PORTFOLIO MANAGEMENT COURSEWORK

MSC MATHEMATICS AND FINANCE

Asset Pricing with Liquidity Risk

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Introduction

The objective of this coursework is to replicate the empirical results of the seminal paper on the Liquidity-Adjusted Capital Asset Pricing Model (CAPM) by Acharya and Pedersen. This study integrates liquidity risk into the asset pricing model, thus extending the traditional CAPM framework.

To achieve this, we will source historical stock prices, trading volumes, and book-to-market ratios from the Center for Research in Security Prices (CRSP) and COMPUSTAT databases. We will focus on stocks listed on the NYSE and AMEX from July 1st, 1962, until December 31st, 1999, excluding Nasdaq-listed stocks to maintain consistency with the original study.

The methodological approach will involve the use of Python, we will process the raw data, compute liquidity measures, and estimate the parameters of the Liquidity-Adjusted CAPM. This will include calculating the illiquidity ratio, constructing the market portfolio, and performing regression analysis to determine the liquidity premium on asset returns.

Our replication effort will be conducted in several stages. We will first access and retrieve the necessary financial data from CRSP and COMPUSTAT. Then we will clean and preprocess the data to align with the requirements of the original study. After that we will code the Liquidity-Adjusted CAPM in Python, ensuring the model captures the essence of the original methodology. At this time we will start the part of our project which consist in analyzing the results and compare them with the findings from the original paper to validate the replication. Finally we will discuss any deviations from the original results and provide possible explanations for such differences.

By following this structured approach, we aim to not only replicate the original findings but also gain insights into the practical challenges and considerations involved in conducting empirical financial research.

1 The liquidity-adjusted Capital Asset Pricing Model

1.1 Presentation of the model and the assumptions

1.1.1 Presentation of the Liquidity-Adjusted CAPM

Acharya and Pedersen address the issue of liquidity in asset pricing. They note that some securities are less liquid and that this illiquidity risk, like market risk, cannot be easily diversified away. Given the greater difficulty in trading illiquid assets, especially during crises, investors require a premium for holding such assets.

The liquidity-adjusted CAPM they propose incorporates transaction costs into asset returns, modeling the relative transaction cost, c_{nt}, as the ratio of the transaction cost C_{nt} to the lagged price P_{t-1}. The model assumes a safe asset with a risk-free rate R_f, and risky assets whose returns, R_{nt}, are affected by transaction costs, C_{nt}, reducing the net return. The adjusted return is given by:

$$R_{nt} = \frac{P_{nt} - P_{n,t-1}}{P_{n,t-1}} - c_{nt} \tag{1}$$

In this model, a new representative investor is born at each time t with some capital that they use to purchase stocks, which they sell in the next period to fund retirement.

1.1.2 Market Liquidity Risk and Asset Returns

To understand the impact of market liquidity on asset returns, let s^n denote the fixed number of outstanding shares of risky asset n. The market portfolio return R_{Mt} is reduced by transaction costs:

$$R_{Mt}^{net} = R_{Mt} - c_{Mt}^{M} = \frac{\sum_n s^n P_{nt} - \sum_n s^n P_{n,t-1}}{\sum_n s^n P_{n,t-1}} - \frac{\sum_n s^n C_{nt}}{\sum_n s^n P_{n,t-1}} \tag{2}$$

The relative market transaction cost c_{Mt}^{M} is defined as:

$$c_{Mt}^{M} = \frac{\sum_n s^n C_{nt}}{\sum_n s^n P_{n,t-1}} \tag{3}$$

The expected return on a risky asset n is then composed of three parts: the risk-free rate R_f, the expected trading cost $E_{t-1}[c_{nt}]$, and the risk premium of the market portfolio adjusted for liquidity costs. This risk premium is multiplied by a liquidity-adjusted beta:

$$E_{t-1}[R_{nt}] = R_f + E_{t-1}[c_{nt}] + \beta_{nt} \left(E_{t-1}[R_{Mt}^{net}] - R_f \right) \tag{4}$$

where the liquidity-adjusted beta β_{nt} accounts for the covariance between the asset's return and the market's liquidity-adjusted return. It can be decomposed into four components that reflect various aspects of market and liquidity risk, as seen in the model.

1.1.3 Assumptions of the Liquidity-Adjusted CAPM

The liquidity-adjusted CAPM assumes a simplified overlapping generations economy. A new generation of agents is born at any discrete time t in $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Each generation t consists of N agents indexed by n, who live for two periods (t and $t+1$) and derive utility from consumption at time $t+1$. The utility function for agent n of generation t, with constant absolute risk aversion A_n, is represented by the expected utility function:

$$-\mathbb{E}_t \left[\exp(-A_n x_{t+1})\right] \tag{5}$$

where x_{t+1} denotes consumption at time $t+1$.

There are I securities indexed by $i = 1, \ldots, I$, each with S_i shares. Each security i at time t pays a dividend D_{it}, has an ex-dividend share price P_{it}, and incurs an illiquidity cost C_{it}. The dividends D_{it} and illiquidity costs C_{it} are random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are measurable with respect to the information filtration $\{\mathcal{F}_t\}$.

