

场论与凝聚态笔记

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1 Invitation: The Cartoon of Confinement

Never see individual quarks.

For separatable particles, like electron charges, their potential is $V(r) \sim \frac{1}{r}$, thus $V(r) - V(r_0)$ is always bounded.

Two quarks forms a pion. They interacts through gluon, and forms a structure called gluon tube or string. The potential is $V(r) \sim r$ and the energy density per length is appropriately constant. To separate a quark pair, the energy inputed $V(r) - V(r_0)$ is unbounded.

Similar phenomenon appears in superconductor(type II). When electronic charge condensed, the interaction of magneticmonopole becomes $V(r) \sim r$. According to EM duality, when magneticmonopole condensed, analogy goes to its counterpart. (Perspective by t'Hooft, Polyakov and Manldstan.)

2 Path Integral for Single Particles

From the two-slit interference, we've known the picture of wave. Whilst, the view of particle could recover the result by computing the phase e^{iS} .

For single particle mechanic, we start from the Schrödinger equation

$$i\hbar\partial_t |\psi(t)\rangle = H(p, x, t) |\psi(t)\rangle. \quad (2.1)$$

We have the time evolution operator

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle, \quad (2.2)$$

which is unitary,

$$U^\dagger(t, t_0)U(t, t_0) = 1. \quad (2.3)$$

If H is time-dependent, we split the time interval into small slices, and we get the infinitesimal U operator as time-independent cases,

$$U(t, t_0) = \prod_{n=0}^{N-1} U(\overbrace{t_{n-1}, t_n}^{\delta t}), \quad (2.4)$$

where $t_n \equiv t_0 + n\delta t$. Note that the product is time ordered.

Another perspective is from the Schrödinger equation, by finite differential,

$$|\psi(t + \delta t)\rangle = \left[1 - \frac{i}{\hbar} H(p, x, t + \frac{\delta t}{2}) \right] |\psi(t)\rangle \quad (2.5)$$

To the order of δt , we have

$$|\psi(t + \delta t)\rangle = e^{-\frac{i}{\hbar} H(p, x, t + \frac{\delta t}{2})} |\psi(t)\rangle \quad (2.6)$$

Next, we put the time evolution operator in spacial basis, considering

$$\langle x' | U(t + \delta t, t) | x \rangle. \quad (2.7)$$

Suppose $H = \frac{p^2}{2m} + V(x)$ for simplicity, we obtain

$$\langle x' | \left[1 - \frac{i\delta t}{\hbar} \left(\frac{p^2}{2m} + V(x, t + \frac{\delta t}{2}) \right) \right] | x \rangle. \quad (2.8)$$

Make a substitution

$$V \rightarrow \frac{V(x', t + \frac{\delta t}{2}) + V(x, t + \frac{\delta t}{2})}{2}, \quad (2.9)$$

and insert a completeness relation of p in each time slice, we arrive at

$$\langle x' | U(t + \delta t, t) | x \rangle = \int dp \frac{1}{2\pi\hbar} e^{\frac{ip(x' - x)}{\hbar}} \exp \left[-\frac{i\delta t}{\hbar} \frac{H(p, x', t + \frac{\delta t}{2}) + H(p, x, t + \frac{\delta t}{2})}{2} \right]. \quad (2.10)$$

written in a more compact form,

$$\langle x' | U(t + \delta t, t) | x \rangle \sim \int dp \frac{1}{2\pi\hbar} e^{i\frac{p\delta x}{\hbar} - i\frac{H\delta t}{\hbar}}. \quad (2.11)$$

The finite-time evolution operator,

$$U(t_N, t_0) = \prod_{n=0}^{N-1} U(t_{n+1}, t_n) \quad (2.12)$$

inserting an identity operator as x basis completeness relation, the element is

$$U(t_{n+2}, t_{n+1}) \underbrace{1}_{\int dx_n |x_n\rangle\langle x_n|} U(t_{n+1}, t_n) \quad (2.13)$$

then we obtain

$$\begin{aligned} \langle x_N | U(t_N, t_0) | x_0 \rangle &= \left(\prod_{n=1}^{N-1} \int dx_n \langle x_{n+1} | U(t_{n+1}, t_n) | x_n \rangle \right) \\ &\times \langle x_1 | U(t_1, t_0) | x_0 \rangle \end{aligned} \quad (2.14)$$

