 6.3.4 Let $\mathbb{P}$ be a probability measure and $\{\mathbb{P}_n\}$ a family of probability measures. Then $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$ if and only if every subsequence $\{\mathbb{P}_{n'}\}$ of $\{\mathbb{P}_n\}$ contains a subsequence $\{\mathbb{P}_{n''}\}$ such that $\mathbb{P}_{n''} \xrightarrow{d} \mathbb{P}$.

Proof. For $\mathbb{P}_n \xrightarrow{d} \mathbb{P}$ it is the case that

$$\int f(x)\, d\mathbb{P}_n(x) \to \int f(x)\, d\mathbb{P}(x)$$

for all bounded and continuous functions f. Clearly, if this holds then any subsequence $\left\{\mathbb{P}_{n'_k}\right\}_{k\in\mathbb{N}}$ converges in distribution to $\mathbb{P}$. Hence, one can simply take the subsequence $n''_k = n'_k$. Conversely, let every subsequence $\left\{\mathbb{P}_{n'_k}\right\}_{k\in\mathbb{N}}$ has a subsequence $\left\{\mathbb{P}_{n''_k}\right\}_{k\in\mathbb{N}}$ such that $\mathbb{P}_{n''} \xrightarrow{d} \mathbb{P}$. Suppose for contradiction that $\mathbb{P}_n \overset{d}{\not\to} \mathbb{P}$, then there exists a bounded and continuous function f such that

$$\left|\int f(x)\, d\mathbb{P}_k(x) - \int f(x)\, d\mathbb{P}\right| \geq \epsilon$$

for infinitely many $k \in \mathbb{N}$. Hence, we can extract a subsequence $\left\{\mathbb{P}_{n'_k}\right\}_{k\in\mathbb{N}}$ such that

$$\left|\int f(x)\, d\mathbb{P}_{n'_k}(x) - \int f(x)\, d\mathbb{P}\right| \geq \epsilon$$

for all $k \in \mathbb{N}$. However, it is clear that this has no convergent subsequence convergent to $\mathbb{P}$, and hence we get a contradiction. ∎

A given family of probability measures $\mathscr{P}$ is not necessarily relatively compact.

Example 6.3.5 Let ξ_n be real-valued random variables and $F_{\xi_n}(x) \to F(x)$ for all $x \in U_F$. Then $F(x)$ is not necessarily a distribution function. Hence, the family $\{\xi_n\}$ is not relatively compact.

1. The variables could run away to infinity. Let ξ_n be $U[n, n+1]$, then $F(x) \equiv 0$.
2. The variables could spread across infinity. Let ξ_n be $U[-n, n]$, then $F(x) \equiv 1/2$.

Let us denote the collection of non-decreasing, right-continuous functions as $\mathscr{G} = \{F : \mathbb{R} \to [0,1]\}$, and refer to it as the collection of generalised distribution functions.

Remark 6.3.6 Distribution functions form a subset of $\mathscr{G}$ for which $F(-\infty) = 0$ and $F(\infty) = 1$.

Theorem 6.3.7 — Helly's Selection Theorem. The collection $\mathscr{G}$ of generalised distribution functions is sequentially compact. That is, for any sequence $(F_n)_{n\in\mathbb{N}} \in \mathscr{G}$, there exists a function $F \in \mathscr{G}$ and a subsequence $F_{n_k} \subseteq \{F_n\}$ such that $F_{n_k}(x) \to F(x)$ for every point $x \in \mathbb{R} \setminus U_F$, where U_F is the set of discontinuities of F.

Proof. Let $(q_k)_{k\geq 1}$ be an enumeration of $\mathbb{Q}$, that is a bijection from $\mathbb{N}$ to $\mathbb{Q}$.

- The sequence $(F_n(q_1))_{n\geq 1}$ is bounded, so by the Bolzano-Weierstrass theorem it has a subsequence $\left(F_{n_k^{(1)}}(q_1)\right)_{k\geq 1}$ which converges to some number $G(q_1) \in [0,1]$.
- Now consider the sequence $\left(F_{n_k^{(1)}}(q_2)\right)_{k\geq 1}$. This is also bounded and so by the Bolzano-Weierstrass theorem it has a subsequence $\left(F_{n_k^{(2)}}(q_2)\right)_{k\geq 1}$ which converges to some number $G(q_2) \in [0,1]$.
- We repeat to extract further subsequences.

We can illustrate the above procedure with the following diagram.

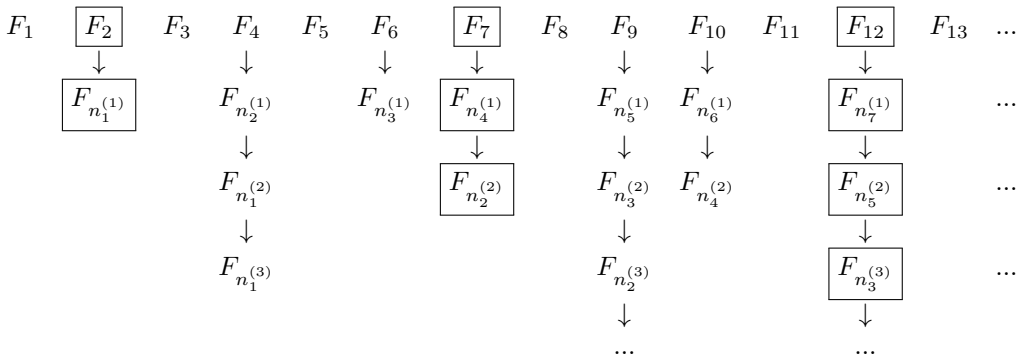


Figure 9: Procedure of extracting a diagonal subsequence of (F_n).

We now extract the diagonal subsequence, that is we let $n_k = n_k^{(k)}$ for all $k \in \mathbb{N}$. Note that for any $m \in \mathbb{N}$ we have $\{n_k\}_{k\in\mathbb{N}} \subseteq \left\{n_k^{(m)}\right\}_{k\in\mathbb{N}}$. Hence, $F_{n_k}(q_m) \to G(q_m)$ as $\left(F_{n_k^{(m)}}\right)_{k\in\mathbb{N}}$ is a converging sequence, and any subsequence of a converging sequence converges to the same limit as the main sequence. Therefore, $F_{n_k}(q) \to G(q)$ for all $q \in \mathbb{Q}$. Moreover, for any $p, q \in \mathbb{Q}$ we have $G(p) \leq G(q)$ by the monotonicity of limits. Now define

$$F(x) = \inf\{f(q) \,:\, q \in \mathbb{Q}, q > x\} = \lim_{q\to x+} G(q).$$

It is clear that $F \in \mathscr{G}$. Moreover, for all $q \in \mathbb{Q}$ we have $F(q) \geq G(q)$ and for $x < q$ we have $F(x) \leq G(q)$. It remains to show that $F_{n_k}(x) \to F(x)$ for all $x \in \mathbb{R} \setminus U_F$, so let us fix $x \in \mathbb{R} \setminus U_F$ and some arbitrary $\epsilon > 0$. By continuity one can choose $y < r < x < q$ with $r, q \in \mathbb{Q}$, such that

$$F(x) - \epsilon < F(y) \leq F(r) = G(r) \leq F(x) \leq F(q) = G(q) < F(x) + \epsilon,$$

and so for sufficiently large k, it follows that $F_{n_k}(r), F_{n_k}(q) \in (F(x) - \epsilon, F(x) + \varepsilon)$ which implies that $F_{n_k}(x) \in (F(x) - \epsilon, F(x) + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. ■

It turns out that the following notion of tightness provides a necessary and sufficient condition for a family of probability measures (or finite measures) to be relatively compact.

Definition 6.3.8 — Tightness. A family of probability measures $\mathscr{P} = \{\mathbb{P}_\alpha\}_{\alpha\in A}$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is **tight** if

for all $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}$ such that

$$\sup_{\alpha \in A} \mathbb{P}_\alpha(\mathbb{R} \backslash K) \leq \varepsilon.$$

Theorem 6.3.9 — Prokhorov's theorem. A family of probability measures $\mathscr{P}$ is tight if and only if it is relatively compact.

Proof. Let us suppose that $\mathscr{P} := \{\mathbb{P}_\alpha\}_{\alpha \in A}$ is a relatively compact but not tight. Then there exists a $\epsilon > 0$ such that for any compact $K \subset \mathbb{R}$ we have $\sup_\alpha \mathbb{P}_\alpha(\mathbb{R} \setminus K) < \epsilon$. Hence, for any n there is a $\mathbb{P}_{\alpha_n}$ such that

$$\mathbb{P}_{\alpha_n}(\mathbb{R} \setminus (-n, n)) > \epsilon. \tag{6.1}$$

By relative compactness, there is a subsequence $(\mathbb{P}_{\alpha_{n_k}})_{k \geq 1}$ such that $\mathbb{P}_{\alpha_{n_k}} \overset{k \to \infty}{\longrightarrow} Q$ weakly, where Q is a probability measure. Therefore, by 6.1.3, it follows that

$$\limsup_{k \to \infty} \mathbb{P}_{\alpha_{n_k}}(\mathbb{R} \setminus (-n, n)) \leq Q(\mathbb{R} \setminus (-n, n)).$$

But the right-hand side $\searrow 0$ as $n \to \infty$, which contradicts (6.1). Hence $\mathscr{P}$ must be tight.

Now let $\mathscr{P}$ be tight, and $(\mathbb{P}_n)_{n \in \mathbb{N}}$ be a sequence of elements in $\mathscr{P}$ with corresponding distribution functions $(F_n)_{n \in \mathbb{N}}$. By Helly's selection theorem, there exists a subsequence $F_{n_k}(x) \to F(x)$ at $x \in \mathbb{R} \backslash U_F$ where F is some generalised distribution function. We now check that $F(-\infty) = 0$ and $F(+\infty) = 1$. Fix $\epsilon > 0$, then from tightness there is an $I = (a, b]$ such that,

$$\sup_n \mathbb{P}_n(\mathbb{R} \setminus I) < \epsilon.$$

Consequently, $1 - \inf_n \mathbb{P}_n(I) > \epsilon$ which implies that $\mathbb{P}_n(I) > 1 - \epsilon$ for all $n \in \mathbb{N}$. Now let $a' < a < b < b'$ where $a', b' \in \mathbb{R} \setminus U_F$, then

$$\begin{aligned} 1 - \epsilon < \mathbb{P}_{n_k}((a, b]) &< \mathbb{P}_{n_k}((a', b']) \\ &= F_{n_k}(b') - F_{n_k}(a') \\ &\overset{k \to \infty}{\longrightarrow} F(b') - F(a'). \end{aligned}$$

This implies that $F(+\infty) - F(-\infty) \geq 1$. Since, $F : \mathbb{R} \to [0, 1]$ we must have $F(+\infty) = 1$ and $F(-\infty) = 0$. Therefore, F is a distribution function which implies that $\mathscr{P}$ is relatively compact. ∎

Remark 6.3.10

1. From Theorem 6.3.9, one see that for a family of random variables, (ξ_n), if

   $$F_{\xi_n}(x) \to F(x)$$

   for all points of continuity of F, where F is a distribution function. Then $(\xi_n)_{n \in \mathbb{N}}$ is tight.

2. Theorem 6.3.9 remains true for measures on $\mathbb{R}^n$, $\mathbb{R}^\infty$ and more generally on any complete separable metric space with a Borel σ-algebra of sets.

Exercise 6.3.11 Let $(\xi_n)_{n \in \mathbb{N}}$ be a family of random variables such that $\mathbb{E}(|\xi_n|) \leq M$ for all $n \in \mathbb{N}$. Show that $(\xi_n)_{n \in \mathbb{N}}$ is tight.

6.4 Solution to Exercises

Exercise 6.1.7

Solution.

