

# The Reduced Tensor Product of Braided Tensor Categories containing a Symmetric Fusion Category

Thomas A. Wasserman

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## Abstract

We detail a construction of a symmetric monoidal structure  $\boxtimes_{\text{red}}^{\mathcal{A}}$  on the 2-category  $\mathbf{BTC}(\mathcal{A})$  of braided tensor categories containing a fixed symmetric fusion subcategory  $\mathcal{A}$ . The construction only depends on the braiding and monoidal structure of the categories involved. The main tool in the construction is an enriching procedure that is shown to give an equivalence between  $\mathbf{BTC}(\mathcal{A})$  and a 2-category  $\mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF}$  of so-called Drinfeld centre crossed braided tensor categories.

As an application of the reduced tensor product, we show that it gives a pairing between minimal modular extensions of braided tensor categories containing  $\mathcal{A}$  as their transparent subcategory.

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# 1 Introduction

## 1.1 Motivation

This paper addresses the following problem: given two tensor categories that carry a central action of a given symmetric fusion category, one can form the balanced tensor product [ENO10, DSPS19] to obtain a tensor category that contains the same symmetric fusion category. However, if the tensor categories were additionally braided, the balanced tensor product will not be braided, unless the symmetric subcategory is transparent (has trivial double braiding

with all other objects) in both tensor categories. The reduced tensor product<sup>1</sup> of braided tensor categories over a symmetric fusion subcategory yields a braided tensor category, also when the symmetric subcategory is not transparent.

The construction in this paper is motivated by the desire to better understand several aspects of braided tensor categories containing a symmetric fusion category. In the particular case where  $\mathcal{A}$  is exactly the transparent subcategory (or Müger centre) of the braided category  $\mathcal{C}$  and Tannakian (has only positive twists), one can produce (see [Bru00, Müg04]) a modular tensor category  $\mathcal{C}/\mathcal{A}$ , by first enriching  $\mathcal{C}$  over  $\mathcal{A}$  (by representing the action of  $\mathcal{A}$  by tensoring on  $\mathcal{C}$ ), and then applying the fibre functor for  $\mathcal{A}$  and idempotent completing. In the super-Tannakian case (with negative twists), this procedure is still well understood, and yields a “super-modular” category. The intermediate step, before applying the fibre functor, is a braided  $\mathcal{A}$ -tensor category, which is “modular” in the sense that the braiding is non-degenerate. In the case where  $\mathcal{A} = \mathbf{Rep}(G)$  and is not transparent in  $\mathcal{C}$ , the same procedure of enriching and then applying the fibre functor now yields a so-called  $G$ -crossed braided tensor category [DGNO10, Müg10, Kir02]. However, the intermediate step is not well-understood, it is still an  $\mathcal{A}$ -tensor category, but no longer braided. This paper fills this gap, by explaining in what sense the intermediate category is a braided object.

Another aspect that motivates the current work is that of recent work on finite gauge theory [LKW17b, LKW17a, BGH<sup>+</sup>17]. For an explanation of the physical reasoning, see these references and [Was17a, Introduction]. From a mathematical standpoint, the message there is that certain physical systems are described by so-called minimal modular extensions, these are modular tensor categories containing a symmetric fusion category  $\mathcal{A}$  (in particular,  $\mathcal{A}$  is not transparent), which are associated to a braided tensor category  $\mathcal{C}$  with Müger centre exactly  $\mathcal{A}$ . Taking a tensor product of such physical systems does not correspond to the usual Deligne tensor product of linear categories, one would like the resulting category to contain just a single copy of  $\mathcal{A}$ . As alluded to above, the naive alternative of taking a balanced tensor product over  $\mathcal{A}$  does not yield a braided tensor category. The reduced tensor product presented here addresses exactly this issue, and corresponds to the product of minimal modular extensions constructed in [LKW17b]. The construction done there relies on Tannaka duality and Ostrik’s results on the correspondence between module categories and algebra objects [Ost03]. Our construction of the reduced tensor product using the braiding and tensor products of the categories involved directly, and furthermore does not rely on a choice of fibre functor for  $\mathcal{A}$ .

## 1.2 Outline

To give some intuition for our construction of the reduced tensor product, let us first take a closer look at the balanced tensor product. One way of constructing

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<sup>1</sup>This term was used by Drinfeld [Dri] to describe this product in the special case where the symmetric subcategory is the category of super vector spaces.

the balanced tensor product over a symmetric fusion category  $\mathcal{A}$  is by using the fact [DSPS14] that we can enrich module categories over  $\mathcal{A}$  to  $\mathcal{A}$ -enriched categories. For categories enriched over a symmetric monoidal category, there is a natural notion of enriched cartesian product, this is the category with as objects pairs of the categories in the product, and as hom-objects the tensor product in  $\mathcal{A}$  of the hom-objects. The balanced tensor product of two module categories can then be formed by enriching both categories in this fashion, taking the enriched cartesian product, and then de-enriching the resulting  $\mathcal{A}$ -enriched category by applying the functor  $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)$  to the hom-objects. Here  $\mathbb{I}_{\mathcal{A}}$  denotes the monoidal unit of  $\mathcal{A}$ . If one is interested in semi-simple and idempotent complete module categories, this construction is followed by a Cauchy completion.

As we will discuss in this paper, the balanced tensor product takes braided tensor categories containing  $\mathcal{A}$  to tensor categories, but not to braided tensor categories. In fact, this is already visible at the level of the enriched categories associated to braided tensor categories containing  $\mathcal{A}$ , these will be “ $\mathcal{A}$ -tensor”, with monoidal structure that factors through the  $\mathcal{A}$ -enriched cartesian product, but not braided. Consequently one should have no hope that the enriched cartesian product of two categories obtained in this way is braided. This in turn means the de-enrichment will not be braided in general.

To address this problem, we introduce the novel idea of taking our enrichment further. We will construct, for each braided tensor category containing  $\mathcal{A}$ , an associated  $\mathcal{Z}(\mathcal{A})$ -enriched category, where  $\mathcal{Z}(\mathcal{A})$  denotes the Drinfeld centre of  $\mathcal{A}$ . We show that this enriched category carries in some sense a braided monoidal structure. Similarly to the classical  $\mathcal{A}$ -enriched case, this enrichment allows an inverse **DE**. This allows us to define the reduced tensor product in terms of the enriched cartesian product of the  $\mathcal{Z}(\mathcal{A})$ -enriched categories.

Recall that the Drinfeld centre, as introduced by Drinfeld and first written down by Majid [Maj91], gives a braided monoidal category  $\mathcal{Z}(\mathcal{M})$  associated to any monoidal category  $\mathcal{M}$ , where the objects are pairs consisting of an object of  $\mathcal{M}$  and a half-braiding between tensoring on the left and tensoring on the right with that object. The monoidal structure  $\otimes_c$  on  $\mathcal{Z}(\mathcal{M})$  is induced from the one on  $\mathcal{M}$ , in particular we tensor the objects in pairs together as objects in  $\mathcal{M}$ . When taking the Drinfeld centre of a symmetric fusion category, the Drinfeld centre carries another, symmetric, tensor product  $\otimes_s$ , constructed in [Was17c]. These two tensor products on the Drinfeld centre are laxly compatible [Was17b].

In doing our enrichment procedure, we find that the composition in our  $\mathcal{Z}(\mathcal{A})$ -enriched category naturally factors through this new symmetric tensor product. That is, the resulting category is enriched over  $(\mathcal{Z}(\mathcal{A}), \otimes_s)$ , and we will denote  $(\mathcal{Z}(\mathcal{A}), \otimes_s)$  by  $\mathcal{Z}(\mathcal{A})_s$  for short. The monoidal structure, however, factors through the usual tensor product  $\otimes_c$ , a phenomenon that we will refer to as having a  $\mathcal{Z}(\mathcal{A})$ -crossed tensor structure. We will additionally show that the categories obtained from braided tensor categories containing  $\mathcal{A}$  by enriching are braided in the appropriate sense. These  $\mathcal{Z}(\mathcal{A})$ -crossed braided categories are introduced in [Was19]. In particular, the appropriate notion of enriched

cartesian product  $\boxtimes_s$  of such categories uses the symmetric tensor product  $\otimes_s$  on hom-objects.

### 1.3 Context

By Tannaka duality [Del90, Del02] we have that  $\mathcal{A}$  is equivalent to the representation category of a finite (super-)group. In [Was19], it is shown that the 2-category  $\mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF}$  of  $\mathcal{Z}(\mathcal{A})$ -crossed tensor categories, equipped with the enriched cartesian product, is, for  $\mathcal{A} = \mathbf{Rep}(G)$ , where  $G$  is a finite group, equivalent to the 2-category  $G\text{-}\mathbf{XBF}$  of  $G$ -crossed braided tensor categories, equipped with the degree-wise Deligne product  $\boxtimes_G$ , and the corresponding statement for the super-group case is also spelled out.

Let us now focus on the case  $\mathcal{A} = \mathbf{Rep}(G)$  for simplicity of phrasing. In [DGNO10], it is established that there is an equivalence, along mutually inverse functors called equivariantisation (**Eq**) and de-equivariantisation (**De – Eq**) between the 2-categories  $\mathbf{BTC}(\mathcal{A})$  and  $G\text{-}\mathbf{XBF}$ . The enrichment procedure given here, together with the equivalence from [Was19], gives a factorisation of these functors, and further shows monoidality of these. In summary, we have a commutative triangle

$$\begin{array}{ccc}
 & (\mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF}, \boxtimes_s) & \\
 \begin{array}{c} \xleftarrow{(-)} \\ \xrightarrow{\overline{(-)}} \end{array} & \begin{array}{c} \text{DeEnrich}(\overline{-}) \\ \text{Fix} \end{array} & \\
 (\mathbf{BTC}(\mathcal{A}), \boxtimes_{\text{red}}^{\mathcal{A}}) & \begin{array}{c} \xleftarrow{\text{De-Eq}} \\ \xrightarrow{\text{Eq}} \end{array} & (G\text{-}\mathbf{XBF}, \boxtimes_G)
 \end{array}$$

of mutually inverse symmetric monoidal equivalences of 2-categories. The proof of commutativity of this triangle is beyond the scope of this work, and can be found in the author's PhD thesis [Was17a]. In this article we will concern ourselves with the left hand side of this triangle.

In the present work, we do not need to refer to Tannaka duality, and all our constructions are independent of a choice of fibre functor on  $\mathcal{A}$ . Indeed, our construction only depends on the monoidal structure and the braiding of the categories involved.

### 1.4 Organisation

The paper is organised as follows. In the first section, we explain the enriching procedure, this is summarised in Theorem 35. We then in the next section proceed to show that this construction has an inverse, this is Theorem 51. In the final section, we construct the reduced tensor product and show it indeed defines a symmetric monoidal structure (Theorem 58). After this, we compute the reduced tensor product in specific examples, and show how it defines a pairing on minimal modular extension.

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## 2 Enriching

### 2.1 Setup

In this section we will introduce the basic objects of study, braided tensor categories containing a fixed symmetric fusion subcategory, and define a 2-category of such categories. Throughout this paper we will work over the complex numbers.

#### 2.1.1 Braided tensor categories containing a symmetric fusion subcategory

In this paper, we assume the reader is familiar with the theory of tensor categories and fusion categories. To avoid any confusion, we will briefly recall the basic definitions here.

By a *tensor category* we will mean a category enriched in the category **Vect** of *finite* dimensional vector spaces, is abelian for this enrichment (i.e. it is a *linear category*), idempotent complete, and carries a monoidal structure that factors through the Deligne tensor product of linear categories and is right exact in both slots. Such a category is called *fusion* if it is rigid, semi-simple with finitely many isomorphism classes of simple objects, and has a simple unit object.

**Definition 1.** Let  $\mathcal{A}$  be a symmetric fusion category and let  $\mathcal{C}$  be a braided tensor category. We say that  $\mathcal{C}$  contains  $\mathcal{A}$  if  $\mathcal{C}$  comes equipped with a braided tensor functor  $\mathcal{A} \subset \mathcal{C}$ .

As  $\mathcal{A}$  is semi-simple, any tensor functor on  $\mathcal{A}$  is automatically faithful. We are not asking that  $\mathcal{A}$  is embedded in  $\mathcal{C}$ , ie. the inclusion need not be full.

Throughout this paper,  $\mathcal{A}$  will denote a fixed symmetric fusion category, and  $\mathcal{C}$  will be used to denote a braided tensor category containing  $\mathcal{A}$ .

### 2.1.2 The 2-category of Braided Tensor Categories Containing $\mathcal{A}$

In defining the 2-category  $\mathbf{BTC}(\mathcal{A})$ , there are several choices to be made, we use the following definition:

**Definition 2.** The 2-category  $\mathbf{BTC}(\mathcal{A})$  of braided tensor categories containing the symmetric fusion category  $\mathcal{A}$  is the 2-category with

- objects: braided tensor categories  $\mathcal{C}$  containing  $\mathcal{A}$ , such that tensoring by  $\mathcal{A}$  is exact,
- morphisms: triples  $(F, \mu_{-1}, \mu_0, \mu_1)$ , where  $(F, \mu_{-1}, \mu_1)$  is a braided tensor functor (that is,  $F: \mathcal{C} \rightarrow \mathcal{C}'$  is a right exact linear functor,  $\mu_{-1}: F(\mathbb{I}_{\mathcal{C}}) \xrightarrow{\cong} \mathbb{I}_{\mathcal{C}'}$  an isomorphism and  $\mu_1: F(- \otimes_{\mathcal{C}} -) \Rightarrow F(-) \otimes_{\mathcal{C}'} F(-)$  a natural isomorphism), that preserves the inclusions of  $\mathcal{A}$  up to a chosen monoidal natural isomorphism  $\mu_0$ ,
- 2-morphisms: monoidal natural transformations  $\eta$  between  $(F, \mu_{-1}, \mu_0, \mu_1)$  and  $(G, \nu_{-1}, \nu_0, \nu_1)$  that satisfy  $\nu_0 \circ \eta|_{\mathcal{A}} = \mu_0$ .

### 2.1.3 The Drinfeld Centre of Symmetric Fusion Category

We recall the definition of the Drinfeld centre of a monoidal category for convenience.

**Definition 3.** Let  $\mathcal{M}$  be a monoidal category. The *Drinfeld centre*  $\mathcal{Z}(\mathcal{M})$  of  $\mathcal{M}$  is the braided monoidal category with objects pairs  $(m, \beta)$  where  $m$  is an object of  $\mathcal{M}$  and  $\beta$  is a natural transformation

$$\beta: - \otimes m \Rightarrow m \otimes -.$$

The  $\beta$  are further required to satisfy

$$\beta_{nn'} = (\beta_n \otimes \text{id}_{n'}) \circ (\text{id}_n \otimes \beta_{n'}), \quad (1)$$

for all  $n, n' \in \mathcal{M}$ .

The morphisms in  $\mathcal{Z}(\mathcal{M})$  are those morphisms in  $\mathcal{M}$  that commute with the half-braidings in the obvious way. The tensor product  $\otimes_c$  is the one on  $\mathcal{M}$  with consecutive half-braiding, and the braiding is the one specified by the half-braidings.

The Drinfeld centre comes with a monoidal *forgetful functor*  $\Phi: \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ , which forgets the half-braiding.

It was shown in [Was17c] that the Drinfeld centre  $\mathcal{Z}(\mathcal{A})$  of a symmetric fusion category carries a second, symmetric, tensor product  $\otimes_s$ , and in [Was17b] that the two tensor products are laxly compatible. In [Was19] this is used to define the notion of a  $\mathcal{Z}(\mathcal{A})$ -crossed braided tensor category. In this paper we will use this notion to produce a symmetric monoidal structure on  $\mathbf{BTC}(\mathcal{A})$ .

## 2.2 Enriching over a symmetric subcategory

It is well known (see for example [Ost03], see also [DSPS14, Proposition 2.15]) that from a module category over a fusion category one can obtain a category enriched over the acting category. Here, we will reproduce this construction, and spell out the special features this has if the acting category is a symmetric category.

### 2.2.1 The enriched category

To start with, we will put ourselves in the situation where we have tensor category with a tensor inclusion of a symmetric fusion category.

**Definition 4.** Let  $\mathcal{C}$  be a tensor category containing a symmetric fusion category  $\mathcal{A}$ . The *left-associated  $\mathcal{A}$ -enriched category*  $\underline{\mathcal{C}}$  has the same objects as  $\mathcal{C}$  and  $\underline{\mathcal{C}}(c, c')$  is defined by

$$\mathcal{A}(a, \underline{\mathcal{C}}(c, c')) = \mathcal{C}(ac, c'). \quad (2)$$

The composition morphisms,

$$\circ: \underline{\mathcal{C}}(c', c'') \otimes \underline{\mathcal{C}}(c, c') \rightarrow \underline{\mathcal{C}}(c, c''),$$

are defined by observing that we have the following string of canonical isomorphisms:

$$\begin{aligned} \mathcal{A}(a, \underline{\mathcal{C}}(c', c'') \otimes \underline{\mathcal{C}}(c, c')) &\cong \mathcal{A}(\underline{\mathcal{C}}(c', c'')^* \otimes a, \underline{\mathcal{C}}(c, c')) \\ &\cong \mathcal{C}(\underline{\mathcal{C}}(c', c'')^* \otimes ac, c') \\ &\cong \mathcal{C}(ac, \underline{\mathcal{C}}(c', c'') \otimes c') \\ &\xrightarrow{\text{ev}} \mathcal{C}(ac, c'') \\ &\cong \mathcal{A}(a, \underline{\mathcal{C}}(c, c'')). \end{aligned} \quad (3)$$

Here  $\text{ev}$  is the unit of the adjunction given by (2), c.f. Definition A.11.

