费曼物理学 (3) 笔记

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第一章　Quantum Mechanic Behaviors

1.1 Two-Slit Interference of Electrons

Both open: $N \neq N_1(x) + N_2(x)$. In reality, it turns out that the pattern is something like

$$N(x) = N_1(x) + N_2(x) + g(x) \sin\left[\omega(x)x\right], \tag{1.1.1}$$

where the last term is a interference term, which satisfies a slow changing condition

$$\frac{1}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}x} \ll \omega, \quad \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}x} \ll \omega. \tag{1.1.2}$$

If we lower the power of electron source so that it emits each electron one by one, thus interactions between electrons will not functional, however, the result o f the experiment recovers. This shows us that the statistical pattern isn't resulted by many-body interactions. So, we come to a ridiculous conclusion, electron must interact with itself passing both hole simultaneously.

Next we build a which-way detector, using a light source for observing whether an electron has passed a hole. The pattern disappears! When we lower the power or enlarging the wavelength, the pattern re-appear gradually.

We have to admit that it is impossible to design an apparatus to determine which hole the electron passes through, that will not at the same time disturb the electrons enough to destroy the interference pattern. That is, **Heisenberg's uncertainty principle**. Hence, in quantum mechanic, we can only make predictions of probability.

Remark that,

1. The evolution of quantum states is **definite** (either by Schrödinger equation or something else).

2. The uncertainty only appears in observations.

3. Quantum mechanic is an **extremely accurate** theory.

第二章　Wave-Particle Duality

When we perform different experiments, electrons behaves differently. The word **duality** was used when we can not obtain a universally description. The concept **state** was invented and complex number was introduced.

We use ψ_1 and ψ_2 to describe the complex amplitude of hole 1 opened and hole 2 opened respectively. Add up the two terms and the tensity is

$$|\psi(x)|^2 = |\psi_1(x) + \psi_2(x)|^2 = |\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^* + \psi_1^*\psi_2. \tag{2.0.1}$$

In the formula above we have implicitly utilized the **Born rule** of probability.

All possible quantum states forms a space, in which some looks like waves and some like particles.

Plank has put forward that $E = \hbar\omega$ and he believes that this property appears only when light interacts with other materials. While Einstein supposed that this it a inner property of light when dealing with photoelectric effect.

We may notice that

$$p^\mu = (E, \vec{p}), \quad k^\mu = (\omega, \vec{k}), \tag{2.0.2}$$

are all Lorentz four vectors, thus $\vec{p} = \hbar\vec{k}$.

2.1 Fourier Transformation

A wave mode with definite $\vec{k}$, we have $\psi_{\vec{k}}(\vec{x}) \sim \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$. For an arbitrary wave packet within some mathematical restriction, it can be written as

$$f(\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k}). \tag{2.1.1}$$

The inverse transformation is

$$\tilde{f}(\vec{k}) = \int \mathrm{d}^3\vec{x} \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} f(\vec{x}). \tag{2.1.2}$$

This is guaranteed by the orthogonal-normalization of plane waves,

$$\begin{aligned}\int \mathrm{d}^3\vec{x}\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}}\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} &= (2\pi)^3\delta^{(3)}(\vec{k}-\vec{q}),\\ \int \frac{\mathrm{d}^3\vec{k}}{2\pi}\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{y}} &= (2\pi)^3\delta^{(3)}(\vec{x}-\vec{y}).\end{aligned} \tag{2.1.3}$$

The Dirac δ function satisfies

$$\delta(x) = 0 \quad \text{if } x \neq 0, \tag{2.1.4}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x) = 1 \tag{2.1.5}$$

$$\Longrightarrow\ \forall f(x),\ \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-y)f(x) = f(y). \tag{2.1.6}$$

2.2 Gaussian Wave Packet

For Gaussian wave packet, $\psi(\vec{x}) = \mathrm{e}^{-\frac{x^2}{4\sigma^2}}$, we would find out how many component of wave-number $\vec{k}$ does it contains.

$$\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\mathrm{e}^{-\frac{|\vec{x}|^2}{4\sigma^2}} = \int \mathrm{d}x\, \mathrm{e}^{\frac{-\left(x+\mathrm{i}2k\sigma^2\right)^2}{4\sigma^2}}\mathrm{e}^{-k^2\sigma^2}. \tag{2.2.1}$$

