M.E. 530.646
UR5 Inverse Kinematics

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Introduction

Figure 1 below illustrates a common assignment of Denavit-Hartenberg convention to the UR5 robot (shown with all joint angles at 0).

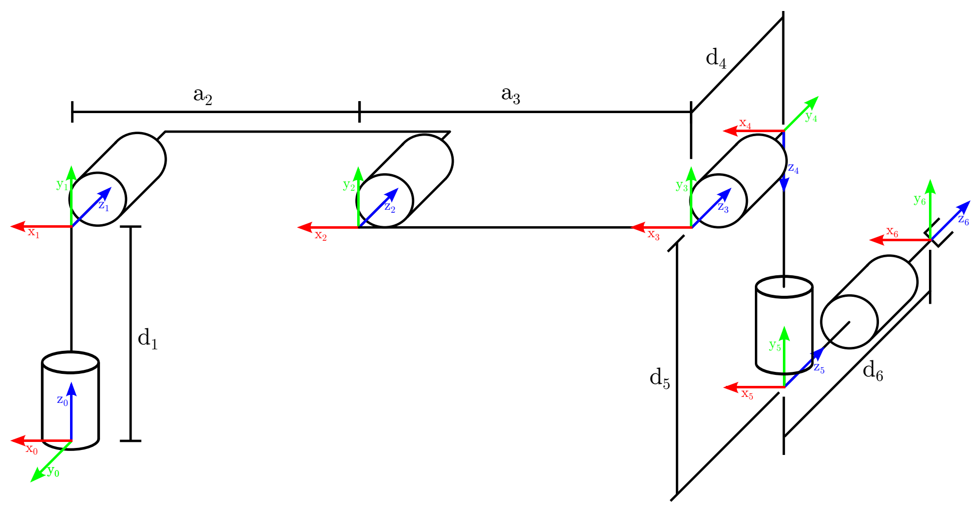


Figure 1: D-H Convention Frame Assignment

Joint	a	α	d	θ
1	0	$\pi/2$	d_1	θ_1
2	$-a_2$	0	0	θ_2
3	$-a_3$	0	0	θ_3
4	0	$\pi/2$	d_4	θ_4
5	0	$-\pi/2$	d_5	θ_5
6	0	0	d_6	θ_6

Table 1: D-H Parameters

As with any 6-DOF robot, the homogeneous transformation from the base frame to the gripper can be defined as follows:

$$T_6^0(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = T_1^0(\theta_1)\, T_2^1(\theta_2)\, T_3^2(\theta_3)\, T_4^3(\theta_4)\, T_5^4(\theta_5)\, T_6^5(\theta_6) \tag{1}$$

Also remember that a homogenous transformation T_j^i has the following form:

$$T_j^i = \begin{bmatrix} R_j^i & \vec{P}_j^i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_x & y_x & z_x & (P_j^i)_x \\ x_y & y_y & z_y & (P_j^i)_y \\ x_z & y_z & z_z & (P_j^i)_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where $\vec{P}_j^i$ is the translation from frame i to frame j and each column of R_j^i is the projection of one of the axes of frame j onto the axes of frame i (i.e. $[x_x, x_y, x_z]^T \equiv x_j^i$).

.[**?**]

Inverse Kinematics

The first step to solving the inverse kinematics is to find θ_1. To do so, we must first find the location of the 5^{th} coordinate frame with respect to the base frame, P_5^0. As illustrated in Figure 2 below, we can do so by translating by d_6 in the negative z direction from the 6^{th} frame.