The illiquidity cost C_{it} represents the per-share cost of selling security i, implying that agents can buy at P_{it} but must sell at $P_{it} - C_{it}$. Short-selling is prohibited. The uncertainty of the illiquidity cost generates liquidity risk in the model. Both D_{it} and C_{it} follow autoregressive processes of order one:

$$D_t = \bar{D} + \rho_D(D_{t-1} - \bar{D}) + \epsilon_t, \tag{6}$$

$$C_t = \bar{C} + \rho_C(C_{t-1} - \bar{C}) + \zeta_t, \tag{7}$$

where $\bar{D}, \bar{C} \in \mathbb{R}^I_+$ are long-term means, $\rho_D, \rho_C \in [0, 1]$ are the autocorrelation coefficients, and (ϵ_t, ζ_t) is an i.i.d. normal process with $\mathbb{E}[\epsilon_t] = \mathbb{E}[\zeta_t] = 0$ and respective variance-covariance matrices Σ_D, Σ_C.

Agents can borrow and lend at a risk-free real return r_f, which is exogenous, and can be interpreted as an inelastic bond market or a production technology available to all.

These model assumptions are made for tractability and to derive closed-form solutions for prices and expected returns.

1.2 Properties of the Liquidity-Adjusted CAPM

The Liquidity-Adjusted Capital Asset Pricing Model developed by Acharya and Pedersen outlines several key properties relating to the risk and return dynamics of assets in the presence of liquidity considerations. These properties are encapsulated in a set of propositions that reflect the equilibrium conditions and the expected returns on assets, as delineated below.

1.2.1 Proposition 1

In the unique linear equilibrium, the conditional expected net return of security i is given by the following relation:

$$E_t(r_{i,t+1} - c_{i,t+1}) = r_f + \lambda \frac{cov_t(r_{i,t+1} - c_{i,t+1}, r_{M,t+1} - c_{M,t+1})}{var_t(r_{M,t+1} - c_{M,t+1})} \tag{8}$$

where λ, the risk premium, is defined as $\lambda = E_t(r_{M,t+1} - c_{M,t+1} - r_f)$. This equation reflects the additional return required by investors to compensate for the risk of liquidity.

Alternatively, the conditional expected gross return can be represented as:

$$E_t(r_{i,t+1}) = r_f + E_t(c_{i,t+1}) + \lambda \left[\beta_i^{'} + \beta_i^{''} - \beta_i^{'''} - \beta_i^{iv}\right] \tag{9}$$

Here, the β terms represent different sensitivities related to market and liquidity risk:

$$\beta_i^{'} = \frac{cov_t(R_{M,t+1}, R_{n,t+1})}{var_t(r_{M,t+1} - c_{M,t+1})} \tag{10}$$

$$\beta_i^{''} = \frac{cov_t(c_{M,t+1}, c_{n,t+1})}{var_t(r_{M,t+1} - c_{M,t+1})} \tag{11}$$

$$\beta_i^{'''} = \frac{cov_t(R_{M,t+1}, c_{n,t+1})}{var_t(r_{M,t+1} - c_{M,t+1})} \tag{12}$$

$$\beta_i^{iv} = \frac{cov_t(c_{M,t+1}, R_{n,t+1})}{var_t(r_{M,t+1} - c_{M,t+1})} \tag{13}$$

1.2.2 Proposition 2

It is posited that the conditional expected return increases with liquidity. If $\rho^c > 0$, and considering a portfolio q with $E_t(p^q_{i,t+1} + D^q_{t+1}) > \rho^c p^q_{i,t+1}$, then the following inequality holds:

$$\frac{\partial E_t(r^q_{i,t+1} - r_f)}{\partial c^q_{i,t+1}} > 0 \tag{14}$$

This highlights that higher liquidity levels, signified by lower $c^q_{i,t+1}$, are associated with an increase in the expected return of the portfolio.

1.2.3 Proposition 3

The third proposition assumes a portfolio such that the covariance of the illiquidity cost $c^q_{i,t+1}$ with the market return $r^q_{M,t+1}$ is negative. In this case, returns are lower when illiquidity increases:

$$cov(c^q_{i,t+1}, r^q_{M,t+1}) < 0 \tag{15}$$

This indicates that assets that become more liquid when the market is down are less risky in terms of liquidity and therefore have lower expected returns.

1.2.4 Unconditional Expected Return Relation

Lastly, the unconditional expected return relation is formalized as follows:

$$E(r_{i,t} - r_f) = E(c_{i,t}) + \lambda \left[\beta^{'}_i + \beta^{''}_i - \beta^{'''}_i - \beta^{iv}_i\right] \tag{16}$$

Where λ is the unconditional risk premium and the β terms are as previously defined. This relation will be studied a lot on the part 2 of the project.

2 The Empirical Results

2.1 Data Collection Summary from CRSP

The data collection process for the study was conducted using the CRSP database. The objective was to replicate the dataset used in a liquidity-adjusted CAPM study, focusing on common shares listed on the NYSE and AMEX from July 1, 1962, to December 31, 1999.

Data Selection Criteria

The following criteria were applied to the data selection:

- Time Frame: Data was collected for the period between July 1, 1962, and December 31, 1999.
- Stock Exchanges: Only common shares listed on NYSE (code 1) and AMEX (code 2) were included.
- Price Range: Stocks with a beginning-of-month price between $5 and $1000 were considered.
- Volume: Stocks with at least 15 trading days of volume data within each month were selected.

