Mid-term Exam: Stochastic Calculus for Finance

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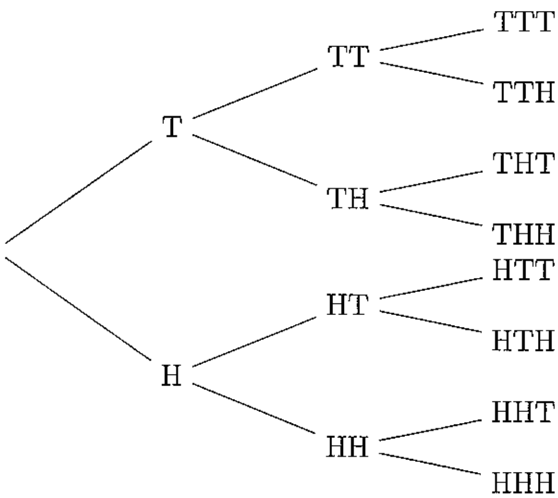
Exercise 1 (5 points)

Consider a coin toss experiment where the outcome can either be Heads (H) or Tails (T). We will flip the coin three times. Below is the tree diagram (not recombinatory) illustrating all possible outcomes of this experiment.



Given this setup, let $X = 2^{\text{number of Heads}} - 2^{\text{number of Tails}}$, where "number of Heads" and "number of Tails" are the exponents to which 2 is raised. Furthermore, define the random variable Y as:

$$Y = \begin{cases} 1 & \text{if the number of Heads} \geq 2 \\ -1 & \text{otherwise} \end{cases}$$

Questions

1. Is Y $\mathcal{F}_2$-measurable, where $\mathcal{F}_2$ is the information obtained after observing the first two outcomes of the coin toss?

2. Prove that Y is $\sigma(X)$-measurable.

3. We assume the coin is fair, meaning the probability of getting Heads or Tails is $\frac{1}{2}$. Prove that $\sigma(X)$ and $\sigma(Y)$ are not independent.

Exercise 2 (3 points)

Consider a simple random walk. It is a process $(S_n)_{n\geq 0}$ where

- $S_0 = 0$ and $S_n = S_{n-1} + X_n$ for $n \geq 1$,

- Each X_n is an independent random variable that takes the value $+1$ with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

Questions

1. Prove that $(S_n)_{n\geq 0}$ is a $\sigma(X)$-martingale, where $X = (X_n)_{n\geq 0}$

2. What is the expected value and variance of S_n? For the variance, you will use that $\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(Y)$ when X and Y are two independent random variable.

3. Now consider a modified random walk $(T_n)_{n\geq 0}$ where $T_0 = 0$ and $T_n = T_{n-1} + Y_n$. Each Y_n takes the value $+2$ with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$. Is $(T_n)_{n\geq 0}$ a martingale, a submartingale, or a supermartingale?

Exercise 3 (2 points)

1. Write the formula for put-call parity. You may use either future values or present values, but be consistent in your notation.

2. A put option is currently selling for \$1, and a call option on that same stock currently costs \$2. One share of the underlying stock is selling for \$5. Both the put option and the call option have a strike price of \$4.05. The risk-free rate of interest is 5%, and both options expire in one year. What can you say about the price of the stock? You will use that $e^{-0,05} = 0,95$.

3. How the stock market should react?

Exercise 4 (5 points)

We suppose that there is no arbitrage opportunity on the market, i.e., (NFL) holds. We denote by B_0 the price at time $t = 0$ of the riskless asset with gain 1 at maturity $t = T$. We also denote by C_0 and P_0 the respective prices at time $t = 0$ of a Call and a Put option on the underlying S with maturity T and strike $K \geq 0$.

1. Prove by an arbitrage argument that $(S_0 - KB_0)^+ \leq C_0 \leq S_0$.

2. Deduce from the previous question that $(KB_0 - S_0)^+ \leq P_0 \leq KB_0$.

3. Prove that the Call option price is a nonincreasing function of the strike K.

4. What can you say about the Put option price as a function of the strike price K?

Exercise 5 (5 points)

Let us take $S_0 = 100$, $r = 0.05$ (discrete rate), and $d = 0.9$, and $u = 1.1$.

1. What is the price and the hedging strategy of a Call option at the money, i.e., $K = S_0 = 100$?

2. What about the Put option at the money?

3. Is the Call-Put parity relation satisfied?

Ex 1.