1. Fix $\epsilon > 0$. As $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$ and $F(x)$ is non-decreasing, there exists an $M \in \mathbb{R}$ such that for $x > M$ we have $F(x) \in \left(1 - \frac{\epsilon}{2}, 1\right]$ and for $x < -M$ we have $F(x) \in [0, \frac{\epsilon}{2})$. Let $x_\epsilon > M$ then there exists an $N_1 \in \mathbb{N}$ such that $|F_n(x_\epsilon) - F(x_\epsilon)| < \frac{\epsilon}{2}$ for $n \geq N_1$. Therefore, for all $x \geq x_\epsilon$ and $n \geq N_1$ it follows that $F_n(x) \in (1 - \epsilon, 1]$, so that $|F_n(x) - F(x)| < \epsilon$. Similarly, there exists an $N_2 \in \mathbb{N}$ such that for $x \leq -x_\epsilon$ and $n \geq N_2$ we have $|F_n(x) - F(x)| < \epsilon$. Then as F is continuous it is uniformly continuous on $[-x_\epsilon, x_\epsilon]$ so that we can choose

$$x_\epsilon = x_0 < x_1 < \cdots < x_k = b$$

such that $|F(x_{i+1}) - F(x_i)| < \frac{\epsilon}{5}$ for each $i \in \{0, \ldots, k\}$. For each $i \in \{0, \ldots, k\}$ let $\tilde{N}_i \in \mathbb{N}$ be such that

$$|F_n(x_i) - F(x_i)| \leq \frac{\epsilon}{5}$$

for all $n \geq \tilde{N}_i$. For $n \geq \tilde{N} = \max_i \left(\tilde{N}_i\right)$ it follows that

$$\begin{aligned} |F_n(x_{i+1}) - F_n(x_i)| &\leq |F_n(x_{i+1}) - F(x_{i+1})| + |F(x_{i+1}) - F(x_i)| + |F(x_i) - F_n(x_i)| \\ &\leq \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\ &\leq \frac{3\epsilon}{5}. \end{aligned}$$

Therefore, for any $x \in [a, b)$, it is true that $x_i \leq x < x_{i+1}$ for some $i \in \{0, \ldots, k-1\}$. By the non-decreasing property of F we know that $F(x_i) \leq F(x) \leq F(x_{i+1})$ and similarly for each F_n. Therefore,

$$\begin{aligned} |F_n(x) - F(x)| &\leq |F_n(x) - F_n(x_i)| + |F_n(x_i) - F(x_i)| + |F(x_i) - F(x)| \\ &\leq |F_n(x_{i+1}) - F_n(x_i)| + |F_n(x_i) - F(x_i)| + |F(x_i) - F(x_{i+1})| \\ &\leq \frac{3\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\ &< \epsilon. \end{aligned}$$

Therefore, for $n \geq N = \max\left(N_1, N_2, \tilde{N}\right)$ we have that

$$|F_n(x) - F(x)| < \epsilon$$

for all $x \in \mathbb{R}$. Hence, $\sup_x |F_n(x) - F(x)| \to 0$ which is equivalent to uniform convergence.

2. Consider the random variable $\xi : \Omega \to \mathbb{R}$ where $\mathbb{P}(\xi = 1) = 1$. The distribution function of ξ is given by

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1. \end{cases}$$

For $n \in \mathbb{N}$, let $\xi_n : \Omega \to \mathbb{R}$ be the random variable where $\mathbb{P}\left(\xi_n = 1 - \frac{1}{n}\right) = 1$. Similarly, the distribution function of ξ_n is given by

$$F_n(x) = \begin{cases} 0 & x < 1 - \frac{1}{n} \\ 1 & x \geq 1 - \frac{1}{n}. \end{cases}$$

Then for $x > 1$ it is clear that $F_n(x) = F(x) = 1$ for all $n \in \mathbb{N}$ which implies that $\lim_{n\to\infty} F_n(x) = F(x)$. For $x < 1$, there exists a $N \in \mathbb{N}$ such that $\frac{1}{N} < 1 - x$. Therefore, for $n \geq N$ we have that $F_n(x) = F(x) = 0$ and so $\lim_{n\to\infty} F_n(x) = F(x)$. As F is only continuous for $\mathbb{R} \setminus \{0\}$ we conclude that $F_n \xrightarrow{d} F$. However, $\sup_x |F_n(x) - F(x)| = 1$ for all $n \in \mathbb{N}$.

3. Consider the setting of the previous part and let $B = (-\infty, 1) \in \mathscr{B}(\mathbb{R})$. Then $\mathbb{P}(\xi_n \in B) = 1$ for all $n \in \mathbb{N}$, however, $\mathbb{P}(\xi \in (0, 1)) = 0$. Therefore, $\mathbb{P}_n(B) \not\to \mathbb{P}(B)$ for all $B \in \mathscr{B}(\mathbb{R})$.

■

Exercise 6.3.11

Solution. For $\epsilon > 0$ let $K = \left[-\frac{2M}{\epsilon}, \frac{2M}{\epsilon}\right]$ then using Markov's inequality it follows that

$$\begin{aligned}\mathbb{P}\left(\xi_n \in \mathbb{R} \setminus K\right) &= \mathbb{P}\left(|\xi_n| > \frac{2M}{\epsilon}\right) \\ &\leq \frac{\mathbb{E}(|\xi_n|)}{\frac{2M}{\epsilon}} \\ &\leq \frac{\epsilon}{2M} M \\ &< \epsilon.\end{aligned}$$

Therefore, the family $(\xi_n)_{n\in\mathbb{N}}$ is tight. ■

7 Convergence of Characteristic Functions

In this chapter, we look at characteristic functions of measures of $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, as well as random variables taking values on this measurable space.

7.1 Definition

Definition 7.1.1 — Characteristic Functions. The characteristic function of a random variable $\xi : \Omega \to \mathbb{R}$ is

$$\varphi(t) := \varphi_\xi(t) \equiv \mathbb{E}\left(e^{it\xi}\right) := \int_\Omega e^{it\xi(\omega)}\,\mathbb{P}(d\omega)$$

for $t \in \mathbb{R}$.

Remark 7.1.2 We may generalise Definition 7.1.1 to random variables defined on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. The characteristic function of a random vector $\xi := (\xi_1, \ldots, \xi_n)$ is

$$\varphi_\xi(t_1, \ldots, t_n) := \mathbb{E}\left(\exp\left(i\sum_{k=1}^n t_k\xi_k\right)\right).$$

The characteristic function of a random variable only depends on its distribution. If $F(x)$ has density $f(x)$, with respect to the Lebesgue measure, then

$$\varphi(t) = \int_{\mathbb{R}^n} e^{i\left(t^\top x\right)} f(x)\mathrm{d}x.$$

Proposition 7.1.3

1. If ξ is a random variable and $\eta = a\xi + b$ for constants a, b. Then $\varphi_\eta(t) = e^{itb}\mathbb{E}\left(e^{iat\xi}\right)$.
2. For a characteristic function φ we have $|\varphi(t)| \leq \varphi(0) = 1$.
3. Let ξ be a random variable. Then $\varphi_\xi(t)$ is uniformly continuous on $\mathbb{R}$.
4. If $\xi_1, \ldots, \xi_n$ are independent random variables and $S = \xi_1 + \cdots + \xi_n$, then

$$\varphi_S(t) = \prod_{j=1}^n \varphi_{\xi_j}(t).$$

Proof.

1. Clear from properties of the expectation.
2. Clear from the definition.
3. Note that
$$|\varphi(t+h) - \varphi(t)| = \left|\mathbb{E}\left(e^{it\xi}\left(e^{ih\xi} - 1\right)\right)\right| \leq \mathbb{E}\left(\left|e^{ih\xi} - 1\right|\right).$$
By dominated convergence theorem we know that $\mathbb{E}\left(\left|e^{ih\xi} - 1\right|\right) \to 0$ as $h \to 0$. Therefore, we have the uniform continuity of φ.
4. Clear from properties of the expectation.

■

The moment-generating function also shares properties 1. and 4., but the lack of properties 2. and 3. means it is preferable to use characteristic functions to establish weak convergence.

Exercise 7.1.4 — Examples of characteristic functions.

1. Let $\xi \sim \mathsf{B}(n, p)$. Then
$$\varphi_\xi(t) = \left(pe^{it} + (1-p)\right)^n.$$

2. Let $\xi \sim \mathsf{N}(m, \sigma^2)$. Then
$$\varphi_\xi(t) = \exp\left(itm - \frac{t^2\sigma^2}{2}\right).$$
3. Let $\xi \sim \mathsf{Po}(\lambda)$. Then
$$\varphi_\xi(t) = e^{-\lambda+\lambda e^{it}}.$$

7.2 Obtaining Moments

The existence of moments for a real-valued random variable is determined by the smoothness of its characteristic function at zero.

Proposition 7.2.1 — Moments. Let ξ be a random variable with a characteristic function φ and distribution function F.

1. If $\mathbb{E}(|\xi|^n) < \infty$ for some $n \geq 1$ then $\varphi^{(\tau)}(t)$ exists for any $0 \leq \tau \leq n$ and the following hold.
   (a) $\varphi^{(\tau)}(t) = \int_{\mathbb{R}} (ix)^\tau e^{itx} \mathrm{d}F(x)$.
   (b) $\mathbb{E}(\xi^\tau) = \frac{\varphi^{(\tau)}(0)}{i^\tau}$.
   (c) $\varphi(t) = \sum_{\tau=0}^{n-1} \frac{(it)^\tau}{\tau!}\mathbb{E}(\xi^\tau) + \frac{(it)^n}{n!}\varepsilon_n(t)$.

   Where $|\varepsilon_n(t)| \leq 3\mathbb{E}(|\xi|^n)$ and $\varepsilon_n(t) \to 0$ as $t \to 0$.
2. If $\mathbb{E}(|\xi|^n) < \infty$ for all $n \geq 1$ and
$$\limsup_{n\to\infty}\left(\frac{(\mathbb{E}(|\xi|^n))^{\frac{1}{n}}}{n}\right) = \frac{1}{e \cdot R} < \infty,$$
   for $R > 0$, then
$$\varphi(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathbb{E}(\xi^n)$$
   converges for all $|t| < R$.

Proof.

1. Since $\mathbb{E}(|\xi|^n) < \infty$, we have $\mathbb{E}(|\xi|^r) < \infty$ for any $r \leq n$ by Lyapunov's inequality. Consider the difference quotient
$$\frac{\varphi(t+h) - \varphi(t)}{h} = \mathbb{E}\left(e^{it\xi}\left(\frac{e^{ih\xi} - 1}{h}\right)\right).$$
   Since
$$\left|\frac{e^{ihx} - 1}{h}\right| \leq |x|,$$
   if $\mathbb{E}(|\xi|) < \infty$, it follows from the dominated convergence theorem that the limit
$$\lim_{h\to 0} \mathbb{E}\left(e^{it\xi}\left(\frac{e^{ih\xi} - 1}{h}\right)\right)$$
   exists as
$$\mathbb{E}\left(e^{it\xi}\lim_{h\to 0}\left(\frac{e^{ih\xi} - 1}{h}\right)\right) = i\mathbb{E}(\xi e^{it\xi}) = i\int_{-\infty}^{\infty} xe^{itx}\mathrm{d}F(x).$$
   Hence $\varphi'(t)$ exists and equals
$$\varphi'(t) = i\mathbb{E}(\xi e^{it\xi}) = i\int_{-\infty}^{\infty} xe^{itx}\mathrm{d}F(x).$$

The existence of the derivatives $\varphi^{(r)}(t), 1 < r \leq n$ follows by induction. Note that (b) follows immediately from (a). To establish the (c), consider

$$e^{iy} = \cos y + i \sin y = \sum_{k=0}^{n-1} \frac{(iy)^k}{k!} + \frac{(iy)^n}{n!}(\cos \theta_1 y + i \sin \theta_2 y)$$

for real y with $|\theta_1| \leq 1$, $|\theta_2| \leq 1$. Therefore,

$$e^{it\xi} = \sum_{k=0}^{n-1} \frac{(i\xi)^k}{k!} + \frac{(i\xi)^n}{n!}(\cos \theta_1 \xi + i \sin \theta_2 \xi)$$

and so

$$\mathbb{E}\left(e^{it\xi}\right) = \sum_{k=0}^{n-1} \frac{(it)^k}{k!}\mathbb{E}\left(\xi^k\right) + \frac{(it)^n}{n!}(\mathbb{E}\left(\xi^n\right) + \varepsilon_n(t)),$$

where

$$\varepsilon_n(t) = \mathbb{E}\left(\xi^n(\cos \theta_1(\omega)t\xi + i \sin \theta_2(\omega)t\xi - 1)\right).$$

It is clear that $|\varepsilon_n(t)| \leq 3\mathbb{E}\left(|\xi^n|\right)$. Then dominated convergence shows that $\varepsilon_n(t) \to 0$ as $t \to 0$.

