Second Quantitative Risk Management Project Report

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Risk Forecasting with Extreme Value Theory

Introduction

In this assessed work for the Quantitative Risk Management module, we embark on a comprehensive analysis of the risk associated with a long equity investment in Tesla (TSLA). This task involves a detailed study of Tesla's stock performance over a decade, from November 26, 2012, to November 25, 2022. The analysis is structured to provide insights into the volatility and risk characteristics of Tesla's stock using advanced statistical methods.
Our approach is multifaceted, beginning with the computation of daily log-returns to capture the stock's price movements. This data is divided into two sets: the training data (prior to November 26, 2021) and the testing data (on or after November 26, 2021). The training data serves as the foundation for our models, aiding in understanding past behaviors and patterns in the stock's performance.
Subsequently, we fit a standard GARCH(1,1) model to the training data, which allows us to analyze and forecast the volatility of Tesla's stock. This step is crucial for understanding the time-varying nature of risk associated with the investment. Additionally, we examine the residuals of the GARCH model to assess their distributional properties. This involves fitting a normalized Student t-distribution and a generalized Pareto distribution (GPD) to the residuals, providing a deeper understanding of the tails of the distribution, which are critical in risk management.
The final part of our analysis focuses on the computation and backtesting of Value at Risk (VaR) and Expected Shortfall (ES) forecasts for the testing data. These risk measures are computed at 95% and 99% confidence levels using different specifications for the white noise component in the GARCH model.

Question 1: Volatility Modeling and Normality Assessment

The empirical analysis starts with a review of Tesla's stock price over the examined period. Figure 1 presents the adjusted closing price of Tesla's stock, highlighting significant growth and volatility, especially in the latter years.

Log Returns Transformation

The first step in our analysis involves transforming the daily closing prices of Tesla's stock into log returns. This transformation is essential for stabilizing the variance (reduce heteroskedasticity)

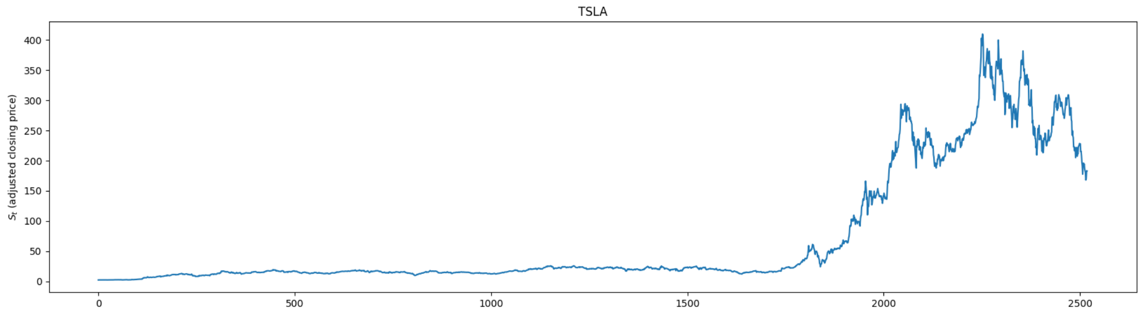

Figure 1: Adjusted Closing Price of Tesla's Stock

and making the time series more suitable for modeling and analysis. The log return for each day is calculated using the formula:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{1}$$

where r_t is the log return on day t, P_t is the closing price of Tesla's stock on day t, and P_{t-1} is the closing price on the previous day. Figure 2 illustrates the daily log returns of Tesla's stock, providing a continuous measure of the stock's performance. The log returns exhibit volatility clustering.

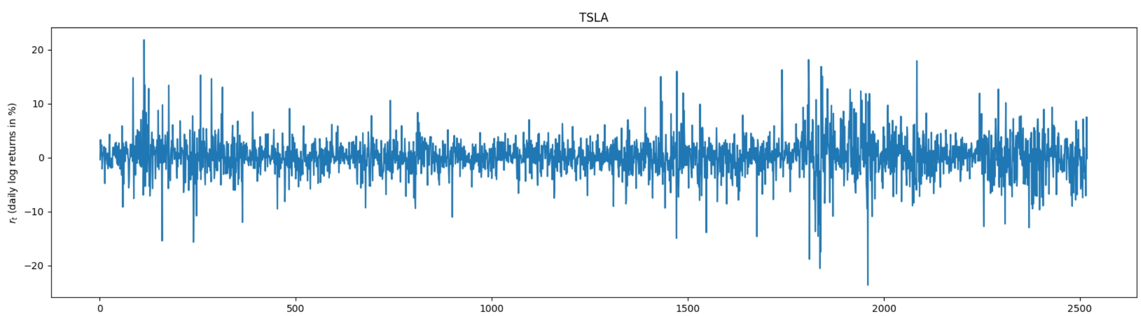

Figure 2: Daily Log Returns of Tesla's Stock

Fitting a GARCH Model

Given the nature of financial data, particularly the clustering of volatility, a GARCH(1,1) model is employed to model the time-varying volatility of Tesla's stock. The GARCH model captures the dynamic nature of the stock's volatility, making it a powerful tool for risk management. The model is specified as follows:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{2}$$

where σ_t^2 is the conditional variance, ϵ_{t-1} is the residual at time $t-1$, and α_0, α_1, and β_1 are parameters to be estimated. The fitting of this model to the training dataset of Tesla's log returns enables us to understand and forecast the stock's volatility. The volatility estimated by the GARCH model is depicted in Figure 3.