Query Variables

The data retrieved from CRSP included various variables necessary for the analysis. The table below summarizes the variables selected during the query process:

Variable	**Description**
Ticker	The ticker symbol for the stock
CRSP Permanent Company Number	A unique identifier for each company
Exchange Code	The code indicating the stock exchange (NYSE or AMEX)
Share Code	The code identifying the type of share
Share Class	The class of share (if applicable)
Price or Bid/Ask Average	The daily closing price or bid/ask average
Volume	The daily trading volume
Returns	The daily returns on the stock
Number of Shares Outstanding	The number of shares outstanding
Factor to Adjust Price	The factor used to adjust stock prices for corporate actions

Dividend Cash Amount	The cash amount of dividends paid

During the data retrieval process, a conditional statement builder was used to apply the selection criteria to the dataset. This ensured that only data meeting the specified conditions were included in the final dataset.
The output format for the query was set to comma-delimited text (`.csv`), suitable for analysis in common data analysis software. The date format was selected as YYYY-MM-DD for consistency and ease of parsing in subsequent analytical processes. The data collection process from CRSP was carefully tailored to match the criteria of the study being replicated.You can see below the different columns of our database :

Column Name	**Description**
PERMNO	A unique identifier for each stock.
SHRCD	Share code, which represent the type of shares.
EXCHCD	Exchange code, indicating the stock exchange where the shares are traded.
TICKER	The stock ticker symbol.
SHRCLS	Share class, indicating different classes of stock for the company.
PERMCO	A company-level identifier.
DIVAMT	Dividend amount, if any, for the stock.
FACPR	Factor price, possibly related to stock splits or adjustments.
PRC	Price of the stock.
VOL	Trading volume, the number of shares traded.
RET	Return, potentially indicating the stock's return over a specified period.
SHROUT	Shares outstanding, the total number of shares currently held by all shareholders.

We can find an extract of the dataset used in this report, with a graph of two variables the return and the volume.

date	PERMNO	PRC	VOL	RET	SHROUT	month	year
1962-07-02	10006	62	1700	0.050847	1453	1962-07	1962
1962-07-03	10006	61.125	2800	-0.014113	1453	1962-07	1962
1962-07-05	10006	61	2200	-0.002045	1453	1962-07	1962
1962-07-06	10006	60.5	2300	-0.008197	1453	1962-07	1962
1962-07-09	10006	61	900	0.008264	1453	1962-07	1962
1988-08-03	71431	11.625	400	-0.03125	0	1988-08	1988
1988-08-04	71431	11.5	2500	-0.010753	0	1988-08	1988
1988-08-05	71431	11.125	2500	-0.032609	0	1988-08	1988
1988-08-08	71431	11.25	6800	0.011236	0	1988-08	1988
1988-08-09	71431	10.75	10100	-0.044444	0	1988-08	1988

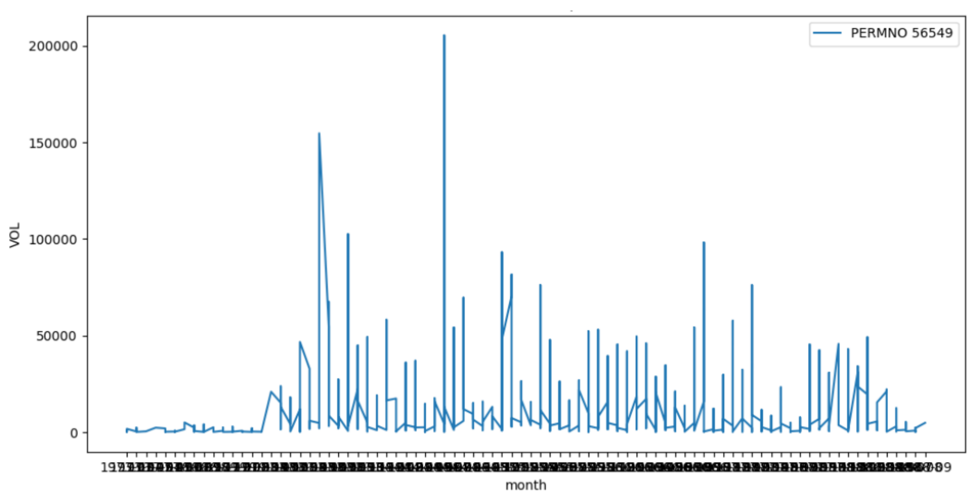

Figure 1: Graph of the volatility for a specifique PERMNO

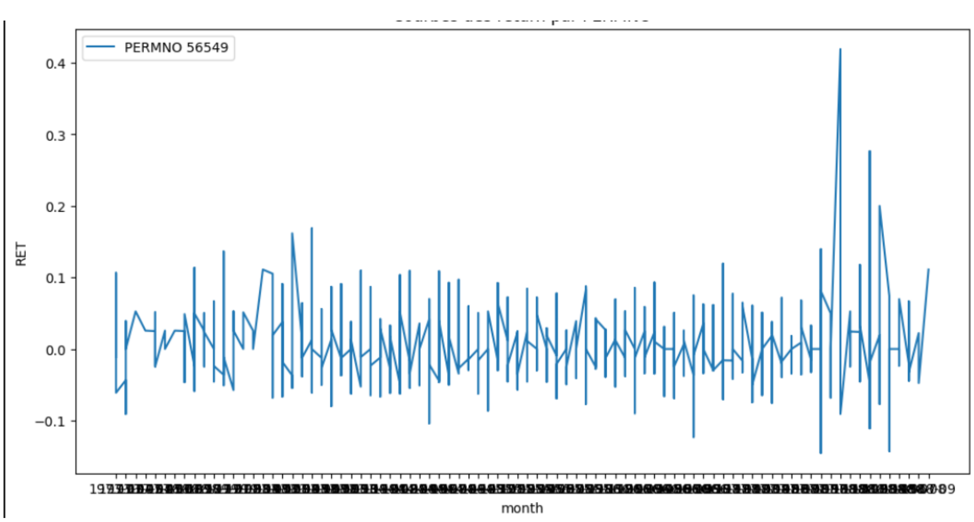

Figure 2: Graph of the return for a specifique PERMNO

2.2 The illiquidity measure

Following the data acquisition from CRSP, our analytical process commenced with the computation of the illiquidity metric for each stock, as stipulated by Amihud (2002). This metric is indispensable for appraising the price impact triggered by trading volumes, thereby encapsulating a crucial aspect of liquidity risk.