2. Let $0 < t_0 < T$. Then,

$$\limsup \frac{(\mathbb{E}\left(|\xi|^n\right))^{1/n}}{n} \leq \frac{1}{et_0}$$

implies that

$$\limsup \frac{(\mathbb{E}\left(|\xi|^n e^n t_0^n\right))^{1/n}}{n} \leq 1.$$

Thus

$$\limsup \left(\frac{\mathbb{E}\left(|\xi|^n e^n t_0^n\right)}{n^n}\right)^{1/n} < 1.$$

By Stirling's formula we have that

$$\limsup \left(\frac{\mathbb{E}\left(|\xi|^n t_0^n\right)}{n!}\right)^{1/n} < \limsup \left(\frac{\mathbb{E}\left(|\xi|^n e^n t_0^n\right)}{n^n}\right)^{1/n} < 1.$$

Consequently, the series $\sum \mathbb{E}\left(\frac{|\xi|^n t_0^n}{n!}\right)$ converges by Cauchy's test and so the series $\sum_{r=0}^{\infty} \frac{(it)^r}{r!}\mathbb{E}\left(\xi^r\right)$ converges for $|t| \leq t_0$. By the previous statement for $n \geq 1$ we know that

$$\varphi(t) = \sum_{r=0}^{n} \frac{(it)^r}{r!}\mathbb{E}\left(\xi^r\right) + R_n(t),$$

where $|R_n(t)| \leq \frac{3|t|^n}{n!}\mathbb{E}\left(|\xi|^n\right)$. Therefore

$$\varphi(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\mathbb{E}\left(\xi^r\right)$$

for all $|t| < T$.

■

Remark 7.2.2 The second part of Proposition 7.2.1 gives a sufficient condition for the moments $\mathbb{E}\left(\xi^n\right)$ to determine $\varphi(t)$ uniquely. Indeed, they already determine $\varphi(t)$ for $-R < t < R$. Take s such that $|s| < \frac{R}{2}$ and follow the proof to obtain that

$$\varphi(t) = \sum_{k=0}^{\infty} i^k \frac{(t-s)^k}{k!}\varphi^{(k)}(s),$$

where

$$\varphi^{(k)}(s) = \mathbb{E}\left(\xi^k e^{is\xi}\right),$$

for $-\frac{R}{2} < s < \frac{R}{2}$, is uniquely determined by $\mathbb{E}(\xi^n)$ for $n \geq 1$. Therefore, the moments uniquely determine $\varphi(t)$ for $|t| < \frac{3}{2}R$. Proceed analogously to increase the domain in which $\varphi(t)$ is defined.

Theorem 7.2.3 — Carleman's test. A sufficient condition for the unique determination of the characteristic function $\varphi(t)$ is

$$\sum_{n=0}^{\infty} \frac{1}{\left(\mathbb{E}\left(\xi^{2n}\right)\right)^{\frac{1}{2n}}} = \infty.$$

Example 7.2.4 If $\mathbb{E}(\xi^n)$ grows too fast, there may be multiple characteristic functions $\varphi(t)$ with the same moments. Consider a random variable distributed as a standard log-normal distribution, that is it has a density

$$f_0(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\ln x)^2}{2}\right),$$

for $x \geq 0$. Consider another random variable with density

$$f_a(x) = f_0(x) \times (1 + a \sin(2\pi \ln x)),$$

for $x \geq 0$ and $a \in [-1, 1]$.

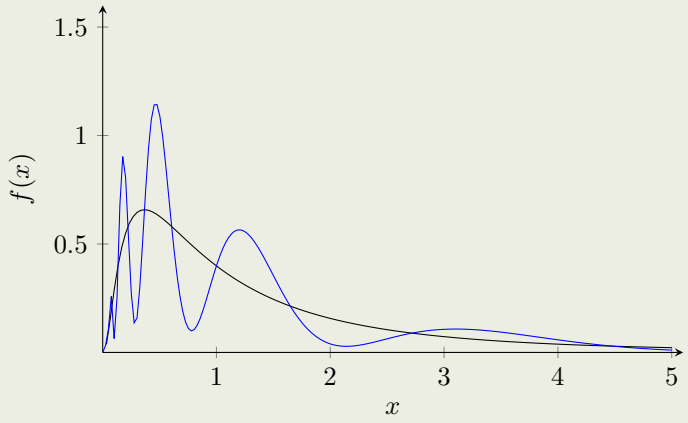


Figure 10: Density of a log-normal distribution and its perturbed version.

These seemingly different random variables have the same r^{th} moment. To see this, it suffices to evaluate the integral

$$\int_0^{\infty} x^{r-1} \exp\left(-\frac{(\ln x)^2}{2}\right) \sin(2\pi \ln x)\, dx,$$

for $r = 0, 1, \ldots$. With a variable substitution of $s = \ln x$, the integral becomes

$$\int_{-\infty}^{\infty} \exp\left((r-1)s\right) \exp\left(-\frac{s^2}{2}\right) \sin(2\pi s)\, dx.$$

The integrand is an L^1 function multiplied by $\sin(2\pi s)$. Therefore, by the Riemann-Lebesgue lemma, the integral is zero, and the two random variables have the same moments.

Exercise 7.2.5 — Computing the moments of log-normal distribution.

1. Verify that if ξ has a standard normal distribution ($\mathsf{N}(0, 1)$), then $\exp(\xi)$ has a standard log-normal distribution.
2. Use LOTUS to show that the r^{th} moment is equal to $\exp\left(\frac{r^2}{2}\right)$.

Notice that the moments grow too fast for the characteristic function to be analytical.

7.3 Inversion Formula

Theorem 7.3.1 — Inversion formula I. Let ξ be a random variable on $(\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with distribution $F(x)$ and characteristic function $\varphi(t)$. If $a < b$ are points of continuity of F then

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt.$$

To prove Theorem 7.3.1 let us define

$$S(T) = \int_0^T \frac{\sin x}{x}\, dx.$$

Note $S(T)$ is a differentiable function with $S(T) > 0$ whenever $T > 0$. Moreover,

$$S(+\infty) = \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}.$$

This can be proven using standard calculus techniques[1]. We therefore know that $S(T)$ is a bounded function and $\sup_{T>0} S(T)$ exists. We also note that

$$\int_0^T \frac{\sin(kx)}{x}\, dx = \int_0^T \frac{\sin(kx)}{kx}\, d(kx) = S(kT)$$

for $k > 0$ and when $k < 0$, we have

$$\int_0^T \frac{\sin(kx)}{x}\, dx = -\int_0^T \frac{\sin(|k|x)}{x}\, dx = -S(|k|T).$$

Equivalently, we can write

$$\int_0^T \frac{\sin(kx)}{x} = \operatorname{sgn}(k) S(|k|T).$$

Moreover, as the integrand is even we know that

$$\int_{-T}^T \frac{\sin(kx)}{x} = 2\operatorname{sgn}(k) S(|k|T).$$

Proof. (Theorem 7.3.1) For fixed T let

$$I_T = \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt = \int_{-T}^T \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} e^{itx}\, dF(x)\, dt$$

We note that the integrand is bounded uniformly in (t, x) by

$$\left| \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \right| \le \left| \int_a^b e^{-its}\, ds \right| \le |b - a|, \tag{7.1}$$

and

$$\int_T^T \int_{\mathbb{R}} |b - a|\, dF(x)\, dt = 2T|b - a| < \infty.$$

Therefore, by Fubini's theorem we may exchange the order of integration so that

$$I_T = \int_{\mathbb{R}} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} e^{itx}\, dt\, dF(x) = \int_{\mathbb{R}} \int_{-T}^T \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\, dt\, dF(x).$$

Since the domain of integration of the inner integral is symmetric, we can ignore the odd parts of the integrand, so that

$$I_T = \int_{\mathbb{R}} \int_{-T}^T \frac{\sin(t(x-a)) - \sin(t(x-b))}{t}\, dt\, dF(x).$$

[1] https://www.wikihow.com/Integrate-the-Sinc-Function

Now let

$$J_{T,x} := \int_{-T}^{T} \frac{\sin(t(x-a)) - \sin(t(x-b))}{t}$$

so that

$$I_T = \int_{\mathbb{R}} J_{T,x}\, dF(x).$$

Note that

$$|J_{T,x}| \leq \left| \int_{-T}^{T} \frac{\sin(t(x-a))}{t} \right| + \left| \int_{-T}^{T} \frac{\sin(t(x-b))}{t} \right| \leq 4 \sup_{T>0} S(T) < \infty,$$

which is integrable with respect to F. Therefore by the dominated convergence theorem we have that

$$I_\infty := \lim_{T\to\infty} I_T = \int_{\mathbb{R}} \lim_{T\to\infty} J_{T,x}.$$

Note that

$$\lim_{T\to\infty} J_{T,x} = \begin{cases} 0 & x \notin [a,b] \\ \pi & x \in \{a,b\} \\ 2\pi & x \in (a,b). \end{cases}$$

Therefore,

$$I_\infty = 2\pi(F(b-) - F(a)) - \pi\left(F(a) - F(a-) - (F(b-) - F(b))\right),$$

but as a and b are points of continuity we conclude that

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt.$$

■

Corollary 7.3.2 There is a one-to-one correspondence between probability distributions on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and characteristic functions.

Proof. Let F and G be probability distribution functions on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with the same characteristic function. By Theorem 7.3.1 we note that $F(b) - F(a) = G(b) - G(a)$ for any $a < b$ that are points of continuity of F and G, which is dense in $\mathbb{R}$. Since the collection of open intervals $\{(a,b) : a < b\}$ generates $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, we must have $F = G$. ■

The next inversion formula concerns the absolute continuity distribution functions with respect to the Lebesgue measure.

Proposition 7.3.3 — Inversion formula II. Let ξ be a random variable on $(\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with distribution $F(x)$ and characteristic function $\varphi(t)$. If $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$, then $F(x)$ is absolutely continuous with density $f(x)$, and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt. \tag{7.2}$$

Proof. Let $f(x)$ be as defined in (7.2). Then for $|h| > 0$ consider

$$f(x+h) - f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-it(x+h)} - e^{-itx}\right) \varphi(t)\, dt.$$

Note that

$$\left|e^{-it(x+h)} - e^{-itx}\right| |\varphi(t)| \leq 2|\varphi(t)|$$

and

$$\left(e^{-it(x+h)} - e^{-itx}\right) \varphi(t) \overset{|h|\to 0}{\longrightarrow} 0.$$

Therefore, as

$$\int_{-\infty}^{\infty} |2\varphi(t)|\, dt = 2 \int_{-\infty}^{\infty} |\varphi(t)|\, dt < \infty$$

we can deduce from the dominated convergence theorem that

$$|f(x+h) - f(x)| \overset{|h|\to 0}{\longrightarrow} 0$$

which implies that f is continuous. Similarly, one can show that f is differentiable. Hence, f is integrable on $[a,b]$ for $a < b$. Therefore, we can deduce that

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^b \int_{-\infty}^{\infty} e^{-itx}\varphi(t)\,dt\,dx \\ &\overset{\text{Fubini.}}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{e^{-ita} - e^{-itb}}{it}\varphi(t)\,dt \\ &\overset{\text{Thm 7.3.1}}{=} F(a) - F(b)\end{aligned}$$

where $F(x) = \int_{-\infty}^{x} f(y)\,dy$ for all $x \in \mathbb{R}$ and so F is absolutely continuous. Suppose F has density g, then

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} g(x)\,dx.$$

As $\varphi(t)$ is integrable we can apply the inversion formula for Fourier transforms to deduce that

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)\,dt = f(x).$$

Therefore, F the density of F is f.

■

7.4 Central Limit Theorems

Theorem 7.4.1 — Levi's Continuity theorem. Let $\varphi_n(t)$ be the characteristic functions of a sequence of distribution functions F_n.

1. If $F_n \to F$ weakly, where F is a distribution function, then $\varphi_n(t) \to \varphi(t)$ pointwise for all $t \in \mathbb{R}$, where φ is the characteristic function of F.
2. If $\lim_{n\to\infty} \varphi_n(t)$ exists for all $t \in \mathbb{R}$, and $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$ is continuous at $t = 0$, then $\varphi(t)$ is a characteristic function of some distribution function F and $F_n \to F$ weakly.
3. If $\varphi_n(t)$ is a characteristic function corresponding to a distribution function F_n and $\varphi(t)$ is a characteristic function corresponding to some distribution function F. Then $\varphi_n(t) \to \varphi(t)$ for all $t \in R$ if and only if $F_n \to F$ weakly.

Statement 1. is a direct consequence of the definition of weak convergence when applied to $\mathrm{Re}\left(e^{it\xi}\right)$, and $\mathrm{Im}\left(e^{it\xi}\right)$.

To prove statements 2. and 3., we need the following estimates.