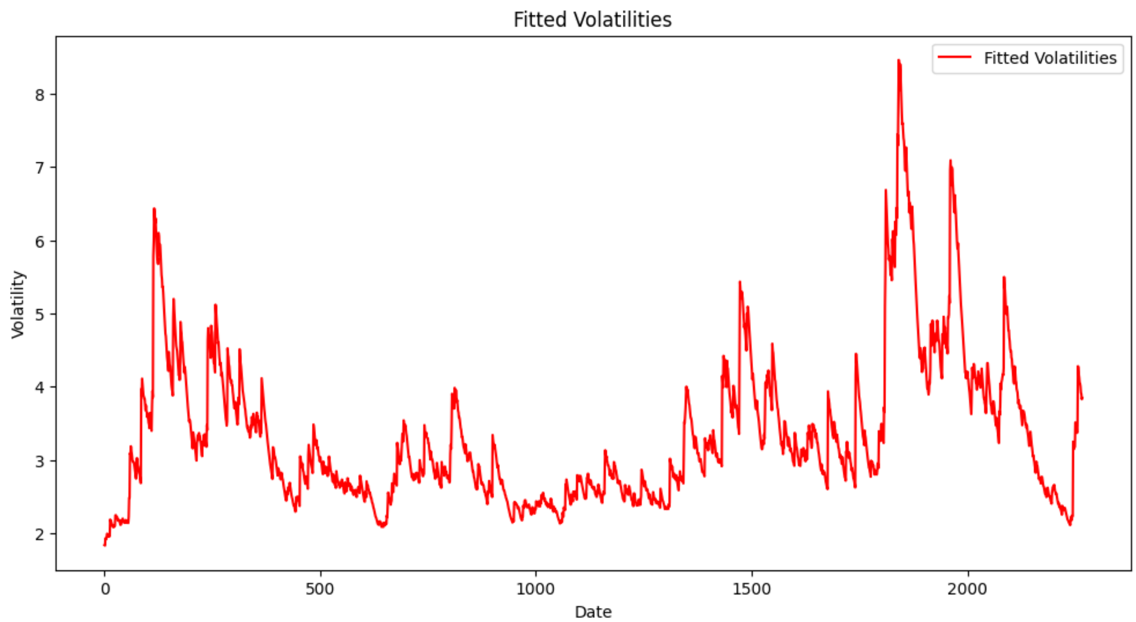

Figure 3: Fitted Volatilities from the GARCH Model

To assess the appropriateness of the GARCH model, the standardized residuals are analyzed, as shown in Figure 4. They appear to exhibit no apparent patterns or systematic structures, indicating that the model is adequate.

Checking for Normality

The final step in our analysis is to assess the normality of the residuals from the GARCH model. This is crucial as many statistical models, including the GARCH model, assume that residuals are normally distributed. The Shapiro-Wilk test is employed for this purpose. The test is defined as:

$$W = \frac{(\sum_{i=1}^{n} a_i X_{(i)})^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \tag{3}$$

where $X_{(i)}$ are the ordered sample values, a_i are calculated using the covariance matrix of the order statistics and the inverse of the variance-covariance matrix of the normal order statistics. These coefficients are dependent only on the sample size n and are designed to provide the best linear unbiased predictions (BLUPs) of the order statistics if the data are normal, and $\bar{X}$ is the sample mean. The outcome of this test provides insight into the distributional characteristics of the residuals, which is important for the accuracy and reliability of our risk forecasts. Figure 5 compares the distribution of standardized residuals with the normal distribution. The Shapiro-Wilk test for the dataset yielded a test statistic of approximately 0.948, indicating a slight deviation