It's still Gaussian in frequency space, and $\sigma_k = \frac{1}{2\sigma}$, thus $\Delta x\Delta k = \frac{1}{2}$, i.e.,

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}. \tag{2.2.2}$$

2.3 Hydrogen Atom

Assuming the electron goes in a circle trajectory with diameter a, according the uncertainty principle, we can write the hamiltonian,

$$E = \frac{p^2}{2m} - \frac{e^2}{a} = \frac{\hbar^2}{2ma^2} - \frac{e^2}{a}, \tag{2.3.1}$$

thus it have a minimum at which $a \neq 0$, the result is $a = \frac{\hbar^2}{me^2} \sim 0.5\,\mathring{A}, E = -13.6eV$

For bounded states, the possible energy levels are always discrete, when it transit from a higher level E_1 to a lower E_2, it emits a photon with frequency $\omega = \frac{E_1 - E_2}{\hbar}$

2.4 The Philosophy of Quantum Mechanic

The observables are the numbers that can be measured in experiments. Physicist works for finding the numerical relations under the observables. The mission of physics is to explain the phenomena of observables qualificationally. There's no need to debate on what is the entity of something, or which conception is more fundamental.

第三章　Probability Amplitude

The superposition law of quantum mechanic imply that there lies a structure of linear algebra beneath the description of quantum mechanic, from which, Schrödinger developed the Wave Mechanic, Heisenberg the Matrix Formalism, and Feynman the Path Integral Methodology.

When we are asked about the probability of a certain process, we compute the magnitude squared of a complex number, that is, *probability amplitude*. This gives the first law

$$\text{Probability} = |\text{amplitude}|^2. \tag{3.0.1}$$

Dirac introduced his notation $\langle A|B\rangle$, which means the amplitude of transferring from the initial state A to a final process B.

The second law

$$\langle B|A\rangle = \langle B|A\rangle_{\text{path 1}} + \langle B|A\rangle_{\text{path 2}}\,. \tag{3.0.2}$$

The third law

$$\langle B|A\rangle_{\text{path 1}} = \langle B|1\rangle\langle 1|A\rangle\,. \tag{3.0.3}$$

Suppose a M-fold $\{H_i\}$ hole interference of electron, using the notation of j_i to represent the j hole of plate i.We can write the amplitude

$$\langle B|A\rangle = \sum_{j_M=1}^{H_M}\sum_{j_{M-1}=1}^{H_{M-1}}\cdots\sum_{j_1=1}^{H_1}\langle B|(j_M)_M\rangle\langle (j_M)_M|(J_{M-1})_{M-1}\rangle\times\cdots\times|(j_1)_1\rangle\langle (j_1)_1|A\rangle\,. \tag{3.0.4}$$

Or, in continuum form,

$$\langle B|A\rangle = \sum_{\text{all paths}}\langle B|A\rangle_{\text{a certain path}}\,. \tag{3.0.5}$$

To some extend, this is quite similar to the path integral, ignoring the fact that we choose $y(x)$ instead of $\vec{r}(t)$ to be the integration variable, which led us to be unable to include the paths in which a electron turns back.

Let us go back to the two-slit interference experiment and consider why observation influences the pattern. There are two detectors, D_1 and D_2, setting up closing to hole 1 and 2, using u to denote the amplitude of electrons passing hole 1 and kicking the photon into D_1, and v for hole-2-electrons kicking photons into hole 1.

Then the amplitude of electrons from A to B through hole 1 is

$$\langle B|A\rangle_{\text{photon to } D_1} = \langle B|1\rangle\, u\, \langle 1|A\rangle + \langle B|2\rangle\, v\, \langle 2|A\rangle\,. \tag{3.0.6}$$

In a valid measurement, which means $|u| \gg |v|$, there will be no interference term $\langle B|1\rangle\, v\, \langle 1|A\rangle\, \langle B|2\rangle\, v\, \langle 2|A\rangle$ in the probability.

Measurement caused the decoherence of electrons, and thus pattern disappears.