Lemma 7.4.2 If $\mathbb{P}$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with characteristic function $\varphi(t)$, then

$$\mathbb{P}\left(\left\{x : |x| \geq \frac{2}{\epsilon}\right\}\right) \leq \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} (1 - \varphi(t))\,dt$$

for all $\epsilon > 0$.

Proof. Note that for all $x \neq 0$, we have

$$\int_{-\epsilon}^{\epsilon} (1 - e^{itx})\,dt = 2\epsilon - \frac{e^{it\epsilon} - e^{-it\epsilon}}{ix} = 2u\left(1 - \frac{\sin \epsilon x}{\epsilon x}\right).$$

Therefore,

$$\begin{aligned}
\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}(1-\varphi(t))\,dt &= \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}\left(1-\int_{\mathbb{R}}e^{itx}\,\mu(dx)\right)\,dt \\
&= \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}}(1-e^{itx})\,\mu(dx)\,dt \\
&\overset{\text{Fubini.}}{=} \int_{\mathbb{R}}\left(\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}(1-e^{itx})\,dt\right)\,\mathbb{P}(dx) \\
&= \int_{\mathbb{R}} 2\underbrace{\left(1-\frac{\sin\epsilon x}{\epsilon x}\right)}_{\geq 0}\,\mathbb{P}(dx) \\
&\geq 2\int_{-2/\epsilon}^{2/\epsilon}\left(1-\frac{\sin\epsilon x}{\epsilon x}\right)\,\mathbb{P}(dx) \\
&\geq 2\int_{-2/\epsilon}^{2/\epsilon}\underbrace{\left(1-\frac{1}{|\epsilon x|}\right)}_{\geq 1/2}\,\mathbb{P}(dx) \\
&\geq \mathbb{P}\left(\left\{x:|x|\geq\frac{2}{\epsilon}\right\}\right).
\end{aligned}$$

■

Lemma 7.4.2 shows that the tail of the measure $\mathbb{P}$, hence the existence of moments, is determined by the smoothness of φ at zero.

Proof. (Theorem 7.4.1 Statements 2. and 3.). Note that the statement 3. follows from statement 2. using Theorem 7.3.1. So it suffices to show statement 2. We first prove that the sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight. As φ is continuous at 0 we know that $\varphi_\infty(0)=1$ and so for all $\epsilon>0$ there is $u>0$ small enough that for all $t\in[-u,u]$ we have $1-\varphi_\infty(t)\leq\frac{\epsilon}{4}$. Hence,

$$\frac{\epsilon}{2}\geq\frac{1}{u}\int_{-u}^{u}(1-\varphi_\infty(t))\,dt\overset{\text{DCT}}{=}\lim_{n\to\infty}\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt.$$

As a result, there is n_0 such that

$$\mathbb{P}_n\left(\mathbb{R}\setminus\left[-\frac{2}{u},\frac{2}{u}\right]\right)\leq\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt\leq\epsilon$$

for all $n\geq n_0$. We may choose smaller u such that the above inequality holds for all $n\geq 1$, and hence we see that $(\mathbb{P}_n)_{n\geq1}$ is tight. By Prokhorov theorem, for any subsequence of $(F_n)_{n\geq1}$, say $(F_{n_k})_{k\geq1}$, there is a further subsequence that converges weakly to F. We note that by statement (1) Theorem 7.4.1 that $\varphi(t)$ is the characteristic function of F, which also shows that the limiting distribution function is the same regardless of the subsequence we choose. Consequently, $F_n\to F$ weakly as otherwise there is a point $y\in\mathbb{R}\setminus U_F$, such that they're exists a subsequence $(F_{n'_k})_{k\geq1}$ with the property $|F_{n'_k}(y)-F(y)|\geq\epsilon$ for all k. However, by the above arguments, there is a further subsequence $(F_{n_{k_j}})_{j\geq1}$ which converges to F, which is a contradiction. ■

Theorem 7.4.3 — Central Limit Theorem for Independent Identically Distributed Random Variables. Let $\xi_1,\xi_2,\ldots$ be a sequence of independent identically distributed nondegenerate random variables with $\mathbb{E}\left(\xi_1^2\right)<\infty$ and $S_n=\xi_1+\cdots+\xi_n$. Then

$$\mathbb{P}\left(\frac{S_n-\mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}}\leq x\right)\longrightarrow\Phi(x)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-u^2/2}\mathrm{d}u$$

as $n\to\infty$ for all $x\in\mathbb{R}$.

Exercise 7.4.4 Show that Theorem 7.4.3 can be written as

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} \xrightarrow{d} \mathsf{N}(0,1).$$

Theorem 7.4.5 — Lindeberg CLT for independent random variables. Let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with finite second moments $\mathbb{E}(\xi_j^2) < \infty$. Let

1. $m_j = \mathbb{E}(\xi_j)$,
2. $\sigma_j^2 = \mathbb{V}(\xi_j) > 0$,
3. $S_n = \xi_1 + \cdots + \xi_n$, and
4. $D_n^2 = \sum_{j=1}^n \sigma_j^2$.

Moreover, suppose that

$$\frac{1}{D_n^2} \sum_{k=1}^n \mathbb{E}\left(|\xi_k - m_k|^2 \chi_{\{|\xi_k - m_k| \geq \varepsilon D_n\}}\right) \overset{n\to\infty}{\longrightarrow} 0 \tag{7.3}$$

for every $\varepsilon > 0$. Then

$$\frac{S_n - \mathbb{E}(S_n)}{D_n} \xrightarrow{d} \mathsf{N}(0,1).$$

Remark 7.4.6 The condition given in equation (7.3) is known as the Lindeberg condition.

We focus on some special cases in which the Lindeberg condition is satisfied and consequently, the central limit theorem is valid. One of the most prominent is the Lyapunov condition.

Corollary 7.4.7 — Lyapunov's CLT. Assume the conditions of Theorem 7.4.5 and in addition assume that the sequence $(\xi_k)_{\geq 1}$ is such that

$$\frac{1}{D_n^{2+\delta}} \sum_{k=1}^n \mathbb{E}\left(|\xi_k - m_k|^{2+\delta}\right) \to 0 \tag{7.4}$$

for some $\delta > 0$ as $n \to \infty$. Then the sequence $(\xi_k)_{\geq 1}$ satisfies the Lindeberg condition, and so the conclusions of Theorem 7.4.3 hold.

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned}\mathbb{E}\left(|\xi_k - m|^{2+\delta}\right) &\geq \mathbb{E}\left(|\xi_k - m_k|^{2+\delta} \chi_{\{|\xi_k - m_k| \geq \varepsilon D_n\}}\right) \\ &\geq \varepsilon^\delta D_n^\delta \mathbb{E}\left(|\xi_k - m_k|^{2+\delta} \chi_{\{|\xi_k - m_k| \geq \varepsilon D_n\}}\right).\end{aligned}$$

Which implies that

$$\frac{1}{D_n^2} \sum_{k=1}^n \int_{\{x : |x - m_k| \geq \varepsilon D_n\}} (x - m_k)^2 \mathrm{d}F_k(x) \leq \frac{1}{\varepsilon^\delta} \frac{1}{D_n^{2+\delta}} \sum_{k=1}^n \mathbb{E}\left(|\xi_k - m_k|^{2+\delta}\right).$$

Consequently, $(\xi_k)_{k\geq 1}$ satisfies the Lindeberg condition. ∎

Remark 7.4.8 The condition of equation (7.4) is known as the Lyapunov condition.

Exercise 7.4.9 Under the settings of Theorem 7.4.7, suppose that there exists K such that

$$|\xi_k| \leq K < \infty$$

for all k, and that $D_n \to \infty$ as $n \to \infty$. Show that the sequence $(\xi_k)_{k\geq 1}$ satisfies the Lindeberg condition.

Theorem 7.4.3 does not hold when $\mathbb{E}\left(\xi_1^2\right) = \infty$. Let $\xi_1, \xi_2, \ldots$ be i.i.d. with Cauchy distribution, that is they have the density

$$f = \frac{\theta}{\pi(x^2 + \theta^2)}$$

for $\theta > 0$. One can see that

$$\varphi_{\xi_1}(t) = e^{-\theta|t|}$$

for $t \in \mathbb{R}$, which implies that

$$\varphi_{\frac{S_n}{n}}(t) = \left(\exp\left(\frac{-\theta|t|}{n}\right)\right)^n = e^{-\theta|t|},$$

Therefore, $\frac{S_n}{n}$ also has a Cauchy distribution.

7.5 Berry-Esseen Inequality

The Central Limit Theorem for iid random variables imply that if $F_n(x)$ is the distribution function of the random variable $(S_n - \mathbb{E}(S_n))/\sqrt{\mathbb{V}(S_n)}$, then

$$\sup_x |F_n(x) - \Phi(x)| \overset{n\to\infty}{\longrightarrow} 0.$$

At what rate does the left-hand side decay? If $(\xi_k)_{k\geq 1}$ are iid with $\xi_1 \in L^3$, then we get bounds on the rate of decay.

Theorem 7.5.1 — Berry-Esseen Inequality. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed random variables with $\mathbb{E}\left(|\xi_1|^3\right) < \infty$. Then

$$\sup_x \left| \mathbb{P}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} \leq x \right) - \Phi(x) \right| \leq C \frac{\mathbb{E}\left(|\xi_1 - \mathbb{E}(\xi_1)|^3\right)}{\sigma^3 \sqrt{n}},$$

where the constant C satisfies

$$\frac{1}{\sqrt{2\pi}} \leq C \leq \frac{1}{2}.$$

Although we do not provide a proof of Theorem 7.5.1, we note that the rate $O\left(\frac{1}{\sqrt{n}}\right)$ is optimal.

Remark 7.5.2 Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed Bernoulli random variables with $\mathbb{P}(\xi_k = 1)$ and $\mathbb{P}(\xi_k = -1) = \frac{1}{2}$. By symmetry, we know that,

$$2\mathbb{P}\left(\sum_{k=1}^{2n} \xi_k < 0\right) + \mathbb{P}\left(\sum_{k=1}^{2n} \xi_k = 0\right) = 1$$

and therefore,

$$\begin{aligned}
\left| \mathbb{P}\left(\sum_{k=1}^{2n} \xi_k < 0\right) - \frac{1}{2} \right| &= \frac{1}{2}\mathbb{P}\left(\sum_{k=1}^{2n} \xi_k = 0\right) \\
&= \frac{1}{2}\binom{2n}{n}\frac{1}{2^{2n}} \\
&\sim \frac{1}{2\sqrt{\pi n}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{2n}}.
\end{aligned}$$

Then $\mathbb{E}(|\xi_1|^3) = 1 = \sigma$, and Theorem 7.5.1 cannot be improved in terms of $O\left(\frac{1}{\sqrt{n}}\right)$ and $C \geq \frac{1}{\sqrt{2\pi}}$.

7.6 Constructing Characteristic Functions

The following theorems determine whether a function φ is a characteristic function of some measure on $\mathbb{R}$, and if so, whether we can easily construct the underlying measure. The constructions are usually difficult and therefore are not usually covered in great detail. Nevertheless, the proofs in this section serve as great examples of using tools developed in the previous chapter. For further discussions refer to [3].

Theorem 7.6.1 — Bochner-Khinchin. Let $\varphi(t)$ be continuous, $t \in \mathbb{R}$, with $\varphi(0) = 1$. A necessary and sufficient condition that $\varphi(t)$ is a characteristic function is that it is positive semi-definite. That is, for all $t_1, \dots, t_n \in \mathbb{R}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, and $n = 1, 2, \dots$ we have

$$\sum_{j,k=1}^{n} \varphi(t_j - t_k)\lambda_j \overline{\lambda}_k \geq 0.$$

To show necessity, we note that if φ is a characteristic function of a real-valued random variable ξ, then

$$\sum_{j,k=1}^{n} \varphi(t_j - t_k)\lambda_j \overline{\lambda}_k = \mathbb{E}\left(\eta\bar{\eta}\right) = \mathbb{E}\left(|\eta|^2\right) \geq 0,$$

where $\eta = \sum_{j=1}^{n} \lambda_j e^{t_j \xi}$.

7.6.1 Polya's Criterion

Theorem 7.6.2 — Polya's criterion. Let a continuous even real-valued function $\varphi(t)$ satisfy $\varphi(t) \geq 0$, $\varphi(0) = 1$, $\varphi(t) \to 0$ as $t \to \infty$ and let $\varphi(t)$ be convex on $0 \leq t < \infty$. Then $\varphi(t)$ is a characteristic function.