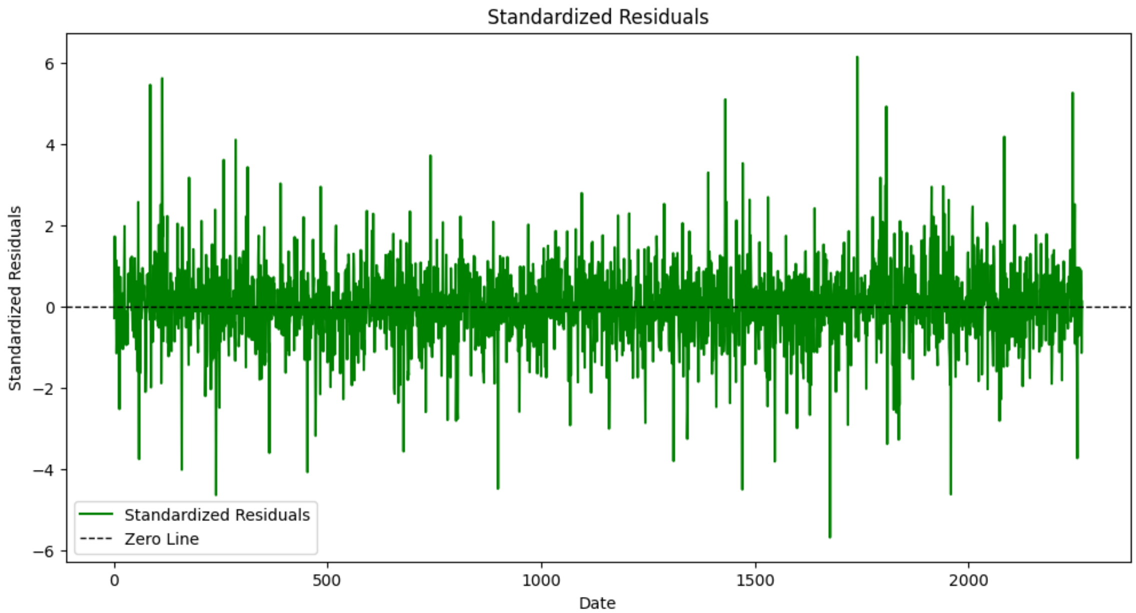

Figure 4: Standardized Residuals from the GARCH Model

from normality. However, the extremely small P-value (approximately 1.14×10^{-27}) provides strong statistical evidence to reject the null hypothesis, suggesting that the data does not follow a normal distribution. This result is significant and implies that the normality assumption for the data is not appropriate.

Conclusion

Through our rigorous quantitative analysis, we have observed that Tesla's stock exhibits significant volatility and non-normal behavior in its returns, as evidenced by the log returns and GARCH model fitting. The application of the Shapiro-Wilk test has led to a rejection of the normality assumption for the residuals of the GARCH model, with a test statistic of 0.948 and an extremely small P-value. These findings underscore the need for non-normal risk assessment methods and have important implications for modeling Tesla's stock volatility, which must account for heavy tails and skewness in the distribution of returns.

Question 2: Student's t-Distribution Fitting

Fitting a Student's t-Distribution to Standardized Residuals

The appropriateness of the normal distribution for the standardized residuals from the GARCH model was evaluated and rejected in favor of a Student's t-distribution. The Shapiro-Wilk test yielded a test statistic of approximately 0.948, with an extremely low P-value, suggesting a significant deviation from normality. In response, we fit a Student's t-distribution, which is renowned for

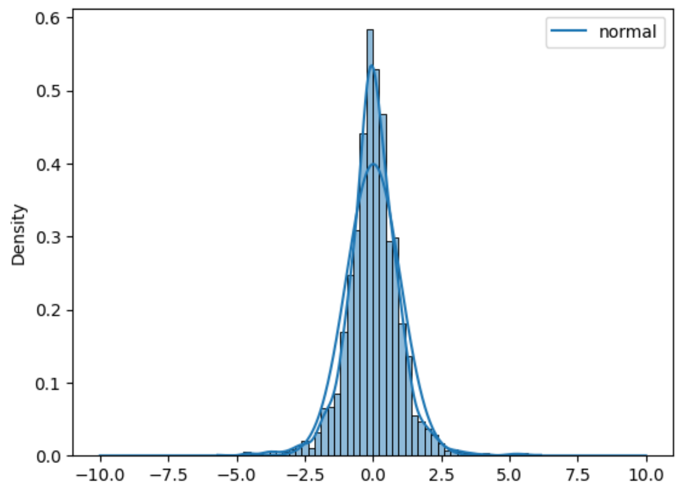

Figure 5: Fit of Standardized Residuals to the Normal Distribution

its ability to accommodate heavy tails in the data.