3.1 Path Integral

For non-relativistic cases, we can define synchronousness, and any particle cannot go backward in time, the amplitude can be written as

$$\langle B(t_2)|A(t_1)\rangle = \sum_{\text{all paths}} \langle B(t_2)|A(t_1)\rangle_{\text{a certain path}}\,. \tag{3.1.1}$$

Note that the amplitude is a complex function of path, when finding how to calculate its value, we need to go back to the classical situation. Feynman gives the result

$$\langle B(t_2)|A(t_1)\rangle_{\text{a certain path}} = \mathrm{e}^{\mathrm{i}S/\hbar}. \tag{3.1.2}$$

We can write in ths form,

$$\langle B(t_2)|A(t_1)\rangle = \int \mathcal{D}x(t)\, \mathrm{e}^{\mathrm{i}S[x(t)]}. \tag{3.1.3}$$

3.1.1 Single Particle Mechanics

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \tag{3.1.4}$$

3.1.2 Electromagnetic Field

Any field configuration can be described by a four-vector field.

$$A_\mu(t, \vec{x}) = (\phi, \vec{A}) \tag{3.1.5}$$

The gauge invariant strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.1.6}$$

The Lagrangian of EM field is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + e j_\mu A^\mu. \tag{3.1.7}$$

We can get the Maxwell equation by variation,

$$-\partial^\nu F_{\mu\nu} + e j_\mu = 0. \tag{3.1.8}$$

3.1.3 Regularization and Renormalization

Some (perhaps most) path integrals are not well defined and we need to discretize the spacetime, or explicitly, describe the divergence in our theory. This process is called *regularization*.

However, the experiment results should have no business of the way of regularization. When we take all factors into consideration, we should get a relationship of the observables, independent of our way of regularization, which is called *Renormalization*.

3.1.4 Verify of Path Integral

Any quantum theory should satisfy two conditions, giving out the right evaluation and going back to classical mechanic when $\hbar \to 0$.

3.2 Neutron Scattering in Crystal Lattice

When we get a interference pattern on the observation screen, surprisingly we find out that except normal spiculate peaks, there are a homogeneous background in some kinds of crystal.

We may write the amplitude of this scattering process on the ith point

$$\langle B|i\rangle \, S \, \langle i|A\rangle \,, \tag{3.2.1}$$

where the S is the scattering amplitude.

Given that a quantum degree called *spin* exists in both neutrons and atoms, several situations are we now faced up with. Scattering by nuclear forces may cause spin flip even if energy is low.

1. All *in* and *out* neutrons are spin-paralleled with atoms.

$$\langle B|A\rangle = \sum_i \langle B|i\rangle \, S \, \langle i|A\rangle \, . \tag{3.2.2}$$

2. All *in* and *out* neutrons are spin-anti-paralleled with atoms. This is the same as former.

3. A non-trivial situation is that when a neutron and atom suffered spin flip in the process, final states are not the same and thus we need to sum up the probability **not the amplitude**.

$$P = \sum_i |\langle B|i\rangle \, R \, \langle i|A\rangle|^2 \, . \tag{3.2.3}$$

   There's no cross term, i.e., interference disappears.

   Another explanation is, when spin flips, we can determine which lattice point the neutron passed. It becomes a which-way detector effectively, similar to electron two-slit interference!

3.3 Identical Particles

In Rutherford scatter experiment, for instance, α particle collide with oxygen atom. We use a detector which can not distinguish α particles and oxygen atoms. In the center-of-mass reference frame, the probability is

$$P = |f(\theta)|^2 + |f(\pi+\theta)|^2. \tag{3.3.1}$$

This result does not hold on when we do the experiment of $\alpha - \alpha$ scattering. The final state are identical, we cannot label the α particles, thus,

$$P = \left|f(\theta) + \mathrm{e}^{\mathrm{i}\delta} f(\pi+\theta)\right|^2 . \tag{3.3.2}$$

第四章　Identical Particles

4.1 Spin-Statistic Relation

At the end of last chapter, we discussed the effect of identity particle in scattering process. The additional phase factor comes from exchanging the pair of identical particles. We find that either $\mathrm{e}^{\mathrm{i}\delta} = 1$ or $\mathrm{e}^{\mathrm{i}\delta} = -1$, we can sort particles into two categories according to this factor, $+1$ called **Boson**, -1 called **Fermion**.

