Source of reference:
John D. Anderson, Jr. (Fundamentals of aerodynamics, fifth edition), 2007

Boundary conditions

From the latter section, considering an incompressible and irrotational flow produced the Laplace equation. This equation has different solutions leading to various flow characteristics such as uniform, source, sink and doublet flow. It was also found that the superposition principle is applicable to the Laplace equation because of it's linearity property. This property allows for simulating more realistic flow types than elementary solutions of the Laplace equation. One can simulate objects placed in the flow-field region by restricting the flow at certain points in the region. This is known as boundary conditions. Boundary conditions give rise to various solutions of the flow-field for specific geometric shapes.

For an solid- body the flow will not be able to flow through the body. The normal velocity must be zero at the surface. In addition, for inviscid fluids the tangential velocity component at the surface of the body is finite. The equations which arise from these boundary conditions can be written mathematically as :

$$\vec{V} \cdot \vec{n} = 0 \tag{1}$$

$\vec{V}$ is the flow velocity and $\vec{n}$ is the normal vector pointing out of the body.

$$\vec{V} \cdot \vec{t} = c \tag{2}$$

$\vec{V}$ is the flow velocity and $\vec{t}$ is the tangential vector. c indicts a constant value for the tangential velocity component. For flow very far away from the body we assume the velocity to be uniform. This can be written as:

$$u = V_\infty \tag{3}$$

u is the velocity in the x-direction and V_∞ is a constant value resulting in a uniform flow in the x-direction.

$$v = 0 \tag{4}$$

The y component of the velocity vector is zero. Note, this is only the case in this scenario and depends on the problem at hand.

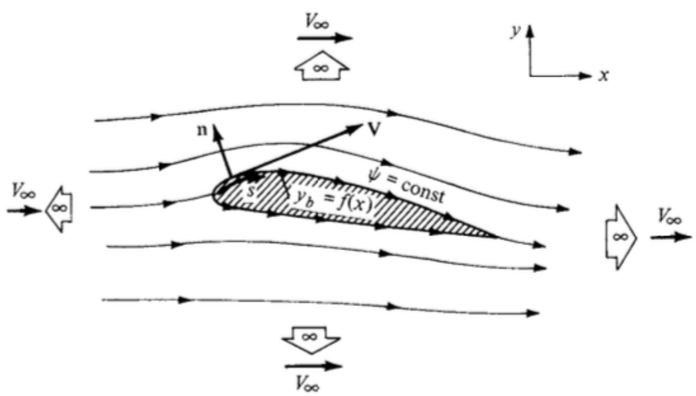

Figure 1: The figure shows flow around an airfoil. Together with vector for boundary conditions on solid body. ψ are streamlines. The uniform flow is the same as the equation above i.e. only contribution in the x-direction. Image from Anderson, 2007

Flow around a cylinder

Write about flow around a circular cylinder using previous text!

Numerical approach

Navier Stokes' equations are nonlinear, integral equations which do not solve analytically. It is therefore commonly applied two approaches in order to solve Navier Stokes' equations. One, is to simplify the problem at hand to extent that the equations become linear and therefore can be solved analytically. The second is, numerical discipline/approach used to solve a set of equations as analytic solutions do not exist. This allows for a more realistic simulations and solutions. This is more commonly known as Computational Fluid Dynamics (CFD). The nonlinear integral terms are replaced by a discretized approach yielding solutions at specified points (or grid).

Consider placing an object in flow environment. Calculating the flow at the walls of the object is done by boundary conditions. This grants the possibility of solving individual flow types for body of different geometries. When it comes to solving a flow characteristic at a specific point, i.e. at

Source panel methods

When solving for the flow around a solid body i.e. the flow around a circular cylinder, the body is predetermined in it's geometric shape. What about when the shapes that are not circular with arbitrary shape? The same principles of finding the velocity and strengths apply to those objects as well, and it would be complicated to guess what combinations of flows would constitute an arbitrary shape. It would be useful to only need the shape the object in order to solve the flow around it. This is where the source panel method is introduced. The source panel is based on dividing a solid body into into sets of panels of the discretized object. The panels contain a panel strength along it's line. These strengths can be represented by the source type flow presented the section before. There are now an infinite number of strengths along the panel to solved for. These panel source strengths make up a source sheet and can be represented by a function $\lambda(s)$ where s is the distance along the sheet and λ its corresponding strength value at the point s. The units of the strength is square meters per second for the strength quantities for source.sink and doublets. So by considering a infinitesimal line of source strength, $\lambda\ ds$ yield the same dimension. For a point located at a point P a distance r away from the small piece of the panel segment ds an infinitesimally small potential dϕ occurs at the point P obtained at ds. This can be written as:

$$d\phi = \frac{\lambda ds}{2\pi} \ln r \tag{5}$$

$r = \sqrt{x^2 + y^2}$ is the distance from ds inducing a dϕ at point p. λ is the source strength. The full source sheet induced at P is found by integrating over the source sheet where the limits are a to b. The velocity potential can further be written as:

$$\phi = \int_a^b \frac{\lambda ds}{2\pi} \ln r \tag{6}$$

To make it clear, the integrand contains the potential equation for a source or sink type flow the integrand can be negative and positive). Early, it was stated that the source strength from a panel was $\lambda(s)$ which gave the equation stated above. Now, by assuming that the source strength from one panel is constant, not a function of the distance s, allocating each panel with an unknown source strength coefficient yields the possibility to approximate the source strengths at each panel. Each panel may have different source strengths λ. For n panels n source flow occur, one for each panel. By combining this with a uniform flow and solving for the source strength λ the desired streamlines is obtained around the object. The objective is to solve each individual panel strength λ_i, $i = 1, 2, 3, ..., n$. The boundary condition for the normal component states that the velocity is zero in that direction. Furthermore, the boundary condition at each *panel point* allows for a set of equations to be expressed. The notation for a panel j giving rise to a potential at a point p from eq(?!?!) above can be written as:

$$\Delta\phi_j = \frac{\lambda_j}{2\pi} \int_j \ln r_{pj} ds_j \tag{7}$$

the distance r_{pj} is now $r_{pj} = \sqrt{(x - x_j)^2 + (y - y_j)^2}$. The point p is located at (x, y). Remember that λ_j is constant over the panel j. For all panels producing a potential at point p the equation above is summed over all panels:

$$\phi(p) = \sum_{j=1}^{n} \Delta\phi_j = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{pj} ds_j \tag{8}$$

Now, move the point p to a panel i and define the point (x_i, y_i) to be the control point of panel i we can express the influence of every panel to the ith panel as:

$$\phi(x_i, y_i) = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij} ds_j \tag{9}$$

With the distance:

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{10}$$

The control point is located at the mid point of each panel. With the location of each panel defined, apply the normal boundary condition at that point. But first, the freestream velocity which was needed to generate a flow around the object is needed to evaluate the correct orientation of the vectors. Let, $\vec{n}_i$ be the normal vector located the control point of panel i. The freestream vector V_∞ can be generated at an angle α relative to the x-axis. This angle to commonly known as the angle of attack. The vector normal to the freestream velocity vector at the control point at the ith panel can be expressed as:

$$V_{\infty,n} = \vec{V_\infty} \cdot \vec{n}_i = V_\infty \cos \beta_i \tag{11}$$

β_i is the angle between $\vec{V_\infty}$ and $\vec{n}_i$. Finding the normal velocity component at panel i produced by every panel is:

$$V_n = \frac{\partial}{\partial n_i} [\phi(x_i, y_i)] \tag{12}$$

We want component to point outward relative the body. V_n is then positive. The derivative $\frac{\partial}{\partial n_i}$ acts on the distance variable r_{ij}. Note that when j = i the calculation the same panel is done on itself and the distance variable becomes zero. The term is now located in the denominator leading to a singularity. When j = i the contribution has been shown to give $\frac{\lambda_i}{2}$. This gives an update of the potential equation to further be expressed as:

$$V_n = \frac{\lambda_i}{2} + \sum_{j=1,(j \neq i)}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \ln r_{ij} ds_j \tag{13}$$

This the normal vector component of the source panel contribution at panel i. The boundary states:

$$V_{\infty,n} + V_n = 0 \tag{14}$$

The flow cannot penetrate the walls of a solid body. The latter equation can be written as:

$$V_\infty \cos \beta_i + \frac{\lambda_i}{2} + \sum_{j=1,(j \neq i)}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \ln r_{ij} ds_j = 0 \tag{15}$$

Substituting the equations V_n and $V_{\infty,n}$ above. This equation has n unknowns. The task now is to gather n equation which in order to solve the source strengths. Solving the source strength coefficients gives rise to the streamlines over the set of panels. This equation can be expressed in matrix form solving the system of equation using numerical methods. This approach makes it possible to control the amount of accuracy needed for the model as the number of panels is used as input to the model.

Tangential velocity calculation

With the source strengths solved. Next in the line will be finding the tangential velocity component. The equation are similar to the normal velocity component. The only change is the direction in which to take derivatives and find the freestream vector. The vector will be evaluated at the control of each panel as for the normal vector. Tangent to the surface the freestream velocity will be:

$$V_{\infty,t} = V_\infty \sin \beta_i \tag{16}$$

In addition, finding the tangential velocity on the panel is found by taking the derivative of eq(9) in the tangential direction. This can be expressed as:

$$V_t = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial t_i} \ln r_{ij} ds_j \tag{17}$$

Adding the above equation the final expression for the tangential velocity is:

$$V_i = V_{\infty,t} + V_t = V_\infty \sin \beta_i + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial t_i} \ln r_{ij} ds_j \tag{18}$$

V_i is the sum over the velocity at the control point i. The extra term for when j = i for the normal component does not occur here as that term is zero when j = i. A sanity check when calculating the source strengths and panels is to sum up the product of all source strengths λ and panels lengths. This sum should be zero. If not then the object would be absorbing mass from the flow.

$$\sum_{j=1}^{n} \lambda_j s_j = 0 \tag{19}$$

S_j is the jth panel length, λ_j is the jth source strength coefficient.