Similarly, we define the *right-associated  $\mathcal{A}$ -enriched category*  $\underline{\mathcal{C}}_*$ , by representing  $a \mapsto \mathcal{C}(ca, c')$ .

Observe that  $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, \underline{\mathcal{C}}(c, c')) = \mathcal{C}(c, c')$ . We can view the mate  $\bar{f}$  (Definition A.4) of  $f: c \rightarrow_a c'$  as a morphism in  $\mathcal{C}$ . In terms of mates and the composition in  $\mathcal{C}$ , the composition of  $f: c \rightarrow_a c'$  and  $f': c' \rightarrow_{a'} c''$  in  $\underline{\mathcal{C}}$  is given by

$$f' \circ f = \bar{f}'(\text{id}_{a'} \otimes \bar{f}), \quad (4)$$



which in string diagrams reads as:

$$\begin{array}{c}
 c'' \\
 \downarrow \\
 \boxed{\bar{f}'} \\
 \swarrow \quad \searrow \\
 a' \quad \quad \boxed{\bar{f}} \\
 \quad \quad \swarrow \quad \searrow \\
 \quad \quad a \quad c
 \end{array}
 \quad . \quad (5)$$

**Remark 5.** Both  $\underline{\mathcal{C}}$  and  $\underline{\mathcal{C}}_{\rightarrow}$  are tensored over  $\mathcal{A}$ . For  $\underline{\mathcal{C}}_{\rightarrow}$ , the tensoring induces a functor  $\mathcal{A}^{\text{mop}} \rightarrow \text{End}(\underline{\mathcal{C}}_{\rightarrow})$ , where  $\mathcal{A}^{\text{mop}}$  denotes the monoidal opposite of  $\mathcal{A}$ .

The  $\mathcal{A}$ -product  $\boxtimes_{\mathcal{A}}$  (Definition A.17) of a  $\mathcal{A}$ -enriched category  $\underline{\mathcal{C}}$  obtained in this way with itself has some nice features. Corresponding to the product  $f_1 \boxtimes f_2$  of  $f_1: c_1 \rightarrow_{a_1} c'_1$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$  there is, by using the tensor structure on the product  $\underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}}$  and the tensor product in  $\mathcal{C}$ , a map  $\bar{f}_1 \otimes_{\mathcal{C}} \bar{f}_2$ . It is tempting to represent this in string diagrams as:

$$\begin{array}{cc}
 c'_1 & c'_2 \\
 \downarrow & \downarrow \\
 \boxed{\bar{f}_1} & \boxed{\bar{f}_2} \\
 \swarrow \quad \searrow & \swarrow \quad \searrow \\
 a_1 \quad c_1 & a_2 \quad c_2
 \end{array}
 \quad .$$

Care should be taken, however, that, by Equation (A.11), the position of the  $a$ 's is immaterial. To avoid confusion, we will therefore always keep the objects of  $\mathcal{A}$  to the left when we are dealing with left enrichments. In drawing string diagrams, this does mean that we need to cross  $\mathcal{A}$ -strands past  $\mathcal{C}$ -strands. To emphasise such crossings are not actual braidings in  $\mathcal{C}$ , we will draw them unresolved as follows:

$$\begin{array}{cc}
 c'_1 & c'_2 \\
 \downarrow & \downarrow \\
 \boxed{\bar{f}_1} & \boxed{\bar{f}_2} \\
 \swarrow \quad \searrow & \swarrow \quad \searrow \\
 a_1 \quad a_2 \quad c_1 & c_2
 \end{array}
 \quad . \quad (6)$$

When considering a morphism  $f: c_1 \boxtimes c_2 \rightarrow_a c'_1 \boxtimes c'_2$ , we will give a string diagram presentation by first picking a factorisation  $(t, f_1, f_2)$ :

$$f: a \xrightarrow{t} a_1 a_2 \xrightarrow{f_1 f_2} \underline{\mathcal{C}}(c_1, c'_1) \underline{\mathcal{C}}(c_2, c'_2), \quad (7)$$

and then using the tensor isomorphism to find mates for  $f_1$  and  $f_2$ . There are many different choices of factorisations for a given  $f$ . In terms of the triples, we have the equivalence relation

$$(t, f_1 \circ g_1, f_2 \circ g_2) \sim (g_1 g_2 \circ t, f_1, f_2),$$

for  $g_i : a'_i \rightarrow a_i$  for  $i = 1, 2$ , with  $t : a \rightarrow a'_1 a'_2$  and  $f_i : a_i c_i \rightarrow c'_i$  with  $i = 1, 2$ . A factorisation  $(t, f_1, f_2)$  can be presented in string diagrams by:

$$(8)$$

Here the trivalent vertex represents the morphism  $t : a \rightarrow a_1 a_2$  from Equation (7).

### 2.2.2 Functors between the associated $\mathcal{A}$ -categories

We want to extend  $\mathcal{C} \mapsto \underline{\mathcal{C}}$  to a 2-functor, so far we have only defined it on the objects of  $\mathbf{BTC}(\mathcal{A})$ .

**Definition 6.** Let  $(F, \mu_{-1}, \mu_0, \mu_1) : \mathcal{C} \rightarrow \mathcal{C}'$  be morphism in  $\mathbf{BTC}(\mathcal{A})$ , then *the associated  $\mathcal{A}$ -enriched functor*

$$\underline{F} : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}',$$

is the functor which acts as  $F$  on objects. On morphisms, we define the morphisms  $\underline{F}_{c,c'}$  as morphisms in  $\mathcal{A}$  by observing that the composite:

$$\mathcal{C}(ac, c') \xrightarrow{F_{ac,c'}} \mathcal{C}'(F(ac), F(c')) \cong \mathcal{C}'(aF(c), F(c')),$$

gives for each  $c, c' \in \mathcal{C}$  a natural transformation from  $\mathcal{C}(-c, c') : \mathcal{A} \rightarrow \mathbf{Vect}$  to  $\mathcal{C}'(-F(c), F(c')) : \mathcal{A} \rightarrow \mathbf{Vect}$ . In this composite the last isomorphism is induced by the composite of  $\mu_1$  and  $\mu_0$ . The natural transformation defined in this way induces a morphism:

$$\underline{F}_{c,c'} : \underline{\mathcal{C}}(c, c') \rightarrow \underline{\mathcal{C}}'(Fc, Fc').$$

This morphism takes the mate  $\bar{f} : ac \rightarrow a$  for a morphism  $f : c \rightarrow_a c'$  to  $\overline{\underline{F}(f)} : aF(c) \xrightarrow{\cong} F(ac) \xrightarrow{\bar{f}} F(c')$ , where the first map is the composite of  $\mu_1$  and  $\mu_0$ .

We need to check that the functor  $\underline{F}$  defined in this way is indeed a  $\mathcal{A}$ -enriched functor:

**Lemma 7.** *The map  $\underline{F}$  defined above respects composition.*

*Proof.* We need to show that for all  $c, c', c'' \in \mathcal{C}$ :

$$\begin{array}{ccc} \underline{\mathcal{C}}(c, c') \underline{\mathcal{C}}(c', c'') & \xrightarrow{\circ} & \underline{\mathcal{C}}(c, c'') \\ \downarrow \underline{F}_{c, c'} \otimes \underline{F}_{c', c''} & & \downarrow \underline{F}_{c, c''} \\ \underline{\mathcal{C}}'(Fc, Fc') \underline{\mathcal{C}}'(Fc', Fc'') & \xrightarrow{\circ} & \underline{\mathcal{C}}'(Fc, Fc''). \end{array}$$

On mates for  $f: c \rightarrow_a c'$  and  $f': c' \rightarrow_{a'} c''$ , the top route computes as:

$$a' a F c \xrightarrow{\cong} F(a' a c) \xrightarrow{\bar{f}' \circ (\text{id}_{a'} \otimes \bar{f})} F(c''),$$

whereas the bottom route becomes:

$$a' a F c \xrightarrow{\cong} a' F(ac) \xrightarrow{\text{id}_{a'} F(\bar{f})} a' F(c') \xrightarrow{\cong} F(a' c') \xrightarrow{F(\bar{f}')} F(c'').$$

Using the fact that the structure isomorphisms for  $F$  are natural, we can exchange the middle two morphisms to get:

$$a' a F(c) \xrightarrow{\cong} F(a' a c) \xrightarrow{F(\bar{f}' \circ (\text{id}_a \otimes \bar{f}'))} F(c''),$$

where we have also used the fact that  $F$  preserves composition, and that the monoidality isomorphisms for  $aF(c)$  and  $a'F(ac)$  compose to the monoidality isomorphism for  $F(a'ac)$ .  $\square$

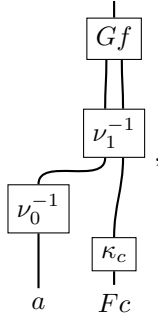
For natural transformations, we use the following.

**Definition 8.** Let  $\kappa: F \Rightarrow G$  be a 2-morphism between two morphisms in  $\mathbf{BTC}(\mathcal{A})$  between  $\mathcal{C}$  and  $\mathcal{C}'$ . Then the *associated  $\mathcal{A}$ -enriched natural transformation*  $\underline{\kappa}: \underline{F} \Rightarrow \underline{G}$  is given by the mate to  $\kappa$ .

As we have added additional morphisms when defining  $\underline{\mathcal{C}}$ , we need to check this definition makes sense:

**Lemma 9.** *The associated  $\mathcal{A}$ -enriched natural transformation  $\underline{\kappa}$  for a 2-morphism  $\kappa: (F, \mu_{-1}, \mu_0, \mu_1) \Rightarrow (G, \nu_{-1}, \nu_0, \nu_1)$  is indeed natural.*

*Proof.* In shorthand notation, we need to check that for any  $f: c \rightarrow_a d$  we have that  $\underline{G}(f) \underline{\kappa}_c = \underline{\kappa}_d \underline{F}(f)$ . In string diagrams, the left hand side is, in terms of mates:



whereas the right hand side gives:

where we have used naturality of  $\kappa$ , monoidality of  $\kappa$ , and the relation  $\nu_0 \kappa|_{\mathcal{A}} = \mu_0$ , consecutively.  $\square$

### 2.2.3 Enriched monoidal structure

Now take  $\mathcal{C}$  to be a braided tensor category containing  $\mathcal{A}$ . The tensor product on  $\mathcal{C}$  together with the braiding between the objects of  $\mathcal{A}$  and those of  $\mathcal{C}$  induces an associated  $\mathcal{A}$ -monoidal structure on  $\underline{\mathcal{C}}$  (and similarly on  $\underline{\mathcal{C}}_r$ ). This  $\mathcal{A}$ -monoidal structure is defined as follows.

**Definition 10.** The *induced  $\mathcal{A}$ -tensor product on  $\underline{\mathcal{C}}$*  is given by  $\otimes_{\mathcal{C}}$  on objects. On morphisms it is given by the map

$$\otimes_{\underline{\mathcal{C}}}: \underline{\mathcal{C}}(c_1, c'_1) \otimes_{\mathcal{A}} \underline{\mathcal{C}}(c_2, c'_2) \rightarrow \underline{\mathcal{C}}(c_1 c_2, c'_1 c'_2), \quad (9)$$

which is obtained from the following composite, writing  $a_i = \underline{\mathcal{C}}(c_i, c'_i)$  for  $i = 1, 2$ :

$$\begin{aligned} \mathcal{A}(a_1, \underline{\mathcal{C}}(c_1, c'_1)) \otimes_{\mathbf{Vect}} \mathcal{A}(a_2, \underline{\mathcal{C}}(c_2, c'_2)) &\cong \mathcal{C}(a_1 c_1, c'_1) \otimes_{\mathbf{Vect}} \mathcal{C}(a_2 c_2, c'_2) \\ &\xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C}(a_1 c_1 a_2 c_2, c'_1 c'_2) \\ &\xrightarrow{(\beta_{a_2, c_1})^*} \mathcal{C}(a_1 a_2 c_1 c_2, c'_1 c'_2), \end{aligned} \quad (10)$$

where in the first line we used the tensor product of the tensor structure on  $\underline{\mathcal{C}}$  with itself, the monoidal structure in  $\mathcal{C}$  in the second line and the braiding between  $a_2$  and  $c_1$  in the last line. We obtain our desired map from Equation (9) as the image of the tensor product of the identities under this morphism.

In terms of mates, this translates to the following. Let  $f_1: c_1 \rightarrow_{a_1} c'_1$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$ , following the above recipe we find:

$$f_1 \otimes_{\underline{\mathcal{C}}} f_2 = \bar{f}_1 \otimes_{\underline{\mathcal{C}}} \bar{f}_2 (\text{id}_{a_1} \otimes \beta_{a_2, c_1} \otimes \text{id}_{c_2}). \quad (11)$$

In string diagrams, this becomes:

$$(12)$$

**Remark 11.** To do the constructions up to this point, it would have sufficed to assume that  $\mathcal{C}$  comes equipped with a central functor  $\mathcal{A} \rightarrow \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ .

**Lemma 12.** *The categories  $\underline{\mathcal{C}}$  and  $\underline{\mathcal{C}}_r$  are  $\mathcal{A}$ -monoidal, with the monoidal structure from Definition 10.*

*Proof.* We will only provide a proof for  $\underline{\mathcal{C}}$ , the case of  $\underline{\mathcal{C}}_r$  is similar. We need to prove the structure above satisfies the interchange law, i.e. that the proposed  $\mathcal{A}$ -monoidal structure is indeed a functor. Checking functoriality boils down to checking that the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathcal{C}}(c_1 \boxtimes c_2, c'_1 \boxtimes c'_2) \otimes \underline{\mathcal{C}}(c'_1 \boxtimes c'_2, c''_1 \boxtimes c''_2) & \xrightarrow{\circ(\underline{\varepsilon} \boxtimes \underline{\varepsilon})} & \underline{\mathcal{C}}(c_1, c'_1) \otimes \underline{\mathcal{C}}(c_2, c''_2) \\ \downarrow \otimes \boxtimes \otimes & & \downarrow \otimes \\ \underline{\mathcal{C}}(c_1 c_2, c'_1 c'_2) \otimes \underline{\mathcal{C}}(c'_1 c'_2, c''_1 c''_2) & \xrightarrow{\circ \underline{\varepsilon}} & \underline{\mathcal{C}}(c_1 c_2, c''_1 c''_2). \end{array}$$

We will do this by checking that the precomposition of the two routes in this diagram with

$$f_1 \otimes f_2 \otimes f'_1 \otimes f'_2: a_1 a_2 a'_1 a'_2 \rightarrow \underline{\mathcal{C}}(c_1, c'_1) \underline{\mathcal{C}}(c_2, c'_2) \underline{\mathcal{C}}(c'_1, c''_1) \underline{\mathcal{C}}(c'_2, c''_2)$$

are the same. This will be the case if and only if their mates are equal. Using Equations (5) and (12) we see that we need to check:

$$(13)$$

and this equation holds by naturality of the braiding in  $\mathcal{C}$ .

The associators in  $\mathcal{C}$  will descend to morphisms in  $\underline{\mathcal{C}}$  and still satisfy the pentagon equations. We have to convince ourselves that these morphisms define

a natural isomorphism, with respect to the extra morphisms in the enriched hom-objects  $\underline{\mathcal{C}}(c, c')$  for  $c, c' \in \mathcal{C}$ . But by (A.2), all these extra morphisms are just morphisms  $ac \rightarrow c'$  for some  $a \in \mathcal{A}$ . Using the pentagon equations on these morphisms, this means the associators from  $\mathcal{C}$  will also be natural for these extra morphisms.  $\square$

#### 2.2.4 A second $\mathcal{A}$ -monoidal structure

Since we made a choice to use  $\beta$  rather than  $\beta^{-1}$  in Definition 10, we also have:

**Definition 13.** We define  $-\otimes^\beta - : \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$ , by taking it to be  $-\otimes_{\mathcal{C}} -$  on objects and on morphisms the image of the identity under the composite

$$\begin{aligned} \mathcal{A}(a_1, \underline{\mathcal{C}}(c_1, c'_1)) \otimes \mathcal{A}(a_2, \underline{\mathcal{C}}(c_2, c'_2)) &= \mathcal{C}(a_1 c_1, c'_1) \otimes_{\mathbf{Vect}} \mathcal{C}(a_2 c_2, c'_2) \\ &\xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C}(a_1 c_1 a_2 c_2, c'_1 c'_2) \\ &\xrightarrow{(\beta_{c_1, a_2}^{-1})^*} \mathcal{C}(a_1 a_2 c_1 c_2, c'_1 c'_2), \end{aligned}$$

where  $a_i = \underline{\mathcal{C}}(c_i, c'_i)$  for  $i = 1, 2$ .

The proof that this indeed specifies an  $\mathcal{A}$ -monoidal structure is analogous to the proof of Lemma 12. In string diagrams for the mates of  $f_1 : c_1 \rightarrow_{a_1} c'_1$ , and  $f_2 : c_2 \rightarrow_{a_2} c'_2$  this monoidal structure gives:

$$(14)$$

This second monoidal structure will be useful below when studying how the braiding on  $\mathcal{C}$  behaves on  $\underline{\mathcal{C}}$ .

### 2.3 Braiding for the Associated $\mathcal{A}$ -Enriched Category

In the previous section, we only used the half-twists  $\beta_{a,c}$  for  $a \in \mathcal{A}$  and  $c \in \mathcal{C}$ , and the braiding in  $\mathcal{A}$ . From here onward, we will need that  $\mathcal{C}$  is itself braided, and that  $\mathcal{A}$  is a symmetric subcategory of  $\mathcal{C}$ .