The illiquidity $ILLIQ_i$ of a stock i within month t is quantified by the average daily ratio of the absolute return to the dollar volume, as follows:

$$ILLIQ_i = \frac{1}{Days_t^i} \sum_{d=1}^{Days_t^i} \frac{\left|R_i^d\right|}{V_i^d} \tag{17}$$

Here, R_i^d and V_i^d denote the return and dollar volume (in millions) of stock i on day d in month t, while $Days_t^i$ is the count of valid trading days for stock i in month t. A stock is deemed illiquid—that is, possessing a high value of $ILLIQ_i$—if the stock's price exhibits considerable fluctuations in response to a comparatively minimal trade volume.

In addition to $ILLIQ_i$, we define a normalized measure of illiquidity c_i' to better compare across stocks and control for market-wide liquidity levels:

$$c_i' = \min\left(0.25 + 0.30\frac{ILLIQ_{i,t-1}}{P_{t-1}^M}, 30.00\right), \tag{18}$$

where P_{t-1}^M represents the ratio of the capitalizations of the market portfolio at the end of month $t-1$ relative to the capitalization at the end of July 1962. This normalization constrains c_i' to a maximum value of 30 to avert any distortions stemming from aberrant observations of $ILLIQ_i$.

Our subsequent analysis entails a rigorous comparison between these calculated illiquidity measures and the corresponding metrics reported in the original study by Acharya and Pedersen. This juxtaposition is pivotal to validate the fidelity of our replication and to critically assess the impact of liquidity risk on asset pricing. Through this scrutiny, we intend to discern the extent to which our findings concur with the antecedent study and to elucidate any divergences therein.

We can find in this report an extract of the database, to provide an exemple for the illiquidity measure.

date	I_i_d	market_cap	normalized_market_cap	c_i_prime	PERMNO
1962-08-31	0.108096	97532.6	0.994444	0.282429	10006
1962-09-30	0.100998	92265.5	0.940741	0.280469	10006
1962-10-31	0.084002	98804	1.00741	0.276788	10006
1962-11-30	0.079943	105887	1.07963	0.273807	10006
1988-04-30	0.482929	0	1	0.394879	71431
1988-05-31	0.838999	0	1	0.5017	71431
1988-06-30	0.22234	0	1	0.316702	71431
1988-07-31	0.623548	0	1	0.437064	71431
1988-08-31	1.4164	0	1	0.674921	71431

2.3 Market and Illiquidity Portfolios

2.3.1 Illiquidity-Weighted Portfolios

We sorted stocks annualy based on their liquidity metric from the preceding year and allocated into 25 distinct portfolios based on these rankings. The liquidity metric itself was derived as the average daily illiquidity throughout the year.

Each portfolio's return for month t was calculated as a value-weighted sum of the returns of its constituent stocks, defined mathematically as:

$$r_{p,t} = \sum_{i \in p} w_{i,p,t} r_{i,t}, \tag{19}$$

where $w_{i,p,t}$ represents the proportionate market value of stock i within portfolio p at time t, and $r_{i,t}$ denotes the return of stock i at time t. The weighting reflects the market capitalization of each stock, thereby weighting returns by the size of each company.

The illiquidity measure for each portfolio was similarly computed on a value-weighted basis, ensuring that larger firms with greater market capitalization had a commensurately larger impact on the portfolio's overall liquidity metric. This approach allowed for a more nuanced interpretation of the portfolios' liquidity characteristics.

In our empirical investigation, these illiquidity-weighted portfolios were analyzed to discern the influence of liquidity on asset returns.

We can find here an example of the value-weighted sum of the returns for the different portfolio.