Now we consider scattering experiment by electrons, note that

$$P_{\uparrow\uparrow} = P_{\downarrow\downarrow} = |f(\theta) - f(\pi + \theta)|^2 \tag{4.1.1}$$

$$P_{\uparrow\downarrow} = P_{\downarrow\uparrow} = |f(\theta)|^2 + |f(\theta + \pi)|^2 \tag{4.1.2}$$

In this experiment, we use a unpolarized source of electron (for instance, a electron tube), then

$$P_1 = \frac{1}{4}\left(P_{\uparrow\uparrow} + P_{\downarrow\downarrow} + P_{\uparrow\downarrow} + P_{\downarrow\uparrow}\right). \tag{4.1.3}$$

Why there are only two possible factor We have two particles in coordinate eigenstate, we now exchange them two times and this operation is effectively an identity operation and thus we have

$$\left(\mathrm{e}^{\mathrm{i}\delta}\right)^2 = 1. \tag{4.1.4}$$

Hence, we proofed the formal statement.

However, we claimed that, "exchange equals to no operation". We have to note that a close path in 3D space is trivial in topology.

4.2 Multi-Particle State

We now consider an experiment with two sources, A and B, emitting particles and there are two detectors, 1 and 2. Denote the amplitude of a particle going from A to 1 by a_1, and so on.

Denote the probability of the two detectors both receiving particles by P.

For identical ones,

$$P = |a_1|^2 |b_2|^2 + |a_2|^2 |b_2|^2 . \tag{4.2.1}$$

For non-identical,

$$P = |a_1 b_2 + a_2 b_1|^2 . \tag{4.2.2}$$

If we move the two detectors close to each other in the limit $1 \to 2$, we will find a difference.

$$P_{\text{identical}} = 2P_{\text{non-identical}}. \tag{4.2.3}$$

4.2.1 Final State Phase Space Integral

When discussing the probability of getting an initial state at θ, we are actually talking about the probability per unit solid angle $\mathrm{d}\Omega = \mathrm{d}\cos\theta \mathrm{d}\phi$.

The probability of detector 1 receiving the particle is

$$P = \int_{S_1} \mathrm{d}S_1 \, |a_1|^2 = |a_1|^2 \Delta S. \tag{4.2.4}$$

Now we repeat the experiment mentioned before.

For different-type particles,

$$P = \int_{S_1} \mathrm{d}S_1 \int_{S_2} \mathrm{d}S_2 \left(|a_1 b_2|^2 + |a_2 b_1|^2 \right) \simeq \left(|a_1 b_2|^2 + |a_2 b_1|^2 \right) \Delta S_1 \Delta S_2 \tag{4.2.5}$$

When we take the limit, we have two divide the measurement by 2 for the overlap of detectors,

$$P = \int_{S_1} \frac{\mathrm{d}S_1 \, \mathrm{d}S_2}{2} \left(|a_1 b_2|^2 + |a_2 b_1|^2 \right). \tag{4.2.6}$$

For identical particles,

$$P = \int_{S_1} \frac{\mathrm{d}S_1 \, \mathrm{d}S_2}{2} \left(|a_1 b_2 + a_2 b_1|^2 \right) \tag{4.2.7}$$

final state of n-Bosons

$$P = \int_{S_1} \frac{\mathrm{d}S_1 \cdots \mathrm{d}S_n}{n!} \left(\left| a_1^{(1)} a_2^{(2)} \cdots a_n^{(n)} + \text{all permutations of subscripts} \right| \right) \tag{4.2.8}$$

$$= n! \left| a_1^{(1)} a_2^{(2)} \cdots a_n^{(n)} \right|^2 (\Delta S_1)^n \tag{4.2.9}$$

4.2.2 Black Body Radiation

We now turn to study a box of photon gas in thermal equilibrium. Photons (almost) do not interact with each other, and when they reach equilibrium, they must interact with atoms on the box.

We choose an atom model which can absorb and emit photons, with the amplitude of a and a^*

We would like to ask the question that what is the number of photons in $[\omega, \omega + \mathrm{d}\omega]$ emitted per unit time.