#### 2.3.1 A Problem with the Braiding

Naively, one might expect  $\underline{\mathcal{C}}$  to be braided if  $\mathcal{C}$  is, with braiding induced by the braiding in  $\mathcal{C}$ . We now pause to show that  $\underline{\mathcal{C}}$  is not a braided  $\mathcal{A}$ -monoidal category. The failure of the braiding on  $\mathcal{C}$  to induce a braiding on  $\underline{\mathcal{C}}$  will

motivate the next step in our construction, where we further enrich  $\underline{\mathcal{C}}$  to a category enriched over the Drinfeld centre of  $\mathcal{A}$ .

When attempting to lift the braiding of  $\mathcal{C}$  to a braiding on  $\underline{\mathcal{C}}$ , one counters the following problem: the braiding will no longer be natural with respect to the additional morphisms. We will show that the naturality diagram

$$\begin{array}{ccc} c_1 c_2 & \xrightarrow{\beta_{c_1, c_2}} & c_2 c_1 \\ a_1 a_2 \downarrow f_1 \otimes f_2 & & a_2 a_1 \downarrow f_2 \otimes f_1 \\ c'_1 c'_2 & \xrightarrow{\beta_{c'_1, c'_2}} & c'_2 c'_1 \end{array}$$

fails to commute in general. Its failure to commute can be seen as follows. In terms of mates, the naturality diagram becomes the outside of:

$$\begin{array}{ccc} a_1 a_2 c_1 c_2 & \xrightarrow{\beta_{a_1, a_2} \otimes \beta_{c_1, c_2}} & a_2 a_1 c_2 c_1 \\ \downarrow \beta_{a_2, c_1} & & \downarrow \beta_{a_1, c_2} \\ a_1 c_1 a_2 c_2 & \xrightarrow{\beta_{a_1 c_1, a_2 c_2}} & a_2 c_2 a_1 c_1 \\ \downarrow \bar{f}_1 \otimes \bar{f}_2 & & \downarrow \bar{f}_2 \otimes \bar{f}_1 \\ c'_1 c'_2 & \xrightarrow{\beta_{c'_1, c'_2}} & c'_2 c'_1 \end{array},$$

here the braiding  $\beta_{a_1, a_2}$  in the top row comes from the switch map for the  $\mathcal{A}$ -product that was implicit in the previous diagram. The map in the middle will help us understand the failure of commutativity. Note that, by naturality of the braiding in  $\mathcal{C}$ , the lower square of the diagram does commute. It therefore suffices to consider the top square, in string diagrams the top and bottom routes read

$$\begin{array}{ccc} \begin{array}{c} a_2 \quad c_2 \quad a_1 \quad c_1 \\ \text{[Diagram 1: } a_2 \text{ and } a_1 \text{ cross, } c_2 \text{ and } c_1 \text{ cross}] \\ a_1 \quad a_2 \quad c_1 \quad c_2 \end{array} & \text{and} & \begin{array}{c} a_2 \quad c_2 \quad a_1 \quad c_1 \\ \text{[Diagram 2: } a_2 \text{ and } c_1 \text{ cross, } a_1 \text{ and } c_2 \text{ cross}] \\ a_1 \quad a_2 \quad c_1 \quad c_2 \end{array}, \end{array} \quad (15)$$

respectively. We see that these diagrams differ from each other by a precomposition with the *braiding monodromy*  $\beta_{c_1, a_2} \beta_{a_2, c_1}$  between  $a_2$  and  $c_1$ .

### 2.3.2 Braiding between the two monoidal structures

In this section, we show how the braiding on  $\mathcal{C}$  can be used to relate the two different monoidal structures  $\otimes$  (Definition 10) and  $\otimes^\beta$  (Definition 13) on  $\underline{\mathcal{C}}$ . We will do this in two ways, the first is by introducing a functor that encodes the braiding monodromy between objects of  $\mathcal{A}$  and objects of  $\mathcal{C}$ , the second is to show that the braiding gives a natural isomorphism between  $\otimes^\beta$  and  $\otimes \circ S$ , where  $S$  is the swap functor for  $\underline{\mathcal{A}}$ . Later, we will use these results to fix the problem with the braiding discussed in the previous section.

**Definition 14.** The *inverse monodromy functor*  $\beta^{-2}: \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}}$  is defined as follows. The functor  $\beta^{-2}$  is the identity on objects. On morphisms, we take mates for  $f: c_1 \boxtimes c_2 \rightarrow_a c'_1 \boxtimes c'_2$  factored over the tensor product between  $f_1: c_1 \rightarrow_{a_1} c'_1$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$ , with  $a_1 a_2 = a$ , and assign:

(16)

We remind the reader of the convention discussed around Equation (6), and emphasise that the double braiding in this diagram really is a double braiding, whereas the unresolved crossings indicate a crossing used to bring all objects of  $\mathcal{A}$  to the left. To justify that this assignment really defines a morphism  $\beta^{-2}(f): a \rightarrow \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}}(c_1 \boxtimes c_2, c'_1 \boxtimes c'_2)$ , we compare with Equation (8). We can interpret the right hand side of Equation (16) as the mate for the tensor product of a morphism  $a_1 a_2 a_2^* \rightarrow \underline{\mathcal{C}}(c_1, c'_1)$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$ , precomposed with the with trivalent vertex  $a_1 a_2 \rightarrow a_1 a_2 a_2^*$  coming from the coevaluation for  $a_2^*$ .

**Lemma 15.** The assignment  $\beta^{-2}$  is an autofunctor of  $\underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}}$ .

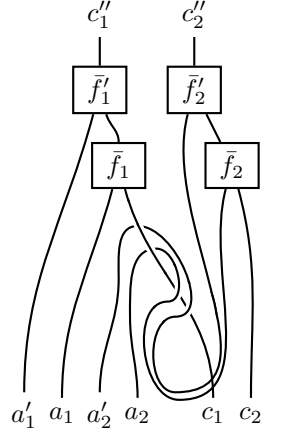
*Proof.* If  $\beta^{-2}$  is indeed a functor, then it is clearly invertible with inverse given by using the opposite crossings in Equation (16). So we need to check that  $\beta^{-2}$  preserves composition. That is, we need to check that the following diagram commutes:

$$\begin{array}{ccc}
 \underline{\mathcal{C}}(c'_1, c''_1) \underline{\mathcal{C}}(c'_2, c''_2) \underline{\mathcal{C}}(c_1, c'_1) \underline{\mathcal{C}}(c_2, c'_2) & \xrightarrow{\circ \underline{\mathcal{C}} \boxtimes_a \underline{\mathcal{C}}} & \underline{\mathcal{C}}(c_1, c'_1) \underline{\mathcal{C}}(c_2, c'_2) \\
 \downarrow \beta^{-2} \otimes \beta^{-2} & & \downarrow \beta^{-2} \\
 \underline{\mathcal{C}}(c'_1, c''_1) \underline{\mathcal{C}}(c'_2, c''_2) \underline{\mathcal{C}}(c_1, c'_1) \underline{\mathcal{C}}(c_2, c'_2) & \xrightarrow{\circ \underline{\mathcal{C}} \boxtimes_a \underline{\mathcal{C}}} & \underline{\mathcal{C}}(c_1, c'_1) \underline{\mathcal{C}}(c_2, c'_2).
 \end{array}$$

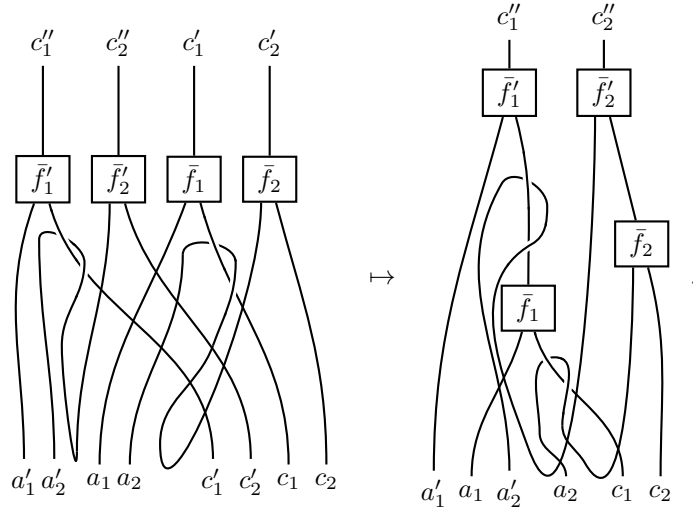
Recall that composition in terms of mates is given by Equation (5). In terms of mates for  $f_1: c_1 \rightarrow_{a_1} c'_1$ ,  $f'_1: c'_1 \rightarrow_{a'_1} c''_1$ ,  $f_2: c_2 \rightarrow_{a_2} c'_2$  and  $f'_2: c'_2 \rightarrow_{a'_2} c''_2$ , the



top route becomes:



while the bottom route becomes, first applying  $\beta^{-2}$ , then the composition:



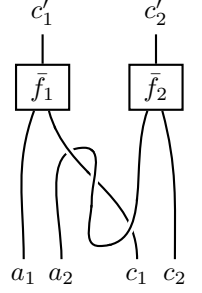
Using the naturality of the braiding, we see that the string diagrams corresponding to the top and bottom routes are indeed equal.  $\square$

The inverse monodromy functor can be used to obtain the two monoidal structures on  $\mathcal{C}$  from each other:

**Lemma 16.** *The functor  $-\otimes^\beta -: \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{C} \rightarrow \mathcal{C}$  is equal to the functor obtained by precomposing  $-\otimes_{\mathcal{C}} -$  with  $\beta^{-2}$ .*

*Proof.* The functors agree on objects by definition, so we only need to check the functors agree on morphisms. Let  $f_1: c_1 \rightarrow_{a_1} c'_1$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$  be

morphisms in  $\underline{\mathcal{C}}$ . The image of their mates under  $-\otimes^\beta-$  is shown in Equation (14). Applying the composite of  $\beta^{-2}$  and  $-\otimes-$  to these mates is given, in string diagrams, by:


(17)

which is indeed equal to Equation (14). □

We will now show that the braiding is a natural transformation between these two monoidal structures on  $\underline{\mathcal{C}}$ , after we compose one with the switch map.

**Lemma 17.** *The braiding in  $\mathcal{C}$  induces a natural isomorphism between the functors  $-\otimes^\beta-: \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$  and the composite of  $-\otimes-: \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$  with the switch map for the  $\mathcal{A}$ -product. This isomorphism satisfies the hexagon equations.*

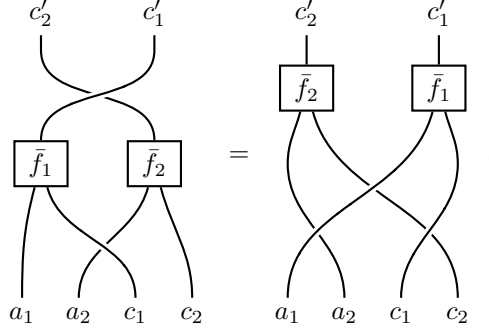
*Proof.* We want to show the diagram

$$\begin{array}{ccc}
 c_1 c_2 & \xrightarrow{\beta_{c_1, c_2}} & c_2 c_1 \\
 a_1 a_2 \downarrow f_1 \otimes^\beta f_2 & & a_2 a_1 \downarrow f_2 \otimes f_1 \\
 c'_1 c'_2 & \xrightarrow{\beta_{c'_1, c'_2}} & c'_2 c'_1
 \end{array}$$

commutes for all  $f_1: c_1 \rightarrow_{a_1} c'_1$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$ . In terms of the mates, this diagram becomes:

$$\begin{array}{ccc}
 a_1 a_2 c_1 c_2 & \xrightarrow{\beta_{a_1, a_2} \otimes \beta_{c_1, c_2}} & a_2 a_1 c_2 c_1 \\
 \downarrow \beta_{a_2, c_1}^{-1} & & \downarrow \beta_{c_2, a_1} \\
 a_1 c_1 a_2 c_2 & & a_2 c_2 a_1 c_1 \\
 \downarrow \bar{f}_1 \otimes \bar{f}_2 & & \downarrow \bar{f}_2 \otimes \bar{f}_1 \\
 c'_1 c'_2 & \xrightarrow{\beta_{c'_1, c'_2}} & c'_2 c'_1
 \end{array} .$$

Writing this in terms of string diagrams:



The hexagon equations follow from the hexagon equations for the braiding in  $\mathcal{C}$ .  $\square$

## 2.4 Towards $\mathcal{Z}(\mathcal{A})$ -Crossed Braided Categories

To ensure that our construction gives a braided object, we will enrich our category  $\mathcal{C}$  further to a  $\mathcal{Z}(\mathcal{A})$ -enriched category  $\mathcal{C}_{\mathcal{Z}}$ , where we will take care to encode the braiding monodromy from Equation (15) into the half-braidings we pick on our hom-objects. As the swap map for the convolution product of  $\mathcal{Z}(\mathcal{A})$ -enriched categories uses these half-braidings, we can use this to cancel the failure of the naturality of the braiding.

### 2.4.1 The $\mathcal{Z}(\mathcal{A})_s$ -enrichment

We will now show that the  $\mathcal{A}$ -enrichment from the previous sections can be pushed to an enrichment over  $(\mathcal{Z}(\mathcal{A}), \otimes_s)$ . That is, we need to define the enriched hom-functor with values in  $\mathcal{Z}(\mathcal{A})$ , the composition, and the identity morphisms. We will denote the resulting  $\mathcal{Z}(\mathcal{A})_s$ -enriched category by  $\mathcal{C}_{\mathcal{Z}}$ .

The aim of this section is to prove:

**Proposition 18.** *Let  $\mathcal{C}$  be a braided tensor category containing a spherical symmetric fusion category  $\mathcal{A}$ . Then the category  $\mathcal{C}_{\mathcal{Z}}$  defined in Section 2.4.1 is a  $\mathcal{Z}(\mathcal{A})_s$ -enriched and tensored category.*

The first step towards enriching  $\mathcal{C}$  over  $\mathcal{Z}(\mathcal{A})_s$  is:

**Definition 19.** Let  $\mathcal{C}$  be a braided tensor category containing  $\mathcal{A}$  as a braided subcategory. The  $\mathcal{Z}(\mathcal{A})_s$ -enriched hom-object  $\mathcal{C}_{\mathcal{Z}}(c, c')$  between  $c, c' \in \mathcal{C}$  is defined as follows. We set:

$$\mathcal{C}_{\mathcal{Z}}(c, c') = (\mathcal{C}(c, c'), \mathbf{b}),$$

where the half-braiding  $\mathbf{b}$  is defined by:

$$a \mathcal{C}_{\mathcal{Z}}(c, c') \xrightarrow{\cong} \mathcal{C}_{\mathcal{Z}}(c, ac') \xrightarrow{(\beta_{a, c'}^{-1} \beta_{c', a}^{-1})^*} \mathcal{C}_{\mathcal{Z}}(c, ac') \xrightarrow{\cong} a \mathcal{C}_{\mathcal{Z}}(c, c') \xrightarrow{s} \mathcal{C}_{\mathcal{Z}}(c, c') a. \quad (18)$$

Here  $s$  denotes the symmetry in  $\mathcal{A}$ .

For this definition to make sense, we need to show that the half-braiding  $\mathbf{b}$  is monoidal (cf. Equation (1)):

**Lemma 20.** *The half-braiding  $\mathbf{b}: \underline{\mathcal{C}}(c, c') \otimes_{\mathcal{A}} - \Rightarrow - \otimes_{\mathcal{A}} \underline{\mathcal{C}}(c, c')$  is a monoidal natural isomorphism between functors  $\mathcal{A} \rightarrow \mathcal{A}$ .*

*Proof.* Using that  $a = \underline{\mathcal{C}}(\mathbb{I}, a)$ , we can unpack the half-braiding from Equation (18) in terms of the mates for  $\text{id}_a: \mathbb{I} \rightarrow_a a$  and  $f_2: c \rightarrow_{a'} c'$  as:

where in the equality we have used the naturality of the braiding that the fact that objects of  $\mathcal{A}$  are transparent to each other. We interpret the last diagram as the mate to the tensor product of a morphism  $aa'a^* \rightarrow \underline{\mathcal{C}}(c, c')$  and  $\text{id}_a: \mathbb{I} \rightarrow a$ . Now one uses that the braiding monodromy between  $a, a' \in \mathcal{A}$  and  $c \in \mathcal{C}$  has the property:

as the objects in  $\mathcal{A}$  are transparent to each other.  $\square$

What we have shown so far is that every hom-object can be viewed as an object in the Drinfeld centre of  $\mathcal{A}$ . The next step is to define a composition morphism. This composition morphism will factor through the symmetric tensor product  $\otimes_s$  for  $\mathcal{Z}(\mathcal{A})$ , see [Was17c, Definition 12].

**Definition 21.** Let  $c, c', c'' \in \mathcal{C}$ , and let  $\underline{\mathcal{C}}(c, c') = (\underline{\mathcal{C}}(c, c'), \mathbf{b}')$  and  $\underline{\mathcal{C}}(c', c'') = (\underline{\mathcal{C}}(c', c''), \mathbf{b}'')$  denote the lifts of the  $\mathcal{A}$ -enriched hom-objects to  $\mathcal{Z}(\mathcal{A})$  from Definition 19. Then we define the *composition morphism for the  $\mathcal{Z}(\mathcal{A})_s$ -enrichment* to be the composite:

$$\Phi(\underline{\mathcal{C}}(c', c'') \otimes_s \underline{\mathcal{C}}(c, c')) \hookrightarrow \underline{\mathcal{C}}(c', c'') \underline{\mathcal{C}}(c, c') \xrightarrow{\circ} \underline{\mathcal{C}}(c, c''), \quad (20)$$

on the underlying objects in  $\mathcal{A}$ .