Thorough derivation of source panel method

From the previous section an overview of the equations for calculating the flow around an arbitrary body was shown. In this section, a more detailed derivation will be done. The integral terms will be calculated together with the geometry of the problem. Finally, this will be set up into a matrix form $Ax = b$ which can be solved thus leading to a solution, namely the source panel method.

Defining geometry

The equations needed for creating the sufficient amount panels is done is here. In addition, some detailed figures will be included for a overview of each variable listed. Each panel can be of different length and therefore each panel needs to be calculated. The starting point needs to be defined and thereafter calculated each panels property in a orderly fashion. This is necessary because the normal vector orientation will be

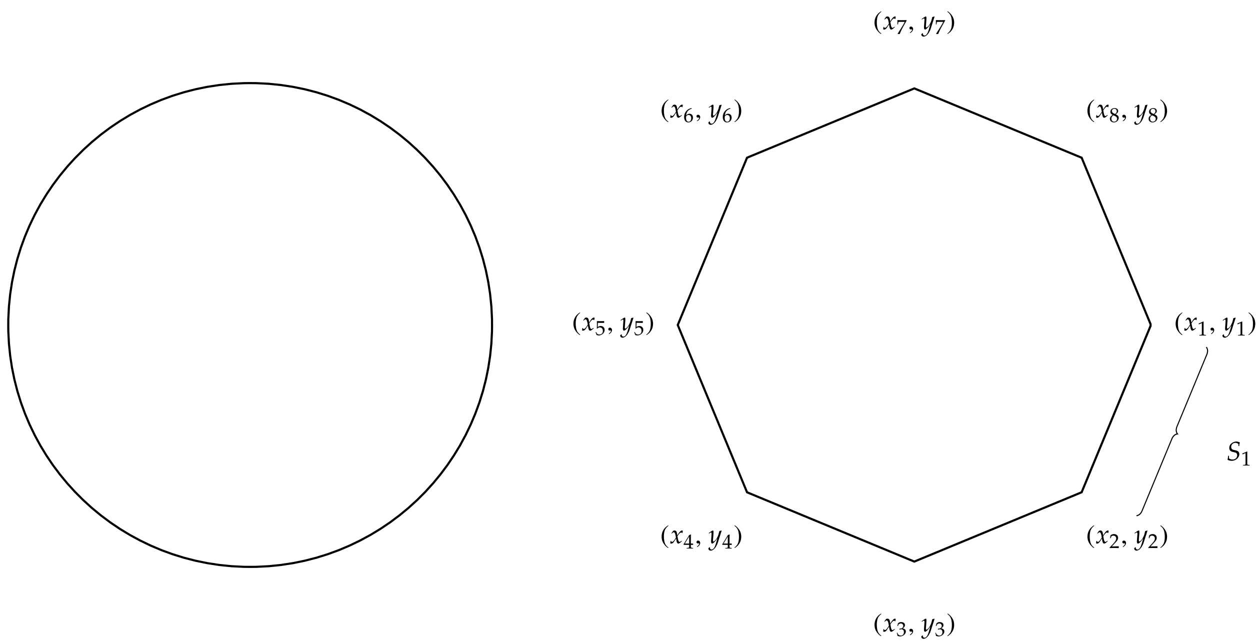

Figure 2: Approximating a circle with panels. N panels are created with boundary points representing the end points of each panel (x_b, y_b). From the figure on the right 8 panels s are created with 9 boundary points ((x_1, y_1) are counted twice for the first and last boundary point). More panels can be included for better approximations.

affected by this choice. The suggested method for calculating panels are clockwise. Below, figure 2 shows 8 panels approximating a circle with boundary points.

$$dx = (x_2 - x_1) \tag{20}$$

$$dy = (y_2 - y_1) \tag{21}$$

$$s_1 = \sqrt{dx^2 + dy^2} \tag{22}$$

Equations (20, 21, 22) for the ith panels have the notation:

$$dx_i = (x_{i+1} - x_i) \tag{23}$$

$$dy_i = (y_{i+1} - y_i) \tag{24}$$

$$s_i = \sqrt{dx_i^2 + dy_i^2} \tag{25}$$

Furthermore, from figure 4 the control points for each panels is needed. This is where the source strength is set up. This point has the relation:

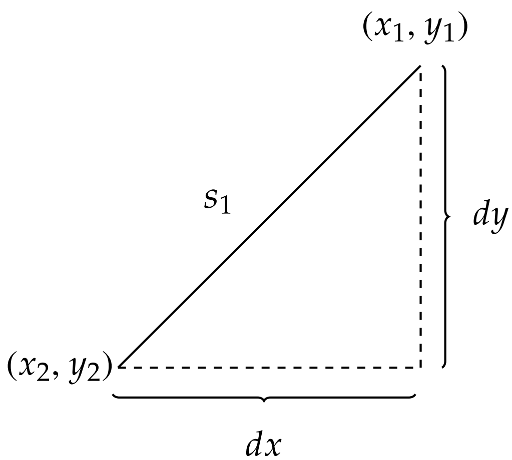

Figure 3: A closer view of panel S_1 from figure 2. The equations needed for the panel calculation can be seen above ref equation numbers!.