As an observation, we note that the function $\varphi(t)$ must be strictly decreasing over $[0, \infty)$. To see this, we let $0 < r < s$, then there is $t_0 > s$ such that $0 < f(t_0) < \frac{f(r)}{2}$. By convexity, we have

$$f(s) \leq \frac{f(t_0) - f(r)}{t_0 - r}(s - r) + f(r) < f(r).$$

Polya's criterion relies on the following simple observation of characteristic functions.

Exercise 7.6.3 — Convex combination of characteristic function. Let $\varphi_k(t)$, $k = 1, 2, \dots$ be characteristic functions and let the nonnegative numbers λ_k satisfy $\sum_{k=1}^{n} \lambda_k = 1$. Show that

$$\sum_{k=1}^{n} \lambda_k \varphi_k(t)$$

for all $n \in \mathbb{Z}_{\geq 1}$ is a characteristic function. Extend the above result for $n = \infty$.

As it turns out, any function $\varphi(t)$ can be approximated by a convex combination of the characteristic function of a Polya distribution $\varphi(t) = (1 - |t|)_+$.

7.6.2 Marcinkiewicz Theorem

Theorem 7.6.4 — Marcinkiewicz's Theorem. If a characteristic function $\varphi(t)$ is of the form $e^{p(t)}$, where $p(t)$ is a polynomial, then this polynomial is of degree at most 2.

Example 7.6.5 As a quick example, e^{-t^4} is not a characteristic function of any real-valued random variables.

7.6.3 Cumulants

Definition 7.6.6 If an expansion

$$\log \varphi_\xi(t) = \sum_{k=0}^{n} \frac{(it)^k}{k!} s_k + o\left(|t|^n\right),$$

exists as $t \to 0$, then the coefficients s_k are called cumulants of ξ.

Exercise 7.6.7 Show that

1. $\mathbb{E}(\xi) = s_1$, and
2. $\mathbb{V}(\xi) = s_2$.

Remark 7.6.8 If $\xi \sim N(m, \sigma^2)$ then

- $s_1 = m$,
- $s_2 = \sigma^2$, and
- $s_k = 0$ for $k \geq 3$.

In general, by Marcinkiewsicz's Theorem if for a random variable ξ there exists n such that $s_k = 0$, for all $k \geq n$, then $s_k = 0$ for all $k \geq 3$ and $\xi \sim N(s_1, s_2)$.

7.6.4 Degenerate distributions

The following theorem shows that a property of the characteristic function of a random variable can lead to a non-trivial conclusion about the nature of the random variable.

Theorem 7.6.9 Let $\varphi(t)$ be a characteristic function of a random variable ξ. If $|\varphi(t_0)| = 1$ for some $t_0 \neq 0$, then ξ is concentrated at the points $a + nh$ and $h = 2\pi/t_0$, for some a. That is,

$$\sum_{n=-\infty}^{\infty} \mathbb{P}(\xi = a + nh) = 1,$$

where a is a constant.

Proof. If $|\varphi(t_0)| = 1$ for some $t_0 \neq 0$ then there is a number a such that $\varphi(t_0) = e^{it_0 a}$. Therefore,

$$e^{it_0 a} = \int_{-\infty}^{\infty} e^{it_0 x}\, dF(x)$$

which implies that

$$1 = \int_{-\infty}^{\infty} e^{it_0(x-a)}\, dF(x).$$

Equating real parts we see that

$$1 = \int_{-\infty}^{\infty} \cos t_0(x-a)\, dF(x)$$

which then implies that

$$\int_{-\infty}^{\infty} 1 - \cos t_0(x-a)\, dF(x) = 0.$$

Since $1 - \cos t_0(x-a) \geq 0$, it follows that

$$1 = \cos t_0(x-a)$$

$\mathbb{P}$-almost surely. That is to say that $\mathbb{P}$ is concentrated at the points $x = a + n\left(\frac{2\pi}{t_0}\right)$ for $n \in \mathbb{Z}$. ∎

Exercise 7.6.10

1. Let $\varphi(t)$ be a characteristic function. Show that the following are also characteristic functions.
   (a) $|\varphi(t)|^2$.
   (b) $e^{\lambda(\varphi(t)-1)}$ for $\lambda \geq 0$.
   (c) $\int_0^1 \varphi(ut)\mathrm{d}u$.
   (d) $\int_0^\infty e^{-u}\varphi(ut)\mathrm{d}u$.
2. Let X and Y be independent identically distributed random variables with zero mean and unit variance. Prove using characteristic functions that if the distribution F of $(X+Y)/\sqrt{2}$ is the same as that of X and Y, then F is the normal distribution.
3. Let ξ be an integer-valued random variable and $\varphi_\xi(t)$ be its characteristic function. Show that
$$\mathbb{P}(\xi = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\varphi_\xi(t)\mathrm{d}t,$$
for $k = 0, \pm 1, \pm 2, \ldots$.
4. Show that if $\varphi(t)$ is a characteristic function, then $\mathrm{Re}(\varphi(t))$ is also a characteristic function, but $\mathrm{Im}(\varphi(t))$ is not.

7.7 Solution to Exercises

Exercise 7.1.4

Solution.

1. Note that if $\xi_1 = \mathsf{B}(1,p)$ the
$$\varphi_{\xi_1} = \mathbb{E}\left(e^{it\xi_1}\right) = e^0(1-p) + e^{it}p.$$
Therefore, by Proposition 7.1.3 4. we deduce that
$$\varphi_\xi(t) = \left(pe^{it} + (1-p)\right)^n.$$
2. Let $\eta = \frac{\xi - m}{\sigma}$. Then $\eta \sim \mathsf{N}(0,1)$ has density
$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$
It is sufficient to show that $\varphi_\eta(t) = e^{-\frac{t^2}{2}}$. We have
$$\begin{aligned}
\varphi_\eta(t) &= \mathbb{E}\left(e^{it\eta}\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx-\frac{x^2}{2}}\mathrm{d}x \\
&= e^{-\frac{t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(x-it)^2}\mathrm{d}x \\
&= e^{-\frac{t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty-it}^{\infty-it} e^{-\frac{z^2}{2}}\mathrm{d}z \\
&\overset{(1)}{=} e^{-\frac{t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\mathrm{d}x \\
&= e^{-\frac{t^2}{2}}.
\end{aligned}$$
We justify this equality using contour integration. Let
$$I_R = \oint_\gamma e^{-\frac{z^2}{2}}\,dz$$
where $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ for

- $\gamma_1 := \{z = -u : -R \leq u \leq R\}$,
- $\gamma_2 := \{z = -R - itu : 0 \leq u \leq 1\}$,
- $\gamma_3 := \{z = u - it : -R \leq u \leq R\}$, and
- $\gamma_4 = \{z = R - it(1-u) : 0 \leq u \leq 1\}$.

Note that

$$\begin{aligned}\int_{\gamma_2} \exp\left(-\frac{z^2}{2}\right) dz &= \int_0^1 \exp\left(-\frac{(-R-itu)^2}{2}\right) dz \\ &= \int_0^1 \exp\left(-\frac{R^2}{2}\right) \exp\left(-Rtui\right) \exp\left(\frac{t^2u^2}{2}\right) du.\end{aligned}$$

Hence, $\int_{\gamma_2} \exp\left(-\frac{z^2}{2}\right) dz \overset{R\to\infty}{\longrightarrow} 0$. Similarly, $\int_{\gamma_4} \exp\left(-\frac{z^2}{2}\right) dz \overset{R\to\infty}{\longrightarrow} 0$.On the other hand,

$$\int_{\gamma_1} \exp\left(-\frac{z^2}{2}\right) dz = \int_{-R}^{R} \exp\left(\frac{u^2}{2}\right) du$$

and

$$\begin{aligned}\int_{\gamma_3} \exp\left(-\frac{z^2}{2}\right) &= \int_{-R}^{R} \exp\left(-\frac{(u-it)^2}{2}\right) du \\ &\overset{v=u-it}{=} \int_{-R-it}^{R-it} \exp\left(-\frac{v^2}{2}\right) dv.\end{aligned}$$

Therefore, as $e^{-\frac{z^2}{2}}$ is analytic in the region defined by γ we deduce that

$$0 = \int_{\gamma} \exp\left(-\frac{z^2}{2}\right) dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}\right) \exp\left(-\frac{z^2}{2}\right) dz.$$

Consequently, as $R \to \infty$ we deduce that

$$0 = \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + \int_{-\infty-it}^{\infty-it} \exp\left(-\frac{v^2}{2}\right) dv$$

which implies that

$$\int_{-\infty}^{\infty} \exp\left(\frac{u^2}{2}\right) du = \int_{-\infty-it}^{\infty-it} \exp\left(-\frac{v^2}{2}\right) dv$$

by using the evenness of the integrand.

3. Proceeding directly we see that

$$\begin{aligned}\varphi_\xi(t) &= \mathbb{E}\left(e^{it\xi}\right) \\ &= e^{-\lambda} \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(e^{it}\lambda\right)^k}{k!} \\ &= e^{-\lambda} e^{\lambda e^{it}} \\ &= e^{-\lambda + \lambda e^{it}}.\end{aligned}$$

■

Exercise 7.2.5

Solution. Let $\eta = \exp(\xi)$. Then as $\exp(\cdot)$ is a strictly increasing function we note that

$$f_\eta(x) = f_\xi\left(\ln(x)\right)\frac{1}{x} = \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{(\ln(x))^2}{2}\right)$$

for $x > 0$. Therefore, η has a standard log-normal distribution. Moreover,

$$\begin{aligned}
\mathbb{E}\left(\eta^r\right) &= \mathbb{E}\left(\exp(\xi))^2\right) \\
&= \int_{-\infty}^{\infty} \exp(rx)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-r)^2}{2} + \frac{r^2}{2}\right)dx \\
&= \exp\left(\frac{r^2}{2}\right)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-r)^2}{2}\right)dx \\
&= \exp\left(\frac{r^2}{2}\right).
\end{aligned}$$

■

Exercise 7.4.4

Solution. Set $m = \mathbb{E}\left(\xi_1\right)$, $\sigma^2 = \mathbb{V}\left(\xi_1\right)$, and $\varphi(t) = \mathbb{E}\left(e^{it(\xi_1 - m)}\right)$. If we put

$$\varphi_n(t) = \mathbb{E}\left(\exp\left(it\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}}\right)\right),$$

by independence

$$\varphi_n(t) = \left(\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n.$$

Since $\mathbb{E}\left(\xi_1^2\right) < \infty$, we have by properties of characteristic functions that

$$\varphi(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$$

as $t \to 0$. So

$$\varphi_n(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \to e^{-\frac{t^2}{2}}$$

for all $t \in \mathbb{R}$. This is the characteristic function of $\mathsf{N}(0,1)$ and so the result follows by Theorem 7.4.1. ■

Exercise 7.4.9

Solution. Note that

$$|m_k| = |\mathbb{E}(\xi_k)| \leq \mathbb{E}(|\xi_k|) \leq K$$

for all k. Therefore, $|\xi_k - m_k| \leq 2K$ for all k. As $D_n \to \infty$, and D_n is monotonically increasing, it follows that for some $N \in \mathbb{N}$ we have that

$$\chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}} = 0$$

for $n \geq N$. Therefore, for $n \geq N$ we have

$$\begin{aligned}
\frac{1}{D_n^2}\sum_{k=1}^{n}\mathbb{E}\left(|\xi_k - m_k|^2\chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}\right) &= \frac{1}{D_n^2}\sum_{k=1}^{N-1}\mathbb{E}\left(|\xi_k - m_k|^2\chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}\right) \\
&\quad + \frac{1}{D_n^2}\sum_{k=N}^{n}\mathbb{E}\left(|\xi_k - m_k|^2\chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}\right) \\
&= \frac{1}{D_n^2}\sum_{k=1}^{N-1}\mathbb{E}\left(|\xi_k - m_k|^2\chi_{\{|\xi_k - m_k| \geq \epsilon D_n\}}\right) \\
&\leq \frac{4K^2(N-1)}{D_n^2}
\end{aligned}$$

$$\xrightarrow{n\to\infty} 0.$$

Hence, Linderberg's condition is satisfied. ■

Exercise 7.6.3

Solution. For $k = 1, \ldots, n$ let ζ_k be a random variable with characteristic function $\phi_k(t)$. Let η be the random variable where $\mathbb{P}(\eta = k) = \lambda_k$ for $k = 1, \ldots, n$. As

$$\zeta_\eta(\omega) = \sum_{k=1}^{n} \zeta_k \mathbf{1}_{\{\eta=k\}}(\omega)$$

we note that ζ_η is the sum of the product of random variables as each ζ_k is a random variable and $\mathbf{1}_{\{\eta=k\}}(\omega)$ is measurable as η is a random variable. Therefore, we can consider the characteristic function of ζ_η. Namely,

$$\begin{aligned}\phi_{\zeta_\eta}(t) &= \mathbb{E}\left(e^{it\zeta_\eta}\right) \\ &= \sum_{k=1}^{n} \mathbb{E}\left(e^{it\zeta_k}\right)\mathbb{P}(\eta = k) \\ &= \sum_{k=1}^{n} \lambda_k \phi_k(t).\end{aligned}$$