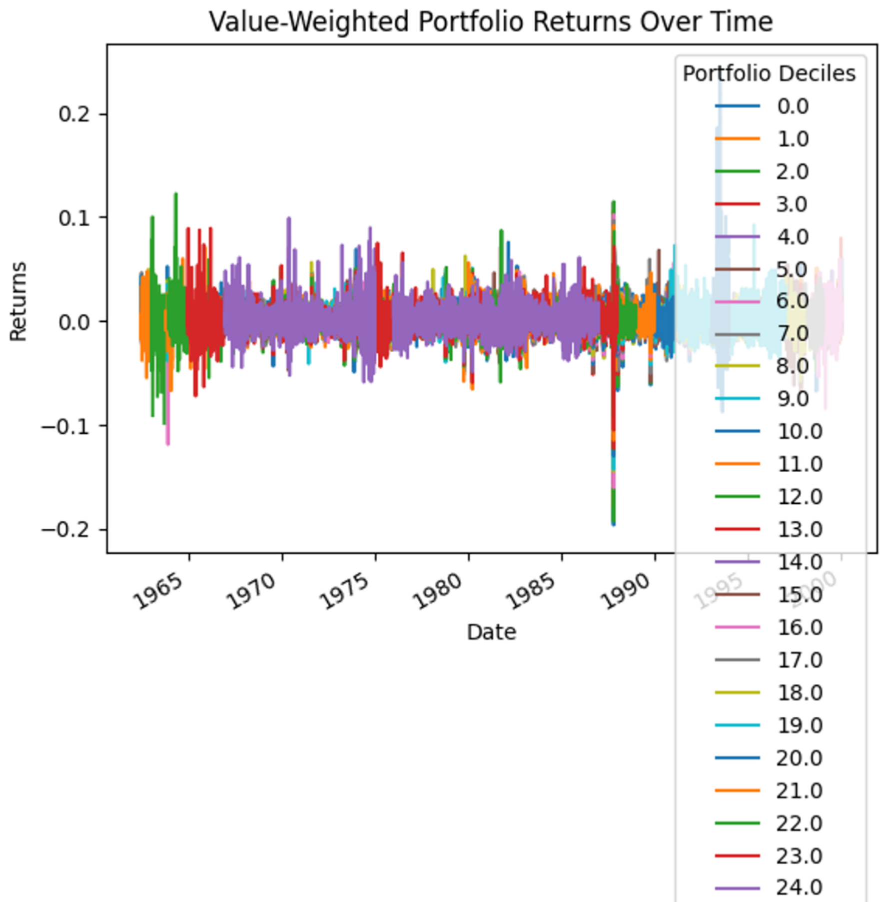

Figure 3: Graph of the value-weighted portfolio returns over time

2.3.2 Market and Illiquidity Portfolios

Employing data from CRSP, we analyzed daily return and volume data from July 1st, 1962, until December 31st, 1999, for all common shares listed on NYSE and AMEX. We excluded Nasdaq-listed shares to ensure consistency across our liquidity measures, as Nasdaq volume data includes interdealer trades and only starts in 1982. Additionally, we utilized book-to-market data based on the COMPUSTAT measure of book value.

We constructed a market portfolio for each month t during this period, focusing on stocks with a beginning-of-month price between \$5 and \$1000 and with at least 15 days of return and volume data in that month. For the creation of illiquidity portfolios, we sorted stocks annually (for each year y from 1964 to 1999) with a beginning-of-year price between \$5 and \$1000 and at least 100 days of return and volume data in the preceding year ($y-1$). These stocks were then divided into 25 portfolios based on their annual illiquidity, calculated as the average daily illiquidity over the entire year $y-1$.

Furthermore, we formed 25 illiquidity-variation portfolios by ranking eligible stocks each year based on the standard deviation of their daily illiquidity measures from the previous year. Additionally, 25 size portfolios were created by ranking stocks based on their market capitalization at the beginning of the year.

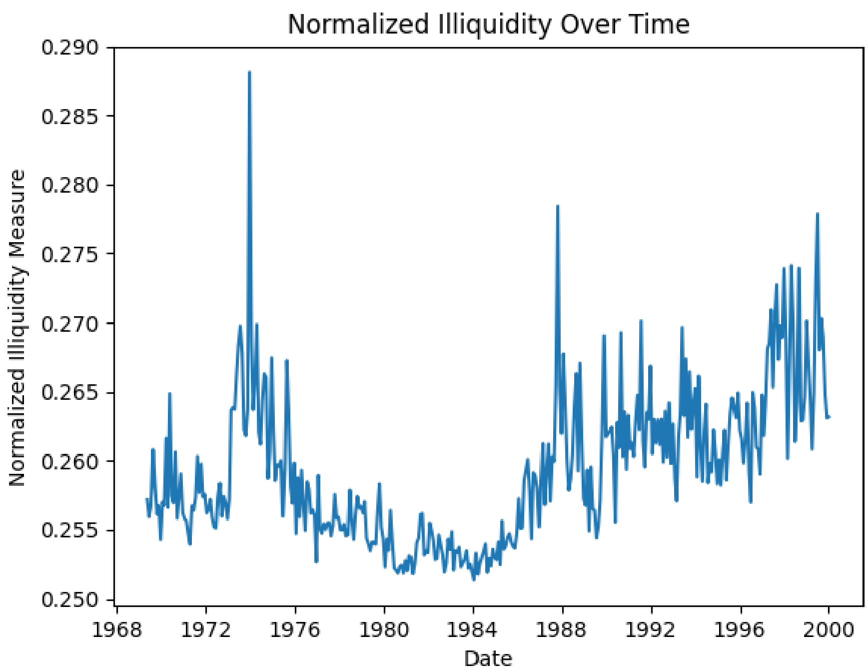

Figure 4: Graph of the normalized illiquidity over time for the JPMorgan shares

2.4 Innovations in illiquidity

Our goal in this part will be to determine the market illiquidity innovation. To do this, we will fit an AR(p) specification to the un-normalized illiquidity, that we will define after, and the residual u_t, of the regression will be interpreted as the market illiquidity innovation,

$$c_t^M - E_{t-1}(c_t^M) = u_t, \tag{20}$$

Fistly, we need to compute un-normalized illiquidity truncated for outliers using our precendent result of part 2.3.2. The formula for this un-normalized illiquidity will be given by

$$\overline{ILLIQ_t^p} = \sum_{i \in p} w_t^{ip} \min\left(ILLIQ_t^i, \frac{30.00 - 0.25}{0.30 P_{t-1}^M}\right), \tag{21}$$

where w_t^{ip} is the portfolio weight. We are doing this normalisation in order to create a stationary variable and to put it on a scale corresponding to the cost of a single trade.

Now that we have our variable we can test different AR(p) specification and compare the results in terms of R^2. This work will be similar to running in regression :

$$(0.25+0.30\overline{ILLIQ_t^M}P_{t-1}^M) = \alpha_0+\alpha_1(0.25+0.30\overline{ILLIQ_{t-1}^M}P_{t-1}^M)+\alpha_2(0.25+0.30\overline{ILLIQ_{t-1}^M}P_{t-1}^M)+u_t, \tag{22}$$

where the three terms inside parentheses in this specification correspond closely to c_t^M, c_{t-1}^M, and c_{t-2}^M respectively.