	X_i	Y_i
TTT	-7	-1
TTH	-2	-1
THT	-2	-1
THH	2	1
HTT	-2	-1
HTH	2	1
HHT	2	1
HHH	7	1

1. no

2. $Y_i = 1 \Leftrightarrow X_i < 0$
   $Y_i = -1 \Leftrightarrow X_i > 0$ $\Leftrightarrow$ Y is $\sigma(X)$ measurable

3. Let Value $A := \{X = 2\}$
   $B := \{Y = 1\}$.

$$\mathbb{P}(A \cap B) = \mathbb{P}(\omega \in \{THH; HTH; HHT\}) = \frac{3}{8}$$

$$\mathbb{P}(A) = \mathbb{P}(\omega \in \{THH; HTH; HHT\}) = \frac{3}{8}$$

$$\mathbb{P}(B) = \frac{1}{2}$$

Hence $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$.

Ex 3.

1. $C_0 - P_0 = S_0 - K B_0$.

2. $S_0 = C_0 - P_0 + K B_0$ (According to 1.)

However $C_0 - P_0 + K B_0 = 4.85 \neq 5 = P_0$.

Hence Stock overvalued.

3) Price of stock will drop on Mkt

Ex 2:

1. ·) $\mathbb{E}[|S_n|] \leq \sum_n \mathbb{E}|X_n| < \infty$

·) S_n - $\mathcal{F}_n$ Measurable with $\mathcal{F}_n = \sigma(X_n)$.

·) $\mathbb{E}[S_n | \mathcal{F}_{n-1}] = \mathbb{E}[S_{n-1} + X_n | \mathcal{F}_{n-1}] = S_{n-1} + \mathbb{E}[X_n | \mathcal{F}_{n-1}]$

with $\mathbb{E}[X | \mathcal{F}_{n-1}] = \mathbb{E}[X] = 0 \Rightarrow$ Martingale

2. $\mathbb{E}[S_n] = \mathbb{E}[\sum X_n] = \sum \mathbb{E}[X_n] = \sum 0 = 0$.

$\operatorname{Var} S_n = \operatorname{Var}(X_1 + X_2 + \dots) = \sum \operatorname{Var} X_i$
$= \sum_{i=1}^{n} 1 = n$.

3. $\mathbb{E}[T_n | \mathcal{F}_{n-1}] = T_{n-1} + \underbrace{\mathbb{E}[Y_n]}_{>0}$

So T_n submartingale

Ex. 4:

1. ∘) Assume $C_0 > S_0$

$C_t = \cancel{\max}(S_t - K)^+$

Hence if $S_t > K$, $C_t = S_t - K < S_t$

if $S_t < K$, $C_t = 0 < S_t$.

Absence of Arb. opportunity $\Rightarrow C_0 < S_0$.

∘) Assume $(S_0 - KB_0)^+ > C_0$.

Because $(S_0 - KB_0)^+ < S_0 - KB_0$.

it means that

$$S_0 - KB_0 > C_0.$$

However $0 > \underbrace{C_0 - S_0 + KB_0}_{= P_0}$.

So it would lead to $0 > P_0$ $\boxed{\text{arbitrage}}$

2. From 1, we have

$$(S_0 - KB_0)^+ \leq C_0 \leq S_0.$$

However, $P_0 = C_0 - (S_0 - KB_0)$

$\Rightarrow$ So $(S_0 - KB_0)^+ - (S_0 - KB_0) \leq P_0 \leq \cancel{S_0} - \cancel{S_0} + KB_0$.

$$(KB_0 - S_0)^+ \leq P_0 \leq KB_0.$$

Ex5:

1.) we need to compute Δ, x

$\bullet$) $q = \dfrac{R-d}{u-d} = \dfrac{1,05-0.9}{0.2} = 0.75$.

$\Delta = \dfrac{C_1^u - C_1^d}{(u-d)S_0} = \dfrac{10}{20} = \dfrac{1}{2}$. $\rightarrow$ Hedging Strat buy $\frac{1}{2}$ shares

$P_0 = \dfrac{q C_1^u + (1-q) C_1^d}{R} = \dfrac{7.5}{1,05} \approx 7.14$ borrow 42.85 in bonds

$P_0 - \Delta S_0 = 7.15 - 50 = -42.85$

2) $\Delta = -\dfrac{1}{2}$ $C_0 = \dfrac{2.5}{1.05} = 2.38$.

Short sell 0.5 shares

invest $2.38 + 50 = 52.38$ in cash

3) $C_0 - P_0 = S_0 - K B_0$.

$C_0 - P_0 = -4,75$

$S_0 - K B_0 = -4.76$ $\Big\} \Rightarrow$ CP parity formula satisfied