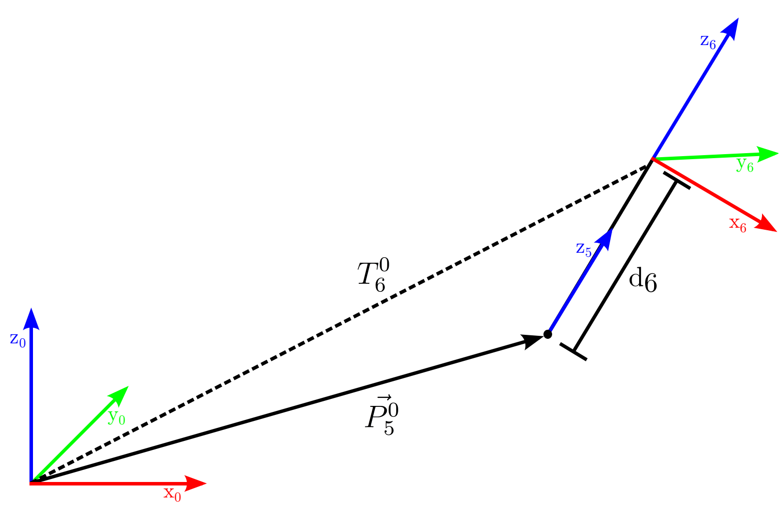


Figure 2: Finding the Origin of the 5^{th} Frame

This is equivalent to the following operation:

$$\vec{P_5^0} = T_6^0 \begin{bmatrix} 0 \\ 0 \\ -d_6 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

With the location of the 5^{th} frame, we can draw an overhead view of the robot:

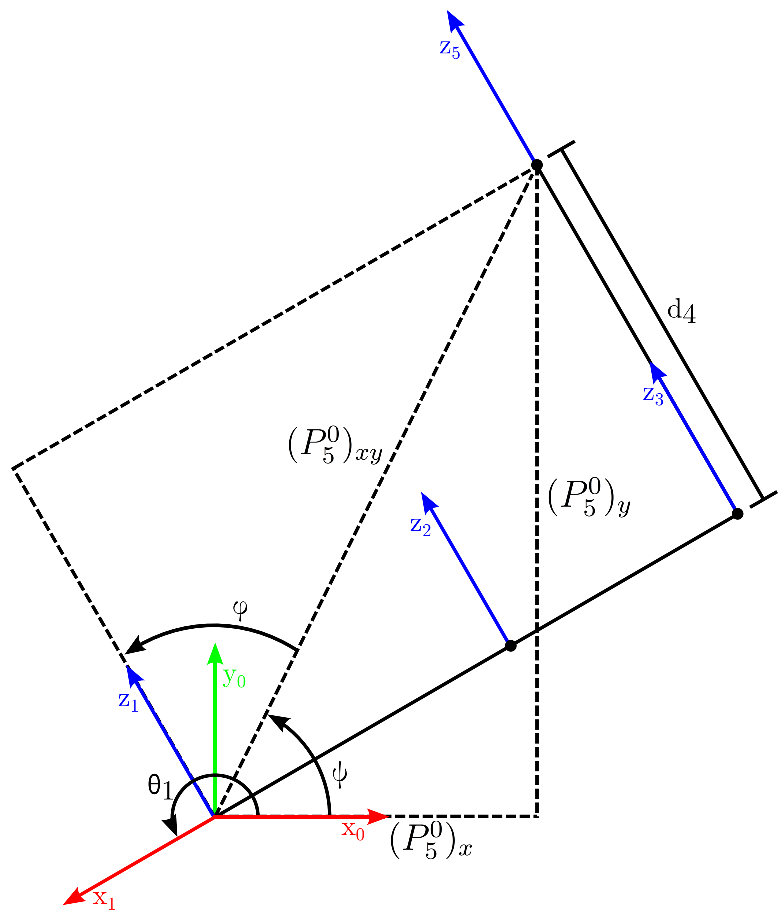


Figure 3: Finding θ_1

From Figure 3, we can see that $\theta_1 = \psi + \phi + \frac{\pi}{2}$ where

$$\psi = \text{atan2}\left((P_5^0)_y, (P_5^0)_x\right) \tag{4}$$

$$\phi = \pm \arccos\left(\frac{d_4}{(P_5^0)_{xy}}\right) = \pm \arccos\left(\frac{d_4}{\sqrt{{(P_5^0)_x}^2 + {(P_5^0)_y}^2}}\right) \tag{5}$$

The two solutions for θ_1 correspond to the shoulder being either “left” or “right,”.