$$x_{ci} = \frac{x_i + x_{i+1}}{2} \tag{26}$$

$$y_{ci} = \frac{y_i + y_{i+1}}{2} \tag{27}$$

$$dxc = x_{i+1} - x_{ci} \tag{28}$$

$$dyc = y_{i+1} - y_{ci} \tag{29}$$

See figure 4 and 5 for a closer view. With the points defined the angle between the panel and x-axis ϕ can be found by:

$$\tan \phi_i = \frac{dyc}{dxc} \tag{30}$$

Furthermore, the normal vector is always perpendicular to the panel. This means that the angle δ_i can be expressed as:

$$\delta_i = \phi_i + \frac{\pi}{2} \tag{31}$$

$$\beta_i = \delta_i - \alpha \tag{32}$$

α is the angle which the freestream vectors occurs. This is constant. The angles are visualized in figure 6. The normal vector components can be expressed as:

$$n_{xi} = x_{ci} + s_i \cos \delta_i \tag{33}$$

$$n_{yi} = y_{ci} + s_i \sin \delta_i \tag{34}$$

$$\vec{n} = n_{xi}\vec{i} + n_{yi}\vec{j} \tag{35}$$

Where i and j are unit vectors in x and y - direction respectively.

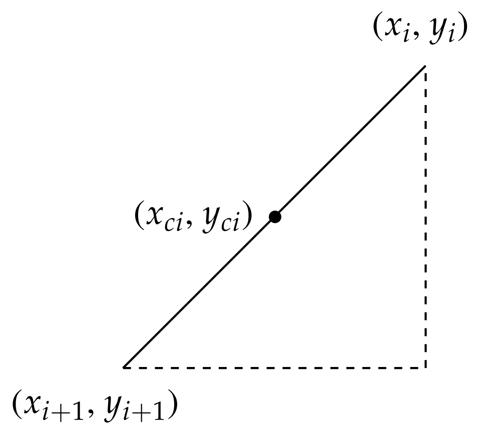

Figure 4: The general case for the ith panel with corresponding boundary points and control points.With the boundary points defined, the control points can be calculated by equations (refrefref).

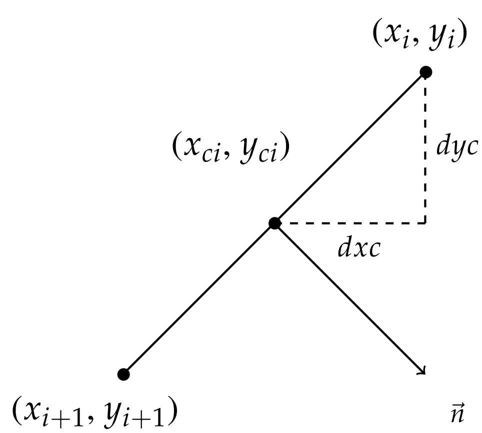

Figure 5: In this figure a normal vector is added at the control point together with the change in x and y direction from the control point (dxc and dyc) which is necessary when calculating angles between panels and normal vectors.

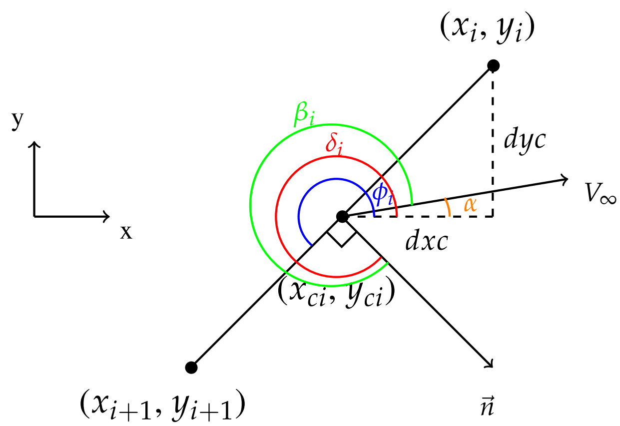

Figure 6: This figure shows the angles needed to compute the normal vectors of each panel derived from the boundary and control points.

Building complex flows

With the panels and angles defined from the boundary and control points, the necessary components are thus obtained. This now leads to the ability to build flows which are independent of the analytic potential flow solutions. Recall, the source velocity potential:

$$\phi_1 = \frac{\lambda_1}{2\pi} \ln r_{1p} \tag{36}$$

Where λ_1 is the source strength from panel S_1 induced at the point p. $r_{1p} = \sqrt{(x_1 - x_p)^2 + (y_1 - y_p)^2}$. With the super position principle we can add together multiple sources. In this case the total amount induced by the panels of the body $\phi_p = \phi_1 + \phi_2 + \phi_3 + ... + \phi_n$. Or expressed in terms of the distance from point P to the panels control point:

$$\phi_p = \frac{\lambda_1}{2\pi} \ln r_{1p} + \frac{\lambda_2}{2\pi} \ln r_{2p} + ... + \frac{\lambda_n}{2\pi} \ln r_{np} \tag{37}$$

Which can be written as:

$$\phi_p = \sum_{i=1}^{n} \frac{\lambda_i}{2\pi} \ln r_{ip} \tag{38}$$

See figure 7 below.