In order for this definition to make sense, we need the composite from Equation (20) to define a morphism in  $\mathcal{Z}(\mathcal{A})$ :

**Lemma 22.** *The composite from Equation (20) defines a morphism in  $\mathcal{Z}(\mathcal{A})$ .*

*Proof.* We need to check that the morphism commutes with the braiding. That is, we need to show that the outside of the diagram

$$\begin{array}{ccccc}
a\Phi(\underline{\mathcal{C}}(c', c'') \otimes_s \underline{\mathcal{C}}(c, c')) & \hookrightarrow & a\underline{\mathcal{C}}(c', c'')\underline{\mathcal{C}}(c, c') & \xrightarrow{\circ} & a\underline{\mathcal{C}}(c, c'') \\
\mathbf{b}_a \downarrow & & s \circ (\text{id} \otimes (\beta_{a, c''}^{-1} \circ \beta_{c'', a}^{-1})_*) \downarrow & & \downarrow s \circ (\beta_{a, c''}^{-1} \circ \beta_{c'', a}^{-1})_* \\
\Phi(\underline{\mathcal{C}}(c', c'') \otimes_s \underline{\mathcal{C}}(c, c'))a & \hookrightarrow & \underline{\mathcal{C}}(c', c'')\underline{\mathcal{C}}(c, c')a & \xrightarrow{\circ} & \underline{\mathcal{C}}(c, c'')a
\end{array}$$

commutes. Here, the leftmost square is the definition of the half-braiding  $\mathbf{b}_a$  [Was17c, Equation 14] on  $\underline{\mathcal{C}}(c', c'') \otimes_s \underline{\mathcal{C}}(c, c')$  in terms of the half-braiding on  $\underline{\mathcal{C}}(c', c'')$  (Equation 18). Note that we have used monoidality of the symmetry in  $\mathcal{A}$  to compose two instances of the symmetry here, and suppressed the isomorphisms between  $a\underline{\mathcal{C}}(c', c'')$  and  $\underline{\mathcal{C}}(c', ac'')$ , here, as well as in the rightmost morphism. As the leftmost square commutes by definition, it suffices to check that the rightmost square commutes. For this we can ignore the symmetry  $s$ , as this is a natural transformation, hence commutes with the morphism  $\circ$  in  $\mathcal{A}$ . The top route is then a composition followed by precomposition, while the bottom is precomposition followed by composition, so they agree by associativity of the composition in  $\mathcal{C}$ .  $\square$

The associativity of this  $\mathcal{Z}(\mathcal{A})_s$ -enriched composition is immediate from the associativity of the composition in  $\underline{\mathcal{C}}$ . To finish setting up the structure of the  $\mathcal{Z}(\mathcal{A})_s$ -enriched category  $\underline{\mathcal{C}}$ , we need to provide, for each  $c \in \mathcal{C}$ , an identity morphism  $1_c: \mathbb{I}_s \rightarrow \underline{\mathcal{C}}(c, c)$ , where  $\mathbb{I}_s$  denotes the unit object for  $\otimes_s$  as introduced in [Was17c, Definition 17]. (Recall that the underlying object in  $\mathcal{A}$  of  $\mathbb{I}_s$  is  $\oplus_{i \in \mathcal{O}(\mathcal{A})} i^* i$ .) We then need to check that this morphism indeed specifies an identity in the sense that the following diagram commutes

$$\begin{array}{ccc}
\underline{\mathcal{C}}(c, c') \otimes_s \mathbb{I}_s & \xrightarrow{\rho} & \underline{\mathcal{C}}(c', c) \\
\text{id} \otimes_s 1_c \downarrow & \nearrow \circ & \\
\underline{\mathcal{C}}(c, c') \otimes_s \underline{\mathcal{C}}(c, c) & & ,
\end{array} \tag{21}$$

where  $\rho$  denotes the right unitor for  $\otimes_s$ , as well as the corresponding diagram for the left unitor.

**Definition 23.** The  $\mathcal{Z}(\mathcal{A})$ -identity morphism  $1_c: \mathbb{I}_s \rightarrow \underline{\mathcal{C}}(c, c)$  for  $c \in \mathcal{C}$  is the

mate for the morphism

$$\mathbb{I}_s \begin{array}{c} c \\ | \\ \text{blue loop} \\ | \\ c \end{array} := \sum_{i \in \mathcal{O}(\mathcal{A})} \begin{array}{c} c \\ | \\ \text{black loop} \\ | \\ i i^* c \end{array} .$$

We need to check that this indeed specifies a morphism in  $\mathcal{Z}(\mathcal{A})$ , and that it satisfies (21).

**Lemma 24.** *The  $\mathcal{Z}(\mathcal{A})$ -identity morphism is a morphism in  $\mathcal{Z}(\mathcal{A})$ , that is:*

$$\begin{array}{ccc} a\mathbb{I}_s & \xrightarrow{1_c} & a\mathcal{C}(c, c) \\ \downarrow \beta_{a, \mathbb{I}_s} & & \downarrow \mathbf{b}_a \\ \mathbb{I}_s a & \xrightarrow{1_c} & \mathcal{C}(c, c)a, \end{array}$$

commutes.

*Proof.* Recalling that the braiding  $\mathbf{b}_a$  was computed in terms of mates in Equation (19), the top and bottom routes compute as

$$\begin{array}{c} \begin{array}{c} c \\ | \\ \text{black loop} \\ | \\ a \mathbb{I}_s c \end{array} \quad \text{and} \quad \begin{array}{c} c \\ | \\ \text{blue loop} \\ | \\ a \mathbb{I}_s c \end{array}, \end{array}$$

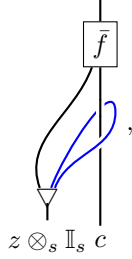
respectively. The latter has summands (using the definition of the half-braiding on  $\mathbb{I}_s$  from [Was17c, Equation (18)]):

$$\begin{array}{c} \begin{array}{c} c \\ | \\ \text{black loop} \\ | \\ \boxed{\phi^*} \\ | \\ \boxed{\phi} \\ | \\ a \ i \ i^* \ c \end{array} = \begin{array}{c} c \\ | \\ \text{black loop} \\ | \\ \boxed{\phi^t} \\ | \\ \boxed{\phi} \\ | \\ a \ i \ i^* \ c \end{array}, \end{array}$$

where the  $\phi$  give a resolution of the identity on  $ai$ . The terms specified by this last diagram sum to the top route, remembering that the objects in  $\mathcal{A}$  are transparent to each other.  $\square$

**Lemma 25.** *The identity morphisms satisfy the triangle equality from Equation (21).*

*Proof.* The unitor for  $\mathcal{Z}(\mathcal{A})_s$  is given in [Was17c, Lemma 19]. Let  $z \in \mathcal{Z}(\mathcal{A})$  and let  $f: z \rightarrow \underline{\mathcal{C}}(c, c')$  be a morphism. The mate for the image of  $f$  under  $\rho$  is:



where we simplified a double symmetry between  $z$  and the summand of the strand, coming from the definition of the braiding on  $\underline{\mathcal{C}}(c, c')$ . On the other hand, the bottom route is the composite of  $f$  with  $1_c$ , which in terms of mates is represented by the same diagram. This shows that the identity morphism indeed satisfies Equation (21).  $\square$

We have now gathered the ingredients to define:

**Definition 26.** Let  $\mathcal{C}$  be a braided tensor category and let  $\mathcal{A}$  be a symmetric subcategory of  $\mathcal{C}$ . Then the *left associated  $\mathcal{Z}(\mathcal{A})_s$ -enriched category  $\underline{\mathcal{C}}$*  for  $\mathcal{C}$  is the  $\mathcal{Z}(\mathcal{A})_s$ -enriched category with objects those of  $\mathcal{C}$ , hom-objects from Definition 19, composition from Definition 21 and identity morphisms from Definition 23.

#### 2.4.2 $\mathcal{Z}(\mathcal{A})_s$ -Tensoring

The category  $\underline{\mathcal{C}}$  produced above is also tensored over  $\mathcal{Z}(\mathcal{A})_s$ .

**Proposition 27.** *Let  $\mathcal{C}$  be a braided tensor category containing  $\mathcal{A}$ . Then for all  $c, c' \in \mathcal{C}$  and  $(a, \beta) \in \mathcal{Z}(\mathcal{A})$ , the subobject  $\pi((a, \beta), c)$  associated to the idempotent*

$$\Pi_{(a, \beta), c} = \sum_{i \in \mathcal{O}(\mathcal{A})} \frac{1}{d_i} \begin{array}{c} \text{Diagram: A box labeled } s_{i,a} \text{ is connected to a box labeled } \beta_i \text{ by a vertical line. A strand from } a \text{ enters the box } \beta_i \text{ from the bottom. A strand from } c \text{ enters the box } s_{i,a} \text{ from the bottom. A curved line connects the top of } \beta_i \text{ to the top of } s_{i,a}. \end{array} i := \begin{array}{c} \text{Diagram: Two vertical lines labeled } a \text{ and } c \text{ are shown. A blue oval encircles both lines.} \end{array}$$

satisfies

$$\mathcal{Z}(\mathcal{A})((a, \beta), \underline{\mathcal{C}}(c, c')) \cong \mathcal{C}(\pi((a, \beta), c), c').$$

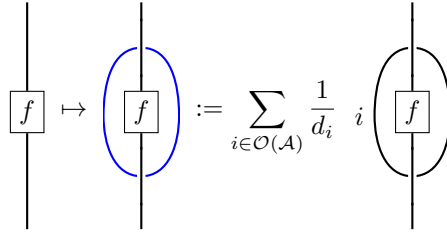
In the shorthand notation for  $\Pi_{(a, \beta), c}$ , care should be taken to remember that the braidings take place in  $\mathcal{Z}(\mathcal{A})$ , viewing  $i$  as an object in  $\mathcal{Z}(\mathcal{A})$  equipped with the half-braiding coming from the symmetry.

*Proof.* By formally the same proof as for [Was17c, Lemma 10] the morphism  $\Pi_{(a, \beta), c}$  is indeed an idempotent on  $(a, \beta)c$ .

To see the isomorphism, we notice that, in  $\mathcal{Z}(\mathcal{A})$ , observe that the hom-object  $\mathcal{C}(\pi((a, \beta), c), c')$  is the equaliser for precomposition with  $\Pi_{(a, \beta), c}$  and the identity on  $\mathcal{C}(ac, c')$ . Precomposition with  $\Pi_{(a, \beta), c}$  takes a morphism  $f: ac \rightarrow c'$  to the morphism



On the other hand, in  $\mathcal{Z}(\mathcal{A})$ , the hom-spaces between  $(a, \beta)$  and  $(a', \beta')$  are the equalisers for the morphism  $\mathcal{A}(a, a') \rightarrow \mathcal{A}(a, a')$  given by



and the identity on  $\mathcal{A}(a, a')$ . This description of the hom-spaces in  $\mathcal{Z}(\mathcal{A})$  is an easy consequence of cloaking ([Was17c, Lemma 9]). Setting  $a' = \underline{\mathcal{C}}(c, c')$  and unpacking the definition of the braiding on  $\underline{\mathcal{C}}(c, c')$  in terms of mates, using the right hand side of Equation (19), we see that this agrees with Equation (22). This shows that the objects in the proposition are equalisers for the same morphisms, and therefore canonically isomorphic.  $\square$

Observe that this  $\mathcal{Z}(\mathcal{A})_s$ -tensoring also provides an action of  $\mathcal{Z}(\mathcal{A})_s$  on the original category  $\mathcal{C}$ . This implies that we could first have provided the  $\mathcal{Z}(\mathcal{A})_s$ -action, and enriched along this in a fashion similar to Definition 4. For the case  $\mathcal{C} = (\mathcal{Z}(\mathcal{A}), \otimes_c)$ , this  $\mathcal{Z}(\mathcal{A})_s$ -action agrees with  $\otimes_s$ .



**Lemma 28.** *The functor  $\mathbb{I}_s \cdot - : \mathcal{C} \rightarrow \mathcal{C}$  induced by the  $\mathcal{Z}(\mathcal{A})$ -tensoring is naturally isomorphic to the identity on  $\mathcal{C}$  along the natural transformation  $\tau$  with components:*

$$\tau_c = \begin{array}{c} c \\ | \\ \text{[Diagram: A blue loop encircling a black vertical line segment.] } \\ | \\ \triangle \\ | \\ \pi(\mathbb{I}_s, c) \end{array},$$

where the triangle denotes the inclusion  $\pi(\mathbb{I}_s, c) \hookrightarrow \mathbb{I}_s c$ .

*Proof.* The proof is analogous to the proof of [Was17c, Lemma 19].

For  $\mathcal{A} \subset \mathcal{Z}(\mathcal{A})_s$ , the tensoring given here relates to the original action of  $\mathcal{A}$  on  $\mathcal{C}$  via the tensor product in  $\mathcal{C}$  in the following way:

**Lemma 29.** *Let  $a \in \mathcal{A}$ , viewed as the object  $(a, s) \in \mathcal{Z}(\mathcal{A})$ , then for any  $c \in \mathcal{C}$  we have:*

$$a \otimes_{\mathcal{C}} c \cong \pi((a, s) \otimes_c \mathbb{I}_s, c).$$

*Proof.* Consider the co-Yoneda embedding of  $\pi((a, s), c)$ :

$$\begin{aligned} \mathcal{C}(\pi((a, s) \otimes_c \mathbb{I}_s, c), c') &\cong \mathcal{Z}(\mathcal{A})((a, s) \otimes_c \mathbb{I}_s, \underline{\mathcal{C}}(c, c')) \\ &\cong \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, (a, s)^* \otimes_c \underline{\mathcal{C}}(c, c')) \\ &\cong \mathcal{A}(\mathbb{I}_A, a^* \otimes_A \underline{\mathcal{C}}(c, c')) \\ &\cong \mathcal{A}(\mathbb{I}_A, \underline{\mathcal{C}}(a \otimes_C c, c')) \\ &\cong \mathcal{C}(a \otimes_C c, c'), \end{aligned}$$

where the first isomorphism is Proposition 27, the third [Was19, Lemma 14], the fourth [Was19, Lemma A.13], and the final isomorphism comes from the defining property of  $\underline{\mathcal{C}}(a \otimes c, c')$ , Definition 2. We see that  $a \otimes c$  and  $\pi((a, s), c)$  have canonically isomorphic co-Yoneda embeddings, so are canonically isomorphic.

The  $\mathcal{Z}(\mathcal{A})_s$ -tensoring allows us to describe morphisms in  $\underline{\mathcal{C}}$  as follows.

**Definition 30.** Suppose  $f: c \rightarrow_z c'$  is a morphism in  $\mathcal{C}_{\overleftarrow{\mathcal{A}}}$ , then its  $\mathcal{Z}(\mathcal{A})$ -mate is the morphism in  $\mathcal{C}$

$$\tilde{f}: \pi(z, c) \rightarrow c',$$

obtained by applying the isomorphism from Proposition 27 to  $f$ .

Conversely, given a morphism  $f: c \rightarrow_a c'$ , we get the *associated*  $\mathcal{Z}(\mathcal{A})$ -*enriched morphism*  $\underline{f}: c \rightarrow_{(a,s) \otimes_c \mathbb{I}_s} c$  in  $\underline{\mathcal{C}}$  given by applying the isomorphism from Proposition 27 to the mate  $\bar{f}: ac \rightarrow c'$  composed with the isomorphism  $ac \cong \pi((a, s) \otimes_c \mathbb{I}_s, c)$  from Lemma 29.

### 2.4.3 Associated $\mathcal{Z}(\mathcal{A})$ -Enriched Functors and Transformations

We want to extend our assignment  $\mathcal{C} \mapsto \underline{\mathcal{C}}$  to a 2-functor. For this, we need to know where to send functors between braided tensor categories containing  $\mathcal{A}$ . Fortunately, this is made easy by the following:

**Lemma 31.** *For any morphism  $F \in \mathbf{BTC}(\mathcal{A})$ , the associated  $\mathcal{A}$ -enriched functor  $\underline{F}$  from Definition 6 lifts to a  $\mathcal{Z}(\mathcal{A})_s$ -enriched functor  $\underline{\underline{F}}$ .*

*Proof.* As  $F$  is braided, the morphism  $\underline{F}_{c,c'}$  will be compatible with the braiding and therefore define a morphism  $\underline{\underline{F}}_{c,c'}$  in  $\mathcal{Z}(\mathcal{A})$ .  $\square$

For natural transformations, we have:

**Lemma 32.** *Let  $\eta : (F, \mu_{-1}, \mu_0, \mu_1) \rightarrow (G, \nu_{-1}, \nu_0, \nu_1)$  be a monoidal natural transformation between functors on  $\mathcal{C} \supset \mathcal{A}$  with  $\nu_0 \eta|_{\mathcal{A}} = \mu_0$ . Setting  $\underline{\eta}_c$  to be the associated  $\mathcal{Z}(\mathcal{A})$ -enriched morphism to the mate to  $\eta_c : F(c) \rightarrow G(c)$ . Then  $\underline{\underline{\eta}} : (\underline{\underline{F}}, \underline{\underline{\mu}}_0, \underline{\underline{\mu}}_1) \Rightarrow (\underline{\underline{G}}, \underline{\underline{\nu}}_0, \underline{\underline{\nu}}_1)$  is a natural transformation between the associated  $\mathcal{Z}(\mathcal{A})$ -enriched functors.*

*Proof.* Naturality follows from Lemma 9.  $\square$

**Proposition 33.** *The assignment  $\underline{\underline{(-)}}$  on  $\mathbf{BTC}(\mathcal{A})$  with values in the 2-category of  $\mathcal{Z}(\mathcal{A})_s$ -enriched categories is functorial.*

*Proof.* We have to check that this assignment preserves composition of functors and of natural transformations. The composite of mates is the mate of composites for degree  $\mathbb{I}_s$ -morphisms, and similarly, as the action of a composition of functors on hom-objects is by the composition of the maps the functors induce on hom-objects, the image of the composition under  $\underline{\underline{(-)}}$  will be the composite of the images.  $\square$

### 2.4.4 Monoidal structure

In this section we examine the sense in which  $\underline{\underline{\mathcal{C}}}$  is monoidal. The monoidal structure on  $\mathcal{C}$  will give rise to a monoidal structure on  $\underline{\underline{\mathcal{C}}}$ , however, this monoidal structure will not factor over the  $(\mathcal{Z}(\mathcal{A}), \otimes_s)$ -product. Rather, it will factor over the convolution tensor product of  $\mathcal{Z}(\mathcal{A})$ -enriched categories, where we use the tensor product  $\otimes_c$  on the  $\mathcal{Z}(\mathcal{A})$ -hom objects. That is, it will be a  $\mathcal{Z}(\mathcal{A})$ -crossed tensor category ([Was19, Definition 30]). As a preparation for showing this, we first establish the existence of a functor which will act as the unit for the crossed tensor structure.

