Homework for Week 1

Topics in Field Theory and Condensed Matter – Fall 2023

due September 26

1. Path Integral without $\int dp$

For $H = \vec{P}^2/2m + V(\vec{X})$, write down the standard path integral with $\int d\vec{p}$ and $\int d\vec{x}$. How to obtain an exactly equivalent path integral with $\int d\vec{x}$ only?

For a more general Hamiltonian, this cannot be done exactly. Usually the saddle point approximation is used, i.e. substitute for $\vec{P} = (\cdots)$ using the classical equation of motion (this is only exact in the case considered above). Try this for a relativistic particle $H = \sqrt{(\vec{P}c)^2 + (mc^2)^2}$.

What happens to a relativistic particle with $m = 0$? (And what is an example of such a particle?) This is one reason why it may be useful to keep $\int dp$ and use the canonical form $pdx - Hdt$ in the path integral. (Another reason is related to the path integral measure.) (10 points)

2. Unitarity

For $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$ and $|\phi(t)\rangle = U(t,t_0)|\phi(t_0)\rangle$, the unitarity of evolution means $\langle\phi(t)|\psi(t)\rangle = \langle\phi(t_0)|\psi(t_0)\rangle$. Show this, both in the operator formalism and in the path integral formalism, for a Hamiltonian of the form $H(X,P) = K(P) + V(X,t)$. (For the general form the path integral derivation for unitarity is more difficult and is not required here.) (10 points)

Remark: What you do here is the basic idea of the Schwinger-Keldysh formalism.

3. Charged Particle

What is the Hamiltonian for a particle of electric charge q in time-dependent electromagnetic field? Derive the path integral for it. Why is the "averaged" $\left(H(x_{n+1}, p_{n+1/2}, t_{n+1/2}) + H(x_n, p_{n+1/2}, t_{n+1/2})\right)/2$ form that we introduced useful in this case?

Then explain why any Hamiltonian $H = H(P,X)$ can be written as a sum of terms of the form $(F(P)G(X) + G(X)F(P))/2$. How is this related to the "averaged" form above? (15 points)

Remark: A closely related concept is Weyl ordering, but it is less convenient to use in a path integral.

4. How Small is δt

When we approximated $i\hbar\partial_t|\psi\rangle = H\psi$ by $|\psi(t+\delta t)\rangle = (1 - \frac{i}{\hbar}\delta t H)|\psi(t)\rangle$, or approximated $U(\delta t) = \exp(-\frac{i}{\hbar}\delta t H)$ by $1 - \frac{i}{\hbar}\delta t H$, there is some error. So δt must be small enough to bound the error.

First consider a Hamiltonian with bounded eigenvalues. What is the error?

Then consider a Hamiltonian with unboundedly large eigenvalues. How to bound the error? *Hint:* This requires some deeper thoughts. The error would depend on $|\psi\rangle$. Think about the properties of $|\langle E|\psi\rangle|$ as a function of energy eigenvalues E. (20 points)