场论与凝聚态笔记

mny

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目录

1 Invitation: The Cartoon of Confinement

Never see individual quarks.

For separatable particles, like electron charges, their potential is $V(r) \sim \frac{1}{r}$, thus $V(r) - V(r_0)$ is always bounded.

Two quarks forms a pion. They interacts through gluon, and forms a structure called gluon tube or string. The potential is $V(r) \sim r$ and the energy density per length is appropriately constant. To separate a quark pair, the energy inputed $V(r) - V(r_0)$ is unbounded.

Similar phenomenon appears in superconductor(type II). When electronic charge condensed, the interaction of magneticmonopole becomes $V(r) \sim r$. According to EM duality, when magneticmonopole condensed, analogy goes to its counterpart. (Perspective by t'Hooft, Polyakov and Manldstan.)

2 Path Integral for Single Particles

From the two-slit interference, we've known the picture of wave. Whilst, the view of particle could recover the result by computing the phase $\mathrm{e}^{\mathrm{i}S}$.

For single particle mechanic, we start from the Schrödinger equation

$$\mathrm{i}\hbar\partial_t \left|\psi(t)\right\rangle = H(p,x,t)\left|\psi(t)\right\rangle . \tag{2.1}$$

We have the time evolution operator

$$\left|\psi(t)\right\rangle = U(t,t_0)\left|\psi(t)\right\rangle , \tag{2.2}$$

which is unitary,

$$U^\dagger(t,t_0)U(t,t_0) = 1. \tag{2.3}$$

If H is time-dependent, we split the time interval into small slices, and we get the infinitesimal U operator as time-independent cases,

$$U(t,t_0) = \prod_{n=0}^{N-1} U(\overbrace{t_{n-1},t_n}^{\delta t}), \tag{2.4}$$

where $t_n \equiv t + n\delta t$. Note that the product is time ordered.

Another perspective is from the Schrödinger equation, by finite differential,

$$\left|\psi(t+\delta t)\right\rangle = \left[1 - \frac{\mathrm{i}}{\hbar}H(p,x,t+\frac{\delta t}{2})\right]\left|\psi(t)\right\rangle \tag{2.5}$$

To the order of δt, we have

$$\left|\psi(t+\delta t)\right\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(p,x,t+\frac{\delta t}{2})}\left|\psi(t)\right\rangle \tag{2.6}$$

Next, we put the time evolution operator in spacial basis, considering

$$\left\langle x'\right| U(t+\delta t,t)\left|x\right\rangle . \tag{2.7}$$

Suppose $H = \frac{p^2}{2m} + V(x)$ for simplicity, we obtain

$$\left\langle x'\right| \left[1 - \frac{\mathrm{i}\delta t}{\hbar}\left(\frac{p^2}{2m} + V(x,t+\frac{\delta t}{2})\right)\right]\left|x\right\rangle . \tag{2.8}$$

Make a substitution

$$V \to \frac{V(x',t+\frac{\delta t}{2}) + V(x,t+\frac{\delta t}{2})}{2}1, \tag{2.9}$$

and insert a completeness relation of p in each time slice, we arrive at

$$\langle x'|\, U(t+\delta t,t)\,|x\rangle = \int \mathrm{d}p\,\frac{1}{2\pi\hbar}\mathrm{e}^{\frac{\mathrm{i}p(x'-x)}{\hbar}}\exp\left[-\frac{i\delta t}{\hbar}\frac{H(p,x',t+\frac{\delta t}{2})+H(p,x,t+\frac{\delta t}{2})}{2}\right]. \tag{2.10}$$

written in a more compact form,

$$\langle x'|\, U(t+\delta t,t)\,|x\rangle \sim \int \mathrm{d}p\,\frac{1}{2\pi\hbar}\mathrm{e}^{\mathrm{i}\frac{p\delta x}{\hbar}-\mathrm{i}\frac{H\delta t}{\hbar}}. \tag{2.11}$$

The finite-time evolution operator,

$$U(t_N,t_0)=\prod_{n=0}^{N-1}U(t_{n-1},t_n) \tag{2.12}$$

inserting an identity operator as x basis completeness relation, the element is

$$U(t_{n+2},t_{n+1})\underbrace{1}_{\int \mathrm{d}x_n\,|x_n\rangle\langle x_n|}U(t_{n+1},t_n) \tag{2.13}$$

then we obtain

$$\begin{aligned}\langle x_N|\, U(t_N,t_0)\,|x_0\rangle = &\left(\prod_{n=1}^{N-1}\int \mathrm{d}x_n\ \langle x_{n+1}|\, U(t_{n+1},t_n)\,|x_n\rangle\right)\\ &\times \langle x_1|\, U(t_1,t_0)\,|x_0\rangle\end{aligned} \tag{2.14}$$