Here is an extract of the database for the innovations for the first portfolio and for the market portfolio.

	DATE	portfolio	value_weighted_ret	ILLIQ_i_t	value_weighted_ret_pred	value_weighted_ret_resid	ILLIQ_i_t_pred	ILLIQ_i_t_resid
0	1962-07	market	1.536892	95576.043307	NaN	NaN	NaN	NaN
1	1962-08	market	1.090426	93288.482461	NaN	NaN	NaN	NaN
2	1962-09	market	-1.274697	84616.542522	1.536892	-2.811589	95576.043307	-10959.500785
3	1962-10	market	-0.562262	99590.060165	1.090426	-1.652687	93288.482443	6301.577722
4	1962-11	market	2.901187	78454.227156	-1.274697	4.175884	84616.542535	-6162.315379
...	...	...	...	...	...	...	...	...
445	1999-08	market	-0.410733	4398.085690	0.848832	-1.259565	4845.234667	-447.148977
446	1999-09	market	-0.246158	3984.621373	-0.078293	-0.167866	3897.353249	87.268124
447	1999-10	market	0.427802	4718.810178	-0.410733	0.838535	4398.085690	320.724488
448	1999-11	market	0.354347	3902.426671	-0.246158	0.600506	3984.621373	-82.194703
449	1999-12	market	1.192534	3444.331717	0.427802	0.764732	4718.810178	-1274.478461

Figure 5: Extract of the database for the Market Portfolio with the predicted values and the residuals

	DATE	portfolio	value_weighted_ret	ILLIQ_i_t	value_weighted_ret_pred	value_weighted_ret_resid	ILLIQ_i_t_pred	ILLIQ_i_t_resid
0	1962-07	6.0	0.062161	886.466185	NaN	NaN	NaN	NaN
1	1962-08	6.0	0.044222	779.654864	NaN	NaN	NaN	NaN
2	1962-09	6.0	-0.029115	634.948341	0.062161	-0.091276	886.466185	-251.517845
3	1962-10	6.0	-0.014583	1074.086688	0.044222	-0.058805	779.654864	294.431824
4	1962-11	6.0	0.097469	575.826531	-0.029115	0.126584	634.948341	-59.121809
...	...	...	...	...	...	...	...	...
445	1999-08	6.0	-0.027589	7.983906	0.031443	-0.059032	7.401806	0.582099
446	1999-09	6.0	-0.013747	8.191396	-0.019862	0.006116	6.888865	1.302531
447	1999-10	6.0	0.029001	8.957325	-0.027589	0.056590	7.983906	0.973419
448	1999-11	6.0	0.003940	8.085166	-0.013747	0.017686	8.191396	-0.106230
449	1999-12	6.0	0.013523	7.797629	0.029001	-0.015477	8.957325	-1.159696

Figure 6: Extract of the database for the 6th Portfolio with the predicted values and the residuals

2.4.1 Liquidity risk

We then, delve deeper int o the liquidity risk. For this, we start by computed the four betas β_p', β_p'', β_p''', β_p'''' for each of our portfolio thanks to the previous equations. Then we compare the result of our βs with the standard deviation of a portfolio illiquidity innovations $\sigma(\Delta c_p)$, the average illiquidity E(c_p), average excess return E$(r^{e,p})$, turnover, size ie market capitalization.

We notice that illiquid stocks (ie stocks with high average illiquidity E(c_p)) tend to exhibit increased stock return volatility, reduced turnover, and diminished market capitalization. As anticipated, our results show that illiquidity stocks have high liquidity risks, more specifically a large (positive) b''_p and large negative b'''_p and b''''_p. In other words, stocks categorized as illiquid in an absolute sense (in c_p) often display a pronounced liquidity relationship with the broader market (cov(c_p, c_M)), enhanced liquidity sensitivity to market return (cov(c_p, r_M)), and lot of return sensitivity to market liquidity (cov(r_p, c_M)).

Stocks with greater illiquidity tend to manifest increased return volatility. Given the lower bound of zero for illiquidity, it follows that illiquid stocks also demonstrate more volatile illiquidity innovations. The table 3 validates our results by showing that the standard deviation of portfolio illiquidity, $\sigma(\Delta c_p)$, escalates consistently with portfolio illiquidity. Nevertheless, they are not the only factors of the positive relationship between illiquidity and liquidity risk. Indeed, the correlation between c_p and c_M is increasing in portfolio illiquidity while the correlations between r_p and c_M and between c_p and r_M diminish. These results are confirmed by computing the correlation among the betas. The collinearity appears also for the stock which lead to an empirical issue : How could we distinguish the effects of illiquidity and individual liquidity betas?