One can readily extend to $n = \infty$ by letting η be a discrete random variable with a countable image. ■

Exercise 7.6.7

Solution. On the one hand,

$$\frac{d}{dt}\log(\varphi_\xi(t)) = \frac{\varphi_\xi'(t)}{\varphi_\xi(t)}$$

and on the other hand,

$$\frac{d}{dt}\log(\varphi_\xi(t)) = \sum_{k=1}^{n} \frac{i(it)^{k-1}}{(k-1)!}s_k + o\left(|t|^{n-1}\right).$$

Substituting $t = 0$ into both expressions we deduce that

$$is_1 = \frac{i\mathbb{E}(\xi)}{(1)} = i\mathbb{E}(\xi)$$

and so $\mathbb{E}(\xi) = s_1$. Similarly,

$$\frac{d^2}{dt^2}\log(\varphi_\xi(t)) = \frac{\varphi_\xi(t)\varphi_\xi''(t) - \left(\varphi_\xi'(t)\right)^2}{\left(\varphi_\xi(t)\right)^2}$$

and

$$\frac{d^2}{dt^2}\log(\varphi_\xi(t)) = \sum_{k=2}^{n} \frac{i^2(it)^{k-2}}{(k-2)!}s_k + o\left(|t|^{n-2}\right).$$

Substituting $t = 0$ into both expressions we deduce that

$$i^2 s_2 = \frac{(1)(i^2\mathbb{E}(\xi^2)) - (i\mathbb{E}(\xi))^2}{1^2} = i^2\mathbb{V}(\xi).$$

Hence, $\mathbb{V}(\xi) = s_2$. ■

Exercise 7.6.10

Solution.

1. (a) Note that $\overline{\varphi(t)} = \varphi(-t)$ and $\varphi(-t)$ is also a characteristic function. Therefore, $|\varphi(t)|^2 = \varphi(t)\overline{\varphi(t)}$ is also a characteristic function.

(b) Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with the characteristic function $\varphi(t)$, and let η be a random variable with Pois(λ) distribution, for $\lambda \geq 0$. Moreover, η is independent of the $(\xi_k)_{k\in\mathbb{N}}$. Let $X = \sum_{k=0}^{\eta} \xi_k$, then

$$\begin{aligned}
\varphi_X(t) &= \mathbb{E}\left(\exp\left(itX\right)\right) \\
&= \mathbb{E}\left(\exp\left(\sum_{k=0}^{\eta} it\xi_k\right)\right) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left(\prod_{k=0}^{n} \exp\left(it\xi_k\right)\right) \mathbb{P}(\eta = n) \\
&\overset{(1)}{=} \sum_{n=0}^{\infty} \prod_{k=0}^{n} \mathbb{E}\left(\exp\left(it\xi_k\right)\right) \frac{e^{-\lambda}\lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\varphi(t)^n \lambda^n}{n!} \\
&= e^{-\lambda} e^{\lambda\varphi(t)} \\
&= e^{\lambda(\varphi(t)-1)}
\end{aligned}$$

where in (1) we have used the independence assumption of the sequence $(\xi_k)_{k\in\mathbb{N}}$. Therefore, $e^{\lambda(\varphi(t)-1)}$ defines a characteristic function.

(c) Let $\psi(t) = \int_0^1 \varphi(ut)\, du$. Then $\psi(0) = \int_0^1 1\, du = 1$. Moreover for all $t_1, \ldots, t_n \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $n = 1, 2, \ldots$ we have that

$$\begin{aligned}
\sum_{j,k=1}^{n} \psi(t_j - t_k)\lambda_j\overline{\lambda}_k &= \sum_{j,k=1}^{n} \left(\int_0^1 \varphi(u(t_j - t_k))\, du\right) \lambda_j\overline{\lambda}_k \\
&= \int_0^1 \sum_{j,k=1} \varphi(t'_j - t'_k)\lambda_j\overline{\lambda}_k\, du
\end{aligned}$$

where $t'_i = ut_i \in \mathbb{R}$. Therefore, as $\varphi(t)$ is a characteristic function we know that

$$\sum_{j,k}^{n} \varphi\left(t'_j - t'_k\right) \lambda_j\overline{\lambda}_k \geq 0$$

by Theorem 7.6.1. This implies that

$$\sum_{j,k=1}^{n} \psi(t_j - t_k)\lambda_j\overline{\lambda}_k \geq 0$$

and so by Theorem 7.6.1 we know that $\psi(t) = \int_0^t \varphi(ut)\, du$ is a characteristic function.

(d) Let $\psi(t) = \int_0^\infty e^{-u}\varphi(ut)\, dt$. Then $\psi(0) = \int_0^\infty e^{-u}\, du = 1$. A similar argument to the above works to conclude that $\psi(t)$ is a characteristic function.

2. Let $\varphi(t)$ be the characteristic function F. Then we have that

$$\varphi(t) = \varphi_{\frac{X+Y}{\sqrt{2}}}(t) = \varphi_X\left(\frac{t}{\sqrt{2}}\right)\varphi_Y\left(\frac{t}{\sqrt{2}}\right) = \varphi\left(\frac{t}{\sqrt{2}}\right)^2.$$

Let $\psi(t) = \log(\varphi(t))$ so that $\psi(t) = 2\psi\left(\frac{t}{\sqrt{2}}\right)$. Consequently, we have that

- $\psi'(t) = \sqrt{2}\psi'\left(\frac{t}{2}\right)$, and
- $\psi'' = \psi''\left(\frac{t}{2}\right)$.

As $\psi''(t)$ is continuous at 0 we must have that $\psi''(t)$ is a constant. In particular, it follows that

$$\psi''(t) = \psi''(0) = \frac{\varphi(t)\varphi''(t) - (\varphi'(t))^2}{\varphi(t)^2} = \frac{i^2\mathbb{V}(X)}{1} = -1.$$

Therefore, $\psi'(t) = -t + c$. As $\psi'(0) = 0$ we know that $c = 0$ and $\psi(t) = -\frac{t^2}{2}$. Hence, $\varphi(t) = \exp\left(-\frac{t}{2}\right)$ and so F is a standard normal distribution.

3. Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\varphi_\xi(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\sum_{m\in\mathbb{Z}} e^{ipt}\mathbb{P}(\xi=m)\,dt.$$

As $\sum_{m\in\mathbb{Z}} e^{imt}\mathbb{P}(\xi=m)$ is absolutely convergent we can interchange the integral and the sum to deduce that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\varphi_\xi(t)\,dt = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi} e^{i(m-k)t}\mathbb{P}(\xi=m)\,dt.$$

Then as

$$\int_{-\pi}^{\pi} e^{i(m-k)t}\,dt = \begin{cases} 2\pi & m=k \\ 0 & \text{otherwise} \end{cases}$$

we conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\varphi_\xi(t)\,dt = \mathbb{P}(\xi=k).$$

4. As $\varphi(0)=0$ it follows that $\mathrm{Im}(\varphi(0))\neq 0$ and so cannot itself be a characteristic function. On the other hand, let ξ be a random variable for which $\varphi(t)$ is the characteristic function. Then it is clear that $-\xi$ is also a random variable and has characteristic function $\varphi(-t)$. In particular, this means that $\varphi(-t)$ is also a random variable and so $\mathrm{Re}(\phi)=\frac{1}{2}(\varphi(t)+\varphi(-t))$ is a characteristic function by Exercise 7.6.3.

■

Part III. Introduction to Stochastic Analysis

8 Conditional Expectation

When studying stochastic processes $(\xi_\alpha)_{\alpha\in A}$, it is natural to determine how different random variables are related. In particular, we want to know if observing one random variable will give more information on the other random variables in the process. For this, we need the notion of conditional probability and conditional expectation.

8.1 Preliminary Measure Theory

To ensure that our notions of conditional probability and conditional expectation are well-defined it will be useful to make note of the following result in measure theory.

Theorem 8.1.1 — Radon-Nikodym Theorem. Let μ be a finite measure on the measure space $(\Omega, \mathscr{F})$. Let λ be a measure on $\mathscr{F}$ that is absolutely continuous with respect to μ. That is, $\lambda(A) = 0$ whenever $\mu(A) = 0$. Then there exists an $\mathscr{F}$-measurable function f such that

$$\lambda(A) = \int_A f \mathrm{d}\mu$$

for all $A \in \mathscr{F}$. Moreover, f is determined uniquely up to sets of measure zero. Consequently, f is called the derivative of λ with respect to μ and is often denoted $f = \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$.

If ξ is a non-negative random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $\mathscr{G} \subset \mathscr{F}$ is a σ-algebra. Then the measure defined by $Q(G) = \int_G \xi \, d\mathbb{P}$ for all $G \in \mathscr{G}$ is an absolutely continuous measure with respect to $\mathbb{P}$. Hence, by Theorem 8.1.1 there exists a $\mathscr{G}$-measurable function, f, such that $Q(G) = \int_G f \, d\mathbb{P}$ for all $G \in \mathscr{G}$. We can extend this naturally to general random variables ξ that are such that $\min\left(\mathbb{E}\left(\xi^+|\mathscr{G}\right), \mathbb{E}\left(\xi^-|\mathscr{G}\right)\right) < \infty$ almost surely.

8.2 Conditional Expectation and Probability

Definition 8.2.1 — Conditional expectation. Let ξ be an integrable random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra. Then there exists a random variable $\mathbb{E}(\xi|\mathscr{G})$, referred to as the conditional expectation of ξ given $\mathscr{G}$, that satisfies the following.

- $\mathbb{E}(\xi|\mathscr{G})$ is $\mathscr{G}$-measurable and integrable.
- For every $G \in \mathscr{G}$ we have

$$\mathbb{E}(\chi_G \mathbb{E}(\xi|\mathscr{G})) = \int_G \mathbb{E}(\xi|\mathscr{G}) \mathrm{d}\mathbb{P} = \int_G \xi \mathrm{d}\mathbb{P} = \mathbb{E}(\chi_G \xi).$$

Remark 8.2.2 Theorem 8.1.1 ensures that the conditional expectation of an integrable random variable is unique up to sets of measure zero.

Definition 8.2.3 Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Then the conditional probability of $B \in \mathscr{F}$ with respect to a σ-algebra $\mathscr{G} \subset \mathscr{F}$ is

$$\mathbb{P}(B|\mathscr{G}) = \mathbb{E}(\chi_B|\mathscr{G}).$$

Note that for a fixed $B \in \mathscr{F}$, the conditional probability $\mathbb{P}(B|\mathscr{G})$ is a $\mathscr{G}$-measurable random variable such that

$$\int_G \mathbb{P}(B|\mathscr{G}) \, d\mathbb{P} = \int_G \chi_B \, d\mathbb{P} = \mathbb{P}(G \cap B)$$

for all $G \in \mathscr{G}$ as we would expect from traditional notions of conditional probabilities. Moreover, let ξ be a random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G} = \sigma(\{D_1, D_2, \dots\})$ where $\{D_1, D_2, \dots\}$ forms a disjoint partition of Ω. That is,

$$\Omega = \bigcup_{i=1}^{\infty} D_i$$

with each D_i disjoint. Moreover, suppose that $\mathbb{P}(D_i) > 0$ for $i = 1, 2, \dots$. Then all $\mathscr{G}$-measurable functions have the form

$$f(\omega) = \sum_{i=1}^{\infty} c_i \chi_{D_i}(\omega)$$

and thus are constant on each D_i. Note that by definition $\mathbb{E}(\xi|\mathscr{G})$ must be of this form. Observe that

$$\begin{aligned}\mathbb{E}(\xi\chi_{D_i}) &= \int_{D_i} \xi \, d\mathbb{P} \\ &= \int_{D_i} \mathbb{E}(\xi|\mathscr{G}) \, d\mathbb{P} \\ &= \mathbb{E}(\xi|D_i)\mathbb{P}(D_i),\end{aligned}$$

that is

$$\mathbb{E}(\xi|D_i) = \frac{\mathbb{E}(\xi\chi_{D_i})}{\mathbb{P}(D_i)}.$$

Consequently, we also note that

$$\begin{aligned}\mathbb{P}(B|D_i) &= \mathbb{E}(\chi_B|D_i) \\ &= \frac{\mathbb{E}(\chi_B\chi_{D_i})}{\mathbb{P}(D_i)} \\ &= \frac{\mathbb{P}(B \cap D_i)}{\mathbb{P}(D_i)}.\end{aligned}$$

Hence,

$$\mathbb{P}(B|\mathscr{G}) = \sum_{i=1}^{\infty} \mathbb{P}(B|D_i)\chi_{D_i}$$

and

$$\mathbb{P}(B|\{\varnothing, \Omega\}) = \mathbb{P}(B).$$

8.3 Properties of Conditional Expectation

Proposition 8.3.1 Let ξ and η be random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G} \subset F$ be a σ-algebra. Then,

$$\mathbb{E}(a\xi + b\eta + c \,|\, \mathscr{G}) = a\mathbb{E}(\xi \,|\, \mathscr{G}) + b\mathbb{E}(\eta \,|\, \mathscr{G}) + c$$

almost surely.