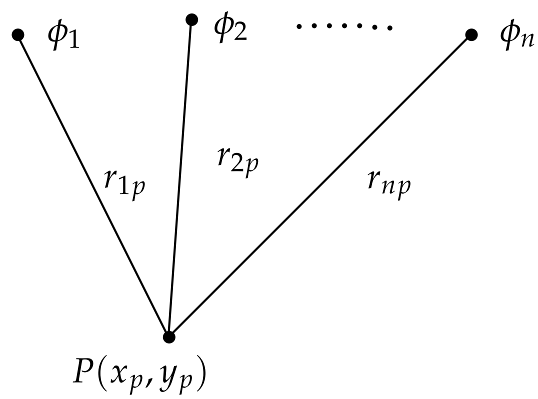

Figure 7: n sources ϕ_n induce the potential ϕ_p at a point P.

Let's now assume that for a curve with infinite source on it's arc. The expression for calculating the velocity potential at a point is given by the integral:

$$\phi_p = \int_a^b \frac{\lambda(s)}{2\pi} \ln r_p(s) ds \tag{39}$$

The integral assumes the source strength over the body' periphery (see figure 8) depends on the distance s along the surface S. By breaking the the surface up into straight lines the integral above can be manipulated into the equation:

$$\phi_p = \int_a^b \frac{\lambda(s_{ab})}{2\pi} \ln r_p(s_{ab}) ds_{ab} + \int_b^c \frac{\lambda(s_{bc})}{2\pi} \ln r_p(s_{bc}) ds_{bc} \tag{40}$$

Figure 9 depicts this integral case. The curve now consists of two lines where each line contains it's own source strength $\lambda(s_{ab}$ and $\lambda(s_{bc})$. The number of panels can be adjusted. For n panels the integral reads:

$$\phi_p = \sum_{j=1}^{n} \int_j \frac{\lambda(s_j)}{2\pi} \ln r_{pj}(s_j) ds_j \tag{41}$$

Now assume the source strength $\lambda(s)$ is constant along each panel. This leads to the source strength at panel j $\lambda(s_j) = \lambda_j$ and the distance $r_{pj}(s_j) = r_{pj}$. With this we write the integral equation above as:

$$\phi_p = \sum_{j=1}^{n} \lambda_j \int_j \frac{\ln r_{pj}}{2\pi} ds_j \tag{42}$$

The source panel strengths needs to be solved. This constitutes the main part of the source panel method. A quick summary of the main features of this section. The goal is to build a flow around a arbitrary object. We found that the object induces a small potential ϕ at a point P in space. Further, a derivation of how to calculate the total

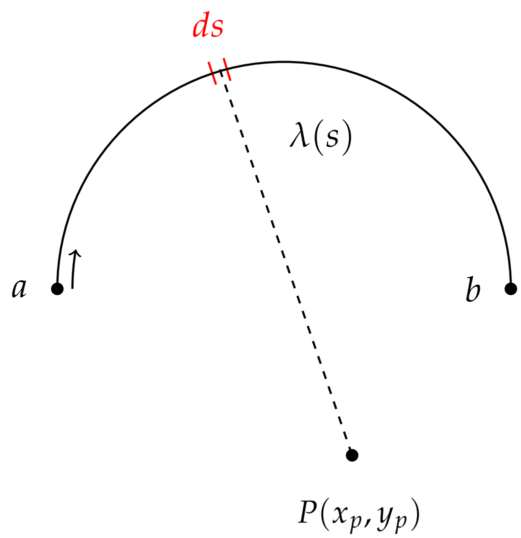

Figure 8: A body with a source strength $\lambda(s)$ induces a potential at point P. Integrating from a to b grants thew total contribution of the potential along the periphery of the surface. The arrow indicates the integration direction.

potential at point P. With this, some simplifications have been considered; breaking a curve of arbitrary shape into n panels consisting of straight lines, assuming the source strength parameter λ is constant along a panel j. Combing this with a freestream velocity, we have all the contributions needed in order to calculate the flow around an object of arbitrary shape. In the next section we dive further into the details how to set up the equations needed for solving the panel source strengths.

Calculating flow around an arbitrary shaped object

An insight into how to solve set up system of equations for source strengths is done. The condition that the normal component at the surface of the object is zero will be used to set up the system of equations. Further to calculate the velocities the tangential component will be used. Finally the potential can be solved for at every grid cell for the defined area of interest.

$$\phi_p = V_\infty \cos(\alpha) x + V_\infty \sin(\alpha) y + \sum_{j=1}^{n} \lambda_j \int_j \frac{\ln r_{pj}}{2\pi} ds_j \tag{43}$$