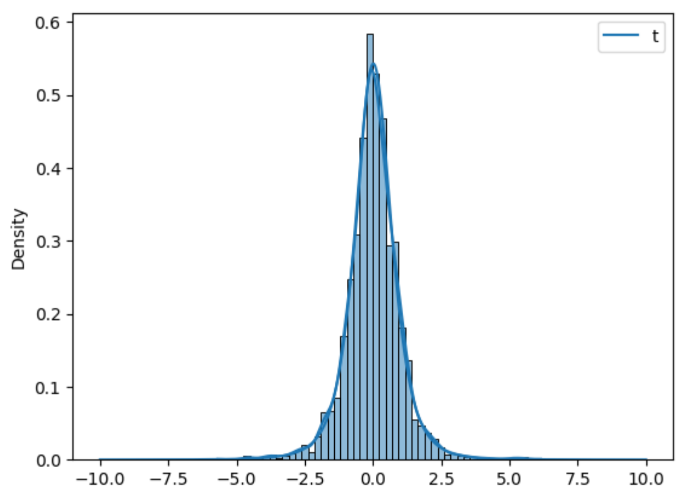

Figure 6: Fit of the Student's t-Distribution to Standardized Residuals

The Student's t-distribution is parameterized by its degrees of freedom, which dictate the heaviness of the tails. A lower degree of freedom corresponds to heavier tails, which is often observed in financial return series and suggests a greater likelihood of extreme values compared to the normal distribution. The estimated degrees of freedom of approximately 3.56 indicate a distribution with heavier tails than the normal distribution, providing a more accurate fit for the standardized residuals from the GARCH model, as depicted in Figure 6.

This finding is critical as it implies that the risk of extreme loss may be underestimated if modeled under the assumption of normality. The Student's t-distribution offers a better representation of the underlying risks by accounting for the observed leptokurtosis - a higher peak and heavier tails - which are characteristic features in the distribution of financial market returns. Consequently, the Student's t-distribution allows for more robust risk estimation, which is particularly relevant for Value at Risk (VaR) and Expected Shortfall (ES) calculations in the field of risk management.

Conclusion

The analysis for Question 2 shows that by accommodating the heavy-tailed nature of the stock's return distribution, the Student's t-distribution provides a more realistic foundation for the subsequent risk measurement and management tasks. The insights gained from fitting this distribution will inform more accurate and prudent risk assessment in the following sections of this study.

Question 3: Generalized Pareto Distribution Fitting

Threshold Selection

The selection of an appropriate threshold u is a critical step in modeling the tail of a distribution using the Generalized Pareto Distribution (GPD). The threshold is chosen by analyzing the sample mean excess function, $e(u)$, defined as:

$$e(u) = E[X - u|X > u] \tag{4}$$

where X represents the standardized residuals. We look for a stable region in the plot of $e(u)$ versus u, where the function begins to level off (see Figure 4 in Appendix). The sample mean excess function plot was utilized to determine an appropriate threshold. A stable region is identified where the mean excess plot $e_n(v)$ displays a 'linear' trend when $v \geq u$, indicating the threshold beyond which the tail behavior can be modeled by the GPD. In what follows, we will select $u = 3.1$

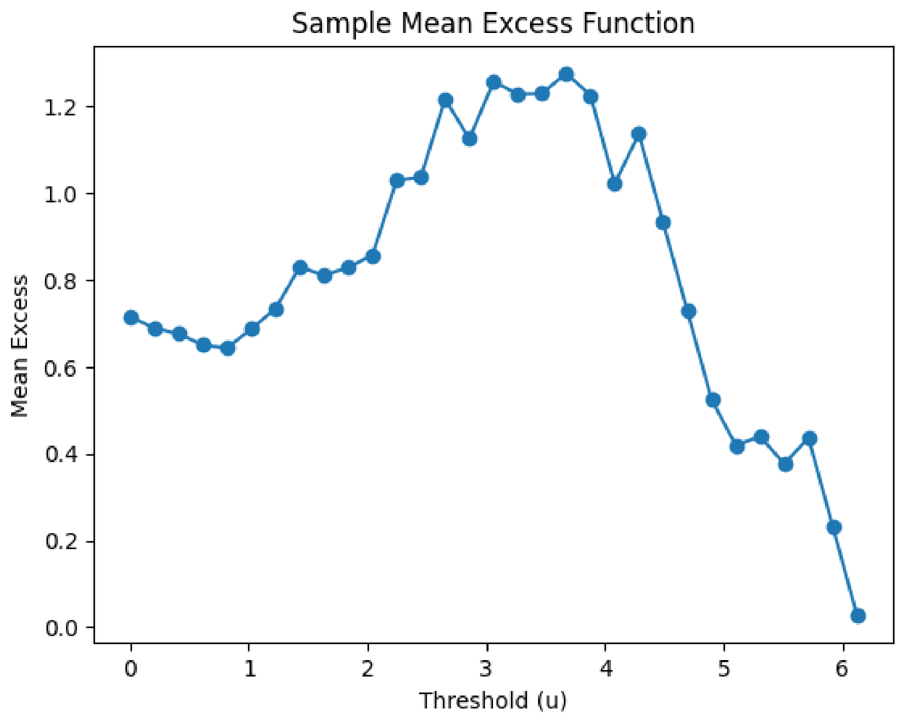

Figure 7: Sample mean excess function plot

GPD Fit

After determining the threshold u, we fit the GPD to the residuals exceeding u. The GPD is characterized by the scale parameter σ and the shape parameter ξ, and its cumulative distribution function is given by:

$$F(x;\xi,\sigma)=\begin{cases}1-\left(1+\xi\frac{x}{\sigma}\right)^{-1/\xi} & \text{for } \xi\neq 0,\\ 1-\exp\left(-\frac{x}{\sigma}\right) & \text{for } \xi=0.\end{cases} \tag{5}$$

where x is the excess over the threshold.