in full,

$$\begin{aligned} \langle x_N | U(t_N, t_0) | x_0 \rangle &= \left(\prod_{n=1}^{N-1} \int \frac{dp_{n+\frac{1}{2}} dx_n}{2\pi\hbar} \right) \int \frac{dp_{\frac{1}{2}}}{2\pi\hbar} \\ &\times \exp \left(\frac{i}{\hbar} \sum_{n=0}^{N-1} \left[p_{n+\frac{1}{2}}(x_{n+1} - x_n) - \delta t \frac{H(p_{n+\frac{1}{2}}, x_{n+1}, t_{n+\frac{1}{2}}) + H(p_{n+\frac{1}{2}}, x_n, t_{n+\frac{1}{2}})}{2} \right] \right) \\ &\sim \left(\prod_{n=0}^{N-1} \int \frac{dp_{n+\frac{1}{2}} dx_n}{2\pi\hbar} \right) \int \frac{dp_{\frac{1}{2}}}{2\pi\hbar} e^{\frac{i}{\hbar} \int p dx - H dt} \end{aligned} \quad (2.15)$$

Insert a operator $\hat{B}(x)$ in between the path integral,

$$\begin{aligned} \langle x_N | U(t_N, t_m) \hat{B}(x) U(t_m, t_0) | x_0 \rangle &= \left(\prod_{n=1}^{N-1} \int \frac{dp_{n+\frac{1}{2}} dx_n}{2\pi\hbar} \right) \int \frac{dp_{\frac{1}{2}}}{2\pi\hbar} B(x_m) \\ &\times \exp \frac{i}{\hbar} \sum_{n=0}^{N-1} \left[p_{\frac{1}{2}}(x_{n+1} - x_n) - \delta t \frac{H(p_{n+\frac{1}{2}}, x_{n+1}, t_{n+\frac{1}{2}}) + H(p_{n+\frac{1}{2}}, x_n, t_{n+\frac{1}{2}})}{2} \right]. \end{aligned} \quad (2.16)$$

We simply need to add the value of the operator as a function of certain space coordinate in the expression of path integral.

3 Observables in QM

$$\langle \psi | A | \psi \rangle, \quad A^\dagger = A. \quad (3.1)$$

Probability of projection,

$$P = |\langle \phi | \psi \rangle|^2 = \langle \psi | \underbrace{|\phi\rangle \langle \phi|}_A | \psi \rangle \quad (3.2)$$

Observables after evaluation,

$$\langle \phi | \underbrace{e^{iHt/\hbar} A e^{-iHt/\hbar}}_{\hat{A}(t)} | \psi \rangle \quad (3.3)$$

Projection after evaluation: scattering

$$P = |\langle \psi | e^{-iHt/\hbar} | \psi \rangle|^2 = \langle \psi | \underbrace{e^{iHt/\hbar} |\phi\rangle \langle \phi| e^{-iHt/\hbar}}_{\hat{A}(t)} | \psi \rangle \quad (3.4)$$

Retarded correlation

$$H(t) = H_0 + \underbrace{a(t)}_{\text{small}} B. \quad (3.5)$$

The contribution of $\langle \psi | U^\dagger(t, 0) A U(t, 0) | \psi \rangle$ to the first order correction in a is

$$-\frac{i}{\hbar} \int dt' a(t') \left[\langle \psi | U_0^\dagger(t, 0) A U_0(t, t') B U_0(t', 0) - U_0^\dagger(t, 0) B U_0(t, t') A U_0(t', 0) | \psi \rangle \right] \quad (3.6)$$

4 Path Integrals for Fields

There's a mattress with springs and massive balls connected. Denoting the offset of each ball as $\phi_{\vec{r}}$ the Hamiltonian is

$$H = \sum_{\vec{r} \text{ on lattice}} \left[\frac{p_{\vec{r}}^2}{2m} + V(\phi_{\vec{r}}) + \sum_{\vec{\tau}=1}^d \frac{k}{2} (\phi_{\vec{r}+\alpha\vec{\tau}} - \phi_{\vec{r}})^2 \right], \quad (4.1)$$

with a commutation relation

$$[\phi_{\vec{r}}, p_{\vec{r}'}] = i\hbar \delta_{\vec{r}, \vec{r}'}. \quad (4.2)$$

The interacting part of the Hamiltonian only involves pairs nearby.

In continuum limit, $H = \int d^d \vec{r} \mathcal{H}(\vec{r})$, and we can write the Hamilton density as

$$\mathcal{H} = \frac{\pi(\vec{r})^2}{2\rho} + \mathcal{V}(\phi(\vec{r})) + \frac{\kappa}{2} [\partial_{\vec{r}} \phi(\vec{r})]^2, \quad (4.3)$$

where $\pi(\vec{r}) = \frac{p_{\vec{r}}}{\alpha^d}$, $\mathcal{V} = \frac{V}{\alpha^d}$, $\rho = \frac{m}{\alpha^d}$, $\kappa = \frac{k\alpha^2}{\alpha^d}$.