Knowing θ_1, we can now solve for θ_5. Once again we draw an overhead view of the robot, but this time we consider location of the 6^{th} frame with respect to the 1^{st}.

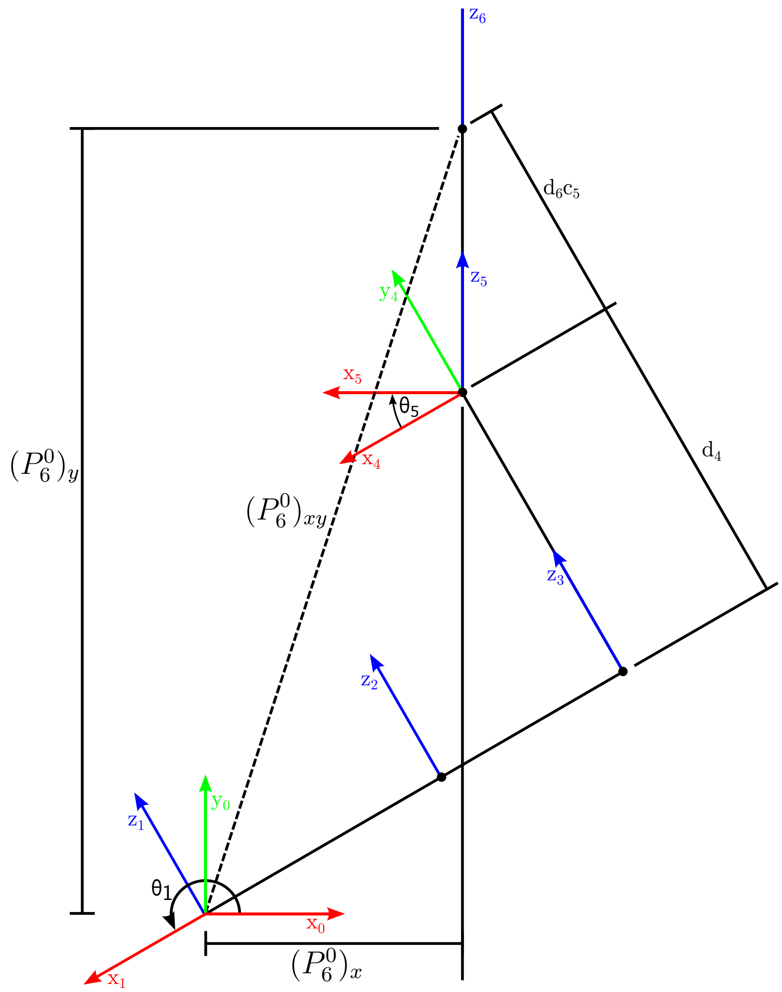


Figure 4: Finding θ_5

We can see that $(P_6^1)_z = d_6 \cos(\theta_5) + d_4$, where $(P_6^1)_z = (P_6^0)_x \sin(\theta_1) - (P_6^0)_y \cos(\theta_1)$. Solving for θ_5,

$$\theta_5 = \pm \arccos\left(\frac{(P_6^1)_z - d_4}{d_6}\right) \tag{6}$$

Once again, there are two solutions. These solutions correspond to the wrist being "down" and "up." Equation 6 also indicates that θ_5 is only well-defined when $\frac{(P_6^1)_z - d_4}{d_6} \leq 1$.

Figure 5 illustrates that, ignoring translations between frames, z_1 can be represented with respect to frame 6 as a unit vector defined with spherical coordinates. We can find the x and y components of of this vector by projecting it onto the x-y plane and then onto the x or y axes.