The two first terms are the freestream velocity vectors in x and y direction respectively. Taking the derivative of the potential ϕ_p in the x and y direction the velocities can be found:

$$V_{x,p} = \frac{\partial \phi_p}{\partial x} \tag{44}$$

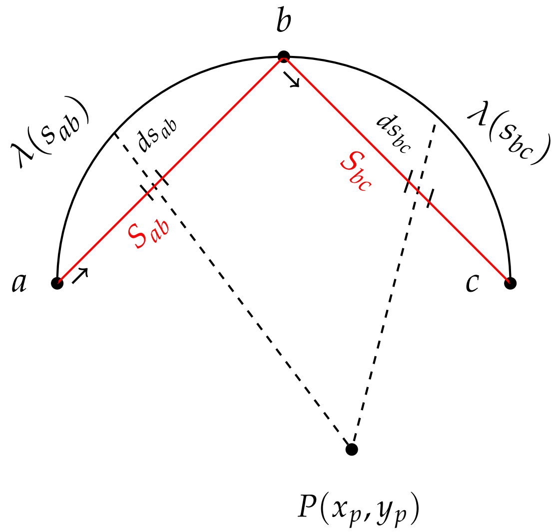

Figure 9: Splitting the body into panels thus adding up the contribution from the panel approximation.

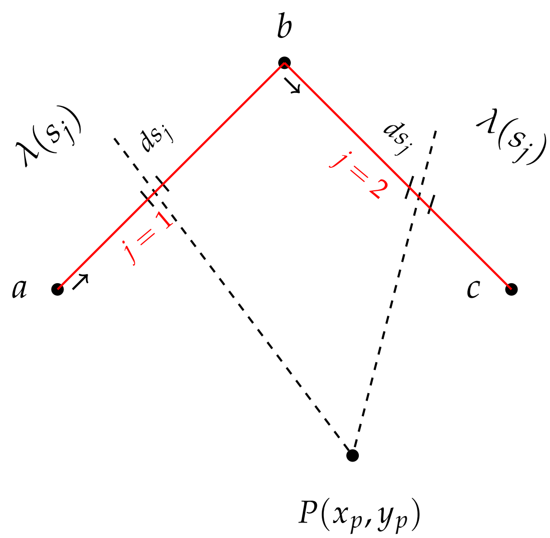

Figure 10: This figure shows the angles needed to compute the normal vectors of each panel derived from the boundary and control points.

$$V_{y,p} = \frac{\partial \phi_p}{\partial y} \tag{45}$$

The normal and tangential velocity deal with the physical part of the solution. They can be expressed as:

$$\vec{V} \cdot \vec{n} = 0 \tag{46}$$

No fluid can pass through the object

$$\vec{V} \cdot \vec{t} = C \tag{47}$$

C is a constant. The flow runs along the surface of the object. By placing the source strengths λ at the control points of each panel the can be done to the normal vector which here we will solve for the source strengths. The velocity potential at panel i can be written using eq(43):

$$\phi_p = V_\infty \cos(\alpha)x + V_\infty \sin(\alpha)y + \sum_{j=1}^{n} \lambda_j \int_j \frac{\ln r_{ij}}{2\pi} ds_j \tag{48}$$

The normal and tangential velocity at panel i:

$$V_{n,i}^{Total} = \frac{\partial \phi_i}{\partial n_i} = V_\infty \cos(\alpha)\frac{\partial x_i}{\partial n_i} + V_\infty \sin(\alpha)\frac{\partial y_i}{\partial n_i} + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \ln r_{ij} ds_j = 0 \tag{49}$$

$$V_{t,i}^{Total} = \frac{\partial \phi_i}{\partial t_i} = V_\infty \cos(\alpha)\frac{\partial x_i}{\partial t_i} + V_\infty \sin(\alpha)\frac{\partial y_i}{\partial t_i} + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial t_i} \ln r_{ij} ds_j \tag{50}$$

Let's start by looking at the first terms in both the normal and tangential vectors. The normal component' terms:

$$V_{n,i}^{freestream} = \frac{\partial \phi_i}{\partial n_i} = V_\infty \cos(\alpha)\frac{\partial x_i}{\partial n_i} + V_\infty \sin(\alpha)\frac{\partial y_i}{\partial n_i} \tag{51}$$

The tangential component:

$$V_{t,i}^{freestream} = \frac{\partial \phi_i}{\partial t_i} = V_\infty \cos(\alpha)\frac{\partial x_i}{\partial t_i} + V_\infty \sin(\alpha)\frac{\partial y_i}{\partial t_i} \tag{52}$$

The change in x-direction with respect to the normal derivatives can be written as:

$$\frac{\partial x_i}{\partial n_i} = \cos(\delta_i) \quad , \quad \frac{\partial y_i}{\partial n_i} = \sin(\delta_i) \tag{53}$$

and the same with tangential:

$$\frac{\partial x_i}{\partial t_i} = \cos(\delta_i - \frac{\pi}{2}) \quad , \quad \frac{\partial y_i}{\partial t_i} = \sin(\delta_i - \frac{\pi}{2}) \tag{54}$$

The setup can seen in figure 11. Inserting equations (52) into eq(51) and using the trig