To proceed, we first construct the "sample excess value above u" from the data. From the original fire claim data $X_1, X_2, ..., X_n$, we extract a subset of them with claim value above u and let's relabel this subset as $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_{N_u}$ where N_u is a random number representing the number of claims above u. Then define

$$Y_i = \tilde{X}_i - u$$

for $i = 1, 2, ..., N_u$. We construct Y as above, which will contain 109 observations only. Next, we use $(Y_i)_{i=1,...,N_u}$ as the available observations to estimate (ξ, β) by the method of MLE. Using the density function of $G_{\xi,\beta}$, we implement the log-likelihood function,

$$\begin{aligned}\log L(\xi, \beta; Y_1, \ldots, Y_{N_u}) &= \sum_{j=1}^{N_u} \log g_{\xi,\beta}(Y_j) \\ &= -N_u \log \beta - \left(1 + \frac{1}{\xi}\right) \sum_{j=1}^{N_u} \log \left(1 + \xi \frac{Y_j}{\beta}\right),\end{aligned}$$

We are now ready to numerically estimate (ξ, β) by maximising $\log L(\xi, \beta; Y_1, ..., Y_{N_u})$. This is done via the Scipy function "minimize".

The fit of the GPD to the data is visualized by comparing the empirical cumulative distribution function of the exceedances is compared to the fitted GPD to assess the adequacy of the fit, highlighting the close fit of the GPD to the tail of the distribution.

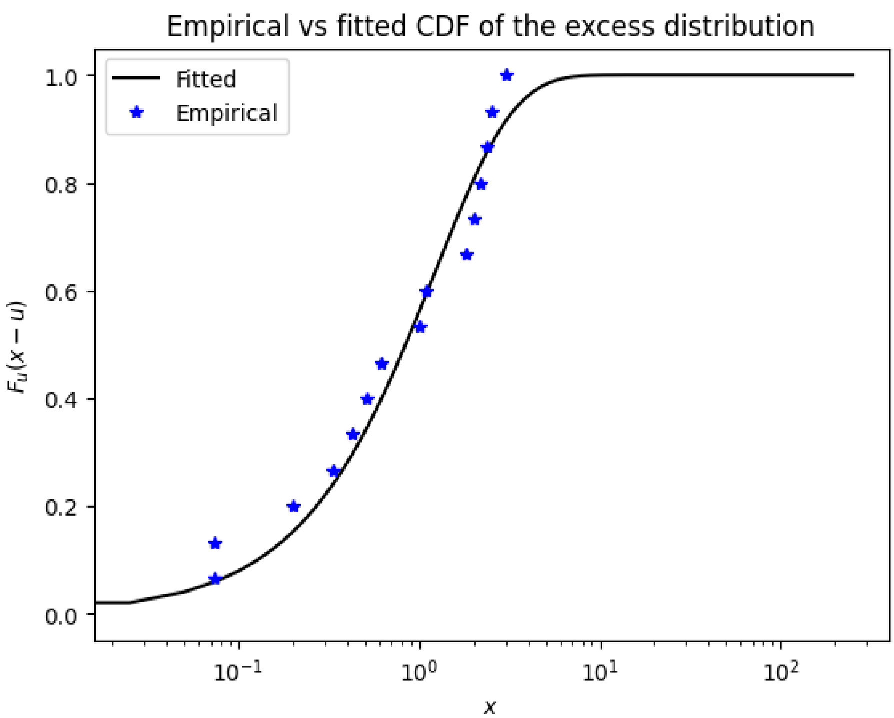

Figure 8: The empirical vs. fitted CDF of the excess distribution

QQ Plot Analysis

The quantile-quantile (QQ) plot is a graphical tool to assess the fit of the GPD to the data. It compares the ordered values of the exceedances to the theoretical quantiles of the GPD. A linear relationship in the QQ plot suggests a good fit. The QQ plot is given by:

$$\text{Ordered Values} = F^{-1}\left(\frac{r_i - 0.5}{n}, \hat{\xi}, \hat{\sigma}\right) \tag{6}$$

where F^{-1} is the inverse CDF of the GPD, r_i is the rank of the ith data point, and n is the total number of exceedances.

The alignment of the points with the red line suggests an appropriate fit for the GPD to the extreme values in the data.