Next we find transformation from frame 6 to frame 1,

$$T_1^6 = ((T_1^0)^{-1}\, T_6^0)^{-1} \tag{7}$$

Remembering the structure the of the first three columns of the homogenous transformation T_1^6(see Equation 2), we can form the following equalities:

$$-\sin(\theta_6)\sin(\theta_5) = z_y \tag{8}$$

$$\cos(\theta_6)\sin(\theta_5) = z_x \tag{9}$$

Solving for θ_6,

$$\theta_6 = \operatorname{atan2}\left(\frac{-z_y}{\sin(\theta_5)}, \frac{z_x}{\sin(\theta_5)}\right) \tag{10}$$

Equation 10 shows that θ_6 is not well-defined when $\sin(\theta_5) = 0$ or when $z_x, z_y = 0$. We can see from Figure 5 that these conditions are actually the same. In this configuration joints 2, 3, 4, and 6 are parallel. As a result, there four degrees of freedom to determine the position and rotation of the end-effector in the plane, resulting in an infinite number of solutions. In this case, a desired value for q_6 can be chosen to reduce the number of degrees of freedom to three.

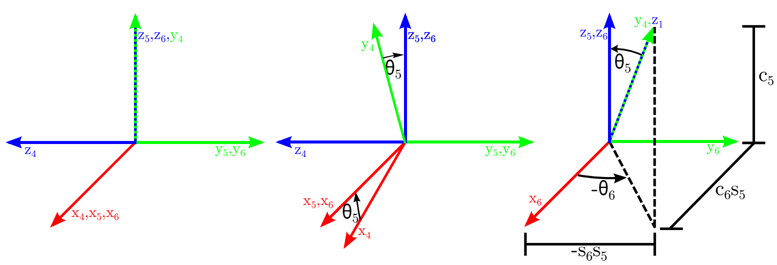


Figure 5: Finding θ_6

We can now consider the three remaining joints as forming a planar 2R manipulator. First we will find the location of frame 3 with respect to frame 1. This is done as follows:

$$T_4^1 = T_6^1\, T_4^6 = T_6^1\, (T_5^4\, T_6^5)^{-1} \tag{11}$$

$$\vec{P_3^1} = T_4^1 \begin{bmatrix} 0 \\ -d_4 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{12}$$

Now we can draw the plane containing frames 1-3, as shown in Figure 6. We can see that

$$\cos(\xi) = \frac{||\vec{P_3^1}||^2 - a_2^2 - a_3^2}{2a_2a_3} \tag{13}$$

with use of the law of cosines.

$$\begin{aligned} \cos(\xi) &= -\cos(\pi - \xi) \\ &= -\cos(-\theta_3) \\ &= \cos(\theta_3) \end{aligned} \tag{14}$$

Combining 13 and 14

$$\theta_3 = \pm \arccos\left(\frac{||\vec{P_3^1}||^2 - a_2^2 - a_3^2}{2a_2a_3}\right) \tag{15}$$

Figure 6 also shows that

$$\theta_2 = -(\delta - \epsilon) \tag{16}$$

where $\delta = \text{atan2}((P_3^1)_y, -(P_3^1)_x)$ and ϵ can be found via law of sines:

$$\frac{\sin(\xi)}{||\vec{P_3^1}||} = \frac{\sin(\epsilon)}{a_3} \tag{17}$$

Combining 16 and 17

$$\theta_2 = -\text{atan2}((P_3^1)_y, -(P_3^1)_x) + \arcsin\left(\frac{a_3\sin(\theta_3)}{||\vec{P_3^1}||}\right) \tag{18}$$