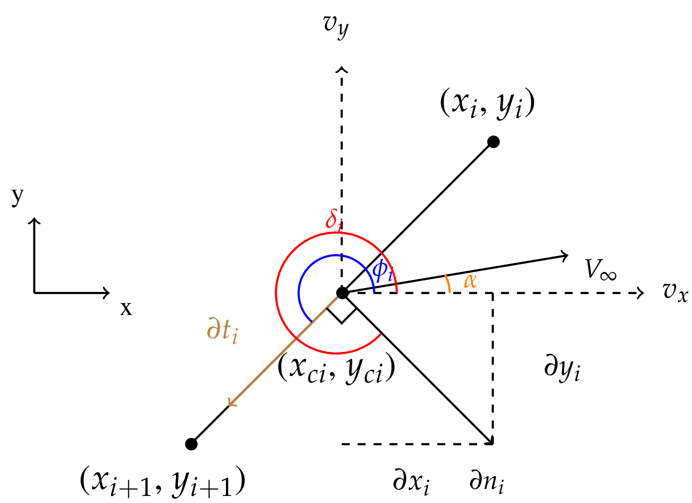

Figure 11: Panel i has both tangential and normal components. These two quantities can be expressed with trigonometric functions.

identity $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ we end up with simpler term for the freestream velocity:

$$V_{n,i}^{freestream} = V_\infty \cos(\beta_i) \tag{55}$$

β_i is the angle from angle of attack α to δ_i $\beta_i = \delta_i - \alpha$. Can be seen in figure 6. Putting eq(55) into eq(49):

$$V_{n,i}^{Total} = \frac{\partial \phi_i}{\partial n_i} = V_\infty \cos(\beta_i) + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \ln r_{ij} ds_j = 0 \tag{56}$$

The same can be done for the tangential term:

$$V_{t,i}^{Total} = \frac{\partial \phi_i}{\partial t_i} = V_\infty \sin(\beta_i) + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial t_i} \ln r_{ij} ds_j \tag{57}$$

Normal vector geometry integral

: In this section we look at the integral part of the total normal velocity induced at panel i. Recall:

$$V_{n,i}^{Total} = \frac{\partial \phi_i}{\partial n_i} = V_\infty \cos(\beta_i) + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \ln r_{ij} ds_j = 0 \tag{58}$$

Let's put our attention at the integral that needs to solved and express the terms with the use of variables mentioned above. This will be done more thoroughly below. Write The integral as:

$$I_{ij} = \int_j \frac{\partial}{\partial n_i} \ln r_{ij} ds_j \tag{59}$$

The logarithm term can be manipulated by the formula $\frac{d\ln(f(x))}{dx} = \frac{1}{f(x)}\frac{df(x)}{dx}$. Applying this to eq (59) gives:

$$I_{ij} = \int_j \frac{1}{r_{ij}} \frac{\partial r_{ij}}{\partial n_i} ds_j \tag{60}$$

Also, recall the distance r_{ij} which is the distance between panel j to panel i:

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{61}$$

Taking derivative of the distance w.r.t to normal component yields the equation:

$$\frac{\partial r_{ij}}{\partial n_i} = \frac{(x_i - x_j)\frac{\partial x_i}{\partial n_i} + (y_i - y_j)\frac{\partial y_i}{\partial n_i}}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} \tag{62}$$

From figure 11 and the equations in (53) we recognize the derivatives in the latter equation. Substituting equations(53):

$$\frac{\partial r_{ij}}{\partial n_i} = \frac{(x_i - x_j)\cos(\delta_i) + (y_i - y_j)\sin(\delta_i)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} \tag{63}$$

Gathering eq(63), eq(61) and inserting this into eq(60) the equation now reads:

$$I_{ij} = \int_j \frac{(x_i - x_j)\cos(\delta_i) + (y_i - y_j)\sin(\delta_i)}{(x_i - x_j)^2 + (y_i - y_j)^2} ds_j \tag{64}$$

Further, we want to express the integrand in terms of s_j as this is the integration variable. We find that x_j and y_j

$$x_j = X_j + s_j\cos(\phi_j), \quad , \quad y_j = Y_j + s_j\sin(\phi_j) \tag{65}$$

Here we represent the point on the jth panel as a parameter representation. X_j and Y_j are starting points for the panel, and S_j is the length of the jth panel. ϕ_j is angle between the x-axis and the panel j. The angle ϕ_j can be derived from the identity $\cos(\delta_i) = -\sin(\phi_i)$ and $\sin(\delta_i) = \cos(\phi_i)$ where the relation of δ_i and ϕ_i are $\delta_i = \phi_i + \pi/2$ as seen in figure 11. Inserting this into I_{ij} the integral now reads:

$$I_{ij} = \int_j \frac{(x_i - x_j)(-\sin(\phi_i)) + (y_i - y_j)\cos(\phi_i)}{(x_i - x_j)^2 + (y_i - y_j)^2} ds_j \tag{66}$$

For the sake tidyness, the numerator will be considered first. Thereafter, the denominator will be considered. From the above equation the numerator reads:

$$(x_i - X_j - S_j\cos(\phi_j))(-\sin(\phi_i)) + (y_i - Y_j - S_j\sin\phi_i)\cos(\phi_i) \tag{67}$$

after inserting eq(65). Further we arrange the terms such that:

$$(X_j - x_i)\sin\phi_i + (y_i - Y_j)\cos\phi_i + s_j(\cos\phi_j\sin\phi_i - \sin\phi_j\sin\phi_i) \tag{68}$$

and use the trig identity $\sin a - b = \sin a \cos b - \cos a \sin b$ on the last term:

$$(X_j - x_i)\sin\phi_i + (y_i - Y_j)\cos\phi_i + s_j \sin(\phi_i - \phi_j) \tag{69}$$

At last we write the numerator in terms of S_j as this is the variable which will be integrated over:

$$s_j A + D \tag{70}$$

where $A = \sin(\phi_i - \phi_j)$ $D = (X_j - x_i)\sin\phi_i + (y_i - Y_j)\cos\phi_i$. The numerator is now simplified and integrable.