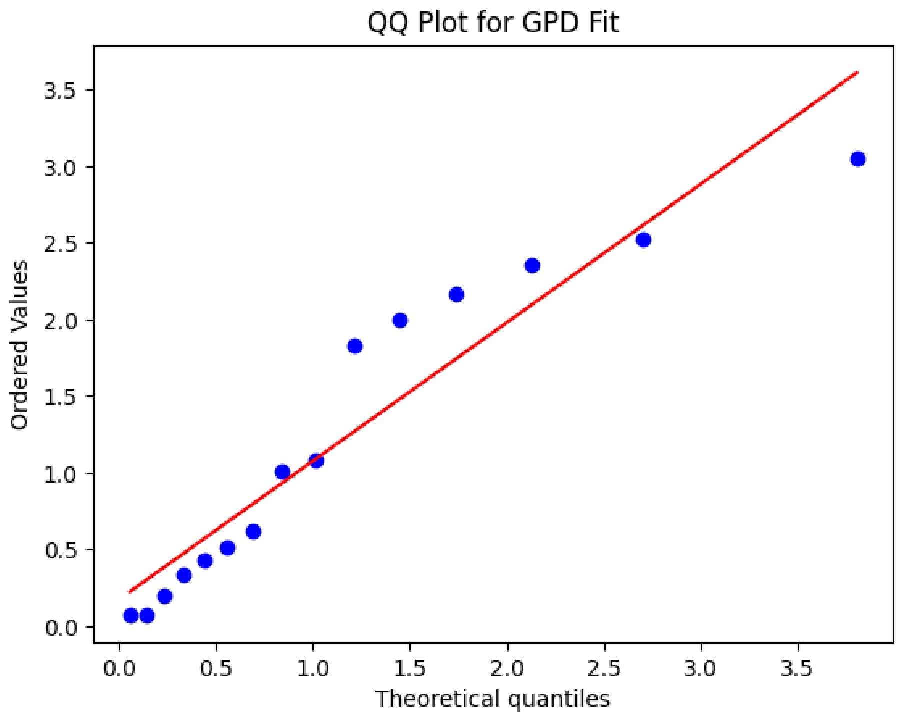

Figure 9: QQ Plot for GPD fit

Conclusion

In conclusion, the fitting of a Generalized Pareto Distribution (GPD) to the standardized residuals of Tesla's stock returns has been successfully executed. By selecting a threshold $u = 3.1$, we were able to focus on the tail behavior of the distribution, the estimates for the GPD parameters, with $\xi \approx 0.01$ and $\beta \approx 1.210$, suggest a distribution with a very light tail. The small value of the shape parameter ξ indicates that the tail decays almost exponentially, which is a characteristic of thin-tailed distributions. This is somewhat atypical for financial return data, which often exhibits heavy tails, and may imply a lower probability of extreme returns than one might expect for a stock like Tesla.

However, the QQ plot analysis and the empirical fit to the cumulative distribution function (CDF) of the excesses demonstrate an adequate fit of the GPD to the extreme values in the data. The relatively straight line observed in the QQ plot (see Figure 9) confirms that the GPD model captures the tail behavior of the standardized residuals effectively.

Question 4: Computation of VaR and ES Forecasts

In the context of this report, we estimate the Value at Risk (VaR) and Expected Shortfall (ES) using three different assumptions for the strict white noise component Z in the GARCH(1,1) model.

Standard Normal Distribution

The assumption here is that Z follows a standard normal distribution. The forecasts for VaR and ES are therefore based on the quantiles of the normal distribution.

- VaR Calculation: The VaR at a specific confidence level α is estimated using the quantile function of the normal distribution, scaled by the conditional volatility provided by the GARCH model:

  $$\mathrm{VaR}_\alpha = -\sigma_t \times \Phi^{-1}(\alpha)$$

  where σ_t is the conditional standard deviation from the GARCH model and Φ^{-1} is the quantile function of the standard normal distribution.

- ES Calculation: The ES is the expected return on the tail beyond the VaR level and is calculated as:

  $$\mathrm{ES}_\alpha = -\sigma_t \times \frac{1}{(1-\alpha)} \int_{-\infty}^{\Phi^{-1}(\alpha)} x \cdot \phi(x) dx$$

  where $\phi(x)$ is the probability density function of the standard normal distribution.

Normalized Student t-Distribution

When Z is modeled to follow a normalized Student t-distribution, we account for heavier tails in the return distribution.

- VaR Calculation: The VaR is estimated using the quantile function of the Student t-distribution with degrees of freedom estimated from the data:

  $$\mathrm{VaR}_\alpha = -\sigma_t \times t_\nu^{-1}(\alpha)$$

  where t_ν^{-1} is the quantile function of the Student's t-distribution with ν degrees of freedom.

- ES Calculation: The ES is calculated using the expected value of the tail distribution beyond the VaR cut-off in the Student's t-distribution:

  $$\mathrm{ES}_\alpha = -\sigma_t \times \frac{1}{(1-\alpha)} \int_{-\infty}^{t_\nu^{-1}(\alpha)} x \cdot f_\nu(x) dx$$

  where $f_\nu(x)$ is the probability density function of the Student's t-distribution.

Generalized Pareto Distribution (GPD)

For modeling the tail behavior, Z is considered such that its excess distribution over a threshold u is described by a GPD.