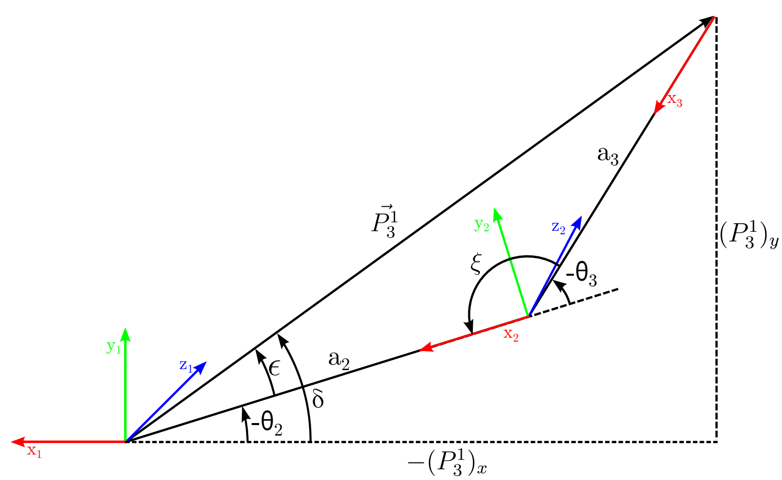


Figure 6: Finding θ_2 and θ_3

Notice that there are two solutions for θ_2 and θ_3. These solutions are known as "elbow up" and "elbow down."

The final step to solving the inverse kinematics is to solve for θ_4. First we want to find T_4^3:

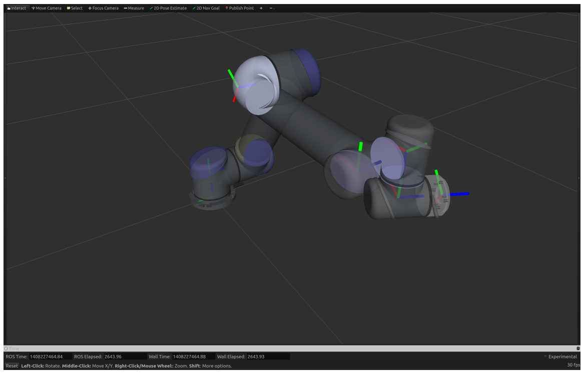
$$T_4^3 = T_1^3 \, T_4^1 = (T_2^1 \, T_3^2)^{-1} T_4^1 \tag{19}$$

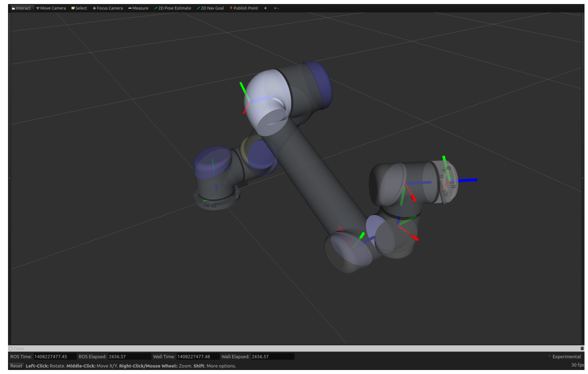
Using the first column of T_4^3,

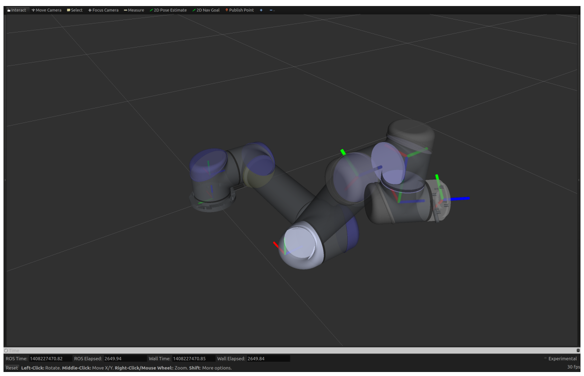
$$\theta_4 = \text{atan2}(x_y, x_x) \tag{20}$$

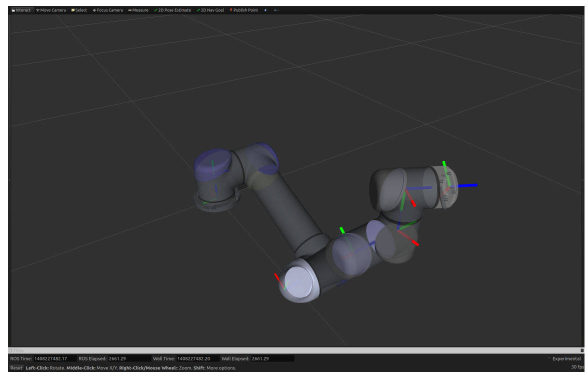
Below are figures which show the eight solutions for one end-effector position/orientation.