**Lemma 34.** *Let  $\mathcal{C}$  be a braided tensor category containing  $\mathcal{A}$ . Then the associated  $\mathcal{Z}(\mathcal{A})_s$ -enriched functor for inclusion  $\mathcal{A} \subset \mathcal{C}$  is a functor*

$$\mathbb{I}_{\mathcal{C}} : \mathcal{A}_{\mathcal{Z}} \rightarrow \underline{\underline{\mathcal{C}}}.$$

Here  $\mathcal{A}_{\mathcal{Z}}$  is the unit for the convolution tensor product (see [Was19, Lemma 22]).

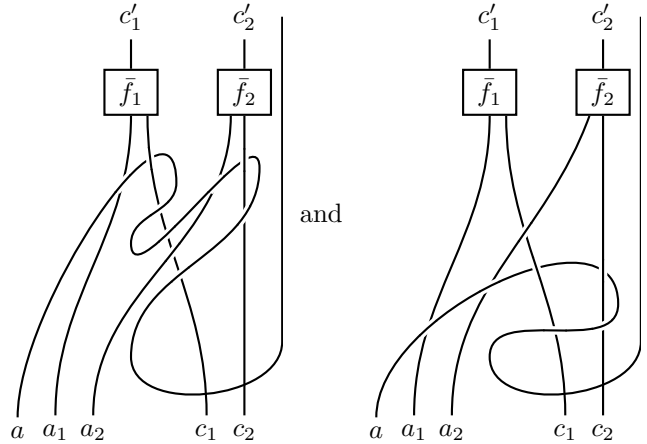
*Proof.* We simply observe that  $\underline{\mathcal{A}} = \mathcal{A}_{\mathcal{Z}}$ .  $\square$

**Proposition 35.** *If  $\mathcal{C}$  is a braided tensor category containing a symmetric spherical fusion category  $\mathcal{A}$ , then  $\underline{\mathcal{C}}$  is  $\mathcal{Z}(\mathcal{A})$ -crossed tensor (see [Was19, Definition 30]), with monoidal structure given in Definition 10 lifted to the  $\mathcal{Z}(\mathcal{A})_s$ -enriched category, with unit functor  $\mathbb{I}_{\mathcal{C}}$  from Lemma 34, and associators and unitors given by the mates to the associators and unitors in  $\mathcal{C}$ .*

*Proof.* As the monoidal structure from Definition 10 is compatible with the composition, and the composition in  $\underline{\mathcal{C}}$  is a restriction of this, the lift of the monoidal structure will be compatible with composition. We still need to show that the morphisms

$$\underline{\mathcal{C}}(c_1, c'_1) \otimes_c \underline{\mathcal{C}}(c_2, c'_2) \xrightarrow{\otimes_{c_1 \boxtimes c_2, c'_1 \boxtimes c'_2}} \underline{\mathcal{C}}(c_1 c_2, c'_1 c'_2)$$

are compatible with the braiding, so that they lift to  $\mathcal{Z}(\mathcal{A})$ . In  $\underline{\mathcal{C}} \boxtimes_c \underline{\mathcal{C}}$ , the left hand object will be equipped with the consecutive braiding on both factors, while the braiding on the right hand side comes from the braiding monodromy of  $c_1 c_2$ . Comparing these braidings with respect to some  $a \in \mathcal{A}$  in terms of mates for  $f_1: c_1 \rightarrow_{a_1} c'_1$  and  $f_2: c_2 \rightarrow_{a_2} c'_2$  gives



for first braiding and then applying  $\otimes$  and the vice versa, respectively. These two are indeed equal.

The associators satisfy the pentagon equations by virtue of the associators in  $\mathcal{C}$  satisfying the pentagon equations. To define the unitors in  $\underline{\mathcal{C}}$ , observe that we have natural isomorphisms between the tensor product composed with the unit functor and the unitor (see [Was19, Lemma 22]) for  $\boxtimes_c$  with components

$$\mathbb{I}_{\mathcal{C}}(a) \otimes c \xrightarrow{\cong} \pi((a, s) \otimes_c \mathbb{I}_s, c),$$

coming from Lemma 29. A similar construction to the one done in Lemma 27 gives a right action of  $\mathcal{Z}(\mathcal{A})_s$  on  $\underline{\underline{\mathcal{C}}}$ , and this action will commute with the left action. Using a result similar to Lemma 29 for this action allows us to define the right unitor. These unitors will satisfy the triangle equation by virtue of the left and right actions commuting.  $\square$

#### 2.4.5 Associated $\mathcal{Z}(\mathcal{A})$ -crossed tensor functors

It turns out that the associated enriched functor to a functor of braided tensor categories containing  $\mathcal{A}$  from Lemma 31 carries naturally the structure of a  $\mathcal{Z}(\mathcal{A})$ -crossed tensor functor (see [Was19, Definition 32]). The  $\mathcal{Z}(\mathcal{A})$ -crossed tensor structure on  $\underline{\underline{F}}$  is obtained as follows:

**Lemma 36.** *The  $\mathcal{Z}(\mathcal{A})_s$ -enriched functor  $\underline{\underline{F}}$  is  $\mathcal{Z}(\mathcal{A})$ -crossed tensor with structure morphisms  $\underline{\underline{\mu}}_0$  and  $\underline{\underline{\mu}}_1$  the enriched natural transformations with components the  $\mathcal{Z}(\mathcal{A})$ -mates to the components of  $\mu_0$  and  $\mu_1$ , respectively.*

*Proof.* Observe that  $\underline{\underline{\mu}}_1$  defined in this way is indeed a  $\mathcal{Z}(\mathcal{A})_s$ -enriched natural isomorphism between  $\underline{\underline{F}}(- \otimes -)$  and  $\underline{\underline{F}}(-) \otimes \underline{\underline{F}}(-)$ , and as the associators come from the associators in the braided tensor categories containing  $\mathcal{A}$ , this natural isomorphism will be compatible with the associators.

Denoting the inclusions of  $\mathcal{A}$  into  $\mathcal{K}$  and  $\mathcal{L}$  by  $i_{\mathcal{K}}$  and  $i_{\mathcal{L}}$ , respectively, the natural isomorphism  $\mu_0$  takes  $F \circ i_{\mathcal{K}}$  to  $\mathcal{L}$ , monoidally. This implies that  $\underline{\underline{\mu}}_0$  gives a natural isomorphism between  $\underline{\underline{F}} \circ \mathbb{I}_{\mathcal{K}}$  and  $\mathbb{I}_{\mathcal{L}}$ , as desired.  $\square$

On the natural transformations, we use the following:

**Lemma 37.** *The associated  $\mathcal{Z}(\mathcal{A})$ -natural transformation  $\underline{\underline{\eta}}$  is a monoidal natural transformation of  $\mathcal{Z}(\mathcal{A})$ -crossed tensor functors (see [Was19, Definition 48]).*

*Proof.* There are two conditions in [Was19, Definition 48]. The first (which is analogous to the usual monoidality axiom) follows directly from the fact that  $\eta$  is monoidal. The second asks for commutativity of:

$$\begin{array}{ccc}
 \underline{\underline{F}} \circ \mathbb{I}_{\mathcal{C}} & & \\
 \parallel & \searrow \underline{\underline{\mu}}_0 & \\
 \underline{\underline{\eta}} \circ \mathbb{I}_{\mathcal{C}} & & \mathbb{I}_{\mathcal{D}} \\
 \downarrow & \nearrow \underline{\underline{\nu}}_0 & \\
 \underline{\underline{G}} \circ \mathbb{I}_{\mathcal{C}} & & 
 \end{array}$$

which follows from the fact that  $\eta$  preserves the inclusion of  $\mathcal{A}$  into  $\mathcal{C}$  and  $\mathcal{D}$ .  $\square$

### 2.4.6 $\mathcal{Z}(\mathcal{A})$ -crossed braiding

We will now show that the braiding for  $\mathcal{C}$  gives rise to a  $\mathcal{Z}(\mathcal{A})$ -crossed braiding, see [Was19, Definition 40]. That is, the braiding will be a monoidal natural isomorphism between  $\otimes_{\underline{\mathcal{C}}}$  and  $\otimes_{\underline{\mathcal{C}}} \circ B$ , where  $B$  is the swap map for  $\boxtimes$ , which uses the braiding on the hom-objects. The first step is to examine what the braiding functor  $B$  from [Was19, Definition 23] becomes for  $\frac{\mathcal{C}}{\underline{\mathcal{C}}} \boxtimes_c \frac{\mathcal{C}}{\underline{\mathcal{C}}}$ .

**Lemma 38.** *Let  $\frac{\mathcal{C}}{\underline{\mathcal{C}}}$  be as above. On a hom-object  $\frac{\mathcal{C}}{\underline{\mathcal{C}}} \boxtimes_c \frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_1 \boxtimes c_2, c_1' \boxtimes c_2')$ , the braiding functor  $B$  (see [Was19, Definition 23]) on  $\frac{\mathcal{C}}{\underline{\mathcal{C}}} \boxtimes_c \frac{\mathcal{C}}{\underline{\mathcal{C}}}$  acts on the underlying objects in  $\mathcal{A}$  of the hom-objects by:*

$$\frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_1, c_1') \frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_2, c_2') \xrightarrow{\beta^{-2}} \frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_1, c_1') \frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_2, c_2') \xrightarrow{s} \frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_2, c_2') \frac{\mathcal{C}}{\underline{\mathcal{C}}}(c_1, c_1'),$$

the composite of the functor  $\beta^{-2}$  (Definition 14) with the symmetry in  $\mathcal{A}$ .

*Proof.* This is immediate from the definition of the half-braidings on the hom-objects (Definition 19).  $\square$

A braiding for a  $\mathcal{Z}(\mathcal{A})$ -crossed tensor category  $\frac{\mathcal{C}}{\underline{\mathcal{C}}}$  is by definition a natural transformation from  $\otimes_{\underline{\mathcal{C}}}$  to  $\otimes_{\underline{\mathcal{C}}} \circ B$ . So, our next step is to compute the composite of  $B$  with the monoidal structure. It turns out that the resulting functor can be viewed as the monoidal structure  $\otimes^\beta$  on  $\frac{\mathcal{C}}{\underline{\mathcal{C}}}$  from Definition 13. A similar argument to the proof of Proposition 35 shows that  $\otimes^\beta$  defines a monoidal structure on  $\frac{\mathcal{C}}{\underline{\mathcal{C}}}$ .

**Proposition 39.** *Let  $\mathcal{C}$  be a braided tensor category containing a spherical symmetric fusion category  $\mathcal{A}$ . Then the category  $\frac{\mathcal{C}}{\underline{\mathcal{C}}}$  is a  $\mathcal{Z}(\mathcal{A})$ -crossed braided tensor category.*

*Proof.* We have already shown in Proposition 35 that  $\frac{\mathcal{C}}{\underline{\mathcal{C}}}$  is  $\mathcal{Z}(\mathcal{A})$ -crossed tensor. We have to show that the braiding for  $\mathcal{C}$  gives a natural transformation between the tensor structure and the composite of  $B$  ([Was19, Definition 23]) with the tensor structure. We start by computing this composite. On objects, it is just the monoidal structure of  $\mathcal{C}$  composed with the switch map. To see what  $\otimes_{\underline{\mathcal{C}}} \circ B$  is on hom-objects, observe the following. By Lemma 38, we know that  $B$  acts on the underlying objects in  $\mathcal{A}$  of the hom-objects as  $\beta^{-2} \circ \text{Switch}_{\mathcal{A}}$ , where  $\text{Switch}_{\mathcal{A}}$  is the switch functor for  $\boxtimes_{\mathcal{A}}$ . So we see that on hom-objects, we have the following equality of morphisms in  $\mathcal{A}$  (with slight abuse of notation):

$$\otimes_{\underline{\mathcal{C}}} \circ B = \otimes_{\underline{\mathcal{C}}} \circ \beta^{-2} \circ \text{Switch}_{\mathcal{A}} = \otimes^\beta \circ \text{Switch}_{\mathcal{A}},$$

where the last equality is Lemma 16. By Proposition 17, the braiding in  $\mathcal{C}$  induces a natural transformation between the last functor and  $\otimes_{\underline{\mathcal{C}}}$ . This implies that the braiding gives a natural isomorphism between  $\otimes_{\underline{\mathcal{C}}} \circ B$  and  $\otimes_{\underline{\mathcal{C}}}$ . Furthermore, the hexagon equations will still be satisfied by virtue of them being satisfied in  $\mathcal{C}$ .  $\square$

### 2.4.7 Associated braided $\mathcal{Z}(\mathcal{A})$ -crossed tensor functors

Now that we have additionally equipped our  $\mathcal{Z}(\mathcal{A})$ -crossed tensor categories with a braiding, we can ask whether the functor from Lemma 36 is also braided in the sense of [Was19, Definition 42].

**Lemma 40.** *Let  $(F, \mu_{-1}, \mu_0, \mu_1) : (\mathcal{C}, \otimes_{\mathcal{C}}, \beta_{\mathcal{C}}) \rightarrow (\mathcal{D}, \otimes_{\mathcal{D}}, \beta_{\mathcal{D}})$  be a morphism in  $\mathbf{BTC}(\mathcal{A})$ , then the associated  $\mathcal{Z}(\mathcal{A})$ -crossed tensor functor  $(\underline{F}, \underline{\mu}_0, \underline{\mu}_1) : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$  from Lemma 36 is  $\mathcal{Z}(\mathcal{A})$ -crossed braided.*

*Proof.* We observe that checking [Was19, Definition 42] boils down to checking  $\mu_1 F(\beta_{\mathcal{C}}) = \beta_{\mathcal{D}} \mu_1$ , which holds by virtue of  $F$  being a braided functor.  $\square$

In summary, we have produced:

**Proposition 41.** *The assignment  $\underline{(-)}$  defines a bifunctor*

$$\underline{(-)} : \mathbf{BFC}/\mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A}) - \mathbf{XBT}.$$

### 2.4.8 Enriching the Commutant of $\mathcal{A}$

We will now, for a category  $\underline{\mathcal{C}}$  obtained by the enriching procedure above, give a characterisation of the neutral subcategory  $\underline{\mathcal{C}}_{\mathcal{A}}$  (the subcategory for which the Yoneda embedding  $\underline{\mathcal{C}}(-, c)$  factors through  $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$ , see [Was19, Definition 17]) in terms of the so-called braided commutant of  $\mathcal{A}$  in  $\mathcal{C}$ .

**Definition 42.** Let  $\mathcal{C}$  be a braided fusion category with braiding  $\beta$  and let  $\mathcal{B}$  be a braided monoidal full subcategory. Then the *braided commutant of  $\mathcal{B}$  in  $\mathcal{C}$*  is the full subcategory with objects

$$\mathcal{Z}_2(\mathcal{B}, \mathcal{C}) = \{c \in \mathcal{C} \mid \beta_{c,b} \circ \beta_{b,c} = \text{id}_{bc} \quad \forall b \in \mathcal{B}\}.$$

When  $\mathcal{B} = \mathcal{C}$ , we will denote this subcategory by  $\mathcal{Z}_2(\mathcal{C})$ , and call it the *Müger centre of  $\mathcal{C}$* .

When  $\mathcal{A}$  is a symmetric subcategory of  $\mathcal{C}$  the commutant  $\mathcal{Z}_2(\mathcal{A}, \mathcal{C})$  contains  $\mathcal{A}$ .

**Proposition 43.** *Denote by  $\underline{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} \subset \underline{\mathcal{C}}$  the full subcategory on the objects of  $\mathcal{Z}_2(\mathcal{A}, \mathcal{C})$ . Then:*

$$\underline{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} = \mathcal{C}_{\mathcal{A}}.$$

*Proof.* As this is a statement about small full subcategories, it suffices to show that  $\underline{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} \subset \mathcal{C}_{\mathcal{A}}$  and  $\underline{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} \supset \mathcal{C}_{\mathcal{A}}$  at the level of objects.

The inclusion  $\underline{\mathcal{Z}_2(\mathcal{A}, \mathcal{C})} \subset \mathcal{C}_{\mathcal{A}}$  follows directly from the way the half-braidings on  $\underline{\mathcal{C}}(c, c')$  are defined in Definition 19: in Equation (18) the morphism  $(\beta_{a,c'}^{-1} \beta_{c',a}^{-1})^*$

is just the identity, so the composite becomes the symmetry in  $\mathcal{A}$  between  $\underline{\mathcal{C}}(c, c')$  and  $a$ .