Table 3: Table for the properties of illiquidity portfolios

Portfolio	β^{1P}	β^{2P}	β^{3P}	β^{4P}
1	2.66E-11	3.21E-13	-1.39E-06	-2.32E-09
3	3.89E-11	3.63E-13	-1.65E-06	-2.45E-09
5	4.01E-11	3.98E-13	-1.98E-06	-2.78E-09
7	4.4E-11	4.12E-13	-2.14E-06	-2.89E-09
9	5.19E-11	4.32E-13	-2.39E-06	-3.12E-09
11	5.28E-11	4.76E-13	-2.43E-06	-3.72E-09
13	5.97E-11	5.21E-13	-2.87E-06	-4.21E-09
15	6.21E-11	7.21E-13	-3.01E-06	-4.75E-09
17	7.32E-11	1.032E-12	-3.7E-06	-4.9E-09
19	8.12E-11	1.189E-12	-4.04E-06	-5.23E-09
21	9.32E-11	1.232E-12	-4.98E-06	-6.12E-09
23	9.54E-11	1.321E-12	-5.23E-06	-6.63E-09
25	9.96E-11	1.489E-12	-6.7E-06	-6.89E-09

2.4.2 How liquidity risk affects returns?

The main part of this article/ work is to study how liquidity risk affects returns. To do this, we run a cross-sectional regressions on our test portfolios using a GMM framework that takes into account the pre-estimation of the beta.

We consider the liquidity-adjusted CAPM for portfolios sorted by illiquidity and the illiquidity variation.

We define the net beta as :

$$b_p^{\text{net}} = b_p' + b_p'' - b_p''' - b_p''''. \tag{23}$$

To impose the model-implied constraint that the risk premia of the different betas is the same.

Therefore, the liquidity-adjusted CAPM becomes:

$$\mathbb{E}(r_{pt} - r_{ft}) = \alpha + \kappa \mathbb{E}(c_{pt}) + \lambda \beta_p^{\text{net}} \tag{24}$$

where we allow a nonzero intercept, α, in the estimation, although the model implies that the intercept is zero.

The average holding duration is empirically determined based on the period when all shares are exchanged once, denoted by κ. Within the liquidity portfolio sample, κ is set at 0.034, translating to an approximate 29-month holding duration. The anticipated illiquidity is symbolized by $\mathbb{E}(c_{pt})$, capturing the portfolio's average liquidity constraint. In regression (25), the net return, $\mathbb{E}(r_{pt} - r_{ft}) - \kappa \mathbb{E}(c_{pt})$, is employed as the dependent variable.
The liquidity-adjusted CAPM (25) has only one risk premium λ, as is the case with the standard CAPM. However, the improvement of the adjusted CAPM lies in the fact that the risk factor is the net beta β^{net}. Regardless of the illiquidity portfolios considered, the risk premium λ remains positive and significant while α is insignificant, aligning with the model's assumptions. Empirically, the R^2 of the liquidity-adjusted CAPM is superior to that of the standard CAPM. It's worth noting that when κ is a free parameter, it results in only very modest variations. We display it in the first and second column of the table.

In order to isolate the effect of liquidity risk ($\beta'', \beta''', \beta''''$) relative to the liquidity level $E[\text{expectation}][c]$ and market risk β', we consider :

$$\mathbb{E}(r_{pt} - r_{ft}) = \alpha + \kappa \mathbb{E}(c_{pt}) + \lambda \beta_p^{\text{net}} + \lambda^1 \beta_p' \tag{25}$$

We estimate with different values of κ. When κ is at its calibrated value, β^{net} is insignificant for the illiquid portfolios but significant for the sigma illiquidity portfolios (illiquid variation). When κ is a free parameter, β^{net} is significant for all portfolios, but κ is estimated as negative for the illiquid portfolios, which is problematic since this parameter should be positive. Thus, we set $\kappa = 0$, and β^{net} is significant in both panels (illiquid portfolio and illiquid variation portfolio). However, due to multicollinearity issues, this result lacks substantial value. We display our result in columns 3, 4 et 5 of the table

Note that a negative coefficient in front of β' does not imply a negative risk premium on market risk, given that β' is also contained within β^{net}. Lastly, with different risk premiums λ^i with fixed κ and then free κ, the equation becomes:

$$\mathbb{E}(r_{pt} - r_{ft}) = \alpha + \kappa\mathbb{E}(c_{pt}) + \lambda^1\beta'_p + \lambda^2\beta''_p + \lambda^3\beta'''_p + \lambda^4\beta''''_p \tag{26}$$

The results are challenging to interpret due to increasing multicollinearity issues. Adjusting for liquidity particularly improves the fit for illiquidity portfolios, aligning with the initial intuition. However, κ doesn't seem to have much significance in the sense that fixing or freeing it doesn't alter the outcomes (column 6 et 7 of the table).

	$E(c_p)$	b'_p	b''_p	b'''_p	b''''_p	b^{net}_p	R^2
1	0.034	-	-	-	-	1.543	0.73
2	0.12	-	-	-	-	1.399	0.76
3	0.034	-1.9	-	-	-	3.78	0.77
4	-0.01	-12.78	-	-	-	14.68	0.79
5	-	- 7.9	-	-	-	9.7	0.814
6	0.034	0.91	-157	6	-16.67	-	8.35
7	1	0.89	-155	5.9	-16.59	-	0.85

Our values are quite close with the values from the article; they consistently have the same signs. However, we have a multiplicative factor of 10^6, but we decided to remove it to adjust our data to the correct unit.

The study highlights the economic importance of liquidity risk. Evaluating differences in expected returns across portfolios based on liquidity reveals a notable annual impact of approximately 1.1%. Among liquidity risk factors examined, the covariance of an asset's illiquidity with market returns stands out with the most significant economic effect. While liquidity risk is impactful, its magnitude appears lower than some previous estimates. The observed correlation between liquidity and liquidity risk suggests that many studies may effectively measure a combined influence of both factors.