- VaR and ES Calculation: Both VaR and ES are estimated based on the parameters of the GPD fitted to the tail of the loss distribution:

$$\mathrm{VaR}_\alpha = u + \frac{\sigma_{GPD}}{\xi_{GPD}} \left[\left(\frac{1}{1-\alpha} \right)^{\xi_{GPD}} - 1 \right]$$

$$\mathrm{ES}_\alpha = \mathrm{VaR}_\alpha + \frac{\sigma_{GPD} - \xi_{GPD} \cdot \mathrm{VaR}_\alpha}{1 - \xi_{GPD}}$$

where σ_{GPD} and ξ_{GPD} are the scale and shape parameters of the GPD, respectively.

Graphical Display of VaR and ES Forecasts

The following figures illustrate the computed VaR and ES forecasts over time, showcasing the implications of each distributional assumption for Z.

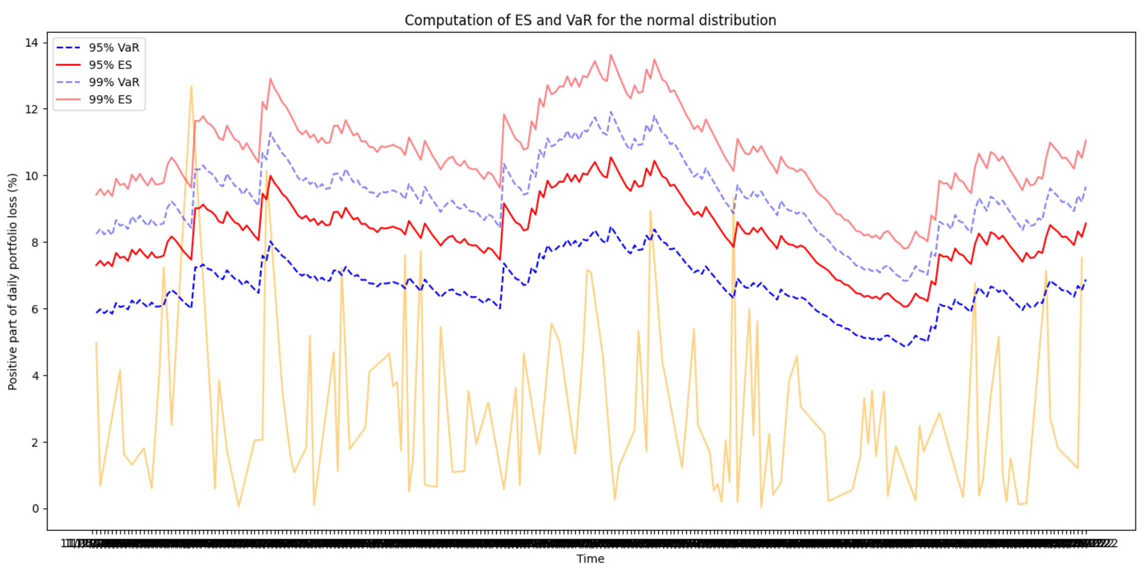

Figure 10: VaR and ES forecasts assuming Z follows a standard normal distribution

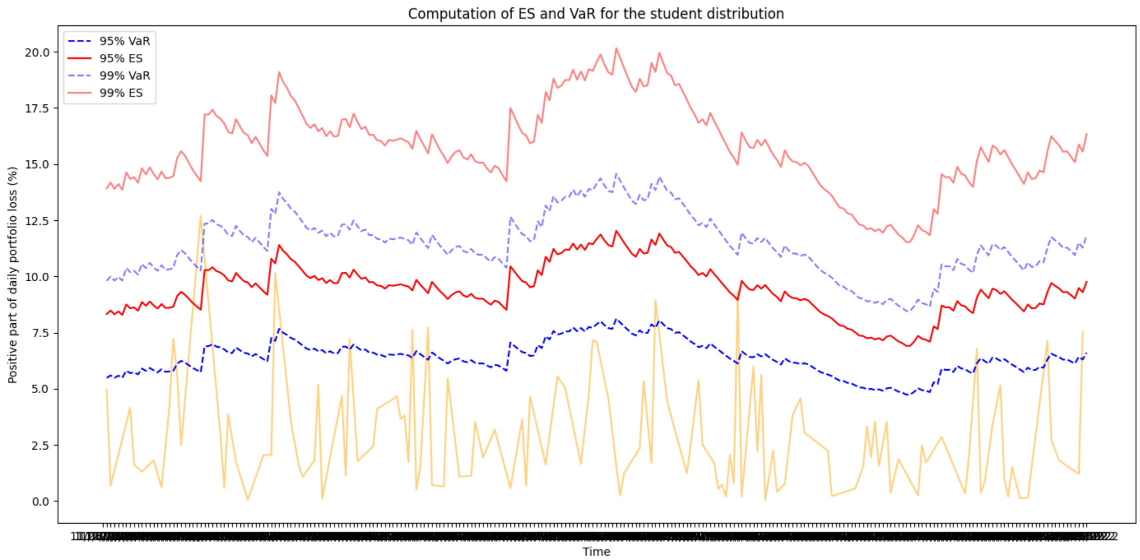

Figure 11: VaR and ES forecasts assuming Z follows a normalized Student t-distribution. The forecasts are higher than the normal distribution, indicating a more conservative estimate of risk due to fatter tails.