For the reverse inclusion, suppose that  $c$  is such that its Yoneda embedding  $\underline{\mathcal{C}}(-, c)$  factors through  $\mathcal{A}$ . This means that for each  $c' \in \mathcal{C}$ , the hom-object  $\underline{\mathcal{C}}(c', c)$  is  $\underline{\mathcal{C}}(c', c)$  equipped with the symmetry in  $\mathcal{A}$ . Looking at the definition (Equation (18)) of the half-braiding, we see that this implies that  $(\beta_{a, c'}^{-1} \beta_{c', a}^{-1})_*$  is the identity on  $\underline{\mathcal{C}}(c', c)$  for all  $c'$ . By the Yoneda lemma this means that  $\beta_{a, c'}^{-1} \beta_{c', a}^{-1}$  is the identity on  $ac$ , which is what we wanted to show.  $\square$

We observe the following, which is immediate from the above proposition combined with the fact that the composite  $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A}) \xrightarrow{\Phi} \mathcal{A}$  of the forgetful functor with the inclusion functor is the identity on  $\mathcal{A}$ :

**Corollary 44.** *Let  $\mathcal{C}$  be a braided fusion category containing  $\mathcal{A}$ , and assume that  $\mathcal{Z}_2(\mathcal{A}, \mathcal{C}) = \mathcal{C}$ . Then:*

$$\overline{\underline{\mathcal{C}}} = \underline{\mathcal{C}},$$

where  $\overline{\mathcal{K}}$  for a  $\mathcal{Z}(\mathcal{A})_s$ -enriched category  $\mathcal{K}$  denotes changing basis along the lax monoidal forgetful functor  $\mathcal{Z}(\mathcal{A})_s \rightarrow \mathcal{A}$ .

### 3 De-enriching

We will now proceed to examine how to take a  $\mathcal{Z}(\mathcal{A})$ -crossed braided tensor category and produce a braided tensor category. This construction will restrict to give a 2-functor from the 2-category  $\mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF}$  of  $\mathcal{Z}(\mathcal{A})$ -crossed braided fusion categories, defined in [Was19] to the 2-category  $\mathbf{BTC}(\mathcal{A})$  of braided fusion categories containing  $\mathcal{A}$ . We will show that this construction defines an inverse to the enriching procedure done in the previous section.

#### 3.1 The De-Enriching 2-Functor

In what follows, we will make use of change of basis along lax monoidal functors for enriched categories. In particular, we will use (a variation of) the statement that change of basis along lax monoidal functors preserves monoidal categories, and, if the lax monoidal functor is braided, also braided monoidal categories. The basics of this are recalled in [Was19, Section A.1.4]. The lax monoidal functors we will use come from instances of the following general statement:

**Lemma 45.** *Let  $\mathcal{V}$  be a symmetric monoidal category, and let  $\mathcal{Z}$  be a symmetric  $\mathcal{V}$ -monoidal  $\mathcal{V}$ -enriched category. That is, the monoidal structure  $\otimes_{\mathcal{Z}}$  factors through the  $\mathcal{V}$ -enriched cartesian product and is symmetric with respect to the swap functor on  $\mathcal{Z} \boxtimes_{\mathcal{V}} \mathcal{Z}$  induced by the symmetry in  $\mathcal{V}$ . Then the functor*

$$\mathcal{Z}(\mathbb{I}_{\mathcal{Z}}, -): \mathcal{Z} \rightarrow \mathcal{V}$$

is symmetric lax monoidal, with lax structure given by:

$$\begin{aligned}\mu_0: \mathbb{I}_{\mathcal{V}} &\xrightarrow{\text{id}_{\mathbb{I}_{\mathcal{Z}}}} \mathcal{Z}(\mathbb{I}_{\mathcal{Z}}, \mathbb{I}_{\mathcal{Z}}) \\ \mu_{z,z'}: \mathcal{Z}(\mathbb{I}_{\mathcal{Z}}, z) \mathcal{Z}(\mathbb{I}_{\mathcal{Z}}, z') &\xrightarrow{\otimes_{\mathcal{Z}}} \mathcal{Z}(\mathbb{I}_{\mathcal{Z}} \mathbb{I}_{\mathcal{Z}}, zz') \xrightarrow[\cong]{(\mathbb{I}_{\mathcal{Z}} \rightarrow \mathbb{I}_{\mathcal{Z}} \mathbb{I}_{\mathcal{Z}})^*} \mathcal{Z}(\mathbb{I}_{\mathcal{Z}}, zz'),\end{aligned}$$

for  $z, z' \in \mathcal{Z}$ .

We omit the proof. We will in particular use this Lemma in the case where  $\mathcal{V} = \mathbf{Vect}$  and  $\mathcal{Z} = \mathcal{A}$  or  $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$ . Using this Lemma, we set:

**Definition 46.** Let  $\mathcal{K}$  be a  $\mathcal{Z}(\mathcal{A})_s$ -enriched category. We will write  $\mathbf{DE}(\mathcal{K})$  for the  $\mathbf{Vect}$ -enriched category obtained from  $\mathcal{K}$  by change of basis along  $\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, -)$ , and will call this the *de-enrichment* of  $\mathcal{K}$ .

To treat  $\mathcal{Z}(\mathcal{A})$ -crossed (braided) tensor categories, we will additionally need a version of Lemma 45 for 2-fold tensor categories.

**Lemma 47.** Let  $(\mathcal{Z}, \otimes_1, \otimes_2)$  be a braided lax 2-fold tensor category [Was17b, Definitions 9, 11 and 12], with, for  $i = 1, 2$ , the braiding for  $\otimes_i$  denoted  $\beta_i$  and the unit denoted by  $\mathbb{I}_i$ . Then the functor

$$\mathcal{Z}(\mathbb{I}_2, -): \mathcal{Z} \rightarrow \mathbf{Vect}$$

is braided lax monoidal with respect to  $\otimes_1$ , with lax structure given by:

$$\begin{aligned}\mu_0: \mathbb{C} &\xrightarrow{\text{id}_{\mathbb{I}_2}} \mathcal{Z}(\mathbb{I}_2, \mathbb{I}_2), \\ \mu_{z,z'}: \mathcal{Z}(\mathbb{I}_2, z) \mathcal{Z}(\mathbb{I}_2, z') &\xrightarrow{\otimes_{\mathcal{Z}}} \mathcal{Z}(\mathbb{I}_2 \otimes_1 \mathbb{I}_2, z \otimes_1 z') \xrightarrow[\cong]{(\mathbb{I}_2 \rightarrow \mathbb{I}_2 \otimes_1 \mathbb{I}_2)^*} \mathcal{Z}(\mathbb{I}_2, z \otimes_1 z'),\end{aligned}$$

for  $z, z' \in \mathcal{Z}$ . Here the morphism  $\mathbb{I}_2 \rightarrow \mathbb{I}_2 \otimes_1 \mathbb{I}_2$  is one of the structure morphisms from [Was17b, Definition 9].

We omit the proof of the lemma, it is a routine adaptation of the proof of Lemma 45.

Using this Lemma, one can prove, with a slight adjustment of the original proof, the following variation of the statement [Was19, Proposition A.27] that change of basis preserves (braided) monoidal categories. The phrasing here is specialised to the case  $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$  that we actually need.

**Proposition 48.** Let  $\mathcal{K}$  be a  $\mathcal{Z}(\mathcal{A})$ -crossed braided tensor category. Then the linear category obtained from  $\mathcal{K}$  by change of basis along  $\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, -)$  is a braided tensor category.

Being additionally tensored over  $\mathcal{Z}(\mathcal{A})_s$  will ensure the resulting category is tensored over  $\mathbf{Vect}$ , by standard arguments. We additionally note that, by definition, the de-enrichment of a  $\mathcal{Z}(\mathcal{A})_s$ -linear category [Was19, Definition 4] is a linear category, and the de-enrichment is semi-simple if the  $\mathcal{Z}(\mathcal{A})_s$ -enriched category is. We have the following immediate corollary of this proposition and the preceding lemmas.



**Corollary 49.** *The de-enrichment of any  $\mathcal{Z}(\mathcal{A})_s$ -fusion category [Was19, Definition 38] is a fusion category.*

It turns out that whenever we start with a  $\mathcal{Z}(\mathcal{A})$ -crossed braided tensor category, the compatibility between the  $\mathcal{Z}(\mathcal{A})_s$ -tensoring, the crossed tensor structure and the braiding will ensure the following:

**Proposition 50.** *Let  $\mathcal{K}$  be a  $\mathcal{Z}(\mathcal{A})$ -crossed braided tensor category. Then the image of the unit functor  $\mathbb{I} : \mathcal{A}_{\mathcal{Z}} \rightarrow \mathcal{K}$  under change of basis along  $\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, -)$  gives a braided monoidal faithful functor from  $\mathcal{A}$  into  $\mathbf{DE}(\mathcal{K})$ .*

*Proof.* Denote the crossed monoidal structure on  $\mathcal{K}$  by  $\otimes_{\mathcal{K}}$ , the  $\mathcal{Z}(\mathcal{A})_s$  tensoring by  $\cdot$ , and the crossed monoidal structure on  $\mathcal{A}_{\mathcal{Z}}$  by  $\otimes_{\mathcal{A}}$ . By definition, the unit functor is such that  $\mathbb{I}(a) \otimes_{\mathcal{K}} k = a \cdot k$ , and the tensoring satisfies  $(a \otimes_{\mathcal{A}} a') \cdot k = a \cdot (a' \cdot k)$ , so, letting  $k = \mathbb{I}(\mathbb{I}_{\mathcal{A}})$ , we have that  $\mathbb{I}(a \otimes_{\mathcal{A}} a') = \mathbb{I}(a) \otimes_{\mathcal{K}} \mathbb{I}(a')$ . The usual argument for compatibility between the unitors and the braiding [JS86, Proposition 1] implies that the unit functor is also braided. This means that the image of  $\mathbb{I}$  under de-enrichment,

$$\mathbf{DE}(\mathbb{I}) : \mathbf{DE}(\mathcal{A}_{\mathcal{Z}}) = \mathcal{A} \rightarrow \mathbf{DE}(\mathcal{K}),$$

is a braided linear functor. Any linear monoidal functor on a fusion category is faithful, monoidality forces non-zero objects to be sent to non-zero objects, so this finishes the proof.  $\square$

### 3.2 Equivalence between braided categories containing $\mathcal{A}$ and $\mathcal{Z}(\mathcal{A})$ -crossed braided categories

The goal of this section is to show that the construction  $\mathcal{C} \mapsto \underline{\underline{\mathcal{C}}}$  outlined in Section 2 above gives an equivalence of 2-categories between the 2-category  $\mathbf{BFC}/\mathcal{A}$  of braided fusion categories (Definition 2) containing  $\mathcal{A}$  and  $\mathcal{Z}(\mathcal{A}) - \mathbf{XBT}$  (see [Was19, Definition 48]), with inverse given by  $\mathbf{DE}(-)$ .

**Theorem 51.** *The bifunctors  $\mathbf{DE}(-)$  and  $\underline{\underline{(-)}}$  are mutually inverse.*

*Proof.* At the level of the objects of the categories that are the objects of the 2-categories, both  $\mathbf{DE}(-)$  and  $\underline{\underline{(-)}}$  are constructions that leave the objects of the categories invariant. So their composites also leave the objects invariant. This means that the components of the natural transformations witnessing that these bifunctors are mutually inverse will be functors that are the identity on objects. In order to prove the theorem, we in particular need to show that these component functors are equivalences, and for this it is enough to show they induce isomorphisms on the hom-objects.

Consider first the composite  $\mathbf{DE}(-) \circ \underline{\underline{(-)}}$ . Let  $\mathcal{C}$  be an object of  $\mathbf{BTC}(\mathcal{A})$ , then the category  $\mathbf{DE}(\underline{\underline{\mathcal{C}}})$  has hom-spaces:

$$\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \underline{\underline{\mathcal{C}}}(c, c')) \cong \mathcal{A}(\mathbb{I}_{\mathcal{A}}, \underline{\underline{\mathcal{C}}}(c, c')) \cong \mathcal{C}(c, c'),$$

where we have used [Was19, Lemma 14], and the definitions of the hom-objects of  $\underline{\mathcal{C}}$  and  $\underline{\mathcal{C}}$ . Taking  $\mathbf{H}_{\mathcal{C}}: \mathbf{DE}(\underline{\mathcal{C}}) \rightarrow \mathcal{C}$  to be the functor that is the identity on objects and the above isomorphism on hom-objects gives us an equivalence  $\mathbf{DE}(\underline{\mathcal{C}}) \cong \mathcal{C}$  for each  $\mathcal{C} \in \mathbf{BTC}(\mathcal{A})$ . To see that these functors combine to a natural transformation, we simply observe that the above isomorphism is the inverse to the isomorphism used to define the action of  $\underline{F}$  on the hom-objects. In particular, the naturality is satisfied on the nose, there is no need for a “naturator” natural isomorphism between the linear functors  $F\mathbf{H}_{\mathcal{C}}$  and  $\mathbf{H}_{\mathcal{D}}\mathbf{DE}(\underline{F})$ .

For the other composite, we observe that, for  $\mathcal{K} \in \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF}$ , the underlying objects in  $\mathcal{A}$  of the hom-objects of  $\underline{\mathbf{DE}}(\mathcal{K})$  are characterised by:

$$\begin{aligned} \mathcal{Z}(\mathcal{A})(z, \underline{\mathbf{DE}}(\mathcal{K})(k, k')) &\cong \mathbf{DE}(\mathcal{K})(z \cdot k, k') \\ &= \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \mathcal{K}(z \cdot k, k')) \\ &\cong \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, z^* \otimes_s \mathcal{K}(k, k')) \\ &\cong \mathcal{Z}(\mathcal{A})(z, \mathcal{K}(k, k')), \end{aligned} \tag{23}$$

where the first isomorphism is the  $\mathcal{Z}(\mathcal{A})$ -tensoring (Proposition 27) of  $\underline{\mathbf{DE}}(\mathcal{K})$ , the second the definition of  $\mathbf{DE}$ , the third is Lemma A.13, and the final equality is the adjunction between  $\otimes_s z$  and  $z^* \otimes_s$  on  $\mathcal{Z}(\mathcal{A})_s$ . These isomorphisms are all natural, so will combine to an equivalence  $\mathbf{J}_{\mathcal{K}}$  between  $\underline{\mathbf{DE}}(\mathcal{K})$  and  $\mathcal{K}$ , which is the identity on objects. Similarly to before, the functors induced by de-enrichment are defined using the isomorphisms above, so naturality of the natural transformation defined by the  $\mathbf{J}_{\mathcal{K}}$  is automatic.  $\square$

## 4 The Reduced Tensor Product

### 4.1 Definition of the Reduced Tensor Product

**Definition 52.** The *reduced tensor product*  $\overset{\mathcal{A}}{\boxtimes}_{\text{red}}$  on the category  $\mathbf{BFC}/\mathcal{A}$  (see Definition 2) is defined to be the composite:

$$\mathbf{BFC}/\mathcal{A} \times \mathbf{BFC}/\mathcal{A} \xrightarrow{\overset{(-) \times (-)}{\xrightarrow{\quad \quad}} \xrightarrow{\quad \quad}} \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \times \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \xrightarrow{\overset{\boxtimes}{\xrightarrow{s}}} \mathcal{Z}(\mathcal{A})\text{-}\mathbf{XBF} \xrightarrow{\mathbf{DE}} \mathbf{BFC}/\mathcal{A},$$

where  $\overset{\boxtimes}{\xrightarrow{s}}$  was defined in Definition 9.

To make  $\mathbf{BTC}(\mathcal{A})$  into a symmetric monoidal 2-category for this reduced tensor product, we need to specify associators. We will do this by using the associators  $\tilde{\alpha}$  for  $\overset{\boxtimes}{\xrightarrow{s}}$ , which in turn are induced from the associators  $\alpha^s$  for  $\otimes_s$ .

**Definition 53.** Let  $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathbf{BTC}(\mathcal{A})$ . Then the *associator* for  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  is the functor

$$A_{\mathcal{C}, \mathcal{D}, \mathcal{E}}: (\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D}) \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} (\mathcal{D} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{E}),$$

given by the composite

$$\begin{aligned} (\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D}) \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{E} &= \mathbf{DE}((\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D}) \boxtimes_s \mathcal{E}) \\ &\xrightarrow{\quad \quad \quad} \mathbf{DE}(\mathbf{DE}(\mathcal{C} \boxtimes_s \mathcal{D}) \boxtimes_s \mathcal{E}) \\ &\xrightarrow{\mathbf{J}_{\mathcal{C} \boxtimes_s \mathcal{D}}} \mathbf{DE}((\mathcal{C} \boxtimes_s \mathcal{D}) \boxtimes_s \mathcal{E}) \\ &\xrightarrow{\tilde{\alpha}} \mathbf{DE}(\mathcal{C} \boxtimes_s (\mathcal{D} \boxtimes_s \mathcal{E})) \\ &\xrightarrow{\mathbf{J}_{\mathcal{D} \boxtimes_s \mathcal{E}}^{-1}} \mathbf{DE}(\mathcal{C} \boxtimes_s \mathbf{DE}(\mathcal{D} \boxtimes_s \mathcal{E})) \\ &\xrightarrow{\quad \quad \quad} \mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} (\mathcal{D} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{E}). \end{aligned}$$

These associators need to satisfy the pentagon equations, possibly up to a invertible 2-cell. Fortunately, we can take this 2-cell to be the identity, as we will show below. To simplify the argument, we first prove the following lemma:

**Lemma 54.** Let  $\mathcal{C} \in \mathbf{BTC}(\mathcal{A})$ . Then

$$\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \mathcal{C}(c, c')) \cong \mathbf{DE}(\mathcal{C})(c, c'),$$

canonically.