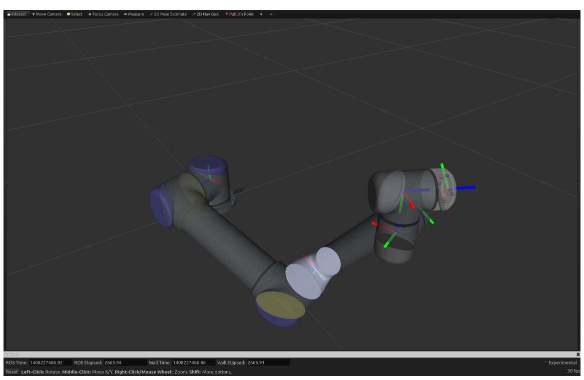
This solution was adapted from an existing solution written by Kelsey P. Hawkins

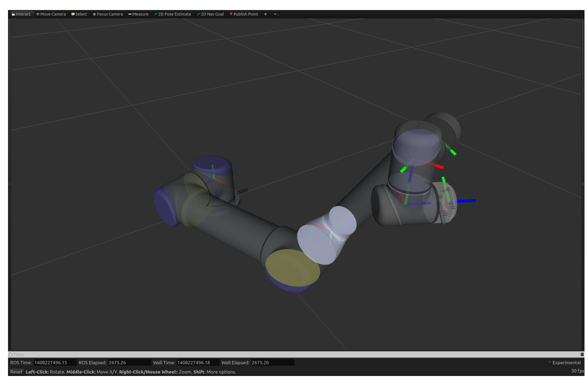

(a) Shoulder Left, Elbow Up, Wrist Down

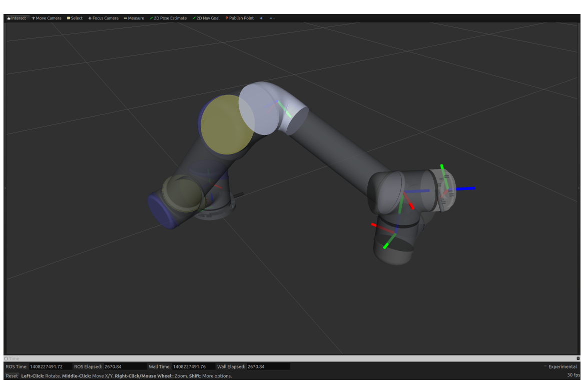

(b) Shoulder Left, Elbow Up, Wrist Up

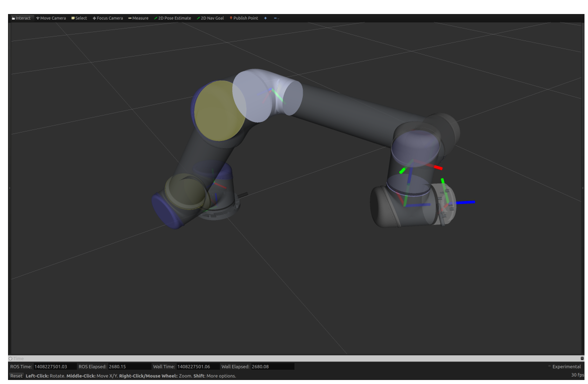

(c) Shoulder Left, Elbow Down, Wrist Down


(d) Shoulder Left, Elbow Down, Wrist Up


(e) Shoulder Right, Elbow Down, Wrist Up


(f) Shoulder Right, Elbow Down, Wrist Down


(g) Shoulder Right, Elbow Up, Wrist Up


(h) Shoulder Right, Elbow Up, Wrist Down