Proof. For all $G \in \mathscr{G}$, we have

$$\begin{aligned}\mathbb{E}\left(\chi_G \mathbb{E}(a\xi + b\eta + c \,|\, \mathscr{G})\right) &= \mathbb{E}(\chi_G(a\xi + b\eta + c)) \\ &= a\mathbb{E}(\chi_G\xi) + b\mathbb{E}(\chi_G\eta) + c\mathbb{E}(\chi_G) \\ &= a\mathbb{E}\left(\chi_G\mathbb{E}(\xi \,|\, \mathscr{G})\right) + b\mathbb{E}(\chi_G\mathbb{E}(\eta \,|\, \mathscr{G})) + c\mathbb{E}(\chi_G) \\ &= \mathbb{E}\left(\chi_G(a\mathbb{E}(\xi \,|\, \mathscr{G}) + b\mathbb{E}(\eta \,|\, \mathscr{G}) + c)\right).\end{aligned}$$

■

Proposition 8.3.2 Let ξ and η be random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that $\xi \leq \eta$ almost surely and let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra. Then,

$$\mathbb{E}(\xi \,|\, \mathscr{G}) \leq \mathbb{E}(\eta \,|\, \mathscr{G})$$

almost surely.

Proof. Consider the set $G = \{\omega : \mathbb{E}(\xi \,|\, \mathscr{G}) > \mathbb{E}(\eta \,|\, \mathscr{G})\} \in \mathscr{G}$. If $\mathbb{P}(G) > 0$ then $\mathbb{E}(\chi_G(\mathbb{E}(\xi \,|\, \mathscr{G}) - \mathbb{E}(\eta \,|\, \mathscr{G}))) > 0$ by our definition of G. But we also know that $\mathbb{E}(\chi_G(\mathbb{E}(\xi \,|\, \mathscr{G}) - \mathbb{E}(\eta \,|\, \mathscr{G}))) = \mathbb{E}(\chi_G(\xi - \eta)) \leq 0$, which is a contradiction. Hence, $\mathbb{P}(G) = 0$, and $\mathbb{E}(\chi_G(\mathbb{E}(\xi \,|\, \mathscr{G}) - \mathbb{E}(\eta \,|\, \mathscr{G}))) = 0$. Therefore,

$$\mathbb{E}((\mathbb{E}(\xi \,|\, \mathscr{G}) - \mathbb{E}(\eta \,|\, \mathscr{G}))) = \mathbb{E}\left(\chi_{G^C}(\mathbb{E}(\xi \,|\, \mathscr{G}) - \mathbb{E}(\eta \,|\, \mathscr{G}))\right) \leq 0,$$

which implies that $\mathbb{E}(\xi \,|\, \mathscr{G}) \leq \mathbb{E}(\eta \,|\, \mathscr{G})$ almost surely. ■

Corollary 8.3.3 In the setting of Proposition 8.3.2 we note that if $\eta \geq 0$ almost surely, then

$$\mathbb{E}(\eta \,|\, \mathscr{G}) \geq 0$$

almost surely.

Exercise 8.3.4 Throughout, let ξ and η be random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra.

1. Show that $\mathbb{E}(\xi|\{\varnothing, \Omega\}) = \mathbb{E}(\xi)$.
2. Show that $\mathbb{E}(\xi \,|\, \mathscr{F}) = \xi$ almost everywhere.
3. Suppose that ξ is independent of $\mathscr{G}$, which means that for all $B \in \mathscr{G}$ we have ξ independent of χ_B, then $\mathbb{E}(\xi \,|\, \mathscr{G}) = \mathbb{E}(\xi)$.

Theorem 8.3.5 Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and ξ is an integrable random variable taking values in an open interval $I \subset \mathbb{R}$. Let $g : I \to \mathbb{R}$ be convex and let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra. If $\mathbb{E}(|g(\xi)|) < \infty$, then

$$\mathbb{E}(\varphi(\xi)|\mathscr{G}) \geq \varphi(\mathbb{E}(\xi|\mathscr{G}))$$

almost surely.

Corollary 8.3.6 Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and ξ is an integrable random variable then

$$\mathbb{E}(|\xi||\mathscr{G}) \geq |\mathbb{E}(\xi|\mathscr{G})|$$

almost surely.

Proposition 8.3.7 Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, ξ an integrable random variable and $\mathscr{F}_1, \mathscr{F}_2 \subset \mathscr{F}$ σ-algebras with $\mathscr{F}_1 \subseteq \mathscr{F}_2$. Then,

$$\mathbb{E}(\mathbb{E}(\xi|\mathscr{F}_2)|\mathscr{F}_1) = \mathbb{E}(\xi|\mathscr{F}_1) = \mathbb{E}(\mathbb{E}(\xi|\mathscr{F}_1)|\mathscr{F}_2) \tag{8.1}$$

almost surely.

Proof. Let $G \in \mathscr{F}_1$, then $G \in \mathscr{F}_2$ and hence

$$\int_G \mathbb{E}(\xi|\mathscr{F}_1)\, \mathrm{d}\mathbb{P} = \int_G \xi\, \mathrm{d}\mathbb{P}$$

by definition of $\mathbb{E}(\xi|\mathscr{F}_1)$. Similarly,

$$\int_A \mathbb{E}\left(\mathbb{E}\left(\xi|\mathscr{F}_2\right)|\mathscr{F}_2\right)\, \mathrm{d}\mathbb{P} = \int_A \mathbb{E}(\xi|\mathscr{F}_2)\, \mathrm{d}\mathbb{P} = \int_A \xi\, \mathrm{d}\mathbb{P}.$$

Hence, the first equality of (8.1) follows. Now let $G \in \mathscr{F}_2$. As $\mathbb{E}(\xi|\mathscr{F}_1)$ is $\mathscr{F}_1$-measurable it follows that it is also $\mathscr{F}_2$ measurable. Hence,

$$\int_G \mathbb{E}(\xi|\mathscr{F}_1)\,\mathrm{d}\mathbb{P} = \int_A \mathbb{E}\left(\mathbb{E}(\xi|\mathscr{F}_1)|\mathscr{F}_2\right).$$

Thus we get the second equality of (8.1). ■

Corollary 8.3.8 For ξ an integrable random variable on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, then

$$\mathbb{E}(\mathbb{E}(\xi\,|\,\mathscr{G})) = \mathbb{E}(\xi).$$

Proof. Take $\mathscr{F}_1 = \{\varnothing, \Omega\}$ and and $\mathscr{F}_2 = \mathscr{G}$ in Proposition 8.3.7 and then use Exercise 8.3.4 to conclude. ■

Proposition 8.3.9 Let $\{\xi_n\}_{n\geq 1}$ be a sequence of random variables.

1. Suppose $|\xi_n| \leq \eta$ where $\mathbb{E}(\eta) < \infty$, and $\xi_n \to \xi$ almost surely. Then,

   $$\mathbb{E}(\xi_n\,|\,\mathscr{G}) \xrightarrow{a.s.} \mathbb{E}(\xi\,|\,\mathscr{G})$$

   and

   $$\mathbb{E}(|\xi_n - \xi||\mathscr{G}) \xrightarrow{a.s.} 0.$$

2. Suppose $\xi_n \geq \eta$ where $\mathbb{E}(eta) > -\infty$, $\xi_n \nearrow \xi$ almost surely and $\mathbb{E}(|\xi|) < \infty$. Then

   $$\mathbb{E}(\xi_n|\mathscr{G}) \nearrow \mathbb{E}(\xi|\mathscr{G})$$

   almost surely.

3. Suppose $\xi_n \leq \eta$ where $\mathbb{E}(\eta) < \infty$, and $\xi_n \searrow \xi$ almost surely. Then

   $$\mathbb{E}(\xi_n|\mathscr{G}) \searrow \mathbb{E}(\xi|\mathscr{G})$$

   almost surely.

4. Suppose $\xi_n \geq \eta$ where $\mathbb{E}(\eta) > -\infty$. Then

   $$\mathbb{E}(\liminf \xi_n|\mathscr{G}) \leq \liminf \mathbb{E}(\xi_n|\mathscr{G})$$

   almost surely.

5. Suppose $\xi_n \leq \eta$ where $\mathbb{E}(\eta) < \infty$. Then

   $$\mathbb{E}(\limsup \xi_n|\mathscr{G}) \leq \limsup \mathbb{E}(\xi_n|\mathscr{G})$$

   almost surely.

6. If $\xi_n \geq 0$ then

   $$\mathbb{E}\left(\sum_{n=1}^{\infty} \xi_n \Big| \mathscr{G}\right) = \sum_{n=1}^{\infty} \mathbb{E}(\xi_n|\mathscr{G})$$

   almost surely.

Proof.

1. Let $\zeta_n = \sup_{m\geq n} |\xi_m - \xi|$. Then $0 \leq |\zeta_n| \leq 2\eta$ and $\zeta_n \to 0$ almost surely, so by the dominated convergence theorem we have $\mathbb{E}(\zeta_n) \xrightarrow{n\to\infty} 0$. Now by the triangle inequality it follows that

   $$0 \leq |\mathbb{E}(\xi_n|\mathscr{G}) - \mathbb{E}(\xi|\mathscr{G})| \leq \mathbb{E}(|\xi_n - \xi|\,|\,\mathscr{G}) \leq \mathbb{E}(\zeta_n|\mathscr{G}).$$

Since the sequence $\mathbb{E}(\zeta_n|\mathscr{G})(\omega)$ is decreasing in n for fixed ω, its limit exists ω-almost surely. In particular, note that

$$0 \leq \mathbb{E}\left(\lim_{n\to\infty} \mathbb{E}(\zeta_n|\mathscr{G})\right) \leq \mathbb{E}\left(\mathbb{E}(\zeta_n|\mathscr{G})\right) = \mathbb{E}(\zeta_n) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, $\lim_{n\to\infty} \mathbb{E}(\zeta_n|\mathscr{G}) = 0$ almost surely which completes the proof.

2. Suppose that $\eta = 0$. Then $\mathbb{E}(\xi_n|\mathscr{G}) \geq 0$ almost surely by Corollary 8.3.3. By the assumption of monotonic convergence almost surely, it follows that for $A_n := \{\mathbb{E}(\xi_n|\mathscr{G}) < \mathbb{E}(\xi_{n-1}|\mathscr{G})\} \in \mathscr{G}$ we have $\mathbb{P}(A_n) = 0$. Let $\tilde{\xi} := \limsup_{n\to\infty} (\mathbb{E}(\xi_n|\mathscr{G}))$ and $A = \bigcup_{n\geq 2} A_n$. Then $A \in \mathscr{G}$ and $\mathbb{P}(A) = 0$, furthermore, for all $\omega \in A^c$ it follows that $\mathbb{E}(\xi_n|\mathscr{G})(\omega) \nearrow \tilde{\xi}(\omega)$. Therefore, for all $G \in G$ we have

$$\begin{aligned}
\mathbb{E}\left(\tilde{\xi}\chi_G\right) &= \mathbb{E}\left(\tilde{\xi}\chi_{G\cap A^c}\right) \\
&\overset{\text{MCT}}{=} \lim_{n\to\infty} \left(\mathbb{E}\left(\mathbb{E}(\xi_n|\mathscr{G})\chi_{G\cap A^c}\right)\right) \\
&\overset{(1)}{=} \lim_{n\to\infty} \left(\xi\chi_{G\cap A^c}\right) \\
&\overset{\text{MCT}}{=} \mathbb{E}\left(\xi\chi_{G\cap A^c}\right) \\
&= \mathbb{E}\left(\xi\chi_G\right),
\end{aligned}$$

where in (1) we have used the fact that $G\cap A^c \in \mathscr{G}$ and the definition of the conditional expectation. Consequently, we see that $\limsup_{n\to\infty} (\mathbb{E}(\xi_n|\mathscr{G}))$ is integrable, and equal to $\mathbb{E}(\xi|\mathscr{G})$ almost surely as desired. When $\eta \neq 0$ we can just consider the random variables $\xi_n + \eta$ instead, which is well-defined as $\mathbb{E}(\eta) > -\infty$.