Assume that the expected photon number on a given state (ω) is $\bar{n}(\omega)$. Then, the amplitude of an atom becoming excited is

$$A_{n \text{ to } n+1} = \sqrt{n+1}a. \tag{4.2.10}$$

The amplitude for de-excitation is

$$A_{n+1 \text{ to } n} = \sqrt{n}a^*. \tag{4.2.11}$$

Given that the atoms are in Boltzmann distribution,

$$N_e = \mathrm{e}^{-\beta(E_e - E_g)} N_g. \tag{4.2.12}$$

When photon gas is in thermal equilibrium, the radiation rate must equal to the absorption rate. And we have

$$\Gamma_r = (n+1)|a|^2 N_e,\ \Gamma_a = n|a|^2 N_g. \tag{4.2.13}$$

Then, by $\Gamma_r = \Gamma_a$, we get

$$\mathrm{e}^{-\frac{\hbar\omega}{kT}} = \frac{N_e}{N_g} = \frac{n}{n+1}. \tag{4.2.14}$$

The result is

$$\bar{n}_\omega = \frac{1}{\mathrm{e}^{\frac{\hbar\omega}{kT}} - 1}. \tag{4.2.15}$$

By consider a box of photon gas, we can get the state density

$$\mathrm{d}N = V\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}, \tag{4.2.16}$$

and it should be multiplied by 2, for two degree of polarize.

We now arrive at the photon number per unit frequency interval

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{\hbar\omega}{\mathrm{e}^{\frac{\hbar\omega}{kT}} - 1} \cdot \frac{V\omega^2}{\pi^2 c^3}. \tag{4.2.17}$$

We now try to quantize the electromagnetic field. We will begin from a one-dimensional scalar field theory on a space interval L.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\phi(t,x)=0. \tag{4.2.18}$$

we can guess the solution of this equation by writing down the Fourier component,

$$\ddot{\phi}_k(t)+\omega^2\phi_k=0, \tag{4.2.19}$$

which is the motion equation of a harmonic oscillator! The average total energy of photons on ω is

$$\langle E\rangle=\frac{\hbar\omega \mathrm{e}^{-\frac{\hbar\omega}{kT}}+2\hbar\omega \mathrm{e}^{-\frac{2\hbar\omega}{kT}}+\cdots}{1+\mathrm{e}^{-\frac{\hbar\omega}{kT}}+\mathrm{e}^{-\frac{2\hbar\omega}{kT}}+\cdots}=\frac{\hbar\omega}{\mathrm{e}^{-\frac{\hbar\omega}{kT}}-1} \tag{4.2.20}$$

The non zero vacuum energy will contribute a force, that is, Casimir effect.

第五章　Spin-1

Stern-Gerlach 实验.

$$\vec{S}^2 = \hbar^2 s(s+1),\ s = 0 \text{ or } \frac{1}{2} \text{ or } 1 \text{ or } \cdots \tag{5.0.1}$$

$$S_z = m\hbar,\ m = -s, -(s-1), \ldots, s \tag{5.0.2}$$

下面我们考虑一个实验, 把一束银原子用磁场分成 z 方向自旋不同的三束, 再合并起来. 我们可以选择用挡板挡住其中两束, 得到一个 projector, 或者什么也不做, 得到一个 identity.

我们用 $|iS\rangle$ 来标记 S 方向自旋为 i 的粒子. orthonormal relation:

$$\langle i|j\rangle = \delta_{ij} \tag{5.0.3}$$

Then, the effect of projector and identity can be written as

$$I : \sum_{k=-1}^{+1} |kT\rangle \langle kT| = 1 \tag{5.0.4}$$

completeness relation:

$$\sum_{j=-1}^{+1} |j\rangle \langle j| = 1. \tag{5.0.5}$$

我们最后可以发现 Dirac bracket 是一种带有内积的向量.

第六章　Spin-$\frac{1}{2}$

A system with spin-$\frac{1}{2}$ is a two-state system. And we may call it a qubit. The two states are usually denoted as $|\uparrow\rangle$ and $|\downarrow\rangle$. It is a building block of any spin system.