Path Integral At t_n , we use $\bigotimes_{\vec{r}} |\phi(\vec{r})\rangle$ basis, and $\bigotimes_{\vec{r}} |\pi(\vec{r})\rangle$ for $t_{n+\frac{1}{2}}$.

Analogously, we get

$$\begin{aligned} \langle \text{end} | U(t, 0) | \text{start} \rangle &= \left(\prod_{t_n, \vec{r}} \frac{dp_{t_n+\frac{1}{2}, \vec{r}} d\phi_{t_n, \vec{r}}}{2\pi\hbar} \right)_{\text{suitable boundary condition}} \\ &\times \exp \frac{i}{\hbar} \sum_{t_n, \vec{r}} \left[p_{t_n+\frac{1}{2}, \vec{r}} (\phi_{t_{n+1}, \vec{r}} - \phi_{t_n, \vec{r}}) - \delta t \left(\frac{p_{t_n+\frac{1}{2}, \vec{r}}^2}{2m} + \frac{k}{2} \sum_{\hat{\tau}=1}^d (\phi_{t_n, \vec{r}+\alpha\hat{\tau}} - \phi_{t_n, \vec{r}})^2 + V(\phi_{t_n, \vec{r}}) \right) \right]. \end{aligned} \quad (4.4)$$

Integrate out p , we obtain

$$\begin{aligned} &\left(\prod_{t_n, \vec{r}} \sqrt{\frac{2\pi\hbar m}{i\delta t}} \frac{d\phi_{t_n, \vec{r}}}{2\pi\hbar} \right)_{\text{suitable boundary condition}} \\ &\times \exp \frac{i\delta}{\hbar} \sum_{t_n, \vec{r}} \left[\frac{m}{2} \left(\frac{\phi_{t_{n+1}, \vec{r}} - \phi_{t_n, \vec{r}}}{\delta t} \right)^2 - \frac{k\alpha^2}{2} \sum_{\hat{\tau}=1}^d \left(\frac{\phi_{t_n, \vec{r}+\alpha\hat{\tau}} - \phi_{t_n, \vec{r}}}{\alpha} \right)^2 - V(\phi_{t_n, \vec{r}}) \right], \end{aligned} \quad (4.5)$$

in which time and space stand in the same place, similar to relativistic K-G field.

5 Free Field Theory

For free fields, $V(\phi_{\vec{r}}) = \frac{u}{2} \phi_{\vec{r}}^2$, it behaves just like coupled simple harmonic oscillators. To find the normal modes, we use Fourier transformation.

$$\phi_{\vec{r}} = \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} \frac{d^d \vec{k}}{(2\pi)^d} e^{i\vec{k} \cdot \vec{r}} \phi_{\vec{k}}, \quad (5.1)$$

likewise for $p_{\vec{r}}$.

$\phi_{\vec{r}}$ is real, thus $\phi_{\vec{r}} = \phi_{\vec{r}}^\dagger \implies \phi_{-\vec{k}} = \phi_{\vec{k}}^\dagger$, and the commutation relation is

$$[\phi_{\vec{k}}, p_{\vec{k}'}] = i\hbar (2\pi)^d \delta^d(\vec{k} + \vec{k}'). \quad (5.2)$$

The Hamiltonian becomes

$$H = \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2} \left[\frac{p_{-\vec{k}} p_{\vec{k}}}{2m} + \left(\frac{k}{2} \sum_{\vec{\tau}=1}^d \left(2 \sin \frac{\alpha k_i}{2} \right)^2 + \frac{u}{2} \right) \phi_{-\vec{k}} \phi_{\vec{k}} \right], \quad (5.3)$$

from which we can directly figure out the frequency $\omega_{\vec{k}} = \omega_{-\vec{k}} = \sqrt{\frac{k \sum_{\vec{\tau}} \left(\sin \frac{\alpha k_i}{2} \right)^2 + U}{m}}$

Next, we need to make a substitution, or Bogoliubov transformation:

$$\phi_{\vec{k}}^c = \frac{\phi_{\vec{k}} + \phi_{-\vec{k}}}{\sqrt{2}}, \quad (5.4)$$

$$\phi_{\vec{k}}^s = i \frac{\phi_{\vec{k}} - \phi_{-\vec{k}}}{\sqrt{2}}. \quad (5.5)$$

the integration range becomes half of \vec{k} .

5.1 Ground State Wave Function and Entanglement

5.2 Energy Gap and Corelation