in full,

$$\begin{aligned}\langle x_N|\, U(t_N,t_0)\,|x_0\rangle &= \left(\prod_{n=1}^{N-1}\int\frac{\mathrm{d}p_{n+\frac{1}{2}}\,\mathrm{d}x_n}{2\pi\hbar}\right)\int\frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar}\\ \times\exp&\left(\frac{\mathrm{i}}{\hbar}\sum_{n=0}^{N-1}\left[p_{n+\frac{1}{2}}(x_{n+1}-x_n)-\delta t\frac{H(p_{n+\frac{1}{2}},x_{n+1},t_{n+\frac{1}{2}})+H(p_{n+\frac{1}{2}},x_n,t_{n+\frac{1}{2}})}{2}\right]\right)\\ &\sim\left(\prod_{n=0}^{N-1}\int\frac{\mathrm{d}p_{n+\frac{1}{2}}\,\mathrm{d}x_n}{2\pi\hbar}\right)\int\frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int p\,\mathrm{d}x-H\,\mathrm{d}t}\end{aligned} \tag{2.15}$$

Insert a operator $\hat{B}(x)$ in between the path integral,

$$\begin{aligned}\langle x_N|\, U(t_N,t_m)\hat{B}(x)U(t_m,t_0)\,|x_0\rangle &= \left(\prod_{n=1}^{N-1}\int\frac{\mathrm{d}p_{n+\frac{1}{2}}\mathrm{d}x_n}{2\pi\hbar}\right)\int\frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar}B(x_m)\\ \times\exp\frac{\mathrm{i}}{\hbar}\sum_{n=0}^{N-1}&\left[p_{\frac{1}{2}}(x_{n-1}-x_n)-\delta t\frac{H(p_{n+\frac{1}{2}},x_{n+1},t_{n+\frac{1}{2}})+H(p_{n+\frac{1}{2}},x_n,t_{n+\frac{1}{2}})}{2}\right].\end{aligned} \tag{2.16}$$

We simply need to add the value of the operator as a function of certain space coordinate in the expression of path integral.

3 Observables in QM

- $$\langle\psi| A |\psi\rangle\,,\ A^\dagger = A. \tag{3.1}$$

- Probability of projection,

$$P = |\,\langle\phi|\psi\rangle\,|^2 = \langle\psi|\underbrace{|\phi\rangle\langle\phi|}_{\hat{A}}|\psi\rangle \tag{3.2}$$

- Observables after evaluation,

$$\langle\phi|\underbrace{\mathrm{e}^{\mathrm{i}Ht/\hbar}A\mathrm{e}^{-iHt/\hbar}}_{\hat{A}(t)}|\psi\rangle \tag{3.3}$$

- Projection after evaluation: scattering

$$P = |\,\langle\psi|\,\mathrm{e}^{-\mathrm{i}Ht/\hbar}\,|\psi\rangle\,|^2 = \langle\psi|\underbrace{\mathrm{e}^{\mathrm{i}Ht/\hbar}\,|\phi\rangle\langle\phi|\,\mathrm{e}^{-iHt/\hbar}}_{\hat{A}(t)}|\psi\rangle \tag{3.4}$$

- Retarded correlation

$$H(t) = H_0 + \underbrace{a(t)}_{\text{small}}B. \tag{3.5}$$

  The contribution of $\langle\psi|\,U^\dagger(t,0)AU(t,0)\,|\psi\rangle$ to the first order correction in a is

$$-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t' a(t')\Big[\langle\psi|\,U_0^\dagger(t,0)AU_0(t,t')BU_0(t',0) - U_0^\dagger(t,0)BU_0(t,t')AU_0(t',0)\,|\psi\rangle\Big] \tag{3.6}$$

4 Path Integrals for Fields

There's a mattress with springs and massive balls connected. Denoting the offset of each ball as $\phi_{\vec{r}}$ the Hamiltonian is

$$H = \sum_{\vec{r}\text{ on lattice}}\left[\frac{p_{\vec{r}}^2}{2m} + V(\phi_{\vec{r}}) + \sum_{\hat{\tau}=1}^{d}\frac{k}{2}\left(\phi_{\vec{r}+\alpha\vec{\tau}} - \phi_{\vec{r}}\right)^2\right], \tag{4.1}$$

with a commutation relation

$$[\phi_{\vec{r}}, p_{\vec{r}'}] = \mathrm{i}\hbar\delta_{\vec{r},\vec{r}'}. \tag{4.2}$$

The interacting part of the Hamiltonian only involves pairs nearby.

In continuum limit, $H = \int \mathrm{d}^d\vec{r}\mathcal{H}(\vec{r})$, and we can write the Hamilton density as

$$\mathcal{H} = \frac{\pi(\vec{r})^2}{2\rho} + \mathcal{V}(\phi(\vec{r})) + \frac{\kappa}{2}\left[\partial_{\vec{r}}\phi(\vec{r})\right]^2, \tag{4.3}$$

where $\pi(\vec{r}) = \frac{p_{\vec{r}}}{\alpha^d}, \mathcal{V} = \frac{V}{\alpha^d}, \rho = \frac{m}{\alpha^d}, \kappa = \frac{k\alpha^2}{\alpha^d}$.