*Proof.* Using Equation (23), it suffices to observe:

$$\mathcal{Z}(\mathcal{A})(z, \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \mathcal{C}(c, c'))) \cong \mathcal{Z}(\mathcal{A})(z, \mathcal{C}(c, c')), \quad (24)$$

which follows from the definition of  $\mathcal{Z}(\mathcal{A})$ .  $\square$

Observe that all functors involved in the definition of the associators are the identity on objects, so it will suffice to consider what happens on morphisms. The lemma allows us to give an alternative expression for the associators:

**Lemma 55.** Let  $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathbf{BTC}(\mathcal{A})$  and, for  $i = 1, 2$ , let  $c_i, d_i, e_i$  be objects of the respective categories. We adopt the shorthand  $\mathcal{C}(c_1, c_2) = \mathcal{C}_{12}$ , with the obvious extension to the other categories, their  $\mathcal{Z}(\mathcal{A})$ -enriched versions and their products. We will further suppress  $\otimes_s$  from the notation for this lemma and its

proof. Note that the action of the associators on the hom-objects between  $c_i, d_i$  and  $e_i$  is a morphism:

$$A_{\mathcal{C}, \mathcal{D}, \mathcal{E}}^{12}: \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \underbrace{\mathbf{DE}(\underbrace{\mathcal{C} \mathcal{D}}_{\leftarrow})}_{\leftarrow} \mathcal{E}_{12}) \rightarrow \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \underbrace{\mathcal{C}_{12} \mathbf{DE}(\underbrace{\mathcal{D}_{12} \mathcal{E}_{12}}_{\leftarrow})}_{\leftarrow}).$$

This morphism is equal to the composite:

$$\begin{aligned} \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \underbrace{\mathbf{DE}(\underbrace{\mathcal{C} \mathcal{D}}_{\leftarrow})}_{\leftarrow} \mathcal{E}_{12}) &\cong \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \underbrace{\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \mathcal{C}_{12} \mathcal{D}_{12})}_{\leftarrow} \mathcal{E}_{12}) \\ &\cong \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, (\mathcal{C}_{12} \mathcal{D}_{12}) \mathcal{E}_{12}) \\ &\xrightarrow{\alpha^s} \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \mathcal{C}_{12} (\mathcal{D}_{12} \mathcal{E}_{12})) \\ &\cong \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, \underbrace{\mathcal{C}_{12} \mathbf{DE}(\underbrace{\mathcal{D}_{12} \mathcal{E}_{12}}_{\leftarrow})}_{\leftarrow}), \end{aligned}$$

where the first isomorphism is Lemma 54, the second Lemma A.13 combined with the definition of  $\mathcal{Z}(\mathcal{A})$ , and the final isomorphism is the inverse of the corresponding isomorphisms.

We are now in a position to verify the pentagon equations.

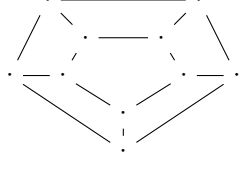
**Proposition 56.** *The associators  $A$  satisfy the pentagon equations on the nose.*

*Proof.* Let  $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathbf{BTC}(\mathcal{A})$  and, for  $i = 1, 2$ , let  $c_i, d_i, e_i$  be objects of the respective categories, and adapt the notation from Lemma 55. It is enough to check the pentagon equation on hom-objects. Using Lemma 55 we get, for each edge of the pentagon, a commutative square, which we will spell out for one edge:

$$\begin{array}{ccc} (((\mathcal{C} \underset{\text{red}}{\boxtimes}^A \mathcal{D}) \underset{\text{red}}{\boxtimes}^A \mathcal{E}) \underset{\text{red}}{\boxtimes}^A \mathcal{F})_{12} & \xrightarrow{A_{\mathcal{C} \underset{\text{red}}{\boxtimes}^A \mathcal{D}, \mathcal{E}, \mathcal{F}}} & ((\mathcal{C} \underset{\text{red}}{\boxtimes}^A \mathcal{D}) \underset{\text{red}}{\boxtimes}^A (\mathcal{E} \underset{\text{red}}{\boxtimes}^A \mathcal{F}))_{12} \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, ((\mathcal{C} \underset{\text{red}}{\boxtimes}^A \mathcal{D}_{12}) \mathcal{E}_{12}) \mathcal{F}_{12}) & \xrightarrow{(\alpha^s_{\mathcal{C} \underset{\text{red}}{\boxtimes}^A \mathcal{D}_{12}, \mathcal{E}_{12}, \mathcal{F}_{12}})} & \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, (\mathcal{C} \underset{\text{red}}{\boxtimes}^A \mathcal{D}_{12}) (\mathcal{E}_{12} \mathcal{F}_{12})) \\ \downarrow \mathbf{J}_{\mathcal{C}_{12} \mathcal{D}_{12}} & & \downarrow \mathbf{J}_{\mathcal{C}_{12} \mathcal{D}_{12}} \\ \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, (\mathcal{C}_{12} \mathcal{D}_{12}) \mathcal{E}_{12}) \mathcal{F}_{12} & \xrightarrow{(\alpha^s_{\mathcal{C}_{12} \mathcal{D}_{12}, \mathcal{E}_{12}, \mathcal{F}_{12}})} & \mathcal{Z}(\mathcal{A})(\mathbb{I}_s, (\mathcal{C}_{12} \mathcal{D}_{12}) (\mathcal{E}_{12} \mathcal{F}_{12})), \end{array}$$

where the top square is commutative by definition, while the bottom square commutes by naturality of  $\alpha^s$ . The squares associated in this way to adjacent edges in the pentagon share a vertical edge, so we get a diagram that looks

schematically like:



where the squares correspond to the ones constructed above, the outer pentagon is the pentagon we want to show commutes, and the inner pentagon is the image under  $\mathcal{Z}(\mathcal{A})(\mathbb{I}_s, -)$  of a pentagon equation for  $\alpha^s$ . All these faces commute, so we conclude that the outer pentagon commutes.  $\square$

#### 4.1.1 The Unit for the Reduced Tensor Product

**Proposition 57.** *Let  $\mathcal{C} \in \mathbf{BTC}(\mathcal{A})$ . Then we have an equivalence*

$$\mathcal{Z}(\mathcal{A}) \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{C} \cong \mathcal{C},$$

*given by the image of the unitor for  $\boxtimes_s$ , as defined in Lemma A.18. We similarly have a right unitor, and together these satisfy the triangle equality.*

*Proof.* It is clear that the functors defined above are equivalences, so we are left with checking the triangle equality. Recall that the unitor for  $\boxtimes_s$  is defined using the  $\mathcal{Z}(\mathcal{A})_s$ -tensoring. On objects, the two routes along the triangle agree up to the natural isomorphism with components coming from Proposition A.19. On hom-objects, we observe that Proposition A.19 implies

$$\underline{\mathcal{C}} \boxtimes_s \underline{\mathcal{D}}(z_1 \cdot c_1 \boxtimes d_1, z_2 \cdot c_2 \boxtimes d_2) \cong \underline{\mathcal{C}} \boxtimes_s \underline{\mathcal{D}}(c_1 \boxtimes z_1 \cdot d_1, c_2 \boxtimes z_1 \cdot d_2),$$

so we are done.  $\square$

In summary:

**Theorem 58.** *The reduced tensor product  $\boxtimes_{\text{red}}^{\mathcal{A}}$  defines a symmetric monoidal structure on the 2-category  $\mathbf{BTC}(\mathcal{A})$ .*

## 4.2 Basic Properties of the Reduced Tensor Product

We will now establish some basic properties of the reduced tensor product, and compute it in some examples. In our computations, the following result, that appears as [Was19, Proposition 19], will be used frequently:

**Proposition 59.** *Let  $\mathcal{K}$  and  $\mathcal{L}$  be  $\mathcal{Z}(\mathcal{A})_s$ -enriched and tensored categories. Then:*

$$(\mathcal{K} \boxtimes_s \mathcal{L})_{\mathcal{A}} \cong \mathcal{K} \boxtimes_s \mathcal{L}_{\mathcal{A}} \cong \mathcal{K}_{\mathcal{A}} \boxtimes_s \mathcal{L} \cong \overline{\mathcal{K}_{\mathcal{A}}} \boxtimes_{\mathcal{A}} \overline{\mathcal{L}_{\mathcal{A}}},$$

*where we view the  $\mathcal{A}$ -enriched and tensored category on the right as  $\mathcal{Z}(\mathcal{A})_s$ -enriched and tensored category by using the symmetric strong monoidal inclusion functor  $\mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$ .*

### 4.2.1 Reduced Tensor Product and the Commutant of $\mathcal{A}$

It is interesting to examine what the reduced tensor product becomes on the commutant (Definition 42) of  $\mathcal{A}$  in  $\mathcal{C}$ . When taking the reduced tensor product, this commutant behaves nicely. We will use the following bit of notation:

**Notation 60.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be braided fusion categories containing  $\mathcal{A}$ . The symbol  $\boxtimes_{\mathcal{A}}$ , with slight abuse of notation, denotes

$$\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D} = \mathbf{DeEnrich}(\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}),$$

where the use of  $\boxtimes_{\mathcal{A}}$  on the right hand side denotes the  $\mathcal{A}$ -product introduced in Definition A.17, and  $\mathbf{DeEnrich}$  denotes change of basis along  $\mathcal{A}(\mathbb{I}_{\mathcal{A}}, -)$ .

**Proposition 61.** *Let  $\mathcal{C}, \mathcal{D} \in \mathbf{BTC}(\mathcal{A})$ . Then the commutant of  $\mathcal{A}$  in  $\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D}$  satisfies:*

$$\mathcal{Z}_2(\mathcal{A}, \mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \boxtimes_{\mathcal{A}} \mathcal{Z}_2(\mathcal{A}, \mathcal{D}).$$

*Proof.* Using Proposition 43, this follows directly from Proposition 19.  $\square$

### 4.2.2 Examples

To give the reader some intuition for the reduced tensor product, we compute some examples.

**Example 62.** Let  $\mathcal{C}$  be a braided fusion category containing a symmetric fusion category  $\mathcal{A}$ . Then the reduced tensor product over  $\mathcal{A}$  of  $\mathcal{C}$  with  $\mathcal{A}$  is given by:

$$\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{A} \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \boxtimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}).$$

To see this, we observe that the neutral subcategory of  $\mathcal{A}$  enriched over itself is all of  $\mathcal{A}$ . Now apply Proposition 19 to get:

$$\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{A} \cong \mathbf{DE}(\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{A}) \cong \mathbf{DeEnrich}(\mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \boxtimes_{\mathcal{A}} \mathcal{A}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}).$$

Here, we have used Corollary 44 and that  $\mathcal{A}$  is the unit for  $\boxtimes_{\mathcal{A}}$ .

**Example 63.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be braided fusion categories containing  $\mathcal{A}$ , and assume that  $\mathcal{D} = \mathcal{Z}_2(\mathcal{A}, \mathcal{D})$ . Then:

$$\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D} \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \boxtimes_{\mathcal{A}} \mathcal{D}.$$

The assumption on  $\mathcal{D}$  means that we have  $\mathcal{D}_{\mathcal{A}} = \mathcal{D}$ , by Proposition 43. Using Proposition 59 and Corollary 44, we get:

$$\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D} \cong \mathbf{DE}(\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}) \cong \mathbf{DeEnrich}(\mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \boxtimes_{\mathcal{A}} \mathcal{D}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \boxtimes_{\mathcal{A}} \mathcal{D}.$$

**Example 64.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be braided fusion categories containing  $\mathcal{A}$ , and assume that  $\mathcal{C} = \mathcal{Z}_2(\mathcal{A}, \mathcal{C})$  and that  $\mathcal{D} = \mathcal{Z}_2(\mathcal{A}, \mathcal{D})$ . Then:

$$\mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D} \cong \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}.$$

The assumption on  $\mathcal{C}$  and  $\mathcal{D}$  means that we have  $\underline{\mathcal{C}}_{\mathcal{A}} = \underline{\mathcal{C}}$  and  $\underline{\mathcal{D}}_{\mathcal{A}} = \underline{\mathcal{D}}$ , by Proposition 43. Using Proposition 59 and Corollary 44, we get:

$$\begin{aligned} \mathcal{C} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{D} &\cong \mathbf{DE}(\underline{\mathcal{C}} \boxtimes_s \underline{\mathcal{D}}) \\ &\cong \mathbf{DeEnrich}(\underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{D}}) \\ &\cong \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}. \end{aligned}$$

#### 4.2.3 Minimal Modular Extensions

In this section we will show that the reduced tensor product between so called minimal modular extensions is again a minimal modular extension. We first recall the definition of a minimal modular extension.

**Definition 65** ([Müg03]). Let  $\mathcal{C} \in \mathbf{BTC}(\mathcal{A})$ , then a *minimal modular extension of  $\mathcal{C}$  over  $\mathcal{A}$*  is a braided tensor category  $\mathcal{M}$  containing  $\mathcal{C}$  with  $\mathcal{Z}_2(\mathcal{M}, \mathcal{M}) = \mathbf{Vect}$  and  $\mathcal{Z}_2(\mathcal{A}, \mathcal{M}) = \mathcal{C}$ . The (possibly empty) set of minimal modular extensions of  $\mathcal{C}$  over  $\mathcal{A}$  will be denoted by  $\text{MME}(\mathcal{C}, \mathcal{A})$ .

The reduced tensor product works particularly well with minimal modular extensions:

**Proposition 66.** Let  $\mathcal{M} \in \text{MME}(\mathcal{C}, \mathcal{A})$  and  $\mathcal{N} \in \text{MME}(\mathcal{D}, \mathcal{A})$  for  $\mathcal{C}, \mathcal{D} \in \mathbf{BTC}(\mathcal{A})$  with  $\mathcal{Z}_2(\mathcal{A}, \mathcal{C}) \cong \mathcal{Z}_2(\mathcal{A}, \mathcal{D}) = \mathcal{A}$ . Then  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N} \in \text{MME}(\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}, \mathcal{A})$ .

*Proof.* We observe that  $\mathcal{Z}_2(\mathcal{A}, \mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}) = \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}$  is immediate from Proposition

61. This leaves showing that  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}$  has  $\mathbf{Vect}$  as its subcategory of transparent objects. To see this, observe that by the double commutant theorem [Müg03], we have that  $\mathcal{Z}_2(\mathcal{Z}_2(\mathcal{A}, \mathcal{M}), \mathcal{M}) = \mathcal{Z}_2(\mathcal{A}, \mathcal{A}) = \mathcal{A}$  and  $\mathcal{Z}_2(\mathcal{Z}_2(\mathcal{A}, \mathcal{N}), \mathcal{N}) = \mathcal{Z}_2(\mathcal{A}, \mathcal{A}) = \mathcal{A}$ . We then have

$$\mathcal{Z}_2(\mathcal{Z}_2(\mathcal{A}, \mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}), \mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}) = \mathcal{Z}_2(\mathcal{Z}_2(\mathcal{A}, \mathcal{M}) \boxtimes_{\mathcal{A}} \mathcal{Z}_2(\mathcal{A}, \mathcal{N}), \mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}) = \mathcal{A} \boxtimes_{\mathcal{A}} \mathcal{A} = \mathcal{A},$$

as the braiding on  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}$  is componentwise. We further have that  $\mathcal{Z}_2(\mathcal{Z}_2(\mathcal{A}, \mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N})) \supset \mathcal{Z}_2(\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}, \mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N})$ , so we are left with establishing which objects  $a$  in  $\mathcal{A}$  are

transparent for all of  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}$ . This is detected by the  $\mathcal{Z}(\mathcal{A})_s$ -tensoring restricted to the non- $\mathcal{A}$  objects of the subcategory  $\mathcal{Z}_s(\langle a \rangle, \mathcal{Z}(\mathcal{A}))$ , where  $\langle a \rangle$  denotes the subcategory spanned by  $a$ : the object  $a$  is transparent if and only if these objects annihilate  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}$ . As, by modularity, no such set of objects of  $\mathcal{Z}(\mathcal{A})$  annihilates  $\mathcal{M}$  or  $\mathcal{N}$ , and therefore no non-unit objects of  $\mathcal{A}$  are transparent in  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}$ , and we conclude that  $\mathcal{M} \boxtimes_{\text{red}}^{\mathcal{A}} \mathcal{N}$  is modular.  $\square$

**Remark 67.** We observe that the reduced tensor product gives  $\text{MME}(\mathcal{A}, \mathcal{A})$  the structure of an abelian group. For the case  $\mathcal{A} = \mathbf{Rep}(G)$  this abelian group was, in [LKW17b], identified with the group  $H^3(G, U(1))$ , and the reduced tensor product corresponds to the pairing given there between sets of minimal modular extensions. The advantage of the approach given here is that the constructions are done purely in terms of the modular structure of the categories involved, using only the braidings and the fusion rules.

## A Appendix

### A.1 Enriched and Tensoried Categories

#### A.1.1 Basics of enriched and tensoried categories

Fix a spherical symmetric fusion category  $\mathcal{A}$  with unit object  $\mathbb{I}$  throughout. We assume the reader is familiar with the basic definition of a category enriched in  $\mathcal{A}$ . This section will deal with categories that are not only enriched, but also tensoried over  $\mathcal{A}$ .

**Notation A.1.** The hom-objects in an  $\mathcal{A}$ -enriched category  $\mathcal{C}$  between  $c, c' \in \mathcal{C}$  will be denoted by  $\underline{\mathcal{C}}(c, c')$ . We will write  $f: c \rightarrow_a c'$  for  $f: a \rightarrow \underline{\mathcal{C}}(c, c')$ . If  $a = \mathbb{I}$ , we will omit it from the notation. Furthermore, we will write  $\mathcal{C}(c, c')$  for  $\mathcal{A}(\mathbb{I}, \underline{\mathcal{C}}(c, c'))$ .