Now, let's focus on the economic significance of the results. To gauge the magnitude of the effect, one needs to calculate the annual return premium required to hold illiquid securities rather than liquid ones, which is computed as the product of the risk premium and the difference in liquidity risk across liquidity portfolios. The impact of liquidity risk is notably different from zero. Notably, among the trio of liquidity risks considered, β'''' stands out for its pronounced economic influence on anticipated returns, specifically reflecting the correlation of an asset's illiquidity with market returns. It's worth highlighting that this particular liquidity risk has remained unexplored both theoretically and empirically in prior studies. Furthermore, the divergence in annualized expected returns between portfolios 1 and 25, attributed to variations in expected illiquidity denoted by $E(c)$, stands at 3.5 %, as indicated by a calibrated coefficient. When synthesizing the effects of both anticipated illiquidity and liquidity risk, the cumulative impact is an annualized 4.6 %. Nevertheless, the magnitude of the liquidity risk is economically significant.

Then the article focuses on robustness, size, and book-to-market. The researchers assess the robustness of their findings by examining different weighting methods: equal and value-weighted. The results show variations between these two methods, but the liquidity-adjusted model stands out with a higher R^2, especially when the weighting is based on value. Similarly, to check for robustness, the model is estimated using portfolios classified by size and book-to-market ratio. Small-sized stocks are found to be both illiquid and associated with high liquidity risk. Although the regressions show coefficients similar to those obtained previously, their statistical significance is reduced. As for the β^{net} coefficient, it remains positive, and the liquidity-adjusted CAPM consistently has a higher R^2 than the standard CAPM.
To further assess the model's ability to explain the effects of size and book-to-market ratio, regressions are conducted while controlling for size and book-to-market ratio. Despite largely similar results, the liquidity risk does not account for the effect of the book-to-market ratio.

Specification Tests of the Liquidity-Adjusted CAPM

To assess the reliability and accuracy of the liquidity-adjusted CAPM, various specification tests were conducted:

1. **Model Restrictions:** Initially, it was observed that the model-implied restriction suggesting $\alpha = 0$ in the liquidity-adjusted CAPM was not statistically significant. This contrasts with the standard CAPM where such restrictions were found to be statistically significant at specific confidence levels.

2. **Wald Test Analysis:** When incorporating an unrestricted risk premium model, the Wald

test did not refute multiple model-implied constraints. Specifically, associated p-values for these constraints were 47% in one scenario and 28% in another. In contrast, for the standard CAPM, these constraints yielded p-values of 15% and 8.7%.

3. **Risk Premium Comparisons:** A significant implication of the model is that the risk premium should be equivalent to the expected net market return minus the risk-free rate. The risk premium estimate for the liquidity-adjusted CAPM exceeded this benchmark, resulting in p-values of 6.6% and 7.3% across distinct regression analyses. In comparison, the standard CAPM presented notably lower p-values.

4. **Pricing Error Analysis:** Rigorous tests were applied to determine if the linear model priced all portfolios without discrepancies. For portfolios with illiquidity, the liquidity-adjusted CAPM's p-values ranged between 8.5% and 9.9% across specific regressions, whereas the standard CAPM recorded a mere 0.5%. A similar trend was evident in size portfolios, emphasizing the liquidity-adjusted CAPM's superior fit.

5. **B/M-by-Size Portfolios Examination:** Despite the liquidity-adjusted CAPM's overall positive results, its performance was relatively subdued when applied to B/M-by-size portfolios. The Wald test returned a p-value of 47%, with specific zero pricing error tests ranging from 15.7% to 85%. In contrast, the standard CAPM had p-values of 23% and 3.2%.

In summary, while the liquidity-adjusted CAPM showcased promising results across various scenarios, its effectiveness displayed variability across different portfolio classifications.

3 Conclusion

In conclusion, this article introduces a straightforward model to account for liquidity risk, notably where the liquidity-adjusted CAPM better explains the data than the standard CAPM. As anticipated, the value of a security increases with the covariance between its liquidity and market liquidity, $\mathrm{cov}_t(c^i_{t+1}, c^M_{t+1})$, as investors demand a higher risk premium for a security that is illiquid when the market is illiquid. Conversely, it decreases with the covariance between its return and market liquidity, $\mathrm{cov}_t(r^i_{t+1}, c^M_{t+1})$, as investors prefer securities with higher returns when the market is illiquid. Additionally, it diminishes with the covariance between its liquidity and market returns, $\mathrm{cov}_t(c^i_{t+1}, r^M_{t+1})$, as investors are willing to pay significantly more (80 %) when the security is liquid when market return is low; this effect is indeed the most significant and predominantly explains the return premium. Furthermore, the model emphasizes that persistent positive shocks to liquidity are associated with low contemporaneous returns and high predicted future returns. Furthermore, we have identified timid evidence suggesting that liquidity risk plays a significant role, beyond the impacts of market risk and the level of liquidity.

Their findings suggest that liquidity risk accounts for approximately 1.1 % of the observed impacts. However, correlations between liquidity and liquidity risk tend to temper the precision of these conclusions. While the model provides valuable insights into the mechanisms of liquidity risk and its effects on expected returns, it remains a simplification. Further studies could therefore refine and enhance this model

4 Références bibliographiques

[1] E. Bocquaire, Les grands principes de l'actuariat.

[2] M. Hanafy, Machine Learning Approaches for Auto Insurance Big Data

[3] McCullagh, P., & Nelder, J. (1989). Generalized linear models (2nd ed.). Chapman and Hall.

[4] L. Guelman (2012). Gradient boosting trees for auto insurance loss cost modeling and prediction.

[5] Brieman, L., Friedman, J., Olshen, R., & Stone, C. (1984). Classification and regression trees. CRC Press