Path Integral At t_n, we use $\bigotimes_{\vec{r}}|\phi(\vec{r})\rangle$ basis, and $\bigotimes_{\vec{r}}|\pi(\vec{r})\rangle$ for $t_{n+\frac{1}{2}}$.

Analogously, we get

$$\begin{aligned}\langle\mathrm{end}|\,U(t,0)\,|\mathrm{start}\rangle = &\left(\prod_{t_n,\vec{r}}\frac{\mathrm{d}p_{t_{n+\frac{1}{2}},\vec{r}}\;\mathrm{d}\phi_{t_n,\vec{r}}}{2\pi\hbar}\right)_{\text{suitable boundary condition}}\\ \times\exp\frac{\mathrm{i}}{\hbar}\sum_{t_n,\vec{r}}&\left[p_{t_{n+\frac{1}{2}},\vec{r}}\left(\phi_{t_{n+1},\vec{r}}-\phi_{t_n,\vec{r}}\right)-\delta t\left(\frac{p^2_{n+\frac{1}{2},\vec{r}}}{2m}+\frac{k}{2}\sum_{\hat{\tau}=1}^{d}\left(\phi_{t_n,\vec{r}+\alpha\hat{\tau}}-\phi_{t_n,\vec{r}}\right)^2+V(\phi_{t_n,\vec{r}})\right)\right].\end{aligned} \tag{4.4}$$

Integrate out p, we obtain

$$\begin{aligned}&\left(\prod_{t_n,\vec{r}}\sqrt{\frac{2\pi\hbar m}{\mathrm{i}\delta t}}\frac{\mathrm{d}\phi_{t_n,\vec{r}}}{2\pi\hbar}\right)_{\text{suitable boundary condition}}\\ \times\exp\frac{i\delta}{\hbar}\sum_{t_n,\vec{r}}&\left[\frac{m}{2}\left(\frac{\phi_{t_{n+1},\vec{r}}-\phi_{t_n,\vec{r}}}{\delta t}\right)-\frac{k\alpha^2}{2}\sum_{\hat{\tau}=1}^{d}\left(\frac{\phi_{t_n,\vec{r}+\alpha\hat{\tau}}-\phi_{t_n,\vec{r}}}{\alpha}\right)^2-V(\phi_{t_n,\vec{r}})\right],\end{aligned} \tag{4.5}$$

in which time and space stand in the same place, similar to relativistic K-G field.

5 Free Field Theory

For free fields, $V(\phi_{\vec{r}}) = \frac{u}{2}\phi^2_{\vec{r}}$, it behaves just like coupled simple harmonic oscillators. To find the normal modes, we use Fourier transformation.

$$\phi_{\vec{r}} = \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}}\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\phi_{\vec{k}}, \tag{5.1}$$

likewise for $p_{\vec{r}}$.

$\phi_{\vec{r}}$ is real, thus $\phi_{\vec{r}} = \phi^\dagger_{\vec{r}} \implies \phi_{-\vec{k}} = \phi^\dagger_{\vec{k}}$, and the commutation relation is

$$\left[\phi_{\vec{k}}, p_{\vec{k}'}\right] = \mathrm{i}\hbar(2\pi)^d\delta^d(\vec{k}+\vec{k}'). \tag{5.2}$$

The Hamiltonian becomes

$$H = \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \frac{1}{2} \left[\frac{p_{-\vec{k}} p_{\vec{k}}}{2m} + \left(\frac{k}{2} \sum_{\vec{\tau}=1}^{d} \left(2 \sin \frac{\alpha k_i}{2} \right)^2 + \frac{u}{2} \right) \phi_{-\vec{k}} \phi_{\vec{k}} \right], \tag{5.3}$$

from which we can directly figure out the frequency $\omega_{\vec{k}} = \omega_{-\vec{k}} = \sqrt{\frac{k \sum_{\vec{\tau}} \left(\sin \frac{\alpha k_i}{2} \right)^2 + U}{m}}$

Next, we need to make a substitution, or Bogoliubov transformation:

$$\phi^c_{\vec{k}} = \frac{\phi_{\vec{k}} + \phi_{-\vec{k}}}{\sqrt{2}}, \tag{5.4}$$

$$\phi^s_{\vec{k}} = \mathrm{i} \frac{\phi_{\vec{k}} - \phi_{-\vec{k}}}{\sqrt{2}}. \tag{5.5}$$

the integration range becomes half of $\vec{k}$.

5.1 Ground State Wave Function and Entanglement

5.2 Energy Gap and Corelation