3. This is equivalent to 2. by considering the sequence $\xi - \xi_n$.
4. This is a direct application of 2. to the sequence $\eta_n = \inf_{k\geq n} \xi_n$, which is increasing.
5. This is equivalent to 4.
6. This is a direct application of 2.

■

Corollary 8.3.10 Let ξ and η be random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with ξ, η and $\xi\eta$ integrable. Let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra and suppose that η is $\mathscr{G}$-measurable. Then

$$\mathbb{E}(\xi\eta|\mathscr{G}) = \eta\mathbb{E}(\xi|\mathscr{G})$$

almost everywhere.

Proof. Note that $\mathbb{E}(\xi|\mathscr{G})$ and η are both $\mathscr{G}$ measurable.
Step 1. Consider $\eta = \chi_A$ for $A \in \mathscr{G}$.
Let $B \in \mathscr{G}$. On the one hand,

$$\begin{aligned}
\mathbb{E}\left(\chi_B\mathbb{E}\left(\xi\eta|\mathscr{G}\right)\right) &= \mathbb{E}\left(\chi_B\xi\eta\right) \\
&= \mathbb{E}\left(\chi_{A\cap B}\xi\right),
\end{aligned}$$

where for the first equality we have used the definition of conditional expectation. On the other hand,

$$\begin{aligned}
\mathbb{E}\left(\chi_B\eta\mathbb{E}(\xi|\mathscr{G}\right) &= \mathbb{E}\left(\chi_{A\cap B}\mathbb{E}\left(\xi|\mathscr{G}\right)\right) \\
&= \mathbb{E}\left(\chi_{A\cap B}\xi\right),
\end{aligned}$$

where for the second equality we have used the fact that $A \cap B \in \mathscr{G}$ and the definition of conditional expectation. Thus, we have proved the result for $\eta = \chi_A$ where $A \in \mathscr{G}$.
Step 2. Consider η to be a simple random variable.
We can extend the result of Step 1 to simple random variables by using Proposition 8.3.1.
Step 3. Consider η a general integrable random variable.
Recall that any integrable random variable η can be approximated by simple functions $(\eta_n)_{n\in\mathbb{N}}$ such that $|\eta_n| \leq \eta$. Moreover, $\eta_n\xi \to \eta\xi$ almost surely, with $|\eta_n\xi| \leq |\eta\xi|$. Therefore, using as $\mathbb{E}(\eta\xi) < \infty$ we can apply Proposition 8.3.9 1. to deduce that $\mathbb{E}(\eta_n\xi|\mathscr{G}) \to \mathbb{E}(\eta\xi|\mathscr{G})$. Using our previous steps we know that $\mathbb{E}(\eta_n\xi|\mathscr{G}) = \eta_n\mathbb{E}(\xi|\mathscr{G})$. Therefore as $\eta_n\mathbb{E}(\xi|G) \to \eta\mathbb{E}(\xi|\mathscr{G})$ we deduce that $\mathbb{E}(\eta\xi|\mathscr{G}) = \eta\mathbb{E}(\xi|\mathscr{G})$. ■

8.4 Conditioning on a Random Variable

Definition 8.4.1 The conditional expectation of a random variable ξ with respect to a random variable η is

$$\mathbb{E}(\xi|\eta) := \mathbb{E}(\xi|\sigma(\eta)),$$

where $\sigma(\eta)$ is the σ-algebra generated by η.

Theorem 8.4.2 Let ξ and η be random variables such that ξ is $\sigma(\eta)$-measurable. Then there exists a Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\xi = f(\eta).$$

In particular, there exists a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}(\xi|\eta) = g(\eta).$$

Proof. Step 1. Consider $\mu = \sum_{i=1}^n c_j \chi_{A_j}$, with $\{A_j\}_{j=1}^n$ partitions Ω.
As μ is $\sigma(\eta)$ it must be the case that $A_j \in \sigma(\eta)$ for all j. Hence, for all j there exists $B \in \mathscr{B}(\mathbb{R})$ such that $\eta^{-1}(B_j) = A_j$. It is clear that $\{B_j\}_{j=1}^n$ partitions $\eta(\Omega)$. Hence, set

$$f(x) = \begin{cases} \sum_{i=1}^n c_j \chi_{B_j}(x) & x \in \bigcup_{j=1}^n B_j \\ 0 & \text{otherwise,} \end{cases}$$

so that $f(\eta(\omega)) = \mu(\omega)$ as required.
Step 2. Consider μ a general random variable.
We can approximate μ with a sequence of simple random variables, $(\mu_n)_{n\in\mathbb{N}}$ such that $\mu_n(\omega) \to \mu(\omega)$ for all $\omega \in \Omega$. By Step 1 we can define Borel-measurable functions f_n such that $\mu_n = f_n(\eta)$. Now set

$$f(x) = \begin{cases} \lim_n f_n(x) & \text{if it exists on } \eta(\Omega) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(x)$ is Borel measurable and

$$\mu(\omega) = \lim_n \mu_n(\omega) = \lim_n F_n(\eta(\omega)) = f(\eta(\omega))$$

as required. ∎

Example 8.4.3 Consider real-valued random variables X and Y defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume the random vector (X, Y) has continuous joint density $f_{X,Y}(x, y) > 0$. Recall that X has density $f_X(x) = \int_\Omega f_{X,Y}(x, y)\, dy$ and Y has density $f_Y(y) = \int_\Omega f_{X,Y}(x, y)\, dx$. Assume $f_X(x), f_Y(y) > 0$ almost everywhere in $\mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function with $\mathbb{E}(|h(X)|) < \infty$. By Theorem 8.4.2, we know that $\mathbb{E}(h(X)\,|\,Y) = \phi(Y)$ for some unique Borel-measurable ϕ, almost everywhere. That is,

$$\mathbb{E}(\chi_A \phi(Y)) = \mathbb{E}(\chi_A h(X))$$

for all $A \in \sigma(Y)$. Since $A \in \sigma(Y) \subseteq \mathscr{F}$, we know that $A = Y^{-1}(B)$ for some $B \in \mathscr{B}(\mathbb{R})$. Then,

$$\begin{aligned} \mathbb{E}(\chi_A h(X)) &= \mathbb{E}(\chi_B(Y) h(X)) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) \chi_B(y) f_{X,Y}(x, y)\, dy\, dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) \chi_B(y) \frac{f_{X,Y}(x, y)}{f_Y(y)} f_Y(y)\, dy\, dx \\ &\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) \chi_B(y) \frac{f_{X,Y}(x, y)}{f_Y(y)} f_Y(y)\, dx\, dy \end{aligned}$$

$$= \int_{\mathbb{R}} \chi_B(y) \left(\int_{\mathbb{R}} h(x) \frac{f_{X,Y}(x,y)}{f_Y(y)} \, dx \right) f_Y(y) dy.$$

So by the uniqueness of ϕ we deduce that

$$\phi(y) = \int_{\mathbb{R}} h(x) \frac{f_{X,Y}(x,y)}{f_Y(y)} \, dx.$$

Exercise 8.4.4 — Conditional expectation of discrete random variables.

1. Let X and Y be random variables taking value in $\mathbb{N}$, with joint mass $p_{X,Y}(x,y)$ for $x, y \in \mathbb{N}$. Assume $h : \mathbb{N} \to \mathbb{R}$ is such that $\mathbb{E}(|h(X)|) < \infty$. Verify that $\mathbb{E}(h(X) \,|\, Y) = \phi(Y)$ where

$$\phi(y) = \sum_{x \in \mathbb{N}} h(x) \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for $p_Y(y) \neq 0$.

2. Consider random variables Z_1, Z_2 on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $Z_1 \sim \mathsf{Po}(\lambda_1)$ and $Z_2 \sim \mathsf{Po}(\lambda_2)$. Assuming $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$, show that

$$\mathbb{P}(Z_1 = k \,|\, Z_1 + Z_2 = n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Proposition 8.4.5 If $\mathbb{E}(\xi^2) < \infty$, then

$$\min_f \mathbb{E}\left((\xi - f(\eta)^2 \right) = \mathbb{E}\left((\xi - \mathbb{E}(\xi|\eta))^2 \right),$$

where the minimum is taken over all $\sigma(\eta)$-measurable functions such that $\mathbb{E}\left(f^2(\eta)\right) < \infty$.

8.5 Solution to Exercises

Exercise 8.3.4

Solution.

1. As $\mathbb{E}(\xi)$ is just a constant it is $\{\varnothing, \Omega\}$-measurable. Moreover,

$$\int_{\varnothing} \xi \, \mathrm{d}\mathbb{P} = 0 = \int_{\varnothing} \mathbb{E}(\xi) \, \mathrm{d}\mathbb{P}$$

and

$$\int_{\Omega} \xi \, \mathrm{d}\mathbb{P} = \mathbb{E}(\xi) = \mathbb{E}(\xi) \int_{\Omega} \mathrm{d}\mathbb{P} = \int_{\Omega} \mathbb{E}(\xi) \, \mathrm{d}\mathbb{P}.$$

It follows that $\mathbb{E}\left(\xi | \{\varnothing, \Omega\}\right) = \mathbb{E}(\xi)$.

2. This is clear as ξ is $\mathscr{F}$-measurable.

3. Let $B \in \mathscr{G}$, then

$$\begin{aligned} \int_B \xi \, \mathrm{d}\mathbb{P} &= \mathbb{E}\left(\xi \chi_B\right) \\ &= \mathbb{E}(\xi) \mathbb{E}\left(\chi_B\right) \\ &= \mathbb{E}(\xi) \int_B \mathrm{d}\mathbb{P} \\ &= \int_B \mathbb{E}(\xi) \, \mathrm{d}\mathbb{P}. \end{aligned}$$

■

Exercise 8.4.4

Solution.

1. It suffices to consider singleton $\{Y = y\}$ sets as Y takes values in $\mathbb{N}$. Proceeding as in Example 8.4.3 we see that

$$\begin{aligned}\mathbb{E}\left(h(X)\chi_{\{Y=y\}}\right) &= \sum_{x\in\mathbb{N}} h(x)p_{X,Y}(x,y)\\ &= \sum_{x\in\mathbb{N}} h(x)\frac{p_{X,Y}(x,y)}{p_y(y)}p_Y(y)\\ &= \phi(y)\mathbb{E}\left(\chi_{\{Y=y\}}\right)\\ &= \mathbb{E}\left(\phi(y)\chi_{\{Y=y\}}\right).\end{aligned}$$

2. Recall that $Z_1 + Z_2 \sim \mathsf{Po}(\lambda_1 + \lambda_2)$. Let $X = Z_1$, $Y = Z_1 + Z_2$ and $h = \chi_{\{Z_1=k\}}$. Then

$$\mathbb{P}(Z_1 = k|Z_1 + Z_2 = n) = \mathbb{E}\left(h(X)|Y\right)(n).$$

Hence,

$$\begin{aligned}\mathbb{P}(Z_1 = k|Z_1 + Z_2 = n) &= \sum_{x\in\mathbb{N}} \chi_{\{Z_1=k\}}\frac{\mathbb{P}(Z_1 = x, Z_1 + Z_2 = n)}{\mathbb{P}(Z_1 + Z_2 = n)}\\ &= \frac{\mathbb{P}(Z_1 = k)\mathbb{P}(Z_2 = n-k)}{\mathbb{P}(Z_1 + Z_2 = n)}\\ &= \frac{\left(\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right)\left(\lambda_2^{n-k}e^{-\lambda_2}\right)}{\left(\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}\right)}\\ &= \frac{n!}{k!(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}.\end{aligned}$$

■

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