Moving on the denominator. The approach will use as with the numerator. From eq(66) write out the terms and insert eq's(65):

$$x_i^2 - 2x_i(X_j + s_j\cos\phi_j) + (X_j + s_j\cos\phi_j)^2 + y_i^2 - 2y_i(Y_j + s_j\sin\phi_j) + (Y_j + s_j\sin(\phi_j)^2 \tag{71}$$

Now gather all s_j terms with same polynomial degree. This leads to:

$$s_j^2 + 2s_j B + C \tag{72}$$

where, $B = X_j\cos\phi_j + Y_j\sin\phi_j - x_i\cos\phi_j - y_i\sin\phi_j$ and $C = (x_i - X_j)^2 + (y_i - Y_j)^2$. The end term with the numerator and denominator put back into eq(66) we finally end up with:

$$I_{ij} = \int_0^{S_j} \frac{s_j A + D}{s_j^2 + 2s_j B + C} ds_j \tag{73}$$

Before integrating we need to complete the square in the denominator.

$$s_j^2 = 2Bs_j + C = s_j^2 + 2Bs_j + B^2 + C - B^2 = (s_j + B)^2 + E^2 \tag{74}$$

where $E = \sqrt{C - B^2}$. Now the integral reads:

$$I_{ij} = \int_0^{S_j} \frac{s_j A + D}{(s_j + B)^2 + E^2} ds_j \tag{75}$$

This is now on the form that can be integrated. Set $u = s_j + B$.

$$I_{ij} = \int_0^{S_j} \frac{(u - B)A + D}{u^2 + E^2} du \tag{76}$$

The integral will be split up into two integrals by separating the numerator.

$$I_{ij} = A\int_0^{S_j} \frac{u}{u^2 + E^2} du + (D - AB)\int_0^{S_j} \frac{1}{u^2 + E^2} du \tag{77}$$

Using substitution $\gamma = u^2 + E^2$ on the first integral again we obtain the solution as:

$$A\int_0^{S_j} \frac{u}{u^2 + E^2} du = \frac{A}{2}\int_0^{S_j} \frac{d\gamma}{\gamma} = \frac{A}{2}[\ln\gamma]_0^{s_j} \tag{78}$$

Now substitute u and γ

$$\frac{A}{2}\Big[\ln\left(S_j + B^2 + E^2\right) - \ln\left(B^2 + E^2\right)\Big] \tag{79}$$

$$\frac{A}{2}\ln\left[\frac{(S_j + B)^2 + E^2}{B^2 + E^2}\right] = \frac{A}{2}\ln\left[\frac{S_j^2 + 2S_jB + B^2 + E^2}{B^2 + E^2}\right] \tag{80}$$

With the final touch of using $E^2 = C - B^2$ we end up with:

$$\frac{A}{2}\ln\left[\frac{S_j + 2S_jB + C}{C}\right] \tag{81}$$

The second integral in eq(77) has the solution:

$$(D - AB)\int_0^{S_j}\frac{1}{u^2 + E^2}du = \frac{(D - AB)}{E}\arctan\left[\frac{u}{E}\right] \tag{82}$$

As with the first integral we substitute back and end up with:

$$\frac{D - AB}{E}\left(\arctan\left[\frac{S_j + B}{E}\right] - \arctan\left[\frac{B}{E}\right]\right) \tag{83}$$

Let's sum up the most important parts of this integration below. The integration gave the solution:

$$I_{ij} = \frac{A}{2}\ln\left[\frac{S_j + 2S_jB + C}{C}\right] + \frac{D - AB}{E}\left(\arctan\left[\frac{S_j + B}{E}\right] - \arctan\left[\frac{B}{E}\right]\right) \tag{84}$$

$$\begin{aligned}
A &= \sin\phi_i - \phi_j \\
B &= (X_j - x_i)\cos\phi_j + (Y_j - y_i)\sin\phi_j \\
C &= (x_i - X_j)^2 + (y_i - Y_j)^2 \\
D &= (X_j - x_i)\sin\phi_i + (y_i - Y_j)\cos\phi_i \\
E &= \sqrt{C - B^2}
\end{aligned}$$

Finally after an amount of mathematical operations the solution I_{ij} is found which means the normal velocity component can be computed by:

$$V_{n,i} = V_\infty\cos\beta_i + \sum_{j=1}^{N}\frac{\lambda_j}{2\pi}I_{ij} \tag{85}$$

This sums up the normal vector computations. Before setting up the system of equations we need to solve the integral in the tangential velocity vector eq(57). The same approach will be applied as for the normal velocity vector.

Tangential vector geometry integral