6.1 Combination of rotation

Rotation is a form of transformation, under which the distance of 3D space is invariant. We only consider the rotations that can go to the identity transformation continuously(parity transformation is excluded).

We use R to denote a rotation operation. The combination is R_1R_2, R_2 first and R_1 latter. The composite of two rotations is not commutable. We now need to find a map from the rotation to a matrix, such that the composite of two rotations is the product of the corresponding matrices. This is called a representation of the rotation.

$$D_{ij}(R) = D_{ij}(R_2R_1) = \sum_{k=1}^{3} D_{ik}(R_2) D_{kj}(R_1). \tag{6.1.1}$$

where D_{ij} is the matrix element of the representation. We have implicitly assumed that the representation is in 3-dimension.

The D matrix is orthogonal, i.e. $D^TD = I$ and the determinant of D is 1.All D matrix forms a group called $SO(3)$.

6.2 Projection Representation

A physical state is determined up to a uncertain phase, so the representation should satisfy

$$D_{ij}(R_2R_1) = \mathrm{e}^{\mathrm{i}\phi(R_2,R_1)} \sum_{k} D_{ik}(R_2) D_{kj}(R_1). \tag{6.2.1}$$

We will use the convention of Euler angle. The rotation is a combination of three rotations,

$$R(\gamma,\beta,\alpha)=R_z(\gamma)R_x(\beta)R_z(\alpha). \tag{6.2.2}$$

Now, in order to find the representation matrix, we will choose a basis for convenient. The basis is chosen as the eigenstates of S_z, i.e. $|\uparrow\rangle$ and $|\downarrow\rangle$. We know that a rotation along z-axis shouldn't change the state, thus it can only bring a phase factor. Denote $C'_j=\langle jT|\phi\rangle$, $C_j=\langle jS|\phi\rangle$, then we have

$$\begin{cases}\left|C'_+\right|=|C_+|\\ \left|C'_-\right|=|C_-|.\end{cases} \tag{6.2.3}$$

$$\Longrightarrow\ C'_+=\mathrm{e}^{\mathrm{i}\lambda}C_+,\ C'_-=\mathrm{e}^{\mathrm{i}\mu}C_-. \tag{6.2.4}$$

Eliminating a phase factor of the whole system, we have

$$C'_\pm=\mathrm{e}^{\pm\mathrm{i}\frac{\lambda-\mu}{2}}C_\pm. \tag{6.2.5}$$

Now we redefine $\frac{\lambda-\mu}{2}\to\lambda$. According to the product rule of the representation matrix, we have

$$\mathrm{e}^{\mathrm{i}\lambda(\phi_1)}\mathrm{e}^{\mathrm{i}\lambda(\phi_2)}=\mathrm{e}^{\mathrm{i}\lambda(\phi_1+\phi_2)}. \tag{6.2.6}$$

thus, λ depend on ϕ linearly. And $\lambda(\phi=0)$ should be zero.

$$\Longrightarrow\ \lambda=m\phi,\ m\in\mathbb{R}. \tag{6.2.7}$$

Notice that the state should come back to itself after a 2π rotation, and shouldn't go back at π, thus

$$m=\frac{1}{2}. \tag{6.2.8}$$

Notice the relation between z-rotation and x-rotation,

$$R_x(\beta)=R_y\left(-\frac{\pi}{2}\right)R_z(\alpha)R_y\left(\frac{\pi}{2}\right). \tag{6.2.9}$$

We now have to figure out the $\frac{\pi}{2}$ rotation along y-axis. We know that

$$D[R_z(\phi)]=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\frac{\phi}{2}} & \\ & \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\end{pmatrix} \tag{6.2.10}$$

Consider a rotation of π along y-axis, we have

$$C'_+=\mathrm{e}^{\mathrm{i}\beta}C_-,\ C'_-=\mathrm{e}^{\mathrm{i}\gamma}C_+. \tag{6.2.11}$$

meanwhile, for rotation of 2π along y-axis, which should equivalent to identity transformation, we have

$$\beta + \gamma = 2\pi n,\ n \in \mathbb{Z}. \tag{6.2.12}$$

we define that $\gamma \equiv 0$, then

$$D\left[R_y\left(\pi\right)\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.2.13}$$