We remind the reader of the following definition.

**Definition A.2.** Let  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be functors of  $\mathcal{A}$ -enriched categories. An *enriched natural transformation* from  $F$  to  $G$  is for each object  $c \in \mathcal{C}$  a morphism  $\eta_c: F(c) \rightarrow_{\mathbb{I}} G(c)$ , that makes the following diagram commute for any  $f: c \rightarrow_a c' \in \mathcal{C}$ :

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ \downarrow F(f) & & \downarrow G(f) \\ F(c') & \xrightarrow{\eta_{c'}} & G(c'). \end{array}$$



**Definition A.3.** Let  $\mathcal{C}$  be a category enriched in  $\mathcal{A}$ . Then  $\mathcal{C}$  is called *tensoried over  $\mathcal{A}$*  if there exists, for every  $c, c' \in \mathcal{C}$  and  $a \in \mathcal{A}$  an object  $a \cdot c$  together with a functorial isomorphism

$$\mathcal{A}(a, \underline{\mathcal{C}}(c, c')) \cong \mathcal{C}(a \cdot c, c'). \quad (\text{A.1})$$

Definition A.3 allows us to write, denoting by  $\mathcal{O}(\mathcal{A})$  a set of representatives for the isomorphism classes of simple objects in  $\mathcal{A}$ :

$$\underline{\mathcal{C}}(c, c') \cong \bigoplus_{a \in \mathcal{O}(\mathcal{A})} \mathcal{C}(ac, c')a. \quad (\text{A.2})$$

This means we can view  $f: c \rightarrow_a c'$  as a morphism  $\tilde{f}: ac \rightarrow c'$ , and the composite of  $f: c \rightarrow_a c'$  and  $g: c' \rightarrow_{a'} c''$  is given by

$$\tilde{g} \circ (\text{id}_{a'} \cdot \tilde{f}) : a'ac \rightarrow c''.$$

**Definition A.4.** The images of morphisms under the isomorphism (A.1) are called *mates*. For  $f: c \rightarrow_a c'$  we will write  $\bar{f}: ac \rightarrow c'$  for its mate, and the mate of  $g: ac \rightarrow c'$  will be denoted by  $\underline{g}: c \rightarrow_a c'$ .

**Remark A.5.** If  $\mathcal{C}$  is enriched over  $\mathcal{A}$ , its category of enriched endofunctors  $\text{End}(\mathcal{C})$  is a tensor category, with the monoidal structure coming from composition. The assignment  $a \mapsto a \cdot -$  extends to a functor  $\mathcal{A} \rightarrow \text{End}(\mathcal{C})$ . This functor is in fact monoidal, cf. Lemma A.10.

Categories enriched and tensoried over  $\mathcal{A}$  form a 2-category, where we do have to take care functors between them are compatible with the tensor structure:

**Definition A.6.** The 2-category of  $\mathcal{A}$ -categories  $\mathcal{ALinCat}$  is the 2-category where

- objects are categories enriched in and tensoried over  $\mathcal{A}$ ,
- morphisms  $\mathcal{A}$ -enriched functors  $F: \mathcal{C} \rightarrow \mathcal{C}'$  equipped with a natural isomorphism

$$F(a \cdot c) \xrightarrow[\cong]{\mu_{a,c}} a \cdot F(c),$$

monoidal in  $a$ , such that the diagrams

$$\begin{array}{ccccc} \mathcal{C}(a \cdot c, c') & \xrightarrow{\quad \cong \quad} & \mathcal{A}(a, \underline{\mathcal{C}}(c, c')) & & \\ \downarrow F & & \downarrow F & & \\ \mathcal{C}'(F(ac), F(c')) & \xleftarrow{\mu} \mathcal{C}'(a \cdot F(c), F(c')) & \xleftarrow[\cong]{} \mathcal{A}(a, \underline{\mathcal{C}}'(c, c')) & & \end{array}$$

commute for all  $a \in \mathcal{A}$  and  $c, c' \in \mathcal{C}$ ,

- and 2-morphisms enriched natural transformations  $\eta: F \Rightarrow G$  that make the diagrams

$$\begin{array}{ccc}
F(ac) & \xrightarrow{\eta_{ac}} & G(ac) \\
\downarrow \cong & & \downarrow \cong \\
a \cdot F(c) & \xrightarrow{\text{id}_a \cdot \eta_c} & a \cdot G(c),
\end{array}$$

commute for all  $a \in \mathcal{A}$  and  $c \in \mathcal{C}$ .

**Remark A.7.** Definition A.6 is the most restrictive of the possible choices for a definition of  $\mathcal{ALinCat}$ . We could also have allowed that there is an auto-equivalence of  $\mathcal{A}$  associated to every morphism between  $\mathcal{A}$ -enriched and tensored categories, and that each 2-morphism comes with a symmetric monoidal transformation between these auto-equivalences. However, Definition A.6 corresponds to the kind of enriched and tensored categories one obtains from module categories over  $\mathcal{A}$ .

**Definition A.8.** The *internal hom* between two objects  $a, a' \in \mathcal{A}$  is the representing object  $\underline{\mathcal{A}}(a, a')$  for the functor  $a'' \mapsto \mathcal{A}(a''a, a')$ . These hom objects make  $\mathcal{A}$  into a category enriched and tensored over itself, i.e. a closed monoidal category.

**Lemma A.9.** *There is a canonical isomorphism*

$$\underline{\mathcal{A}}(\mathbb{I}, a) \cong a. \quad (\text{A.3})$$

for all objects  $a \in \mathcal{A}$ .

*Proof.* Consider that

$$\mathcal{A}(a', \underline{\mathcal{A}}(\mathbb{I}, a)) \cong \mathcal{A}(a', a),$$

so  $\underline{\mathcal{A}}(\mathbb{I}, a)$  and  $a$  are canonically isomorphic under the Yoneda embedding.  $\square$

The tensor structure of  $\mathcal{C}$  over  $\mathcal{A}$  induces an enriched natural isomorphism  $\eta$  with components

$$\eta_{a,c,c'} : \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c')) \rightarrow \underline{\mathcal{C}}(ac, c'). \quad (\text{A.4})$$

To see this, observe that, given the natural transformation from (A.1), the definition of the enriched hom for  $\mathcal{A}$  gives

$$\mathcal{A}(a', \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c'))) \cong \mathcal{A}(a'a, \underline{\mathcal{C}}(c, c')) \quad (\text{A.5})$$

$$\cong \mathcal{A}(a', \underline{\mathcal{C}}(ac, c')) \quad (\text{A.6})$$

where the second line uses the isomorphism from (A.1). The preimage of this isomorphism under the Yoneda embedding is the desired isomorphism.

**Lemma A.10.** *Suppose  $\mathcal{C}$  is tensored over  $\mathcal{A}$ . Then the functor  $\mathcal{A} \rightarrow \text{End}(\mathcal{C})$  taking  $a$  to  $a \cdot -$  is a tensor functor.*

*Proof.* We only prove that there exist isomorphisms  $a \cdot (a' \cdot c) \cong aa' \cdot c$  and omit checking the triangle and pentagon equations. We observe that

$$\begin{aligned} \mathcal{C}(a \cdot (a' \cdot c), c') &\cong \mathcal{A}(a, \underline{\mathcal{C}}(a'c, c')) \\ &\cong \mathcal{A}(a, \underline{\mathcal{A}}(a', \underline{\mathcal{C}}(c, c'))) \\ &\cong \mathcal{A}(aa', \underline{\mathcal{C}}(c, c')) \\ &\cong \mathcal{C}(aa'c, c'), \end{aligned} \tag{A.7}$$

for all  $c' \in \mathcal{C}$ .  $\square$

**Definition A.11.** We let

$$\text{ev}: \underline{\mathcal{C}}(c, c') \cdot c \rightarrow c'. \tag{A.8}$$

be the unit of the adjunction between  $\underline{\mathcal{C}}(c, -): \mathcal{C} \rightarrow \mathcal{A}$  and  $- \cdot c: \mathcal{A} \rightarrow \mathcal{C}$  from (A.1).

**Remark A.12.** The evaluation morphism allows us to rewrite the defining diagram for an enriched natural transformation (Definition A.2) as follows:

$$\begin{array}{ccc} \underline{\mathcal{C}}(c, c') \cdot F(c) & \xrightarrow{\eta_c} & \underline{\mathcal{C}}(c, c') \cdot G(c) \\ \downarrow F \cdot \text{id} & & \downarrow G \cdot \text{id} \\ \underline{\mathcal{C}}(Fc, Fc') \cdot F(c) & & \underline{\mathcal{C}}(c, c') \cdot G(c) \\ \downarrow \text{ev} & & \downarrow \text{ev} \\ F(c') & \xrightarrow{\eta_c} & G(c'). \end{array}$$

**Lemma A.13.** *There exists a canonical isomorphism*

$$a\underline{\mathcal{C}}(c, c') \rightarrow \underline{\mathcal{C}}(c, ac').$$

*Proof.* We construct an isomorphism between the images of  $a\underline{\mathcal{C}}(c, c')$  and  $\underline{\mathcal{C}}(c, ac')$  under the Yoneda embedding:

$$\begin{aligned} \mathcal{A}(a', a\underline{\mathcal{C}}(c, c')) &\cong \mathcal{A}(a^*a', \underline{\mathcal{C}}(c, c')) \\ &\cong \mathcal{C}(a^*a'c, c') \\ &\cong \mathcal{C}(a'c, ac') \\ &\cong \mathcal{A}(a', \underline{\mathcal{C}}(c, ac')). \end{aligned} \tag{A.9}$$

$\square$

### A.1.2 Tensor product of enriched tensored categories

**Definition A.14.** The *enriched cartesian product*  $\underline{\mathcal{C}} \hat{\boxtimes}_{\mathcal{A}} \mathcal{D}$  of two  $\mathcal{A}$ -enriched categories  $\mathcal{C}$  and  $\mathcal{D}$  is the  $\mathcal{A}$ -enriched category whose objects are symbols  $c \boxtimes_{\mathcal{A}} d$  with  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$ , and whose hom-objects are given by:

$$\underline{\mathcal{C}} \hat{\boxtimes}_{\mathcal{A}} \mathcal{D}(c \boxtimes d, c' \boxtimes d') := \underline{\mathcal{C}}(c, c') \otimes \underline{\mathcal{D}}(d, d'), \tag{A.10}$$

where  $\otimes$  is the tensor product in  $\mathcal{A}$ . Composition is given by first applying the braiding in  $\mathcal{A}$  and then the compositions in  $\mathcal{C}$  and  $\mathcal{D}$ .

Definition A.14 has an undesirable feature: if  $\mathcal{C}$  and  $\mathcal{D}$  in the above definition are idempotent complete,  $\mathcal{C} \hat{\boxtimes}_{\mathcal{A}} \mathcal{D}$  in general will not be. Another, more prosaic, problem is that this notion of tensor product is not compatible with direct sums, we will fix this momentarily.

**Definition A.15.** The *Cauchy completion* of a  $\mathcal{A}$ -enriched category  $\mathcal{C}$  is the category with objects  $n$ -tuples of objects from  $\mathcal{C}$  together with a matrix of morphisms from  $\mathcal{C}$  that is idempotent as morphism from the  $n$ -tuple to itself. Morphisms are matrices of morphisms from  $\mathcal{C}$  that commute with the idempotents.

**Remark A.16.** Considering  $n$ -tuples of objects ensures compatibility with direct sums, picking idempotents ensures the category is idempotent complete. Note that  $\mathcal{C}$  includes into its Cauchy completion. Any functor of  $\mathcal{A}$ -enriched categories induces a functor between the Cauchy completions, and the Cauchy completion of any category tensored over  $\mathcal{A}$  is also tensored over  $\mathcal{A}$ .

**Definition A.17.** The  $\mathcal{A}$ -product  $\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}$  of two  $\mathcal{A}$ -enriched categories  $\mathcal{C}$  and  $\mathcal{D}$  is the Cauchy completion of  $\mathcal{C} \hat{\boxtimes}_{\mathcal{A}} \mathcal{D}$ .

**Lemma A.18.** *The unit for the enriched cartesian product of enriched and tensored categories is  $\mathcal{A}$  enriched over itself, denoted by  $\underline{\mathcal{A}}$ , with unitor given by*

$$\begin{aligned} \underline{\mathcal{A}} \boxtimes_{\mathcal{A}} \mathcal{C} &\rightarrow \mathcal{C} \\ a \boxtimes c &\mapsto ac, \end{aligned}$$

where  $ac$  denotes the action of  $\mathcal{A}$  on  $\mathcal{C}$ .

*Proof.* As  $\mathbb{I}_{\mathcal{A}}$  acts as the identity on  $\mathcal{C}$ , setting  $a = \mathbb{I}_{\mathcal{A}}$  shows this functor is essentially surjective. For full faithfulness, we compute:

$$\underline{\mathcal{A}}(a, a') \otimes_{\mathcal{A}} \mathcal{C}(c, c') \cong \mathcal{C}(c, a^* a' c') \cong \mathcal{C}(ac, a' c'),$$

which is what we wanted to show.  $\square$

**Proposition A.19.** *If  $\mathcal{C}, \mathcal{D} \in \mathcal{ALinCat}$ , then  $\mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D}$  is tensored over  $\mathcal{A}$  with tensoring,*

$$a \cdot (c \boxtimes d) \cong (a \cdot c) \boxtimes d \cong c \boxtimes (a \cdot d), \quad (\text{A.11})$$

and we have isomorphisms:

$$(a \cdot c) \boxtimes d \cong c \boxtimes (a \cdot d).$$

*Proof.* For the first part, recall, from Definition A.3, that it is enough to show that  $c \boxtimes (a \cdot d)$  satisfies:

$$\underline{\mathcal{A}}(a, \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{D}}(c \boxtimes d, c' \boxtimes d')) \cong \underline{\mathcal{C}} \boxtimes_{\mathcal{A}} \underline{\mathcal{D}}(c \boxtimes (ad), c' \boxtimes d'). \quad (\text{A.12})$$

As  $a \cdot (c \boxtimes d)$  is characterised by this equation, this will both establish existence of the tensor structure and  $a \cdot (c \boxtimes d) \cong c \boxtimes (a \cdot d)$ .

Substituting in the definition of the hom-objects in the  $\mathcal{A}$ -product, we see we are trying to find

$$\underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c') \otimes \underline{\mathcal{D}}(d, d')) \cong \underline{\mathcal{C}}(c, c') \otimes \underline{\mathcal{D}}(ad, d').$$

Applying Lemma A.13 to  $\mathcal{A}$  viewed as a category tensored over itself, we see that the left hand side reads:

$$\underline{\mathcal{C}}(c, c') \cdot \underline{\mathcal{A}}(a, \underline{\mathcal{D}}(d, d')) \cong \underline{\mathcal{C}}(c, c') \otimes \underline{\mathcal{D}}(ad, d'), \quad (\text{A.13})$$

where the last isomorphism is (A.4). This gives us the desired isomorphism (A.12).

To establish the remaining assertion, observe that besides (A.13) we also have, after applying the symmetry in  $\mathcal{A}$ :

$$\begin{aligned} \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c') \otimes \underline{\mathcal{D}}(d, d')) &\cong \underline{\mathcal{D}}(d, d') \cdot \underline{\mathcal{A}}(a, \underline{\mathcal{C}}(c, c')) \\ &\cong \underline{\mathcal{C}}(ac, c') \otimes \underline{\mathcal{D}}(d, d') \\ &\cong \underline{\mathcal{C}} \hat{\boxtimes}_{\mathcal{A}} \underline{\mathcal{D}}((ac) \boxtimes_{\mathcal{A}} d, c' \boxtimes_{\mathcal{A}} d'), \end{aligned}$$

where we used the symmetry in  $\mathcal{A}$  again in the penultimate line.  $\square$

The  $\mathcal{A}$ -product is symmetric in the sense that we have a switch functor:

**Definition A.20.** Let  $\mathcal{C}, \mathcal{D} \in \mathcal{ALinCat}$ , then the *switch functor*  $S: \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{D} \rightarrow \mathcal{D} \boxtimes_{\mathcal{A}} \mathcal{C}$  is defined by

$$c \boxtimes_{\mathcal{A}} d \mapsto d \boxtimes_{\mathcal{A}} c$$

at the level of objects and

$$\underline{\mathcal{C}}(c, c') \otimes \underline{\mathcal{D}}(d, d') \xrightarrow{s} \underline{\mathcal{D}}(d, d') \otimes \underline{\mathcal{C}}(c, c'),$$

where  $s$  is the symmetry in  $\mathcal{A}$ .

As the monoidal structure and the symmetry in  $\mathcal{A}$  satisfy the appropriate coherence equations, the  $\mathcal{A}$ -product and the switch functor will strictly satisfy the coherence equations for a symmetric monoidal structure on the 2-category of categories enriched in and tensored over  $\mathcal{A}$ . That is,  $(\mathcal{ALinCat}, \boxtimes_{\mathcal{A}}, S)$  is a (strict) symmetric monoidal 2-category.

Given this  $\mathcal{A}$ -product, we can define:

**Definition A.21.** Let  $\mathcal{C}$  be an  $\mathcal{A}$ -enriched category. Then a  $\mathcal{A}$ -*tensor structure* is a pair of functors:

$$\otimes: \mathcal{C} \boxtimes_{\mathcal{A}} \mathcal{C} \rightarrow \mathcal{C}, \quad \mathbb{I}: \underline{\mathcal{A}} \rightarrow \mathcal{C},$$

equipped with associators and unitors satisfying the usual coherence conditions.

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