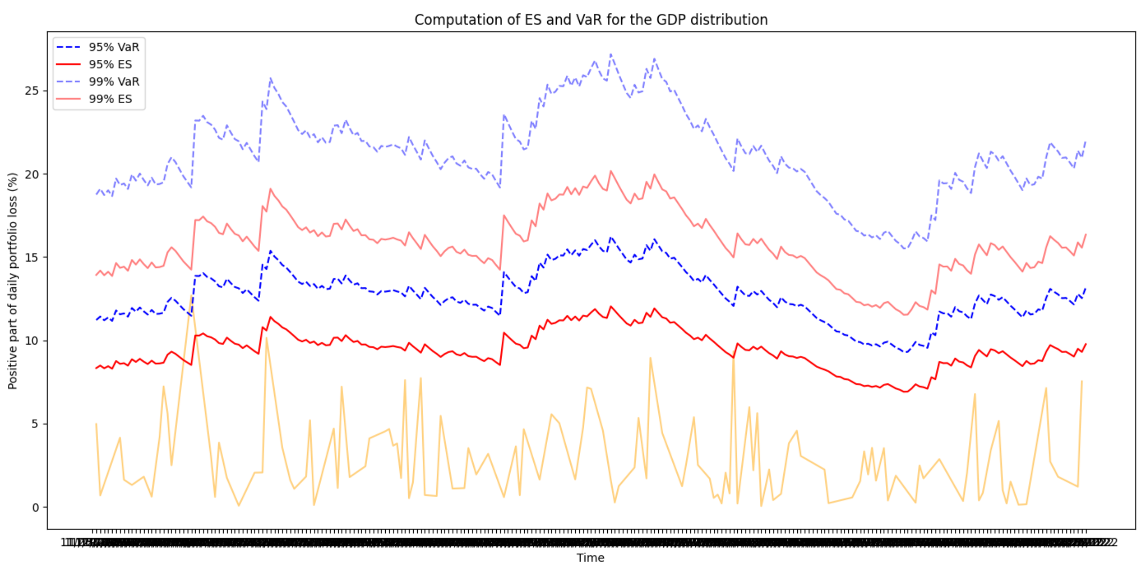

Figure 12: VaR and ES forecasts with Z modeled by a GPD for tail distribution

Unconditional Coverage Test (Kupiec's Test)

The Unconditional Coverage Test, also known as Kupiec's Test, evaluates the accuracy of the VaR model in terms of predicting the correct number of exceedances or violations. It specifically tests the null hypothesis that the frequency of VaR exceedances matches the expected frequency at a given confidence level α.

For a VaR model at confidence level α, the expected number of violations over N observations should be $N(1-\alpha)$. The test statistic for Kupiec's Test is calculated as:

$$LR_{uc} = -2\log\left((1-\alpha)^{N-n} \cdot \alpha^n\right) + 2\log\left(\left(\frac{N-n}{N}\right)^{N-n} \cdot \left(\frac{n}{N}\right)^n\right)$$

where N is the total number of observations, n is the number of observed violations, and α is the confidence level. The test statistic follows a chi-square distribution with one degree of freedom under the null hypothesis.

The backtest results for our VaR forecasts are as follows:

Method	P-value	Violations
Normal VaR 95%	0.6468	11
Normal VaR 99%	0.7373	2
Student VaR 95%	0.6468	11
Student VaR 99%	0.2756	1
GPD VaR 95%	0.0845	1
GPD VaR 99%	0.3103	0

Table 1: Unconditional Coverage Test (Kupiec's Test) Results

The table displays the number of observed violations and the corresponding P-values from Kupiec's Test for VaR forecasts at 95% and 99% confidence levels across different distributional assumptions for the white noise component Z in the GARCH model.

At the 95% confidence level, both the normal and Student t-distribution models exhibit a number of violations in line with expectations given the P-values, suggesting no evidence against the null hypothesis. However, the GPD model shows a P-value close to the conventional significance level, indicating potential underestimation of risk, though not at a statistically significant level.

At the 99% confidence level, all models show fewer violations than expected, with the GPD model showing no violations. The P-values indicate that there is no evidence to reject the null hypothesis, suggesting that the models may be overestimating risk at this confidence level.

These results indicate that for the 95% confidence level, the normal and Student's t-distributions may be appropriate for VaR estimation, whereas the GPD model may require further evaluation. At the 99% confidence level, the conservative nature of the VaR estimates is evident, particularly with the GPD model, which may be too conservative as it predicts no violations.